ON A HYPERGEOMETRIC IDENTITY OF GELFAND, GRAEV AND RETAKH

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Abstract. A hypergeometric identity equating a triple sum to a single sum, originally found by Gelfand, Graev and Retakh [Russian Math. Surveys **47** (1992), 1–88] by using systems of differential equations, is given hypergeometric proofs. As a bonus, several q-analogues can be derived.

1. Introduction

In a recent talk at the Institute of Mathematics and Its Applications, Minnesota, Richard Askey posed several problems for the audience, one of which concerned the following identity due to Gelfand, Graev and Retakh [3, p. 67],

$$\sum_{j,k,m\geq 0}\frac{(\alpha)_j\,(\beta)_k\,(1-\gamma)_m\,(\gamma)_{j+k-m}}{j!\,k!\,m!\,(\alpha+\beta)_{j+k-m}}x^jy^kz^m \tag{1.1}$$

$$=\frac{(1-z)^{\alpha+\beta-1}\,(1-xz)^{\gamma-\alpha-\beta}}{(1-x)^{\gamma-\beta}\,(1-y)^{\beta}}\,{}_2F_1\left[\begin{matrix}\beta,\alpha+\beta-\gamma\\ \alpha+\beta\end{matrix};\frac{(x-y)(1-z)}{(1-y)(1-xz)}\right] \tag{1.2}$$

$$=\frac{(1-z)^{\alpha+\beta-1}\,(1-xz)^{\gamma-\alpha}}{(1-x)^{\gamma}\,(1-yz)^{\beta}}\,{}_2F_1\left[\begin{matrix}\beta,\gamma\\ \alpha+\beta\end{matrix};\frac{(y-x)\,(1-z)}{(1-x)\,(1-yz)}\right], \tag{1.3}$$

where the Pochhammer symbol $(a)_k$ is defined by $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$ if $k>0$, $(a)_0=1$, $(a)_k=1/(a+k)_{-k}$ if $k<0$, and the hypergeometric series ${}_rF_s$ is

2000 *Mathematics Subject Classification.* Primary 33C70; Secondary 33C20, 33D70.

Key words and phrases. Multiple hypergeometric series, basic hypergeometric series, Chu–Vandermonde summation, Pfaff–Saalschütz summation, Ramanujan's ${}_1\psi_1$ summation.

† Research partially supported by the Austrian Science Foundation FWF, grant P12094-MAT, and by EC's IHRP Programme, grant HPRN-CT-2001-00272.

defined by

$$ {}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{\ell=0}^{\infty}\frac{(a_1)_\ell\cdots(a_r)_\ell}{\ell!\,(b_1)_\ell\cdots(b_s)_\ell}z^\ell\ . $$

This triple sum – single sum identity is derived in [3] as a special case of a general reduction formula for hypergeometric functions connected with the Graßmannian $G_{N,K}$, and is derived there by exploiting systems of differential equations. Richard Askey posed the problem of finding a purely hypergeometric proof of this identity (i.e., one that uses classical summation and transformation formulas for hypergeometric series), with the ulterior motive that such a proof will make it possible to find a q-analogue of the identity, which was not found until then.

The purpose of this note is to provide such a hypergeometric proof for this identity, which we give in the following section. Indeed, as a bonus we are able to derive several q-analogues of the formula, see Section 3. As we are going to outline, the finding of this proof was greatly facilitated by the *Mathematica* package HYP [4] developed by the first author. However, it must be noted that this proof yields exclusively *formal* identities, i.e., identities for formal power series in x,y,z; the identities are either meaningless or wrong analytically. To remedy this fact, in Section 4 we provide a second, direct approach to find q-analogues of the equality (1.1)=(1.2)=(1.3). This approach is based on Ramanujan's ${}_1\psi_1$ summation formula. As a result, we obtain q-analogues which are valid as analytic as well as formal identities. This proof yields, as a by-product, also a hypergeometric proof of the identity of Gelfand, Graev and Retakh. However, it is a *genuine basic* hypergeometric proof, i.e., it seems that the use of the base q cannot be avoided in that proof.

2. The proof

We remark that the expressions (1.2) and (1.3) are equal because of the transformation formula (cf. [5, (1.7.1.3)])

$$ {}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix};z\right]=(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a,c-b\\ c\end{matrix};-\frac{z}{1-z}\right]. $$

(In fact, the equality which is given explicitly in [3] is the equality between (1.1) and (1.3).) Hence it would suffice to establish one of the equalities (1.1)=(1.2) or (1.1)=(1.3). However, since we eventually want to extend our proofs to the q-case, we shall provide *direct* proofs for *both* equalities.

