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GENERATING FUNCTIONS FOR PLANE PARTITIONS OF A GIVEN SHAPE

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Abstract. For fixed integers α and β, planar arrays of integers of a given shape, in which the entries decrease at least by α along rows and at least by β along columns, are considered. For various classes of these (α,β)-plane partitions we compute three different kinds of generating functions. By a combinatorial method, determinantal expressions are obtained for these generating functions. In special cases these determinants may be evaluated by a simple determinant lemma. All known results concerning plane partitions of a given shape are included. Thus our approach to computation of generating functions for plane partitions of a given shape provides a uniform proof method and yields numerous generalizations of known results.

1. Introduction. Let D_r denote the set of all r-tupels $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ of integers with $\lambda_1\ge\lambda_2\ge\dots\ge\lambda_r$. For $\lambda,\mu\in D_r$ we write $|\lambda|$ for $\lambda_1+\lambda_2+\dots+\lambda_r$, $\lambda+\mu$ for $(\lambda_1+\mu_1,\dots,\lambda_r+\mu_r)$ and $\lambda\ge\mu$ if $\lambda_i\ge\mu_i$ for all $i=1,2,\dots r$.

Let α,β be arbitrary integers and $\lambda,\mu\in D_r$ with $\lambda\ge\mu$. We call an array π of integers of the form

$$\begin{array}{ccccccc}
 & & \pi_{1,\mu_1+1} & \pi_{1,\mu_1+2} & \cdots\cdots\cdots & & \pi_{1,\lambda_1}\\
 & \pi_{2,\mu_2+1}\ \cdots & \pi_{2,\mu_1+1} & \pi_{2,\mu_1+2} & \cdots\cdots & \pi_{2,\lambda_2} & \\
 \cdot^{\cdot^{\cdot}} & & & & & \cdot^{\cdot^{\cdot}} & \\
\pi_{r,\mu_r+1} & \cdots\cdots\cdots & & & \pi_{r,\lambda_r} & &
\end{array} \qquad (1.1)$$

an *(α,β)-plane partition of shape λ/μ* if

$$\pi_{ij} \ge \pi_{i,j+1}+\alpha \qquad \text{for} \quad 1\le i\le r,\ \mu_i<j<\lambda_i \qquad (1.2)(a)$$

and

$$\pi_{ij} \geq \pi_{i+1,j}+\beta \qquad \text{for} \quad 1\le i<r,\ \mu_i<j\le\lambda_{i+1} \ . \tag{1.2)(b}$$

If $\mu=0$ we shortly say, π is of shape λ. The entries of π are called *parts* of π. The sum of all parts of π, $\sum \pi_{ij}$, is called the *norm* of π, for which we write $n(\pi)$. To give an example, the array π_0 in Figure 1 below is a (0,2)-plane partition of shape (8,6,6,2)/(3,2,0,-1). For the

```
        7 6 6 3 1
      8 5 4 3
  3 3 2 2 1 1
4 1 1
```

Figure 1

norm of π_0 we have $n(\pi_0)=61$. Obviously, a (0,0)-plane partition of shape λ/μ containing only positive parts is an ordinary (skew) plane partition, a (0,1)-plane partition containing only positive parts is a column-strict (skew) plane partition, a (1,0)-plane partition containing only positive parts is a row-strict (skew) plane partition, and a (1,1)-plane partition containing only positive parts is a row and column-strict (skew) plane partition. (See [21] for the terminology concerning plane partitions.)

In this paper we compute generating functions for (α,β)-plane partitions subject to various restrictions. We shall consider three types of generating functions, the *part generating function,* the *norm generating function* and the *trace generating function.* The part generating function for a certain class of (α,β)-plane partitions is

$$\sum_{\pi} \prod x_{\pi_{ij}} \tag{1.3}$$

where the sum is over all π being an element of that class of (α,β)-plane partitions and the product is over all parts π_{ij} of π. Thus, the plane partition π_0 in Figure 1 would contribute the term $x_8x_7x_6^2x_5x_4^2x_3^4x_2^2x_1^5$ to the sum in (1.3). The norm generating function for a certain class of (α,β)-plane partitions is

$$\sum_{\pi} x^{n(\pi)} \tag{1.4}$$

where the sum is over all π being an element of that class. For the trace generating function (introduced by Gansner [4,5]) we need some more definitions. The *l-diagonal* of an array π of the form (1.1) is the sequence of parts π_{ij} of π with $j-i=l$. The *l-trace* of π, denoted $t_l(\pi)$, is the sum of the parts of the l-diagonal of π. In our example π_0 in Figure 1 the 3-diagonal would be the set $\{7,4,1\}$, hence $t_3(\pi_0)=12$. The trace generating function for a certain class of (α,β)-plane partitions is

$$\sum_{\pi} \prod_{l \in \mathbf{Z}} x_l^{t_l(\pi)} \tag{1.5}$$

where the sum is over all π being an element of that class ($\mathbf{Z}$ is the set of integers). Of course, (1.4) is a special case of both (1.3) and (1.5) ($x_i=x$ for all $i \in \mathbf{Z}$).

With the help of an involution (given in section 4) and a combinatorial lemma (Lemma 2.1), we obtain determinantal expressions for these three types of generating functions for (α,β)-plane partitions of a given shape, where the parts are limited in each row by different lower and upper bounds. (An expression for the part generating function is obtained only if $\alpha+\beta=1$.) In special cases these determinants may be evaluated by a simple determinant lemma (Lemma 2.2). Thus in particular, all known results about generating functions for the various kinds of plane partitions of a given shape (see [1,2,3,4,5,14,20,21,22,24]) can be derived. These results were originally obtained by a variety of methods (q-difference equations [1, section 11.2; 3; 8,proof of Theorem 1; 14,sections IX,X], symmetric functions [13,21], matrix correspondences [2], the Hillman-Grassl correspondence [4,9], bijective proofs [17,18], which are based on the Gessel-Viennot theory of nonintersecting paths [6,7;23, section 4.5], Lie theory [15,16]). Unfortunately, these methods only apply for some classes of plane partitions, while they are not applicable for other classes. Our approach offers a uniform method of proving these plane partition results. Moreover, it may be called "direct" in that it works with tableaux (see section 3) only. We also give several generalizations and new results.

Our derivation of the determinant formulas for the generating functions for (α,β)-plane partitions is related to Gessel's and Viennot's elegant method [7], working with nonintersecting paths. Their paper bases on a counting theorem for those nonintersecting paths. With the help of this, they find a determinant formula for the generating functions of, what they call, R-tableaux, a special case of which are $(\alpha,1-\alpha)$-reverse plane partitions. (We remark that a modification of our involution of section 4 also could be used to prove this result about R-tableaux.) Using simple combinatorial arguments, determinant formulas for (α,β)-reverse and ordinary plane partitions can be derived. The determinant lemmas which they use are not so powerful as our Lemma 2.2, therefore they do not obtain new closed forms for *generating functions* of special classes of plane partitions. But they succeed in finding some new evaluations for the *cardinality* of some special classes of plane partitions. With the help of Lemma 2.2 we are able to generalize these results. This will be the object of a forthcoming paper [12]. (Besides, among other results, Gessel's and Viennot's paper [7] contains a lot of interesting combinatorial interpretations of special numbers, such as Fibonomial coefficients, Bernoulli numbers, Stirling numbers, Salié and Faulhaber numbers, etc.)

The outline of the paper is as follows. Section 2 contains the combinatorial Lemma 2.1 and the determinant Lemma 2.2, which were mentioned above. In section 3 we extend the notion of (α,β)-plane partitions to α-tableaux and give generating functions for them. The involution acting on α-tableaux which we use to obtain the determinantal expressions for the generating functions is explained and discussed in section 4. Section 5 is devoted to results about part generating functions. In fact the main result of this section, Theorem 5.1, generalizes the Jacobi-Trudi identity for Schur functions [13,(5.4),(5.5); 24,Theorems 3.5 and 3.5*]. In section 6 we obtain results for the norm generating function. Here, as special cases the determinant formulas of MacMahon [14,sec.494,490], Carlitz [3,(6.12)], Stanley [20], and the hook length formulae of Stanley [21,Theorem 15.3, Propositions 18.3-18.5] are contained. Finally, in section 7 we consider the trace generating function, thus extending the work of Gansner [4,5] and Stanley [22].

2. Preliminaries. Let S_r denote the symmetric group of order r. If M is a set with weight function w, acting on M, we write $F(M;w)$ for the generating function

$$\sum_{\pi\in M} w(\pi) .$$

LEMMA 2.1. *Let* $\{M_\sigma\colon \sigma\in S_n\}$ *be a collection of pairwise disjoint sets with a weight function* w *acting on* $\bigcup_{\sigma\in S_n} M_\sigma$. *Let* M_{id} *be partitioned into two disjoint subsets,* M^+_{id} *and* M^-_{id} , *with* $M_{\mathrm{id}} = M^+_{\mathrm{id}} \cup M^-_{\mathrm{id}}$. *Assume a weight-preserving involution* φ *acting on* $M^-_{\mathrm{id}} \cup \bigcup_{\sigma\neq\mathrm{id}} M_\sigma$ *satisfying the following property, holding for all* $\sigma\in S_n$: *For every* π *on which* φ *is applicable,* $\pi\in M_\sigma$, *there exists a transposition* (i,j), $1\le i,j\le n$, *for* *which* $\varphi(\pi) \in M_{\sigma(i,j)}$. *Then*

$$F(M^+_{\mathrm{id}};w) = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma\, F(M_\sigma;w) .$$

PROOF. By definition, φ maps an element of $M^-_{\mathrm{id}} \cup \bigcup_{\substack{\sigma \text{ even}\\ \sigma\neq \mathrm{id}}} M_\sigma$ onto an element of $\bigcup_{\sigma \text{ odd}} M_\sigma$, and vice versa. Since the M_σ's are pairwise disjoint and φ is weight preserving, we have

$$F(M^-_{\mathrm{id}};w) + \sum_{\substack{\sigma \text{ even}\\ \sigma\neq \mathrm{id}}} F(M_\sigma;w) = \sum_{\sigma \text{ odd}} F(M_\sigma;w) .$$

This leads to

$$\begin{aligned} F(M^+_{\mathrm{id}};w) &= F(M_{\mathrm{id}};w) - F(M^-_{\mathrm{id}};w) \\ &= F(M_{\mathrm{id}};w) + \sum_{\sigma\neq \mathrm{id}} \operatorname{sgn}\sigma\, F(M_\sigma;w) \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}\sigma\, F(M_\sigma;w) . \end{aligned}$$

∎

Next we prove the promised determinant lemma.

