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Heine transformations for a new kind of basic hypergeometric series in $U(n)$

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Abstract

Heine transformations are proved for a new kind of multivariate basic hypergeometric series which had been previously introduced by Krattenthaler in connection with generating functions for nonintersecting lattice paths. As a consequence, a q-Gauss and q-Chu–Vandermonde sum are proved and also a generalization of Ramanujan's ${}_1\psi_1$ sum.

Keywords: Basic hypergeometric series in $U(n)$; Heine transformation; Bilateral basic hypergeometric series; Good's identity

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1. Introduction and statement of results

The classical basic hypergeometric series (with notation as in [2]) is defined by

$$ {}_2\varphi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(q;q)_n (c;q)_n}, $$

where

$$ (a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1,2,\ldots, \end{cases} $$

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is the *q-shifted factorial* and it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \ldots$, and the series converges absolutely if $|q| < 1$ and $|z| < 1$. We will also use the notation

$$(a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n \equiv (a_1, \ldots, a_k; q)_n \equiv (a_1, \ldots, a_k)_n,$$

where we assume that the base q is fixed throughout.

The study of the properties of such a ${}_2\varphi_1$ series was initiated by Heine [4, 5] who proved the following transformation formulas:

$$ {}_2\varphi_1(a, b; c; q, z) = \frac{(a, bz)_\infty}{(c, z)_\infty}\, {}_2\varphi_1(c/a, z; bz; q, a) \tag{1.1}$$

$$ = \frac{(c/b, bz)_\infty}{(c, z)_\infty}\, {}_2\varphi_1(abz/c, b; az; q, c/b) \tag{1.2}$$

$$ = \frac{(abz/c)_\infty}{(z)_\infty}\, {}_2\varphi_1\left(c/a, c/b; c; q, \frac{abz}{c}\right). \tag{1.3}$$

These transformations can be iterated and it was Rogers [10] who observed how to simply describe the symmetry of the ${}_2\varphi_1$ function under the symmetry group generated by these transformations. A description of Roger's result and how it became a starting point in Roger's further investigation on q-Hermite polynomials and partition identities is given in the second chapter of [1].

We will prove multivariate generalizations of the three transformations (1.1)–(1.3) involving the following extension of the classical ${}_2\varphi_1$. For a positive integer r and $A, B, C, Z, X_1, \ldots, X_r \in \mathbb{C}$, and for convergence assume $|Z| < |q|^{r-1} < 1$, define

$$\begin{aligned} &{}_2\varphi_1^{(r)}(X_1, \ldots, X_r; A, B; C; q, Z) \\ &= \sum_{k_1, \ldots, k_r \geqslant 0} \left\{ \prod_{i=1}^{r} \left(q^{k_i(1-i)} Z^{k_i} \frac{(A)_{k_i} (BX_i)_{k_i}}{(q)_{k_i} (CX_i)_{k_i}} \right) \times \prod_{1 \leqslant i < j \leqslant r} \frac{1 - q^{k_j - k_i} X_j / X_i}{1 - X_j / X_i} \right\} \\ &= \sum_{k_1, \ldots, k_r \geqslant 0} \prod_{1 \leqslant i < j \leqslant r} \frac{(X_i^{-1} q^{-k_i} - X_j^{-1} q^{-k_j})}{(X_i^{-1} - X_j^{-1})} \times \prod_{i=1}^{r} Z^{k_i} \frac{(A)_{k_i} (BX_i)_{k_i}}{(q)_{k_i} (CX_i)_{k_i}}. \end{aligned} \tag{1.4}$$

Remark 1. The multivariate hypergeometric series (1.4) first occurred in connection with certain generating functions for nonintersecting lattice paths [8]. The ${}_2\varphi_1^{(r)}$ series is a new kind of series associated to the group $U(r)$ (or root system A_{r-1}). It is closely related to Milne's basic hypergeometric series in $U(n)$ [9, Definition 1.39], which is in turn a q-analog of the ordinary hypergeometric series in $U(n)$ introduced by Holman [7]. The main difference between the ${}_2\varphi_1^{(r)}$ series and one of Milne's series $[F]^{(r)}$ are the factors $(A)_{k_i}/(q)_{k_i}$ appearing in (1.4).

Our generalizations of (1.1)–(1.3) read as follows.

