ON A LINEAR FORM FOR CATALAN'S CONSTANT

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Dedicated to George Andrews

Abstract. It is shown how Andrews' multidimensional extension of Watson's transformation between a very-well-poised ${}_8\phi_7$-series and a balanced ${}_4\phi_3$-series can be used to give a straightforward proof of a conjecture of Zudilin and the second author on the arithmetic behaviour of the coefficients of certain linear forms of 1 and Catalan's constant. This proof is considerably simpler and more stream-lined than the first proof, due to the second author.

1. Introduction

Andrews' multidimensional extension [1, Theorem 4] of Watson's transformation between a very-well-poised ${}_8\phi_7$-series and a balanced ${}_4\phi_3$-series [6, (2.5.1); Appendix (III.18)] in its full beauty reads

$$\sum_{k=0}^{n} \frac{(a;q)_k\,(q\sqrt{a};q)_k\,(-q\sqrt{a};q)_k\,(b_1;q)_k\,(c_1;q)_k\cdots(b_{m+1};q)_k\,(c_{m+1};q)_k\,(q^{-n};q)_k}{(\sqrt{a};q)_k\,(-\sqrt{a};q)_k\,(qa/b_1;q)_k\,(qa/c_1;q)_k\cdots(qa/b_{m+1};q)_k\,(qa/c_{m+1};q)_k\,(q^{n+1}a;q)_k}$$
$$\cdot\left(\frac{a^{m+1}q^{m+1+n}}{b_1c_1\cdots b_{m+1}c_{m+1}}\right)^k$$
$$= \frac{(qa;q)_n\,(qa/b_{m+1}c_{m+1};q)_n}{(qa/b_{m+1};q)_n\,(qa/c_{m+1};q)_n} \sum_{0\le i_1\le i_2\le\cdots\le i_m\le n} \frac{a^{i_1+\cdots+i_{m-1}}q^{i_1+\cdots+i_m}}{(b_2c_2)^{i_1}\cdots(b_mc_m)^{i_{m-1}}}$$
$$\cdot\frac{(q^{-n};q)_{i_m}}{(b_{m+1}c_{m+1}/aq^n;q)_{i_m}} \prod_{k=1}^{m} \frac{(qa/b_kc_k;q)_{i_k-i_{k-1}}\,(b_{k+1};q)_{i_k}\,(c_{k+1};q)_{i_k}}{(q;q)_{i_k-i_{k-1}}\,(qa/b_k;q)_{i_k}\,(qa/c_k;q)_{i_k}}, \quad (1.1)$$

where, by definition, $i_0 := 0$. Here, $(\alpha;q)_k = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{k-1})$ if $k \ge 1$ and $(\alpha;q)_0 = 1$. This formula has found important applications to the theory of partition identities (see [1]).

Remarkably, Andrews' formula has started a surprising new life recently. Its utility for proving arithmetic properties of coefficients of certain linear forms for values of the Riemann zeta function at integers was discovered by the authors in [9], and was also exploited in [10] for proving the equality of certain multiple integrals and hypergeometric series. Closely

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related are the applications given by Zudilin in [16, 17]. The afore-mentioned articles make actually "only" use of the $q = 1$ special case of (1.1) (see (4.2) below for the explicit statement of that special case). The line of argument developed in [9] has been extended to the q-case by Jouhet and Mosaki in [8] to establish irrationality results for values of a q-analogue of the zeta function. Moreover, Guo, Jouhet and Zeng [7] have extended Zudilin's work in [16] to the q-case, together with further applications of Andrews' formula (1.1). In a completely different field, Beliakova, Bühler and Lê [3, 4, 11] have exploited (1.1) in the study of quantum invariants of manifolds. Finally, Andrews himself returned to his identity after over 30 years to prove deep partition theorems in [2].

The purpose of the present paper is to add another item to this list of applications of Andrews' formula. More precisely, we show how the ideas from [9] lead to an alternative proof of a conjecture from [13] on the arithmetic behaviour of the coefficients in certain linear forms of 1 and *Catalan's constant* $G = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}$. It is considerably simpler and more stream-lined than the first proof [12] by one of the authors, which used a somewhat indirect method based on Padé approximations. A partial, "asymptotic," proof had been given earlier by Zudilin in [15].

We give a precise statement of the conjecture in the next section, where we also derive explicit expressions for the coefficients $\mathbf{a}_n$ and $\mathbf{b}_n$ in the linear forms of 1 and Catalan's constant. The arithmetic claim for the coefficient $\mathbf{a}_n$ is then proved in Section 3 with the help of a limit case of Whipple's transformation between a very-well-poised ${}_7F_6$-series and a balanced ${}_4F_3$-series (the latter being the $q = 1$ special case of the afore-mentioned transformation formula of Watson). The arithmetic claim for the coefficient $\mathbf{b}_n$ is proved in Section 4 with the help of the $q = 1$ special case of Andrews' formula (1.1), given explicitly in (4.2).