LEMMA 2.2. *Let* $X_1,\dots,X_r,A_2,\dots,A_r,B_2,\dots,B_r$ *be indeterminates. Then*

$$\det_{1\le s,t\le r}\Big((X_s+A_r)\cdots(X_s+A_{t+1})(X_s+B_t)\cdots(X_s+B_2)\Big) = \\ = \prod_{1\le i<j\le r}(X_i-X_j) \prod_{2\le i\le j\le r}(B_i-A_j) ,$$

with the convention that empty products (like $(X_s+A_r)\cdots(X_s+A_{t+1})$ *for* $t=r$*) are set equal to one.*

PROOF. Subtract the $(r-1)$'th column from the r'th, the $(r-2)$'th from the $(r-1)$'th,...,the first from the second to obtain

$$\det\Big((X_s+A_r)\cdots(X_s+A_{t+1})(X_s+B_t)\cdots(X_s+B_2)\Big) = \\ = (B_r-A_r)(B_{r-1}-A_{r-1})\cdots(B_2-A_2)\det\Big((X_s+A_r)\cdots(X_s+A_{t+1})(X_s+B_{t-1})\cdots(X_s+B_2)\Big).$$

Next, in the determinant on the right-hand side of this equation subtract the $(r-1)$'th column from the r'th,...,the second from the third. This time the expression $(B_{r-1}-A_r)(B_{r-2}-A_{r-1})\cdots(B_2-A_3)$ can be factored out on the right-hand side. This process is continued until

$$\det\Big((X_s+A_r)\cdots(X_s+A_{t+1})(X_s+B_t)\cdots(X_s+B_2)\Big) = \\ = \prod_{2\le i\le j\le r}(B_i-A_j)\cdot\det\Big((X_s+A_r)\cdots(X_s+A_{t+1})\Big) .$$

The determinant on the right-hand side of this equation is a polynomial in $X_1,X_2,\dots,X_r$ with degree $\binom{r}{2}$. Since for $X_{s_1}=X_{s_2}$, where $1\le s_1<s_2\le r$,

the determinant vanishes, it may be factorized

$$\det\left((X_s+A_r)\cdots(X_s+A_{t+1})\right) = \prod_{1\le i<j\le r} (X_i-X_j) \cdot p(X_1,\dots,X_r) ,$$

where $p(X_1,\dots,X_r)$ is some polynomial in $X_1,\dots,X_r$. But the degree of $\prod_{1\le i<j\le r} (X_i-X_j)$ is $\binom{r}{2}$, therefore $p(X_1,\dots,X_r)$ is identical to a constant, which is easily seen to be equal to one. ▮

It is convenient to state two limiting cases of Lemma 2.2 separatedly.

LEMMA 2.2.1.

$$\det_{1\le s,t\le r}\left((X_s+A_r)\cdots(X_s+A_{t+1})\right) = \prod_{1\le i<j\le r} (X_i-X_j) .$$

PROOF. Divide both sides of Lemma 2.2 by $\prod_{2\le i\le j\le r} (B_i-A_j)$ and let $B_j \longrightarrow \infty$ for $j=2,3,\dots,r$. ▮

LEMMA 2.2.2.

$$\det_{1\le s,t\le r}\left((X_s+B_t)\cdots(X_s+B_2)\right) = \prod_{1\le i<j\le r} (X_j-X_i) .$$

PROOF. Divide both sides of Lemma 2.2 by $\prod_{2\le i\le j\le r} (A_j-B_i)$ and let $A_j \longrightarrow \infty$ for $j=2,3,\dots,r$. ▮

3. α-Tableaux. In the proofs of the generating function theorems we shall work with "α-tableaux". By this notion we mean the folllowing extension of the notion "(α,β)-plane partition".

DEFINITION 3.1. *Let λ,μ be arbitrary r-tupels of integers with $\lambda\ge\mu$. An array π of the form* (1.1) *is called an α-tableaux of shape λ/μ if the condition* (1.2)(a) *holds. The entries of an α-tableaux are called parts. All other notions, such as norm, norm generating function, part generating function, trace generating function, etc., are also extended to α-tableaux (see section* 1).

Thus, for α-tableaux we only have a restriction on the rows, but none on the columns. Note that in Definition 3.1 we did not require that $\lambda,\mu \in D_r$. To give an example, the array in Figure 2

$$\begin{array}{cccc} & 6 & 6 & 7 & 5 \\ 1 & 2 & 3 & & \\ & 4 & & & \end{array}$$

Figure 2

is a (-1)-tableaux of shape (1,0,1)/(5,3,2).

The difficulty in counting (i.e. finding generating functions for) (α,β)-plane partitions lies in the fact that there are restrictions on both rows *and* columns ((1.2)(a) and (b)). However, in an α-tableaux the rows are mutually independent, since (1.2)(b) must not hold. Therefore it is easy to find generating functions for α-tableaux of a given shape. The relevant results which we need in the following sections are summarized in the propositions below.

For any integer α and $\mathbf{x}=(\ldots x_{-1},x_0,x_1,\ldots)$ let

$$h_n^{(\alpha)}(\mathbf{x};A,B) = \sum x_{i_1}\cdots x_{i_n}$$

where the sum is over all $i_1,\ldots i_n$ satisfying $i_1\le A$, $i_n\ge B$, and $i_j\ge i_{j+1}+\alpha$ for $j=1,2,\ldots n-1$. Obviously, $h_n^{(0)}(\mathbf{x};A,B)$ and $h_n^{(1)}(\mathbf{x};A,B)$ are the complete symmetric function of degree n and the elementary symmetric function of degree n in the variables $x_B,\ldots,x_A$, respectively. Then we have

PROPOSITION 3.2. *The part generating function for α-tableaux τ of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\prod_i h_{\lambda_i-\mu_i}^{(\alpha)}(\mathbf{x};a_i,b_i) \ . \qquad \blacksquare$$

Next, for the indeterminate x we introduce the "x-notations" $[\beta]=1-x^\beta$, $[n]!=[1][2]\cdots[n]$, $[0]!=1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \ ,$$

$$(a;x)_\infty = \prod_{j=0}^{\infty}(1-ax^j) \ ,$$

and

$$(a;x)_\beta = (a;x)_\infty \ / \ (ax^\beta;x)_\infty \ .$$

Then we have

PROPOSITION 3.3. *The norm generating function for α-tableaux τ of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\prod_i x^{b_i(\lambda_i-\mu_i)+\alpha\binom{\lambda_i-\mu_i}{2}} \begin{bmatrix} a_i-b_i-\alpha(\lambda_i-\mu_i-1)+\lambda_i-\mu_i \\ \lambda_i-\mu_i \end{bmatrix} .$$

PROOF. In Proposition 3.2 set $x_i=x^i$ for all integers i. We have

$$h^{(\alpha)}_{\lambda_i-\mu_i}((\ldots,x^l,\ldots);a_i,b_i) = \sum x^{j_1+\cdots+j_{\lambda_i-\mu_i}} \tag{3.3.1}$$

where the sum is over all $j_1,\ldots,j_{\lambda_i-\mu_i}$ with $j_1\le a_i$, $j_{\lambda_i-\mu_i}\ge b_i$ and $j_l\ge j_{l+1}+\alpha$. Performing the index transformation $J_l:=j_l+\alpha(l-\lambda_i+\mu_i)-b_i$ we get the conditions

$$J_1\le a_i-\alpha(\lambda_i-\mu_i-1)-b_i \ , \ J_{\lambda_i-\mu_i}\ge 0 \text{ and } J_l\ge J_{l+1} \ . \tag{3.3.2}$$

Hence, the right-hand side of (3.3.1) becomes

$$x^{b_i(\lambda_i-\mu_i)+\alpha\binom{\lambda_i-\mu_i}{2}} \sum x^{J_1+\cdots J_{\lambda_i-\mu_i}} ,$$

where the J_l's satisfy (3.3.2). But, because of (3.3.2), the sum in this expression is the generating function for linear partitions with at most $(\lambda_i-\mu_i)$ parts and largest part $\le (a_i-\alpha(\lambda_i-\mu_i-1)-b_i)$. This proves the proposition using [1, Theorem 3.1]. ∎

Let us write $x(m,\infty)$ for the formal product $\prod_{l=m+1}^{\infty} x_l$, and $x(m,n)$ for $x(m,\infty)/x(n,\infty)$. Then we have

PROPOSITION 3.4. *The trace generating function for α-tableaux τ of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\prod_i \left(x(\mu_i-i,\lambda_i-i)^{b_i} \left(\prod_{l=\mu_i-i}^{\lambda_i-i-1} x(\mu_i-i,l)^{\alpha}\right) t(\lambda_i-i/\mu_i-i;a_i-\alpha(\lambda_i-\mu_i-1)-b_i)\right) ,$$

where $t(p/q;m)$ is the coefficient of z^m in

$$1 \Big/ \prod_{l=q}^{p} (1-x(q,l)z) \ .$$

PROOF. Referring to the definition of the trace generating function we obtain for the trace generating function for α-tableaux the expression

$$\prod_i \sum x_{\mu_i-i+1}^{j_1} \cdots x_{\lambda_i-i}^{j_{\lambda_i-\mu_i}} , \tag{3.4.1}$$

where the sum is over all $j_1,\ldots,j_{\lambda_i-\mu_i}$ with $j_1\le a_i$, $j_{\lambda_i-\mu_i}\ge b_i$ and $j_l\ge j_{l+1}+\alpha$, such as in Proposition 3.3. Perform the same index transformation. Then the right-hand side of (3.4.1) may be rewritten in the form

$$\prod_i \left(x(\mu_i-i,\lambda_i-i)^{b_i} \prod_{l=\mu_i-i}^{\lambda_i-i-1} x(\mu_i-i,l)^{\alpha} \sum x_{\mu_i-i+1}^{J_1} \cdots x_{\lambda_i-i}^{J_{\lambda_i-\mu_i}} \right) ,$$

where the sum is over all $J_1,\dots,J_{\lambda_i-\mu_i}$ satisfying (3.3.2). It is a simple matter of fact that this sum is equal to the coefficient of $z^{a_i-\alpha(\lambda_i-\mu_i-1)-b_i}$ in

$$1 \Big/ \prod_{l=\mu_i-i}^{\lambda_i-i} (1-x(\mu_i-i,l)z) . \qquad \blacksquare$$

When there is no upper bound on the parts of the α-tableaux, we obtain a simpler expression.

