An advanced application of HYP

Dick Askey, in his talk on the last day of the conference, asked for a hypergeometric proof of the following identity due to Gelfand, Graev and Retakh:

$$\sum_{j,k,m\geq 0} \frac{(\alpha)_j\,(\beta)_k\,(1-\gamma)_m\,(\gamma)_{j+k-m}}{j!\,k!\,m!\,(\alpha+\beta)_{j+k-m}} x^j y^k z^m$$
$$= (1-x)^{\beta-\gamma}(1-y)^{-\beta}(1-z)^{\alpha+\beta-1}(1-xz)^{\gamma-\alpha-\beta}{}_2F_1\left[\begin{matrix}\beta, \alpha+\beta-\gamma\\ \alpha+\beta\end{matrix}; \frac{(x-y)(1-z)}{(1-y)(1-xz)}\right].$$

Such a proof shall be given now.

First I load my package.

```
In[1]:= << hyp.m
```

The way I proceed is that I compare coefficients of x^X*y^Y*z^Z on both sides of the identity.

Then, on the left-hand side, I get just a single term, whereas on the right-hand side I will be faced with a three-fold sum. The summand in the three-fold sum will be the following:

```
In[2]:= (-1)^(i + j + k + l + m)Binomial[be - ga, i]
          Binomial[-be - n, j]
          Binomial[al + be - 1 + n, k]*
          Binomial[ga - al - be - n, l]p[be, n]
          p[al + be - ga, n]/p[al + be, n]/n!
          Binomial[n, m]
```

Out[2]=

$$\frac{1}{n!\,((al+be)_n)}\left((-1)^{i+j+k+l+m}\binom{be-ga}{i}\binom{-be-n}{j}\binom{-al-be+ga-n}{l}\binom{n}{m}\binom{-1+al+be+n}{k}((be)_n)\,((al+be-ga)_n)\right)$$

provided that

```
In[3]:= Solve[{i + l + n - m == X, j + m == Y, k + l == Z},
          {l, m, n}]
Out[3]= {{l → -k + Z, m → -j + Y,
          n → -i - j + k + X + Y - Z}}
```

That is, the sumand is

```
In[4]:= %2/.%[[1]]
```

Out[4]=

$$\left((-1)^{i+Y+Z}\binom{be-ga}{i}\binom{-i-j+k+X+Y-Z}{-j+Y}\binom{-1+al+be-i-j+k+X+Y-Z}{k}\binom{-be+i+j-k-X-Y+Z}{j}\binom{-al-be+ga+i+j-k-X-Y+Z}{-k+Z}((be)_{-i-j+k+X+Y-Z})((al+be-ga)_{-i-j+k+X+Y-Z})\right)\Big/\left((-i-j+k+X+Y-Z)!\,((al+be)_{-i-j+k+X+Y-Z})\right)$$

and it has to be summed over i,j,k.

We start with the sum over j:

```
In[5]:= SUM[%, {j, 0, ∞}]
```

Out[5]=

$$\sum_{j=0}^{\infty} \left((-1)^{i+Y+Z} \left(\binom{be-ga}{i} \right) \left(\binom{-i-j+k+X+Y-Z}{-j+Y} \right) \left(\binom{-1+al+be-i-j+k+X+Y-Z}{k} \right) \left(\binom{-be+i+j-k-X-Y+Z}{j} \right) \left(\binom{-al-be+ga+i+j-k-X-Y+Z}{-k+Z} \right) ((be)_{-i-j+k+X+Y-Z}) ((al+be-ga)_{-i-j+k+X+Y-Z}) \right) \Big/ \left((-i-j+k+X+Y-Z)! \, ((al+be)_{-i-j+k+X+Y-Z}) \right)$$

In hypergeometric notation this is

In[6]:= **%/.SUMF**

`Is  i + Y + Z  even, odd, or neither of both?`

Out[6]=

$$\Bigg((-1)^{i+Y+Z}\left({}_2F_1\left[\begin{matrix}-Y,\ 1-al-be+i-X-Y+Z\\ 1-al-be+ga+i-X-Y\end{matrix};\ 1\right]\right)((be)_{-i+k+X+Y-Z})\,((al+be-ga)_{-i+k+X+Y-Z})\,((1+be-ga-i)_{i})\,((1-al-be+ga+i-X-Y)_{-k+Z})\,((1-i+k+X-Z)_{Y})\,((al+be-i+X+Y-Z)_{k})\Bigg)\Big/\,(((1)_{i})\,((1)_{k})\,((1)_{Y})\,((1)_{-i+k+X+Y-Z})\,((1)_{-k+Z})\,((al+be)_{-i+k+X+Y-Z}))$$

Clearly, we can sum this 2F1 by means of Chu-Vandermonde summation.

