VICIOUS WALKERS, FRIENDLY WALKERS AND YOUNG TABLEAUX III: BETWEEN TWO WALLS

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Abstract. We derive exact and asymptotic results for the number of star and watermelon configurations of vicious walkers confined to lie between two impenetrable walls, as well as for the analogous problem for ∞-friendly walkers. Our proofs make use of results from symmetric function theory and the theory of basic hypergeometric series.

1. Introduction

This is the third paper in a series studying vicious and friendly walkers. In the first paper [7] it was shown how certain results from the theory of Young tableaux, and related results in algebraic combinatorics enabled one to readily prove an expression for the number of star configurations of vicious walkers on a d-dimensional lattice as well as proving the corresponding result for the number of watermelon configurations.

In the second paper [11], we showed how some results from the theory of symmetric functions could be used to prove analogous results for the more difficult case of walkers in the presence of an impenetrable wall. We also gave rigorous asymptotic results.

In that paper we also developed the theory of n-friendly walkers, introduced in [8] and [10]. The two models differ slightly. In [8], the "vicious" constraint is systematically relaxed, so that any two walks (but not more than two) may stay together for up to n lattice sites in a row, but may never swap sides. We refer to this as the n-friendly walker model. In the limit as $n \to \infty$ we obtain the ∞-friendly walker model in which two walkers may share an arbitrary number of steps. The Tsuchiya-Katori model [10], by contrast, corresponds to a variant of the ∞-friendly walker model which allows *any number* of walkers to share any number of lattice sites, whereas in the Guttmann-Vöge definition [8], only two walkers may share a lattice site. We subsequently refer to these two models as the TK and GV models respectively. Thus the number of TK friendly walk configurations gives an upper bound on the number of ∞-friendly walk configurations in the definition of GV. We make use of this observation in subsequent proofs.

In this, the final paper in the series, we address the problem of vicious and friendly walkers confined to a finite strip — or, equivalently, confined to lie between two parallel walls. A star configuration is shown in Figure 1.

Vicious walkers describes the situation in which two or more walkers arriving at the same lattice site annihilate one another. Accordingly, the only configurations we consider in that case are those in which such contacts are forbidden. Alternatively expressed, we consider mutually self-avoiding networks of lattice walks which also model

[1] Research partially supported by EC's IHRP Programme, grant RTN2-2001-00059.

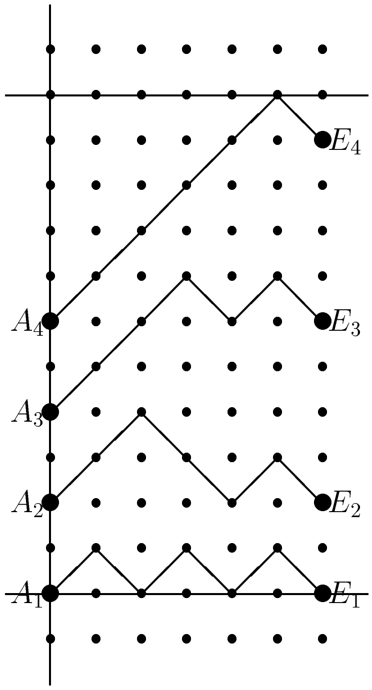

FIGURE 1. A star of $p = 4$ vicious walkers, of length $m = 6$, confined to a strip of width $h = 11$.

directed polymer networks. The connection of these vicious walker problems to the 5 and 6 vertex model of statistical mechanics was also discussed in [7].

The problem, together with a number of physical applications, was first introduced by Fisher [1]. The general model is one of p random walkers on a d-dimensional lattice who at regular time intervals simultaneously take one step with equal probability in the direction of one of the allowed lattice vectors such that at no time do two walkers occupy the same lattice site.

The two standard topologies of interest are that of a *star* and a *watermelon*. Consider a directed square lattice, rotated 45° and augmented by a factor of $\sqrt{2}$, so that the "unit" vectors on the lattice are $(1,1)$ and $(1,-1)$. Both configurations consist of p branches of length m (the lattice paths along which the walkers proceed) which start at $(0,0),(0,2),(0,4),\dots,(0,2p-2)$. The watermelon configurations end at $(m,k),(m,2+k),(m,4+k),\dots,(m,k+2p-2)$, for some k. For stars, the end points of the branches all have x-coordinate equal to m, but the y coordinates are unconstrained, apart from the ordering imposed by the non-crossing condition. Thus if the end points are $(m,e_1),(m,e_2),(m,e_3),\cdots,(m,e_p)$, then $e_1 < e_2 < e_3 < \cdots < e_p \le 2p-2+m$. In the problem considered here, the additional constraint of impenetrable walls imposes the conditions that at no stage may any walker step to a point with negative y-coordinate, or to a point with y-cordinate greater than the strip width, h. We can also consider *displaced configurations,* in which the starting points are $(0,a),(0,a+2),(0,a+4),\dots,(0,a+2p-2)$.