PROPOSITION 3.5. *The trace generating function for α-tableaux τ of shape λ/μ in which the last part in row i is at least b_i is*

$$\prod_i \left(x(\mu_i-i,\lambda_i-i)^{b_i} \prod_{l=\mu_i-i}^{\lambda_i-i-1} x(\mu_i-i,l)^{\alpha} \prod_{l=\mu_i-i+1}^{\lambda_i-i} (1-x(\mu_i-i,l))^{-1} \right) . \qquad \blacksquare$$

4. The involution. In this section we construct and investigate the involution which is crucial to apply Lemma 2.1 to the computation of plane partition generating functions. It was inspired by Zeilberger's [25] reflection proof for the n-candidate ballot problem and the author's paper [11] about generating functions for plane partitions with unrestricted part magnitude. Subsequently, in Theorem 4.2 we give the properties of this involution which we use in the following sections.

DEFINITION 4.1. *Let α,β be arbitrary integers, ν be an r-tupel of integers and $\lambda\in D_r$. For an α-tableaux τ of shape λ/ν the map $\varphi_{\alpha,\beta}$ is defined in three steps:*

(A) *Consider all pairs $(\tau_{ij},\tau_{i+1,j})$ of parts of τ with*

$$\tau_{ij}<\tau_{i+1,j}+\beta . \tag{4.4.1}$$

(By convention this inequality is also considered to be valid if $\tau_{i+1,j}$ does not exist and $j=\mu_{i+1}$.) Let J be the largest integer with $\tau_{iJ}<\tau_{i+1,J}+\beta$ for some i. Let I be the smallest integer with $\tau_{IJ}<\tau_{I+1,J}+\beta$. (The idea is to search for the right-most and, among them, highest pair satisfying (4.4.1) in the tableaux τ.)

(B) (a) $\nu_I \geq \nu_{I+1}$. *Let K be the smallest integer with* $\tau_{I,K} < \tau_{I+1,K} + \beta$.

(b) $\nu_I < \nu_{I+1}$. *Let K be the smallest integer with* $\tau_{I,K-1} > \tau_{I+1,K+1} + 2\alpha + \beta - 2$. *(By convention this inequality is also considered to be valid if $\tau_{I+1,K+1}$ does not exist and $K=\lambda_{I+1}$.)*

(C) *The image of τ under application of $\varphi_{\alpha,\beta}$ is the tableaux τ', which is obtained by leaving all rows of τ unchanged with the exception of the I'th and $(I+1)$'th row. Here we set*

$$\tau'_{Ij} = \begin{cases} \tau_{Ij} & j \geq K \\ \tau_{I+1,j+1} - (1-\alpha-\beta) & j < K \end{cases}$$

and

$$\tau'_{I+1,j} = \begin{cases} \tau_{I+1,j} & j > K \\ \tau_{I,j-1} + (1-\alpha-\beta) & j \leq K \end{cases} .$$

Geometrically, τ' is obtained from τ the following way:

$$\begin{array}{lll} & \tau_{I,\nu_I+1} \cdots \tau_{I,K-1} & \tau_{IK} \cdots\cdots \\ \tau_{I+1,\nu_{I+1}+1} \cdots\cdots & \tau_{I+1,K} & \tau_{I+1,k+1} \cdots \end{array}$$

$$\Big\downarrow \varphi_{\alpha,\beta}$$

$$\begin{array}{lll} \tau_{I+1,\nu_{I+1}+1} - (1-\alpha-\beta) \cdots\cdots & \tau_{I+1,K} - (1-\alpha-\beta) & \tau_{IK} \cdots\cdots \\ \quad \tau_{I,\nu_I+1} + (1-\alpha-\beta) \cdots\cdots & \tau_{I,K-1} + (1-\alpha-\beta) & \tau_{I+1,K+1} \cdots \end{array}$$

Following Zeilberger [25], for $\sigma \in S_r$ we define $\mathbf{e}_\sigma := (1-\sigma(1), 2-\sigma(2), \ldots, r-\sigma(r))$. If $a=(a_1,\ldots,a_r)$ we write $a_\sigma = (a_{\sigma(1)},\ldots,a_{\sigma(r)})$. The set of α-tableaux of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is denoted by $T_\alpha(\lambda/\mu;a,b)$, where $b=(b_1,\ldots,b_r)$. The subset of $T_\alpha(\lambda/\mu;a,b)$ containing all (α,β)-plane partitions of $T_\alpha(\lambda/\mu;a,b)$ is denoted by $T^+_{\alpha,\beta}(\lambda,\mu;a,b)$, its complement by $T^-_{\alpha,\beta}(\lambda/\mu;a,b)$. It will be proved in the theorem below, that $\varphi_{\alpha,\beta}$ satisfies all conditions required to apply Lemma 2.1 with $M_\sigma = T_\alpha(\lambda/\mu_\sigma + \mathbf{e}_\sigma; a_\sigma + (1-\alpha-\beta)\mathbf{e}_\sigma, b)$, $M^+_{\mathrm{id}} = T^+_{\alpha,\beta}(\lambda/\mu;a,b)$ and $M^-_{\mathrm{id}} = \,= T^-_{\alpha,\beta}(\lambda/\mu;a,b)$. We give three weights for which $\varphi_{\alpha,\beta}$ is weight-preserving, w_p (for $\alpha+\beta=1$), w_n, and w_t, respectively. Application of Lemma 2.1 then leads to the results concerning part generating functions (section 5), norm generating functions (section 6), and trace generating functions (section 7), respectively.

THEOREM 4.2. *Let α,β be integers and λ,μ,a,b be r-tupels of integers such that $\lambda,\mu \in D_r$ and*

$$a_i+\alpha(\mu_i-\mu_{i+1})+(1-\beta) \ge a_{i+1} \text{ and } b_i+\alpha(\lambda_i-\lambda_{i+1})+(1-\beta) \ge b_{i+1} \tag{4.2.1}$$

for $i=1,2,\dots,r-1$. $\varphi_{\alpha,\beta}$ satisfies the following properties:

(A) *$\varphi_{\alpha,\beta}$ is an involution on*

$T^-_{\alpha,\beta}(\lambda/\mu;a,b) \cup \bigcup_{\substack{\sigma\in S_r\\ \sigma\ne \mathrm{id}}} T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b)$. If τ is an element of the domain of $\varphi_{\alpha,\beta}$, $\tau \in T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b)$, there exists a transposition $(I,I+1)$ with

$\varphi_{\alpha,\beta}(\tau) \in T_\alpha(\lambda/\mu_{\sigma(I,I+1)}+\mathbf{e}_{\sigma(I,I+1)};a_{\sigma(I,I+1)}+(1-\alpha-\beta)\mathbf{e}_{\sigma(I,I+1)},b)$.

(B) *$\varphi_{\alpha,1-\alpha}$ is weight-preserving with respect to the weight*

$$w_p(\tau) = \prod x_{\tau_{ij}},$$

where the product is over all parts τ_{ij} of τ.

(C) *$\varphi_{\alpha,\beta}$ is weight-preserving with respect to the weight*

$$w_n(\tau) = x^{n(\pi)+(1-\alpha-\beta)\sum_{s=1}^{r}\binom{\nu_s}{2}},$$

where λ/ν is the shape of τ.

(D) *$\varphi_{\alpha,\beta}$ is weight-preserving with respect to the weight*

$$w_t(\tau) = \prod_{s=1}^{r} x(\nu_s-s,\infty)^{s(\alpha+\beta-1)} \prod_{l\in\mathbb{Z}} x_l^{t_l(\pi)},$$

where λ/ν is the shape of τ.

PROOF. (A) Recalling Definition 4.1.(A), $\varphi_{\alpha,\beta}$ is applicable on α-tableaux violating the column-condition $\tau_{ij} \ge \tau_{i+1,j}+\beta$ at least once, and on α-tableaux of shape λ/ν with $\nu \notin D_r$. (The second fact is due to the convention we made in interpreting the inequality $\tau_{ij}<\tau_{i+1,j}+\beta$.) Let $\tau \in T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b)$ and I be the index which was defined in Definition 4.1.(A). From Definition 4.1.(C) it is clear that the shape of $\varphi_{\alpha,\beta}(\tau)$ is λ/ν, where ν is obtained from $\mu_\sigma+\mathbf{e}_\sigma$ by leaving all coordinates of $\mu_\sigma+\mathbf{e}_\sigma$ unchanged with the exception of the I'th and $(I+1)$'th, which are given by $\nu_I=[\mu_\sigma+\mathbf{e}_\sigma]_{I+1}-1$ and $\nu_{I+1}=[\mu_\sigma+\mathbf{e}_\sigma]_I+1$. Hence, $\nu_I=\mu_{\sigma(I,I+1)}+\mathbf{e}_{\sigma(I,I+1)}$. Arguing similarly, we obtain the part restrictions for $\varphi_{\alpha,\beta}(\tau)$. Of course we only have to concentrate on the I'th and $(I+1)$'th row. The parts in the I'th row of $\varphi_{\alpha,\beta}(\tau)$ are at most $[a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma]_{I+1}-(1-\alpha-\beta)$ and at least b_I, the parts in the $(I+1)$'th row are at most $[a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma]_I+(1-\alpha-\beta)$ and at least b_{I+1}, which after a little calculation leads to the desired row bounds stated above. (At this point the inequalities (4.2.1) were needed.) Hence, $\varphi_{\alpha,\beta}(\tau)\in T_\alpha(\lambda/\mu_{\sigma(I,I+1)}+\mathbf{e}_{\sigma(I,I+1)};a_{\sigma(I,I+1)}+(1-\alpha-\beta)\mathbf{e}_{\sigma(I,I+1)},b)$. That $\varphi_{\alpha,\beta}$ is an involution is clear from the definition.

(B) In case $\beta=1-\alpha$ we have $(1-\alpha-\beta)=0$, therefore the multiset of parts of $\varphi_{\alpha,\beta}(\tau)$ is identically with that of τ.