Theorem 2. *With notation as above and* $|Z| < |q|^{r-1}$ *and* $|q| < 1$,

$$
\begin{aligned}
&{}_2\varphi_1^r(X_1, \dots, X_r; A, B; C; q, Z)\\
&= \prod_{i=1}^{r} \frac{(Aq^{i-r})_\infty (BZX_i)_\infty}{(Zq^{i-r})_\infty (CX_i)_\infty} \, {}_2\varphi_1^{(r)}(X_1, \dots, X_r; Z, C/A; BZ; q, A) &(1.5)\\
&= \prod_{i=1}^{r} \frac{(Cq^{i-r}/B)_\infty (BZX_i)_\infty}{(Zq^{i-r})_\infty (CX_i)_\infty} \, {}_2\varphi_1^{(r)}(X_1, \dots, X_r; ABZ/C, B; BZ; q, C/B) &(1.6)\\
&= \prod_{i=1}^{r} \frac{(ABZq^{i-r}/C)_\infty}{(Zq^{i-r})_\infty} \, {}_2\varphi_1^{(r)}(X_1, \dots, X_r; C/B, C/A; C; q, ABZ/C). &(1.7)
\end{aligned}
$$

Theorem will be proved in Section 2.
By specializing $Z = C/AB$ in (1.6), one obtains a generalization of the q-Gauss sum.

Corollary 3. *With notation as above and assuming convergence,*

$$
{}_2\varphi_1^{(r)}(X_1, \dots, X_r; A, B; C; q, C/AB) = \prod_{i=1}^{r} \frac{(Cq^{i-r}/B)_\infty (CX_i/A)_\infty}{(Cq^{i-r}/AB)_\infty (CX_i)_\infty}. \tag{1.8}
$$

Setting $A = q^{-n}$ for some nonnegative integer n and reversing the series on the left-hand side of (1.8), one finds a generalization of the q-Chu–Vandermonde sum:

$$
{}_2\varphi_1^{(r)}(X_1, \dots, X_r; q^{-n}, B; C; q, q) = q^{n\binom{r}{2}} \prod_{i=1}^{r} \frac{(Cq^{i-r}/B)_n (BX_i)^n}{(CX_i)_n}. \tag{1.9}
$$

Remark 4. Identity (1.8) was first discovered by counting nonintersecting lattice paths [8, identity (4.3.12)] and identity (1.6) was used in the same paper for rewriting certain generating functions for nonintersecting lattice paths.

One can also give a natural generalization of the bilateral ${}_1\psi_1$ hypergeometric series:

$$
{}_1\psi_1^{(r)}(X_1, \dots, X_r; A, B; q, Z) = \sum_{k_1, \dots, k_r = -\infty}^{\infty} \prod_{1 \leqslant i < j \leqslant r} \left(\frac{X_i^{-1} q^{-k_i} - X_j^{-1} q^{-k_j}}{(X_i^{-1} - X_j^{-1})} \right) \prod_{i=1}^{r} \frac{(A)_{k_i}}{(B)_{k_i}} Z^{k_i}, \tag{1.10}
$$

which converges when $|B/A| < |Z| < |q|^{r-1} < 1$. There is the following generalization of Ramanujan's ${}_1\psi_1$ sum (which includes the q-binomial theorem as a special case). The proof is included below.

Theorem 5. *With notation and assumptions as above,*

$$ {}_1\psi_1^{(r)}(X_1, \dots, X_r; A, B; q, z) = \prod_{i=1}^{r} \frac{(q)_\infty (B/A)_\infty (q^{1+r-i}/AZ)_\infty (AZq^{i-r})_\infty}{(B)_\infty (q/A)_\infty (q^{r-i}B/AZ)_\infty (Zq^{i-r})_\infty}. \tag{1.11} $$

Proof. Expand the sum on the left-hand side of (1.11) using the classical $r = 1$ case and the Vandermonde determinant. We find

$$ \begin{aligned} {}_1\psi_1^{(r)}(X_1, \dots, X_r; A, B; q, z) = & \prod_{1 \leqslant i < j \leqslant r} (X_r^{-1} - X_j^{-1})^{-1} \sum_{\sigma \in S_r} \varepsilon(\sigma) \\ & \times \prod_{i=1}^{r} \left\{ X_i^{\sigma(i)-r} \cdot \sum_{k_i=-\infty}^{\infty} \frac{(A)_{k_i}}{(B)_{k_i}} (Zq^{\sigma(i)-r})^{k_i} \right\} \end{aligned} \tag{1.12} $$