Vicious walkers confined to lie between two walls can be seen in a different way, namely as random walks in an alcove of an affine Weyl group of type C if the set of allowed steps is appropriately chosen. In this form they are considered by Grabiner in [6, Sec. 5]. The topic of the paper [6] is the exact enumeration of random walks in alcoves of affine Weyl groups of types A, C and D. One of the problems posed in [6] is to find asymptotic formulas for these random walks (when the length of the walks goes to infinity). We solve this problem for the random walks in an alcove of type C that correspond to our vicious walkers between two walls. Asymptotic results for the other enumeration problems considered in [6] will appear in the forthcoming paper [13].

It is intuitively clear that the asymptotic growth of the number of vicious walkers within a strip must be exponential, with the base of the exponential depending on the width of the strip and the number of walkers, but not on the starting and end points of the walkers. This is confirmed by our results. (The same must of course be true for n-friendly walkers, although we are only able to rigorously confirm this for ∞-friendly walkers in the TK model, by deriving explicit formulas.) Thus, for example, the asymptotics of stars and watermelons in the same strip will be exponential with the same base, and will only differ in the constant by which the exponential is multiplied.

Our paper is organised as follows. In Section 2 we provide exact formulas for the number of vicious walkers between two walls with arbitrary starting and end points. With the exception of one, these appear already in [6], in equivalent forms. These results follow from the Lindström–Gessel–Viennot theorem on nonintersecting lattice paths and known results for lattice paths between two parallel lines. They express the number of vicious walkers within a strip as determinants. In Section 3 we address the asymptotics of these formulas when the length of the branches of the vicious walkers goes to infinity. In Theorem 4 we give the asymptotics for vicious walkers within a strip for arbitrary (but fixed) starting and end points. By specializing the starting and end points, we obtain asymptotic results for watermelons within a strip, see Corollary 5. In order to obtain asymptotic results for stars within a strip, the formula in Theorem 4 has to be summed over all possible end points. The task to carry out this summation is highly nontrivial. It requires some symmetric function theory, in particular certain relations between Schur functions and symplectic and orthogonal characters, and a summation theorem for a basic hypergeometric series. The final result is given in Theorem 6. This theorem is then specialized to obtain the asymptotics for stars within a strip, see Corollaries 7 and 8.

2. General networks and stars

The Lindström–Gessel–Viennot determinant [14, 5] in the case of the presence of two walls yields the following result. It appears already in [6, (13)], in an equivalent form.

Theorem 1. *Let $0 \le a_1 < a_2 < \cdots < a_p \le h$, all a_i's of the same parity, and $0 \le e_1 < e_2 < \cdots < e_p \le h$, all e_i's of the same parity, such that $a_i + e_i \equiv m \pmod 2$, $i = 1, 2, \dots, p$. The number of vicious walkers with p branches of length m, the i-th branch running from $A_i = (0, a_i)$ to $E_i = (m, e_i)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h$, is given by*

$$\det_{1\le s,t\le p}\left(\sum_{k=-\infty}^{\infty}\left(\binom{m}{\frac{m+e_t-a_s}{2}+k(h+2)}-\binom{m}{\frac{m+e_t+a_s}{2}+k(h+2)+1}\right)\right). \tag{2.1}$$

Proof. According to the main theorem of non-intersecting lattice paths [5, Cor. 2] (see [18, Theorem 1.2]), the number of vicious walkers in question equals

$$\det_{1\le s,t\le p}\left(\left|\mathcal{P}^{++}(A_t\to E_s)\right|\right), \tag{2.2}$$

where $\mathcal{P}^{++}(A\to E)$ denotes the set of all lattice paths from A to E which do not go below the x-axis nor above the line $y=h$. There is a well-known formula (see [16, (1.7)]) for the latter number, which is obtained by an iterated reflection principle. Substitution of this formula into (2.2) immediately gives (2.1). □

By bringing the sums in (2.1) outside the determinant (using the multi-linearity of the determinant), the number of these vicious walkers can be described as a multiple sum of determinants. In some cases, such as for certain watermelons and stars, the determinants can be evaluated. In those cases, a multiple hypergeometric sum is obtained.

