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Q-ANALOGUE OF A TWO VARIABLE INVERSE PAIR OF SERIES WITH APPLICATIONS TO BASIC DOUBLE HYPERGEOMETRIC SERIES

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1. Introduction. Let

$$f(z_1,z_2) = \big(f_1(z_1,z_2), f_2(z_1,z_2)\big)$$

be a pair of a formal series (fps) in z_1 and z_2 of the form

$$(1.1) \qquad f_i(z_1,z_2) = z_i/\psi_i(z_1,z_2) \quad \text{for } i = 1,2,$$

where $\psi_i(z_1,z_2)$ is an fps with $\psi_i(0,0) \neq 0$ for $i = 1,2$. Then there exists a unique pair of fps

$$F(z_1,z_2) = \big(F_1(z_1,z_2), F_2(z_1,z_2)\big),$$

which is also of the form (1.1), with

$$(1.2) \qquad f_i\left(F_1(z_1,z_2), F_2(z_1,z_2)\right) = z_i \quad \text{for } i = 1,2.$$

This pair is called the *inverse* of $f(z_1,z_2)$.

For $\mathbf{k},\mathbf{l} \in \mathbf{Z}^2$ (pairs of integers), $\mathbf{k} = (k_1,k_2)$ and $\mathbf{l} = (l_1,l_2)$, we adopt the familiar multidimensional notations, $|\mathbf{k}| = k_1 + k_2$, $\mathbf{k}+\mathbf{l} = (k_1+l_1, k_2+l_2)$, $\mathbf{k} \geqq \mathbf{l}$ if and only if $k_1 \geqq l_1$ and $k_2 \geqq l_2$, $\mathbf{0} = (0,0)$,

$$\mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2},$$

and

$$f^{\mathbf{k}}(z_1,z_2) = f_1^{k_1}(z_1,z_2) f_2^{k_2}(z_1,z_2).$$

By two-variable Lagrange inversion the coefficients of $F_i(z_1,z_2)$ or, even more generally, the coefficients of integral powers of $F_i(z_1,z_2)$, $i = 1,2$, may be evaluated (see for example [**7**, (4.5) with $\phi(\mathbf{x}) = \mathbf{x}^{\mathbf{l}}$]):

$$(1.3) \qquad \langle \mathbf{z}^{\mathbf{k}} \rangle F^{\mathbf{l}}(z_1,z_2) = \langle \mathbf{z}^{-\mathbf{l}} \rangle f^{-\mathbf{k}}(z_1,z_2) D(f)(z_1,z_2),$$

where

$$D(f)(z_1,z_2) = \psi_1(z_1,z_2)\psi_2(z_1,z_2) \frac{\partial f}{\partial z}(z_1,z_2),$$

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with $\frac{\partial f}{\partial z}(z_1, z_2)$ the Jacobian of $f(z_1, z_2)$. ($\langle \mathbf{z}^{\mathbf{k}} \rangle a(z_1, z_2)$ means the coefficient of $\mathbf{z}^{\mathbf{k}}$ in the formal Laurent series (fLs) $a(z_1, z_2)$.)

An example of such an inverse pair is

$$\text{(1.4)(a)} \quad f_0(z_1, z_2) = \left(\frac{z_1}{1 - z_2}, \frac{z_2}{1 - z_1} \right)$$

and

$$\text{(1.4)(b)} \quad F_0(z_1, z_2) = \left(\frac{z_1(1 - z_2)}{1 - z_1 z_2}, \frac{z_2(1 - z_1)}{1 - z_1 z_2} \right).$$

With the help of Lagrange inversion the coefficient of $\mathbf{z}^{\mathbf{k}}$ in $F_0^{\mathbf{l}}(z_1, z_2)$, and, consequently, of

$$(1 - z_2)^{l_1}(1 - z_1)^{l_2}/(1 - z_1 z_2)^{l_1 + l_2},$$

can be evaluated. Essentially, this was done in [**6**, Theorem 1] and [**7**, p.190]. In addition, Evans et al. [**6**, Theorem 9] give the following q-analogue of their formula, which may be written as

$$\begin{aligned} \text{(1.5)} \qquad & \langle \mathbf{z}^{\mathbf{k}} \rangle \frac{(qz_1z_2; q)_{\alpha+\beta}}{(qz_2; q)_\alpha (qz_1; q)_\beta} \\ &= q^{k_1 k_2} \frac{(q^{\beta - k_2 + 1}; q)_{k_1 - 1} (q^{\alpha - k_1 + 1}; q)_{k_2 - 1}}{(q; q)_{k_1} (q; q)_{k_2}} \\ &\times \left((q^{k_1} - q^\alpha)(q^{k_2} - q^\beta) - (1 - q^{k_1})(1 - q^{k_2}) \right). \end{aligned}$$

(For definition of the symbol $(a; q)_\beta$ see (4.2).) They prove it by a basic hypergeometric transformation formula and put the question if a proof by two-variable q-Lagrange inversion could be given. The first approach towards multivariable q-Lagrange formulas was made by the author in [**16**]. In the present paper we give a new two-variable q-Lagrange formula (Theorem 3), a special case of which helps to establish a proof of (1.5). As a by-product, in Theorem 5, a q-analogue of the inverse pair (1.4) is obtained.

The coefficient matrices of the members of the pair which will be given in (4.13) of Theorem 5 are inverses of each other (with respect to matrix multiplication). In [**8**, **10**] Gessel and Stanton, using certain pairs of matrices, which are inverses of each other, systematically derive one-variable basic hypergeometric summations and transformations. Applying the same method to the inverse matrices determined by the pair in (4.13), we are able to deduce a number of two-variable basic hypergeometric summations and transformations, all of which appear to be new. (F. H. Jackson [**12**, **13**] was the first to treat basic double hypergeometric series systematically. While there exists an extensive theory on one-variable basic hypergeometric series, until now there have not appeared many results on basic double series. In particular, the number of summation theorems is not very large. A collection of papers dealing with basic double series is included in the references, although we do not claim that this list is complete.)

Our paper is organized as follows. Section 2 contains a short outline of [**16**, Section 3] in order to explain how two-variable q-Lagrange inversion should be understood. In Section 3 the q-Lagrange formula is given which we need to prove the coefficient theorem (1.5). The proof of (1.5) is done in Section 4 by use of the inverse pair (4.13). Finally, Section 5 is devoted to the derivation of basic double hypergeometric summations and transformations.

2. Preliminaries. Unless otherwise stated, in this paper we shall always consider fps (fLs) in the indeterminates z_1 and z_2 of the form

$$\sum_{\mathbf{i} \geqq \mathbf{m}} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}, \quad \text{for some } \mathbf{m} \in \mathbf{Z}^2,$$

whose coefficients a_i are rational functions in the indeterminate q.

A sequence

$$f = \big(f_{\mathbf{k}}(z_1, z_2)\big)_{\mathbf{k} \in \mathbf{Z}^2}$$

of fLs satisfying

$$f_{\mathbf{k}}(z_1, z_2) = \sum_{\mathbf{n} \geqq \mathbf{k}} f_{\mathbf{nk}} \mathbf{z}^{\mathbf{n}} \quad \text{and} \quad f_{\mathbf{kk}} \neq 0 \tag{2.1}$$

is called a *diagonal sequence*. The sequence of powers of

$$f(z_1, z_2) = \big(f_1(z_1, z_2), f_2(z_1, z_2)\big)$$

satisfying (1.1), $\big(f^{\mathbf{k}}(z_1, z_2)\big)_{\mathbf{k} \in \mathbf{Z}^2}$, is an example of a diagonal sequence. Given another sequence

$$g = \big(g_{\mathbf{l}}(z_1, z_2)\big)_{\mathbf{l} \in \mathbf{Z}^2},$$

where

$$g_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{k} \geqq \mathbf{l}} g_{\mathbf{kl}} \mathbf{z}^{\mathbf{k}},$$

the substitution of f into g is defined by

$$g(f) = \big(h_{\mathbf{l}}(z_1, z_2)\big)_{\mathbf{l} \in \mathbf{Z}^2}$$

with

$$h_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{k} \geqq \mathbf{l}} g_{\mathbf{kl}} f_{\mathbf{k}}(z_1, z_2),$$

or more precisely

$$h_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{n}} \left(\sum_{\mathbf{n} \geqq \mathbf{k} \geqq \mathbf{l}} f_{\mathbf{nk}} g_{\mathbf{kl}} \right) \mathbf{z}^{\mathbf{n}}.$$

The sequence

$$\mathcal{F} = \left(F_{\mathbf{l}}(z_1, z_2)\right)_{\mathbf{l} \in \mathbf{Z}^2}$$

is called *inverse* of f if

$$f(\mathcal{F}) = (\mathbf{z}^{\mathbf{k}})_{\mathbf{k} \in \mathbf{Z}^2},$$

i.e., if and only if

$$\sum_{\mathbf{n} \geqq \mathbf{k}} f_{\mathbf{nk}} F_{\mathbf{n}}(z_1, z_2) = \mathbf{z}^{\mathbf{k}} \quad \text{for all } \mathbf{k} \in \mathbf{Z}^2 \tag{2.2}$$

or, equivalently

$$\sum_{\mathbf{m} \geqq \mathbf{n} \geqq \mathbf{k}} F_{\mathbf{mn}} f_{\mathbf{nk}} = \delta_{\mathbf{mk}} \quad \text{for all } \mathbf{m}, \mathbf{k} \in \mathbf{Z}^2, \tag{2.3}$$

where we have set

$$F_{\mathbf{l}}(z_1, z_2) = \sum_{k \geqq \mathbf{l}} F_{\mathbf{kl}} \mathbf{z}^{\mathbf{k}}.$$

($\delta_{\mathbf{mk}}$ is the Kronecker-delta.) That means, f and $\mathcal{F}$ are inverses of each other if and only if the corresponding coefficient matrices $(f_{\mathbf{nk}})$ and $(F_{\mathbf{kl}})$ are inverses of each other. This fact will be of importance in Section 5. Equation (2.2) is the analogue for (1.2) in the setting of sequences. Obviously, for any diagonal sequence f there exists a uniquely determined inverse sequence $\mathcal{F}$.