In[7]:= **%/.S2101**

Out[7]=

$$((-1)^{i+Y+Z}\,((be)_{-i+k+X+Y-Z})\,((al+be-ga)_{-i+k+X+Y-Z})\,((1+be-ga-i)_{i})\,((1-al-be+ga+i-X-Y)_{-k+Z})\,((ga-Z)_{Y})\,((1-i+k+X-Z)_{Y})\,((al+be-i+X+Y-Z)_{k}))\Big/\,(((1)_{i})\,((1)_{k})\,((1)_{Y})\,((1)_{-i+k+X+Y-Z})\,((1)_{-k+Z})\,((al+be)_{-i+k+X+Y-Z})\,((1-al-be+ga+i-X-Y)_{Y}))$$

Now we form the sum over k.

In[8]:= **SUM[%, {k, 0, ∞}]**

Out[8]=

$$\sum_{k=0}^{\infty}((-1)^{i+Y+Z}\,((be)_{-i+k+X+Y-Z})\,((al+be-ga)_{-i+k+X+Y-Z})\,((1+be-ga-i)_{i})\,((1-al-be+ga+i-X-Y)_{-k+Z})\,((ga-Z)_{Y})\,((1-i+k+X-Z)_{Y})\,((al+be-i+X+Y-Z)_{k}))\Big/\,(((1)_{i})\,((1)_{k})\,((1)_{Y})\,((1)_{-i+k+X+Y-Z})\,((1)_{-k+Z})\,((al+be)_{-i+k+X+Y-Z})\,((1-al-be+ga+i-X-Y)_{Y}))$$

In hypergeometric notation this is

```
In[9]:= %/.SUMF

Is  i + Y + Z  even, odd, or neither of both?
                            [                        ]
              /           / |  -Z, be - i + X + Y - Z  |  \
Out[9]=  | (-1)^(i+Y+Z)  | 2F1|                    ; 1 |   |
              \           \ |      1 - i + X - Z       |  /
                            [                        ]

        ((be)_(-i+X+Y-Z)) ((al + be - ga)_(-i+X+Y-Z))
        ((1 + be - ga - i)_i)
        ((1 - al - be + ga + i - X - Y)_Z)

        ((ga - Z)_Y) ((1 - i + X - Z)_Y) ) /

     (((1)_i) ((1)_Y) ((1)_(-i+X+Y-Z))
        ((1)_Z) ((al + be)_(-i+X+Y-Z))
        ((1 - al - be + ga + i - X - Y)_Y))
```

which can again be summed by means of the Chu-Vandermonde summation.

```
In[10]:= %/.S2101
Out[10]= ((-1)^(i+Y+Z) ((be)_(-i+X+Y-Z))
           ((al + be - ga)_(-i+X+Y-Z)) ((1 + be - ga - i)_i)
           ((1 - be - Y)_Z) ((1 - al - be + ga + i - X - Y)_Z)
           ((ga - Z)_Y) ((1 - i + X - Z)_Y)) /
         (((1)_i) ((1)_Y) ((1)_(-i+X+Y-Z))
           ((1)_Z) ((al + be)_(-i+X+Y-Z))
           ((1 - al - be + ga + i - X - Y)_Y)
           ((1 - i + X - Z)_Z))
```