2. A linear form for Catalan's constant

Let us consider the series

$$\mathbf{G}_n = n! \sum_{k=1}^{\infty} (-1)^k \left(k + \frac{n-1}{2}\right) \frac{(k-n)_n (k+n)_n}{\left(k - \frac{1}{2}\right)_{n+1}^3}, \tag{2.1}$$

where the *Pochhammer symbol* $(\alpha)_k$ is defined by $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ if $k \geq 1$ and $(\alpha)_0 = 1$. By applying a partial fraction decomposition with respect to k to the summand, and by performing the appropriate summations, it is not difficult to see (cf. [5, Sec. 1.4] for details on this kind of calculation) that

$$\mathbf{G}_n = \mathbf{a}_n G - \mathbf{b}_n,$$

where

$$\mathbf{a}_n = 4(-1)^{n-1} \sum_{j=0}^{n} \frac{\partial}{\partial \varepsilon} \Bigg(\left(\frac{n}{2} - j + \varepsilon\right) \left(\frac{n!}{(1-\varepsilon)_j \, (1+\varepsilon)_{n-j}} \right)^3 \cdot \binom{n+j-\varepsilon-\frac{1}{2}}{n} \binom{2n-j+\varepsilon-\frac{1}{2}}{n} \Bigg) \Bigg|_{\varepsilon=0}, \qquad (2.2)$$

and

$$\mathbf{b}_n = (-1)^n \sum_{j=0}^{n} \sum_{e=1}^{3} \frac{1}{(3-e)!} \frac{\partial^{3-e}}{\partial j^{3-e}} \Bigg(\left(\frac{n}{2} - j + \varepsilon\right) \left(\frac{n!}{(1-\varepsilon)_j \, (1+\varepsilon)_{n-j}} \right)^3 \cdot \binom{n+j-\varepsilon-\frac{1}{2}}{n} \binom{2n-j+\varepsilon-\frac{1}{2}}{n} \Bigg) \Bigg|_{\varepsilon=0} \sum_{k=1}^{j} \frac{(-1)^k}{\left(k-\frac{1}{2}\right)^e}. \qquad (2.3)$$

Writing d_n for $\operatorname{lcm}(1, 2, \dots, n)$, it is easy to see by a standard approach (see [13, Sec. 5]) that $2^{4n} \mathrm{d}_{2n} \mathbf{a}_n$ and $2^{4n} \mathrm{d}_{2n}^3 \mathbf{b}_n$ are integers. Based on computer calculations, the second author and Zudilin conjectured however (cf. [13, p. 720]) that in fact even $2^{4n} \mathbf{a}_n$ and $2^{4n} \mathrm{d}_{2n}^2 \mathbf{b}_n$ are integers. While this is still too weak for proving the irrationality of Catalan's constant G, it is nevertheless an interesting and non-obvious observation which we shall prove in the two subsequent sections. This proof makes use of identities for (*generalised*) *hypergeometric series*, the latter being defined by

$${}_{q+1}F_q\left[\begin{matrix} \alpha_0, \alpha_1, \dots, \alpha_q \\ \beta_1, \dots, \beta_q \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k \, (\alpha_1)_k \cdots (\alpha_q)_k}{k! \, (\beta_1)_k \cdots (\beta_q)_k} z^k.$$

As we already mentioned in the Introduction, an earlier (but more involved) proof is due to one of the authors [12].

3. The coefficient $\mathbf{a}_n$

The purpose of this section is to prove the following theorem.

Theorem 1. *For all positive integers n, the number $2^{4n} \mathbf{a}_n$ is an integer.*

For accomplishing the proof of this theorem (as well as the proof of Theorem 2 in the following section), we need the following two arithmetic auxiliary facts (cf. [14, Sec. 7] and [13, Lemma 6], respectively). Following [14] (where this is attributed to Nesterenko), we shall call the expressions $R_1(\alpha, \beta; t)$ and $R_2(\alpha, \beta; t)$ in the two lemmas below *elementary bricks*.

