SOME COMPOSITION DETERMINANTS

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Abstract. We compute several parametric determinants in which rows and columns are indexed by compositions, where the entries are either products of binomial coefficients or products of powers. These results generalize previous determinant evaluations due to the first and fourth author [*SIAM J. Matrix Anal. Appl.* **23** (2001), 459–471] and ["A polynomial generalization of the power-compositions determinant," *Linear Multilinear Algebra* (to appear)], and they prove two conjectures of the second author ["Advanced determinant calculus: a complement," preliminary version].

1. Introduction

A composition of a non-negative integer n is a vector $(\alpha_1, \alpha_2, \dots, \alpha_k)$ of non-negative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, for some k. For a fixed k, let $\mathcal{C}(n,k)$ denote the corresponding set of compositions of n. While working on a problem in global optimisation, two of the authors [1] discovered the following surprising determinant evaluation. It allowed them to show how to explicitly express a multivariable polynomial as a *difference of convex functions*. In the statement, we use standard multi-index

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notation: if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_k)$ are two compositions, we let

$$\boldsymbol{\alpha}^{\boldsymbol{\beta}} := \alpha_1^{\beta_1} \alpha_2^{\beta_2} \cdots \alpha_k^{\beta_k},$$

where 0^0 is interpreted as 1.

Theorem 1. *For any positive integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left(\boldsymbol{\alpha}^{\boldsymbol{\beta}}\right) = n^{\binom{n+k-1}{k}+k-1} \prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+k-i-1}{k-2}}. \tag{1.1}$$

In the preliminary version [5] of [6], the second author observed empirically that there seemed to be a polynomial generalisation of this theorem.

Conjecture 2 ([5, Conjecture 57])**.** *For any positive integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left((x+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right) = (kx+n)^{\binom{n+k-1}{k}} n^{k-1} \prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+k-i-1}{k-2}}, \tag{1.2}$$

where x is a variable, and $x + \boldsymbol{\alpha}$ is short for $(x + \alpha_1, x + \alpha_2, \dots, x + \alpha_k)$.

Morover, he also worked out a binomial version of this conjecture. Extending our multi-index notation, let

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_k}{\beta_k}.$$

Conjecture 3 ([5, Conjecture 58])**.** *For any positive integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left(\binom{x+\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\right) = \frac{\prod_{i=0}^{n-1} (kx+n+k+i)^{\binom{k+i-1}{k-1}}}{\prod_{i=1}^{n} i^{\binom{n+k-i-1}{k-1}}}, \tag{1.3}$$

where x is a variable, and $x+\boldsymbol{\alpha}+\boldsymbol{\beta}$ is short for $(x+\alpha_1+\beta_1, x+\alpha_2+\beta_2, \dots, x+\alpha_k+\beta_k)$.

In the recent paper [2], the first and fourth author succeeded to prove Conjecture 2. In fact, they established the following multivariable generalisation.

Theorem 4. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ be a vector of indeterminates. Then, for any positive integers n and k, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right) = (|\mathbf{x}|+n)^{\binom{n+k-1}{k}} \prod_{i=1}^{n} i^{(k-1)\binom{n+k-i-1}{k-1}}, \tag{1.4}$$

where $\mathbf{x}+\boldsymbol{\alpha}$ is short for $(x_1+\alpha_1, x_2+\alpha_2, \dots, x_k+\alpha_k)$, and where $|\mathbf{x}| = x_1+x_2+\cdots+x_k$.

The purpose of this paper is to, in some sense, explain the miraculous existence of all these formulae. We do this by introducing further variables, $\lambda_1, \lambda_2, \dots, \lambda_k$, in the binomial determinant in (1.3), and by proving an evaluation theorem for the resulting determinant. All the afore-mentioned determinant evaluations are then special cases, respectively limit cases, of this new theorem. To be precise, the main result of this paper is the following determinant evaluation.