We start with the proof of the equality (1.1)=(1.2). The strategy that we pursue is as follows: We compare coefficients of $x^Xy^Yz^Z$ in (1.1) and (1.2). The corresponding coefficient in (1.1) is the compact expression

$$ \frac{(\alpha)_X\,(\beta)_Y\,(1-\gamma)_Z\,(\gamma)_{X+Y-Z}}{X!\,Y!\,Z!\,(\alpha+\beta)_{X+Y-Z}}, \tag{2.1} $$

while the corresponding coefficient in (1.2) is the multiple sum

$$\sum_{i,j,k,l,m,n}{}' (-1)^{i+j+k+l+m} \frac{(\beta)_n\,(\alpha+\beta-\gamma)_n}{n!\,(\alpha+\beta)_n} \cdot \binom{\beta-\gamma}{i}\binom{-\beta-n}{j}\binom{\alpha+\beta+n-1}{k}\binom{\gamma-\alpha-\beta-n}{l}\binom{n}{m}, \tag{2.2}$$

where the sum $\sum'$ is subject to

$$i+l+n-m=X, \quad j+m=Y \quad \text{and} \quad k+l=Z. \tag{2.3}$$

Because of the linear dependencies of the 6 parameters i,j,k,l,m,n, the sum (2.2) can be written as a triple sum, once one expresses all the parameters in terms of three fixed ones out of them.

The task is then to simplify this triple sum, by using known hypergeometric summation and transformation formulas, interchange of summations, and similar manipulations, until one arrives at the compact expression (2.1). Clearly, in view of the many possibilities there are to choose 3 out of 6 parameters, and the many possibilities to proceed afterwards, this is a daunting task if one is to do this by hand. However, the use of the computer will greatly facilitate the search for a feasible path. Indeed, by using the *Mathematica* package HYP [4], it did not take us more than three attempts to find such a classical hypergeometric proof of the equality of (2.1) and (2.2). As a matter of fact, all the subsequent calculations were first carried out on the computer by using HYP, and were then directly transformed into TEX-code using the *Mathematica* command `TeXForm`.[1]

We are going to express all the parameters in terms of i, j and k. That is, we write

$$l=-k+Z, \quad m=-j+Y, \quad n=-i-j+k+X+Y-Z,$$

and substitute this in (2.2). Making the sum over j the inner-most sum, and writing it in hypergeometric notation, we obtain

$$\sum_{i,k\ge 0}\Bigg((-1)^{Y+Z}\frac{(\beta)_{k+X+Y-Z-i}\,(\alpha+\beta-\gamma)_{k+X+Y-Z-i}\,(\gamma-\beta)_i}{i!\,k!\,Y!\,(k+X-Z-i)!\,(Z-k)!} \cdot \frac{(1-\alpha-\beta+\gamma+i-X-Y)_{Z-k}}{(\alpha+\beta)_{X+Y-Z-i}}\,{}_2F_1\!\left[\begin{matrix}1-\alpha-\beta+i-X-Y+Z,-Y\\ 1-\alpha-\beta+\gamma+i-X-Y\end{matrix};1\right]\Bigg).$$

The ${}_2F_1$-series can be summed by means of the Chu–Vandermonde summation (see [5, (1.7.7), Appendix (III.4)]),

$${}_2F_1\!\left[\begin{matrix}a,-N\\ c\end{matrix};1\right]=\frac{(c-a)_N}{(c)_N}, \tag{2.4}$$

where N is a nonnegative integer, because Y is a nonnegative integer. We substitute the result and now make the sum over k the inner-most sum. If it is written in

[1]The corresponding *Mathematica* Notebook is available at:
`http://euler.univ-lyon1.fr/home/kratt/artikel/gelfand1.html`.

hypergeometric notation, we get

$$\sum_{i\ge0}\left(\frac{(\beta)_{X+Y-Z-i}\,(\alpha+\beta-\gamma)_{X-i}\,(\gamma-\beta)_i}{i!\,Y!\,Z!}\right.\\ \left.\cdot\frac{(\gamma-Z)_Y}{(X-Z-i)!\,\,(\alpha+\beta)_{X+Y-Z-i}}\,{}_2F_1\!\left[\begin{matrix}\beta-i+X+Y-Z,-Z\\ 1-i+X-Z\end{matrix};1\right]\right).$$

Again, the ${}_2F_1$-series can be summed by means of the Chu–Vandermonde summation (2.4), since Z is a nonnegative integer. Upon substituting the result, and writing the remaining sum over i in hypergeometric notation, we arrive at the expression

$$\frac{(\beta)_{X+Y-Z}\,(\alpha+\beta-\gamma)_X\,(1-\beta-Y)_Z\,(\gamma-Z)_Y}{X!\,Y!\,Z!\,(\alpha+\beta)_{X+Y-Z}}\\ \times{}_3F_2\!\left[\begin{matrix}1-\alpha-\beta-X-Y+Z,-\beta+\gamma,-X\\ 1-\beta-X-Y+Z,1-\alpha-\beta+\gamma-X\end{matrix};1\right].$$

This time, the Pfaff–Saalschütz summation formula (see [5, (2.3.1.3), Appendix (III.2)]),

$${}_3F_2\!\left[\begin{matrix}a,b,-N\\ c,1+a+b-c-N\end{matrix};1\right]=\frac{(c-a)_N\,(c-b)_N}{(c)_N\,(c-a-b)_N},\tag{2.5}$$

where N is a nonnegative integer, applies, because X is a nonnegative integer. Some simplification of the result finally yields (2.1), which finishes the proof of the equality (1.1)=(1.2).