Lemma 1. *Given integers α and β, let*

$$R_1(\alpha, \beta; t) = \begin{cases} \dfrac{(t+\beta)_{\alpha-\beta}}{(\alpha-\beta)!} & \text{if } \alpha \ge \beta, \\[2ex] \dfrac{(\beta-\alpha-1)!}{(t+\alpha)_{\beta-\alpha}} & \text{if } \alpha < \beta. \end{cases}$$

Then, for all integers α, β, k, H with $\alpha \ge \beta$ and $H \ge 0$, the number

$$\mathrm{d}_{\alpha-\beta}^{H} \cdot \frac{1}{H!}\frac{\partial^H}{\partial t^H} R_1(\alpha,\beta;t)\Big|_{t=-k}$$

is an integer. Furthermore, for all integers α, β, k, H with $\alpha \le k \le \beta - 1$ and $H \ge 0$, the number

$$\mathrm{d}_{\beta-\alpha-1}^{H} \cdot \frac{1}{H!}\frac{\partial^H}{\partial t^H} R_1(\alpha,\beta;t)(t+k)\Big|_{t=-k}$$

is an integer.

Lemma 2. *Given integers α and β with $\alpha \ge \beta$, let*

$$R_2(\alpha,\beta;t) = 2^{2(\alpha-\beta)}\frac{(t+\beta-\frac{1}{2})_{\alpha-\beta}}{(\alpha-\beta)!}.$$

Then, for all integers k and H with $H \ge 0$, the number

$$\mathrm{d}_{2(\alpha-\beta)}^{H} \cdot \frac{1}{H!}\frac{\partial^H}{\partial t^H} R_2(\alpha,\beta;t)\Big|_{t=-k}$$

is an integer.

In order to apply these two lemmas, we need an alternative expression for the coefficient $\mathbf{a}_n$, see the lemma below. The expression in (3.1) was already given in [12, Sec. 4.1]. Again, it was obtained there in a somewhat roundabout way. Here, the equality in the next-to-last displayed equation in [12, Sec. 4.2] is explained directly.

Lemma 3. *For all non-negative integers n, we have*

$$\mathbf{a}_n = -4\sum_{j=0}^{n}\binom{n}{j}\binom{n-\frac{1}{2}}{j}\binom{n+j-\frac{1}{2}}{j}. \tag{3.1}$$

Proof. We loosely follow analogous considerations in [9, Lemme 7].

Let H_m denote the m-th harmonic number, defined by $H_m = \sum_{j=1}^{m}\frac{1}{j}$. By abuse of notation, we “extend” harmonic numbers to half-integers m by defining $H_m = \sum_{j=1}^{\lceil m\rceil}\frac{1}{m-j+1}$. For example,

$$H_{5/2} = \frac{1}{5/2} + \frac{1}{3/2} + \frac{1}{1/2}.$$

We rewrite the expression for $\mathbf{a}_n$ given in (2.2) in the form

$$\mathbf{a}_n = 4(-1)^{n-1}\sum_{j=0}^{n}\left(\frac{n}{2}-j\right)\binom{n}{j}^3\binom{n+j-\frac12}{n}\binom{2n-j-\frac12}{n}$$
$$\cdot\left(\frac{1}{\frac{n}{2}-j}+3H_j-3H_{n-j}+H_{2n-j-\frac12}-H_{n+j-\frac12}-H_{n-j-\frac12}+H_{j-\frac12}\right)$$
$$=4(-1)^{n-1}\lim_{\varepsilon\to0}\frac{2}{\varepsilon}\sum_{j=0}^{\infty}\left(\frac{n}{2}+\frac{\varepsilon}{2}-j\right)\binom{n}{j}$$
$$\cdot\frac{(n-j-\varepsilon+1)_j}{(1-\varepsilon)_j}\frac{(n-j+\varepsilon+1)_j}{(1-2\varepsilon)_j}\frac{(j+\frac12)_n}{(1-\varepsilon)_n}\frac{(n-j+\varepsilon+\frac12)_n}{(1+\varepsilon)_n}.$$

In hypergeometric notation, this reads

$$\mathbf{a}_n = 4(-1)^{n-1}\lim_{\varepsilon\to0}\frac{(n+\varepsilon)\,(\frac12)_n\,(n+\varepsilon+\frac12)_n}{\varepsilon(1-\varepsilon)_n\,(1+\varepsilon)_n}$$
$$\times\,{}_6F_5\left[\begin{matrix}-n-\varepsilon,1-\frac{n}{2}-\frac{\varepsilon}{2},n+\frac12,-n,-n+\varepsilon,\frac12-n-\varepsilon\\ -\frac{n}{2}-\frac{\varepsilon}{2},\frac12-2n-\varepsilon,1-\varepsilon,1-2\varepsilon,\frac12\end{matrix};-1\right].$$