(C) Let $\tau\in T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b)$ and I be the index defined in Definition 4.1.1. Then we have

$$\begin{aligned} n(\varphi_{\alpha,\beta}(\tau)) &= n(\tau)-(1-\alpha-\beta)([\mu_\sigma+\mathbf{e}_\sigma]_I-[\mu_\sigma+\mathbf{e}_\sigma]_{I+1}+1)\\ &= n(\tau)-(1-\alpha-\beta)(\mu_{\sigma(I)}-\sigma(I)-\mu_{\sigma(I+1)}+\sigma(I+1)) . \end{aligned}$$

Therefore we get

$$\begin{aligned} n(\varphi_{\alpha,\beta}(\tau)) &+ (1-\alpha-\beta)\sum_{s=1}^{r}\binom{[\mu_{\sigma(I,I+1)}+\mathbf{e}_{\sigma(I,I+1)}]_s}{2} = \\ &= n(\tau)-(1-\alpha-\beta)(\mu_{\sigma(I)}-\sigma(I)-\mu_{\sigma(I+1)}+\sigma(I+1)) + (1-\alpha-\beta)\sum_{s=1}^{r}\binom{[\mu_\sigma+\mathbf{e}_\sigma]_s}{2}+\\ &\quad +(1-\alpha-\beta)(\mu_{\sigma(I)}+I-\sigma(I))-(1-\alpha-\beta)(\mu_{\sigma(I+1)}+I-\sigma(I+1))\\ &= n(\tau)+\sum_{s=1}^{r}\binom{[\mu_\sigma+\mathbf{e}_\sigma]_s}{2} , \end{aligned}$$

hence $w_n(\varphi_{\alpha,\beta}(\tau)) = w_n(\tau)$.

(D) With the assumptions of (C) we have

$$\prod_{l\in\mathbb{Z}} x_l^{t_l(\varphi_{\alpha,\beta}(\tau))} = x(\mu_{\sigma(I+1)}-\sigma(I+1),\mu_{\sigma(I)}-\sigma(I))^{\alpha+\beta-1}\cdot\prod_{l\in\mathbb{Z}}\left(x_l^{t_l(\tau)}\right)$$

Therefore we get

$$\begin{aligned} w_t(\varphi_{\alpha,\beta}(\tau)) &= \frac{x(\mu_{\sigma(I)}-\sigma(I),\infty)^{\alpha+\beta-1}}{x(\mu_{\sigma(I+1)}-\sigma(I+1),\infty)^{\alpha+\beta-1}}\cdot\prod_{s=1}^{r} x(\mu_{\sigma(s)}-\sigma(s),\infty)^{s(\alpha+\beta-1)}\\ &\qquad\cdot x(\mu_{\sigma(I+1)}-\sigma(I+1),\mu_{\sigma(I)}-\sigma(I))^{\alpha+\beta-1}\cdot\prod_{l\in\mathbb{Z}}\left(x_l^{t_l(\tau)}\right)\\ &= w_t(\tau) . \end{aligned}$$

5. Part generating functions. Let $\beta=1-\alpha$. Then Theorem 4.2.(B) can be applied to compute part generating functions.

THEOREM 5.1. *Let $\lambda,\mu \in D_r$ and a,b be r-tupels of integers satisfying*

$$a_i+\alpha \geq a_{i+1} \quad \text{and} \quad b_i+\alpha \geq b_{i+1} \quad \text{for } i=1,2,\dots,r-1.$$

The part generating function for $(\alpha,1-\alpha)$-plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is

$$\det_{1\le s,t\le r}\left(h^{(\alpha)}_{\lambda_s-s-\mu_t+t}(\mathbf{x};a_t,b_s)\right) .$$

PROOF. We apply Lemma 2.1 with $M_\sigma=T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma,b)$, $M^+_{\mathrm{id}}=T^+_{\alpha,1-\alpha}(\lambda/\mu;a,b)$, $M^-_{\mathrm{id}}=T^-_{\alpha,1-\alpha}(\lambda/\mu;a,b)$, $\varphi=\varphi_{\alpha,\beta}$, and $w=w_p$. Because of Theorem 4.2.(A),(B) all assumptions of Lemma 2.1 are satisfied. Since by Proposition 3.2

$$F(T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma,b);w_p) = \prod_{s=1}^{r} h_{\lambda_s-s-\mu_{\sigma(s)}+\sigma(s)}(\mathbf{x};a_{\sigma(s)},b_s) ,$$

the theorem follows. ∎

For $\alpha=0$ and $\alpha=1$ we obtain the generalized Jacobi-Trudi identities for flagged Schur functions [24,Theorems 3.5 and 3.5*]:

COROLLARY 5.2. *Let $\lambda,\mu,a,b \in D_r$. The part generating function for column-strict plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\det_{1\le s,t\le r}\left(h_{\lambda_s-s-\mu_t+t}(\mathbf{x};a_t,b_s)\right) ,$$

where $h_n(\mathbf{x};A,B)$ is the complete symmetric function of order n in the variables $x_B,\dots,x_A$. ∎

COROLLARY 5.3. *Let $\lambda,\mu \in D_r$ and a,b be r-tupels of integers satisfying*

$$a_i+1 \geq a_{i+1} \quad \text{and} \quad b_i+1 \geq b_{i+1} \quad \text{for } i=1,2,\dots,r-1.$$

The part generating function for row-strict plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is

$$\det_{1\le s,t\le r}\left(e_{\lambda_s-s-\mu_t+t}(\mathbf{x};a_t,b_s)\right) ,$$

where $e_n(\mathbf{x};A,B)$ is the elementary symmetric function of order n in the variables $x_B,\dots,x_A$. ▮

6. Norm generating functions. This section contains applications of Theorem 4.2.(C). Our first result generalizes Stanley's [20,Prop. 21.2], MacMahon's [14,sec.494], Carlitz's [3,(6.12)] and Gordon's [8,Theorem 1] determinant formulas.

THEOREM 6.1. *Let $\lambda,\mu \in D_r$ and a,b be r-tupels of integers satisfying (4.2.1). The norm generating function for (α,β)-plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\det_{1\le s,t\le r}\left(x^{N_1(s,t)}\begin{bmatrix}(1-\alpha)(\lambda_s-\mu_t)-\beta(s-t)+a_t-b_s+\alpha\\ \lambda_s-s-\mu_t+t\end{bmatrix}\right) ,$$

where

$$N_1(s,t) = b_s(\lambda_s-s-\mu_t+t)+(1-\alpha-\beta)\left[\binom{\mu_t+s-t}{2}-\binom{\mu_t}{2}\right]+\alpha\binom{\lambda_s-s-\mu_t+t}{2} .$$

PROOF. We apply Lemma 2.1 with $M_\sigma=T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b)$, $M^+_{\mathrm{id}}=T^+_{\alpha,\beta}(\lambda/\mu;a,b)$, $M^-_{\mathrm{id}}=T^-_{\alpha,\beta}(\lambda/\mu;a,b)$, $\varphi=\varphi_{\alpha,\beta}$ and $w=w_n$. Because of Theorem 4.2.(A),(C) all assumptions of Lemma 2.1 are satisfied. Since by Proposition 3.3

$$F(T_\alpha(\lambda/\mu_\sigma+\mathbf{e}_\sigma;a_\sigma+(1-\alpha-\beta)\mathbf{e}_\sigma,b);w_n) =$$
$$= x^{\sum_{s=1}^{r}\binom{\mu_{\sigma(s)}+s-\sigma(s)}{2}}\prod_{s=1}^{r} x^{N_2(s,\sigma(s))}\begin{bmatrix}(1-\alpha)(\lambda_s-\mu_{\sigma(s)})-\beta(s-\sigma(s))+a_{\sigma(s)}-b_s+\alpha\\ \lambda_s-s-\mu_{\sigma(s)}+\sigma(s)\end{bmatrix} ,$$

where

$$N_2(s,t) = b_s(\lambda_s-s-\mu_t+t)+\alpha\binom{\lambda_s-s-\mu_t+t}{2} ,$$

the theorem follows. ▮

For $\lambda,\mu \in D_r$ we define an (α,β)-*reverse* plane partition of shape λ/μ to be an array π of integers of the form (1.1) with

$$\pi_{ij}+\alpha \le \pi_{i,j+1} \qquad \text{for} \quad 1\le i\le r,\ \mu_i<j<\lambda_i \tag{6.2)(a}$$

and

$$\pi_{ij}+\beta \le \pi_{i+1,j} \qquad \text{for} \quad 1\le i<r,\ \mu_i<j\le\lambda_{i+1} . \tag{6.2)(b}$$

Let ρ be the rotation which maps π onto π' by

$$\pi'_{ij} = \pi_{r-i+1,r-j+1} . \tag{6.3}$$

If π is an (α,β)-reverse plane partition of shape λ/μ in which the first part in row i is at least b_i and the last part in row i is at most a_i, we obtain $\rho(\pi)$ to be an (α,β)-plane partition of shape λ'/μ' in which the first part in row i is at most a_{r-i+1} and the last part in row i is at least b_{r-i+1}, where $\lambda'=(r-\mu_r,\dots,r-\mu_2,r-\mu_1)$ and $\mu'=(r-\lambda_r,\dots,r-\lambda_2,r-\lambda_1)$. Thus Theorem 6.1 implies:

COROLLARY 6.4. *Let $\lambda,\mu \in D_r$ and a,b be r-tupels of integers satisfying*

$$a_i+\alpha(\lambda_{i-1}-\lambda_i)+(1-\beta) \geq a_{i-1} \text{ and } b_i+\alpha(\mu_{i-1}-\mu_i)+(1-\beta) \geq b_{i-1} \tag{6.4.1}$$

for $i=2,3,\dots,r$. The norm generating function for (α,β)-reverse plane partitions of shape λ/μ in which the first part in row i is at least b_i and the last part in row i is at most a_i is

$$\det_{1\le s,t\le r}\left(x^{N_3(s,t)}\begin{bmatrix}(1-\alpha)(\lambda_s-\mu_t)-\beta(s-t)+a_s-b_t+\alpha\\ \lambda_s-s-\mu_t+t\end{bmatrix}\right),$$

where

$$N_3(s,t) = b_t(\lambda_s-s-\mu_t+t)+(1-\alpha-\beta)\left[\binom{-\lambda_s+t-s}{2}-\binom{-\lambda_s}{2}\right]+\alpha\binom{\lambda_s-s-\mu_t+t}{2}.\blacksquare$$

REMARK. We also could prove Corollary 6.4 by modifying the involution $\varphi_{\alpha,\beta}$ suitably. For sake of simplicity, we decided to derive it from Theorem 6.1.