By analogy with (1.3), we call an identity of the form

$$\langle \mathbf{z}^{\mathbf{k}} \rangle F_{\mathbf{l}}(z_1, z_2) = \langle \mathbf{z}^{-\mathbf{l}} \rangle \tilde{f}_{\mathbf{k}}(z_1, z_2) \tag{2.4}$$

a *Lagrange formula*, where the sequence of fLs $\tilde{f} = \left(\tilde{f}_{\mathbf{k}}(z_1, z_2)\right)_{\mathbf{k} \in \mathbf{Z}^2}$ is expressed in terms of f. Equation (2.4) immediately implies

$$\tilde{f}_{\mathbf{k}}(z_1, z_2) = \sum_{\mathbf{l} \leqq \mathbf{k}} F_{\mathbf{kl}} \mathbf{z}^{-\mathbf{l}}, \tag{2.5}$$

and therefore

$$\langle \mathbf{z}^{\mathbf{0}} \rangle f_{\mathbf{k}}(z_1, z_2) \tilde{f}_{\mathbf{n}}(z_1, z_2) = \delta_{\mathbf{nk}}. \tag{2.6}$$

In [**16**] the author gave a method for finding $\tilde{f}$ for a given sequence f. For fLs $a(z_1, z_2)$ and $b(z_1, z_2)$ we introduce a bilinear form $\langle\ ,\ \rangle$ by

$$\langle a(z_1, z_2), b(z_1, z_2)\rangle = \langle \mathbf{z}^{\mathbf{0}}\rangle a(z_1, z_2) b(z_1, z_2). \tag{2.7}$$

Given any linear operator L mapping fLs into fLs, L^* denotes the adjoint of L with respect to $\langle\ ,\ \rangle$, meaning

$$\langle La(z_1, z_2), b(z_1, z_2)\rangle = \langle a(z_1, z_2), L^* b(z_1, z_2)\rangle$$

for all $a(z_1, z_2)$ and $b(z_1, z_2)$. What we need is the following special case of [**16**, Theorem 1].

LEMMA 1. *Let*

$$f = \left(f_{\mathbf{k}}(z_1, z_2)\right)_{\mathbf{k}\in\mathbf{Z}^2}$$

be a diagonal sequence satisfying the system of equations

$$U_i f_{\mathbf{k}}(z_1, z_2) = c_i(\mathbf{k}) V f_{\mathbf{k}}(z_1, z_2) \quad i = 1, 2, \tag{2.8}$$

where U_i and V are linear operators (acting on fLs*), V being bijective and $c_i(\mathbf{k})$, $i = 1, 2$, are functions from $\mathbf{Z}^2$ into the field of rational functions of q satisfying the property that if $m \neq n$ there exists a j ($j = 1$ or 2) for which $c_j(\mathbf{m}) \neq c_j(\mathbf{n})$. Let*

$$\left(h_{\mathbf{k}}(z_1, z_2)\right)_{\mathbf{k}\in\mathbf{Z}^2}$$

be a non-trivial solution (i.e., $h_k \not\equiv 0$ for all $\mathbf{k} \in \mathbf{Z}^2$) of the dual system

$$U_i^* h_{\mathbf{k}}(z_1, z_2) = c_i(\mathbf{k}) V^* h_{\mathbf{k}}(z_1, z_2) \quad i = 1, 2, \tag{2.9}$$

then

$$\tilde{f}_{\mathbf{k}}(z_1, z_2) = \langle f_{\mathbf{k}}(z_1, z_2), V^* h_{\mathbf{k}}(z_1, z_2)\rangle^{-1} V^* h_{\mathbf{k}}(z_1, z_2). \tag{2.10}$$

3. The Lagrange formula. Recall the definition of q-powers due to Hofbauer [**11**, **15**].

Definition 2. The fps $\varphi_\alpha(t)$ in the indeterminate t, $\alpha \in \mathbf{R}$ (real numbers), are called q-powers for a fixed fps $\varphi(t)$ if $\varphi_\alpha(0) \neq 0$ for all $\alpha \in \mathbf{R}$ and

$$\varphi_\alpha(qt) = \left(1 + (q^\alpha - 1)t\varphi(t)\right) \varphi_\alpha(t). \tag{3.1}$$

It is easy to see that for $q \longrightarrow 1$ we have

$$\varphi_\alpha(t) \longrightarrow \bar{\varphi}^\alpha(t)$$

where

$$\bar{\varphi}'(t)/\bar{\varphi}(t) = \varphi(t).$$

Let the operators ϵ_1, ϵ_2 be defined by

$$\epsilon_i \mathbf{z}^{\mathbf{n}} = q^{n_i} \mathbf{z}^{\mathbf{n}} \quad i = 1, 2.$$

The next theorem gives a q-analogue of the two-variable Lagrange inversion for that special case of $f(z_1, z_2)$ of Section 1, where in (1.1) ψ_1 depends only on z_2 and ψ_2 only on z_1.

THEOREM 3. *Let $\varphi_\alpha(t)$ and $\phi_\alpha(t)$ be q- powers for $\varphi(t)$ and $\phi(t)$, respectively. For the inverse sequence $\left(F_{\mathbf{l}}(z_1, z_2)\right)_{\mathbf{l} \in \mathbf{Z}^2}$ of $\left(f_{\mathbf{k}}(z_1, z_2)\right)_{\mathbf{k} \in \mathbf{Z}^2}$, where*

$$f_{\mathbf{k}}(z_1, z_2) = z_1^{k_1} z_2^{k_2} / \varphi_{k_1+\lambda}(qz_2) \phi_{k_2+\mu}(qz_1) \tag{3.2}$$

with $\lambda, \mu \in \mathbf{R}$, we have that

$$\begin{aligned} \langle \mathbf{z}^{\mathbf{k}} \rangle F_{\mathbf{l}}(z_1, z_2) = \langle \mathbf{z}^{-\mathbf{l}} \rangle &(1 - q^{\lambda+\mu} z_1 \phi(z_1) z_2 \varphi(z_2) \epsilon_1^{-1} \epsilon_2^{-1}) \\ &\times \mathbf{z}^{-\mathbf{k}} \varphi_{k_1+\lambda}(qz_2) \phi_{k_2+\mu}(qz_1). \end{aligned} \tag{3.3}$$

Proof. By (3.1) we get

$$\begin{cases} \left(1 + (q^{k_2+\mu} - 1) qz_1 \phi(qz_1)\right) \epsilon_1 f_{\mathbf{k}} = q^{k_1} f_{\mathbf{k}} \\ \left(1 + (q^{k_1+\lambda} - 1) qz_2 \varphi(qz_2)\right) \epsilon_2 f_{\mathbf{k}} = q^{k_2} f_{\mathbf{k}}, \end{cases}$$

and after a short calculation

$$\begin{cases} \left[\left(1 - qz_1 \phi(qz_1)\right) \epsilon_1 + q^\mu qz_1 \phi(qz_1) \epsilon_1 \left(1 - qz_2 \varphi(qz_2)\right) \epsilon_2\right] f_{\mathbf{k}} \\ \qquad = q^{k_1} (1 - q^{\lambda+\mu} qz_2 \varphi(qz_2) \epsilon_2 qz_1 \phi(qz_1) \epsilon_1) f_{\mathbf{k}} \\ \left[q^\lambda qz_2 \varphi(qz_2) \epsilon_2 \left(1 - qz_1 \phi(qz_1)\right) \epsilon_1 + \left(1 - qz_2 \varphi(qz_2)\right) \epsilon_2\right] f_{\mathbf{k}} \\ \qquad = q^{k_2} (1 - q^{\lambda+\mu} qz_2 \varphi(qz_2) \epsilon_2 qz_1 \phi(qz_1) \epsilon_1) f_{\mathbf{k}} \end{cases}$$

This is a system of "eigenvalue" equations in the sense of (2.8). Thus the dual system for the auxiliary sequence

$$\left(h_{\mathbf{k}}(z_1, z_2)\right)_{\mathbf{k} \in \mathbf{Z}^2}$$

reads, by use of $\epsilon_i^* = \epsilon_i^{-1} (i = 1, 2)$ and $a(z_1, z_2)^* = a(z_1, z_2)$ for any multiplication operator $a(z_1, z_2)$,

$$\begin{cases} \left[\epsilon_1^{-1} \left(1 - qz_1 \phi(qz_1)\right) \right. \\ \qquad \left. + q^\mu \epsilon_2^{-1} \left(1 - qz_2 \varphi(qz_2)\right) \epsilon_1^{-1} qz_1 \phi(qz_1)\right] h_{\mathbf{k}} \\ \qquad = q^{k_1} \left(1 - q^{\lambda+\mu} \epsilon_1^{-1} qz_1 \phi(qz_1) \epsilon_2^{-1} qz_2 \varphi(qz_2)\right) h_{\mathbf{k}} \\ \left[q^\lambda \epsilon_1^{-1} \left(1 - qz_1 \phi(qz_1)\right) \epsilon_2^{-1} qz_2 \varphi(qz_2) \right. \\ \qquad \left. + \epsilon_2^{-1} \left(1 - qz_2 \varphi(qz_2)\right)\right] h_{\mathbf{k}} \\ \qquad = q^{k_2} \left(1 - q^{\lambda+\mu} \epsilon_1^{-1} qz_1 \phi(qz_1) \epsilon_2^{-1} qz_2 \varphi(qz_2)\right) h_{\mathbf{k}}, \end{cases}$$

which is equivalent to

$$\begin{cases} \left(1+(q^{k_2+\mu}-1)z_1\phi(z_1)\right)\epsilon_1^{-1}h_{\mathbf{k}} = q^{k_1}h_{\mathbf{k}} \\ \left(1+(q^{k_1+\lambda}-1)z_2\varphi(z_2)\right)\epsilon_2^{-1}h_{\mathbf{k}} = q^{k_2}h_{\mathbf{k}}. \end{cases}$$

A solution of this system is

$$h_{\mathbf{k}}(z_1,z_2) = \mathbf{z}^{-\mathbf{k}}\varphi_{k_1+\lambda}(qz_2)\phi_{k_2+\mu}(qz_1),$$

hence, by (2.10),

$$\tilde{f}_{\mathbf{k}}(z_1,z_2) = (1-q^{\lambda+\mu}z_1\phi(z_1)z_2\varphi(z_2)\epsilon_1^{-1}\epsilon_2^{-1}) \times \mathbf{z}^{-\mathbf{k}}\varphi_{k_1+\lambda}(qz_2)\phi_{k_2+\mu}(qz_1), \tag{3.4}$$

which together with (2.4) proves (3.3).

