A DETERMINANT EVALUATION AND SOME ENUMERATION RESULTS FOR PLANE PARTITIONS

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Abstract. For fixed integers α and β, planar arrays of integers of a given shape, in which the entries decrease at least by α along rows and at least by β along columns, are considered. The number of these (α, β)–plane partitions in which the first part and the last part in the i'th row are bounded by some integers a_i and b_i, respectively, can be given by a determinant. We give a new evaluation of this determinant for some special choices of the row bounds a_i and b_i.

1. Introduction. When enumerating plane partitions (we give all relevant definitions in §2 below), one is lead to certain determinantal expressions, which one succeeds to simplify in some special cases. The origins of this kind of work may be found in the work of MacMahon [10]. Since then, many authors have contributed important advances in the theory of plane partitions. Roughly speaking, there are three different kinds of enumeration problems. Firstly, counting plane partitions *of* a given shape subject to some restrictions (some references are [4,5,6,8,10,15]), secondly, counting plane partitions *contained* in a given shape subject to some restrictions, and, thirdly, counting *symmetry classes* of plane partitions.

In this paper we deal with the first type of enumeration problems. Recently, Gessel and Viennot [6] and the author [8] proved determinant formulas for various generating functions of rather general classes of plane partitions of a given shape. As mentioned above, there are several special cases where those determinants may be evaluated in order to result in closed expressions. Important examples are the hook length and hook–content formulae of Stanley [13] and Gansner [4,5].

It was discovered in [8] that a single, very simple determinant lemma [8, Lemma 2.2] serves to do all known evaluations for type one–problems. Besides, it could be applied to obtain some new evaluations. (The situation is quite different for the second and third type of enumeration problems, which are related in some sense. Here

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one has to encounter very intricate evaluations of Gaussian binomial determinants. The above cited determinant lemma does not apply in this case. But nevertheless there are beautiful results and beautiful conjectured results [1,2,3,7,9,11,12,13,14], see also the surveys in [16,17]).

Gessel's and Viennot's paper also contains three very nice, new evaluations of binomial deteminants [6, Theorems $21, 22, 23$]. It turned out that our deteminant lemma may be applied to derive these results, too. This was the motivation to write this paper. In Theorem 8 below, we are able to give a common generalization of Gessel's and Viennot's evaluations. This theorem concerns the enumeration of *reverse* plane partitions. There is also a counterpart for the enumeration of plane partitions (Theorem 5). It generalizes the well–known hook–content formula for the number of column–strict plane partitions, given in this form by Stanley [15, Theorem 15.3 for $x = 1$]. This fact becomes clear in Theorem 7. A special case (Corollary 10) of Theorem 5 gives the number of ordinary plane partitions of a (what we call) "staircase" shape with parts not exceeding a given integer. This result also seems to be new. Moreover, the "staircase" shape and the rectangular shape (up to now) are the only ones for which MacMahon's determinant ([10, sec.494], for a clearer presentation of this formula see [8, Theorem 6.1 for $\alpha = 0$, $\beta = 1$, $b_i = 1$, $a_i = a$, $\mu = 0$, and $x = 1$]) could be evaluated. (For rectangular shapes even the norm generating function has a closed expression ([10, sec.429], see also [15, Theorem 18.1]).)

Another unsolved problem is to give a closed expression for the number of row and column–strict plane partitions of a given shape with parts not exceeding a given integer. A solution for staircase shapes is given in Corollary 11, which also is an implication of Theorem 5.

2. Definitions and Assumptions. This section contains all relevant definitions. Besides, we recapitulate the determinant lemma (Lemma 4 below) and the enumeration results (Propositions 1,2 below) of [8] which we shall need in the sequel.

An r–tupel $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers satisfying $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ is called a *partition*. For a partition λ we write $|\lambda|$ for $\lambda_1 + \lambda_2 + \cdots + \lambda_r$.