Theorem 5. *Let* $\mathbf{x} = (x_1, x_2, \dots, x_k)$ *and* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *be vectors of indeterminates. Then, for any positive integers* n *and* k*, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}} \frac{\displaystyle\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n+\sum_{j=1}^{k}\left(\frac{x_j}{\lambda_j}-\frac{\varepsilon_j}{\lambda_j}\right)\right)}{\displaystyle\prod_{i=1}^{n} i^{\binom{n+k-i-1}{k-1}}}, \tag{1.5}$$

where $\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}$ *is short for* $(x_1+\lambda_1\alpha_1, x_2+\lambda_2\alpha_2, \dots, x_k+\lambda_k\alpha_k)$.

Using the elementary property $\binom{X}{m} = (-1)^m\binom{-X+m-1}{m}$ of binomial coefficients, we show that an equivalent way to write the same result is as follows.

Theorem 6. *Let* $\mathbf{x} = (x_1, x_2, \dots, x_k)$ *and* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *be vectors of indeterminates. Then, with notation as in Theorem* 5*, for any positive integers* n *and* k *we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}} \frac{\displaystyle\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n+\sum_{j=1}^{k}\left(\frac{x_j+1}{\lambda_j}+\frac{\varepsilon_j}{\lambda_j}\right)\right)}{\displaystyle\prod_{i=1}^{n} i^{\binom{n+k-i-1}{k-1}}}. \tag{1.6}$$

In order to see how Conjecture 3 is implied by these results, it is convenient to first state separately the special cases of Theorems 5 and 6 where all the λ_i's are identical. In that case, the product in the numerator on the right-hand sides of (1.5) and (1.6) can be rearranged by grouping together the factors corresponding to compositions $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ of $n-i$, which become identical. Taking into account that the number of such compositions is $\binom{n-i+k-1}{k-1}$, this yields the following two corollaries.

Corollary 7. *Let* $\mathbf{x} = (x_1, x_2, \dots, x_k)$ *be a vector of indeterminates, and let* λ *be an indeterminate. Then, for any positive integers* n *and* k*, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\binom{\mathbf{x}+\lambda\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) = \lambda^{(k-1)\binom{n+k-1}{k}}\prod_{i=1}^{n}\left(\frac{|\mathbf{x}|+(\lambda-1)n+i}{i}\right)^{\binom{n+k-i-1}{k-1}}, \tag{1.7}$$

where $\mathbf{x}+\lambda\boldsymbol{\alpha}$ *is short for* $(x_1+\lambda\alpha_1, x_2+\lambda\alpha_2, \dots, x_k+\lambda\alpha_k)$*, and where* $|\mathbf{x}| = x_1+x_2+\dots+x_k$*, as before.*

Corollary 8. *Let* $\mathbf{x} = (x_1, x_2, \dots, x_k)$ *be a vector of indeterminates, and let* λ *be an indeterminate. Then, with notation as in Corollary* 7*, for any positive integers* n *and* k *we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\binom{\mathbf{x}+\lambda\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\right) = \lambda^{(k-1)\binom{n+k-1}{k}}\prod_{i=1}^{n}\left(\frac{|\mathbf{x}|+(\lambda+1)n+k-i}{i}\right)^{\binom{n+k-i-1}{k-1}}. \tag{1.8}$$

Clearly, Conjecture 3 is the special case of the above corollary where $\lambda = 1$ and $x_i = x$ for all i.

Theorem 4 is also implied by Theorem 5. To see this, we shall show that, by extracting the highest homogeneous component in (1.5) (this could also be realised by an appropriate limit), we obtain the following corollary.

Corollary 9. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be vectors of indeterminates. Then, with notation as in Theorem 5, for any positive integers n and k we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left((\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}} \left(n+\sum_{j=1}^{k}\frac{x_j}{\lambda_j}\right)^{\binom{n+k-1}{k}} \prod_{i=1}^{n} i^{(k-1)\binom{n+k-i-1}{k-1}}. \quad (1.9)$$

The corresponding special case where all the λ_i's are identical is the following.

Corollary 10. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ be a vector of indeterminates, and let λ be an indeterminate. Then, with notation as in Corollary 7, for any positive integers n and k we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)} \left((\mathbf{x}+\lambda\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right) = \lambda^{(k-1)\binom{n+k-1}{k}} \left(|\mathbf{x}|+\lambda n\right)^{\binom{n+k-1}{k}} \prod_{i=1}^{n} i^{(k-1)\binom{n+k-i-1}{k-1}}. \quad (1.10)$$

Clearly, Theorem 4 is the special case $\lambda = 1$ of this corollary.