Theorem 2. *Let $0\le e_1<e_2<\cdots<e_p\le h$ with $e_i\equiv m \pmod 2$, $i=1,2,\dots,p$. The number of vicious walkers with p branches of length m, the i-th branch running from $A_i=(0,2i-2)$ to $E_i=(m,e_i)$, which do not go below the x-axis nor above the line $y=h$, $i=1,2,\dots,p$, is given by*

$$\sum_{k_1,\dots,k_p=-\infty}^{\infty} 2^{p-p^2}\prod_{s=1}^{p}\frac{(e_s+2k_s(h+2)+1)\,(m+2s-2)!}{\left(\frac{m+e_s}{2}+k_s(h+2)+p\right)!\,\left(\frac{m-e_s}{2}-k_s(h+2)+p-1\right)!}$$
$$\times\prod_{1\le s<t\le p}(e_t-e_s+2(h+2)(k_t-k_s))(e_t+e_s+2(h+2)(k_t+k_s)+2). \tag{2.3}$$

Proof. As described above the statement of the theorem, we first write the expression (2.1), with $a_i=2i-2$, as a sum of determinants,

$$\sum_{k_1,\dots,k_p=-\infty}^{\infty}\det_{1\le s,t\le p}\left(\binom{m}{\frac{m+e_t}{2}+k_t(h+2)-s+1}-\binom{m}{\frac{m+e_t}{2}+k_t(h+2)+s}\right). \tag{2.4}$$

Suppose that, initially, we disregard the terms $k_t(h+2)$ in the determinant, then it simplifies to

$$\det_{1\le s,t\le p}\left(\binom{m}{\frac{m+e_t}{2}-s+1}-\binom{m}{\frac{m+e_t}{2}+s}\right). \tag{2.5}$$

Gessel–Viennot theory (again) says that this determinant counts vicious walkers with p branches of length m, the i-th branch running from $A_i=(0,2i-2)$ to $E_i=(m,e_i)$, which do not go below the x-axis. By Theorem 6 of [11], the number of these vicious walkers is given by

$$2^{-p^2+p}\prod_{s=1}^{p}\frac{(e_s+1)\,(m+2s-2)!}{\left(\frac{m+e_s}{2}+p\right)!\,\left(\frac{m-e_s}{2}+p-1\right)!}\prod_{1\le s<t\le p}(e_t-e_s)(e_s+e_t+2). \tag{2.6}$$

Hence, the determinant in (2.5) must equal the expression (2.6). In fact, as the equality between (2.5) and (2.6) can be reduced to an equation which is polynomial in $e_1,e_2\dots,e_p$, the equality is true for *any* choice of $e_1,e_2,\dots,e_p$. In particular, it remains true if we replace e_i by $e_i+2k_i(h+2)$, $i=1,2,\dots,p$. However, the determinant in (2.5) under these replacements becomes the determinant in (2.4). Thus, if we substitute the expression (2.6), with these replacements, into (2.4), we immediately obtain (2.3). □

For the subsequent asymptotic calculations, however, we need a different type of expression for the number of vicious walkers under consideration. This expression can be easily derived by a combination of the Lindström–Gessel–Viennot theorem and an alternative expression for the number of lattice paths between two parallel boundaries in terms of sines and cosines. It appears already in [6, (18)], in an equivalent form.

Theorem 3. *Let $0 \le a_1 < a_2 < \cdots < a_p \le h$, all a_i's of the same parity, and $0 \le e_1 < e_2 < \cdots < e_p \le h$, all e_i's of the same parity, such that $a_i + e_i \equiv m \pmod 2$, $i = 1, 2, \dots, p$. The number of vicious walkers with p branches of length m, the i-th branch running from $A_i = (0, a_i)$ to $E_i = (m, e_i)$, $i = 1, 2, \dots, p$, which do not go below the x-axis and not above the line $y = h$, is given by*

$$\frac{2^p}{(h+2)^p} \sum_{k_1,\dots,k_p=1}^{h+1} \left(2^p \prod_{s=1}^{p} \cos \frac{k_s \pi}{h+2}\right)^m \prod_{t=1}^{p} \sin \frac{\pi k_t (e_t+1)}{h+2} \det_{1\le s,t\le p} \left(\sin \frac{\pi k_t (a_s+1)}{h+2}\right). \tag{2.7}$$

Proof. We already know that, by the Lindström–Gessel–Viennot theorem, the number in question is given by (2.2). Instead of using the iterated reflection formula for $\mathcal{P}^{++}(A \to E)$, we now apply the (equally well-known) alternative formula (see [9, §184, Ex. 1, Eq. (9)])

$$\mathcal{P}^{++}(A \to E) = \frac{2}{h+2} \sum_{k=1}^{h+1} \left(2 \cos \frac{\pi k}{h+2}\right)^m \sin \frac{\pi k (a+1)}{h+2} \cdot \sin \frac{\pi k (e+1)}{h+2},$$

given that $A = (0, a)$ and $E = (m, e)$. Substituting this into (2.2), and bringing the summations and a few factors outside of the determinant utilising the multi-linearity of the determinant, we get (2.7). □

3. The asymptotics of vicious walkers between two walls

Theorem 3 now enables us to derive asymptotic formulae for the number of vicious walkers between two walls, for arbitrary starting and end points.