Let α, β be arbitrary integers and λ a partition. We call an array π of integers of the form

$$\begin{array}{llllll} \pi_{11} & \pi_{12} & \dots\dots\dots\dots & & & \pi_{1,\lambda_1} \\ \pi_{21} & \pi_{22} & \dots\dots\dots & & \pi_{2,\lambda_2} & \\ \vdots & & & \iddots & & \\ \pi_{r1} & \dots & \pi_{r,\lambda_r} & & & \end{array} \tag{2.1}$$

an *(α, β)–plane partition of shape λ* if

$$\pi_{ij} \ge \pi_{i,j+1} + \alpha \qquad \text{for} \quad 1 \le i \le r,\ 0 < j < \lambda_i \tag{2.2(a)}$$

and

$$\pi_{ij} \ge \pi_{i+1,j} + \beta \qquad \text{for} \quad 1 \le i \le r,\ 0 < j \le \lambda_{i+1}\ . \tag{2.2(b)}$$

To give an example, the array in Figure 1 below is a $(0,2)$–plane partition of shape (5,3,3,1).

$$\begin{array}{rrrrr} 7 & 6 & 6 & 3 & 1 \\ 5 & 4 & 3 & & \\ 2 & 1 & 1 & & \\ -1 & & & & \end{array}$$

Figure 1

The entries of π are called *parts* of π. A $(0,0)$–plane partition containing only positive parts is an ordinary plane partition, a $(0,1)$–plane partition containing only positive parts is a column–strict plane partition, a $(1,0)$–plane partition is a row–strict plane partition, and a $(1,1)$–plane partition is a row and column–strict plane partition.

An (α,β)*–reverse plane partition of shape* λ is an array π of the form (2.1) satisfying

$$\pi_{ij} \le \pi_{i,j+1} + \alpha \qquad \text{for} \quad 1 \le i \le r,\ 0 < j < \lambda_i \tag{2.3)(a}$$

and

$$\pi_{ij} \le \pi_{i+1,j} + \beta \qquad \text{for} \quad 1 \le i \le r,\ 0 < j \le \lambda_{i+1}\ . \tag{2.3)(b}$$

If λ_i' is the number of parts in the i'th column of an array π of the form (2.1) (i.e. of shape λ), we call $\lambda' = (\lambda_1', \lambda_2', \dots, \lambda_{\lambda_1}')$ the *conjugate partition* to λ.

For a partition λ with $|\lambda| = p$ define the *hook lengths* of λ to be the integers $\lambda_i + \lambda_j' - i - j + 1$, for $1 \le i \le r$ and $1 \le j \le \lambda_i$. (With other words, only pairs (i,j) are allowed for which π_{ij} is a part in an array π of shape λ.) We write $h_1, h_2, \dots, h_p$ for the hook lengths of λ (in some order). Similarly, we define the *contents* of λ to be the integers $j - i$, for $1 \le i \le r$ and $1 \le j \le \lambda_i$. We write $c_1, c_2, \dots, c_p$ for the contents of λ (in some order). For instance, for the shape $(5,3,3,1)$ of the array in Figure 1, the hook lengths and contents are as follows:

$$\begin{array}{lllll} 8 & 6 & 5 & 2 & 1 \\ 5 & 3 & 2 & & \\ 4 & 2 & 1 & & \\ 1 & & & & \\ \multicolumn{5}{l}{\text{hook lengths}} \end{array} \qquad\qquad \begin{array}{rrrrr} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & & \\ -2 & -1 & 0 & & \\ -3 & & & & \\ \multicolumn{5}{c}{\text{contents}} \end{array}$$

We refer the reader to [15] for a detailed exposition of the terminology concerning plane partitions.

Next we come to enumeration results for (α,β)–plane partitions of shape λ subject to certain part restrictions. The first proposition is the special case $\mu = 0$, $x = 1$ of [8, Theorem 6.1].