Finally, we sum our intermediate result over i:

```
In[11]:= SUM[%, {i, 0, ∞}]
            ∞
          -----
           \
Out[11]=    >      ((-1)^(i+Y+Z)
           /
          -----
           i = 0

              ((be)_(-i+X+Y-Z)) ((al + be - ga)_(-i+X+Y-Z))
              ((1 + be - ga - i)_i) ((1 - be - Y)_Z)
              ((1 - al - be + ga + i - X - Y)_Z)
              ((ga - Z)_Y) ((1 - i + X - Z)_Y)) /
           (((1)_i) ((1)_Y) ((1)_(-i+X+Y-Z))
              ((1)_Z) ((al + be)_(-i+X+Y-Z))
              ((1 - al - be + ga + i - X - Y)_Y)
              ((1 - i + X - Z)_Z))
```

and convert it to hypergeometric notation.

```
In[12]:= %/.SUMF
Is  Y + Z  even, odd, or neither of both?
Out[12]= ((-1)^(Y+Z)
            (3F2[1 - al - be - X - Y + Z, -be + ga, -X;
                 1 - be - X - Y + Z, 1 - al - be + ga - X; 1] ((be)_(X+Y-Z)) ((al + be - ga)_(X+Y-Z))
            ((1 - be - Y)_Z) ((1 - al - be + ga - X - Y)_Z)
            ((ga - Z)_Y) ((1 + X - Z)_Y)) /
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z) ((al + be)_(X+Y-Z))
            ((1 - al - be + ga - X - Y)_Y) ((1 + X - Z)_Z))
```

This time, the Pfaff-Saalschütz summation can be applied:

```
In[13]:= %/.S3201
Out[13]= ((-1)^(Y+Z) ((al)_X) ((be)_(X+Y-Z))
            ((al + be - ga)_(X+Y-Z)) ((1 - be - Y)_Z)
            ((1 - al - be + ga - X - Y)_Z) ((ga - Z)_Y)
            ((1 + X - Z)_Y) ((1 - ga - X - Y + Z)_X))/
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z) ((al + be)_(X+Y-Z))
            ((al + be - ga)_X) ((1 - al - be + ga - X - Y)_Y)
            ((1 + X - Z)_Z) ((1 - be - X - Y + Z)_X))
```

The only remaining task is to convert this expression to the coefficient of xˆX*yˆY*zˆZ in the left-hand side sum:

```
In[14]:= %//.zus1
Out[14]= ((-1)^(Y+Z) ((al)_X) ((be)_(X+Y-Z))
            ((al + be - ga)_(X+Y-Z)) ((1 - be - Y)_Z)
            ((1 - al - be + ga - X - Y)_Z) ((ga - Z)_Y)
            ((1 + X - Z)_Y) ((1 - ga - X - Y + Z)_X))/
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z) ((al + be)_(X+Y-Z))
            ((al + be - ga)_X) ((1 - al - be + ga - X - Y)_Y)
            ((1 + X - Z)_Z) ((1 - be - X - Y + Z)_X))
In[15]:= %//.zus2
```

```
Out[15]= ((-1)^(Y+Z) ((al)_X) ((be)_(X+Y-Z))
            ((1 - al - be + ga - X)_(-Y+Z))
            ((al + be - ga + X)_(Y-Z)) ((1 - be - Y)_Z)
            ((ga - Z)_Y) ((1 - ga - X - Y + Z)_X))/
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z) ((al + be)_(X+Y-Z))
            ((1 + X + Y - Z)_(-Y+Z)) ((1 - be - X - Y + Z)_X))

In[16]:= %//.zus3
Out[16]= ((-1)^(Y+Z) ((al)_X) ((be)_(X+Y-Z))
            ((1 - al - be + ga - X)_(-Y+Z))
            ((al + be - ga + X)_(Y-Z)) ((1 - be - Y)_(-X+Z))
            ((ga - Z)_Y) ((1 - ga - X - Y + Z)_X))/
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z)
            ((al + be)_(X+Y-Z)) ((1 + X + Y - Z)_(-Y+Z)))

In[17]:= PosListe[%]
Out[17]= {{(-1)^(Y+Z), {{1}}}, {1/(1)_Y, {{2}}},
         {1/(1)_(X+Y-Z), {{3}}}, {1/(1)_Z, {{4}}},
         {(al)_X, {{5}}}, {(be)_(X+Y-Z), {{6}}},
         {1/(al + be)_(X+Y-Z), {{7}}},
         {(1 - al - be + ga - X)_(-Y+Z), {{8}}},
         {(al + be - ga + X)_(Y-Z), {{9}}},
         {(1 - be - Y)_(-X+Z), {{10}}},
         {(ga - Z)_Y, {{11}}},
         {1/(1 + X + Y - Z)_(-Y+Z), {{12}}},
         {(1 - ga - X - Y + Z)_X, {{13}}}}

In[18]:= Ers[%%, zerl, {6}]
Out[18]= ((-1)^(Y+Z) ((al)_X)
            ((be)_Y) ((1 - al - be + ga - X)_(-Y+Z))
            ((al + be - ga + X)_(Y-Z))
            ((1 - be - Y)_(-X+Z)) ((be + Y)_(X-Z))
            ((ga - Z)_Y) ((1 - ga - X - Y + Z)_X))/
         (((1)_Y) ((1)_(X+Y-Z)) ((1)_Z)
            ((al + be)_(X+Y-Z)) ((1 + X + Y - Z)_(-Y+Z)))

In[19]:= PosListe[%]
```