In the next section, we give proofs of Theorems 5 and 6, and of Corollary 9 (and, thus, of Corollaries 7, 8 and 10 also). In contrast to the inductive procedure in [1, 2] that was used in the original proofs of Theorems 1 and 4, our proof is based on the "identification of factors" technique (see [4, Sec. 2.4]). As it turns out, the crucial identity in our proof is the multivariate version of the Chu–Vandermonde summation formula (see Lemma 11). Finally, in the last section, we derive analogues of Theorems 5 and 6, and of Corollary 9 for the subdeterminants in which we restrict the rows and columns to compositions of n with exactly k *positive* summands.

2. The proofs

Lemma 11. *Let $\mathbf{x} = (x_1, x_2, \dots, x_k)$ be a vector of indeterminates, and let n and k be non-negative integers. Then*

$$\sum_{\boldsymbol{\delta}\in\mathcal{C}(n,k)} \binom{\mathbf{x}}{\boldsymbol{\delta}} = \sum_{\delta_1+\cdots+\delta_k=n} \binom{\mathbf{x}}{\boldsymbol{\delta}} = \binom{|\mathbf{x}|}{n}.$$

Proof. The Chu–Vandermonde summation formula (see e.g. [3, Sec. 5.1, (5.27)]) reads

$$\sum_{r=0}^{s} \binom{M}{r}\binom{N}{s-r} = \binom{M+N}{s}.$$

On the basis of this formula, the assertion of the lemma is easily proved by induction on k. □

Proof of Theorem 5. We prove the theorem by the identification of factors method described in [4, Sec. 2.4]. For convenience, let us write $M(n,k)$ for the matrix of which we want to compute the determinant, that is,

$$M(n,k)=\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right)_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}.$$

Step 1. The term

$$T(n,k,\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon}):=n\prod_{j=1}^{k}\lambda_j+\sum_{t=1}^{k}x_t\prod_{\substack{j=1\\ j\neq t}}^{k}\lambda_j-\sum_{t=1}^{k}\varepsilon_t\prod_{\substack{j=1\\ j\neq t}}^{k}\lambda_j$$

divides $\det M(n,k)$ *for any composition* $\boldsymbol{\varepsilon}\in\mathcal{C}(n-i,k)$*,* $1\le i\le n$*.* (It should be noted that $T(n,k,\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})$ is the factor corresponding to $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k$ in the product in the numerator on the right-hand side of (1.5), up to multiplication by $\prod_{j=1}^{k}\lambda_j$.) To prove this assertion, we find a vector in the kernel of

$$M(n,k)\Big|_{T(n,k,\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})=0}. \tag{2.1}$$

This kernel lives in the free vector space generated by the compositions in $\mathcal{C}(n,k)$. Given $\boldsymbol{\delta}\in\mathcal{C}(n,k)$, let us denote the corresponding element ("unit vector") in this vector space by $e_{\boldsymbol{\delta}}$. Then we claim that the vector

$$v_{\boldsymbol{\varepsilon}}:=\sum_{\boldsymbol{\delta}\in\mathcal{C}(i,k)}\left(\sum_{t=1}^{k}\frac{\delta_t}{\lambda_t}\right)\binom{\boldsymbol{\delta}+\boldsymbol{\varepsilon}}{\boldsymbol{\delta}}e_{\boldsymbol{\delta}+\boldsymbol{\varepsilon}} \tag{2.2}$$

is in the kernel of the matrix (2.1). To see this, we calculate, using Lemma 11 and the notation $\mathbf{u}_t=(0,\dots,0,1,0,\dots,0)$ (with the 1 in position t),

