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On the q-log-Concavity of Gaussian Binomial Coefficients

By

Christian Krattenthaler, Wien

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Abstract. We give a combinatorial proof that $\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - \begin{bmatrix} a \\ k-1 \end{bmatrix}_q \begin{bmatrix} b \\ l+1 \end{bmatrix}_q$ is a polynomial in q with nonnegative coefficients for nonnegative integers a, b, k, l with $a \geqslant b$ and $l \geqslant k$. In particular, for $a = b = n$ and $l = k$, this implies the q-log-concavity of the Gaussian binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which was conjectured by Butler (Proc. Amer. Math. Soc. 101 (1987), 771—775).

1. Introduction

A sequence $(p_k(q))_{k \in \mathbb{Z}}$ of polynomials $p_k(q)$ in q is called q-log-concave if $p_k(q)^2 - p_{k-1}(q)\, p_{k+1}(q)$ is a polynomial with nonnegative coefficients. In a recent paper Butler [2] conjectured that the rank numbers of the lattice of subgroups of a finite abelian q-group are q-log-concave. Even in the special case of the q-group being of type $\lambda = (1^n)$ this conjecture was not settled. ((1^n) is the partition consisting of n parts equal to 1.) Here we have to prove that the Gaussian binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},$$

where $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$, are q-log-concave. That means we have to show that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^2 - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$$

is a polynomial in q with nonnegative coefficients.

The first proof of q-log-concavity of Gaussian binomial coefficients was found by Butler herself [3]. When being confronted with Butler's (combinatorial) proof, SAGAN [5] supplied an inductive proof by extending his work begun in [4]. Being unaware of both, we devised an alternative combinatorial proof, which we present in this paper. Moreover, we prove the following stronger theorem:

Theorem 1. *Let a, b, k, l be nonnegative integers with $a \geqslant b$ and $l \geqslant k$. Then*

$$\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - \begin{bmatrix} a \\ k-1 \end{bmatrix}_q \begin{bmatrix} b \\ l+1 \end{bmatrix}_q \tag{1.1}$$

is a polynomial in q with nonnegative coefficients.

Once having proved Theorem 1 (which is done in section 2), the special case $a = b = n$, $l = k$ furnishes q-log-concavity of Gaussian binomial coefficients. Besides, writing

$$\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - \begin{bmatrix} a \\ k-r \end{bmatrix}_q \begin{bmatrix} b \\ l+r \end{bmatrix}_q =$$

$$= \sum_{i=0}^{r-1} \left(\begin{bmatrix} a \\ k-i \end{bmatrix}_q \begin{bmatrix} b \\ l+i \end{bmatrix}_q - \begin{bmatrix} a \\ k-i-1 \end{bmatrix}_q \begin{bmatrix} b \\ l+i+1 \end{bmatrix}_q \right)$$

we obtain from Theorem 1:

Corollary 2. *Let a, b, k, l, r be nonnegative integers with $a \geqslant b$ and $l \geqslant k$. Then*

$$\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - \begin{bmatrix} a \\ k-r \end{bmatrix}_q \begin{bmatrix} b \\ l+r \end{bmatrix}_q \tag{1.2}$$

is a polynomial in q with nonnegative coefficients.

As is well-known (cf. e.g. [1, p. 48]) the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a symmetric, unimodal polynomial with degree $k(n-k)$. The product of symmetric, unimodal polynomials again is symmetric and unimodal [1, Theorem 3.9], hence both expressions in (1.2) are symmetric and unimodal, and the degree of the first expression exceeds the degree of the second by $2r(l-k+r) + r(a-b)$. Therefore we may generalize Corollary 2 to

Corollary 3. *Let a, b, k, l, r, s be nonnegative integers with $a \geqslant b$, $l \geqslant k$ and $s \leqslant 2r(l-k+r)+r(a-b)$. Then*

$$\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - q^s \begin{bmatrix} a \\ k-r \end{bmatrix}_q \begin{bmatrix} b \\ l+r \end{bmatrix}_q \tag{1.3}$$

is a polynomial in q with nonnegative coefficients.

Both, BUTLER's [3] and SAGAN's [5] papers, contain the case $a = b$ of Corollary 3, and besides, the discussion of related problems.