Proposition 1. *Let λ be a partition and a, b be r–tupels of integers satisfying*

$$a_i + (1-\beta) \ge a_{i+1} \quad \text{and} \quad b_i + \alpha(\lambda_i - \lambda_{i+1}) + (1-\beta) \ge b_{i+1} \ . \tag{2.4}$$

The number of (α,β)–plane partitions of shape λ in which the first part in row i is at most a_i and the last part in row i is at least b_i is

$$\det_{1\le s,t\le r}\left(\binom{(1-\alpha)\lambda_s - \beta(s-t) + a_t - b_s + \alpha}{\lambda_s - s + t}\right) . \quad \square \tag{2.5}$$

The analogue for (α,β)–reverse plane partitions is the special case $\mu = 0$, $x = 1$ of [8, Corollary 6.4].

Proposition 2. *Let λ be a partition and a, b be r–tupels of integers satisfying*

$$a_i + \alpha(\lambda_{i-1} - \lambda_i) + (1-\beta) \ge a_{i-1} \quad \text{and} \quad b_i + (1-\beta) \ge b_{i-1} \ . \tag{2.6}$$

The number of (α,β)–reverse plane partitions of shape λ in which the first part in row i is at least b_i and the last part in row i is at most a_i is

$$\det_{1\le s,t\le r}\left(\binom{(1-\alpha)\lambda_s - \beta(s-t) + a_s - b_t + \alpha}{\lambda_s - s + t}\right) . \quad \square \tag{2.7}$$

We are interested in finding closed forms for these determinants. Of course, in general this will not be possible. Therefore, we have to restrict ourselves to suitable choices of the row bounds a_i and b_i. In [8] some work in this direction was done for the even more general problem of evaluating the determinants for the *generating functions* of those classes of (α,β)–plane partitions. In this paper we consider only determinant evaluations for the *number* of (α,β)–plane partitions, restricted by some special row bounds, because it does not seem to be possible to obtain closed expressions for the corresponding generating functions.

To get a flavour of how such results look like, we cite the celebrated hook–content formula for the number of column–strict plane partitions [15, Theorem 15.3 for $x = 1$].

Proposition 3. *Let λ be a partition with $|\lambda| = p$ and a a positive integer. The number of column–strict plane partitions of shape λ with parts $\le a$ is*

$$\frac{(a+c_1)(a+c_2)\cdots(a+c_p)}{h_1\cdot h_2\cdots h_p} , \tag{2.8}$$

where the h_i's and c_i's are the hook lengths and contents of λ_i, respectively. $\square$

The determinant which has to be evaluated to obtain (2.8) is the determinant in (2.5) for $\alpha = 0$, $\beta = 1$, $a_i = a$, $b_i = 1$. This evaluation may be performed using the following determinant lemma, which also serves for the evaluation which we give in §3.

Lemma 4. ([8, Lemma 2.2]) *Let $X_1,\dots,X_r,A_2,\dots,A_r,B_2,\dots,B_r$ be indeterminates. Then*

$$\det_{1\le s,t\le r}\Big((X_s+A_r)\cdots(X_s+A_{t+1})(X_s+B_t)\cdots(X_s+B_2)\Big)=$$
$$=\prod_{1\le i<j\le r}(X_i-X_j)\prod_{2\le i\le j\le r}(B_i-A_j)\ .$$

with the convention that empty products (like $(X_s+A_r)\cdots(X_s+A_{t+1})$ for $t=r$) are set equal to one. $\square$

3. The Determinant evaluation.

Theorem 5. *Let λ be a partition and B be a positive real number such that $b+(B-\alpha)\lambda_i+(1-\beta-B)i$ is an integer for $i=1,2,\dots,r$. The number of (α,β)–plane partitions π of shape λ in which the first part in row i is at most $a-\beta i$ (or equivalently $\pi_{11}\le a-\beta$) and the last part in row i is at least $b+(B-\alpha)\lambda_i+(1-\beta-B)i$ is*

$$\prod_{i=1}^{r}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!}\cdot\prod_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1}$$
$$\times\frac{\prod\limits_{1\le i<j\le r}(\lambda_i-i-\lambda_j+j)}{\prod\limits_{i=1}^{r}(\lambda_i-i+r)!}\,,\tag{3.1}$$

where the upper factorial $(A)_n$ is defined by $A(A+1)\cdots(A+n-1)$.