(where S_r is the permutation group on r letters and $\varepsilon(\sigma)$ is the sign of the permutation σ)

$$ \begin{aligned} = & \prod_{1 \leqslant i < j \leqslant r} (X_i^{-1} - X_j^{-1})^{-1} \sum_{\sigma \in S_r} \varepsilon(\sigma) \\ & \times \prod_{i=1}^{r} X_{\sigma(i)}^{i-r} \frac{(q)_\infty (B/A)_\infty (q^{1+r-i}/AZ)_\infty (AZq^{i-r})_\infty}{(B)_\infty (q/A)_\infty (q^{r-i}B/AZ)_\infty (Zq^{i-r})_\infty} \end{aligned} \tag{1.13} $$

$$ = \prod_{i=1}^{r} \frac{(q)_\infty (B/A)_\infty (q^{1+r-i}/AZ)_\infty (AZq^{i-r})_\infty}{(B)_\infty (q/A)_\infty (q^{r-i}B/AZ)_\infty (Zq^{i-r})_\infty}. \tag{1.14} $$

This completes the proof. $\square$

2. Proof of Theorem 2

We will prove identity (1.5) by induction on r. Identity (1.6) is proved by an entirely similar argument and (1.7) follows by equating the right-hand sides of (1.5) and (1.6).

The case $r = 1$ of (1.5) is just the classical result (1.1). For the general case we will use Good's identity [3, 11], [6, p. 61]:

$$ \begin{aligned} 1 &= \sum_{i=1}^{r} \prod_{\substack{k=1 \\ k \neq i}}^{r} (1 - y_i/y_k)^{-1} \\ &= \sum_{i=1}^{r} \prod_{\substack{k=1 \\ k \neq i}}^{r} \frac{y_i^{1-r}}{(y_i^{-1} - y_k^{-1})}. \end{aligned} \tag{2.1} $$

Setting $y_i = X_i q^{k_i}$, use (2.1) to expand the series on the left-hand side of (1.5),

$$ \begin{aligned} & {}_2\varphi_1^{(r)}(X_1, \dots, X_r; A, B; C; q, Z) \\ & = \sum_{i=1}^{r} \frac{X_i^{1-r}}{\prod_{k=1}^{r} (X_1^{-1} - X_k^{-1})} \, {}_2\varphi_1(A, BX_i; CX_i; q, q^{1-r}Z) \\ & \quad \times {}_2\varphi_1^{(r-1)}(X_1, \dots, \hat{X}_i, \dots, X_r; A, B; C; q, z), \end{aligned} \tag{2.2} $$

where $\hat{X}_i$ means omit X_i. Then by induction and (1.1) we have

$$= \sum_{i=1}^{r} \frac{(A)_\infty (BX_iZq^{1-r})_\infty}{(CX_i)_\infty (Zq^{1-r})_\infty} \prod_{j=1}^{r-1} \frac{(Aq^{1-j})_\infty}{(Zq^{1-j})_\infty}$$

$$\times \prod_{\substack{j=1\\ j\neq i}}^{r} \frac{(BX_jZ)_\infty}{(CX_j)_\infty} \frac{X_i^{1-r}}{\prod_{\substack{k=1\\ k\neq i}}^{r}(X_i^{-1}-X_k^{-1})}$$

$$\times {}_2\varphi_1(CX_i/A, Zq^{1-r}; BZX_iq^{1-r}; q, A)$$

$$\times {}_2\varphi_1^{(r-1)}(X_1, \ldots, \hat{X}_i, \ldots, X_r; Z, C/A; BZ; q, A) \tag{2.3}$$

$$= \prod_{j=1}^{r} \frac{(Aq^{j-r})_\infty (BZX_j)_\infty}{(Zq^{j-r})_\infty (CX_j)_\infty} \sum_{j=1}^{r} \left\{ \frac{X_i^{1-r}}{\prod_{\substack{j=1\\ j\neq i}}^{r}(X_i^{-1}-X_j^{-1})} \right.$$

$$\times \frac{(BX_iZq^{1-r})_{r-1}}{(Aq^{1-r})_{r-1}} \sum_{k=0}^{\infty} \frac{(CX_i/A)_k (Zq^{1-r})_k}{(q)_k (BZX_iq^{1-r})_k} A^k$$

$$\left. \times {}_2\varphi_1^{(r-1)}(X_1, \ldots, \hat{X}_i, \ldots, X_r; Z, C/A; BZ; q, A) \right\}. \tag{2.4}$$