$$\text{coefficient of } e_\alpha \text{ in } M(n,k)\cdot v_{\boldsymbol{\varepsilon}}=\sum_{\boldsymbol{\delta}\in\mathcal{C}(i,k)}\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\delta}+\boldsymbol{\varepsilon}}\left(\sum_{t=1}^{k}\frac{\delta_t}{\lambda_t}\right)\binom{\boldsymbol{\delta}+\boldsymbol{\varepsilon}}{\boldsymbol{\delta}}$$

$$=\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\varepsilon}}\sum_{t=1}^{k}\sum_{|\boldsymbol{\delta}|=i}\frac{\delta_t}{\lambda_t}\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}-\boldsymbol{\varepsilon}}{\boldsymbol{\delta}}$$

$$=\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\varepsilon}}\sum_{t=1}^{k}\frac{1}{\lambda_t}(x_t+\lambda_t\alpha_t-\varepsilon_t)\sum_{|\boldsymbol{\delta}|=i}\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}-\boldsymbol{\varepsilon}-\mathbf{u}_t}{\boldsymbol{\delta}-\mathbf{u}_t}$$

$$=\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\varepsilon}}\left(n+\sum_{t=1}^{k}\left(\frac{x_t}{\lambda_t}-\frac{\varepsilon_t}{\lambda_t}\right)\right)\binom{|\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}|-(n-i)-1}{i-1}.$$

Since $i\ge 1$, the occurrence of the factor in the middle implies

$$M(n,k)\Big|_{T(n,k,\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})=0}\cdot v_{\boldsymbol{\varepsilon}}=0.$$

Step 2. Comparison of degrees. By inspection, the (total) degree in the x_i's and λ_i's of the determinant on the left-hand side of (1.5) is at most $n\cdot|\mathcal{C}(n,k)|=n\binom{n+k-1}{n}$. On the other hand, the degree in the x_i's and λ_i's of the right-hand side of (1.5) is equal to $k\binom{n+k-1}{k}=n\binom{n+k-1}{n}$.

Since the degree bound on the determinant is the same as the degree of the right-hand side of (1.5), the determinant must be equal to the right-hand side up to a possible multiplicative constant which is independent of the x_i's and λ_i's. We conclude that

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right)=\operatorname{const}(n,k)\left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}}\\ \times\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n+\sum_{j=1}^{k}\left(\frac{x_j}{\lambda_j}-\frac{\varepsilon_j}{\lambda_j}\right)\right), \tag{2.3}$$

where $\operatorname{const}(n,k)$ is independent of $\mathbf{x}$ and $\boldsymbol{\lambda}$.

Step 3. Computation of the multiplicative constant. If we set $x_i=0$ and $\lambda_i=1$ for all i, then the determinant on the left-hand side of (1.5) becomes triangular with 1s on the diagonal. Thus, we obtain that

$$1=\operatorname{const}(n,k)\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n-\sum_{j=1}^{k}\varepsilon_j\right)=\operatorname{const}(n,k)\prod_{i=1}^{n}i^{|\mathcal{C}(n-i,k)|}.$$

This implies that

$$\operatorname{const}(n,k)=\prod_{i=1}^{n}i^{-\binom{n+k-i-1}{k-1}},$$

completing the proof of the theorem. □

Proof of the equivalence of Theorem 5 and 6. If we replace λ_i by $-\lambda_i$ and x_i by $-x_i-1$ for all i in (1.5), and then use the identity $\binom{X}{m}=(-1)^m\binom{-X+m-1}{m}$, then we obtain

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left((-1)^n\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\right)=\\ =(-1)^{k\binom{n+k-1}{k}}\left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}}\frac{\displaystyle\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n+\sum_{j=1}^{k}\left(\frac{x_j+1}{\lambda_j}+\frac{\varepsilon_j}{\lambda_j}\right)\right)}{\displaystyle\prod_{i=1}^{n}i^{\binom{n+k-i-1}{k-1}}}. \tag{2.4}$$

Except for the signs, this is exactly (1.6). However, we have

$$n\cdot|\mathcal{C}(n,k)|+k\binom{n+k-1}{k}=n\binom{n+k-1}{k-1}+k\binom{n+k-1}{k}=2\frac{(n+k-1)!}{(k-1)!\,(n-1)!}.$$