Proof. Because of $B\ge 0$, the row bounds satisfy (2.4). Hence we may apply Proposition 1 with $a_i=a-\beta i$ and $b_i=b+(B-\alpha)\lambda_i+(1-\beta-B)i$ to obtain for the desired number the expression

$$\det_{1\le s,t\le r}\left(\binom{(1-B)(\lambda_s-s)+a+\alpha-b}{\lambda_s-s+t}\right)\ .$$

Taking some factors out of the determinant leads to

$$\det_{1\le s,t\le r}\left(\binom{(1-B)(\lambda_s-s)+a+\alpha-b}{\lambda_s-s+t}\right)=$$
$$=\frac{\prod\limits_{i=1}^{r}((1-B)(\lambda_i-i)+a+\alpha-b)!}{\prod\limits_{i=1}^{r}((-B)(\lambda_i-i)+a-\alpha-b-1)!\,\prod\limits_{i=1}^{r}(\lambda_i-i+r)!}$$
$$\times\det_{1\le s,t\le r}\Big((\lambda_i-i+r)\cdots(\lambda_i-i+t+1)$$
$$\cdot((-B)(\lambda_i-i)+a+\alpha-b-t+1)\cdots((-B)(\lambda_i-i)+a+\alpha-b-1)\Big)\ .$$

If the determinant in this expression is rewritten in the form

$$(-B)^{\binom{r}{2}} \det_{1\le s,t\le r}\Big((\lambda_i - i+r)\cdots(\lambda_i-i+t+1)\cdot$$
$$\cdot(\lambda_i-i-\frac{a+\alpha-b-t+1}{B})\cdots(\lambda_i-i-\frac{a+\alpha-b-1}{B})\Big)\ ,$$

Lemma 4 with $X_i=\lambda_i-i$, $A_i=i$ and $B_i=-(a+\alpha-b-i+1)/B$ can be applied. Thus we get

$$\det_{1\le s,t\le r}\left(\binom{(1-B)(\lambda_s-s)+a+\alpha-b}{\lambda_s-s+t}\right)=$$
$$=\frac{\prod_{i=1}^{r}((1-B)(\lambda_i-i)+a+\alpha-b)!}{\prod_{i=1}^{r}((-B)(\lambda_i-i)+a-\alpha-b-1)!}\frac{\prod_{1\le i<j\le r}(\lambda_i-i-\lambda_j+j)}{\prod_{i=1}^{r}(\lambda_i-i+r)!}$$
$$\times\prod_{2\le i\le j\le r}\left(-\frac{a+\alpha-b-i+1}{B}-j\right)\cdot(-B)^{\binom{r}{2}}\ .$$

Finally, simple manipulation turns the right–hand side of this identity into the expression (3.1). □

There is also a "hook–content formulation" for this theorem. The hook lengths are the same, but we have to generalize the contents of a partition $\lambda=(\lambda_1,\ldots,\lambda_r)$ with $|\lambda|=p$. If (i,j) is a pair of integers such that $1\le i\le r$ and $1\le j\le\lambda_i$ we define a "generalized content" $d(i,j)$ as follows:

$$d(i,j)=\begin{cases}-B\lambda_i+(B-1)i+j & \text{if } i\le j\\ B\lambda'_j-i-(B-1)j+B & \text{if } i>j\ .\end{cases}\tag{3.2}$$

We write $d_1,d_2,\ldots,d_p$ for these generalized contents of λ (in some order). Obviously, for $B=0$, the d_i's are identical with the ordinary contents c_i.

For the proof of the hook–content form of Theorem 5 we need the following lemma:

Lemma 6. *Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ be a partition. Let s be the largest integer for which $\lambda_s-s\ge 0$. Then the r numbers*

$$r+1+\lambda_i-i\quad(s+1\le i\le r),\qquad r+j-\lambda'_j\quad(1\le j\le s)\tag{3.3}$$

are a permutation of $\{1,2,\ldots,r\}$.