4. Inverse pairs of sequences. We use the standard notations

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j), \tag{4.1}$$

for arbitrary $\beta \in \mathbf{R}$

$$(a;q)_\beta = (a;q)_\infty/(aq^\beta;q)_\infty, \tag{4.2}$$

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,\dots,a_{r+1}; \\ b_1,\dots,b_r; \end{matrix} q,z\right] = \sum_{j=0}^{\infty}\frac{(a_1;q)_j\cdots(a_{r+1};q)_j}{(b_1;q)_j\cdots(b_r;q)_j}\frac{z^j}{(q;q)_j}, \tag{4.3}$$

and

$$\prod\left[\begin{matrix} a_1,\dots,a_r; \\ b_1,\dots,b_s; \end{matrix} q\right] = \frac{(a_1;q)_\infty\cdots(a_r;q)_\infty}{(b_1;q)_\infty\cdots(b_s;q)_\infty}. \tag{4.4}$$

We shall frequently write $(a)_\infty$ or $(a)_\beta$ instead of $(a;q)_\infty$ or $(a;q)_\beta$, respectively. That is to say, the base of such an expression is q unless otherwise stated.

By the q-binomial theorem [**17**, Appendix (IV. 11)]

$${}_1\phi_0\left[\begin{matrix} a; \\ -; \end{matrix} q,z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{4.5}$$

(which we use as an identity of fps in z; for an fps-definition of $(z;q)_\infty$ see [**8**, Theorem 3.13]) we have

$$(z;q)_\beta = \sum_{j=0}^{\infty}\frac{(q^{-\beta};q)_j}{(q;q)_j}q^{\beta j}z^j. \tag{4.6}$$

Letting $q \longrightarrow 1$ we see that $(z;q)_\beta$ is a q-analogue for $(1-z)^\beta$.

We shall freely make use of the relations between expressions of the type $(a;q)_\beta$, contained in [**17**, Appendix II], and the "flip q into q^{-1}"-idea, made precise in [**8**, Theorem 3.13]. In particular,

$$(4.7)\qquad (z;q^{-1})_\infty = 1/(qz;q)_\infty$$

and

$$(4.8)\qquad (z;q^{-1})_\beta = 1/(qz;q)_{-\beta}.$$

To obtain the promised inverse pair of sequences, we require the following basic double summation.

LEMMA 4. *For $r_1, r_2 \in \mathbf{R}$ holds*

$$(4.9)\qquad \sum_{\mathbf{j}\geqq \mathbf{0}} q^{j_1 j_2} \frac{(q^{r_2})_{j_1}(q^{r_1})_{j_2}}{(q)_{j_1}(q)_{j_2}} \mathbf{z}^{\mathbf{j}} \frac{(z_2)_{r_1+j_1}(z_1)_{r_2+j_2}}{(z_1z_2)_{r_1+r_2+j_1+j_2}} = 1$$

Proof. By two-fold use of the q-analogue of Gauss's theorem [**17**, Appendix (IV.2)]

$$(4.10)\qquad {}_2\phi_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; q, c/ab\right] = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty},$$

(which, just as (4.5) we use in a formal sense) we evaluate

$$\begin{aligned}
&\sum_{\mathbf{j}\geqq \mathbf{0}} q^{j_1 j_2} \frac{(q^{r_2})_{j_1}(q^{r_1})_{j_2}}{(q)_{j_1}(q)_{j_2}} \mathbf{z}^{\mathbf{j}} \frac{(z_2)_{r_1+j_1}(z_1)_{r_2+j_2}}{(z_1z_2)_{r_1+r_2+j_1+j_2}} \\
&= \sum_{j_2\geqq 0} z_2^{j_2} \frac{(q^{r_1})_{j_2}(z_1)_{r_2+j_2}(z_2)_{r_1}}{(q)_{j_2}(z_1z_2)_{r_1+r_2+j_2}} {}_2\phi_1\left[\begin{matrix} z_2q^{r_1}, q^{r_2}; \\ z_1z_2q^{r_1+r_2+j_2}; \end{matrix}\; q, z_1q^{j_2}\right] \\
&= \sum_{j_2\geqq 0} z_2^{j_2} \frac{(q^{r_1})_{j_2}(z_1)_{r_2+j_2}(z_2)_{r_1}}{(q)_{j_2}(z_1z_2)_{r_1+r_2+j_2}} \frac{(z_1q^{r_2+j_2})_\infty (z_1z_2q^{r_1+j_2})_\infty}{(z_1z_2q^{r_1+r_2+j_2})_\infty (z_1q^{j_2})_\infty} \\
&= \frac{(z_2)_{r_1}}{(z_1z_2)_{r_1}} {}_2\phi_1\left[\begin{matrix} z_1, q^{r_1}; \\ z_1z_2q^{r_1}; \end{matrix}\; q, z_2\right] \\
&= 1.
\end{aligned}$$

In (4.9) perform the substitutions $r_1 = k_1 + \lambda$, $r_2 = k_2 + \mu$, $j_1 = n_1 - k_1$ and $j_2 = n_2 - k_2$. Multiplying the resulting identity by $\mathbf{z}^{\mathbf{k}}$ we get

$$\begin{aligned}
(4.11)\qquad &\sum_{\mathbf{n}\geqq \mathbf{k}} q^{(n_1-k_1)(n_2-k_2)} \frac{(q^{k_2+\mu})_{n_1-k_1}(q^{k_1+\lambda})_{n_2-k_2}}{(q)_{n_1-k_1}(q)_{n_2-k_2}} \\
&\times \mathbf{z}^{\mathbf{n}} \frac{(z_2)_{n_1+\lambda}(z_1)_{n_2+\mu}}{(z_1z_2)_{n_1+n_2+\lambda+\mu}} = \mathbf{z}^{\mathbf{k}}.
\end{aligned}$$

After replacing z_1 by $z_1q^{-\mu}$ and z_2 by $z_2q^{-\lambda}$ in (4.11), turning q into q^{-1} leads to

$$(4.12)\quad \sum_{\mathbf{n}\geqq\mathbf{k}} q^{k_1k_2-n_1n_2+|\mathbf{n}-\mathbf{k}|}\frac{(q^{k_2+\mu})_{n_1-k_1}(q^{k_1+\lambda})_{n_2-k_2}}{(q)_{n_1-k_1}(q)_{n_2-k_2}} \times \mathbf{z}^{\mathbf{n}}\frac{(z_2q^{\lambda};q^{-1})_{n_1+\lambda}(z_1q^{\mu};q^{-1})_{n_2+\mu}}{(z_1z_2q^{\lambda+\mu};q^{-1})_{n_1+n_2+\lambda+\mu}} = \mathbf{z}^{\mathbf{k}},$$

valid for all $\mathbf{k}\in\mathbf{Z}^2$. By comparing this system of equations with (2.2), we obtain

THEOREM 5. *The sequences*

$$g = \left(g_{\mathbf{k}}(z_1,z_2)\right)_{\mathbf{k}\in\mathbf{Z}^2} \quad \textit{and} \quad \mathcal{G} = \left(G_{\mathbf{l}}(z_1,z_2)\right)_{\mathbf{l}\in\mathbf{Z}^2}$$

are inverses of each other where

$$(4.13)(a)\quad g_{\mathbf{k}}(z_1,z_2) = \epsilon_{12}^{-1}\left(q^{k_1k_2}\mathbf{z}^{\mathbf{k}}/(qz_2;q)_{k_1+\lambda}(qz_1;q)_{k_2+\mu}\right)$$

and

$$(4.13)(b)\quad G_{\mathbf{l}}(z_1,z_2) = \mathbf{z}^{\mathbf{l}}\frac{(z_2q^{\lambda};q^{-1})_{l_1+\lambda}(z_1q^{\mu};q^{-1})_{l_2+\mu}}{(z_1z_2q^{\lambda+\mu};q^{-1})_{l_1+l_2+\lambda+\mu}};$$

ϵ_{12} *is the linear operator defined by*

$$\epsilon_{12}\mathbf{z}^{\mathbf{n}} = q^{n_1n_2}\mathbf{z}^{\mathbf{n}}.$$

Proof. By the q-binomial theorem (4.5) we get

$$\begin{aligned}\sum_{\mathbf{n}\geqq\mathbf{k}} q^{k_1k_2-n_1n_2+|\mathbf{n}-\mathbf{k}|}\frac{(q^{k_2+\mu})_{n_1-k_1}(q^{k_1+\lambda})_{n_2-k_2}}{(q)_{n_1-k_1}(q)_{n_2-k_2}}\mathbf{z}^{\mathbf{n}} &= q^{k_1k_2}\epsilon_{12}^{-1}\left(\mathbf{z}^{\mathbf{k}}/(qz_2)_{k_1+\lambda}(qz_1)_{k_2+\mu}\right)\\ &= g_{\mathbf{k}}(z_1,z_2).\end{aligned}$$

Using this, comparison of (2.2) and (4.12) completes the proof.