2. Proof of Theorem 1

Since the cases $k = 0$ and $l \geqslant b$ are trivial we may concentrate on $0 < k \leqslant l < b$.

Let $P_k(n)$ denote the set of k-element subsets of $\{1, 2, \ldots, n\}$. For $S \in P_k(n)$ we write $\| S \|$ for the sum of all the elements of S. Then it is well-known (this is seen e.g. equating coefficients in [1, (3.3.6)]) that

$$\sum_{S \in P_k(n)} q^{\| S \|} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{2.1}$$

For pairs of integer subsets (C, D) let $\|(C, D)\| = \|C\| + \|D\|$.

We are going to construct an injection φ from $P_{k-1}(a) \times P_{l+1}(b)$ into $P_k(a) \times P_l(b)$ with weight property

$$\|\varphi((A, B))\| = \|(A, B)\| - (l - k + 1) \tag{2.2}$$

for $(A, B) \in P_{k-1}(a) \times P_{l+1}(b)$. Let us write $F(\mathscr{M})$ for the generating function $\sum q^{\|(C,D)\|}$ where the sum is over all $(C, D) \in \mathscr{M}$. Suppose φ given, then (2.1) would imply

$$\begin{aligned} q^{\binom{k}{2}+\binom{l+2}{2}} &\left(\begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b \\ l \end{bmatrix}_q - \begin{bmatrix} a \\ k-1 \end{bmatrix}_q \begin{bmatrix} b \\ l+1 \end{bmatrix}_q \right) = \\ &= q^{l-k+1} F(P_k(a) \times P_l(b)) - F(P_{k-1}(a) \times P_{l+1}(b)) = \\ &= q^{l-k+1} F(P_k(a) \times P_l(b) \setminus \mathrm{im}_\varphi P_{k-1}(a) \times P_{l+1}(b)) + \\ &\quad + q^{l-k+1} F(\mathrm{im}_\varphi P_{k-1}(a) \times P_{l+1}(b)) - F(P_{k-1}(a) \times P_{l+1}(b)), \end{aligned} \tag{2.3}$$

where $\mathrm{im}_\varphi P_{k-1}(a) \times P_{l+1}(b)$ is the image of $P_{k-1}(a) \times P_{l+1}(b)$ under application of φ. But, by (2.2) the expressions in the last line of (2.3) cancel, hence

$$q^{\binom{k}{2}+\binom{l+2}{2}}\left(\begin{bmatrix}a\\k\end{bmatrix}_q\begin{bmatrix}b\\l\end{bmatrix}_q-\begin{bmatrix}a\\k-1\end{bmatrix}_q\begin{bmatrix}b\\l+1\end{bmatrix}_q\right)=$$

$$= q^{l-k+1}F(P_k(a)\times P_l(b)\setminus \mathrm{im}_\varphi P_{k-1}(a)\times P_{l+1}(b)), \tag{2.4}$$

which proves that (1.1) is a polynomial with nonnegative coefficients.

Leaves to construct the injection φ. First, for a given pair (C, D) of integer subsets, we introduce an integer-valued function $k_{(C,D)}$ acting on positive integers by

$$k_{(C,D)}(m) = \sum_{i=1}^{m}\chi(i\in C) - \sum_{i=1}^{m}\chi(i\in D), \tag{2.5}$$

where $\chi(\mathscr{A}) = 1$ if $\mathscr{A}$ is true, $\chi(\mathscr{A}) = 0$ otherwise.

Suppose $(A, B)\in P_{k-1}(a)\times P_{l+1}(b)$. The simplest way, one can imagine, in obtaining an injection satisfying (2.2) is removing an element from B, subtracting $(l-k+1)$ from it and putting it into the set A. Of course, there are some difficulties. First of all, we have to take an element, say e, of B which is larger than $(l-k+1)$. Secondly, $e-(l-k+1)$ must not occur in A. And last, but really not least, this has to become an injection. But nevertheless, we succeed in finding such a procedure, which we introduce in three steps.

Step 1. Add $(l-k+1)$ to each element of A, thus obtaining the pair (A_1, B_1), where $B_1 = B$.