It seems unlikely that the determinants in Theorem 6.1 or Corollary 6.4 can be simplified significantly in general. But there are several special cases in which Lemma 2.2 may be applied to obtain closed forms for these determinants. The first such evaluation yields a generalization of Stanley's celebrated hook-content formula [21,Theorem 15.3]. For definition of hook lengths and contents of a shape we refer the reader to [21,Definition 15.1].

THEOREM 6.5. *Let $\lambda\in D_r$ and $\lambda\ge0$. The norm generating function for (α,β)-plane partitions of shape λ in which the first part in row i is at most $a+(1-\beta)i$ (or equivalently, $\pi_{11}\le a+1-\beta$) and the last part in row i is at least $b-\alpha\lambda_i+(1-\beta)i$ is*

$$x^{N_4}\frac{[a+\alpha-b+1+c_1]\cdots[a+\alpha-b+1+c_p]}{[h_1]\cdots[h_p]},$$

where $p=|\lambda|$, $N_4=\sum_{i=1}^{r}\lambda_i(-\alpha\lambda_i/2+(2-\beta)i+b-\alpha/2-1)$, and c_i and h_i are the contents and hook lengths of the shape λ, respectively.

PROOF. Evidently, the row bounds satisfy (4.2.1). Setting $\mu=0$,

$a_i=a+(1-\beta)i$, and $b_i=b-\alpha\lambda_i+(1-\beta)i$ in Theorem 6.1 we obtain for the desired generating function the expression

$$x^{N_5}\cdot\det\left(\begin{bmatrix}\lambda_s-s+t+a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right),$$

where $N_5=N_4-\sum_{i=1}^{r}\lambda_i(i-1)$. Taking some factors out of the determinant leads to

$$\det\left(\begin{bmatrix}\lambda_s-s+t+a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right)=\frac{\prod_{i=1}^{r}[\lambda_i-i+1+a+\alpha-b]!}{[a+\alpha-b]!^r\prod_{i=1}^{r}[\lambda_i-i+r]!}\times$$

$$\times\det([\lambda_s-s+r]\cdots[\lambda_s-s+t+1][\lambda_s-s+t+a+\alpha-b]\cdots[\lambda_s-s+2+a+\alpha-b])$$

$$=\frac{\prod_{i=1}^{r}[\lambda_i-i+1+a+\alpha-b]!}{[a+\alpha-b]!^r\prod_{i=1}^{r}[\lambda_i-i+r]!}\,x^{\sum_{i=1}^{r}(\lambda_i-i)(r-1)}\times$$

$$\times\det((x^{-\lambda_s+s}-x^{r})\cdots(x^{-\lambda_s+s}-x^{t+1})(x^{-\lambda_s+s}-x^{t+a+\alpha-b})\cdots(x^{-\lambda_s+s}-x^{2+a+\alpha-b})).$$

Now Lemma 2.2 with $X_i=x^{-\lambda_i+i}$, $A_i=-x^{i}$, and $B_i=-x^{i+a+\alpha-b}$ can be applied to get

$$\det\left(\begin{bmatrix}\lambda_s-s+t+a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right)=\frac{\prod_{i=1}^{r}[\lambda_i-i+1+a+\alpha-b]!}{[a+\alpha-b]!^r\prod_{i=1}^{r}[\lambda_i-i+r]!}\,x^{\sum_{i=1}^{r}(\lambda_i-i)(r-1)}\times$$

$$\times\prod_{1\le i<j\le r}(x^{\lambda_i-i}-x^{\lambda_j-j})\prod_{2\le i\le j\le r}(x^{j}-x^{i+a+\alpha-b})$$

$$=x^{\sum_{i=1}^{r}\lambda_i(i-1)}\frac{\prod_{1\le i<j\le r}[\lambda_i-i-\lambda_j+j]}{\prod_{i=1}^{r}[\lambda_i-i+r]!}\;\frac{\prod_{i=1}^{r}[\lambda_i-i+a+\alpha-b+1]!}{\prod_{i=1}^{r}[a+\alpha-b+1-i]!}.$$

The first quotient in the last line is equal to $1/[h_1]\cdots[h_p]$, (using [13,p.9,Example 1(4)]), the latter is equal to $[a+\alpha-b+1+c_1]\cdots$ $\cdots[a+\alpha-b+1+c_p]$, (using [13,p.10,Example 3]). ∎

Stanley's theorem [21,Theorem 15.3] is the special case $\alpha=0$, $\beta=1$, $b=1$ of Theorem 6.5.

There is another case in which the determinant in Theorem 6.1 can be evaluated. It generalizes one of the main results of MacMahon on plane partitions [14,sec.429,proof in sec.494].

THEOREM 6.6. *Let* $\lambda\in D_r$ *and* $\lambda\ge0$. *The norm generating function for* (α,β)*-plane partitions of shape* λ *in which the first part in row* i *is* *at most* $a-\beta i$ *(or equivalently,* $\pi_{11}\le a-\beta$*) and the last part in row* i *is at least* $b+(1-\alpha)\lambda_i-\beta i$ *is*

$$x^{N_6}\,\frac{[a+\alpha-b-c_1]\cdots[a+\alpha-b-c_p]}{[h_1]\cdots[h_p]},$$

where $p=|\lambda|$, $N_6=\sum_{i=1}^{r}\lambda_i((1-\alpha/2)\lambda_i-\beta i+b-\alpha/2)$, *and* c_i *and* h_i *are the contents and hook lengths of the shape* λ *, respectively.*

PROOF. Evidently, the row bounds satisfy (4.2.1). Setting $\mu=0$, $a_i=a-\beta i$ and $b_i=b+(1-\alpha)\lambda_i-\beta i$ in Theorem 6.1 we obtain for the desired generating function the expression

$$x^{N_7}\cdot\det\left(x^{t(\lambda_s-s)}\begin{bmatrix}a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right),$$

where $N_7=N_6-\sum_{i=1}^{r}i(\lambda_i-i)$. Taking some factors out of the determinant we get

$$\det\left(x^{t(\lambda_s-s)}\begin{bmatrix}a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right)=\frac{[a+\alpha-b]!^r}{\prod_{i=1}^{r}[\lambda_i-i+r]!\,\prod_{i=1}^{r}[-\lambda_i+i-1+a+\alpha-b]!}\times$$
$$\times\det(x^{t(\lambda_s-s)}[\lambda_s-s+r]\cdots[\lambda_s-s+t+1][-\lambda_s+s-t+1+a+\alpha-b]\cdots[-\lambda_s+s-1+a+\alpha-b])$$
$$=\frac{[a+\alpha-b]!^r}{\prod_{i=1}^{r}[\lambda_i-i+r]!\,\prod_{i=1}^{r}[-\lambda_i+i-1+a+\alpha-b]!}\cdot x^{\sum_{i=1}^{r}(r(\lambda_i-i)+(a+\alpha-b-i+1)(r-i+1))}(-1)^{\binom{r}{2}}\times$$
$$\times\det((x^{-\lambda_s+s}-x^r)\cdots(x^{-\lambda_s+s}-x^{t+1})(x^{-\lambda_s+s}-x^{t-1-a-\alpha+b})\cdots(x^{-\lambda_s+s}-x^{1-a-\alpha+b})).$$

Application of Lemma 2.2 with $X_i=x^{-\lambda_i+i}$, $A_i=-x^i$, and $B_i=-x^{i-1-a-\alpha+b}$ gives

$$\det\left(x^{t(\lambda_s-s)}\begin{bmatrix}a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right)=$$
$$=\frac{[a+\alpha-b]!^r}{\prod_{i=1}^{r}[\lambda_i-i+r]!\,\prod_{i=1}^{r}[-\lambda_i+i-1+a+\alpha-b]!}\cdot x^{\sum_{i=1}^{r}(r(\lambda_i-i)+(a+\alpha-b-i+1)(r-i+1))}(-1)^{\binom{r}{2}}\times$$
$$\times\prod_{1\le i<j\le r}(x^{\lambda_i-i}-x^{\lambda_j-j})\prod_{2\le i\le j\le r}(x^j-x^{i-1-a-\alpha+b})$$
$$=x^{\sum_{i=1}^{r}i(\lambda_i-i)}\,\frac{\prod_{1\le i<j\le r}[\lambda_i-i-\lambda_j+j]}{\prod_{i=1}^{r}[\lambda_i-i+r]!}\;\frac{[a+\alpha-b]!^r\prod_{2\le i\le j\le r}[a+\alpha-b+1+j-i]}{\prod_{i=1}^{r}[a+\alpha-b-\lambda_i+i-1]!}.$$

Again, the first quotient in the last line is equal to $1/[h_1]\cdots[h_p]$. By considerations which are similar to those in [13,p.10,Example 3], it can be shown that the second quotient is equal to

$[a+\alpha-b-c_1]\cdots[a+\alpha-b-c_p]$, which completes the proof. ∎

The special case $\alpha=\beta=0$, $\lambda_i=c$, $b=1-c$ of Theorem 6.6 gives MacMahon's formula [14,sec.429], see also [21,Theorem 18.1; 3,(6.12)] for other proofs and other formulations of the resulting product. If we set $\alpha=1$, $\beta=0$, $b=1$, we obtain Stanley's hook-content formula [21,Theorem 15.3] a second time (but formulated with the conjugate shape; see [21,p.167] for definition of conjugation). This is because the reflection at the main diagonal is a norm- (since even part-) preserving bijection between row-strict and column-strict plane partitions of conjugate shape.

Next we turn our attention to cases of simplification of the determinant in Corollary 6.4. Our first result unifies two further results of Stanley [21,Propositions 18.4 and 18.5].

THEOREM 6.7. *Let $\lambda\in D_r$ and $\lambda\geq 0$. The norm generating function for (α,β)-reverse plane partitions of shape λ in which the first part in row i is at least $b-(1-\beta)i$ (or equivalently, $\pi_{11}\geq b-1+\beta$) and the last part in row i is at most $a+\alpha\lambda_i-(1-\beta)i$ is*

$$x^{N_8}\frac{[a+\alpha-b+1+c_1]\cdots[a+\alpha-b+1+c_p]}{[h_1]\cdots[h_p]},$$

where $p=|\lambda|$, $N_8=\sum_{i=1}^{r}\lambda_i(\alpha\lambda_i/2+\beta i+b-\alpha/2-1)$, and c_i and h_i are the contents and hook lengths of the shape λ, respectively.