For $\lambda=\mu=0$ the sequences g and $\mathcal{G}$ are the q-analogues of the powers of $f_0(z_1,z_2)$ and $F_0(z_1,z_2)$ respectively, given in (1.4). In fact, for $\lambda=\mu=0$, when $q\longrightarrow 1$ we have

$$g_{\mathbf{k}}(z_1,z_2)\longrightarrow \left(z_1/(1-z_2)\right)^{k_1}\left(z_2/(1-z_1)\right)^{k_2}$$

and

$$G_{\mathbf{l}}(z_1,z_2)\longrightarrow \left(\frac{z_1(1-z_2)}{1-z_1z_2}\right)^{l_1}\left(\frac{z_2(1-z_1)}{1-z_1z_2}\right)^{l_2}.$$

Another formulation of Theorem 5 is

COROLLARY 6. *The sequences $\bar{g}$ and $\bar{G}$ are inverses of each other where*

$$\text{(4.14)(a)}\quad \bar{g}_{\mathbf{k}}(z_1,z_2)=\mathbf{z}^{\mathbf{k}}/(qz_2;q)_{k_1+\lambda}(qz_1;q)_{k_2+\mu}$$

and

$$\text{(4.14)(b)}\quad \bar{G}_{\mathbf{l}}(z_1,z_2)=\epsilon_{12}\left(q^{-l_1l_2}\mathbf{z}^{\mathbf{l}}\frac{(z_2q^{\lambda};q^{-1})_{l_1+\lambda}(z_1q^{\mu};q^{-1})_{l_2+\mu}}{(z_1z_2q^{\lambda+\mu};q^{-1})_{l_1+l_2+\lambda+\mu}}\right).$$

In the proof of Theorem 5 we saw that the coefficients of $\mathbf{z}^{\mathbf{n}}$ in $g_{\mathbf{k}}(z_1,z_2)$ are

$$\text{(4.15)}\quad g_{\mathbf{nk}}=q^{k_1k_2-n_1n_2+|\mathbf{n}-\mathbf{k}|}\frac{(q^{k_2+\mu};q)_{n_1-k_1}(q^{k_1+\lambda};q)_{n_2-k_2}}{(q;q)_{n_1-k_1}(q;q)_{n_2-k_2}}.$$

Using a variation of the q-Lagrange formula of Theorem 3, we compute the coefficients of $G_{\mathbf{l}}(z_1,z_2)$.

THEOREM 7. *The coefficient of $\mathbf{z}^{\mathbf{k}}$ in $G_{\mathbf{l}}(z_1,z_2)$ is given by*

$$\text{(4.16)}\quad G_{\mathbf{kl}}=q^{l_1l_2-k_1k_2}\frac{(q^{k_2+\mu};q^{-1})_{k_1-l_1}(q^{k_1+\lambda};q^{-1})_{k_2-l_2}}{(q^{-1};q^{-1})_{k_1-l_1}(q^{-1};q^{-1})_{k_2-l_2}}\times\left(1-q^{\lambda+\mu+k_1+k_2}\frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right).$$

Proof. It is a simple fact that the fps $(z;q)_\alpha$ are q-powers for $-1/(1-z)$. Hence, using (4.14)(a) and (3.4) with $\varphi(z)=\phi(z)=-1/(1-z)$,

$$\tilde{\bar{g}}_{\mathbf{k}}(z_1,z_2)=\left(1-q^{\lambda+\mu}\frac{z_1z_2}{(1-z_1)(1-z_2)}\epsilon_1^{-1}\epsilon_2^{-1}\right)\mathbf{z}^{-\mathbf{k}}(qz_2)_{k_1+\lambda}(qz_1)_{k_2+\mu}.$$

Because of (2.6) we get, since $\epsilon_{12}^*=\epsilon_{12}$,

$$\tilde{g}_{\mathbf{k}}(z_1,z_2)=\epsilon_{12}q^{-k_1k_2}\tilde{\bar{g}}_{\mathbf{k}}(z_1,z_2).$$

Therefore the Lagrange formula (2.4) for $F_{\mathbf{l}}=G_{\mathbf{l}}$, etc., reads

$$\begin{aligned}&\langle\mathbf{z}^{\mathbf{k}}\rangle G_{\mathbf{l}}(z_1,z_2)\\&=\langle\mathbf{z}^{-\mathbf{l}}\rangle\epsilon_{12}\Big(q^{-k_1k_2}\left(1-q^{\lambda+\mu}\frac{z_1z_2}{(1-z_1)(1-z_2)}\epsilon_1^{-1}\epsilon_2^{-1}\right)\\&\times\mathbf{z}^{-\mathbf{k}}(qz_2)_{k_1+\lambda}(qz_1)_{k_2+\mu}\Big).\end{aligned}$$

By the q-binomial theorem (4.5), with q replaced by q^{-1}, the right-hand side of the last equation is equal to

$$q^{l_1 l_2 - k_1 k_2} \frac{(q^{k_2+\mu}; q^{-1})_{k_1-l_1} (q^{k_1+\lambda}; q^{-1})_{k_2-l_2}}{(q^{-1}; q^{-1})_{k_1-l_1} (q^{-1}; q^{-1})_{k_2-l_2}}$$
$$\times \left(1 - q^{\lambda+\mu+k_1+k_2} \frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right),$$

which establishes (4.16).

The coefficient evaluation (1.5) of Evans et al. [**6**] is only a reformulation of Theorem 7.

COROLLARY 8. *For $\alpha, \beta \in \mathbf{R}$ we have that*

$$\text{(4.17)} \quad \frac{(qz_1z_2)_{\alpha+\beta}}{(qz_2)_\alpha (qz_1)_\beta} = \sum_{\mathbf{k} \geqq \mathbf{0}} q^{k_1 k_2} \frac{(q^{\beta-k_2+1})_{k_1-1} (q^{\alpha-k_1+1})_{k_2-1}}{(q)_{k_1} (q)_{k_2}}$$
$$\times \left((q^{k_1} - q^\alpha)(q^{k_2} - q^\beta) - (1-q^{k_1})(1-q^{k_2})\right) \mathbf{z}^{\mathbf{k}}.$$

Proof. By (4.13)(b) and (4.16) we have

$$\frac{(z_2 q^\lambda; q^{-1})_{l_1+\lambda} (z_1 q^\mu; q^{-1})_{l_2+\mu}}{(z_1 z_2 q^{\lambda+\mu}; q^{-1})_{l_1+l_2+\lambda+\mu}}$$
$$= \sum_{\mathbf{k} \geqq \mathbf{l}} q^{l_1 l_2 - k_1 k_2} \frac{(q^{k_2+\mu}; q^{-1})_{k_1-l_1} (q^{k_1+\lambda}; q^{-1})_{k_2-l_2}}{(q^{-1}; q^{-1})_{k_1-l_1} (q^{-1}; q^{-1})_{k_2-l_2}}$$
$$\times \left(1 - q^{\lambda+\mu+k_1+k_2} \frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right) \mathbf{z}^{\mathbf{k}-\mathbf{l}}.$$

In this identity substitute (in order) $\mathbf{k}+\mathbf{l}$ for $\mathbf{k}$, $-\beta-\mu$ for l_2, $-\alpha-\lambda$ for l_1, $z_2 q^{-\lambda}$ for z_2 and $z_1 q^{-\mu}$ for z_1 thus obtaining

$$\frac{(z_2; q^{-1})_{-\alpha} (z_1; q^{-1})_{-\beta}}{(z_1 z_2; q^{-1})_{-\alpha-\beta}}$$
$$= \sum_{\mathbf{k} \geqq \mathbf{0}} q^{-k_1 k_2 + \alpha k_2 + \beta k_1} \frac{(q^{k_2-\beta}; q^{-1})_{k_1} (q^{k_1-\alpha}; q^{-1})_{k_2}}{(q^{-1}; q^{-1})_{k_1} (q^{-1}; q^{-1})_{k_2}}$$
$$\times \left(1 - q^{k_1+k_2-\alpha-\beta} \frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{k_1-\alpha})(1-q^{k_2-\beta})}\right) \mathbf{z}^{\mathbf{k}}.$$

Regarding (4.8), a short calculation leads to

$$\frac{(qz_1z_2; q)_{\alpha+\beta}}{(qz_2; q)_\alpha (qz_1; q)_\beta} = \sum_{\mathbf{k} \geqq \mathbf{0}} q^{k_1 k_2 + k_1 + k_2} \frac{(q^{\beta-k_2}; q)_{k_1} (q^{\alpha-k_1}; q)_{k_2}}{(q; q)_{k_1} (q; q)_{k_2}}$$
$$\times \left(1 - \frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{\alpha-k_1})(1-q^{\beta-k_2})}\right) \mathbf{z}^{\mathbf{k}},$$

which is equivalent to (4.17).

Remark. Our approach to the proof of (4.17) relies heavily on Gessel's [**7**, p. 160] $q=1$-proof. For example, the system of equations (4.12) is the q-analogue for the two equations at the top of p. 160 in [**7**] (for $a=d=0$).

From Corollary 8 we may deduce another identity of [**6**, (6.1)].

