AN ENTRY OF RAMANUJAN ON HYPERGEOMETRIC SERIES IN HIS NOTEBOOKS

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Abstract. Example 7, after Entry 43, in Chapter XII of the first Notebook of Srinivasa Ramanujan is proved and, more generally, a summation theorem for ${}_3F_2(a, a, x; 1 + a, 1 + a + N; 1)$, where N is a non-negative integer, is derived.

1. Introduction

In the Notebooks of Ramanujan, identities for hypergeometric series occupy a prominent part (see [3, Chapters X, XI]). In fact, Ramanujan discovered for himself all the (now classical) summation theorems established by Gauß, Chu-Vandermonde, Kummer, Pfaff–Saalschütz, Dixon and Dougall. For example, Dougall's summation theorem (of which all the others mentioned are special cases), which was discovered by Dougall [6] in 1907, was independently found by Ramanujan during 1910–1912 (see Entry 1 in Chapter XII of Notebook 1 and in the corresponding Entry 1 in Chapter X of Notebook 2). We cannot assert an exact date, because there are no dates anywhere in the Notebooks of Ramanujan. In any case, Ramanujan rediscovered not only all that was known in Europe on hypergeometric series at that time, but he also discovered several new theorems, and, in particular, theorems on products of hypergeometric series [12],

2000 *Mathematics Subject Classification.* Primary 33C20; Secondary 33C05, 33B15.

Key words and phrases. Hypergeometric functions, gamma function, digamma function, Ramanujan's Notebooks, summation theorems.

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as well as several types of asymptotic expansions.

On the other hand, it is well known [3, 5] that Ramanujan did not publish any of his results from his Chapters on hypergeometric series that are to be found in his Notebooks. The Chapters on hypergeometric series, in particular, Chapters X and XI, in his *second* Notebook, have been extensively studied by Hardy [9] and Berndt [3], since the Chapters of Ramanujan's second Notebook are considered as *revised, enlarged* versions of the Chapters in his *first* Notebook. However, Example 7, after Entry 43, in Chapter XII of the first Notebook did not find a place in the "corresponding" examples after Entry 10 in Chapter X of his second Notebook, and is therefore not discussed in [3].[1]

The purpose of this article is to provide a proof for Example 7, after Entry 43, in Chapter XII, (XII, 43, Ex. 7), in the first Notebook of Ramanujan, using well-known transformation and summation theorems of hypergeometric series. In fact, we prove a more general identity (which may have been the identity that Ramanujan actually had, and from which he recorded the most elegant specializations into his Notebook — a conjecture consistent with a wide spread belief that that was Ramanujan's style).

Our paper is organized as follows: In Section 2, Entry (XII, 43, Ex. 7) in the first Notebook of Ramanujan and some related entries are presented in Ramanujan's notation, and subsequently translated into current notation. This leads us to the statement of a summation theorem for a ${}_3F_2(x, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; 1)$ hypergeometric series, for $\Re x < 2$. In Section 3, the statement is generalized to a summation theorem for the ${}_3F_2(a, a, x; 1+a, 1+a+N; 1)$ series, where N is a non-negative integer, and it is proved. Finally, in Section 4, some remarks regarding special cases of the theorem are made.

2. Ramanujan's Entry

In Ramanujan's notation, Example 7, after Entry 43, in Chapter XII, of the first Notebook reads as follows:

$$\frac{\pi}{\tan(\pi x)} \frac{\underline{|2x}}{\left(2^x \underline{|x}\right)^2 (1-2x)} \left(\sum \frac{1}{2x} - \frac{1}{2} \sum \frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2} \tan(\pi x) \right)$$
$$= \frac{1}{1^2} + \frac{x}{\underline{|1}} \cdot \frac{1}{3^2} + \frac{x(x+1)}{\underline{|2}} \cdot \frac{1}{5^2} + \&c. \qquad \text{(XII, 43, Ex.7)}$$