To the ${}_6F_5$-series we apply the transformation formula (see [6, (3.10.4), $q\to1$])

$$ {}_6F_5\left[\begin{matrix}a,1+\frac{a}{2},b,x,y,-N\\ \frac{a}{2},1+a-b,1+a-x,1+a-y,1+a+N\end{matrix};-1\right]$$
$$=\frac{(1+a)_N\,(1+a-x-y)_N}{(1+a-x)_N\,1+a-y)_N}\,{}_3F_2\left[\begin{matrix}-N,x,y\\ -a-N+x+y,1+a-b\end{matrix};1\right],\quad(3.2)$$

where N is a non-negative integer. Thus, we obtain

$$\mathbf{a}_n = 4(-1)^{n-1}\lim_{\varepsilon\to0}\frac{(-1)^n\,n!}{(1-\varepsilon)_n}\,{}_3F_2\left[\begin{matrix}-n,n+\frac12,\frac12-n-\varepsilon\\ 1,1-2\varepsilon\end{matrix};1\right]$$
$$=-4\sum_{j=0}^{n}\binom{n}{j}\binom{n-\frac12}{j}\binom{n+j-\frac12}{j},$$

as we claimed. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1. By Lemma 2 with $\alpha=j$, $\beta=H=0$, and $k=-n$ respectively $k=-n-j$, the numbers $2^{2n}\binom{n-\frac12}{j}$ and $2^{2n}\binom{n+j-\frac12}{j}$ are integers. Given the expression for $\mathbf{a}_n$ in Lemma 3, this implies the assertion of the theorem. □

4. The coefficient $\mathbf{b}_n$

The purpose of this section is to prove the following theorem.

Theorem 2. *For all positive integers n, the number $2^{4n}\mathrm{d}_{2n}^2\mathbf{b}_n$ is an integer.*

Proof. This proof follows loosely analogous considerations in [9, Prop. 7]. It depends on an arithmetic fact which is stated and proved separately in Lemma 4 below.

Let us start by reordering the summations in (2.3) to obtain

$$\mathbf{b}_n = (-1)^n \sum_{e=1}^{3} \sum_{k=1}^{n} \frac{(-1)^k}{\left(k-\frac{1}{2}\right)^e} \frac{1}{(3-e)!} \frac{\partial^{3-e}}{\partial \varepsilon^{3-e}} \left(\sum_{j=k}^{n} \left(\frac{n}{2} - j + \varepsilon\right) \left(\frac{n!}{(1-\varepsilon)_j\,(1+\varepsilon)_{n-j}}\right)^3 \cdot \binom{n+j-\varepsilon-\frac{1}{2}}{n} \binom{2n-j+\varepsilon-\frac{1}{2}}{n} \right) \Bigg|_{\varepsilon=0}. \quad (4.1)$$

Also for $\mathbf{b}_n$, we need an alternative expression. It is provided for by the $q=1$ special case of Andrews' identity (1.1). More precisely, in (1.1) on replaces a by q^a, b_i by q^{b_i}, c_i by q^{c_i}, $i=1,2,\dots,m+1$, and then lets q tend to 1. As a result, one obtains the transformation formula

$$ {}_{2m+5}F_{2m+4}\left[\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \dots, b_{m+1}, c_{m+1}, -n \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \dots, 1+a-b_{m+1}, 1+a-c_{m+1}, 1+a+n \end{matrix} ; 1\right] $$
$$= \frac{(1+a)_n\,(1+a-b_{m+1}-c_{m+1})_n}{(1+a-b_{m+1})_n\,(1+a-c_{m+1})_n} \sum_{0\le i_1\le i_2\le\dots\le i_m\le n} \frac{(-n)_{i_m}}{(b_{m+1}+c_{m+1}-a-n)_{i_m}} \cdot \left(\prod_{k=1}^{m} \frac{(1+a-b_k-c_k)_{i_k-i_{k-1}}\,(b_{k+1})_{i_k}\,(c_{k+1})_{i_k}}{(i_k-i_{k-1})!\,(1+a-b_k)_{i_k}\,(1+a-c_k)_{i_k}} \right), \quad (4.2)$$

where again, by definition, $i_0 := 0$. In this formula we put $m=3$, $a=-n+2k-2\varepsilon$, $b_1=-n+k-\varepsilon$, $b_2=-n+k-\varepsilon+\frac{1}{2}$, $c_2=n+k-\varepsilon+\frac{1}{2}$, $b_3=-n+k-\varepsilon$, $c_3=k-2\varepsilon-\delta+1$, $b_4=-n+k-\varepsilon$, $c_4=1$, $N=n-k$, and then let δ tend to 0. This leads to the identity

