Alternative proof of a proposition on cylindric partitions by Alexei Borodin

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Abstract. It is shown how a generating function formula for a certain class of cylindric partitions originally due to Alexei Borodin can be derived by applying Gustafson's $SU(r)$-extension of Bailey's ${}_6\psi_6$-summation to a specialisation of an earlier generating function result on cylindric partition due to Gessel and the author.

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be r-tuples of integers, and let d be an integer. According to [2], a *cylindric partition of shape* $\boldsymbol{\lambda}/\boldsymbol{\mu}/d$ is a planar array of non-negative integers of the form

$$\begin{array}{ccccccccc}
 & & & & \pi_{1,\mu_1+1} & \cdots\cdots\cdots\cdots & & \pi_{1,\lambda_1} \\
 & & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots\cdots\cdots & \pi_{2,\lambda_2} & \\
 & \iddots & & & \vdots & & \iddots & \\
\pi_{r,\mu_r+1} & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \pi_{r,\lambda_r} & & &
\end{array}$$

such that

$$\begin{aligned}
\pi_{i,j} &\ge \pi_{i,j+1}\\
\pi_{i,j} &\ge \pi_{i+1,j}
\end{aligned}$$

and

$$\pi_{r,j} \ge \pi_{1,j+d}.$$

The sum of all the $\pi_{i,j}$'s is denoted by $|\pi|$.

In [1], Borodin proves the following theorem (in a different but equivalent form).

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Theorem ([1, Proposition 5.1]). *Let r, d be given positive integers, $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_r)$, $d > -r$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r \ge \mu_1 - d$. The generating function $\sum_\pi q^{|\pi|}$, where the sum is over all cylindric partitions of shape $(\infty^r)/\boldsymbol{\mu}/d$, is equal to*

$$\frac{(q^{r+d}; q^{r+d})_\infty^{r-1}}{(q;q)_\infty^r} \prod_{1\le i<j\le r} (q^{\mu_i - i - \mu_j + j}; q^{r+d})_\infty \, (q^{r+d+\mu_j - j - \mu_i + i}; q^{r+d})_\infty. \tag{1}$$

This theorem is proved in [1] by making use of a formula for the partition function of the periodic Schur process, which is the main subject of [1]. The purpose of this note is to show that this theorem follows straightforwardly when a result on cylindric partitions due to Gessel and the author is combined with Gustafson's $SU(r)$-extension of Bailey's ${}_6\psi_6$-summation.

Proof of the Theorem. We let $c \to \infty$ in [2, Theorem 5]. This yields that the generating function that one is interested in is given by

$$\sum_{k_1+\cdots+k_r=0} q^{r(r+d)\sum_{i=1}^r k_i^2/2 + d\sum_{i=1}^r ik_i + r\sum_{i=1}^r \mu_i k_i} \\ \cdot \frac{\prod_{1\le i<j\le r}(1 - q^{\mu_i + (r+d)k_i - i - \mu_j - (r+d)k_j + j})}{(q;q)_\infty^r}. \tag{2}$$

The sum over $k_1 + \cdots + k_r = 0$ can be evaluated by means of the $SU(r)$-extension of the ${}_6\psi_6$-summation of Bailey, due to Gustafson [3, Theorem 1.15; 4, Theorem 4.1] (see also [2, (7.9)]):

$$\sum_{k_1+\cdots+k_r=0} q^{\sum_{i=1}^r ik_i} \prod_{1\le i<j\le r} \frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \prod_{i,j=1}^r \frac{(a_i x_j; q)_{k_j}}{(b_i x_j; q)_{k_j}} \\ = \frac{(q;q)_\infty^{r-1} \prod_{i,j=1}^r (b_i/a_j; q)_\infty \prod_{1\le i<j\le r} (qx_i/x_j; q)_\infty \, (qx_j/x_i; q)_\infty}{(q^{1-r}\prod_{i=1}^r (b_i/a_i); q)_\infty \prod_{i,j=1}^r (q/a_i x_j; q)_\infty \, (b_i x_j; q)_\infty} \\ \cdot \left(q/\prod_{i=1}^r a_i x_i; q\right)_\infty \left(q^{1-r}\prod_{i=1}^r b_i x_i; q\right)_\infty.$$

To be precise, one replaces q by q^{r+d}, one sets $x_i = q^{\mu_i - i}$, $b_i = 0$, and one performs the limit $a_i \to \infty$, $i = 1, 2, \dots, r$. In this way, one arrives at

$$\sum_{k_1+\cdots+k_r=0} q^{r(r+d)\sum_{i=1}^r k_i^2/2 + d\sum_{i=1}^r ik_i + r\sum_{i=1}^r \mu_i k_i} \prod_{1\le i<j\le r} \frac{1 - q^{(r+d)k_i - (r+d)k_j + \mu_i - i - \mu_j + j}}{1 - q^{\mu_i - i - \mu_j + j}} \\ = (q^{r+d}; q^{r+d})_\infty^{r-1} \prod_{1\le i<j\le r} (q^{r+d+\mu_i - i - \mu_j + j}; q^{r+d})_\infty \, (q^{r+d+\mu_j - j - \mu_i + i}; q^{r+d})_\infty.$$

Using this equation, we see that the sum in (2) simplifies to (1), as desired. $\square$

References

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