Proof. First we have to show that these numbers actually are elements of $\{1,2,\ldots,r\}$. This is clear for $r+1+\lambda_i-i$, since for $s+1\le i\le r$ we assumed $\lambda_i-i<0$. Considering the numbers $r+j-\lambda'_j$ observe that $j-\lambda'_j$ is increasing for $j=1,2,\ldots,s$. But since $\lambda_s\ge s$, also by assumption, obviously also $\lambda'_s\ge s$ holds. Therefore we have $j-\lambda'_j\le s-\lambda'_s\le 0$ for $j=1,2,\ldots,s$.

The fact that the numbers in (3.3) are pairwise distinct is an easy consequence of [9, p.3, (1.7)] with $n=r+1$ and $m=r$. □

Theorem 7. *Let λ be a partition with $|\lambda| = p$ and B be a positive real number such that $b + (B-\alpha)\lambda_i + (1-\beta-B)i$ is an integer for $i = 1, 2, \dots, r$. The number of (α,β)–plane partitions π of shape λ in which the first part in row i is at most $a-\beta i$ (or equivalently $\pi_{11} \le a-\beta$) and the last part in row i is at least $b+(B-\alpha)\lambda_i+(1-\beta-B)i$ is*

$$\frac{(a+\alpha-b+d_1)\cdots(a+\alpha-b+d_p)}{h_1\cdots h_p}, \tag{3.3}$$

where the h_i's are the hook lengths of λ and the d_i's are the generalized contents of λ, which were introduced in (3.2).

Proof. It is a simple matter of fact that

$$\frac{\prod\limits_{1\le i<j\le r}(\lambda_i - i - \lambda_j + j)}{\prod\limits_{i=1}^{r}(\lambda_i - i + r)!} = \frac{1}{h_1\cdots h_p}$$

[9, p.9, Example 1(4)]. Leaves to transform the expression in the first line of (3.1) into the numerator of (3.3). Such as in Lemma 6, let s be the largest integer for which $\lambda_s - s \ge 0$. Breaking the left product of that expression into two factors, we are able to perform some cancellations:

$$\begin{aligned}
&\prod_{i=1}^{r}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!}\cdot\prod_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1} = \\
&\quad= \prod_{i=1}^{s}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!}\prod_{i=s+1}^{r}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!} \\
&\qquad\qquad \times\prod_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1} \\
&\quad= \prod_{i=1}^{s}(a+\alpha-b-B(\lambda_i-i))_{\lambda_i-i+1} \\
&\qquad\qquad \times\frac{\prod\limits_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1}}{\prod\limits_{i=s+1}^{r}(a+\alpha-b+(B-1)(i-\lambda_i)+1)_{i-\lambda_i-1}} \\
&\quad= \prod_{i=1}^{s}(a+\alpha-b-B(\lambda_i-i))_{\lambda_i-i+1}\prod_{\substack{j=1\\ j\ne k-\lambda_k\\ \text{for all } k}}^{r}(a+\alpha-b+(B-1)j+1)_{j-1}
\end{aligned}$$

The expression in the last line may be rewritten in the form

$$\prod_{1\le i\le j\le\lambda_i}(a+\alpha-b-B(\lambda_i-i)+j-i)\prod_{\substack{1\le j\le i<r\\ j\ne r+1+\lambda_k-k\\ \text{for all } k}}(a+\alpha-b+(B-1)(r+1-j)+r-i)\,.$$

Because of Lemma 6, the index j of the second product only attains the values $r+l-\lambda'_l$, $l=1,2,\dots,s$. Therefore the last expression is equal to

$$\prod_{1\le i\le j\le\lambda_i}(a+\alpha-b-B(\lambda_i-i)+j-i)\prod_{\substack{1\le j\le s\\ r+j-\lambda'_j\le i<r}}(a+\alpha-b+(B-1)(\lambda'_j-j+1)+r-i)\,.$$

Replacing i by $i+r-\lambda'_j-1$ in the second product, we obtain after some additional manipulation

$$\prod_{i=1}^{r}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!}\cdot\prod_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1}=$$
$$=\prod_{1\le i\le j\le\lambda_i}(a+\alpha-b-B\lambda_i+(B-1)i+j)\prod_{1\le j<i\le\lambda'_j}(a+\alpha-b+B\lambda'_j-i-(B-1)j+B)\,,$$

which is equivalent with our claim. $\square$

Proposition 3 is the special case $\alpha=0$, $\beta=1$, $B=0$, $b=1$, $a\to a+1$ of the above theorem.

