THE NUMBER OF CENTERED LOZENGE TILINGS OF A SYMMETRIC HEXAGON

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Abstract. Propp conjectured [15] that the number of lozenge tilings of a semiregular hexagon of sides $2n-1$, $2n-1$ and $2n$ which contain the central unit rhombus is precisely one third of the total number of lozenge tilings. Motivated by this, we consider the more general situation of a semiregular hexagon of sides a, a and b. We prove explicit formulas for the number of lozenge tilings of these hexagons containing the central unit rhombus, and obtain Propp's conjecture as a corollary of our results.

1. Introduction. Let a, b and c be positive integers, and consider a semiregular hexagon of sides a, b and c (i.e., all angles have 120 degrees and the sides have, in order, lengths a, b, c, a, b, c). By a well-known bijection [2], the number of tilings of this hexagon by rhombi of unit edge-length and angles of 60 and 120 degrees (we call such a rhombus a *lozenge* and such tilings *lozenge tilings*) is equal to the number $P(a,b,c)$ of plane partitions contained in an $a\times b\times c$ box. In turn, by a famous result of MacMahon [12], the latter is given by the product

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$$P(a,b,c)=\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{1.1}$$

The starting point of this paper is a conjecture of Propp [15] stating that for a semiregular hexagon of sides $2n-1$, $2n-1$ and $2n$, precisely one third of its lozenge tilings contain the central lozenge. Call the lozenge tilings with this property *centered*. In this paper we consider the following more general problem: for a semiregular hexagon of sides a, a and b, how many of its tilings are centered? It is easy to see that such a hexagon has a central lozenge only if a and b have opposite parity. The two cases are addressed in Theorems 1 and 2 below.

For any nonnegative integer m and positive integer n define

$$Q(m,n)=\frac{(2n)!^2\,(2m)!\,(m+2n-1)!}{2\cdot n!^2\,m!\,(2m+4n-2)!}\left(\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}\right), \tag{1.2}$$

where the shifted factorial $(a)_k$ is defined by $(a)_k:=a(a+1)\cdots(a+k-1)$, $k\geq 1$, and $(a)_0:=1$.

Theorem 1. *Let m be a nonnegative integer and n a positive integer. The number of centered lozenge tilings of a semiregular hexagon with sides $2n-1$, $2n-1$ and $2m$ is $Q(m,n)P(2n-1,2n-1,2m)$.*

Theorem 2. *Let m and n be positive integers. The number of centered lozenge tilings of a semiregular hexagon with sides $2n$, $2n$ and $2m-1$ is $Q(m,n)P(2n,2n,2m-1)$.*

In the case when m equals n, the expression $Q(m,n)$ evaluates to $1/3$ (this is due to a remarkable simplification of the sum in (1.2) in this case). Thus, the statement in Propp's conjecture follows from Theorem 1.

Corollary 3. *Let n be a positive integer. Exactly one third of the lozenge tilings of a semiregular hexagon with sides $2n-1$, $2n-1$ and $2n$ contain the central lozenge. The same is true for a semiregular hexagon with sides $2n$, $2n$ and $2n-1$.*

Theorems 1 and 2 have also been found (in an equivalent form) by Helfgott and Gessel [5, Theorem 15], using a completely different method. They also obtained Corollary 3 as a corollary of their results.

For $m\neq n$ the sum in (1.2) does not seem to simplify. However, if m and n approach infinity so that their ratio approaches some non-negative real number a, $Q(m,n)$ turns out to approach the value $\frac{2}{\pi}\arcsin(1/(a+1))$.

Corollary 4. *Let a be any nonnegative real number. For $m\sim an$, the proportion of the rhombus tilings that contain the central rhombus in the total number of rhombus tilings of a semiregular hexagon with sides $2n-1$, $2n-1$ and $2m$ is $\sim\frac{2}{\pi}\arcsin(1/(a+1))$ as n tends to infinity. The same is true for a semiregular hexagon with sides $2n$, $2n$ and $2m-1$.*

Remark. Using the bijection [2] between lozenge tilings and plane partitions, the statement of Theorem 1 can be interpreted as follows. Let $\mathcal{P}$ be the set of plane partitions (a_{ij}) of square shape $(2n-1)^{2n-1}$, with entries between 0 and $2m$. Let $\mathcal{P}_k$ be the subset consisting of the plane partitions for which $a_{n+k,n+k} = m+k$, for $-\min(n-1,m) \le k \le \min(n-1,m)$. Then the number of elements in the union of the $\mathcal{P}_k$'s is $Q(m,n)P(2n-1,2n-1,2m)$ (this union is clearly disjoint).

A similar interpretation can be given to the statement of Theorem 2.

The rest of the paper is devoted to giving proofs of Theorems 1 and 2, and of Corollaries 3 and 4. In Section 2 we provide proofs of Corollaries 3 and 4, and outline the proofs of Theorems 1 and 2, the latter consisting of several steps. The details of these steps are then given in detail in the subsequent sections. These steps are the following. First, an application of the first author's Matchings Factorization Theorem [1, Theorem 1.2] allows to reduce our problem to the enumeration of lozenge tilings of *simply*-connected regions. This is described in Section 3. Then, in Section 4, lozenge tilings are translated into nonintersecting lattice paths. By the main theorem on nonintersecting lattice paths [3, 4], the number(s) of nonintersecting lattice paths that we are interested in can be immediately written down in form of a determinant (see Lemmas 5, 6, 7). Finally, in Section 5, these determinants are evaluated (see Lemmas 9 and 10).

2. Outline of proofs. Here we outline the proofs of Theorems 1 and 2 and we deduce Corollaries 3 and 4. We fill in the details in the subsequent sections. Denote by $L(R)$ the number of lozenge tilings of the region R.

Proof of Theorem 1. In Section 3 it is shown that the number of centered lozenge tilings of a semiregular hexagon with sides $2n-1$, $2n-1$ and $2m$ equals 2^{2n-2} times the product of the number of lozenge tilings of two regions of the triangular lattice, H^+ and H^- (see (3.2) and Figure 3.3). Then, in Section 4 we use the Gessel-Viennot method of nonintersecting lattice paths to obtain determinantal expressions for $L(H^+)$ (see Lemma 5) and $L(H^-)$ (see Lemma 7). Finally, in Section 5 we evaluate these determinants (see Lemmas 9 (with $N = 2n-2$) and 10). After some manipulation of the expressions on the right-hand sides of (5.2) and (5.3) one obtains the statement of the Theorem. $\square$

Proof of Theorem 2. We proceed along the same lines as in the proof of Theorem 1. In Section 3 we show that the number of centered lozenge tilings of a semiregular hexagon with sides $2n$, $2n$ and $2m-1$ equals 2^{2n-1} times the product of the number of lozenge tilings of two regions of the triangular lattice, $\bar{H}^+$ and H^- (see (3.3) and Figure 3.4). The region H^- is the same as the one appearing in the proof of Theorem 1, while $\bar{H}^+$ differs from H^+ only in the sizes of its sides. The determinant evaluations in Lemmas 9 (with $N = 2n$ and m replaced by $m-1$) and 16 lead, after some manipulation of the expressions involved, to the statement of the Theorem. $\square$

Proof of Corollary 3. We have to compute the value of the expression on the right-hand side of (1.2) for $m = n$. Clearly, except for trivial manipulations, we will be done once we are able to evaluate the sum in (1.2) for $m = n$.