$$\sum_{j=k}^{n} \left(\frac{n}{2}-j+\varepsilon\right) \left(\frac{n!}{(1-\varepsilon)_j\,(1+\varepsilon)_{n-j}}\right)^3 \binom{n+j-\frac{1}{2}-\varepsilon}{n} \binom{2n-j-\frac{1}{2}+\varepsilon}{n}$$
$$= -\frac{1}{2}\left(k-\varepsilon-\frac{1}{2}\right) \sum_{0\le i_1\le i_2\le i_3\le n-k} (-1)^{i_2} \frac{i_3!}{i_1!\,(i_2-i_1)!\,(i_3-i_2)!} \frac{(\frac{1}{2}-\varepsilon)_n}{(\frac{1}{2}-\varepsilon)_k\,(1+\varepsilon)_{n-k}}$$
$$\cdot \frac{(n-\varepsilon+\frac{1}{2})_{k+i_1}}{(1-\varepsilon)_{k+i_1}} \frac{(n+i_1-i_2+\varepsilon+\frac{1}{2})_{n-k-i_1}}{(1+\varepsilon)_{n-k-i_1}} \frac{n!}{(1-\varepsilon)_{k+i_2}\,(1+\varepsilon)_{n-k-i_2}}$$
$$\cdot \frac{(\frac{1}{2}+n+i_1-i_2)_{i_2-i_1}}{(\frac{1}{2}+n+\varepsilon+i_1-i_2)_{i_2-i_1}} \frac{(n-\frac{1}{2}-i_3+\varepsilon)_{i_3+1}}{(n-\frac{1}{2}-i_3-\varepsilon)_{i_3+1}}$$
$$\cdot \frac{(\varepsilon)_{i_3-i_2}\,(1-2\varepsilon)_{k+i_2}\,(\frac{1}{2}+\varepsilon)_{n-i_3-1}}{(1-2\varepsilon)_{k-1}\,(\frac{1}{2}+\varepsilon)_{n-k-i_3}\,(1-\varepsilon)_{k+i_3}}. \quad (4.3)$$

Using the notations $R_1(\alpha,\beta;t)$ and $R_2(\alpha,\beta;t)$ for elementary bricks that were introduced in Lemmas 1 and 2, and the notations

$$R_3(n,i_1,i_2,\varepsilon)=\frac{(\frac12+n+i_1-i_2)_{i_2-i_1}}{(\frac12+n+\varepsilon+i_1-i_2)_{i_2-i_1}},$$
$$R_4(n,i_3,\varepsilon)=\frac{(n-\frac12-i_3+\varepsilon)_{i_3+1}}{(n-\frac12-i_3-\varepsilon)_{i_3+1}},$$
$$R_5(n,k,i_2,i_3,\varepsilon)=2^{2(k-1)}\frac{(\varepsilon)_{i_3-i_2}\,(1-2\varepsilon)_{k+i_2}\,(\frac12+\varepsilon)_{n-i_3-1}}{(1-2\varepsilon)_{k-1}\,(\frac12+\varepsilon)_{n-k-i_3}\,(1-\varepsilon)_{k+i_3}}$$

for the *special bricks* $R_3(n,i_1,i_2,\varepsilon)$, $R_4(n,i_3,\varepsilon)$, and $R_5(n,k,i_2,i_3,\varepsilon)$, use of (4.3) in (4.1) yields

$$\begin{aligned}2^{4n}\mathrm{d}_{2n}^2\mathbf{b}_n&=-(-1)^n\mathrm{d}_{2n}^2\sum_{e=1}^{3}\sum_{k=1}^{n}\frac{(-1)^k}{\big(k-\frac12\big)^e}\frac{1}{(3-e)!}\frac{\partial^{3-e}}{\partial\varepsilon^{3-e}}\Bigg((2k-2\varepsilon-1)\\
&\times\sum_{0\le i_1\le i_2\le i_3\le n-k}(-1)^{i_2}\frac{i_3!}{i_1!\,(i_2-i_1)!\,(i_3-i_2)!}\cdot R_2(n,k;1-\varepsilon)\cdot\varepsilon\cdot R_1(0,n+1-k;\varepsilon)\\
&\cdot R_2(k+i_1,0;n-\varepsilon+1)\cdot(-\varepsilon)\cdot R_1(0,k+i_1;-\varepsilon)\\
&\cdot R_2(n-k-i_1,0;n+i_1-i_2+1)\cdot\varepsilon\cdot R_1(0,n-k-i_1;\varepsilon)\\
&\cdot(-1)^{k+i_2}\varepsilon\cdot R_1(-k-i_2,n-k-i_2+1;\varepsilon)\cdot R_3(n,i_1,i_2,\varepsilon)\cdot R_4(n,i_3,\varepsilon)\cdot R_5(n,k,i_2,i_3,\varepsilon)\Bigg)\Bigg|_{\varepsilon=0}.\end{aligned}\tag{4.4}$$