Observe that

$$(Zq^{1-r})_k = \frac{(Zq^{1-r})_{r-1}}{(Zq^{k-r+1})_{r-1}} (Z)_k \tag{2.5}$$

and by the q-Chu–Vandermonde sum, we also have

$$q^{k(r-1)}(Zq^{1-r})_{r-1} = \sum_{\ell=0}^{r-1} \left\{ \frac{(Zq^{k-r+1})_\ell}{(q)_\ell} (q^{k-r+\ell+2})_{r-\ell-1} (q^{1-r})_\ell (-1)^{r-1} q^{\binom{r-1}{2}} q^\ell \right\}. \tag{2.6}$$

Substituting (2.5) and (2.6) into the right-hand side of (2.4) we find that

$${}_2\varphi_1^{(r)}(X_1, \ldots, X_r; A, B; C; q, Z)$$

$$= \prod_{i=1}^{r} \frac{(Aq^{j-r})_\infty (BZX_j)_\infty}{(Zq^{j-r})_\infty (CX_j)_\infty}$$

$$\times \sum_{j=1}^{r} \left\{ \frac{X_i^{1-r}}{\prod_{\substack{j=1\\ j\neq i}}^{r}(X_i^{-1}-X_k^{-1})} \frac{(BX_iZq^{1-r})_{r-1}}{(Aq^{1-r})_{r-1}} \right.$$

$$\times {}_2\varphi_1^{(r-1)}(X_1, \ldots, \hat{X}_i, \ldots, X_r; Z, C/A; BZ; q, A)$$
$$\times \sum_{k=0}^{\infty} \sum_{\ell=0}^{r-1} \frac{q^{-k(r-1)}}{(Zq^{k-r+1})_{r-1}} \frac{(Z)_k}{(q)_k} \frac{(Zq^{k-r+1})_\ell}{(BZX_i q^{1-r})_k}$$
$$\times \frac{(CX_i/A)_k (q^{k-r+\ell+2})_{r-\ell-1}}{(q)_\ell} (q^{1-r})_\ell (-1)^{r-1} q^{\binom{r-1}{2}} q^\ell A^k \Bigg\}. \tag{2.7}$$

Note that if $0 \leqslant k \leqslant r - \ell - 1$ then the factor $(q^{k-r+\ell+2})_{r-\ell-1}$ vanishes. Hence in the right-hand side of (2.7) we may replace $k - r + \ell + 1$ by m and sum over $m \geqslant 0$ instead of $k \geqslant 0$. Also observe that

$$\frac{(Z)_{m+r-\ell-1}(Zq^{m-\ell})_\ell}{(Zq^{m-\ell})_{r-1}} = (Z)_m \tag{2.8a}$$

and

$$\frac{(q^{m+1})_{r-\ell-1}}{(q)_{m+r-\ell-1}} = \frac{1}{(q)_m}, \tag{2.8b}$$

so we have

$$\begin{aligned}
&{}_2\varphi_1^{(r)}(X_1, \ldots, X_r; A, B; C; q, Z)\\
&\quad = \prod_{j=1}^{r} \frac{(Aq^{j-r})_\infty (BZX_j)_\infty}{(Zq^{j-r})_\infty (CX_j)_\infty}\\
&\qquad \times \sum_{i=1}^{r} \Bigg\{ \frac{X_i^{1-r}}{\prod_{j=1, j\neq i}^{r}(X_i^{-1} - X_j^{-1})} \frac{(BX_i Z q^{1-r})_{r-1}}{(Aq^{1-r})_{r-1}}\\
&\qquad\qquad \times {}_2\varphi_1^{(r-1)}(X_1, \ldots, \hat{X}_i, \ldots, X_r; Z, C/A; BZ; q, A)\\
&\qquad\qquad \times \sum_{m=0}^{\infty} \sum_{\ell=0}^{r-1} \frac{(-1)^{r-1} q^{(\ell-m)(r-1)} q^{-(r-1)^2} q^{\binom{r-1}{2}} q^\ell (CX_i q^m/A)_{r-\ell-1}}{(BZX_i q^{1-r})_{r-1} (BZX_i q^m)_{-\ell}}\\
&\qquad \times (A)^{r-\ell-1} \frac{(q^{1-r})_\ell}{(q)^\ell} \frac{(Z)_m (CX_i/A)_m}{(q)_m (BZX_i)_m} A^m \Bigg\}
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\quad = \frac{(-1)^{r-1} q^{-\binom{r}{2}}}{(Aq^{1-r})_r} \prod_{1 \leqslant i < j \leqslant r} (X_i^{-1} - X_j^{-1})^{-1}\\
&\qquad \times \prod_{j=1}^{r} \frac{(Aq^{j-r})_\infty (BZX_j)_\infty}{(Zq^{j-r})_\infty (CX_j)_\infty}\\
&\qquad \times \sum_{m_1, \ldots, m_r \geqslant 0} \sum_{\ell=0}^{r} \Bigg\{ \frac{A^{r-\ell-1} q^{\ell r} (q^{1-r})_\ell}{(q)_\ell}
\end{aligned}$$