The counterparts of Theorems 5 and 7 for (α,β)–reverse plane partitions read as follows:

Theorem 8. *Let λ be a partition with $|\lambda|=p$ and B be a positive real number such that $a-(B-\alpha)\lambda_i-(1-\beta-B)i$ is an integer for $i=1,2,\dots,r$. The number of (α,β)–reverse plane partitions π of shape λ in which the first part in row i is at least $b+\beta i$ (or equivalently $\pi_{11}\ge b+\beta$) and the last part in row i is at most $a-(B-\alpha)\lambda_i-(1-\beta-B)i$ is*

$$\prod_{i=1}^{r}\frac{(a+\alpha-b-(B-1)(\lambda_i-i))!}{(a+\alpha-b-1-B(\lambda_i-i))!}\cdot\prod_{i=1}^{r}(a+\alpha-b+(B-1)i+1)_{i-1}$$
$$\times\frac{\prod_{1\le i<j\le r}(\lambda_i-i-\lambda_j+j)}{\prod_{i=1}^{r}(\lambda_i-i+r)!}\,,\tag{3.5}$$

or, in terms of hooks and contents,

$$\frac{(a+\alpha-b+d_1)\cdots(a+\alpha-b+d_p)}{h_1\cdots h_p}\,,\tag{3.6}$$

where the h_i's are the hook lengths of λ and the d_i's are the generalized contents of λ, which were introduced in (3.2).

PROOF. Because of $B \geq 0$, the row bounds satisfy (2.6). Hence we may apply Proposition 2 with $a_i = a - (B-\alpha)\lambda_i - (1-\beta-B)i$ and $b_i = b + \beta i$ to obtain for the desired number the expression

$$\det_{1 \le s,t \le r} \left(\binom{(1-B)(\lambda_s - s) + a + \alpha - b}{\lambda_s - s + t} \right) .$$

But this expression has already been evaluated in the proof of Theorem 5. The hook–content formulation follows from the proof of Theorem 7. $\square$

Gessel's and Viennot's theorems, which were mentioned in the Introduction, are special cases of Theorem 8. To obtain [6, Theorem 21] we have to take $\alpha = 0$, $\beta = 1$, $B = 2$, $a = r - 1$ and $b = 0$. Indeed, our generalized contents for $B = 2$ differ from those given by Gessel and Viennot, but this is not an essential difference. If we consider the array $\big(d(i,j)\big)$ of shape λ $\quad (1 \le i \le r,\ 1 \le j \le \lambda_i)$, where $d(i,j)$ are the $B = 2$ –numbers of (3.2), then, to obtain Gessel's and Viennot's contents from ours, above the main diagonal we only have to reverse the order of elements in each row, and below we have to reverse the order of elements in each column. More precisely, for $i \ge j$ let $D(i,j) := d(i, \lambda_i + i - j)$, and for $i < j$ let $D(i,j) := d(\lambda'_j + j - i + 1, j)$. Then $D(i,j)$ is identical with Gessel's and Viennot's definition of contents (save to a constant, which is simply 1). We have chosen the form (3.2) for our definition, because for $B = 0$ our $d(i,j)$'s reduce to ordinary contents.

Setting $\alpha = 1$, $\beta = 0$, $B = 2$, $b = 0$, $a = 2k$ or $2k - 1$ in (3.5), respectively, we get Theorems 22 and 23 of [6], though in "conjugate" formulation. We remark that Gessel and Viennot used a different determinant to obtain these theorems. Not surprisingly, this determinant can also be computed by Lemma 4. The hook–content formula (3.6) in this special case is also new.