COROLLARY 9. *For $\alpha, \beta \in \mathbf{R}$ we have that*

$$\frac{(z_1z_2)_{\alpha+\beta-1}}{(z_2)_\alpha(z_1)_\beta} = \sum_{\mathbf{k}\geqq\mathbf{0}} q^{k_1k_2}\frac{(q^{\beta-k_2})_{k_1}(q^{\alpha-k_1})_{k_2}}{(q)_{k_1}(q)_{k_2}}\mathbf{z}^{\mathbf{k}}. \tag{4.18}$$

Proof. In (4.17) replace z_1 by z_1/q and z_2 by z_2/q. Multiplying the resulting identity by

$$(1-z_1z_2/q)^{-1} = \sum_{j=0}^{\infty}(z_1z_2)^j q^{-j}$$

and collecting the terms on the right-hand side yield (4.18).

Just as the pair (4.13) corresponds to the identity (4.17), there exists an inverse pair of sequences corresponding to (4.18), which we will state without proof.

THEOREM 10. *The sequence of* fLs

$$\epsilon_{12}^{-1}\left(q^{k_1k_2}(1-q^{k_1+\lambda}z_2-q^{k_2+\mu}z_1)\frac{\mathbf{z}^{\mathbf{k}}}{(z_2;q)_{k_1+\lambda+1}(z_1;q)_{k_2+\mu+1}}\right) \tag{4.19(a)}$$

is the inverse of the sequence

$$\mathbf{z}^{\mathbf{l}}\frac{(z_2q^{\lambda-1};q^{-1})_{l_1+\lambda}(z_1q^{\mu-1};q^{-1})_{l_2+\mu}}{(z_1z_2q^{\lambda+\mu-1};q^{-1})_{l_1+l_2+\lambda+\mu+1}}. \tag{4.19(b)}$$

5. Transformations and summations. Suppose that the infinite lower-triangular matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbf{Z}^2}$ and $(F_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbf{Z}^2}$ are inverses of each other. Then the "inverse relations"

$$b_{\mathbf{n}} = \sum_{\mathbf{n}\geqq\mathbf{k}\geqq\mathbf{0}} f_{\mathbf{nk}}a_{\mathbf{k}}, \quad \mathbf{n}\in\mathbf{Z}^2 \tag{5.1}$$

and

$$a_{\mathbf{k}} = \sum_{\mathbf{k}\geqq\mathbf{l}\geqq\mathbf{0}} F_{\mathbf{kl}}b_{\mathbf{l}}, \quad \mathbf{k}\in\mathbf{Z}^2 \tag{5.2}$$

are equivalent, where $(a_{\mathbf{k}}) = (a_{\mathbf{k}}^{(q)})$ and $(b_{\mathbf{n}}) = (b_{\mathbf{n}}^{(q)})$ are sequences of rational functions in q. For the choice

$$f_{\mathbf{nk}} = q^{-k_1k_2+n_1n_2-|\mathbf{n}-\mathbf{k}|}g_{\mathbf{nk}} \quad \text{and}$$

$$F_{\mathbf{kl}} = q^{k_1k_2-l_1l_2-|\mathbf{k}-\mathbf{l}|}G_{\mathbf{kl}},$$

by (4.15) and (4.16) the equations (5.1) and (5.2) may be written in the form

$$\begin{aligned}(5.3)\qquad b_{\mathbf{n}} = {}& \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}} \sum_{\mathbf{k}\geqq\mathbf{0}} (-1)^{|\mathbf{k}|} q^{n_1k_1+n_2k_2-\binom{k_1}{2}-\binom{k_2}{2}} \\ &\times \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(A)_{k_1}(B)_{k_2}} a_{\mathbf{k}}\end{aligned}$$

and

$$\begin{aligned}(5.4)\qquad a_{\mathbf{k}} = {}& (-1)^{|\mathbf{k}|} q^{\binom{k_1}{2}+\binom{k_2}{2}} \frac{(Bq^{k_2-k_1})_{k_1}(Aq^{k_1-k_2})_{k_2}}{(q)_{k_1}(q)_{k_2}} \\ &\times \Bigg[\frac{(1-Aq^{k_1})(1-Bq^{k_2})}{(1-Aq^{k_1-k_2})(1-Bq^{k_2-k_1})} \\ &\times \sum_{\mathbf{l}\geqq\mathbf{0}} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2-k_1+1}B)_{l_1}(q^{k_1-k_2+1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}} \\ &- \frac{AB(1-q^{k_1})(1-q^{k_2})}{(1-Aq^{k_1-k_2})(1-Bq^{k_2-k_1})} \\ &\times \sum_{\mathbf{l}\geqq\mathbf{0}} \frac{(q^{-k_1+1})_{l_1}(q^{-k_2+1})_{l_2}}{(q^{k_2-k_1+1}B)_{l_1}(q^{k_1-k_2+1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}} \Bigg],\end{aligned}$$

where we have set $A = q^\lambda$ and $B = q^\mu$. Setting

$$X_{\mathbf{k}} = \sum_{\mathbf{l}\geqq\mathbf{0}} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2-k_1+1}B)_{l_1}(q^{k_1-k_2+1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}},$$

(5.4) becomes a recursion for $X_{\mathbf{k}}$, namely,

$$\begin{aligned}&(-1)^{|\mathbf{k}|} q^{-\binom{k_1}{2}-\binom{k_2}{2}} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2-k_1+1}B)_{k_1}(q^{k_1-k_2+1}A)_{k_2}} a_{\mathbf{k}} \\ &= X_{\mathbf{k}} - \frac{AB(1-q^{k_1})(1-q^{k_2})}{(1-Aq^{k_1})(1-Bq^{k_2})} X_{\mathbf{k}-\mathbf{e}},\end{aligned}$$

where $\mathbf{e} = (1, 1)$. The solution of this recursion is

$$X_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2-k_1+1}B)_{k_1}(q^{k_1-k_2+1}A)_{k_2}} \times \sum_{j\geq 0} q^{-\binom{k_1-j}{2}-\binom{k_2-j}{2}}(AB)^j a_{\mathbf{k}-j\mathbf{e}},$$

consequently,

$$\text{(5.5)} \quad \sum_{\mathbf{l}\geq \mathbf{0}} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2-k_1+1}B)_{l_1}(q^{k_1-k_2+1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}} = (-1)^{|\mathbf{k}|} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2-k_1+1}B)_{k_1}(q^{k_1-k_2+1}A)_{k_2}} \times \sum_{j\geq 0} q^{-\binom{k_1-j}{2}-\binom{k_2-j}{2}}(AB)^j a_{\mathbf{k}-j\mathbf{e}}.$$

Multiplication of (5.1) (where $f_{\mathbf{nk}} = q^{-k_1k_2+n_1+n_2-|\mathbf{n}-\mathbf{k}|}g_{\mathbf{nk}}$) by $\mathbf{z}^{\mathbf{n}}$ and summing up both sides with respect to $\mathbf{n}$, by use of (4.13)(a) and (4.15), yield the transformation

$$\text{(5.6)} \quad \sum_{\mathbf{n}\geq \mathbf{0}} b_{\mathbf{n}} z_1^{n_1} z_2^{n_2} = \frac{(Bz_1)_\infty (Az_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{\mathbf{k}\geq \mathbf{0}} a_{\mathbf{k}} \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}}.$$

Equation (5.6) holds if (5.3) or (5.4) is satisfied. Similarly, multiplying (5.2) (where $F_{\mathbf{kl}} = q^{k_1k_2-l_1l_2-|\mathbf{k}-\mathbf{l}|}G_{\mathbf{kl}}$) by $q^{-k_1k_2}\mathbf{z}^{\mathbf{k}}$ and summing with respect to $\mathbf{k}$, we get by (4.13)(b) and (4.16)

$$\sum_{\mathbf{k}\geq \mathbf{0}} q^{-k_1k_2} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} = \frac{(Bz_1/q;q^{-1})_\infty (Az_2/q;q^{-1})_\infty (z_1z_2/q^2;q^{-1})_\infty}{(z_1/q;q^{-1})_\infty (z_2/q;q^{-1})_\infty (ABz_1z_2/q^2;q^{-1})_\infty} \times \sum_{\mathbf{l}\geq \mathbf{0}} q^{-l_1l_2} b_{\mathbf{l}} \mathbf{z}^{\mathbf{l}} \frac{(z_2/q;q^{-1})_{l_1}(z_1/q;q^{-1})_{l_2}}{(z_1z_2/q^2;q^{-1})_{l_1+l_2}}.$$

After replacing z_1 by qz_1, z_2 by qz_2 and q by q^{-1}, we obtain the transformation

$$\text{(5.7)} \quad \sum_{\mathbf{k}\geq \mathbf{0}} q^{k_1k_2-k_1-k_2} a_{\mathbf{k}}^{(1/q)} z_1^{k_1} z_2^{k_2} = \frac{(Bz_1)_\infty (Az_2)_\infty (z_1z_2)_\infty}{(z_1)_\infty (z_2)_\infty (ABz_1z_2)_\infty} \times \sum_{\mathbf{l}\geq \mathbf{0}} q^{l_1l_2-l_1-l_2} b_{\mathbf{l}}^{(1/q)} z_1^{l_1} z_2^{l_2} \frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}.$$

Again, (5.7) holds if (5.3) or (5.4) is satisfied. Different choices of $(a_{\mathbf{k}})$ and $(b_{\mathbf{n}})$ which satisfy (5.3) or (5.4) yield basic double hypergeometric summations and transformations when substituted into (5.5), (5.6) or (5.7).