Step 2. Let $L(A, B)$ be the largest integer greater than $(l-k)$ where $k_{(A_1,B_1)}$ reaches its largest value. To be precise, for an integer n, $n > l-k$, we have

$$n > L(A,B) \text{ implies } k_{(A_1,B_1)}(n) < k_{(A_1,B_1)}(L(A,B)),$$

$$n \leqslant L(A,B) \text{ implies } k_{(A_1,B_1)}(n) \leqslant k_{(A_1,B_1)}(L(A,B)).$$

Obviously, by definition of $L(A,B)$, we have $(L(A,B)+1)\in B_1\setminus A_1$. Removing $(L(A,B)+1)$ from B_1 and putting it into A_1, we get the pair (A_2, B_2) where $A_2 = A_1\cup\{L(A,B)+1\}$ and $B_2 = B_1\setminus\{L(A,B)+1\}$.

Step 3. Subtract $(l-k+1)$ from each element of A_2.

To give an example take $a = 11$, $b = 10$, $l = 6$, $k = 5$ and $(A,B) = (\{2,4,6,11\}, \{1,2,4,5,7,8,10\})$. Performing Step 1 we obtain $(A_1,B_1) = (\{4,6,8,13\}, \{1,2,4,5,7,8,10\})$. Here the largest integer greater than $l-k=1$ where $k_{(A_1,B_1)}$ reaches its largest value is $L(A,B)=6$ (since $k_{(A_1,B_1)}(i)=-2$ for $i=2,3,4,6$, $k_{(A_1,B_1)}(i)=$ $=-3$ for $i=5,7,8,9,13$ and $k_{(A_1,B_1)}(i)=-4$ for $i=10,11,12$). Therefore $(A_2,B_2)=(\{4,6,7,8,13\},\ \{1,2,4,5,8,10\})$. Finally, by Step 3, we obtain $\varphi(\{2,4,6,11\},\{1,2,4,5,7,8,10\})=(\{2,4,5,6,11\},$ $\{1,2,4,5,8,10\})$.

Because of

$$k_{(A_1,B_1)}(l-k+1) \geqslant -(l-k+1) > k-l-2 = \\ = k_{(A_1,B_1)}(a+l-k+1),$$

the largest value of $k_{(A_1,B_1)}$ is attained for integers being smaller than $(a+l-k+1)$ only. This shows the existence of $L(A,B)$ for all $(A,B)\in P_{k-1}(a)\times P_{l+1}(b)$. Therefore Step 2 always can be performed.

Obviously φ, by definition, satisfies (2.2).

In order to show that φ is injective, we claim that the image of φ is given by the set of all pairs $(C,D)\in P_k(a)\times P_l(b)$ which satisfy the following condition:

(C) There exists an integer j, $l-k+1<j\leqslant b$ with

$$k_{(C_1,D_1)}(j) > -\sum_{i=1}^{l-k+1} \chi(i\in D_1),$$

where (C_1,D_1) comes out of (C,D) by adding $(l-k+1)$ to each element of C, and $D_1=D$.

In our preceding example, $a=11$, $b=10$, $l=6$, $k=5$, $(C,D)=$ $=(\{2,4,5,6,11\},\{1,2,4,5,8,10\})$ we have $(C_1,D_1)=(\{4,6,7,8,13\},$ $\{1,2,4,5,8,10\})$. Indeed $k_{(C_1,D_1)}(7)=-1>-2=\sum_{i=1}^{2}\chi(i\in D_1)$.

Our claim may be settled by establishing the inverse map $\bar{\varphi}$ of φ. Again, this is done in three steps. Consider $(C,D)\in P_k(a)\times P_l(b)$ which satisfies condition (C).

Step 1. Add $(l-k+1)$ to each element of C, thus obtaining the pair (C_1,D_1), where $D_1=D$.