PROOF. Evidently, the row bounds satisfy (6.4.1). Setting $\mu=0$, $a_i=a+\alpha\lambda_i-(1-\beta)i$, and $b_i=b-(1-\beta)i$ in Corollary 6.4 we obtain for the desired generating function the expression

$$x^{N_9}\cdot\det\left(\begin{bmatrix}\lambda_s-s+t+a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right),$$

where $N_9=N_8-\sum_{i=1}^{r}\lambda_i(i-1)$. This determinant has already been evaluated in the proof of Theorem 6.5. ∎

Stanley's result about the norm generating function for column-strict reverse plane partitions [21,Proposition 18.4] comes out of the above theorem setting $\alpha=0$, $\beta=1$, and $b=1$. His generating function for row and column-strict reverse plane partitions [21,Proposition 18.5] is the limiting case $\alpha=\beta=1$, $b=1$, $a\to\infty$ of Theorem 6.7.

THEOREM 6.8. *Let $\lambda\in D_r$ and $\lambda\geq 0$. The norm generating function for (α,β)-reverse plane partitions of shape λ in which the first part in row i is at least $b+\beta i$ (or equivalently, $\pi_{11}\geq b+\beta$) and the last part* *in row i is at most $a-(1-\alpha)\lambda_i+\beta i$ is*

$$x^{N_{10}}\frac{[a+\alpha-b-c_1]\cdots[a+\alpha-b-c_p]}{[h_1]\cdots[h_p]},$$

where $p=|\lambda|$, $N_{10}=\sum_{i=1}^{r}\lambda_i(\alpha\lambda_i/2+\beta i+b-\alpha/2)$, and c_i and h_i are the contents and hook lengths of the shape λ , respectively.

PROOF. Evidently, the row bounds satisfy (6.4.1). Setting $\mu=0$, $a_i=a-(1-\alpha)\lambda_i+\beta i$, and $b_i=b+\beta i$ in Corollary 6.4 we get the expression

$$x^{N_{11}}\cdot\det\left(x^{t(\lambda_s-s)}\begin{bmatrix}a+\alpha-b\\ \lambda_s-s+t\end{bmatrix}\right),$$

where $N_{11}=N_{10}-\sum_{i=1}^{r}i(\lambda_i-i)$. This determinant has been evaluated in Theorem 6.6. ∎

For $\alpha=1$, $\beta=0$, and $b=1$ we obtain Stanley's result about column-strict reverse plane partitions [21,Proposition 18.4] another time, this time in "conjugate" formulation. The hook formula for the norm generating function for reverse plane partitions with unrestricted part magnitude [21,Proposition 18.3] is derived letting $\alpha=\beta=0$, $b=1$ and $a\to\infty$.

Concluding this section we derive a nice result of Bender and Knuth [2,Theorem 2*]. In our context the proof again degenerates to a determinant evaluation.

THEOREM 6.9. *The norm generating function for r-rowed plane partitions with largest part $\le m$ and exactly k parts in the last row is*

$$x^{rk}\begin{bmatrix}k+m-1\\ k\end{bmatrix}\prod_{i=0}^{r-2}\frac{[i]!}{[m+i]!}.$$

PROOF. In Theorem 6.1 let $\alpha=\beta=0$, $\lambda_i\to\infty$ for $i=1,\dots,r-1$, and $\lambda_r=k$, $\mu=0$, $a_i=m$, $b_i=1$ for $i=1,\dots,r-1$, and $b_r=1$. Thus we obtain the determinant

$$x^{k-r}\cdot\det\begin{pmatrix} x^{\binom{s-t}{2}}\dfrac{1}{[m+s-t]!} & \text{row } s,\ s<r \\ x^{\binom{r-t}{2}+t}\begin{bmatrix}k+m-1\\ k-r+t\end{bmatrix} & \text{row } r\end{pmatrix}.$$

For $t=r,r-1,\dots,2$ subtract the $(t-1)$'th column times $x^{t-1}[m-t+2]$ from the t'th. The effect of this is that the first row becomes $(1/[m]!,0,\dots,0)$. Therefore the above expression may be written in the form

$$x^{k-r-r+2}\,\frac{[r-2]!}{[m]!}\cdot\det\begin{pmatrix} x^{\binom{s-t}{2}}\dfrac{1}{[m+s-t+1]!} & \text{row } s,\ s<r-1 \\ \\ x^{\binom{r-t}{2}+2t}\left(\begin{bmatrix}k+m-1\\k-r+t+1\end{bmatrix}-x^{r-2}[m-t+1]\begin{bmatrix}k+m-1\\k-r+t\end{bmatrix}\right) & \text{row } r-1\end{pmatrix}.$$

This process is iterated and after some manipulation finally results in the expression

$$x^{k-r-\binom{r-1}{2}}\prod_{i=0}^{r-2}\frac{[i]!}{[m+i]!}\,x^{\binom{r-1}{2}+r}\times$$

$$\times\sum_{i=0}^{r-1}(-1)^{r+i+1}x^{\binom{r-i-1}{2}}\begin{bmatrix}k+m-1\\k-r+i+1\end{bmatrix}\begin{bmatrix}r-1\\i\end{bmatrix}[m]\cdots[m+r-2-i]=$$

$$=x^{k}\prod_{i=0}^{r-2}\frac{[i]!}{[m+i]!}\begin{bmatrix}k+m-1\\k\end{bmatrix}\sum_{i=0}^{r-1}\begin{bmatrix}r-1\\i\end{bmatrix}(x^{k}-1)\cdots(x^{k}-x^{r-i-2})\ .$$

The sum in the last line is equal to $x^{(r-1)k}$, which is easily seen by q-Vandermonde summation [19,App.IV.1]. ∎

Similarly we could derive Gansner's theorem [5,Theorem 4.5], which is a generalization of the above theorem.

7. Trace generating functions. Trace generating functions were first considered by Stanley [21,22]. He weighted each plane partition by its norm and its 0-trace (the sum of the parts on the main diagonal, cf. section 1). He also considered enumeration by "conjugate trace" (we give the definition below). Gansner [4,5] generalized Stanley's trace definitions and obtained nice expressions for the (generalized) trace generating function (as defined in section 1) for several classes of plane partitions.

In this section we give determinant formulas for the trace generating function of (α,β)-plane partitions of a given shape. Stanley's and Gansner's results are obtained as special cases, again when applying Lemma 2.2 (or its sublemmas 2.2.1 or 2.2.2, respectively). Also contained are some new computations for the 0-trace generating function.

We begin with the general result

THEOREM 7.1. *Let $\lambda,\mu\in D_r$ and a,b be r-tupels of integers satisfying* (4.2.1). *The trace generating function for (α,β)-plane partitions of* *shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i is*

$$\det_{1\le s,t\le r}\left(x(\mu_t-t,\lambda_s-s)^{b_s}x(\mu_t-t,\mu_s-s)^{s(\alpha+\beta-1)}\prod_{\ell=\mu_t-t}^{\lambda_s-s-1}x(\mu_t-t,\ell)^{\alpha}\times\right.$$
$$\left.\times\ t(\lambda_s-s/\mu_t-t;-\alpha(\lambda_s-\mu_t)+(1-\beta)(s-t)+a_t-b_s+\alpha)\right).$$

($t(p/q;m)$ was defined in Proposition 3.4.) The trace generating function for (α,β)-plane partitions of shape λ in which the last part in row i is at least b_i is

$$\det_{1\le s,t\le r}\left(x(\mu_t-t,\lambda_s-s)^{b_s}x(\mu_t-t,\mu_s-s)^{s(\alpha+\beta-1)}\prod_{\ell=\mu_t-t}^{\lambda_s-s-1}x(\mu_t-t,\ell)^{\alpha}\times\right.$$
$$\left.\times\prod_{\ell=\mu_t-t+1}^{\lambda_s-s}(1-x(\mu_t-t,\ell))^{-1}\right).$$

PROOF. The arguing is just as in the proof of Theorem 5.1, with the exception that here instead of Theorem 4.2.(C) we have to use 4.2.(D), and instead of Proposition 3.3 we have to use Propositions 3.4 and 3.5, respectively. ∎

Consider an (α,β)-plane partition π and the action of the rotation ρ on π, which was defined in (6.3). ρ is "almost" trace-preserving, in the sense that $t_l(\rho(\pi))=t_{-l}(\pi)$ for all integers l. Respecting this matter of fact, Theorem 7.1 can be converted in the analogous result for (α,β)-reverse plane partitions, such as before Corollary 6.4 was derived from Theorem 6.1.

THEOREM 7.2. *Let $\lambda,\mu\in D_r$ and a,b be r-tupels of integers satisfying (6.4.1). The trace generating function for (α,β)-reverse plane partitions of shape λ/μ in which the first part in row i is at least b_i and the last part in row i is at most a_i is*

$$\det_{1\le s,t\le r}\left(x(\mu_t-t,\lambda_s-s)^{b_t}x(\lambda_t-t,\lambda_s-s)^{t(1-\alpha-\beta)}\prod_{\ell=\lambda_t-t+1}^{\lambda_s-s}x(\ell,\lambda_s-s)^{\alpha}\times\right.$$
$$\left.\times\ t'(\lambda_s-s/\mu_t-t;-\alpha(\lambda_s-\mu_t)+(1-\beta)(s-t)+a_s-b_t+\alpha)\right),$$

where $t'(p/q;m)$ is the coefficient of z^m in

$$1\Big/\prod_{\ell=q}^{p}(1-x(\ell,p)z).$$

The trace generating function for (α,β)-reverse plane partitions of shape λ in which the first part in row i is at least b_i is

$$\det_{1\le s,t\le r}\left(x(\mu_t-t,\lambda_s-s)^{b_t}x(\lambda_t-t,\lambda_s-s)^{t(1-\alpha-\beta)}\prod_{\ell=\mu_t-t+1}^{\lambda_s-s}x(\ell,\lambda_s-s)^{\alpha}\times\right.$$
$$\left.\times\prod_{\ell=\mu_t-t}^{\lambda_s-s-1}(1-x(\ell,\lambda_s-s))^{-1}\right).\blacksquare$$

Since they are easier to handle, we turn our attention first to determinant evaluations for (α,β)-reverse plane partitions.