Next we prove the equality (1.1)=(1.3), following an analogous approach. If we compare coefficients of $x^Xy^Yz^Z$ on both sides, we see that we have to prove the equality of (2.1) with

$$\sum_{i,j,k,l,m,n}{}^{'}\,(-1)^{i+j+k+l+m}\,\frac{(\beta)_n\,(\gamma)_n}{n!\,(\alpha+\beta)_n}\\ \cdot\binom{-\gamma-n}{i}\binom{-\beta-n}{j}\binom{\alpha+\beta+n-1}{k}\binom{\gamma-\alpha}{l}\binom{n}{m},\tag{2.6}$$

where the sum $\sum'$ is subject to

$$i+l+m=X,\quad j+n-m=Y\quad\text{and}\quad j+k+l=Z.$$

Here, we express all the parameters in terms of i, k and l. That is, we write

$$j=-k-l+Z,\quad m=-i-l+X,\quad n=-i+k+X+Y-Z,\tag{2.7}$$

and substitute this in (2.6). Making the sum over k the inner-most sum, and writing it in hypergeometric notation, we obtain

$$\sum_{i,l\ge0}\left((-1)^{X+l+i}\frac{(\beta)_{X+Y-i-l}\,(\alpha-\gamma)_l\,(\gamma)_{X+Y-Z}}{i!\,l!\,\,(X-i-l)!\,\,(Y+l-Z)!\,\,(Z-l)!\,(\alpha+\beta)_{X+Y-Z-i}}\right.\\ \left.\cdot\,{}_2F_1\!\left[\begin{matrix}\gamma+X+Y-Z,l-Z\\ 1+l+Y-Z\end{matrix};1\right]\right).$$

The ${}_2F_1$-series can be summed by means of the Chu–Vandermonde summation (2.4), because $Z-l$ is a nonnegative integer. We substitute the result and now make the sum over i the inner-most sum. If it is written in hypergeometric notation, we get

$$\sum_{l\ge 0}\Bigg((-1)^{X+l}\frac{(\beta)_{X+Y-l}\,(\alpha-\gamma)_l\,(\gamma)_{X+Y-Z}\,(1-\gamma+l-X)_{Z-l}}{l!\;(X-l)!\,Y!\;(Z-l)!\,(\alpha+\beta)_{X+Y-Z}} \cdot {}_2F_1\left[\begin{matrix}1-\alpha-\beta-X-Y+Z,l-X\\ 1-\beta+l-X-Y\end{matrix};1\right]\Bigg).$$

Again, the ${}_2F_1$-series can be summed by means of the Chu–Vandermonde summation (2.4), since $X-l$ is a nonnegative integer. Upon substituting the result, and writing the remaining sum over l in hypergeometric notation, we arrive at the expression

$$(-1)^X\frac{(\beta)_{X+Y}\,(\gamma)_{X+Y-Z}\,(1-\gamma-X)_Z\,(\alpha-Z)_X}{X!\,Y!\,Z!\,(\alpha+\beta)_{X+Y-Z}\,(1-\beta-X-Y)_X}\,{}_3F_2\left[\begin{matrix}\alpha-\gamma,-Z,-X\\ 1-\gamma-X,\alpha-Z\end{matrix};1\right].$$

As in the previous derivation, it is now the Pfaff–Saalschütz summation formula (2.5) which can be applied, because X is a nonnegative integer. Some simplification of the result finally yields (2.1), which finishes the proof of the equality (1.1)=(1.3).

3. Formal q-analogues

Once a path of proof is found for a hypergeometric identity, it may hint at a way to find a q-analogue. This principle constituted the original motivation of Richard Askey to pose his problem, as we already mentioned in the Introduction. Indeed, since the only identities that we used in the derivation in Section 2 were the Chu–Vandermonde summation and the Pfaff–Saalschütz summation, it is not difficult to come up with a q-analogue. What one has to do is to replace Pochhammer symbols by q-shifted factorials

$$(a;q)_n:=\begin{cases}\prod_{i=0}^{n-1}(1-aq^i), & n>0,\\ 1, & n=0,\\ \prod_{i=0}^{-n-1}(1-aq^{n+i})^{-1}, & n<0,\end{cases}$$

binomials by q-binomials, and finally to insert the "right" powers of q. Since there are two q-analogues of the Chu–Vandermonde summation formula, there are in fact several possible q-analogues, the derivation of one of which we shall describe below. While we did not succeed to find a q-analogue of the equality (1.1)=(1.2) which looks equally elegant as the original identity, we did succeed for the equality (1.1)=(1.3).