Since this is an even number, the signs in (2.4) do indeed cancel. □

Proof of Corollary 9. The right-hand and left-hand sides of (1.5) are both polynomials in the λ_i's and the x_i's. As we already observed in Step 2 of the first proof of Theorem 5, the (total) degree in the λ_i's and the x_i's of the determinant on the left-hand side is equal to

$$n \cdot |\mathcal{C}(n,k)| = n\binom{n+k-1}{k-1} = k\binom{n+k-1}{k},$$

and the degree in the λ_i's and the x_i's of the expression on the right-hand side is exactly the same value. Therefore, if we extract the homogeneous parts in the λ_i's and the x_i's of degree $k\binom{n+k-1}{k}$ in (1.5), we obtain

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left(\frac{(\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha})^{\boldsymbol{\beta}}}{\boldsymbol{\beta}!}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}} \frac{\prod\limits_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n}}\left(n+\sum\limits_{j=1}^{k}\frac{x_j}{\lambda_j}\right)}{\prod\limits_{i=1}^{n} i^{\binom{n+k-i-1}{k-1}}},$$

where $\boldsymbol{\beta}! = \prod_{i=1}^{k}\beta_i!$, or, equivalently,

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n,k)}\left((\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n+k-1}{k}} \frac{\left(n+\sum_{j=1}^{k}\frac{x_j}{\lambda_j}\right)^{\sum_{i=0}^{n-1}|\mathcal{C}(i,k)|}}{\prod_{i=1}^{n} i^{\binom{n+k-i-1}{k-1}}}\left(\prod_{\boldsymbol{\beta}\in\mathcal{C}(n,k)}\boldsymbol{\beta}!\right).$$

Since

$$\sum_{i=0}^{n-1}|\mathcal{C}(i,k)| = \sum_{i=0}^{n-1}\binom{i+k-1}{k-1} = \binom{n+k-1}{k},$$

the only missing piece for the proof of the corollary is the verification of the identity

$$\prod_{\boldsymbol{\beta}\in\mathcal{C}(n,k)}\boldsymbol{\beta}! = \prod_{\beta_1+\dots+\beta_k=n}\beta_1!\cdots\beta_k! = \prod_{i=1}^{n} i^{k\binom{n+k-i-1}{k-1}}. \tag{2.5}$$

This can, for example, be done by induction on $n+k$, by using the obvious recurrence

$$\Pi(n,k) = \prod_{i=0}^{n}\left(i!^{|\mathcal{C}(n-i,k-1)|}\,\Pi(n-i,k-1)\right),$$

where

$$\Pi(n,k) = \prod_{\beta_1+\dots+\beta_k=n}\beta_1!\cdots\beta_k!.$$

$\square$

3. Determinants for compositions with only positive parts

Let $\mathcal{C}^*(n,k)$ denote the set of all compositions of n with exactly k positive summands. Then we have the following theorem.

Theorem 12. *Let $\mathbf{x}=(x_1,x_2,\dots,x_k)$ and $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_k)$ be vectors of indeterminates. Then, for any positive integers n and k, $n\ge k$, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}^*(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n-1}{k}}\left(\prod_{i=1}^{k}\prod_{j=1}^{n-k+1}\left(\frac{x_i+\lambda_i j}{j}\right)^{\binom{n-j-1}{k-2}}\right)$$

$$\times \frac{\displaystyle\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n-k}} \left(n+\sum_{j=1}^k\left(\frac{x_j-1}{\lambda_j}-\frac{\varepsilon_j}{\lambda_j}\right)\right)}{\displaystyle\prod_{i=1}^n i^{\binom{n-i-1}{k-1}}}. \tag{3.1}$$