We claim that

$$\sum_{i=0}^{n-1} \frac{(-1)^{n-i-1}}{(2n-2i-1)} \frac{(2n-i)_{2i}}{i!^2} = 3^{n-1} \frac{\prod_{i=1}^{n-1}(6i-1)(6i+1)}{(2n-1)!!^2} \tag{2.1}$$

(where the empty product is defined to be 1). Let us denote the sum by $S(n)$ and its summand by $F(n,i)$. We use the Gosper–Zeilberger algorithm [14, 19, 20] to obtain the relation

$$n(2n+1)^2 F(n+1,i) - 3n\,(6n-1)\,(6n+1)\,F(n,i) = G(n,i+1) - G(n,i), \tag{2.2}$$

with

$$G(n,i) = (-1)^{n-i}\left(-3+9\,i-6\,i^2-30\,n+62\,in-28\,i^2n-104\,n^2+104\,in^2-112\,n^3\right) \\ \times \frac{i^2\,(2n-i+2)_{2i-2}}{(2n-2i+1)\;i!^2}.$$

Summation of the relation (2.2) from $i=0$ to $i=n$, little rearrangement, and division by n on both sides, leads to the recurrence

$$(2n+1)^2 S(n+1) - 3\,(6n-1)\,(6n+1)\,S(n) = 0$$

for the sum in (2.1). (Paule and Schorn's [13] *Mathematica* implementation of the Gosper–Zeilberger algorithm, which is the one we used, gives this recurrence directly.) Since $S(1)=1$, and since the right-hand side of (2.1) satisfies the same recurrence, equation (2.1) is proved, and, thus, the Corollary also. $\square$

Proof of Corollary 4. First let $a>0$. We have to determine the limit of $Q(m,n)$ as n tends to infinity, and where the relation between m and n is fixed by $m\sim an$. Clearly, the "difficult" part of this asymptotic computation is to find the asymptotics of the sum in (1.2). It turns out that it is convenient to manipulate this sum first, before taking the limit $n\to\infty$. We reverse the order of summation in the sum in (1.2), and then are able to rewrite the sum using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{k!\,(b_1)_k\cdots(b_s)_k}z^k\ . \tag{2.3}$$

(The reader should notice that the sum in (1.2) cannot be directly converted into a hypergeometric series since the upper bound on the summation index, $n-1$, is "artificial", i.e., the sum is changed if we extend the range of summation to all nonnegative i. If however the order of summation is reversed, this problem is removed because of the presence of the term $i!$ in the denominator of the summand.) Thus, expression (1.2) is converted into

$$\frac{(2n)!^2\,(2m)!\,(m+2n-1)!}{2\cdot n!^2\,m!\,(2m+4n-2)!}\,\frac{(m+1)_{2n-2}}{(n-1)!^2}\,{}_4F_3\left[\begin{matrix}1,\frac{1}{2},1-n,1-n\\ 1+m,2-m-2\,n,\frac{3}{2}\end{matrix};1\right].$$

Next we apply Bailey's transformation for a balanced ${}_4F_3$-series (see [17, (4.3.5.1)]),

$${}_4F_3\left[\begin{matrix}a,b,c,-N\\ e,f,1+a+b+c-e-f-N\end{matrix};1\right] \\ \frac{(e-a)_N\,(f-a)_N}{(e)_N\,(f)_N}\,{}_4F_3\left[\begin{matrix}-N,a,1+a+c-e-f-N,1+a+b-e-f-N\\ 1+a+b+c-e-f-N,1+a-e-N,1+a-f-N\end{matrix};1\right],$$

where N is a nonnegative integer. Thus we obtain the expression

$$\frac{(2n)!^2\,(2m)!\,(m+2n-1)!}{2\cdot n!^2\,m!\,(2m+4n-2)!}\,\frac{(m)_{n-1}\,(m+n+1)_{n-1}}{(n-1)!^2}\,{}_4F_3\left[\begin{matrix}1-n,1,1,\frac{1}{2}+n\\ \frac{3}{2},2-m-n,1+m+n\end{matrix};1\right].$$

Now we substitute $m\sim an$ and perform the limit $n\to\infty$. Using Stirling's formula, it is easy to determine the limit for the quotient in front of the ${}_4F_3$-series. It is

$$\frac{2\sqrt{a(a+2)}}{\pi(a+1)^2}. \tag{2.4}$$

For the ${}_4F_3$-series itself, we may exchange limit and summation, because of uniform convergence. This gives

$$\lim_{n\to\infty}{}_4F_3\left[\begin{matrix}1-n,1,1,\frac{1}{2}+n\\ \frac{3}{2},2-m-n,1+m+n\end{matrix};1\right]={}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};\frac{1}{(a+1)^2}\right]. \tag{2.5}$$

(Recall that $m\sim an$.) Combining (2.4) and (2.5), and using the identity (see [16, p. 463, (133)])

$${}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};z\right]=\frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}}$$

in (2.5), we obtain the desired limit $\frac{2}{\pi}\arcsin(1/(a+1))$.

Finally we address the case $a=0$, which means that $m\sim 0$. Then the proportion (1.2) is arbitrarily close to the expression which results from (1.2) when $m\to 0$. In particular, according to Theorem 1, this expression gives the proportion of the rhombus tilings that contain the central rhombus in the total number of rhombus tilings of the semiregular hexagon with sides $2n-1$, $2n-1$ and 0. That proportion is simply 1 since, trivially, there is exactly one such rhombus tiling, and it does contain the central rhombus. The value of 1 agrees with the claimed expression $\frac{2}{\pi}\arcsin(1/(a+1))$ evaluated at $a=0$. $\square$

3. Reduction to simply-connected regions. One useful way to approach certain tiling enumeration problems is to biject them with nonintersecting lattice paths, and then use the Gessel-Viennot determinant theorem [3, 4]. This approach seems to be especially appropriate if the entries of the Gessel-Viennot matrix have a simple expression. In the case of the $(2n-1)\times(2n-1)\times 2m$ semiregular hexagon with the central lozenge removed (whose tilings can clearly be identified with the centered tilings we are concerned with) this is not quite the case. However, one can get around this using the Factorization Theorem for perfect matchings presented in [1, Theorem 1.2].