First take

$$(5.8)\qquad a_{\mathbf{k}} = q^{k_1k_2}B^{k_1}A^{k_2}\frac{(A)_{k_1}(C)_{k_2}}{(q)_{k_1}(q)_{k_2}}.$$

By two-fold use of the q-Vandermonde summation [**17**, Appendix (IV.1)]

$$(5.9)\qquad {}_2\phi_1\left[\begin{matrix}a, q^{-n};\\ b;\end{matrix}\ q, q\right] = \frac{(b/a;q)_n\, a^n}{(b;q)_n},$$

we have by (5.3) and a bit of manipulation

$$\begin{aligned}
b_{\mathbf{n}} &= \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}}\sum_{k_2\geqq 0}\frac{(q^{-n_2})_{k_2}(q^{n_1}B)_{k_2}(C)_{k_2}}{(q^{1-n_2}/A)_{k_2}(B)_{k_2}(q)_{k_2}}\\
&\times q^{k_2}{}_2\phi_1\left[\begin{matrix}q^{n_2-k_2}A, q^{-n_1};\\ q^{1-n_1-k_2}/B;\end{matrix}\ q, q\right]\\
&= \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}}\sum_{k_2\geqq 0}\frac{(q^{-n_2})_{k_2}(q^{n_1}B)_{k_2}(C)_{k_2}}{(q^{1-n_2}/A)_{k_2}(B)_{k_2}(q)_{k_2}}q^{k_2}\\
&\times \frac{(q^{1-n_1-n_2}/AB)_{n_1}}{(q^{1-n_1-k_2}/B)_{n_1}}q^{n_1(n_2-k_2)}A^{n_1}\\
&= (-AB)^{n_1}q^{n_1n_2+\binom{n_1}{2}}\frac{(A)_{n_2}(q^{1-n_1-n_2}/AB)_{n_1}}{(q)_{n_1}(q)_{n_2}}{}_2\phi_1\left[\begin{matrix}C, q^{-n_2};\\ q^{1-n_2}/A;\end{matrix}\ q, q\right]\\
&= (-AB)^{n_1}q^{n_1n_2+\binom{n_1}{2}}\frac{(A)_{n_2}(q^{1-n_1-n_2}/AB)_{n_1}}{(q)_{n_1}(q)_{n_2}}\\
&\times \frac{(q^{1-n_2}/AC)_{n_2}}{(q^{1-n_2}/A)_{n_2}}C^{n_2},
\end{aligned}$$

and finally

$$(5.10)\qquad b_{\mathbf{n}} = \frac{(q^{n_2}AB)_{n_1}(AC)_{n_2}}{(q)_{n_1}(q)_{n_2}}.$$

The resulting evaluation (5.3) reads

$$\sum_{\mathbf{k}\geqq\mathbf{0}} \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(C)_{k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(B)_{k_2}(q)_{k_1}(q)_{k_2}} \times q^{k_1k_2-\binom{k_1}{2}-\binom{k_2}{2}}(-Bq^{n_1})^{k_1}(-Aq^{n_2})^{k_2} = \frac{(q^{n_2}AB)_{n_1}(AC)_{n_2}}{(B)_{n_1}(A)_{n_2}}. \tag{5.11}$$

For $B = C$ we get the symmetric formula

$$\sum_{\mathbf{k}\geqq\mathbf{0}} \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}} \times q^{k_1k_2-\binom{k_1}{2}-\binom{k_2}{2}}(-Bq^{n_1})^{k_1}(-Aq^{n_2})^{k_2} = \frac{(AB)_{n_1+n_2}}{(B)_{n_1}(A)_{n_2}}. \tag{5.12}$$

A standard notation for a q-Appell type basic double hypergeometric series is

$$\phi^{p:r;u}_{m:s;v}\left[\begin{matrix} e_1,\dots,e_p : a_1,\dots,a_r; c_1,\dots,c_u; \\ f_1,\dots,f_m : b_1,\dots,b_s; d_1,\dots,d_v; \end{matrix} q; z_1, z_2\right] = \sum_{\mathbf{k}\geqq\mathbf{0}} \frac{\prod_{j=1}^{p}(e_j;q)_{k_1+k_2}\prod_{j=1}^{r}(a_j;q)_{k_1}\prod_{j=1}^{u}(c_j;q)_{k_2}}{\prod_{j=1}^{m}(f_j;q)_{k_1+k_2}\prod_{j=1}^{s}(b_j;q)_{k_1}\prod_{j=1}^{v}(d_j;q)_{k_2}} z_1^{k_1}z_2^{k_2}. \tag{5.13}$$

The dual evaluation (5.5) for the above choice of $(a_{\mathbf{k}})$ and $(b_{\mathbf{n}})$ can be written as

$$\begin{aligned}
&\phi^{1:1;2}_{0:1;2}\left[\begin{matrix} AB : q^{-k_1}; \quad AC, q^{-k_2}; \\ - : q^{k_2-k_1+1}B; \ AB, q^{k_1-k_2+1}A; \end{matrix} q; q, q\right] \\
&= (-1)^{|\mathbf{k}|}q^{k_1k_2-\binom{k_1}{2}-\binom{k_2}{2}}B^{k_1}A^{k_2}\frac{(A)_{k_1}(C)_{k_2}}{(Bq^{k_2-k_1+1})_{k_1}(Aq^{k_1-k_2+1})_{k_2}} \\
&\quad\times {}_3\phi_2\left[\begin{matrix} q^{k_1}, q^{k_2}, q^{-1}; \\ q^{k_1-1}A, q^{k_2-1}C; \end{matrix} q^{-1}, q^{-1}\right] \\
&= (-1)^{|\mathbf{k}|}q^{k_1k_2-\binom{k_1}{2}-\binom{k_2}{2}}B^{k_1}A^{k_2}\frac{(A)_{k_1}(C)_{k_2}}{(Bq^{k_2-k_1+1})_{k_1}(Aq^{k_1-k_2+1})_{k_2}} \\
&\quad\times {}_3\phi_2\left[\begin{matrix} q^{-k_1}, q^{-k_2}, q; \\ q^{1-k_1}/A, q^{1-k_2}/C; \end{matrix} q, q/AC\right].
\end{aligned} \tag{5.14}$$

By (5.6) we get the transformation

$$\sum_{\mathbf{n}\geqq\mathbf{0}} \frac{(q^{n_2}AB)_{n_1}(AC)_{n_2}}{(q)_{n_1}(q)_{n_2}} z_1^{n_1} z_2^{n_2} \tag{5.15}$$
$$= \frac{(Bz_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty} \sum_{\mathbf{k}\geqq\mathbf{0}} q^{k_1k_2}B^{k_1}A^{k_2}$$
$$\times \frac{(A)_{k_1}(C)_{k_2}}{(q)_{k_1}(q)_{k_2}} \frac{z_1^{k_1}z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}}.$$

The special case $B = C$

$$\sum_{\mathbf{n}\geqq\mathbf{0}} \frac{(AB)_{n_1+n_2}}{(q)_{n_1}(q)_{n_2}} z_1^{n_1} z_2^{n_2} \tag{5.16}$$
$$= \frac{(Bz_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty} \sum_{\mathbf{k}\geqq\mathbf{0}} q^{k_1k_2}B^{k_1}A^{k_2}$$
$$\times \frac{(A)_{k_1}(B)_{k_2}}{(q)_{k_1}(q)_{k_2}} \frac{z_1^{k_1}z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}}$$

turns out to be a q-analogue of

$$(1-z_1-z_2)^{-a-b} = (1-z_1)^{-b}(1-z_2)^{-a}\left(1-\frac{z_1}{1-z_2}\right)^{-a}\left(1-\frac{z_2}{1-z_1}\right)^{-b},$$

when setting $A = q^a$, $b = q^b$ and letting $q \longrightarrow 1$.

Finally, the dual transformation (5.7) reads

$$\sum_{k\geqq\mathbf{0}} \frac{(A;q^{-1})_{k_1}(C;q^{-1})_{k_2}}{(q^{-1};q^{-1})_{k_1}(q^{-1};q^{-1})_{k_2}} (B/q)^{k_1}(A/q)^{k_2} z_1^{k_1} z_2^{k_2}$$
$$= \frac{(Bz_1)_\infty(Az_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(ABz_1z_2)_\infty} \sum_{\mathbf{l}\geqq\mathbf{0}} q^{l_1l_2-l_1-l_2}$$
$$\times \frac{(q^{-l_2}AB;q^{-1})_{l_1}(AC;q^{-1})_{l_2}}{(q^{-1};q^{-1})_{l_1}(q^{-1};q^{-1})_{l_2}}$$
$$\times z_1^{l_1} z_2^{l_2} \frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}.$$

The left-hand side can be summed by the q-binomial theorem (4.5). In basic hypergeometric notation we obtain after a short calculation

$$\phi_{1:0;1}^{1:1;2}\left[\begin{matrix}1/AB : z_2; 1/AC, z_1; \\ z_1z_2 : -; 1/AB;\end{matrix} q; z_1, z_2\right] = \prod\left[\begin{matrix}ABz_1z_2, z_1, z_2; \\ z_1z_2, ABz_1, ACz_2;\end{matrix} q\right],$$

or, after setting $F = 1/AB$ and $E = 1/AC$,

$$\phi_{1:0;1}^{1:1;2}\left[\begin{matrix}F \quad : z_2; E, z_1; \\ z_1z_2 : -; F;\end{matrix} q; z_1, z_2\right] = \prod\left[\begin{matrix}z_1z_2/F, z_1, z_2; \\ z_1z_2, z_1/F, z_2/E;\end{matrix} q\right], \tag{5.17}$$

As the cases $z_1 = 0$ and $z_2 = 0$, respectively, show, this is a generalization of the q-binomial theorem (4.5).