$$\times \prod_{j=1}^{r} \frac{(Z)_{m_j}(CX_j/A)_{m_j}A^{m_j}}{(q)_{m_j}(BZX_j)_{m_j}}$$

$$\times \sum_{\sigma\in S_r} \varepsilon(\sigma)\Bigg[(BZX_{\sigma(1)}q^{m_{\sigma(1)}-\ell})_\ell (CX_{\sigma(1)}q^{m_{\sigma(1)}}/A)_{r-\ell-1}$$

$$\times \prod_{i=1}^{r} (X_{\sigma(i)}^{-1}q^{-m_{\sigma(i)}})^{r-i}\Bigg]\Bigg\}. \tag{2.10}$$

where the Vandermonde determinant is used, S_r is the symmetric group on r letters, and $\varepsilon(\sigma)$ is the sign of the permutation $\sigma \in S_r$.

We expand the product

$$(BZX_{\sigma(1)}q^{m_{\sigma(1)}-\ell})_\ell (CX_{\sigma(1)}q^{m_{\sigma(1)}}/A)_{r-\ell-1} = \sum_{k=0}^{r-1} d_k (X_{\sigma(1)}q^{m_{\sigma(1)}})^k, \tag{2.11}$$

where $d_0 = 1$ and d_k is independent of $X_{\sigma(1)}$ and $q^{m_{\sigma(1)}}$ for $0 \leqslant k \leqslant r-1$. It follows that

$$\sum_{\sigma\in S_r} \varepsilon(\sigma)\Bigg[(BZX_{\sigma(1)}q^{m_{\sigma(1)}-\ell})_\ell (CX_{\sigma(1)}q^{m_{\sigma(1)}}/A)_{r-\ell-1}$$

$$\times \prod_{i=1}^{r} (X_{\sigma(i)}^{-1}q^{-m_{\sigma(i)}})^{r-i}\Bigg]$$

$$= \sum_{k=0}^{r-1} d_k \sum_{\sigma\in S_r} \varepsilon(\sigma)(X_{\sigma(1)}q^{m_{\sigma(1)}})^{k-r+1}$$

$$\times \sum_{j=2}^{r} (X_{\sigma(j)}q^{m_{\sigma(j)}})^{j-r}$$

$$= \sum_{\sigma\in S_r} \varepsilon(\sigma) \prod_{i=1}^{r} (X_{\sigma(i)}q^{m_{\sigma(i)}})^{i-r}, \tag{2.12}$$

since the only nonvanishing term in the sum over k is the $k = 0$ term. It follows that

$${}_2\varphi_1^{(r)}(X_1, \dots, X_r; A, B; C; q, Z) = \frac{(-1)^{r-1}q^{-\binom{r}{2}}}{(Aq^{1-r})_r} \prod_{j=1}^{r} \frac{(Aq^{j-r})_\infty (BZX_j)_\infty}{(Zq^{j-r})_\infty (CX_j)_\infty}$$

$$\times A^{r-1}\,{}_2\varphi_1^{(r)}(X_1, \dots, X_r; Z, C/A; BZ; q, A)$$

$$\times \sum_{\ell=0}^{r} \frac{(q^{1-r})_\ell}{(q)_\ell}\left(\frac{q^r}{A}\right)^\ell. \tag{2.13}$$

The proof of Theorem 2 is completed by applying the q-binomial theorem to the sum over ℓ in (2.13) and simplifying.

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