Theorem 4. *Let $0 \le a_1 < a_2 < \cdots < a_p \le h$, all a_i's of the same parity, and $0 \le e_1 < e_2 < \cdots < e_p \le h$, all e_i's of the same parity, such that $a_i + e_i \equiv m \pmod 2$, $i = 1, 2, \dots, p$. The number of vicious walkers with p branches of length m, the i-th branch running from $A_i = (0, a_i)$ to $E_i = (m, e_i)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h$, is asymptotically*

$$\frac{4^{p^2}}{(h+2)^p} \left(2^p \prod_{s=1}^{p} \cos \frac{s\pi}{h+2}\right)^m \prod_{1\le s<t\le p} \sin \frac{\pi(a_t - a_s)}{2(h+2)} \cdot \sin \frac{\pi(e_t - e_s)}{2(h+2)}$$
$$\times \prod_{1\le s\le t\le p} \sin \frac{\pi(a_t + a_s + 2)}{2(h+2)} \cdot \sin \frac{\pi(e_t + e_s + 2)}{2(h+2)}. \tag{3.1}$$

Remark. What this theorem says is that vicious walkers in a strip of width h grow exponentially like $\left(2^p \prod_{s=1}^{p} \cos \frac{s\pi}{h+2}\right)^m$, everything else is just the multiplicative constant.

Proof. Theorem 3 tells us that the number of vicious walkers that we wish to estimate can be written in the form of a finite sum $\sum_\ell c_\ell b_\ell^m$, where the c_ℓ's and b_ℓ's are independent of m. Hence, what we have to find is $b=\max_\ell b_\ell$. Then, asymptotically, the number of vicious walkers is $b^m\sum_{\ell:\ b_\ell=b} c_\ell$.

The b_ℓ's have the form

$$2^p\prod_{s=1}^p \cos\big(k_s\pi/(h+2)\big). \tag{3.2}$$

To find the largest among these, we have to choose k_s close to the lower limit of the summation in (2.7), which is 1, or close to the upper limit, $h+1$. However, we are not allowed to make a choice such that $k_s=k_t$ for some $s\ne t$, because in that case the determinant in (2.7) vanishes. For the same reason a choice such that $k_s=h+2-k_t$ for some s and t is forbidden. Therefore the expression (3.2) will be maximal if the set $\{k_1,k_2\dots,k_p\}$ is chosen from $\{1,2,\dots,p,h+2-p,\dots,h,h+1\}$, subject to the two restrictions mentioned above. These are exactly 2^p different choices. As a short calculation shows, the sum of the corresponding summands in (2.7) equals

$$\frac{4^p}{(h+2)^p}\left(2^p\prod_{s=1}^p\cos\frac{s\pi}{h+2}\right)^m\det_{1\le s,t\le p}\left(\sin\frac{\pi t(a_s+1)}{h+2}\right)\det_{1\le s,t\le p}\left(\sin\frac{\pi t(e_s+1)}{h+2}\right),$$

or, equivalently,

$$\frac{(-1)^p}{(h+2)^p}\left(2^p\prod_{s=1}^p\cos\frac{s\pi}{h+2}\right)^m\det_{1\le s,t\le p}\left(e^{\frac{\pi it}{h+2}(a_s+1)}-e^{-\frac{\pi it}{h+2}(a_s+1)}\right)$$
$$\times\det_{1\le s,t\le p}\left(e^{\frac{\pi it}{h+2}(e_s+1)}-e^{-\frac{\pi it}{h+2}(e_s+1)}\right). \tag{3.3}$$

Both determinants are easily evaluated by means of the determinant identity

$$\det_{1\le i,j\le N}\left(x_i^j-x_i^{-j}\right)=(x_1x_2\cdots x_N)^{-N}\prod_{1\le i<j\le N}(x_i-x_j)(1-x_ix_j)\prod_{i=1}^N(x_i^2-1), \tag{3.4}$$

which may be readily proved by the standard argument that proves Vandermonde-type determinant identities.

A little manipulation then leads to (3.1). □

If we now specialize a_i to $a+2i-2$ and e_i to $e+2i-2$ in Theorem 4, we then obtain the asymptotics for watermelons between two walls.