Consider the tiling of the plane by unit equilateral triangles, illustrated in Figure 3.1. Define a *region* to be the union of finitely many such unit triangles. Suppose the region R is symmetric with respect to the horizontal symmetry axis l. Suppose further that the unit triangles of R crossed by l can be grouped in pairs such that the two triangles in a pair share an edge, forming a rhombic tile. Let $T_1,\dots,T_k$ be these rhombi. Let P be the zig-zag lattice path that borders the tiles T_i on their upper boundary (see Figure 3.2). Define R^+ and R^-

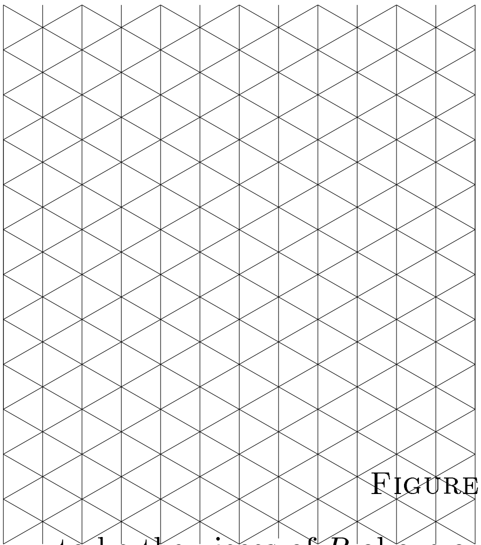


FIGURE 3.1

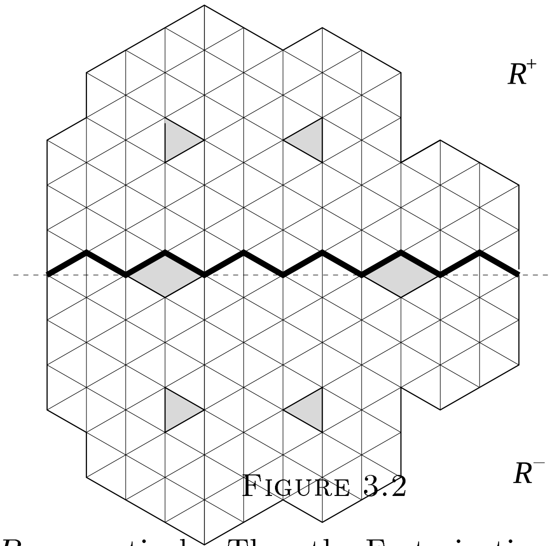


FIGURE 3.2

to be the pieces of R above and below P, respectively. Then the Factorization Theorem of [1] implies

$$L(R) = 2^k L(R^+) L^*(R^-), \tag{3.1}$$

where $L(R)$ is the number of lozenge tilings of R, and $L^*(R^-)$ is the weighted count of the lozenge tilings of R^- assigning weight 2^{-i} to a lozenge tiling containing i of the rhombi $T_1, \dots, T_k$ (a similar corollary of the Factorization Theorem is given in Remark 2.3 of [1] for the square lattice).

Let H be the region obtained from a semiregular hexagon with side-lengths $2n-1, 2n-1, 2m$ by removing the central lozenge (see Figure 3.3). By (3.1) we obtain

$$L(H) = 2^{2n-2} L(H^+) L^*(H^-), \tag{3.2}$$

where the regions H^+ and H^- are indicated in Figure 3.3 (H^+ is obtained from the piece above the zig-zag path by removing $2m$ forced lozenges, which are indicated by a shading). Similarly, applying (3.1) to the hexagonal region $\bar{H}$ (illustrated in Figure 3.4) with side-lengths $2n, 2n, 2m-1$ and central lozenge removed we obtain

$$L(\bar{H}) = 2^{2n-1} L(\bar{H}^+) L^*(H^-). \tag{3.3}$$

(Indeed, the region obtained from the bottom piece in Figure 3.4 after removing the forced tiles, which are again indicated by a shading, is the same as the region H^- in (3.2); this explains the last factor in (3.3); $\bar{H}^+$ is shown in Figure 3.4).

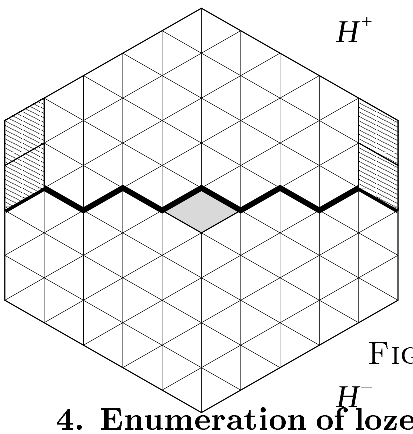


FIGURE 3.3

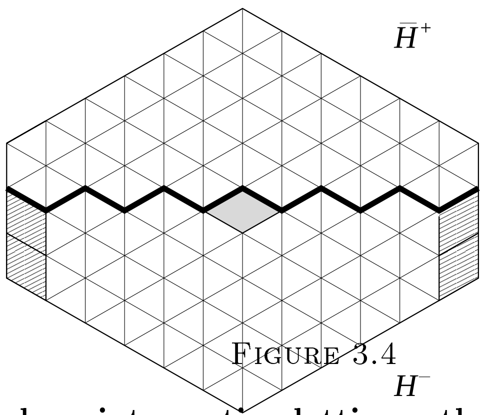


FIGURE 3.4

4. Enumeration of lozenge tilings and nonintersecting lattice paths. In this section we transform the problem of enumerating lozenge tilings of the regions that arose in Section 2 into a problem of enumerating families of nonintersecting lattice paths. Thus we derive determinantal formulas for the number of lozenge tilings we are interested in.

Our next Lemma exhibits an expression for $L(H^+)$.

Lemma 5. *We have*

$$L(H^+) = \det_{1\le i,j\le 2n-2}\left(\binom{2n+m-i-1}{m+i-j}\right). \tag{4.1}$$

Proof. Using the correspondence with nonintersecting lattice paths, the number of lozenge tilings of the upper "half hexagon" H^+ (see Figure 3.3) equals the number of families $(P_1, P_2, \dots, P_{2n-2})$ of nonintersecting lattice paths (consisting of horizontal positive unit steps and vertical negative unit steps) in which P_i runs from $A_i = (2i, i+m)$ to $E_i = (2n+i-1, i)$, $i = 1, 2, \dots, 2n-2$. By the main theorem of nonintersecting lattice paths [4, Cor. 2; 18, Theorem 1.2], this number is given by the determinant

$$\det_{1\le i,j\le 2n-2}\big(|\mathcal{P}(A_i \to E_j)|\big),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E and $|\mathcal{M}|$ is the cardinality of the set $\mathcal{M}$. Obviously, the number $|\mathcal{P}(A_i \to E_j)|$ of paths from A_i to E_j equals the binomial $\binom{2n+m-i-1}{m+i-j}$. This establishes the Lemma. $\square$

In the same way, we may derive a determinantal formula for $L(\bar{H}^+)$. As we have already noted in Section 2, the only difference between the regions H^+ and $\bar{H}^+$ is in their side lengths. To be precise, the vertical sides of H^+ have length m, while those of $\bar{H}^+$ have length $m-1$. On the other hand, the slanted sides of H^+ have length $2n-2$, while those of $\bar{H}^+$ have length $2n$. Hence, we will get a formula for $L(\bar{H}^+)$ by replacing m by $m-1$ and n by $n+1$ in the formula for $L(H^+)$.