Another choice is

$$a_{\mathbf{k}} = \delta_{k_1k_2} q^{2\binom{k_1}{2}} \frac{(A)_{k_1}(B)_{k_1}}{(C)_{k_1}(q)_{k_1}} C^{k_1}. \tag{5.18}$$

By the q-Gauss sum (4.10) we obtain from (5.3)

$$b_{\mathbf{n}} = \frac{(C)_{n_1+n_2}(B)_{n_1}(A)_{n_2}}{(C)_{n_1}(C)_{n_2}(q)_{n_1}(q)_{n_2}}. \tag{5.19}$$

By (5.5) we get

$$\begin{aligned}\phi_{0:2;2}^{1:2;2}&\left[\begin{matrix}C : B, q^{-k_1}; \qquad A, q^{-k_2}; \\ - : C, q^{k_2-k_1+1}B; C, q^{k_1-k_2+1}A;\end{matrix} q; q, q\right] \\ &= \delta_{k_1k_2} C^{k_1} \frac{(1-A)(1-B)}{(1-q^{k_1}A)(1-q^{k_1}B)} \frac{(q)_{k_1}}{(C)_{k_1}} \\ &\times {}_3\phi_2\left[\begin{matrix}q^{-k_1}, q^{1-k_1}/C, q; \\ q^{1-k_1}/A, q^{1-k_1}/B;\end{matrix} q, q\right].\end{aligned} \tag{5.20}$$

For the special case $C = qAB$ the Pfaff-Saalschütz summation [**17**, Appendix (IV.4)]

$${}_3\phi_2\left[\begin{matrix}a, b, q^{-n}; \\ c, d;\end{matrix} q, q\right] = \frac{(c/a; q)_n(c/b; q)_n}{(c; q)_n(c/ab; q)_n}, \tag{5.21}$$

provided $cd = abq^{1-n}$, can be applied to find

$$(5.22)\quad \phi_{0:2;2}^{1:2;2}\left[\begin{matrix} qAB : B, q^{-k_1}; & A, q^{-k_2}; \\ - : qAB, q^{k_2-k_1+1}B; & qAB, q^{k_1-k_2+1}A; \end{matrix} q; q, q\right] = \delta_{k_1k_2}(AB)^{k_1}\frac{(q)_{k_1}}{(qAB)_{k_1}}.$$

Another example is when $B = C$ and the right-hand side of (5.20) may be summed. This time by the q-Vandermonde summation (5.9)

$$(5.23)\quad \phi_{0:1;2}^{1:1;2}\left[\begin{matrix} B : q^{-k_1}; & A, q^{-k_2}; \\ - : q^{k_2-k_1+1}B; & B, q^{k_1-k_2+1}A; \end{matrix} q; q, q\right] = \delta_{k_1k_2}B^{k_1}\frac{(q)_{k_1}}{(qB)_{k_1}}.$$

The transformation (5.6) is

$$(5.24)\quad \phi_{0:1;1}^{1:1;1}\left[\begin{matrix} C : B; A; \\ - : C; C; \end{matrix} q; z_1, z_2\right] = \frac{(Bz_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{k\geqq 0} q^{k^2-k}C^k\frac{(A)_k(B)_k}{(C)_k(q)_k}\frac{(z_1z_2)^k}{(Az_2)_k(Bz_1)_k}.$$

The special case $A = B = C$

$$(5.25)\quad \sum_{\mathbf{n}\geqq\mathbf{0}}\frac{(A)_{n_1+n_2}}{(q)_{n_1}(q)_{n_2}}z_1^{n_1}z_2^{n_2} = \frac{(Az_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{k\geqq 0} q^{k^2-k}A^k\frac{(A)_k}{(q)_k}\frac{(z_1z_2)^k}{(Az_2)_k(Az_1)_k}$$

is a q-analogue of

$$(1 - z_1 - z_2)^{-a} = (1 - z_1)^{-a}(1 - z_2)^{-a}\left(1 - \frac{z_1z_2}{(1 - z_1)(1 - z_2)}\right)^{-a}.$$

By (5.7) the dual transformation is

$$\sum_{k\geqq 0} q^{-k}\frac{(A;q^{-1})_k(B;q^{-1})_k}{(C;q^{-1})_k(q^{-1};q^{-1})_k}(Cz_1z_2)^k = \frac{(Bz_1)_\infty(Az_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(ABz_1z_2)_\infty}\sum_{\mathbf{l}\geqq\mathbf{0}} q^{l_1l_2-l_1-l_2} \times \frac{(C;q^{-1})_{l_1+l_2}(B;q^{-1})_{l_1}(A;q^{-1})_{l_2}}{(C;q^{-1})_{l_1}(C;q^{-1})_{l_2}(q^{-1};q^{-1})_{l_1}(q^{-1};q^{-1})_{l_2}} \times z_1^{l_1}z_2^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}},$$

which after setting $E = 1/A$, $D = 1/B$, $F = 1/C$ is transformed into

$$(5.26)\quad \phi^{1:2;2}_{1:1;1}\left[\begin{matrix} F : D, z_2; E, z_1; \\ z_1z_2 : F; F; \end{matrix} q; z_1/D, z_2/E\right]$$
$$= \prod\left[\begin{matrix} z_1, z_2, z_1z_2/DE; \\ z_1/D, z_2/E, z_1z_2; \end{matrix} q\right] {}_2\phi_1\left[\begin{matrix} D, E; \\ F; \end{matrix} q, z_1z_2/DE\right].$$

If $D = F$ the ${}_2\phi_1$ reduces to a ${}_1\phi_0$, which is summable by the q-binomial theorem (4.5), thus arriving at (5.17) a second time. For $D = E = F$, again by (4.5), we deduce from (5.26)

$$(5.27)\quad \phi^{1:1;1}_{1:0;0}\left[\begin{matrix} D : z_2; z_1; \\ z_1z_2 : -; -; \end{matrix} q; z_1/D, z_2/D\right]$$
$$= \prod\left[\begin{matrix} z_1, z_2, z_1z_2/D; \\ z_1/D, z_2/D, z_1z_2; \end{matrix} q\right].$$

This is a q-analogue of

$$\left(1 - \frac{z_2(1-z_1)}{1-z_1z_2} - \frac{z_1(1-z_2)}{1-z_1z_2}\right)^{-d}$$
$$= (1-z_1)^{-d}(1-z_2)^{-d}(1-z_1z_2)^{d}.$$

Also of interest is the limiting case $F \longrightarrow \infty$ in (5.26),

$$(5.28)\quad \sum_{\mathbf{l}\geqq\mathbf{0}} q^{l_1l_2}\frac{(D)_{l_1}(E)_{l_2}}{(q)_{l_1}(q)_{l_2}}(z_1/D)^{l_1}(z_2/E)^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}$$
$$= \prod\left[\begin{matrix} z_1, z_2, z_1z_2/DE; \\ z_1/D, z_2/E, z_1z_2; \end{matrix} q\right],$$

it is a q-analogue of

$$\left(1 - \frac{z_1(1-z_2)}{1-z_1z_2}\right)^{-d}\left(1 - \frac{z_2(1-z_1)}{1-z_1z_2}\right)^{-e}$$
$$= (1-z_1)^{-d}(1-z_2)^{-e}(1-z_1z_2)^{d+e}.$$

Equations (5.27) and (5.28) could also be obtained as limiting cases of Al-Salam's [**1**] q-Saalschützian double series theorems.

We have only proved (5.26) in a formal sense. But it is easy to deduce that

(5.26) remains valid for $|q| < 1$, $|z_1| < |D|$ and $|z_2| < |E|$, when it is interpreted as an identity for basic hypergeometric functions. Setting $z_1 = q^a$, $z_2 = q^b$, $D = q^d$, $E = q^e$, $F = q^f$ and letting $q \longrightarrow 1$, the resulting ${}_2F_1$ on the right-hand side of (5.26) is summable by ordinary Gauss-summation [**17**, Appendix (III.3)], which leads to

$$(5.29)\quad F^{1:1;2}_{1:1;1}\left[\begin{matrix} f: d, b; e, a; \\ a+b: f; f; \end{matrix} 1, 1\right] = \Gamma\left[\begin{matrix} a-d, b-e, a+b, f, f-d-e \\ a, b, a+b-d-e, f-d, f-e \end{matrix}\right],$$

subject to suitable restrictions on a, b, d, e, f such that convergence is provided. This identity is due to Carlitz [**5**, (1.7)].