Corollary 5. *Let a and e be integers with $0\le a,e\le h-2p+2$ and $a+e\equiv m$ (mod 2). The number of watermelons with p branches of length m, in which the lowest branch starts at height a and terminates at height e, which do not go below the x-axis nor above the line $y=h$, is asymptotically*

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^p\cos\frac{s\pi}{h+2}\right)^m\prod_{1\le s<t\le p}\sin^2\frac{\pi(t-s)}{(h+2)}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a+t+s-1)}{(h+2)}\cdot\sin\frac{\pi(e+t+s-1)}{(h+2)}. \tag{3.5}$$

Let a and e be integers with $0 \le a, e \le h - 2p + 2$ and $a + e \equiv m \pmod 2$. The number of ∞-friendly watermelons in the TK model with p branches of length m, in which the lowest branch starts at height a and terminates at height e, which do not go below the x-axis nor above the line $y = h$, is asymptotically

$$\frac{4^{p^2}}{(h+2p)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2p}\right)^m \prod_{1\le s<t\le p}\sin^2\frac{2\pi(t-s)}{(h+2p)} \\ \times \prod_{1\le s\le t\le p}\sin\frac{\pi(a+2t+2s-3)}{(h+2p)}\cdot\sin\frac{\pi(e+2t+2s-3)}{(h+2p)}. \quad (3.6)$$

Proof. There is nothing to say about the first claim, which follows immediately from the theorem. To establish the second claim, we shift the i-th branch of the ∞-friendly watermelon by $2(i-1)$ units up, as in the proof of Theorem 4 of [11]. We transform ∞-friendly watermelons into families of non-intersecting lattice paths by shifting the i-th path up by $2(i-1)$ units. Thus we obtain a set of vicious walkers with p branches of length m, the i-th branch starting from $A_i = (0, a + 4i - 4)$ and terminating at $E_i = (m, e + 4i - 4)$, $i = 1, 2, \dots, p$, which do not go below the x-axis and not above the line $y = h + 2p - 2$ (!). Hence, Theorem 4 with $a_i = a + 4i - 4$, $e_i = e + 4i - 4$, $i = 1, 2, \dots, p$, and h replaced by $h + 2p - 2$ immediately gives the desired asymptotics. □

Clearly, by performing the obvious summations of (3.5) over e, respectively a, we could also obtain the asymptotics for watermelons of arbitrary deviation. The resulting sums do not appear to simplify however. Nevertheless, since the summations are over finite sets (depending only on the width h of the strip and the number p of walkers), it is obvious that the order of the asymptotic growth is again $\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m$ for vicious walkers and $\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2p}\right)^m$ for ∞-friendly walkers in the TK model.

It should be noted, however, that in contrast to watermelons without restriction and with the restriction of *one* wall, as considered in our previous papers, the situation considered here, that is in the presence of the restriction of *two* walls, the asymptotics of (ordinary) watermelons and ∞-friendly watermelons (compare the bases of the exponentials in the two statements in Corollary 5) is of a different order of magnitude (except in the case of single branch watermelons, of course). Hence, from these considerations, it is impossible to conclude whether n-friendly watermelons restricted by two walls will have the same order of magnitude as (ordinary) watermelons, or not.

Let us turn now to the asymptotics for stars. We begin with a general theorem, which solves the problem, posed in [6], of computing the asymptotics for the number of random walks in an alcove of an affine Weyl group of type C if the allowed steps are of the form $\frac{1}{2}(\pm1, \pm1, \dots, \pm1)$.

Theorem 6. *Let $0 \le a_1 < a_2 < \cdots < a_p \le h$ be integers, all of the same parity. The number of vicious walkers with p branches of length m, the i-th branch starting from $A_i = (0, a_i)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h$,* *is asymptotically*

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m\prod_{1\le s<t\le p}\sin\frac{\pi(a_t-a_s)}{2(h+2)}\cdot\sin\frac{\pi(t-s)}{h+2}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a_t+a_s+2)}{2(h+2)}\cdot\sin\frac{\pi(t+s)}{h+2}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor(h+2)/2\rfloor)}{h+2}}{\sin\frac{\pi(t-s+p)}{h+2}}$$
$$\times\prod_{s=1}^{p}\frac{\sin\frac{(s+\lfloor(h+2)/2\rfloor-p)\pi}{h+2}}{\sin\frac{(2s+\lfloor(h+2)/2\rfloor-p)\pi}{h+2}},\quad(3.7)$$

if $m+a_i$ *is even, and*

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m\prod_{1\le s<t\le p}\sin\frac{\pi(a_t-a_s)}{2(h+2)}\cdot\sin\frac{\pi(t-s)}{h+2}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a_t+a_s+2)}{2(h+2)}\cdot\sin\frac{\pi(t+s)}{h+2}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor(h+1)/2\rfloor)}{h+2}}{\sin\frac{\pi(t-s+p)}{h+2}},\quad(3.8)$$

if $m+a_i$ *is odd.*

Proof. It is obvious that, in view of Theorem 4, we have to compute the sum of (3.1) over all possible choices of $e_1<e_2<\dots<e_p$. Here we have to distinguish between two cases, depending on whether $m+a_i$ is even or odd.