Lemma 6. *We have*

$$L(\bar{H}^+) = \det_{1\le i,j\le 2n}\left(\binom{2n+m-i}{m+i-j-1}\right). \tag{4.2}$$

□

Next we turn to the number $L(H^-)$. In analogy to the preceding, we derive also a determinantal expression for $L(H^-)$, using nonintersecting lattice paths. However, the resulting determinant is not as "regular" as the preceding one, and is therefore harder to evaluate.

Lemma 7. *We have*

$$L(H^-) = \det_{1\le i,j\le 2n-1}\left(\frac{(2n+m-i-1)!}{(m+i-j)!\,(2n-2i+j)!}\begin{Bmatrix}(n+m-\frac{j}{2}) & \text{if } i\ne n\\ j & \text{if } i=n\end{Bmatrix}\right). \tag{4.3}$$

Proof. Using the correspondence with nonintersecting lattice paths, the number of lozenge tilings of the lower "half hexagon" H^- (see Figure 3.3) equals the number of families $(P_1,P_2,\dots,P_{2n-1})$ of nonintersecting lattice paths (consisting of horizontal positive unit steps and vertical negative unit steps) in which P_i runs from $A_i=(2i,i+m)$ to $E_i=(2n+i,i)$, $i=1,2,\dots,n-1,n+1,\dots,2n-1$, whereas P_n runs from $A_n=(2n+1,n+m)$ to $E_n=(3n,n)$ (i.e., it is the starting point A_n which deviates slightly from the "general" rule). In this count, horizontal steps originating from any A_i, $i\ne n$, count with weight $1/2$. Again, by the main theorem of nonintersecting lattice paths, this number is given by the determinant

$$\det_{1\le i,j\le 2n-2}\left(|\mathcal{P}(A_i\to E_j)|_w\right),$$

where $|.|_w$ denotes this weighted count. It is not difficult to see that the weighted count $|\mathcal{P}(A_i\to E_j)|_w$ of paths from A_i to E_j equals $\frac{(n+m-j/2)\,(2n+m-i-1)!}{(m+i-j)!\,(2n-2i+j)!}$ if $i\ne n$ and $\binom{n+m-1}{m+n-j}=\frac{j\,(n+m-1)!}{(m+n-j)!\,j!}$ otherwise. This establishes the Lemma. □

5. Determinant evaluations. Here we evaluate the determinants which appear in Lemmas 5, 6 and 7.

In order to compute the determinants in (4.1) and (4.2), we utilize the following determinant lemma from [6, Lemma 2.2].

Lemma 8. *Let $X_1,\dots,X_N$, $A_2,\dots,A_N$, and $B_2,\dots B_N$ be indeterminates. Then there holds*

$$\det_{1\le i,j\le N}\Big((X_i+A_N)\cdots(X_i+A_{j+1})(X_i+B_j)\cdots(X_i+B_2)\Big)$$
$$=\prod_{1\le i<j\le N}(X_i-X_j)\prod_{2\le i\le j\le N}(B_i-A_j). \quad (5.1)$$

□

Now we can state, and prove, the evaluations of the determinants in (4.1) and (4.2), in a unified fashion.

Lemma 9. *For any positive integer n there holds*

$$\det_{1\le i,j\le N}\left(\binom{N+m-i+1}{m+i-j}\right)=\prod_{i=1}^{N}\frac{(N+m-i+1)!\,(i-1)!\,(2m+i+1)_{i-1}}{(m+i-1)!\,(2N-2i+1)!}. \quad (5.2)$$

Proof. We take $(N+m-i+1)!/\big((m+i-1)!\,(2N-2i+1)!\big)$ out of the i-th row of the determinant in (5.2), $i=1,2,\dots,N$. Thus we obtain

$$\prod_{i=1}^{N}\frac{(N+m-i+1)!}{(m+i-1)!\,(2N-2i+1)!}$$
$$\times\det_{1\le i,j\le N}\big((m+i-j+1)\cdots(m+i-1)(N-2i+j+2)\cdots(2N-2i+1)\big).$$

Taking $(-2)^{N-j}$ out of the j-th column, $j=1,2,\dots,N$, we may write this as

$$(-2)^{\binom N2}\prod_{i=1}^{N}\frac{(N+m-i+1)!}{(m+i-1)!\,(2N-2i+1)!}$$
$$\times\det_{1\le i,j\le N}\left(\Big(i-\frac{2N+1}{2}\Big)\cdots\Big(i-\frac{N+j+2}{2}\Big)(i+m-j+1)\cdots(i+m-1)\right).$$

Now Lemma 8 can be applied with $X_i=i$, $A_j=-(N+j+1)/2$, $B_j=m-j+1$. After some simplification one arrives at the right-hand side of (5.2). □

The determinant in (4.3) evaluates as follows.

Lemma 10. *For any positive integer n there holds*

$$\det_{1\le i,j\le 2n-1}\left(\frac{(2n+m-i-1)!}{(m+i-j)!\,(2n-2i+j)!}\begin{Bmatrix}(n+m-\frac{j}{2}) & \text{if } i\ne n\\ j & \text{if } i=n\end{Bmatrix}\right)$$
$$=\frac{1}{2^{3n-3}\,(n-1)!}\prod_{i=1}^{n}(2i-1)!^2\prod_{i=1}^{2n-1}\frac{(2n+m-i-1)!}{(m+i-1)!\,(4n-2i-1)!}\prod_{i=1}^{2n-2}(2m+i+1)_i$$
$$\times\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}. \quad (5.3)$$

Proof. The method that we use for this proof is also applied successfully in [10, 7, 8, 9, 11] (see in particular the tutorial description in [9, Sec. 2]).