We start by proving the following q-analogue of the equality of (1.1) and (1.2):

$$\sum_{j,k,m\ge 0} q^{-\binom{k}{2}-\binom{m}{2}}\alpha^{-m}\,\beta^{j-m}\,\gamma^{2m-j-k}\,\frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}}x^jy^kz^m$$
$$=\sum_{i,j,k,l,m,n\ge 0}\Bigg(q^{mn-kl-jm-ln-\binom{j}{2}-\binom{k}{2}-\binom{l}{2}-\binom{m}{2}}\,\alpha^{-l}\,\beta^{i-l}\,\gamma^{-i-j+k+l}\,\frac{(\beta;q)_n\,(\alpha\beta/\gamma;q)_n}{(\alpha\beta;q)_n\,(q;q)_n}$$
$$\frac{(\gamma/\beta;q)_i\,(\beta q^n;q)_j\,(q^{1-n}/\alpha\beta;q)_k\,(\alpha\beta q^n/\gamma;q)_l\,(q^{-n};q)_m}{(q;q)_i\,(q;q)_j\,(q;q)_k\,(q;q)_l\,(q;q)_m}\,x^{i+l+n-m}y^{j+m}z^{k+l}\Bigg). \quad (3.1)$$

The reader must be warned at this point that the only way to give this identity a meaningful interpretation is as a formal power series in x, y, z, and our proof will adopt this point of view. (Analytically, the series on the left-hand side of (3.1) diverges if $|q| < 1$ because of the quadratic powers of q. If $|q| > 1$ the right-hand side diverges.)

For the proof of (3.1) we proceed in complete analogy to the previous section. It is needless to say that all the subsequent calculations were again first carried out on the computer, this time using the "q-analogue" of HYP, HYPQ [4], after which they were transformed into TEX-code using the *Mathematica* command `TeXForm`.[2]

As before, we compare coefficients of $x^X y^Y z^Z$ on both sides of (3.1). We start with the corresponding coefficient on the right-hand side. In analogy with (2.2), it is expressed in terms of a multiple sum over i, j, k, l, m, n subject to (2.3). We express all the parameters i, j, k, l, m, n in terms of i, j and k, make the sum over j the inner-most sum, and write it in the standard basic hypergeometric notation

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell \cdots (a_{r+1};q)_\ell}{(q;q)_\ell\,(b_1;q)_\ell \cdots (b_r;q)_\ell} z^\ell \, . $$

Thus we obtain for the coefficient of $x^X y^Y z^Z$ on the right-hand side of (3.1) the expression

$$ \sum_{i,k\ge 0} \Bigg((-1)^{k+Y} q^{\binom{k+1}{2}+\binom{Z+1}{2}+Z(i-k-X-Y)}\, \alpha^{-Z}\, \beta^{i-Z}\, \gamma^{Z-i} \\ \cdot \frac{(\beta;q)_{k+X+Y-Z-i}\,(\alpha\beta/\gamma;q)_{X+Y-i}\,(\gamma/\beta;q)_i}{(q;q)_i\,(q;q)_k\,(q;q)_Y\,(q;q)_{k+X-Z-i}\,(q;q)_{Z-k}\,(\alpha\beta;q)_{X+Y-Z-i}} \\ \cdot {}_2\phi_1\left[\begin{matrix} q^{1+i-X-Y+Z}/\alpha\beta, q^{-Y} \\ \gamma q^{1+i-X-Y}/\alpha\beta \end{matrix}; q, q\right] \Bigg). $$

The ${}_2\phi_1$-series can be summed by means of the following q-analogue of the Chu–Vandermonde summation (see [2, (1.5.3); Appendix (II.6)]),

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-N} \\ c \end{matrix}; q, q\right] = \frac{a^N\,(c/a;q)_N}{(c;q)_N}, \tag{3.2} $$

where N is a nonnegative integer. We substitute the result and now make the sum over k the inner-most sum. If it is written in basic hypergeometric notation, we get

$$ \sum_{i\ge 0} \Bigg(q^{-\binom{Y}{2}+\binom{Z+1}{2}+Z(i-X)}\, \alpha^{-Z}\, \beta^{i-Z}\, \gamma^{-i-Y+Z} \frac{(\beta;q)_{X+Y-Z-i}\,(\alpha\beta/\gamma;q)_{X-i}}{(q;q)_i\,(q;q)_Y\,(q;q)_Z} \\ \cdot \frac{(\gamma/\beta;q)_i\,(\gamma/q^Z;q)_Y}{(q;q)_{X-Z-i}\,(\alpha\beta;q)_{X+Y-Z-i}}\, {}_2\phi_1\left[\begin{matrix} \beta q^{-i+X+Y-Z}, q^{-Z} \\ q^{1-i+X-Z} \end{matrix}; q, q\right] \Bigg). $$

Again, the ${}_2\phi_1$-series can be summed by means of the q-Chu–Vandermonde summation (3.2). Upon substituting the result, and writing the remaining sum over i in basic

[2]The corresponding *Mathematica* Notebook is available at:
`http://euler.univ-lyon1.fr/home/kratt/artikel/gelfand1.html`.