THEOREM 7.3. *Let $\lambda\in D_r$ and $\lambda\ge 0$. The trace generating function for (α,β)-reverse plane partitions of shape λ in which the first part in row i is at least $b-(1-\beta)i$ is*

$$\prod_{i=1}^{r}\left(x(-i,\lambda_i-i)^{b+\beta i-1}\prod_{\ell=-i+1}^{\lambda_i-i}x(\ell,\lambda_i-i)^{\alpha}\right)\frac{\prod\limits_{1\le i<j\le r}(1-x(\lambda_j-j,\lambda_i-i))}{\prod\limits_{i=1}^{r}\prod\limits_{\ell=-r}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}.$$

PROOF. In the second expression of Theorem 7.2 set $\mu=0$ and $b_i=b-(1-\beta)i$. Thus we obtain for the desired generating function the expression

$$\det_{1\le s,t\le r}\left(x(-t,\lambda_s-s)^{b-(1-\beta)t}x(\lambda_t-t,\lambda_s-s)^{t(1-\alpha-\beta)}\prod_{\ell=-t+1}^{\lambda_s-s}x(\ell,\lambda_s-s)^{\alpha}\times\right.$$
$$\left.\times\prod_{\ell=-t}^{\lambda_s-s-1}(1-x(\ell,\lambda_s-s)^{-1}\right).$$

Taking some factors out of the determinant yields

$$\prod_{i=1}^{r}\left(x(-i,\lambda_i-i)^{b-(1-\beta)i}x(0,\lambda_i-i)^{r-1}\prod_{\ell=-i+1}^{\lambda_i-i}x(\ell,\lambda_i-i)^{\alpha}\prod_{\ell=-r}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))^{-1}\right)\times$$
$$\times\det((x(\lambda_s-s,0)-x(-r,0))\cdots(x(\lambda_s-s,0)-x(-t-1,0))x(\lambda_s-s,0)^{t-1}).$$

Once again this determinant can be evaluated by Lemma 2.2, this time taking $X_i=x(\lambda_i-i,0)$, $A_i=-x(-i,0)$, and $B_i=0$. $\blacksquare$

THEOREM 7.4. *Let $\lambda\in D_r$ and $\lambda\ge 0$. The trace generating function for (α,β)-reverse plane partitions of shape λ in which the first part in row i is at least $b+\beta i$ is*

$$\prod_{i=1}^{r}\left(x(-i,\lambda_i-i)^{b+\beta i}\prod_{\ell=-i+1}^{\lambda_i-i}x(\ell,\lambda_i-i)^{\alpha}\right)\frac{\prod\limits_{1\le i<j\le r}(1-x(\lambda_j-j,\lambda_i-i))}{\prod\limits_{i=1}^{r}\prod\limits_{\ell=-r}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}.$$

PROOF. Here, in the second expression of Theorem 7.2 set $\mu=0$ and $b_i=b+\beta i$. Then for the desired generating function we get the expression

$$\det_{1\le s,t\le r}\left(x(-t,\lambda_s-s)^{b+\beta t}x(\lambda_t-t,\lambda_s-s)^{t(1-\alpha-\beta)}\prod_{\ell=-t+1}^{\lambda_s-s}x(\ell,\lambda_s-s)^{\alpha}\times\right.$$
$$\left.\times\prod_{\ell=-t}^{\lambda_s-s-1}(1-x(\ell,\lambda_s-s)^{-1}\right).$$

After taking some factors out of the determinant we obtain

$$\prod_{i=1}^{r}\left(x(-i,\lambda_i-i)^{b+\beta i}x(0,\lambda_i-i)^{r-i}\prod_{\ell=-i+1}^{\lambda_i-i}x(\ell,\lambda_i-i)^{\alpha}\prod_{\ell=-r}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))^{-1}\right)\times$$
$$\times\det(x(\lambda_s-s,0)-x(-r,0))\cdots(x(\lambda_s-s,0)-x(-t-1,0))).$$

The determinant is evaluated using Lemma 2.2.1 with $X_i=x(\lambda_i-i,0)$ and $A_i=-x(-i,0)$. ∎

For $\beta=b=1$ Theorem 7.3 contains Gansner's result [4,Theorem 5.1], it is also a special case of Theorem 7.4 ($\beta=1$, $b=0$).

Determinant evaluations for trace generating functions for (α,β)-plane partitions of shape λ become very intricate. This is caused by the term $\prod_{\ell=-t+1}^{\lambda_s-s}(1-x(-t,\ell))^{-1}$ occuring in the second expression of Theorem 7.1 for $\mu=0$. Only for rectangular shapes we succeed in finding a closed form by applying one of our determinant lemmas, thus generalizing Gansner's [5,Theorem 4.2].

THEOREM 7.5. *The trace generating function for (α,β)-plane partitions consisting of r rows and c columns in which the last part in row i is at least $b-\beta i$ is*

$$\prod_{i=1}^{r}\left((x(-i,c-i)^{b-\beta i}\prod_{\ell=-i}^{c-i-1}x(-i,\ell)^{\alpha}\right)\prod_{\substack{1\le i\le r\\1\le j\le c}}(1-x(-i,j-1))^{-1}.$$

PROOF. Setting $\lambda=(c^r)$, $\mu=0$, and $b_i=b-\beta i$ in Theorem 7.1 we obtain for the desired generating function the expression

$$\det_{1\le s,t\le r}\left(x(-t,c-s)^{b-\beta s}x(-t,-s)^{s(\alpha+\beta-1)}\prod_{\ell=-t}^{c-s-1}x(-t,\ell)^{\alpha}\prod_{\ell=-t+1}^{c-s}(1-x(-t,\ell)^{-1}\right).$$

Taking some factors out of the determinant gives

$$\prod_{i=1}^{r}\left(x(-i,c-i)^{b-\beta i}x(-i,0)^{i-1}\prod_{\ell=-i}^{c-i-1}x(0,-\ell)^{\alpha}\prod_{\ell=-i+1}^{c-1}(1-x(-i,\ell))^{-1}\right)\times$$
$$\times\det\Big((x(0,-t)-x(0,c-1))\cdots(x(0,-t)-x(0,c-s+1))\Big).$$

The computation is completed by applying Lemma 2.2.2 with $X_i=x(0,-i)$ and $B_i=-x(0,c-i+1)$. ∎

A little bit more is possible if we concentrate on the norm and 0-trace only. In (1.5) set $x_i=x$ for all integers i different from zero and $x_0=xy$. Then in the generating function (1.5) the coefficient of $x^n y^t$ counts plane partitions with norm n and 0-trace t.

THEOREM 7.6. *Let $\lambda\in D_r$, $\lambda\ge 0$, with $\lambda_i=k$ for $i>k$. The norm and 0-trace generating function for (α,β)-plane partitions of shape λ in which the last part in row i is at least $b-\alpha\lambda_i+(1-\beta)i$ is*

$$y^{k(b+r)-(\alpha+\beta)\binom{k+1}{2}}x^{N_4}\prod_{i=1}^{k}\frac{(y;x)_i}{(y;x)_{\lambda_i-i+r+1}}\;\frac{\prod_{1\le i<j\le r}[\lambda_i-i-\lambda_j+j]}{\prod_{i=1}^{r}[i-1]!},$$

with the N_4 of Theorem 6.5.

PROOF. In the second expression of Theorem 7.1 we set $x_i=x$ for $i\neq 0$, $x_0=xy$, $\mu=0$, and $b_i=b-\alpha\lambda_i+(1-\beta)i$. After some simplification we obtain for the desired generating function the expression

$$y^{bk+(1-\alpha-\beta)\binom{k+1}{2}}x^{N_5}\prod_{i=1}^{r}[i-1]!^{-1}\det_{1\le s,t\le r}\left(1/(y^{\chi(\lambda_s\ge s)}x^t;x)_{\lambda_s-s+1}\right), \tag{7.6.1}$$

where N_5 comes from the proof of Theorem 6.5 and $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise. To evaluate the determinant we first reduce it by column operations. For $t=r,r-1,\dots,2$ subtract the $(t-1)$'th column from the t'th. The effect is that the $(k+1)$'th row becomes $(1,0,\dots,0)$ (since $\lambda_{k+1}=k$). After expanding the determinant according to the $(k+1)$'th row and taking some factors out of the determinant we get

$$\det_{1\le s,t\le r}\begin{pmatrix}\dfrac{1}{(yx^t;x)_{\lambda_s-s+1}} & s\le k\\[2ex] \dfrac{1}{(x^t;x)_{k-s+1}} & s>k\end{pmatrix}=$$

$$=y^k x^{\binom{r}{2}}\prod_{i=1}^{k}[\lambda_i-i+1]\prod_{i=k+2}^{r}(x^{k-i+1}-1)\det_{1\le s,t\le r-1}\begin{pmatrix}\dfrac{1}{(yx^t;x)_{\lambda_s-s+2}} & s\le k\\[2ex] \dfrac{1}{(x^t;x)_{k-s+1}} & s>k\end{pmatrix}.$$

This reduction is continued. After $(r-k)$ steps we arrive at

$$\det = y^{k(r-k)}x^{\sum_{i=k+1}^{r}\binom{i}{2}}\prod_{j=1}^{r-k}\left(\prod_{i=1}^{k}[\lambda_i-i+j]\prod_{i=k+2}^{r-j+1}(x^{k-i+1}-1)\right)\times$$

$$\times\det_{1\le s,t\le k}\left(1/(yx^t;x)_{\lambda_s-s+r-k+1}\right)$$

$$= y^{k(r-k)}x^{N_{12}}\prod_{i=1}^{k}\frac{[\lambda_i-i+r-k]!}{[\lambda_i-i]!}\prod_{i=1}^{r-k-1}[i]!\prod_{i=1}^{k}\frac{(y;x)_i}{(y;x)_{\lambda_i-i+r+1}}\times$$
$$\times\det_{1\le s,t\le k}\left((x^{-k}-yx^{\lambda_s-s+r-k})\cdots(x^{-t-1}-yx^{\lambda_s-s+r-k})\right),$$

where $N_{12}=\sum_{i=k+1}^{r}\binom{i}{2}-\sum_{i=1}^{r-k}\binom{i}{2}+\sum_{i=1}^{k}i(i-1)$. The determinant in the last line is computed by Lemma 2.2.1 with $X_i=-yx^{\lambda_i-i+r-k}$ and $A_i=x^{-i}$. Some simplification finally yields the claimed expression. ∎

The companion of Theorem 7.6 is

THEOREM 7.7. *Let $\lambda\in D_r$, $\lambda\ge0$, with $\lambda_i=k$ for $i>k$. The norm and 0-trace generating function for (α,β)-plane partitions of shape λ in which the last part in row i is at least $b+(1-\alpha)\lambda_i-\beta i$ is*

$$y^{bk-(\alpha+\beta)\binom{k+1}{2}+\sum_{i=1}^{k}\lambda_i}x^{N_6}\prod_{i=1}^{k}\frac{(y;x)_i}{(y;x)_{\lambda_i-i+r+1}}\frac{\prod_{1\le i<j\le r}[\lambda_i-i-\lambda_j+j]}{\prod_{i=1}^{r}[i-1]!},$$

with the N_6 of Theorem 6.6.