Step 2. Let $S(C,D)$ be the smallest integer with $l-k<$ $<S(C,D)\leqslant b$ where $k_{(C_1,D_1)}$ reaches its largest value. To be precise, for all integers n, with $l-k<n\leqslant b$, we have

$$n < S(C,D) \text{ implies } k_{(C_1,D_1)}(n) < k_{(C_1,D_1)}(S(C,D)),$$
$$n \geqslant S(C,D) \text{ implies } k_{(C_1,D_1)}(n) \leqslant k_{(C_1,D_1)}(S(C,D)).$$

Removing $S(C,D)$ from C_1 and putting it into D_1, we get (C_3, D_3) where $C_3 = C_1 \setminus \{S(C,D)\}$ and $D_3 = D_1 \cup \{S(C,D)\}$.

Step 3. Subtract $(l-k+1)$ from each element of C_3.

In our example, $a = 11$, $b = 10$, $l = 6$, $k = 5$, $(C,D) = = (\{2,4,5,6,11\}, \{1,2,4,5,8,10\})$, we get $(C_1, D_1) = (\{4,6,7,8,13\}, \{1,2,4,5,8,10\})$, $S(C,D) = 7$ (since $k_{(C_1,D_1)}(i) = -1$ for $i = 7,8,9$, $k_{(C_1,D_1)}(i) = -2$ for $i = 2,3,4,6,10$ and $k_{(C_1,D_1)}(5) = -3$) and therefore, by Step 2 and 3,

$$\bar{\varphi}(\{2,4,5,6,11\}, \{1,2,4,5,8,10\}) = (\{2,4,6,11\}, \{1,2,4,5,7,8,10\}).$$

Step 2, for pairs (C,D) satisfying condition (C), always can be performed, since (C) and the definition of $S(C,D)$ guarantee $S(C,D) \in C_1 \setminus D_1$. Hence $\bar{\varphi}((C,D)) \in P_{k-1}(a) \times P_{l+1}(b)$ for $(C,D) \in \in P_k(a) \times P_l(b)$ and satisfying (C).

Given $(A,B) \in P_{k-1}(a) \times P_{l+1}(b)$ we have to prove that $\varphi((A,B))$ satisfies (C). Because of

$$k_{(A_1,B_1)}(l-k+1) = -\sum_{i=1}^{l-k+1} \chi(i \in B_1),$$

we must have

$$\begin{aligned} k_{(A_2,B_2)}(L(A,B)+1) &= k_{(A_1,B_1)}(L(A,B)+1) + 2 = \\ &= k_{(A_1,B_1)}(L(A,B)) + 1 > -\sum_{i=1}^{l-k+1} \chi(i \in B_1), \end{aligned}$$

the inequality being true because of maximality of $k_{(A_1,B_1)}(L(A,B))$. But, since $L(A,B) + 1 > l - k + 1$ this implies that $\varphi((A,B))$ satisfies condition (C).

$\bar{\varphi} \circ \varphi = \mathrm{id}$ is shown by observing $S(\varphi((A,B))) = L(A,B) + 1$. The details are left to the reader.

Summarizing, we have φ which maps elements of $P_{k-1}(a) \times P_{l+1}(b)$ into elements of $P_k(a) \times P_l(b)$ which satisfy condition (C). We have $\bar{\varphi}$ mapping elements of $P_k(a) \times P_l(b)$ satisfying (C) into elements of $P_{k-1}(a) \times P_{l+1}(b)$, and $\bar{\varphi} \circ \varphi = \mathrm{id}$. Hence, φ is a bijection between $P_{k-1}(a) \times P_{l+1}(b)$ and $\mathrm{im}_\varphi P_{k-1}(a) \times P_{l+1}(b)$, and therefore, in particular, injective, which completes the proof of the Theorem.

References

[1] ANDREWS, G. E.: The Theory of Partitions. Encyclopedia of Mathematics and its Applications, Vol 2. Reading, Mass.: Addison-Wesley. 1976.

[2] BUTLER, L. M.: A unimodality result in the enumeration of subgroups of a finite abelian group. Proc. Amer. Math. Soc. **101**, 771—775 (1987).

[3] BUTLER, L. M.: The q-log concavity of q-binomial coefficients. J. Combin. Theory A. To appear.

[4] SAGAN, B. E.: Inductive and injective proofs of log concavity results. Discrete Math. **68**, 281—292 (1988).

[5] SAGAN, B. E.: Inductive proofs of q-log concavity. Preprint.

CH. KRATTENTHALER
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien, Austria