hypergeometric notation, we arrive at the expression

$$q^{-\binom{Y}{2}-\binom{Z}{2}+YZ}\,\alpha^{-Z}\,\gamma^{-Y+Z}\,\frac{(\beta;q)_{X+Y-Z}\,(\alpha\beta/\gamma;q)_X\,(q^{1-Y}/\beta;q)_Z\,(\gamma/q^Z;q)_Y}{(q;q)_X\,(q;q)_Y\,(q;q)_Z\,(\alpha\beta;q)_{X+Y-Z}}\\ \times {}_3\phi_2\left[\begin{matrix}q^{1-X-Y+Z}/\alpha\beta,\gamma/\beta,q^{-X}\\ q^{1-X-Y+Z}/\beta,\gamma q^{1-X}/\alpha\beta\end{matrix};q,q\right].$$

The ${}_3\phi_2$-series can be summed by means of the q-analogue of the Pfaff–Saalschütz summation formula (see [2, (1.7.2); Appendix (II.12)]),

$$ {}_3\phi_2\left[\begin{matrix}a,b,q^{-N}\\ c,abq^{1-N}/c\end{matrix};q,q\right]=\frac{(c/a;q)_N(c/b;q)_N}{(c;q)_N(c/ab;q)_N}, \tag{3.3}$$

where N is a nonnegative integer. Some simplification then yields the expression

$$q^{-\binom{Y}{2}-\binom{Z}{2}}\alpha^{-Z}\,\beta^{X-Z}\,\gamma^{2Z-X-Y}\,\frac{(\alpha;q)_X\,(\beta;q)_Y\,(q/\gamma;q)_Z\,(\gamma;q)_{X+Y-Z}}{(q;q)_X\,(q;q)_Y\,(q;q)_Z\,(\alpha\beta;q)_{X+Y-Z}},$$

which is exactly the coefficient of $x^Xy^Yz^Z$ on the left-hand side of (3.1), thus establishing our claim.

In a completely analogous manner, we can prove the following q-analogue of the equality of (1.1) and (1.3):

$$\sum_{j,k,m\ge0}\frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}}x^jy^kz^m\\ =\sum_{i,j,k,l,m,n\ge0}\Bigg(q^{kn-in+m}\,\alpha^k\,\beta^{k-i-m}\gamma^{-k}\frac{(\beta;q)_n\,(\gamma;q)_n}{(\alpha\beta;q)_n\,(q;q)_n}\\ \cdot\frac{(\gamma q^n;q)_i\,(\beta q^n;q)_j\,(q^{1-n}/\alpha\beta;q)_k\,(\alpha/\gamma;q)_l\,(q^{-n};q)_m}{(q;q)_i\,(q;q)_j\,(q;q)_k\,(q;q)_l\,(q;q)_m}x^{i+l+m}y^{j+n-m}z^{j+k+l}\Bigg). \tag{3.4}$$

That is, as in the proof of the equality (1.1)=(1.3) in the previous section, we compare coefficients of $x^Xy^Yz^Z$ on both sides, then start with the the coefficient of the right-hand side, substitute the relations (2.7), evaluate the sums over k and i (in that order) by means of the q-analogue (3.2) of the Chu–Vandermonde summation, and finally evaluate the sum over l by means of the the q-analogue (3.3) of the Pfaff–Saalschütz summation. However, again, this is an identity which makes sense only as a formal power series in x,y,z. (Analytically, the right-hand side never converges.)

Remarkably, in this case, the right-hand side of (3.4) can be simplified. To be precise, the sums over i,j,k,l,m can be evaluated with the help of the q-binomial theorem (see [2, (1.3.2); Appendix (II.3)])

$$\sum_{\ell=0}^{\infty}\frac{(a;q)_\ell}{(q;q)_\ell}z^\ell=\frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.5}$$

If the remaining sum over n is written in basic hypergeometric notation, the result is the compact identity

$$\sum_{j,k,m\ge 0} \frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}} x^j y^k z^m \\ = \frac{(\gamma x/\beta;q)_\infty}{(x/\beta;q)_\infty} \frac{(qz/\gamma;q)_\infty}{(\alpha\beta z/\gamma;q)_\infty} \frac{(\alpha xz/\gamma;q)_\infty}{(xz;q)_\infty} \frac{(\beta yz;q)_\infty}{(yz;q)_\infty} \\ \times {}_4\phi_3\left[\begin{matrix}\beta,\alpha\beta z/\gamma,\beta y/x,\gamma\\ \alpha\beta,\beta yz,q\beta/x\end{matrix};q,q\right], \quad (3.6)$$

which is a much more elegant q-analogue of the equality (1.1)=(1.3) than the identity (3.1) is a q-analogue of the equality (1.1)=(1.2). (We remark that the ${}_4\phi_3$-series on the right-hand side is balanced, i.e., the product of the lower parameters is equal to q times the product of the upper parameters.) We repeat that this is an identity for *formal power series in* x, y, z. It is in fact wrong (!) if one would interpret the left- and right-hand sides as analytic series. Although both sides make perfect sense analytically, on the right-hand side there is a term missing as we are going to show in the next section (cf. (4.6)). The problem, when the derivation of (3.6) is regarded in the analytic sense, arises when in (3.4) we use the q-binomial theorem (3.5) to evaluate the sum over i. For, we would have to apply (3.5) with $z = x/q^n$. If $|q| < 1$, then the left-hand side of (3.5) converges only if $|z| < 1$, but we cannot have $|x/q^n| < 1$ for arbitrarily large n (unless $x = 0$). There are similar problems if $|q| > 1$.