Proof. Let $\mathbf{1}$ be the k-vector with all entries equal to 1. The mapping $\mathcal{C}^*(n,k)\to\mathcal{C}(n-k,k)$ defined by $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\mapsto\boldsymbol{\alpha}-\mathbf{1}=(\alpha_1-1,\dots,\alpha_k-1)$ is bijective. Thus, we have

$$\begin{aligned}
\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}^*(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) &= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}^*(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}+\boldsymbol{\lambda}(\boldsymbol{\alpha}-\mathbf{1})}{\boldsymbol{\beta}-\mathbf{1}+\mathbf{1}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n-k,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}+\mathbf{1}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n-k,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}-\mathbf{1}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\prod_{i=1}^k\frac{x_i+\lambda_i+\lambda_i\alpha_i}{\beta_i+1}\right)\\
&= \left(\prod_{\boldsymbol{\alpha}\in\mathcal{C}(n-k,k)}\prod_{i=1}^k\frac{x_i+\lambda_i(\alpha_i+1)}{\alpha_i+1}\right)\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}(n-k,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}-\mathbf{1}+\boldsymbol{\lambda}\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right).
\end{aligned}$$

In the product, a ratio $(x_i+\lambda_i(j+1))/(j+1)$ appears as many times as there are compositions in $\mathcal{C}(n-k-j,k-1)$, i.e., $\binom{n-j-2}{k-2}$ times. Thus,

$$\begin{aligned}
\prod_{\boldsymbol{\alpha}\in\mathcal{C}(n-k,k)}\prod_{i=1}^k\frac{x_i+\lambda_i(\alpha_i+1)}{\alpha_i+1} &= \prod_{i=1}^k\prod_{j=0}^{n-k}\left(\frac{x_i+\lambda_i(j+1)}{j+1}\right)^{\binom{n-j-2}{k-2}}\\
&= \prod_{i=1}^k\prod_{j=1}^{n-k+1}\left(\frac{x_i+\lambda_i j}{j}\right)^{\binom{n-j-1}{k-2}}.
\end{aligned}$$

Substituting this and using Theorem 5 with n replaced by $n-k$ and $\mathbf{x}$ replaced by $\mathbf{x}+\boldsymbol{\lambda}-\mathbf{1}$ to evaluate the determinant, we obtain the desired formula. $\square$

By replacing λ_i by $-\lambda_i$ and x_i by $-x_i-1$ for all i in (3.1), and then using the identity $\binom{X}{m}=(-1)^m\binom{-X+m-1}{m}$, we obtain the following equivalent form of Theorem 12.

Theorem 13. *Let $\mathbf{x}=(x_1,x_2,\dots,x_k)$ and $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_k)$ be vectors of indeterminates. Then, for any positive integers n and k, $n\ge k$, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}^*(n,k)}\left(\binom{\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\beta}}\right) = \left(\prod_{j=1}^k\lambda_j\right)^{\binom{n-1}{k}}\left(\prod_{i=1}^k\prod_{j=1}^{n-k+1}\left(\frac{x_i+\lambda_i j+1}{j}\right)^{\binom{n-j-1}{k-2}}\right)$$

$$\times \frac{\displaystyle\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n-k}} \left(n+\sum_{j=1}^{k}\left(\frac{x_j+2}{\lambda_j}+\frac{\varepsilon_j}{\lambda_j}\right)\right)}{\displaystyle\prod_{i=1}^{n} i^{\binom{n-i-1}{k-1}}}. \quad (3.2)$$

Extracting the highest homogeneous component in Theorem 12, we obtain the following analogue of Corollary 10.

Corollary 14. *Let* $\mathbf{x}=(x_1,x_2,\dots,x_k)$ *and* $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_k)$ *be vectors of indeterminates. Then, for any positive integers* n *and* k*,* $n\ge k$*, we have*

$$\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{C}^*(n,k)}\left((\mathbf{x}+\boldsymbol{\lambda}\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right)=\left(\prod_{j=1}^{k}\lambda_j\right)^{\binom{n-1}{k}}\left(\prod_{i=1}^{k}\prod_{j=1}^{n-k+1}(x_i+\lambda_i j)^{\binom{n-j-1}{k-2}}\right)$$
$$\times\left(\prod_{\substack{\varepsilon_1,\dots,\varepsilon_k\ge 0\\ \varepsilon_1+\dots+\varepsilon_k<n-k}}\left(n+\sum_{j=1}^{k}\frac{x_j}{\lambda_j}\right)\right)\prod_{i=1}^{n-k} i^{(k-1)\binom{n-i-1}{k-1}}. \quad (3.3)$$

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