We can rewrite this in the form

$$\begin{aligned}2^{4n}\mathrm{d}_{2n}^2\mathbf{b}_n&=-(-1)^n\mathrm{d}_{2n}^2\sum_{e=1}^{3}\sum_{k=1}^{n}\frac{(-1)^k}{\big(k-\frac12\big)^e}\frac{1}{(3-e)!}\frac{\partial^{3-e}}{\partial\varepsilon^{3-e}}\Bigg\{(2k-2\varepsilon-1)\\
&\cdot\sum_{0\le i_1\le i_2\le i_3\le n-k}C(i_1,i_2,i_3)\cdot R_5(n,k,i_2,i_3;\varepsilon)\prod_{h=1}^{M}t_h(n,k,i_1,i_2,i_3;\varepsilon)\Bigg\}\Bigg|_{\varepsilon=0},\end{aligned}$$

where each $C(i_1,i_2,i_3)$ is an integer and each t_h is an expression $R_1(\alpha,\beta;\pm\varepsilon+K)$ with $\alpha\ge\beta$, an expression $R_1(\alpha,\beta;\pm\varepsilon)$ multiplied by $\pm\varepsilon$ with $\alpha<\beta$, an expression $R_2(\alpha,\beta;\pm\varepsilon+K)$ with $\alpha\ge\beta$, or one of $R_3(n,i_1,i_2,\varepsilon)$ and $R_4(n,i_3,\varepsilon)$.

By Leibniz's formula, this last expression can be expanded into

$$\begin{aligned}2^{4n}\mathrm{d}_{2n}^2\mathbf{b}_n&=-(-1)^n\sum_{e=1}^{3}\sum_{k=1}^{n}\frac{2(-1)^k\mathrm{d}_{2n}^{e-1}}{\big(k-\frac12\big)^{e-1}}\\
&\cdot\mathrm{d}_{2n}^{3-e}\Bigg\{\sum_{\ell_0+\dots+\ell_M=3-e}\frac{1}{\ell_0!\,\ell_1!\cdots\ell_M!}\sum_{0\le i_1\le i_2\le i_3\le n-k}C(i_1,i_2,i_3)\end{aligned}$$

$$\left.\cdot \frac{\partial^{\ell_0}}{\partial\varepsilon^{\ell_0}} R_5(n,k,i_2,i_3;\varepsilon) \prod_{h=1}^{M} \frac{\partial^{\ell_h}}{\partial\varepsilon^{\ell_h}} t_h(n,k,i_1,i_2,i_3;\varepsilon)\right\}\Bigg|_{\varepsilon=0}$$

$$+ (-1)^n \sum_{e=1}^{3}\sum_{k=1}^{n} \frac{2(-1)^k \mathrm{d}_{2n}^e}{\left(k-\frac{1}{2}\right)^e}$$

$$\cdot\, \mathrm{d}_{2n}^{2-e} \Bigg\{ \sum_{\ell_0+\cdots+\ell_M=2-e} \frac{1}{\ell_0!\,\ell_1!\cdots\ell_M!} \sum_{0\le i_1\le\cdots\le i_3\le n-k} C_2(i_1,i_2,i_3)$$

$$\left.\cdot \frac{\partial^{\ell_0}}{\partial\varepsilon^{\ell_0}} R_5(n,k,i_2,i_3;\varepsilon) \prod_{h=1}^{M} \frac{\partial^{\ell_h}}{\partial\varepsilon^{\ell_h}} t_h(n,k,i_1,i_2,i_3;\varepsilon)\right\}\Bigg|_{\varepsilon=0}. \tag{4.5}$$

Now, for any h with $1 \le h \le M$, we claim that

$$\frac{\mathrm{d}_{2n}^{\ell_h}}{\ell_h!} \frac{\partial^{\ell_h}}{\partial\varepsilon^{\ell_h}} t_h(n,k,i_1,i_2,i_3;\varepsilon)\Big|_{\varepsilon=0}$$

is an integer. Indeed, if $t_h(n,k,i_1,i_2,i_3;\varepsilon)$ is one of the elementary bricks $R_1(\dots)$ (possibly multiplied by $\pm\varepsilon$) or $R_2(\dots)$, then this follows directly from Lemmas 1 and 2. If $t_h(n,k,i_1,i_2,i_3;\varepsilon)$ is one of the special bricks $R_3(\dots)$ or $R_4(\dots)$, this can be seen directly. Since $2(k-\frac{1}{2})$ divides d_{2n}, Identity (4.5) would imply the assertion of the theorem once we could prove that

$$\frac{\mathrm{d}_{2n}^{\ell_0}}{\ell_0!} \frac{\partial^{\ell_0}}{\partial\varepsilon^{\ell_0}} R_5(n,k,i_2,i_3;\varepsilon)\Big|_{\varepsilon=0} \tag{4.6}$$

is an integer as well.