In the next section, we shall not only derive a q-analogue of (1.1)=(1.3) which does not have these problems, i.e., which is valid in the formal *as well as* in the analytic sense, but as well a more elegant q-analogue of (1.1)=(1.2), which is also valid in both senses.

4. Analytic q-analogues

In this section we outline a different method to obtain q-analogues of the equality (1.1)=(1.2)=(1.3). It is based on Ramanujan's ${}_1\psi_1$ summation (see [2, Ex. 1.6(ii); Appendix (II.5)])

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q;q)_\infty\,(b/a;q)_\infty\,(az;q)_\infty\,(q/az;q)_\infty}{(b;q)_\infty\,(q/a;q)_\infty\,(z;q)_\infty\,(b/az;q)_\infty}, \qquad |b/a| < |z| < 1, \qquad (4.1)$$

or equivalently,

$$\frac{(a;q)_n}{(b;q)_n} = \frac{1}{2\pi i} \int \frac{(q;q)_\infty\,(b/a;q)_\infty\,(az;q)_\infty\,(q/az;q)_\infty}{(b;q)_\infty\,(q/a;q)_\infty\,(z;q)_\infty\,(b/az;q)_\infty}\, z^{-n-1}\,dz,$$

where the integration is over a contour encircling the origin counter-clockwise inside the annulus of convergence. Thus,

$$\begin{aligned}
&\sum_{j,k,m\geq 0} \frac{(a;q)_j\,(b;q)_k\,(c;q)_m\,(e;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(f;q)_{j+k-m}}\,x^j y^k z^m \\
&\quad= \sum_{j,k,m\geq 0} \frac{(a;q)_j\,(b;q)_k\,(c;q)_m}{(q;q)_j\,(q;q)_k\,(q;q)_m}\,x^j y^k z^m \\
&\qquad\qquad \cdot \frac{1}{2\pi i}\int \frac{(q;q)_\infty\,(f/e;q)_\infty\,(et;q)_\infty\,(q/et;q)_\infty}{(f;q)_\infty\,(q/e;q)_\infty\,(t;q)_\infty\,(f/et;q)_\infty}\,t^{m-j-k-1}\,dt \\
&\quad= \frac{1}{2\pi i}\int \frac{(q;q)_\infty\,(f/e;q)_\infty\,(et;q)_\infty\,(q/et;q)_\infty\,(ax/t;q)_\infty\,(by/t;q)_\infty\,(czt;q)_\infty}{(f;q)_\infty\,(q/e;q)_\infty\,(t;q)_\infty\,(f/et;q)_\infty\,(x/t;q)_\infty\,(y/t;q)_\infty\,(zt;q)_\infty}\,\frac{dt}{t},
\end{aligned}$$

where we used the q-binomial theorem (3.5) to evaluate the sums over j, k, and m. This requires the convergence conditions $\max(|x|,|y|,|f/e|) < \min(1,1/|z|)$. If we in addition assume $|ce|<1$, the value of the integral is given by [2, (4.10.9)] as

$$\begin{aligned}
&\frac{(ax;q)_\infty\,(by;q)_\infty\,(cz;q)_\infty\,(e;q)_\infty}{(x;q)_\infty\,(y;q)_\infty\,(z;q)_\infty\,(f;q)_\infty}\,{}_4\phi_3\left[\begin{matrix}x,y,q/cz,f/e\\ ax,by,q/z\end{matrix};q,ce\right] \\
&\qquad + \frac{(axz;q)_\infty\,(byz;q)_\infty\,(c;q)_\infty\,(e/z;q)_\infty\,(qz/e;q)_\infty\,(f/e;q)_\infty}{(xz;q)_\infty\,(yz;q)_\infty\,(1/z;q)_\infty\,(f;q)_\infty\,(q/e;q)_\infty\,(fz/e;q)_\infty} \\
&\qquad\qquad \times {}_4\phi_3\left[\begin{matrix}xz,yz,q/c,fz/e\\ axz,byz,qz\end{matrix};q,ce\right].
\end{aligned}$$

Until now, nothing "special" has been used. Indeed, using (4.1) instead of the q-binomial theorem, we get the integral representation