First let $m+a_i$ be odd. This implies that all the e_i's are odd as well, so that we have to compute the sum of (3.1) over all possible choices of $1\le e_1<e_2<\dots<e_p\le h$, with $e_i=2e_i'-1$ for some integer e_i', $i=1,2,\dots,p$. If we remember that expression (3.1) came from (3.3), we see that this is

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m\prod_{1\le s<t\le p}\sin\frac{\pi(a_t-a_s)}{2(h+2)}\prod_{1\le s\le t\le p}\sin\frac{\pi(a_t+a_s+2)}{2(h+2)}$$
$$\times(-i)^p\sum_{1\le e_1'<\dots<e_p'\le\lfloor(h+1)/2\rfloor}\det_{1\le s,t\le p}\left(e^{\frac{2\pi it}{h+2}e_s'}-e^{-\frac{2\pi it}{h+2}e_s'}\right).\quad(3.9)$$

In order to evaluate the sum in the last line, we rewrite it in terms of symplectic characters $sp_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_p^{\pm1})$, which are defined by (see [2, (24.18)])

$$sp_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_p^{\pm1})=\frac{\det_{1\le s,t\le p}\left(x_t^{\lambda_s+p-s+1}-x_t^{-(\lambda_s+p-s+1)}\right)}{\det_{1\le s,t\le p}\left(x_t^{p-s+1}-x_t^{-(p-s+1)}\right)}.\quad(3.10)$$

Therefore, writing q for $e^{2\pi i/(h+2)}$ and H for $\lfloor (h+1)/2\rfloor$, the sum in the last line of (3.9) equals

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{1\le e_1'<\dots<e_p'\le H} sp_{(e_p'-p,\dots,e_1'-1)}(q^{\pm1},q^{\pm2},\dots,q^{\pm p})$$
$$=\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{\nu\subseteq((H-p)^p)} sp_\nu(q^{\pm1},q^{\pm2},\dots,q^{\pm p}). \quad (3.11)$$

Now we appeal to the formula (see [12, (3.4)]),

$$s_{(c^r)}(x_1,x_1^{-1},\dots,x_p,x_p^{-1},1)=\sum_{\nu\subseteq(c^r)} sp_\nu(x_1^{\pm1},\dots,x_p^{\pm1}). \quad (3.12)$$

Use of this formula in (3.11) gives

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right) s_{((H-p)^p)}(q^{-p},\dots,q^{-1},1,q,\dots,q^p). \quad (3.13)$$

Clearly, the determinant is easily evaluated by means of (3.4), whereas the specialized Schur function can be evaluated by means of the hook-content formula (see [15, I, Sec. 3, Ex. 1], [2, Ex. A.30, (ii)])

$$s_\lambda(q^{-L},q^{-L+1},\dots,q^{-L+P})=q^{\sum_{l\ge1}(l-L-1)\lambda_l}\prod_{\rho\in\lambda}\frac{1-q^{P+c_\rho}}{1-q^{h_\rho}}, \quad (3.14)$$

where c_ρ and h_ρ are the *content* and the *hook length* of the cell ρ. Substitution of all this in (3.9) and some manipulation then leads to (3.8).

Now let $m+a_i$ be even. This implies that all the e_i's are even as well, so that we have to compute the sum of (3.1) over all possible choices of $0\le e_1<e_2<\dots<e_p\le h$, with $e_i=2e_i'-2$ for some integer e_i', $i=1,2,\dots,p$. Again, if we remember that expression (3.1) came from (3.3), we see that this is

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m\prod_{1\le s<t\le p}\sin\frac{\pi(a_t-a_s)}{2(h+2)}\prod_{1\le s\le t\le p}\sin\frac{\pi(a_t+a_s+2)}{2(h+2)}$$
$$\times(-i)^p\sum_{1\le e_1'<\dots<e_p'\le\lfloor(h+2)/2\rfloor}\det_{1\le s,t\le p}\left(e^{\frac{2\pi it}{h+2}(e_s'-\frac12)}-e^{-\frac{2\pi it}{h+2}(e_s'-\frac12)}\right). \quad (3.15)$$