To accomplish this, we distinguish between two cases. If $i_2 = i_3$, then $R_5(n,k,i_2,i_3;\varepsilon)$ can be factored as follows:

$$\begin{aligned} R_5(n,k,i_2,i_3;\varepsilon) &= R_5(n,k,i_3,i_3;\varepsilon)\\ &= 2^{2(k-1)} \frac{(1-2\varepsilon)_{k+i_3}\,(\frac{1}{2}+\varepsilon)_{n-i_3-1}}{(1-2\varepsilon)_{k-1}\,(1-\varepsilon)_{k+i_3}\,(\frac{1}{2}+\varepsilon)_{n-k-i_3}}\\ &= R_1(k+i_3,0;1-2\varepsilon)\\ &\qquad \cdot(-\varepsilon)\cdot R_1(0,k+i_3+1;-\varepsilon)\cdot R_2(k-1,0;n-k-i_3+\varepsilon)\\ &\qquad \cdot(-2\varepsilon)\cdot R_1(0,k;-2\varepsilon). \end{aligned}$$

Another application of Leibniz's formula and of Lemmas 1 and 2 show that (4.6) is an integer for $i_2 = i_3$.

If $i_2 < i_3$, one observes that

$$R_5(n,k,i_2,i_3;\varepsilon) = \varepsilon \cdot R_6(n,k,i_2,i_3;\varepsilon),$$

where $R_6(\dots)$ is the special brick defined in Lemma 4. Consequently, for $\ell_0 \ge 1$, we have

$$\frac{1}{\ell_0!}\frac{\partial^{\ell_0}}{\partial\varepsilon^{\ell_0}} R_5(n,k,i_2,i_3;\varepsilon)\Big|_{\varepsilon=0} = \frac{1}{(\ell_0-1)!}\frac{\partial^{\ell_0-1}}{\partial\varepsilon^{\ell_0-1}} R_6(n,k,i_2,i_3;\varepsilon)\Big|_{\varepsilon=0}.$$

The above relation together with Lemma 4 with $m_1 = i_3$ and $m_2 = i_2$ then shows that (4.6) is also an integer for $i_2 < i_3$.

This completes the proof of the theorem. □

Lemma 4. *Let*

$$R_6(n,k,m_1,m_2;\varepsilon) = 2^{2(k-1)} \frac{(1+\varepsilon)_{m_1-m_2-1}\,(1-2\varepsilon)_{k+m_2}\,(\frac{1}{2}+\varepsilon)_{n-m_1-1}}{(1-2\varepsilon)_{k-1}\,(1-\varepsilon)_{k+m_1}\,(\frac{1}{2}+\varepsilon)_{n-k-m_1}}.$$

Then, for all integers n,k,m_1,m_2,H with $H\ge 0$ and $0 \le m_2 < m_1 \le n-k$, the number

$$\mathrm{d}_{2n}^{H+1}\cdot\frac{1}{H!}\frac{\partial^H}{\partial\varepsilon^H} R_6(n,k,m_1,m_2;\varepsilon)\Big|_{\varepsilon=0} \tag{4.7}$$

is an integer.

Proof. We loosely follow analogous arguments in the proof of [9, Lemme 11]. In fact, the arguments given in the last paragraph here show that that proof could have been simplified.

We shall show that, for all integers $1 \le f_1 \le f_2 \le \dots \le f_{H+1} \le 2n$, the number

$$\mathrm{d}_{2n}^{H+1}\cdot\frac{1}{H!}\,2^{2(k-1)}\frac{(m_1-m_2-1)!\,(k+m_2)!\,(\frac{1}{2})_{n-m_1-1}}{(k-1)!\,(k+m_1)!\,(\frac{1}{2})_{n-k-m_1}}\frac{1}{f_1 f_2\cdots f_H} \tag{4.8}$$

is an integer. In view of the definition of $R_6(n,k,m_1,m_2;\varepsilon)$, this implies that (4.7) is an integer.