First of all, as in the proof of Lemma 9, we take appropriate factors out of the determinant. To be precise, we take $(2n+m-i-1)!/\big((m+i-1)!\,(4n-2i-1)!\big)$ out of the i-th row of the determinant in (5.3), $i=1,2,\dots,2n-1$. Thus we obtain

$$\prod_{i=1}^{2n-2}\frac{(2n+m-i-1)!}{(m+i-1)!\,(4n-2i-1)!}\\ \times\det_{1\le i,j\le 2n-1}\Bigg((m+i-j+1)\cdots(m+i-1)(2n-2i+j+1)\cdots(4n-2i-1)\\ \cdot\begin{Bmatrix}(n+m-\frac{j}{2}) & \text{if } i\ne n\\ j & \text{if } i=n\end{Bmatrix}\Bigg). \tag{5.4}$$

Let us denote the determinant in (5.4) by $D(m;n)$. Using the notation of shifted factorials, this means that

$$D(m;n):=\det_{1\le i,j\le 2n-1}\left((m+i-j+1)_{j-1}\,(2n-2i+j+1)_{2n-j-1}\begin{Bmatrix}(n+m-\frac{j}{2}) & \text{if } i\ne n\\ j & \text{if } i=n\end{Bmatrix}\right). \tag{5.5}$$

Comparison of (5.3) and (5.4) yields that (5.3) will be proved once we are able to establish the determinant evaluation

$$D(m;n)=2^{2(n-1)(n-2)}\frac{\prod_{i=1}^{n}(2i-1)!^2}{(n-1)!}\prod_{i=1}^{n-1}\big((m+i)_{2n-2i}\,(m+i+1/2)_{n-1}\big)\\ \times\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}. \tag{5.6}$$

For the proof of (5.6) we proceed in several steps. An outline is as follows. In the first step we show that $\prod_{i=1}^{n-1}(m+i)_{2n-2i}$ is a factor of $D(m;n)$ as a polynomial in m. In the second step we show that $\prod_{i=1}^{n-1}(m+i+1/2)_{n-1}$ is a factor of $D(m;n)$. In the third step we determine the maximal degree of $D(m;n)$ as a polynomial in m, which turns out to be $(2n+1)(n-1)$. From a combination of these three steps we are forced to conclude that

$$D(m;n)=\prod_{i=1}^{n-1}\big((m+i)_{2n-2i}\,(m+i+1/2)_{n-1}\big)P(m;n), \tag{5.7}$$

where $P(m;n)$ is a polynomial in m of degree at most $2n-2$. Then, in the fourth step we show that $P(m;n)=P(1-2n-m;n)$. And, in the fifth step, we evaluate $P(m;n)$ at $m=0,-1,\dots,-n+1$. Namely, for $m=0,-1,\dots,-n+1$ we show that

$$P(m;n)=(-1)^n\,2^{2(n-2)(n-1)}\frac{\prod_{i=1}^{n}(2i-1)!^2}{(n-1)!}\frac{(\frac{1}{2}+n)_{m+n-1}}{2(\frac{1}{2}-n)_{m+n}}. \tag{5.8}$$

Clearly the latter two properties determine a polynomial of maximal degree $2n-2$ uniquely. As is easy to check, the sum in (5.6) has the first property, too, namely that it is invariant under replacement of m by $1-2n-m$. Since in the sixth step we prove that for $m=0,-1,\dots,-n+1$ we also have

$$\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}=(-1)^n\frac{(\frac{1}{2}+n)_{m+n-1}}{2(\frac{1}{2}-n)_{m+n}}$$

we are forced to conclude that

$$P(m;n)=2^{2(n-2)(n-1)}\frac{\prod_{i=1}^{n}(2i-1)!^2}{(n-1)!}\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}. \tag{5.9}$$

This would finish the proof of the Lemma since a combination of (5.7) and (5.9) gives (5.6), and thus (5.3), as we already noted.

Step 1. $\prod_{i=1}^{n-1}(m+i)_{2n-2i}$ *is a factor of* $D(m;n)$. For i between 1 and $n-1$ let us consider row $2n-i$ of the determinant $D(m;n)$. Recalling the definition (5.5) of $D(m;n)$, we see that the j-th entry in this row has the form

$$(m+2n-i-j+1)_{j-1}\,(-2n+2i+j+1)_{2n-j-1}\binom{n+m-\frac{j}{2}}{}.$$

Since $(-2n+2i+j+1)_{2n-j-1}=0$ for $j=1,2,\dots,2n-2i-1$, the first $2n-2i-1$ entries in this row vanish. Therefore $(m+i)_{2n-2i}$ is a factor of each entry in row $2n-i$, $i=1,2,\dots,n-1$. Hence, the complete product $\prod_{i=1}^{n-1}(m+i)_{2n-2i}$ divides $D(m;n)$.

Step 2. $\prod_{i=1}^{n-1}(m+i+1/2)_{n-1}$ *is a factor of* $D(m;n)$. Let us concentrate on a typical factor $(m+j+l+1/2)$, $1\le j\le n-1$, $0\le l\le n-2$. We claim that for each such factor there is a linear combination of the rows that vanishes if the factor vanishes. More precisely, we claim that for any j,l with $1\le j\le n-1$, $0\le l\le n-2$ there holds

$$\begin{aligned}(-1)^{j-1}&\frac{(n-j-l-\frac{1}{2})_j}{4^j(n-j-l)_j}\\ &\qquad\cdot(\text{column }(2n-2j-2l-1)\text{ of }D(-j-l-1/2;n))\\ &+\sum_{s=2n-j-2l-1}^{2n-2l-1}\binom{j}{s+j+2l-2n+1}\cdot(\text{column }s\text{ of }D(-j-l-1/2;n))=0\end{aligned}\tag{5.10}$$

if $j+l<n$, and

$$\begin{aligned}\sum_{s=n-l}^{2j+1}&\binom{2j+l-n+1}{s+l-n}\cdot(\text{column }s\text{ of }D(-j-l-1/2;n))\\ &+\frac{(-1)^{n-l}}{4^{n-l-1}}\frac{(j+l-n+\frac{3}{2})_{n-l-1}}{(j+l-n+2)_{n-l-1}}\\ &\times\sum_{s=1}^{2j+2l-2n+3}\binom{2j+2l-2n+2}{s-1}\cdot(\text{column }s\text{ of }D(-j-l-1/2;n))=0\end{aligned}\tag{5.11}$$

if $j+l\ge n$.

In order to verify (5.10), for $j+l<n$ we have to check

$$\sum_{s=2n-j-2l-1}^{2n-2l-1}\binom{j}{s+j+2l-2n+1}\\ \cdot\left(n-j-l-\frac{s}{2}-\frac{1}{2}\right)(i-j-l-s+\tfrac{1}{2})_{s-1}\,(2n-2i+s+1)_{2n-s-1}=0,$$

which is (5.10) restricted to the i-th row, $i\ne n$ (note that the entry in column $2n-2j-2l-1$ of $D(-j-l-1/2;n)$ vanishes in such a row), and

$$(-1)^{j-1}\frac{(j+l-n+\frac{3}{2})_{2n-j-2l-2}\,(2n-2j-2l-1)_{2j+2l+1}}{4^j\,(n-j-l)_j}\\ +\sum_{s=2n-j-2l-1}^{2n-2l-1}\binom{j}{s+j+2l-2n+1}(n-j-l-s+\tfrac{1}{2})_{s-1}\,(s)_{2n-s}=0,$$

which is (5.10) restricted to the n-th row. Equivalently, using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{k!\,(b_1)_k\cdots(b_s)_k}z^k\ ,$$

this means to check

$$\frac{j}{2}(i+l-2n+\tfrac{3}{2})_{2n-j-2l-2}\,(4n-2i-j-2l)_{j+2l}\ {}_3F_2\left[\begin{matrix}1+j,-\frac{1}{2}-i-l+2n,-j\\ j,-2i-j-2l+4n\end{matrix};1\right]=0, \tag{5.12}$$

and

$$(-1)^{j-1}\frac{(j+l-n+\frac{3}{2})_{2n-j-2l-2}\,(2n-2j-2l-1)_{2j+2l+1}}{4^j\,(n-j-l)_j}\\ +(l-n+\tfrac{3}{2})_{2n-j-2l-2}\,(2n-j-2l-1)_{j+2l+1}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2}-l+n,-j\\ -1-j-2l+2n\end{matrix};1\right]=0. \tag{5.13}$$