Out[19]=

$$\left\{\left\{(-1)^{Y+Z},\{\{1\}\}\right\},\left\{\frac{1}{(1)_Y},\{\{2\}\}\right\},\left\{\frac{1}{(1)_{X+Y-Z}},\{\{3\}\}\right\},\left\{\frac{1}{(1)_Z},\{\{4\}\}\right\},\{(al)_X,\{\{5\}\}\},\{(be)_Y,\{\{6\}\}\},\left\{\frac{1}{(al+be)_{X+Y-Z}},\{\{7\}\}\right\},\{(1-al-be+ga-X)_{-Y+Z},\{\{8\}\}\},\{(al+be-ga+X)_{Y-Z},\{\{9\}\}\},\{(1-be-Y)_{-X+Z},\{\{10\}\}\},\{(be+Y)_{X-Z},\{\{11\}\}\},\{(ga-Z)_Y,\{\{12\}\}\},\left\{\frac{1}{(1+X+Y-Z)_{-Y+Z}},\{\{13\}\}\right\},\{(1-ga-X-Y+Z)_X,\{\{14\}\}\}\right\}$$

In[20]:= **Ers[%%, zerl, {3}]**

Out[20]=

$$((-1)^{Y+Z}((al)_X)((be)_Y)((1-al-be+ga-X)_{-Y+Z})((al+be-ga+X)_{Y-Z})((1-be-Y)_{-X+Z})((be+Y)_{X-Z})((ga-Z)_Y)((1-ga-X-Y+Z)_X))/(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z})((1+X)_{Y-Z})((1+X+Y-Z)_{-Y+Z}))$$

In[21]:= **PosListe[%]**

Out[21]=

$$\left\{\left\{(-1)^{Y+Z},\{\{1\}\}\right\},\left\{\frac{1}{(1)_X},\{\{2\}\}\right\},\left\{\frac{1}{(1)_Y},\{\{3\}\}\right\},\left\{\frac{1}{(1)_Z},\{\{4\}\}\right\},\{(al)_X,\{\{5\}\}\},\{(be)_Y,\{\{6\}\}\},\left\{\frac{1}{(al+be)_{X+Y-Z}},\{\{7\}\}\right\},\{(1-al-be+ga-X)_{-Y+Z},\{\{8\}\}\},\left\{\frac{1}{(1+X)_{Y-Z}},\{\{9\}\}\right\},\{(al+be-ga+X)_{Y-Z},\{\{10\}\}\},\{(1-be-Y)_{-X+Z},\{\{11\}\}\},\{(be+Y)_{X-Z},\{\{12\}\}\},\{(ga-Z)_Y,\{\{13\}\}\},\left\{\frac{1}{(1+X+Y-Z)_{-Y+Z}},\{\{14\}\}\right\},\{(1-ga-X-Y+Z)_X,\{\{15\}\}\}\right\}$$

In[22]:= **Ers[%%, erw1, {9}]**

Out[22]=

$$((-1)^{Y+Z}((al)_X)((be)_Y)((1-al-be+ga-X)_{-Y+Z})((al+be-ga+X)_{Y-Z})((1-be-Y)_{-X+Z})((be+Y)_{X-Z})((ga-Z)_Y)((1-ga-X-Y+Z)_X))/(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z}))$$