We prove the above claim by verifying that the p-adic valuation of (4.8) is non-negative for all prime numbers p. Writing $[\alpha]$ for the greatest integer less than or equal to α, this p-adic valuation is equal to

$$\begin{aligned}(H+1)\cdot[\log_p(2n)]+\sum_{\ell=1}^{\infty}\Bigg(\left[\frac{k+m_2}{p^\ell}\right]+\left[\frac{m_1-m_2-1}{p^\ell}\right]+\left[\frac{2n-2m_1-2}{p^\ell}\right]-\left[\frac{n-m_1-1}{p^\ell}\right]\\ -\left[\frac{k-1}{p^\ell}\right]-\left[\frac{k+m_1}{p^\ell}\right]-\left[\frac{2n-2k-2m_1}{p^\ell}\right]+\left[\frac{n-k-m_1}{p^\ell}\right]\Bigg)-\sum_{h=1}^{H} v_p(f_h)\end{aligned} \tag{4.9}$$

for *any* prime number p (also for $p = 2$!). If $p > 2n$, it is obvious that this expression is non-negative since all terms vanish. Hence, from now on we assume that $p \le 2n$.

In fact, the conditions on k, n, m_1, m_2 imply that the terms of the infinite series in (4.9) vanish for $\ell > [\log_p(2n)]$. The expression (4.9) can therefore be rewritten in the form

$$[\log_p(2n)] + \sum_{\ell=1}^{[\log_p(2n)]} \left(\left[\frac{k+m_2}{p^\ell}\right] + \left[\frac{m_1-m_2-1}{p^\ell}\right] + \left[\frac{2n-2m_1-2}{p^\ell}\right] - \left[\frac{n-m_1-1}{p^\ell}\right] \right.$$
$$\left. - \left[\frac{k-1}{p^\ell}\right] - \left[\frac{k+m_1}{p^\ell}\right] - \left[\frac{2n-2k-2m_1}{p^\ell}\right] + \left[\frac{n-k-m_1}{p^\ell}\right] \right) - \sum_{h=1}^{H} \big(v_p(f_h) - [\log_p(2n)]\big)\,. \tag{4.10}$$

Since, by definition, $1 \le f_h \le 2n$ for all h, the terms in the sum over h are non-positive. Hence, it suffices to show that the summands in the sum over ℓ are all at least -1.

In order to accomplish this, we write $N = \{n/p^\ell\}$, $K = \{k/p^\ell\}$, $M_1 = \{m_1/p^\ell\}$, $M_2 = \{m_2/p^\ell\}$ for the fractional parts of n/p^ℓ, k/p^ℓ, m_1/p^ℓ and m_2/p^ℓ, respectively. With these notations, the summand of the sum over ℓ becomes

$$[K+M_2] + \left[M_1 - M_2 - \frac{1}{p^\ell}\right] + \left(\left[2N - 2M_1 - \frac{2}{p^\ell}\right] - \left[N - M_1 - \frac{1}{p^\ell}\right]\right)$$
$$- \left[K - \frac{1}{p^\ell}\right] - [K+M_1] - \big([2N-2K-2M_1] - [N-K-M_1]\big). \tag{4.11}$$

We first discuss the case $K = 0$. For this special choice of K, the expression in (4.11) reduces to

$$\left[M_1 - M_2 - \frac{1}{p^\ell}\right] + \left(\left[2N - 2M_1 - \frac{2}{p^\ell}\right] - \left[N - M_1 - \frac{1}{p^\ell}\right]\right)$$
$$+ 1 - \big([2N-2M_1] - [N-M_1]\big). \tag{4.12}$$

Since, by elementary properties of the (weakly) increasing function $x \mapsto [2x] - [x]$, we have

$$\left(\left[2N - 2M_1 - \frac{2}{p^\ell}\right] - \left[N - M_1 - \frac{1}{p^\ell}\right]\right) - \big([2N-2M_1] - [N-M_1]\big) \ge -1,$$

the expression in (4.12) is indeed ≥ -1.

From now on let $K > 0$, i.e., $K \ge \frac{1}{p^\ell}$. In this case, clearly, $\left[K - \frac{1}{p^\ell}\right] = 0$ and

$$\left(\left[2N - 2M_1 - \frac{2}{p^\ell}\right] - \left[N - M_1 - \frac{1}{p^\ell}\right]\right) - \big([2N-2K-2M_1] - [N-K-M_1]\big) \ge 0.$$

Hence, if the expression in (4.11) wants to be ≤ -2, then we must have $[K + M_2] = 0$, $\left[M_1 - M_2 - \frac{1}{p^\ell}\right] = -1$ and $[K + M_1] = 1$, that is

$$K + M_2 < 1, \tag{4.13}$$

$$M_1 - M_2 - \frac{1}{p^\ell} < 0, \tag{4.14}$$

$$K + M_1 \ge 1. \tag{4.15}$$

But a combination of (4.13) and (4.15) yields $M_1 - M_2 > 0$, which contradicts (4.14) since the denominators of the rational numbers M_1 and M_2 are both p^ℓ. □

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