In order to verify (5.11), for $j+l\ge n$ we have to check

$$\sum_{s=n-l}^{2j+1}\binom{2j+l-n+1}{s+l-n}\left(n-j-l-\frac{s}{2}-\frac{1}{2}\right)\\ \cdot(i-j-l-s+\tfrac{1}{2})_{s-1}\,(1-2i+2n+s)_{2n-s-1}\\ +\frac{(-1)^{n-l}}{4^{n-l-1}}\frac{(j+l-n+\frac{3}{2})_{n-l-1}}{(j+l-n+2)_{n-l-1}}\sum_{s=1}^{2j+2l-2n+3}\binom{2j+2l-2n+2}{s-1}\\ \cdot\left(n-j-l-\frac{s}{2}-\frac{1}{2}\right)(i-j-l-s+\tfrac{1}{2})_{s-1}\,(1-2i+2n+s)_{2n-s-1},$$

which is (5.11) restricted to the i-th row, $i \neq n$, and

$$\sum_{s=n-l}^{2j+1} \binom{2j+l-n+1}{s+l-n} (n-j-l-s+\tfrac{1}{2})_{s-1}\,(s)_{2n-s}$$
$$+\frac{(-1)^{n-l}}{4^{n-l-1}} \frac{(j+l-n+\frac{3}{2})_{n-l-1}}{(j+l-n+2)_{n-l-1}}$$
$$\times \sum_{s=1}^{2j+2l-2n+3} \binom{2j+2l-2n+2}{s-1} (n-j-l-s+\tfrac{1}{2})_{s-1}\,(s)_{2n-s},$$

which is (5.11) restricted to the n-th row. Equivalently, using hypergeometric notation, this means to check

$$\frac{(n-2j-l-1)}{2}(\tfrac{1}{2}+i-j-n)_{n-l-1}\,(3n-2i-l+1)_{n+l-1}$$
$$\times {}_3F_2\left[\begin{matrix} 2+2j+l-n, \frac{1}{2}-i+j+n, -1-2j-l+n \\ 1+2j+l-n, 1-2i-l+3n \end{matrix}; 1\right]$$
$$+\frac{(-1)^{n-l}}{4^{n-l-1}} \frac{(n-j-l-1)\,(\frac{3}{2}+j+l-n)_{n-l-1}\,(2n-2i+2)_{2n-2}}{(2+j+l-n)_{n-l-1}}$$
$$\times {}_3F_2\left[\begin{matrix} 3+2j+2l-2n, \frac{3}{2}-i+j+l, -2-2j-2l+2n \\ 2+2j+2l-2n, 2-2i+2n \end{matrix}; 1\right]=0 \qquad (5.14)$$

and

$$(\tfrac{1}{2}-j)_{n-l-1}\,(n-l)_{n+l}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}+j, -1-2j-l+n \\ -l+n \end{matrix}; 1\right]$$
$$+\frac{(-1)^{n-l}}{4^{n-l-1}} \frac{(1)_{2n-1}\,(\frac{3}{2}+j+l-n)_{n-l-1}}{(2+j+l-n)_{n-l-1}}$$
$$\times {}_2F_1\left[\begin{matrix} \frac{3}{2}+j+l-n, -2-2j-2l+2n \\ 1 \end{matrix}; 1\right]=0. \qquad (5.15)$$

We start with the proof of (5.12). We apply the contiguous relation

$${}_3F_2\left[\begin{matrix} a, A_1, A_2 \\ B_1, B_2 \end{matrix}; z\right] = {}_3F_2\left[\begin{matrix} a-1, A_1, A_2 \\ B_1, B_2 \end{matrix}; z\right] + z\frac{A_1 A_2}{B_1 B_2}\,{}_3F_2\left[\begin{matrix} a, A_1+1, A_2+1 \\ B_1+1, B_2+1 \end{matrix}; z\right]$$

to the ${}_3F_2$-series in (5.12). Since in this case $a-1=B_1$, parameters cancel inside the two ${}_3F_2$-series on the right hand side of the contiguous relation, leaving two ${}_2F_1$-series instead. Thus, (5.12) is turned into

$$\frac{j}{2}(i+l-2n+\tfrac{3}{2})_{2n-j-2l-2}\,(4n-2i-j-2l)_{j+2l}\Bigg({}_2F_1\left[\begin{matrix} 2n-i-l-\frac{1}{2}, -j \\ 4n-2i-j-2l \end{matrix}; 1\right]$$
$$-\frac{(2n-i-l-\frac{1}{2})}{4n-2i-j-2l}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}-i-l+2n, 1-j \\ 1-2i-j-2l+4n \end{matrix}; 1\right]\Bigg)=0.$$

Each of the two ${}_2F_1$-series can be evaluated by means of the Chu–Vandermonde summation (see [17, (1.7.7); Appendix (III.4)]),

$$ {}_2F_1\left[\begin{matrix} a, -N \\ c \end{matrix}; 1\right] = \frac{(c-a)_N}{(c)_N}, \tag{5.16}$$

where N is a nonnegative integer, and thus (5.12) follows upon minor simplification (the terms in big parentheses cancel each other).

To the ${}_2F_1$-series in (5.13) Chu–Vandermonde summation (5.16) can be applied directly, and it yields the desired result.

The verifications of (5.14) and (5.15) are similar. The reader will have no difficulties to fill in the details.

This finishes the proof that the product $\prod_{i=1}^{n-1}(m+i+1/2)_{n-1}$ divides $D(m;n)$.

Step 3. $D(m;n)$ is a polynomial in m of maximal degree $(2n+1)(n-1)$. Obviously, the degree in m of the (i,j)-entry in the determinant $D(m;n)$ is j for $i \neq n$, while it is $j-1$ for $i=n$. Hence, in the defining expansion of the determinant, each term has degree $\left(\sum_{j=1}^{2n-1} j\right) - 1 = \binom{2n}{2} - 1 = (2n+1)(n-1)$.

Step 4. $P(m;n) = P(1-2n-m;n)$. We claim that there holds the relation

$$D(m;n) = (-1)^{n-1} D(1-2n-m;n). \tag{5.17}$$

It is clear from the definition (5.7) of $P(m;n)$ that (5.17) immediately implies the desired relation $P(m;n) = P(1-2n-m;n)$.