This time, it is possible to rewrite the sum in the last line in terms of odd orthogonal characters. The odd orthogonal characters $so_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_m^{\pm1},1)$ where $x_1^{\pm1}$ is a shorthand notation for x_1,x_1^{-1}, etc., and λ is an m-tuple $(\lambda_1,\lambda_2,\dots,\lambda_m)$ of integers, or of half-integers, are defined by

$$so_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_m^{\pm1},1)=\frac{\det\limits_{1\le i,j\le m}\left(x_j^{\lambda_i+m-i+1/2}-x_j^{-(\lambda_i+m-i+1/2)}\right)}{\det\limits_{1\le i,j\le m}\left(x_j^{m-i+1/2}-x_j^{-(m-i+1/2)}\right)}. \quad (3.16)$$

While Schur functions are polynomials in $x_1,x_2,\dots,x_m$, odd orthogonal characters $so_\lambda(x_1^{\pm1},x_2^{\pm1},\dots,x_m^{\pm1},1)$ are polynomials in $x_1,x_1^{-1},x_2,x_2^{-1},\dots,x_m,x_m^{-1},1$. They have combinatorial descriptions in terms of certain tableaux as well, see [3, Sec. 2], [17, Sec. 6–8], [19, Theorem 2.3].

Now, writing q for $e^{2\pi i/(h+2)}$ and H for $\lfloor (h+2)/2\rfloor$, the sum in the last line of (3.15) equals

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{1\le e_1'<\dots<e_p'\le H} so_{(e_p'-p,\dots,e_1'-1)}(q^{\pm1},q^{\pm2},\dots,q^{\pm p},1)$$
$$=\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{\nu\subseteq((H-p)^p)} so_\nu(q^{\pm1},q^{\pm2},\dots,q^{\pm p},1). \quad (3.17)$$

Again there is a formula which allows us to evaluate the sum in the last line (see [12, (3.2)]),

$$s_{((c^{r-\ell},(c-1)^\ell)}(x_1,x_1^{-1},\dots,x_p,x_p^{-1},1)=\sum_{\substack{\nu\subseteq(c^r)\\ \operatorname{oddrows}\left((c^r)/\nu\right)=\ell}} so_\nu(x_1^{\pm1},\dots,x_p^{\pm1},1), \quad (3.18)$$

where $\operatorname{oddrows}\big((c^r)/\nu\big)=\ell$ means that the number of rows of odd length in the skew shape $(c^r)/\nu$ equals exactly ℓ. Use of this formula in (3.17) gives

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{\ell=0}^{p} s_{((H-p)^{p-\ell},(H-p-1)^\ell)}(q^{-p},\dots,q^{-1},1,q,\dots,q^p). \quad (3.19)$$

Again, the hook-content formula (3.14) applies and yields

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)\sum_{\ell=0}^{p} q^{-(H-p)\binom{p+1}{2}+\binom{\ell+1}{2}}\frac{\prod_{s=1}^{H-p-1}(q^{p+s+1};q)_p}{\prod_{s=1}^{H-p}(q^s;q)_p}$$
$$\times\frac{(q^{H+\ell+1};q)_{p-\ell}\,(q^{\ell+1};q)_{p-\ell}\,(q^{H-p};q)_\ell}{(q;q)_{p-\ell}}$$

for the expression (3.19). Here we used the standard notation for shifted q-factorials, $(a;q)_k:=(1-a)(1-aq)\cdots(1-aq^{k-1})$, $k\ge1$, $(a;q)_0:=1$. In terms of the standard basic hypergeometric notation

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]=\sum_{\ell=0}^{\infty}\frac{(a_1;q)_\ell\cdots(a_r;q)_\ell}{(q;q)_\ell(b_1;q)_\ell\cdots(b_s;q)_\ell}\left((-1)^\ell q^{\binom{\ell}{2}}\right)^{s-r+1}z^\ell\ ,$$

this can be written in the form

$$\det_{1\le s,t\le p}\left(q^{st}-q^{-st}\right)q^{-(H-p)\binom{p+1}{2}}\left(\prod_{s=1}^{H-p}\frac{(q^{p+s+1};q)_p}{(q^s;q)_p}\right){}_2\phi_1\left[\begin{matrix}q^{H-p},q^{-p}\\ q^{H+1}\end{matrix};q,-q^{p+1}\right].$$

The ${}_2\phi_1$-series in this expression can be summed with the help of the q-analogue of Kummer's summation (see [4, Appendix (II.9)]),

$${}_2\phi_1\left[\begin{matrix}a,b\\ aq/b\end{matrix};q,-q/b\right]=\frac{(-q;q)_\infty\,(aq;q^2)_\infty\,(aq^2/b^2;q^2)_\infty}{(-q/b;q)_\infty\,(aq/b;q)_\infty}\ .$$

Thus we have finally evaluated the sum in (3.15). From there, it is then routine to arrive at the expression (3.7). □

Clearly, if we specialize Theorem 6 to $a_i=a+2i-2$, $i=1,2,\dots,p$, we obtain the asymptotics for stars between two walls.