In[23]:= **PosListe[%]**

Out[23]= $$\left\{\{(-1)^{Y+Z},\{\{1\}\}\},\left\{\frac{1}{(1)_X},\{\{2\}\}\right\},\left\{\frac{1}{(1)_Y},\{\{3\}\}\right\},\left\{\frac{1}{(1)_Z},\{\{4\}\}\right\},\{(\mathrm{al})_X,\{\{5\}\}\},\{(\mathrm{be})_Y,\{\{6\}\}\},\left\{\frac{1}{(\mathrm{al}+\mathrm{be})_{X+Y-Z}},\{\{7\}\}\right\},\{(1-\mathrm{al}-\mathrm{be}+\mathrm{ga}-X)_{-Y+Z},\{\{8\}\}\},\{(\mathrm{al}+\mathrm{be}-\mathrm{ga}+X)_{Y-Z},\{\{9\}\}\},\{(1-\mathrm{be}-Y)_{-X+Z},\{\{10\}\}\},\{(\mathrm{be}+Y)_{X-Z},\{\{11\}\}\},\{(\mathrm{ga}-Z)_Y,\{\{12\}\}\},\{(1-\mathrm{ga}-X-Y+Z)_X,\{\{13\}\}\}\right\}$$

In[24]:= **Ers[%%, neg2, {8}]**

Out[24]= $$((-1)^{2Y}((\mathrm{al})_X)((\mathrm{be})_Y)((1-\mathrm{be}-Y)_{-X+Z})((\mathrm{be}+Y)_{X-Z})((\mathrm{ga}-Z)_Y)((1-\mathrm{ga}-X-Y+Z)_X))\Big/(((1)_X)((1)_Y)((1)_Z)((\mathrm{al}+\mathrm{be})_{X+Y-Z}))$$

In[25]:= **PosListe[%]**

Out[25]= $$\left\{\{(-1)^{2Y},\{\{1\}\}\},\left\{\frac{1}{(1)_X},\{\{2\}\}\right\},\left\{\frac{1}{(1)_Y},\{\{3\}\}\right\},\left\{\frac{1}{(1)_Z},\{\{4\}\}\right\},\{(\mathrm{al})_X,\{\{5\}\}\},\{(\mathrm{be})_Y,\{\{6\}\}\},\left\{\frac{1}{(\mathrm{al}+\mathrm{be})_{X+Y-Z}},\{\{7\}\}\right\},\{(1-\mathrm{be}-Y)_{-X+Z},\{\{8\}\}\},\{(\mathrm{be}+Y)_{X-Z},\{\{9\}\}\},\{(\mathrm{ga}-Z)_Y,\{\{10\}\}\},\{(1-\mathrm{ga}-X-Y+Z)_X,\{\{11\}\}\}\right\}$$

In[26]:= **Ers[%%, neg2, {8}]**

Out[26]= $$((-1)^{X+2Y-Z}((\mathrm{al})_X)((\mathrm{be})_Y)((\mathrm{ga}-Z)_Y)((1-\mathrm{ga}-X-Y+Z)_X))\Big/(((1)_X)((1)_Y)((1)_Z)((\mathrm{al}+\mathrm{be})_{X+Y-Z}))$$

In[27]:= **PosListe[%]**

Out[27]= $$\left\{\{(-1)^{X+2Y-Z},\{\{1\}\}\},\left\{\frac{1}{(1)_X},\{\{2\}\}\right\},\left\{\frac{1}{(1)_Y},\{\{3\}\}\right\},\left\{\frac{1}{(1)_Z},\{\{4\}\}\right\},\{(\mathrm{al})_X,\{\{5\}\}\},\{(\mathrm{be})_Y,\{\{6\}\}\},\left\{\frac{1}{(\mathrm{al}+\mathrm{be})_{X+Y-Z}},\{\{7\}\}\right\},\{(\mathrm{ga}-Z)_Y,\{\{8\}\}\},\{(1-\mathrm{ga}-X-Y+Z)_X,\{\{9\}\}\}\right\}$$