We prove (5.17) by, up to sign, transforming the determinant $D(m;n)$ into the determinant $D(1-2n-m;n)$ by a sequence of elementary column operations (which, of course, leave the value of the determinant invariant). To be precise, for $j = 2n-1, 2n-2, \dots, 2$, we add

$$\sum_{k=1}^{j-1} \binom{j-1}{k-1} \cdot (\text{column } k \text{ of } D(m;n))$$

to column j. Thus, in the new determinant, $D_1(m;n)$ say, the (i,j)-entry is

$$\sum_{k=1}^{j} \binom{j-1}{k-1} \left(m+n-\frac{k}{2}\right) (m+i-k+1)_{k-1}\,(2n-2i+k+1)_{2n-k-1}$$

for $i \neq n$, and

$$\sum_{k=1}^{j} \binom{j-1}{k-1} (m+n-k+1)_{k-1}\,(k)_{2n-k}$$

for $i=n$. Using hypergeometric notation, the (i,j)-entry of $D_1(m;n)$ is

$$\left(m+n-\frac{1}{2}\right)(2-2i+2n)_{2n-2}\ {}_3F_2\left[\begin{matrix} 2-2m-2n, 1-i-m, 1-j \\ 1-2m-2n, 2-2i+2n \end{matrix}; 1\right] \tag{5.18}$$

for $i \neq n$, and

$$(1)_{2n-1}\ {}_2F_1\left[\begin{matrix}1-m-n, 1-j\\ 1\end{matrix}; 1\right] \tag{5.19}$$

for $i = n$. The ${}_3F_2$-series in (5.18) can be evaluated in the same way as we evaluated the ${}_3F_2$-series in (5.12). The ${}_2F_1$-series in (5.19) is easily evaluated by Chu–Vandermonde summation (5.16). Thus, we obtain that the (i,j)-entry of $D_1(m;n)$ is given by

$$(-1)^j \left(1 - \frac{j}{2} - m - n\right)(2+i-j-m-2n)_{j-1}\,(1-2i+j+2n)_{2n-j-1} \tag{5.20}$$

for $i \neq n$, and by

$$(-1)^{j-1} j\,(2-j-m-n)_{j-1}\,(j+1)_{2n-j-1} \tag{5.21}$$

for $i = n$. Now, the expression (5.20) is exactly $(-1)^j$ times the (i,j)-entry of $D(1-2n-m;n)$, while the expression (5.21) is exactly $(-1)^{j-1}$ times the (n,j)-entry of $D(1-2n-m;n)$. Hence, relation (5.17) follows immediately, implying $P(m;n) = P(1-2n-m;n)$, as we already noted.

Step 5. Evaluation of $P(m;n)$ at $m = 0, -1, \dots, -n+1$. The polynomial $P(m;n)$ is defined by means of (5.7),

$$D(m;n) = \prod_{i=1}^{n-1} \left((m+i)_{2n-2i}\,(m+i+1/2)_{n-1}\right) P(m;n). \tag{5.22}$$

So, what we would like to do is to set $m = -e$, e being one of $0, 1, \dots, n-1$, evaluate $D(-e;n)$, divide both sides of (5.22) by the product on the right-hand side of (5.22), and get the evaluation of $P(m;n)$ at $m = -e$. However, the product on the right-hand side of (5.22) unfortunately (usually) *is zero* for $m = -e$, $0 \le e \le n-1$. Therefore we have to find a way around this difficulty.

Fix an e with $0 \le e \le n-1$. Before setting $m = -e$ in (5.22), we have to cancel $(m+e)^e$ on the right-hand side of (5.22). To accomplish this, we have to "generate" these factors on the left-hand side. We do this by adding

$$\sum_{k=j+1}^{2e+2j-2n} \binom{2e+j-2n}{k-j} \cdot (\text{column } k \text{ of } D(m;n))$$

to column j, $j = 2n-2e, 2n-2e+1, \dots, 2n-e-1$. Thus, in the new determinant the entry in the i-th row in such a column is

$$\sum_{k=j}^{2e+2j-2n} \left(m+n-\frac{k}{2}\right)\binom{2e+j-2n}{k-j}(m+i-k+1)_{k-1}\,(2n-2i+k+1)_{2n-k-1}$$

if $i \neq n$, and

$$\sum_{k=j}^{2e+2j-2n} \binom{2e+j-2n}{k-j}(m+n-k+1)_{k-1}\,(k)_{2n-k}$$

if $i=n$. In hypergeometric terms this is

$$\Big(m+n-\frac{j}{2}\Big)(m+i-j+1)_{j-1}\,(2n-2i+j+1)_{2n-j-1}\\ \times {}_3F_2\left[\begin{matrix}1+j-2m-2n,-i+j-m,-2e-j+2n\\ j-2m-2n,1-2i+j+2n\end{matrix};1\right]$$

if $i\ne n$, and

$$(m+n-j+1)_{j-1}\,(j)_{2n-j}\,{}_2F_1\left[\begin{matrix}j-m-n,-2e-j+2n\\ j\end{matrix};1\right]$$

if $i=n$. Again, the ${}_3F_2$-series can be evaluated in the same way as before the ${}_3F_2$-series in (5.12), while the ${}_2F_1$-series is evaluated by means of the Chu–Vandermonde summation (5.16). Thus, we obtain that the (i,j)-entry, $2n-2e\le j\le 2n-e-1$, of the modified determinant is given by

$$(m+e)\,(m+i-j+1)_{j-1}\,(m+2n-i)_{2e+j-2n}\,(2e-2i+2j+1)_{4n-2e-2j-1}$$

if $i\ne n$, and by

$$(m+e)\ (m+n-j+1)_{e+j-n-1}\,(m+e+1)_{e+j-n-1}\,(2e+2j-2n)_{4n-2e-2j}$$

if $i=n$. Clearly, $(m+e)$ is a factor of each entry in the j-th column of the modified determinant, $2n-2e\le j\le 2n-e-1$. Therefore, we may take $(m+e)$ out of the j-th column, $j=2n-2e,2n-2e+1,\dots,2n-e-1$. The remaining determinant, $D_2(m;n)$ say, is then defined as $D_2(m;n)=\det_{1\le i,j\le 2n-1}(E_{ij})$, where for $1\le j\le 2n-2e-1$ and for $2n-e\le j\le 2n-1$ the entry E_{ij} is given by

$$E_{ij}=(m+i-j+1)_{j-1}\,(2n-2i+j+1)_{2n-j-1}\begin{cases}(n+m-\frac{j}{2}) & \text{if } i\ne n\\ j & \text{if } i=n\end{cases} \tag{5.23}$$