PROOF. Setting $x_i=x$ for $i\ne0$, $x_0=xy$, $\mu=0$ and $b_i=b+(1-\alpha)\lambda_i-\beta i$ in the second expression of Theorem 7.1 after some manipulation yields the expression

$$y^{k(b-1)+(1-\alpha-\beta)\binom{k+1}{2}}x^{N_7-\binom{r+1}{2}-\sum_{i=1}^{r}\binom{\lambda_i-i+1}{2}}(-1)^{\sum_{i=1}^{r}(\lambda_i-i+1)}\prod_{i=1}^{r}[i-1]!^{-1}\times$$
$$\times\det_{1\le s,t\le r}\left(1/(y^{-\chi(\lambda_s\ge s)}x^{-t};x^{-1})_{\lambda_s-s+1}\right),$$

where N_7 appears in the proof of Theorem 6.6. The determinant is the same as that in (7.6.1), but with x and y being replaced by their reciprocals. It has already been evaluated in the proof of Theorem 7.6. ∎

There is not much hope to find a simple closed form for the norm and 0-trace generating function for the classes of (α,β)-plane partitions considered in Theoremm 7.6 or 7.7, respectively, when the shape λ is arbitrary. As the example $r=3$, $\lambda=(\lambda_1,\lambda_2,1)$, $\lambda_2\ge2$, shows, it is impossible to find a product similar to those in Theorem 7.6 or 7.7, respectively, for the generating functions. But, at least, the $r\times r$-determinant obtained by Theorem 7.1 can be reduced to a $k\times k$-determinant.

THEOREM 7.8. *Let $\lambda\in D_r$, $\lambda\ge0$, with $\lambda_i<i$ for $i>k$. The norm and 0-trace* *generating function for (α,β)-plane partitions of shape λ in which the last part in row i is at least $b-\alpha\lambda_i+(1-\beta)i$ is*

$$y^{N_{13}}x^{N_{14}}(-1)^{\binom{k}{2}}\frac{\prod_{i=k+1}^{r}[i-\lambda_i-1]!}{\prod_{i=1}^{r}[i-1]!\prod_{i=1}^{k}(yx;x)_{\lambda_i-i+1}}\det_{1\le s,t\le k}\left(\frac{(x^{\lambda_s-s+1};x)_{g(t)}}{(yx^{\lambda_s-s+2};x)_{g(t)}}\right),$$

where $N_{13}=k(b+k)-(\alpha+\beta)\binom{k+1}{2}+\sum_{i=k+1}^{r}\lambda_i$, $N_{14}=\sum_{i=1}^{r}\lambda_i(-\alpha\lambda_i/2+(1-\beta)i+b-\alpha/2)+\binom{r+1}{3}-\sum_{i=k+1}^{r}\binom{i-\lambda_i}{2}$, and $g(t)=t-1+\sum_{i=k+1}^{r}\chi(t\ge k-\lambda_i+1)$.

PROOF. The second expression of Theorem 7.1 with $x_i=x$ if $i\neq 0$, $x_0=xy$, $\mu=0$, and $b_i=b-\alpha\lambda_i+(1-\beta)i$ implies that the desired generating function is equal to

$$y^{bk+(1-\alpha-\beta)\binom{k+1}{2}}x^{N_5}\prod_{i=1}^{r}[i-1]!^{-1}\det\left(1/(y^{\chi(\lambda_s\ge s)}x^t;x)_{\lambda_s-s+1}\right),$$

where N_5 appears in the proof of Theorem 6.5. For $t=r,r-1,\dots,2$ we subtract the $(t-1)$'th column from the t'th in the determinant. We get

$$\det_{1\le s,t\le r}\left(1/(y^{\chi(\lambda_s\ge s)}x^t;x)_{\lambda_s-s+1}\right)=\det_{1\le s,t\le r}\left(\frac{y^{\chi(\lambda_s\ge s)}(x^{\lambda_s-s+t}-x^{t-1})}{(y^{\chi(\lambda_s\ge s)}x^{t-1};x)_{\lambda_s-s+2}}\right)$$

$$=x^{\binom{r}{2}}\det_{1\le s,t\le r}\left(\frac{y^{\chi(\lambda_s\ge s)}(x^{\lambda_s-s+1}-1)}{(y^{\chi(\lambda_s\ge s)}x^{t-1};x)_{\lambda_s-s+2}}\right).$$

For $t=r,r-1,\dots,3$, etc., we iterate this procedure. This furnishes

$$\det_{1\le s,t\le r}\left(1/(y^{\chi(\lambda_s\ge s)}x^t;x)_{\lambda_s-s+1}\right)=$$

$$=x^{\binom{r+1}{3}}\det_{1\le s,t\le r}\left(\frac{(-1)^{t-1}y^{\chi(\lambda_s\ge s)(t-1)}(x^{\lambda_s-s+1};x)_{t-1}}{(y^{\chi(\lambda_s\ge s)}x;x)_{\lambda_s-s+t}}\right).$$

Let $s>k$. Then, by the assumptions of the Theorem, we have $\lambda_s<s$. It is easy to see that because of this all entries in the s'th row are zero except the $(s-\lambda_s)$'th, which is equal to $x^{-\binom{s-\lambda_s}{2}}[s-\lambda_s-1]!$. Next the determinant is expanded according to the rows $k+1,k+2,\dots,r$. Thus, a $k\times k$-determinant times some polynomial in x is obtained. The final step is to take some factors out of the determinant. ∎

The companion of Theorem 7.8 is derived similarly as Theorem 7.7 was derived from the proof of Theorem 7.6.

THEOREM 7.9. *Let* $\lambda\in D_r$, $\lambda\geq0$, *with* $\lambda_i<i$ *for* $i>k$. *The norm and 0-trace generating function for* (α,β)*-plane partitions of shape* λ *in which the last part in row* i *is at least* $b+(1-\alpha)\lambda_i-\beta i$ *is*

$$y^{N_{15}}x^{N_{16}}(-1)^{\binom{k}{2}}\frac{\prod_{i=k+1}^{r}[i-\lambda_i-1]!}{\prod_{i=1}^{r}[i-1]!\prod_{i=1}^{k}(yx;x)_{\lambda_i-i+1}}\det_{1\leq s,t\leq k}\left(\frac{(x^{\lambda_s-s+1};x)_{g(t)}}{(yx^{\lambda_s-s+2};x)_{g(t)}}\right),$$

where $N_{15}=bk-(\alpha+\beta)\binom{k+1}{2}+\sum_{i=1}^{k}\lambda_i$ *and* $N_{16}=\sum_{i=1}^{r}\lambda_i((1-\alpha)\lambda_i/2-\beta i+b-\alpha/2-1/2)+$ $+\binom{k}{2}+\sum_{i=1}^{k}\binom{\lambda_i-i+2}{2}+\sum_{i=k+1}^{r}\lambda_i$. $g(t)$ *was defined in the previous Theorem.* ∎

In [22] Stanley introduced the notion of "conjugate trace" of a plane partition. The *conjugate trace* $t^*(\pi)$ of a plane partition π is the number of parts π_{ij} of π with $\pi_{ij}\geq i$. An immediate check shows that the involution $\phi_{0,0}$ (introduced in Definition 4.1) is conjugate trace-preserving. Intending to enumerate plane partitions by their norm and conjugate trace, for a plane partition π we define

$$w_{t^*}(\pi) = x^{n(\pi)}y^{t^*(\pi)} .$$

Since for the w_{t^*}-generating function for r-rowed 0-tableaux we have

$$F(T_0(\infty/0;a,0);w_{t^*}) = \prod_{i=1}^{r}[i-1]!^{-1}\left((y^{\chi(a_i\geq i)}x^{i};x)_{a_i-i+1}\right)^{-1},$$

it is no difficulty to obtain a determinantal expression for the norm and conjugate trace generating function for r-rowed plane partitions with largest part $\leq a$. (Here a is an r-tupel of integers, the shape $\infty/0$ consists of r rows unrestricted to the right-hand side.) The arguments in the proof of Theorem 6.1 only have to be modified. The resulting determinant then is computed as usually, thus establishing an alternative proof of Stanley's result [22,Theorem 2.2].

THEOREM 7.10. *The norm and conjugate trace generating function for* r*-rowed plane partitions with largest part* $\leq a$ *is*

$$1/\prod_{i=1}^{a}(yx^{i};x)_r .$$

PROOF. Regarding Theorem 4.2.(C) for $\alpha=\beta=0$ and the considerations above, it is seen that $\phi_{0,0}$ is weight-preserving with respect to the weight

$$w'_{t^*}(\pi) = x^{n(\pi)+\binom{s-t}{2}}y^{t^*(\pi)} .$$

This enables us to apply Lemma 2.1 with $M_\sigma=T_0(\infty/e_\sigma;(a,\dots,a)+e_\sigma,0)$, etc., $\phi=\phi_{0,0}$. We obtain

$$\det_{1\le s,t\le r}\left(x^{\binom{s-t}{2}}\Big/[s-1]!\left((y^{\chi(a\ge t)}x^{s};x)_{a-t+1}\right)^{-1}\right)$$

for the desired generating function. This determinant has already been evaluated in the proof of Theorem 7.7 (set $\lambda_i=a$). ▮

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