In[28]:= **Ers[%%, zerl, {8}]**

Out[28]= $$((-1)^{X+2Y-Z}((\mathrm{al})_X)((\mathrm{be})_Y)((\mathrm{ga})_{Y-Z})((\mathrm{ga}-Z)_Z)((1-\mathrm{ga}-X-Y+Z)_X))\Big/(((1)_X)((1)_Y)((1)_Z)((\mathrm{al}+\mathrm{be})_{X+Y-Z}))$$

In[29]:= **PosListe[%]**

Out[29]= $\left\{\{(-1)^{X+2Y-Z}, \{\{1\}\}\}, \left\{\frac{1}{(1)_X}, \{\{2\}\}\right\}, \left\{\frac{1}{(1)_Y}, \{\{3\}\}\right\}, \left\{\frac{1}{(1)_Z}, \{\{4\}\}\right\}, \{(al)_X, \{\{5\}\}\}, \{(be)_Y, \{\{6\}\}\}, \left\{\frac{1}{(al+be)_{X+Y-Z}}, \{\{7\}\}\right\}, \{(ga)_{Y-Z}, \{\{8\}\}\}, \{(ga-Z)_Z, \{\{9\}\}\}, \{(1-ga-X-Y+Z)_X, \{\{10\}\}\}\right\}$

In[30]:= **Ers[%%, erw1, {8}]**

Out[30]= $((-1)^{X+2Y-Z}((al)_X)((be)_Y)((ga)_{X+Y-Z})((ga-Z)_Z)((1-ga-X-Y+Z)_X))\Big/(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z})((ga+Y-Z)_X))$

In[31]:= **PosListe[%]**

Out[31]= $\left\{\{(-1)^{X+2Y-Z}, \{\{1\}\}\}, \left\{\frac{1}{(1)_X}, \{\{2\}\}\right\}, \left\{\frac{1}{(1)_Y}, \{\{3\}\}\right\}, \left\{\frac{1}{(1)_Z}, \{\{4\}\}\right\}, \{(al)_X, \{\{5\}\}\}, \{(be)_Y, \{\{6\}\}\}, \left\{\frac{1}{(al+be)_{X+Y-Z}}, \{\{7\}\}\right\}, \{(ga)_{X+Y-Z}, \{\{8\}\}\}, \{(ga-Z)_Z, \{\{9\}\}\}, \left\{\frac{1}{(ga+Y-Z)_X}, \{\{10\}\}\right\}, \{(1-ga-X-Y+Z)_X, \{\{11\}\}\}\right\}$

In[32]:= **Ers[%%, zerl, {11}]**

Out[32]= $((-1)^{X+2Y-Z}((al)_X)((be)_Y)((1-ga)_{-Y+Z})((ga)_{X+Y-Z})((ga-Z)_Z)((1-ga-X-Y+Z)_{X+Y-Z}))\Big/(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z})((ga+Y-Z)_X))$