(i.e., E_{ij} equals the (i,j)-entry of $D(m;n)$, as given by (5.5), in that case), and for $2n-2e\le j\le 2n-e-1$ the entry E_{ij} is given by

$$E_{ij}=\begin{cases}(m+i-j+1)_{j-1}\,(m+2n-i)_{2e+j-2n}\,(2e-2i+2j+1)_{4n-2e-2j-1} & \text{if } i\ne n\\ (m+n-j+1)_{e+j-n-1}\,(e+m+1)_{e+j-n-1}\,(2e+2j-2n)_{4n-2e-2j} & \text{if } i=n.\end{cases} \tag{5.24}$$

Due to the manipulations that we did, the new determinant $D_2(m;n)$ is related to the original determinant $D(m;n)$ by

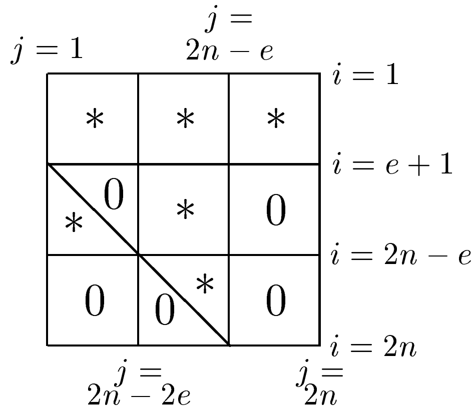
$$D(m;n)=(m+e)^e\,D_2(m;n).$$

Substituting this into (5.22), and rearranging terms, we get

$$P(m;n)=D_2(m;n)\prod_{i=1}^{e}\left((m+i)_{e-i}\,(m+e+1)_{2n-i-e-1}\,(m+i+1/2)_{n-1}\right)^{-1}\\ \times\prod_{i=e+1}^{n-1}\left((m+i)_{2n-2i}\,(m+i+1/2)_{n-1}\right)^{-1}. \tag{5.25}$$

Now we may safely set $m = -e$. So, what we need in order to obtain the evaluation of $P(m;n)$ at $m = -e$ is the evaluation of the determinant $D_2(-e;n)$.

In order to determine the evaluation of $D_2(-e;n)$, we observe that $D_2(-e;n)$ has a block form which is sketched in Figure 10.



The block form of $D_2(-e;n)$

Figure 10

The figure has to be read according to the following convention: If a block is bounded by horizontal lines marked as $i = h_1$ and $i = h_2$ and vertical lines marked as $j = v_1$ and $j = v_2$, then the block consists of the entries that are in rows $i = h_1, h_1 + 1, \dots, h_2 - 1$ and columns $j = v_1, v_1 + 1, \dots, v_2 - 1$. It is an easy task to check from the definitions (5.23) and (5.24) of the entries of $D_2(m;n)$ that indeed the lower left block of $D_2(-e;n)$, consisting of the entries in rows $i = 2n - e, 2n - e + 1, \dots, 2n - 1$ and columns $j = 1, 2, \dots, 2n - 2e - 1$, and the lower right block, consisting of the entries in rows $i = e+1, e+2, \dots, 2n-1$ and columns $j = 2n - e, 2n - e + 1, \dots, 2n - 1$ are blocks of zeroes. Hence, the determinant $D_2(-e;n)$ factors into the product

$$\det(B_1)\,\det(B_2)\,\det(B_3),$$

where B_1 is the middle left block, consisting of the entries in rows $i = e+1, e+2, \dots, 2n-e-1$ and columns $j = 1, 2, \dots, 2n - 2e - 1$, where B_2 is the lower middle block, consisting of the entries in rows $i = 2n-e, 2n-e+1, \dots, 2n-1$ and columns $j = 2n-2e, 2n-2e+1, \dots, 2n-e-1$, and where B_3 is the upper right block, consisting of the entries in rows $i = 1, 2, \dots, e$ and columns $j = 2n - e, 2n - e + 1, \dots, 2n - 1$.

As is indicated in Figure 10, the blocks B_1 and B_2 are lower and upper triangular matrices, respectively. Hence, their determinants are easily computed.

The determinant of the block B_3 is the determinant

$$\det_{1 \le i,j \le e} \left((-2n + i - j + 2)_{2n-e+j-2}\,(4n - e - 2i + j)_{e-j} \left(\frac{1 - e - j}{2} \right) \right).$$

(All the entries of B_3 are given by the first case of formula (5.23) because $e \le n - 1$.) We take $(-2n + i + 1)_{2n-e-1}$ out of the i-th row of this determinant, $i = 1, 2, \dots, e$, and

$(-2)^{e-j}\left(\frac{1-e-j}{2}\right)$ out of the j-th column, $j=1,2,\dots,e$. Thus we obtain

$$(-2)^{\binom{e}{2}}\prod_{i=1}^{e}\left(\frac{1-e-i}{2}\right)(-2n+i+1)_{2n-e-1}$$
$$\times\det_{1\le i,j\le e}\left(\left(i-2n+\frac{1}{2}\right)\cdots\left(i-2n+\frac{e-j}{2}\right)(i-2n-j+2)\cdots(i-2n)\right).$$

The latter determinant is easily evaluated using Lemma 8 with $N=e$, $X_i=i$, $A_j=-2n+(e-j+1)/2$, $B_j=-2n-j+2$.

This finishes the desired evaluation of $D_2(-e;n)$, and, via (5.25), of $P(-e;n)$ for $e=0,1,\dots,n-1$. If everything is put together and simplified, the result is exactly (5.8).

Step 6. Evaluation of the sum on the right-hand side of (5.3) *at* $m=0,-1,\dots,-n+1$. We claim that if $0\le e\le n-1$ we have

$$\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(n-e-i)_{2i}}{i!^2}=(-1)^n\frac{\left(\frac{1}{2}+n\right)_{n-e-1}}{2\left(\frac{1}{2}-n\right)_{n-e}}. \tag{5.26}$$

This is seen by first rewriting the sum in (5.26) as

$$\frac{(-1)^n}{(1-2n)}\sum_{i=0}^{n-1}\frac{\left(\frac{1}{2}-n\right)_i(1+e-n)_i\,(n-e)_i}{(1)_i^{\,2}\left(\frac{3}{2}-n\right)_i}.$$

Since $0\le e\le n-1$, the term $(1+e-n)_i$ will make the sum terminate at $i=n-e-1$, and so at $i=n-1$ latest. Therefore we may extend the range of summation to *all* nonnegative numbers i, without altering the sum. Then we can write the sum in hypergeometric notation as

$$\frac{(-1)^n}{(1-2n)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}-n,n-e,1+e-n\\ 1,\frac{3}{2}-n\end{matrix};1\right].$$

This ${}_3F_2$-series can be evaluated by means of the Pfaff–Saalschütz summation (see [17, (2.3.1.3); Appendix (III.2)]),

$${}_3F_2\left[\begin{matrix}a,b,-N\\ c,1+a+b-c-N\end{matrix};1\right]=\frac{(c-a)_N(c-b)_N}{(c)_N(c-a-b)_N},$$

where N is a nonnegative integer. Thus we arrive at the right-hand side of (5.26).

This completes the proof of the Lemma. $\square$

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