Corollary 7. *Let a be an integer with $0 \le a \le h-2p+2$. The number of stars with p branches of length m, the i-th branch starting from $A_i = (0, a+2i-2)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h$, is asymptotically*

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m \prod_{1\le s<t\le p}\sin^2\frac{\pi(t-s)}{(h+2)}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a+t+s-1)}{(h+2)}\cdot\sin\frac{\pi(t+s)}{h+2}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor (h+2)/2\rfloor)}{h+2}}{\sin\frac{\pi(t-s+p)}{h+2}}$$
$$\times\prod_{s=1}^{p}\frac{\sin\frac{(s+\lfloor (h+2)/2\rfloor-p)\pi}{h+2}}{\sin\frac{(2s+\lfloor (h+2)/2\rfloor-p)\pi}{h+2}}, \quad (3.20)$$

if $m+a$ is even, and

$$\frac{4^{p^2}}{(h+2)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2}\right)^m \prod_{1\le s<t\le p}\sin^2\frac{\pi(t-s)}{(h+2)}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a+t+s-1)}{(h+2)}\cdot\sin\frac{\pi(t+s)}{h+2}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor (h+1)/2\rfloor)}{h+2}}{\sin\frac{\pi(t-s+p)}{h+2}}, \quad (3.21)$$

if $m+a$ is odd.

A different specialization yields the asymptotics for ∞-friendly stars between two walls.

Corollary 8. *Let a be an integer with $0 \le a \le h-2p+2$. The number of ∞-friendly stars in the TK model with p branches of length m, the i-th branch starting from $A_i = (0, a+2i-2)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h$, is asymptotically*

$$\frac{4^{p^2}}{(h+2p)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2p}\right)^m \prod_{1\le s<t\le p}\sin\frac{2\pi(t-s)}{(h+2p)}\cdot\sin\frac{\pi(t-s)}{h+2p}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a+2t+2s-3)}{(h+2p)}\cdot\sin\frac{\pi(t+s)}{h+2p}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor (h+2p)/2\rfloor)}{h+2p}}{\sin\frac{\pi(t-s+p)}{h+2p}}$$
$$\times\prod_{s=1}^{p}\frac{\sin\frac{(s+\lfloor (h+2p)/2\rfloor-p)\pi}{h+2p}}{\sin\frac{(2s+\lfloor (h+2p)/2\rfloor-p)\pi}{h+2p}}, \quad (3.22)$$

if $m+a$ is even, and

$$\frac{4^{p^2}}{(h+2p)^p}\left(2^p\prod_{s=1}^{p}\cos\frac{s\pi}{h+2p}\right)^m \prod_{1\le s<t\le p}\sin\frac{2\pi(t-s)}{(h+2p)}\cdot\sin\frac{\pi(t-s)}{h+2p}$$
$$\times\prod_{1\le s\le t\le p}\sin\frac{\pi(a+2t+2s-3)}{(h+2p)}\cdot\sin\frac{\pi(t+s)}{h+2p}\prod_{s=0}^{p}\prod_{t=1}^{p}\frac{\sin\frac{\pi(t-s+\lfloor (h+1)/2\rfloor)}{h+2p}}{\sin\frac{\pi(t-s+p)}{h+2p}}, \quad (3.23)$$

if $m+a$ is odd.

Proof. As in the proof of equation (3.6), we shift the i-th branch up by $2(i-1)$ units. Thus we obtain a set of vicious walkers with p branches of length m, the i-th branch starting from $A_i = (0, a+4i-4)$, $i = 1, 2, \dots, p$, which do not go below the x-axis nor above the line $y = h + 2p - 2$. Hence, Theorem 6 with $a_i = a + 4i - 4$, $i = 1, 2, \dots, p$, and h replaced by $h + 2p - 2$ immediately gives the desired asymptotics. □

As we noted in the case of watermelons, the restriction to stars confined between two walls, in contrast to stars without restriction and with the restriction of *one* wall, the asymptotics of (ordinary) stars and ∞-friendly stars (compare the bases of the exponentials in Corollaries 7 and 8) is of a different order of magnitude (except in the case of single branch stars, of course). Hence, again, from these considerations, it is impossible to conclude whether n-friendly stars restricted by two walls will have the same order of magnitude as (ordinary) stars, or not.

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