In[33]:= **PosListe[%]**

Out[33]= $\left\{\{(-1)^{X+2Y-Z}, \{\{1\}\}\}, \left\{\frac{1}{(1)_X}, \{\{2\}\}\right\}, \left\{\frac{1}{(1)_Y}, \{\{3\}\}\right\}, \left\{\frac{1}{(1)_Z}, \{\{4\}\}\right\}, \{(al)_X, \{\{5\}\}\}, \{(be)_Y, \{\{6\}\}\}, \left\{\frac{1}{(al+be)_{X+Y-Z}}, \{\{7\}\}\right\}, \{(1-ga)_{-Y+Z}, \{\{8\}\}\}, \{(ga)_{X+Y-Z}, \{\{9\}\}\}, \{(ga-Z)_Z, \{\{10\}\}\}, \left\{\frac{1}{(ga+Y-Z)_X}, \{\{11\}\}\right\}, \{(1-ga-X-Y+Z)_{X+Y-Z}, \{\{12\}\}\}\right\}$

```
In[34]:= Ers[%%, trans, {12}]
Out[34]= ((-1)^(2X+3Y-2Z) ((al)_X) ((be)_Y)
            ((1 - ga)_(-Y+Z)) ((ga)_(X+Y-Z))^2 ((ga - Z)_Z)) /
         (((1)_X) ((1)_Y) ((1)_Z) ((al + be)_(X+Y-Z))
            ((ga + Y - Z)_X))
In[35]:= PosListe[%]
Out[35]= {{(-1)^(2X+3Y-2Z), {{1}}},
          {1/(1)_X, {{2}}}, {1/(1)_Y, {{3}}},
          {1/(1)_Z, {{4}}}, {(al)_X, {{5}}},
          {(be)_Y, {{6}}}, {1/(al + be)_(X+Y-Z), {{7}}},
          {(1 - ga)_(-Y+Z), {{8}}}, {((ga)_(X+Y-Z))^2, {{9}}},
          {(ga - Z)_Z, {{10}}}, {1/(ga + Y - Z)_X, {{11}}}}
In[36]:= Ers[%%, zerl, {8}]
Out[36]= ((-1)^(2X+3Y-2Z) ((al)_X) ((be)_Y) ((1 - ga)_Z)
            ((ga)_(X+Y-Z))^2 ((ga - Z)_Z) ((1 - ga + Z)_(-Y))) /
         (((1)_X) ((1)_Y) ((1)_Z) ((al + be)_(X+Y-Z))
            ((ga + Y - Z)_X))
In[37]:= PosListe[%]
Out[37]= {{(-1)^(2X+3Y-2Z), {{1}}},
          {1/(1)_X, {{2}}}, {1/(1)_Y, {{3}}},
          {1/(1)_Z, {{4}}}, {(al)_X, {{5}}},
          {(be)_Y, {{6}}}, {1/(al + be)_(X+Y-Z), {{7}}},
          {(1 - ga)_Z, {{8}}}, {((ga)_(X+Y-Z))^2, {{9}}},
          {(ga - Z)_Z, {{10}}}, {1/(ga + Y - Z)_X, {{11}}},
          {(1 - ga + Z)_(-Y), {{12}}}}
In[38]:= Ers[%%, trans, {12}]
Out[38]= ((-1)^(2X+2Y-2Z) ((al)_X) ((be)_Y) ((1 - ga)_Z)
            ((ga)_(X+Y-Z))^2 ((ga - Z)_Z) ((ga + Y - Z)_(-Y))) /
         (((1)_X) ((1)_Y) ((1)_Z) ((al + be)_(X+Y-Z))
            ((ga + Y - Z)_X))
In[39]:= PosListe[%]
```

Out[39]= $\left\{\{(-1)^{2X+2Y-2Z}, \{\{1\}\}\},\right.$
$\left\{\frac{1}{(1)_X}, \{\{2\}\}\right\}, \left\{\frac{1}{(1)_Y}, \{\{3\}\}\right\},$
$\left\{\frac{1}{(1)_Z}, \{\{4\}\}\right\}, \{(al)_X, \{\{5\}\}\},$
$\{(be)_Y, \{\{6\}\}\}, \left\{\frac{1}{(al+be)_{X+Y-Z}}, \{\{7\}\}\right\},$
$\{(1-ga)_Z, \{\{8\}\}\}, \{((ga)_{X+Y-Z})^2, \{\{9\}\}\},$
$\{(ga-Z)_Z, \{\{10\}\}\}, \left\{\frac{1}{(ga+Y-Z)_X}, \{\{11\}\}\right\},$
$\left.\{(ga+Y-Z)_{-Y}, \{\{12\}\}\}\right\}$

In[40]:= **%%/.zus3**

Out[40]= $((-1)^{2X+2Y-2Z}((al)_X)((be)_Y)$
$((1-ga)_Z)((ga)_{Y-Z})((ga)_{X+Y-Z})$
$((ga-Z)_Z)((ga+Y-Z)_{-Y}))/$
$(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z}))$

In[41]:= **%//.zus1**

Out[41]= $((-1)^{2X+2Y-2Z}((al)_X)$
$((be)_Y)((1-ga)_Z)((ga)_{X+Y-Z}))/$
$(((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z}))$

In[42]:= **%/.MinusOne**

Is 2 (X + Y - Z) even, odd, or neither of both?

Out[42]= $\frac{((al)_X)((be)_Y)((1-ga)_Z)((ga)_{X+Y-Z})}{((1)_X)((1)_Y)((1)_Z)((al+be)_{X+Y-Z})}$