Finally we turn our attention to (α,β)–plane partitions of some particular shape, which we name "staircase" shape. Let L be a nonnegative integer and $\lambda_i = \Lambda - Li$, $i = 1, 2, \ldots, r$, Λ sufficiently large, such that λ_i is positive for $i = 1, 2, \ldots, r$. The shape λ given by the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) = (\Lambda - L, \Lambda - 2L, \ldots, \Lambda - Lr)$ we call a *staircase shape*. The reason is that a typical staircase shape looks like a reversed staircase:

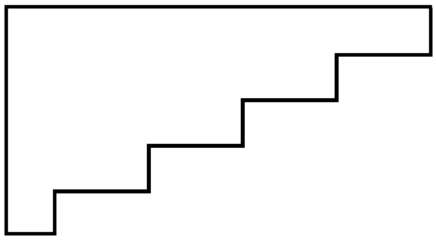

Figure 2

(Here we chose $\Lambda = 11$, $L = 2$, and $r = 5$.)

In Theorem 5 for $\lambda = (\Lambda - L, \ldots, \Lambda - Lr)$ take $B = \frac{1-\beta+L\alpha}{L+1}$ and $b = 1 - \frac{1-\alpha-\beta}{L+1}\Lambda$ in order to get $b_i = 1$ for all i. If we also replace a by $a + \beta$ we obtain

Corollary 9. *Let λ be a staircase shape, $\lambda = (\Lambda - L, \Lambda - 2L, \dots, \Lambda - Lr)$, with L satisfying $Lr < \Lambda$ and $L\alpha - \beta + 1 \geq 0$. The number of (α,β)–plane partitions π of shape λ in which $\pi_{11} \leq a$ and the last part in each row is positive is*

$$\prod_{i=1}^{r} \frac{(a+\alpha+\beta-1+\Lambda(1-\alpha)-(\beta-L\alpha+L)i)!}{(a+\alpha+\beta-2-\alpha\Lambda-(\beta-L\alpha-1)i)!} \times \tag{3.7}$$
$$\times \prod_{i=1}^{r} \frac{\left(a+\alpha+\beta+\frac{1-\alpha-\beta}{L+1}\Lambda-\frac{\beta-L\alpha+L}{L+1}i\right)_{i-1}}{(r+\Lambda-i(L+1))!} \cdot (L+1)^{\binom{r}{2}} 2^{r-2} 3^{r-3} \cdots (r-1)^1 \ . \quad \square$$

For $\alpha = \beta = 0$ this result gives the number of plane partitions of staircase shape with parts $\leq a$.

Corollary 10. *Let λ be a staircase shape, $\lambda = (\Lambda - L, \Lambda - 2L, \dots, \Lambda - Lr)$, with L satisfying $Lr < \Lambda$. The number of plane partitions of shape λ with largest part $\leq a$ is*

$$\prod_{i=1}^{r} \frac{(a+\Lambda-Li-1)!}{(a+i-2)!} \prod_{i=1}^{r} \frac{\left(a+\frac{\Lambda-Li}{L+1}\right)_{i-1}}{(r+\Lambda-i(L+1))!} \cdot (L+1)^{\binom{r}{2}} 2^{r-2} 3^{r-3} \cdots (r-1)^1 \ . \quad \square \tag{3.8}$$

Another interesting special case of Corollary 9 is the case $\alpha = \beta = 1$.

Corollary 11. *Let λ be a staircase shape, $\lambda = (\Lambda - L, \Lambda - 2L, \dots, \Lambda - Lr)$, with L satisfying $Lr < \Lambda$. The number of row and column–strict plane partitions of shape λ with largest part $\leq a$ is*

$$\prod_{i=1}^{r} \frac{(a+1-i)!}{(a-\Lambda+Li)!} \prod_{i=1}^{r} \frac{\left(a+2-\frac{\Lambda+i}{L+1}\right)_{i-1}}{(r+\Lambda-i(L+1))!} \cdot (L+1)^{\binom{r}{2}} 2^{r-2} 3^{r-3} \cdots (r-1)^1 \ . \quad \square \tag{3.9}$$

Clearly, there are also analogues of these corollaries for reverse plane partitions. We omit them here for sake of brevity.

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