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=-\infty}^{\infty} \frac{(a;q)_{k_1+\dots+k_n}}{(c;q)_{k_1+\dots+k_n}}\prod_{j=1}^n \frac{(b_j;q)_{k_j}}{(d_j;q)_{k_j}}\,x_j^{k_j} \\
&\qquad = \frac{1}{2\pi i}\int \frac{(q;q)_\infty\,(c/a;q)_\infty\,(at;q)_\infty\,(q/at;q)_\infty}{(c;q)_\infty\,(q/a;q)_\infty\,(t;q)_\infty\,(c/at;q)_\infty} \\
&\qquad\qquad \times \prod_{j=1}^n \frac{(q;q)_\infty\,(d_j/b_j;q)_\infty\,(b_jx_j/t;q)_\infty\,(qt/b_jx_j;q)_\infty}{(d_j;q)_\infty\,(q/b_j;q)_\infty\,(x_j/t;q)_\infty\,(d_jt/b_jx_j;q)_\infty}\,\frac{dt}{t},
\end{aligned}$$

where $\max(c/a,x_1,\dots,x_n) < |t| < \min(1,x_1b_1/d_1,\dots,x_nb_n/d_n)$ on the contour of integration. This type of integral may be expressed as a finite sum of basic hypergeometric series, cf. [2, (4.10.8), (4.10.9)]. The case considered above is the special case $n=3$, $d_1=d_2=q$, $b_3=1$. The case $d_1=\dots=d_n=q$ gives Andrews's identity [1]

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{\infty} \frac{(a;q)_{k_1+\dots+k_n}}{(c;q)_{k_1+\dots+k_n}}\prod_{j=1}^n \frac{(b_j;q)_{k_j}}{(q;q)_{k_j}}\,x_j^{k_j} \\
&\qquad = \frac{(c;q)_\infty}{(d;q)_\infty}\prod_{j=1}^n \frac{(a_jx_j;q)_\infty}{(x_j;q)_\infty}\,{}_{n+1}\phi_n\left[\begin{matrix}d/c,x_1,\dots,x_n\\ ax_1,\dots,ax_n\end{matrix};q,c\right].
\end{aligned}$$

Returning to the special case under consideration, we choose $ab=f$ and $ce=q$. This is the condition that the ${}_4\phi_3$-series are balanced (i.e., that the product of the lower parameters is equal to q times the product of the upper parameters). They may then be combined to a very-well-poised ${}_8\phi_7$-series using [2, (2.10.10), Appendix (III.36)]. After replacing a by α, b by β, and e by γ, the conclusion is that

$$\sum_{j,k,m\ge0}\frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}}\,x^jy^kz^m$$
$$=\frac{(\alpha x;q)_\infty}{(x;q)_\infty}\frac{(\beta y;q)_\infty}{(y;q)_\infty}\frac{(\gamma y;q)_\infty}{(\alpha\beta y;q)_\infty}\frac{(qz/\gamma;q)_\infty}{(\alpha\beta z/\gamma;q)_\infty}\frac{(\alpha\beta yz/\gamma;q)_\infty}{(yz;q)_\infty}$$
$$\times{}_8\phi_7\left[\begin{matrix}\alpha\beta y/q,\sqrt{\alpha\beta qy},-\sqrt{\alpha\beta qy},\alpha,y,\gamma/z,\alpha\beta/\gamma,\beta y/x\\ \sqrt{\alpha\beta y}/\sqrt q,-\sqrt{\alpha\beta y}/\sqrt q,\beta y,\alpha\beta,\alpha\beta yz/\gamma,\gamma y,\alpha x\end{matrix};q,xz\right].\quad(4.2)$$

Finally, we apply Bailey's very-well-poised ${}_8\phi_7$ transformation (see [2, (2.10.1); Appendix (III.23)])

$${}_8\phi_7\left[\begin{matrix}a,\sqrt aq,-\sqrt aq,b,c,d,e,f\\ \sqrt a,-\sqrt a,aq/b,aq/c,aq/d,aq/e,aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right]$$
$$=\frac{(aq;q)_\infty\,(aq/ef;q)_\infty\,(a^2q^2/bcde;q)_\infty\,(a^2q^2/bcdf;q)_\infty}{(aq/e;q)_\infty\,(aq/f;q)_\infty\,(a^2q^2/bcd;q)_\infty\,(a^2q^2/bcdef;q)_\infty}$$
$$\times{}_8\phi_7\left[\begin{matrix}a^2q/bcd,aq^{3/2}/\sqrt{bcd},-aq^{3/2}/\sqrt{bcd},aq/cd,aq/bd,aq/bc,e,f\\ a\sqrt q/\sqrt{bcd},-a\sqrt q/\sqrt{bcd},aq/b,aq/c,aq/d,a^2q^2/bcde,a^2q^2/bcdf\end{matrix};q,\frac{aq}{ef}\right].\quad(4.3)$$

As a result, we obtain the identity

$$\sum_{j,k,m\ge0}\frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}}\,x^jy^kz^m$$
$$=\frac{(\gamma x/\beta;q)_\infty}{(x;q)_\infty}\frac{(\beta y;q)_\infty}{(y;q)_\infty}\frac{(qz/\gamma;q)_\infty}{(\alpha\beta z/\gamma;q)_\infty}\frac{(\alpha\beta xz/\gamma;q)_\infty}{(xz;q)_\infty}\frac{(\beta yz;q)_\infty\,(\alpha\beta yz/\gamma;q)_\infty}{(yz;q)_\infty\,(\alpha\beta^2yz/\gamma;q)_\infty}$$
$$\times{}_8\phi_7\left[\begin{matrix}\alpha\beta^2yz/\gamma q,\beta\sqrt{\alpha qyz}/\sqrt\gamma,-\beta\sqrt{\alpha qyz}/\sqrt\gamma,\alpha\beta z/\gamma,\beta yz/\gamma,\beta,\alpha\beta/\gamma,\beta y/x\\ \beta\sqrt{\alpha yz}/\sqrt{\gamma q},-\beta\sqrt{\alpha yz}/\sqrt{\gamma q},\beta y,\alpha\beta,\alpha\beta yz/\gamma,\beta yz,\alpha\beta xz/\gamma\end{matrix};q,\frac{\gamma x}{\beta}\right],\quad(4.4)$$