Of course it is possible first to choose $(b_{\mathbf{n}})$ and, by (5.4), compute the $a_{\mathbf{k}}$'s. If we take

$$(5.30)\quad b_{\mathbf{n}} = \frac{(C)_{n_1}(D)_{n_2}}{(q)_{n_1}(q)_{n_2}},$$

two-fold application of q-Vandermonde summation (5.9) yields

$$(5.31)\quad a_{\mathbf{k}} = (-1)^{|\mathbf{k}|} q^{\binom{k_1}{2}+\binom{k_2}{2}} \times \frac{(q^{k_2-k_1+1}B/C)_{k_1}(q^{k_1-k_2+1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}} \times C^{k_1}D^{k_2}\left(1 - \frac{AB}{CD}\frac{(1-q^{k_1})(1-q^{k_2})}{(1-q^{k_2}B/C)(1-q^{k_1}A/D)}\right).$$

The evaluation (5.3) becomes

$$\begin{aligned} \frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} &= \sum_{\mathbf{k}\geqq\mathbf{0}}\Bigg(q^{n_1k_1+n_2k_2}C^{k_1}D^{k_2} \\ &\quad\times \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{k_2-k_1+1}B/C)_{k_1}}{(B)_{k_2}(A)_{k_1}} \\ &\quad\times \frac{(q^{k_1-k_2+1}A/D)_{k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}}\Bigg) \\ &\quad - \frac{AB}{CD}\sum_{\mathbf{k}\geqq\mathbf{0}}\Bigg(q^{n_1k_1+n_2k_2}C^{k_1}D^{k_2} \\ &\quad\times \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{k_2-k_1+1}B/C)_{k_1-1}(q^{k_1-k_2+1}A/D)_{k_2-1}}{(B)_{k_2}(A)_{k_1}} \\ &\quad\times \frac{(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1-1}(q)_{k_2-1}}\Bigg). \end{aligned}$$

Let the first of the two sums on the right-hand side of this identity be abbreviated by $Y_{\mathbf{n}}(A,B,C,D)$. Thus we may rewrite it as

$$\frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} = Y_{\mathbf{n}}(A,B,C,D) - \frac{(1-q^{n_1})(1-q^{n_2})}{(1-A)(1-B)} AB Y_{\mathbf{n}-\mathbf{e}}(qA,qB,qC,qD).$$

The solution of this recursive relation is

$$Y_{\mathbf{n}}(A,B,C,D) = \frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} \sum_{j\geqq 0} q^{j^2-j}(AB)^j \times \frac{(q^{n_1};q^{-1})_j(q^{n_2};q^{-1})_j}{(C;q)_j(D;q)_j},$$

consequently we obtain the transformation

$$\sum_{\mathbf{k}\geqq\mathbf{0}} \Bigg(q^{n_1k_1+n_2k_2} C^{k_1} D^{k_2} \times \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{k_2-k_1+1}B/C)_{k_1}}{(A)_{k_1}(B)_{k_2}} \times \frac{(q^{k_1-k_2+1}A/D)_{k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}} \Bigg) = \frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} {}_3\phi_2 \left[\begin{matrix} q^{-n_1}, q^{-n_2}, q; \\ C, D; \end{matrix} \; q, q^{n_1+n_2}AB \right]. \tag{5.32}$$

The transformation (5.6) becomes

$$\sum_{\mathbf{n}\geqq\mathbf{0}} \frac{(C)_{n_1}(D)_{n_2}}{(q)_{n_1}(q)_{n_2}} z_1^{n_1} z_2^{n_2} = \frac{(Bz_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty} \sum_{\mathbf{k}\geqq\mathbf{0}} \Bigg[(-1)^{|\mathbf{k}|} q^{\binom{k_1}{2}+\binom{k_2}{2}} \times \frac{(q^{k_2-k_1+1}B/C)_{k_1}(q^{k_1-k_2+1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}} \times C^{k_1}D^{k_2} \left(1 - \frac{AB}{CD} \frac{(1-q^{k_1})(1-q^{k_2})}{(1-q^{k_2}B/C)(1-q^{k_1}A/D)} \right) \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}} \Bigg].$$

The left-hand side is summable by the q-binomial theorem (4.5). Replacing z_1 by z_1/B, z_2 by z_2/A, C/B by E and D/A by F, we obtain after some manipulation

$$\frac{(Ez_1)_\infty(Fz_2)_\infty}{(z_1)_\infty(z_2)_\infty} = \sum_{\mathbf{k}\geqq\mathbf{0}} \Bigg[q^{2k_1k_2} \frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}} \times \left(1 - \frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{-k_2}E)(1-q^{-k_1}F)} \right) \frac{z_1^{k_1} z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}} \Bigg]. \tag{5.33}$$

In view of (4.17), this identity is a q-analogue for

$$(1-z_1)^{-e}(1-z_2)^{-f} = \frac{\left(1-\dfrac{z_1}{1-z_2}\dfrac{z_2}{1-z_1}\right)^{e+f}}{\left(1-\dfrac{z_1}{1-z_2}\right)^{e}\left(1-\dfrac{z_2}{1-z_1}\right)^{f}}.$$

Setting

$$R(z_1,z_2) = \sum_{\mathbf{k}\geqq\mathbf{0}} q^{2k_1k_2}\frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}}\frac{z_1^{k_1}z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}}$$

(5.33) may be rewritten as

$$\frac{(Ez_1)_\infty(Fz_2)_\infty}{(z_1)_\infty(z_2)_\infty} = R(z_1,z_2) - \frac{z_1z_2}{(1-z_1)(1-z_2)}R(qz_1qz_2).$$

Thus, we have

$$R(z_1,z_2) = \frac{(Ez_1)_\infty(Fz_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{j\geqq 0} q^{j^2-j}\frac{(z_1z_2)^j}{(Ez_1)_j(Fz_2)_j},$$

hence,

$$\text{(5.34)}\quad \sum_{\mathbf{k}\geqq\mathbf{0}} q^{2k_1k_2}\frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}}\frac{z_1^{k_1}z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}} = \frac{(Ez_1)_\infty(Fz_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{j\geqq 0} q^{j^2-j}\frac{(z_1z_2)^j}{(Ez_1)_j(Fz_2)_j}.$$

From (4.18) we see that this is a q-analogue of

$$\frac{\left(1-\dfrac{z_1}{1-z_2}\dfrac{z_2}{1-z_1}\right)^{e+f-1}}{\left(1-\dfrac{z_1}{1-z_2}\right)^{e}\left(1-\dfrac{z_2}{1-z_1}\right)^{f}} = (1-z_1)^{-e}(1-z_2)^{-f}\left(1-\frac{z_1z_2}{(1-z_1)(1-z_2)}\right)^{-1}.$$

Finally, we turn to the dual transformation (5.7),

$$\sum_{\mathbf{k}\geq\mathbf{0}}\Bigg[(-1)^{|\mathbf{k}|}q^{k_1k_2-\binom{k_1+1}{2}-\binom{k_2+1}{2}}$$
$$\times\frac{(q^{-k_2+k_1-1}B/C;q^{-1})_{k_1}(q^{-k_1+k_2-1}A/D;q^{-1})_{k_2}}{(q^{-1};q^{-1})_{k_1}(q^{-1};q^{-1})_{k_2}}$$
$$\times C^{k_1}D^{k_2}\left(1-\frac{AB}{CD}\frac{(1-q^{-k_2})(1-q^{-k_1})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)}\right)z_1^{k_1}z_2^{k_2}\Bigg]$$
$$=\frac{(Bz_1)_\infty(Az_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(ABz_1z_2)_\infty}$$
$$\times\sum_{\mathbf{l}\geq\mathbf{0}}\Bigg[q^{l_1l_2-l_1-l_2}\frac{(C;q^{-1})_{l_1}(D;q^{-1})_{l_2}}{(q^{-1};q^{-1})_{l_1}(q^{-1};q^{-1})_{l_2}}$$
$$\times z_1^{l_1}z_2^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}\Bigg]$$

or, equivalently,

$$\sum_{\mathbf{k}\geq\mathbf{0}}\Bigg[q^{k_1k_2}\frac{(q^{-k_2}B/C)_{k_1}(q^{-k_1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}}$$
$$\times\left(1-\frac{AB}{CD}\frac{(1-q^{-k_2})(1-q^{-k_1})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)}\right)$$
$$\times(Cz_1)^{k_1}(Dz_2)^{k_2}\Bigg]=\frac{(Bz_1)_\infty(Az_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(ABz_1z_2)_\infty}$$
$$\times\sum_{\mathbf{l}\geq\mathbf{0}}q^{l_1l_2}\frac{(1/C)_{l_1}(1/D)_{l_2}}{(q)_{l_1}(q)_{l_2}}$$
$$\times(Cz_1)^{l_1}(Dz_2)^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}. \tag{5.35}$$

By use of (4.18), the left-hand side turns out to be equal to

$$\frac{(CDz_1z_2)_\infty(Az_2)_\infty(Bz_1)_\infty}{(q^{-1}ABz_1z_2)_\infty(Cz_1)_\infty(Dz_2)_\infty}(1-q^{-1}ABz_1z_2), \tag{5.36}$$

therefore from (5.35) we derive

$$\sum_{\mathbf{l}\geq\mathbf{0}}q^{l_1l_2}\frac{(1/C)_{l_1}(1/D)_{l_2}}{(q)_{l_1}(q)_{l_2}}(Cz_1)^{l_1}(Dz_2)^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}$$
$$=\frac{(Cz_1)_\infty(Dz_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(CDz_1z_2)_\infty}. \tag{5.37}$$

This identity appeared before in (5.28). Another identity, which was proved "en passant" by (5.36) is

$$\text{(5.38)}\quad \frac{(CDz_1z_2)_\infty(Az_2)_\infty(Bz_1)_\infty}{(ABz_1z_2)_\infty(Cz_1)_\infty(Dz_2)_\infty}=\sum_{\mathbf{k}\geqq\mathbf{0}}\left[q^{k_1k_2}\frac{(q^{-k_2}B/C)_{k_1}(q^{-k_1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}}\times\left(1-\frac{AB}{CD}\frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)}\right)\times(Cz_1)^{k_1}(Dz_2)^{k_2}\right].$$

After replacing z_1 by z_1/C, z_2 by z_2/D, B/C by q^α and A/D by q^β, we get by (4.2)

$$\text{(5.39)}\quad \frac{(z_1z_2)_{\alpha+\beta}}{(z_2)_\alpha(z_1)_\beta}=\sum_{\mathbf{k}\geqq\mathbf{0}}\left(q^{k_1k_2}\frac{(q^{\alpha-k_2+1})_{k_1-1}(q^{\beta-k_1+1})_{k_2-1}}{(q)_{k_1}(q)_{k_2}}\times\left((1-q^{\alpha-k_2})(1-q^{\beta-k_1})-q^{\alpha+\beta}(1-q^{-k_1})(1-q^{-k_2})\right)\right),$$

which also appears in [**6**, Theorem 10].

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