valid if $\max(|x|,|y|,|\alpha\beta/\gamma|)<\min(1,1/|z|)$ and $|\gamma x/\beta|<1$, which is a perfect q-analogue of the equality (1.1)=(1.2). It is valid both analytically and as a formal power series in x,y,z.

On the other hand, if we apply the transformation formula (4.3) to the ${}_8\phi_7$-series in (4.2) where the lower and upper parameters are in the order

$${}_8\phi_7\left[\begin{matrix}\alpha\beta y/q,\sqrt{\alpha\beta qy},-\sqrt{\alpha\beta qy},\alpha,\alpha\beta/\gamma,y,\beta y/x,\gamma/z\\ \sqrt{\alpha\beta y}/\sqrt q,-\sqrt{\alpha\beta y}/\sqrt q,\beta y,\gamma y,\alpha\beta,\alpha x,\alpha\beta yz/\gamma\end{matrix};q,xz\right],$$

then the result is the identity

$$\sum_{j,k,m\ge 0} \frac{(\alpha;q)_j\,(\beta;q)_k\,(q/\gamma;q)_m\,(\gamma;q)_{j+k-m}}{(q;q)_j\,(q;q)_k\,(q;q)_m\,(\alpha\beta;q)_{j+k-m}}\,x^j y^k z^m$$
$$= \frac{(\gamma x;q)_\infty}{(x;q)_\infty}\frac{(qz/\gamma;q)_\infty}{(\alpha\beta z/\gamma;q)_\infty}\frac{(\alpha xz/\gamma;q)_\infty}{(xz;q)_\infty}\frac{(\beta yz;q)_\infty}{(yz;q)_\infty}\frac{(\beta y;q)_\infty\,(\gamma y;q)_\infty}{(\beta\gamma y;q)_\infty\,(y;q)_\infty}$$
$$\times\,{}_8\phi_7\left[\begin{matrix}\beta\gamma y/q, \sqrt{\beta\gamma q y}, -\sqrt{\beta\gamma q y}, \gamma, \beta, \gamma y/\alpha, \beta y/x, \gamma/z\\ \sqrt{\beta\gamma y}/\sqrt{q}, -\sqrt{\beta\gamma y}/\sqrt{q}, \beta y, \gamma y, \alpha\beta, \gamma x, \beta y z\end{matrix}; q, \frac{\alpha x z}{\gamma}\right], \quad (4.5)$$

which is another q-analogue of the equality (1.1)=(1.3). Again, it is not only valid analytically, but also as a formal power series in x, y, z.

We remark that, by another application of [2, (III.36)], the left-hand side of (4.5) can alternatively be written

$$\frac{(\gamma x/\beta;q)_\infty\,(qz/\gamma;q)_\infty\,(\alpha xz/\gamma;q)_\infty\,(\beta yz;q)_\infty}{(x/\beta;q)_\infty\,(\alpha\beta z/\gamma;q)_\infty\,(xz;q)_\infty\,(yz;q)_\infty}\,{}_4\phi_3\left[\begin{matrix}\beta, \gamma, \alpha\beta z/\gamma, \beta y/x\\ \alpha\beta, \beta y z, q\beta/x\end{matrix}; q, q\right]$$
$$+ \frac{(qz/\gamma;q)_\infty\,(\beta;q)_\infty\,(\gamma;q)_\infty\,(\beta y/x;q)_\infty\,(\alpha x;q)_\infty\,(xyz;q)_\infty}{(x;q)_\infty\,(y;q)_\infty\,(xz;q)_\infty\,(yz;q)_\infty\,(\alpha\beta;q)_\infty\,(\beta/x;q)_\infty}$$
$$\times\,{}_4\phi_3\left[\begin{matrix}x, y, \gamma x/\beta, \alpha x z/\gamma\\ \alpha x, xyz, qx/\beta\end{matrix}; q, q\right]. \quad (4.6)$$

Note that the first term is precisely the (analytically false) expression given in (3.6).

An interesting aspect of this proof is that it provides in particular a proof of the equality (1.1)=(1.2)=(1.3) (by doing the replacements $\alpha \to q^\alpha$, $\beta \to q^\beta$, $\gamma \to q^\gamma$, and then performing the limit $q \to 1$). However, because of the use of Ramanujan's ${}_1\psi_1$ summation, it seems impossible to make it into a proof "with $q = 1$," i.e., into a proof that uses only identities for *ordinary* hypergeometric series. On the other hand, we have given such a proof in Section 2.

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