[1]It is, however, mentioned in [4, p. 410] as "incorrect." Although this is, in the literal sense of the word, true, it is not the whole truth, as we are going to show in this paper. (Ramanujan missed only a multiplicative factor of x.) In [4, p. 410], it is also mentioned that "A correct formula can be derived by replacing x by $-x$ and setting $n = 1/2$ in Example 5 of Section 10 of Chapter 10 of Ramanujan's second Notebook (see [3, p. 26])." (This may have been the reason that Ramnujan did not include Example 7, after Entry 43, in Chapter XII of the first Notebook in the second Notebook, while all the other examples after Entry 43 found their way into the second Notebook.) We are going to derive an equivalent statement, and, in fact, an identity that is more general than Example 5 of Section 10 of Chapter 10 of Ramanujan's second Notebook. Our proof is independent of the proof of that Example in [3, p. 26].

Here, $\lfloor\underline{x}$ is Ramanujan's notation for the *gamma function* $\Gamma(x+1)$, which, for him, was a function over real numbers x (see [11]). We, of course, adopt the contemporary point of view and regard the gamma function as a function over the complex numbers.

The factor on the left-hand side of (XII, 43, Ex. 7):

$$\frac{\pi}{\tan(\pi x)} \frac{\lfloor\underline{2x}}{\left(2^x \lfloor\underline{x}\right)^2 (1-2x)} \tag{1}$$

is the same factor that appears on the left-hand side of (XII, 43, Ex. 4) which, in Ramanujan's first Notebook is

$$\frac{\pi}{\tan(\pi x)} \frac{\lfloor\underline{2x}}{\left(2^x \lfloor\underline{x}\right)^2 (1-2x)} = 1 + \frac{x}{\lfloor\underline{1}} \cdot \frac{1}{3} + \frac{x(x+1)}{\lfloor\underline{2}} \cdot \frac{1}{5} + \&c. \qquad \text{(XII, 43, Ex.4)}$$

This is given in Ramanujan's second Notebook as:

$$\frac{\sqrt{\pi}\lfloor\underline{n}}{2\lfloor\underline{n+\frac{1}{2}}} = 1 - \frac{n}{\lfloor\underline{1}} \cdot \frac{1}{3} + \frac{n(n-1)}{\lfloor\underline{2}} \cdot \frac{1}{5} + \&c. \qquad \text{(X, 10, Ex.4)}$$

As pointed out by Berndt [3], the Entry (X, 10, Ex. 4), or, (XII, 43, Ex. 4), is the special case of (X, 10), or, (XII, 43), where we do the replacements $n \to \frac{1}{2}$ and $x \to n$. The factor (1) which occurs on the left-hand side of (XII, 43, Ex. 4) can be shown to be equal to:

$$\frac{\sqrt{\pi}\,\Gamma(1-x)}{2x\,\Gamma(\frac{3}{2}-x)} = \frac{\sqrt{\pi}\lfloor\underline{-x}}{2x\lfloor\underline{-x+\frac{1}{2}}}\ , \tag{2}$$

after using the reflection formula: $\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z)$, and the duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$ and some algebraic manipulations.

When written in the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k} z^k\ ,$$

where the Pochhammer symbol $(\alpha)_k$ is defined by $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$, $k > 0$, $(\alpha)_0 = 1$, the series on the right-hand side of (XII, 43, Ex. 4) is:

$${}_2F_1\left[\begin{matrix} \frac{1}{2}, x \\ \frac{3}{2} \end{matrix}; 1\right],$$

which by the Gauß summation theorem (see [12, (1.7.6); Appendix (III.3)]):

$${}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; 1\right] = \frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)} \quad \text{valid for } \Re(C-A-B) > 0, \tag{3}$$

becomes:

$${}_2F_1\left[\begin{matrix} \frac{1}{2}, x \\ \frac{3}{2} \end{matrix}; 1\right] = \frac{\sqrt{\pi}\,\Gamma(1-x)}{2\,\Gamma(3/2-x)} = \frac{\sqrt{\pi}}{2} \frac{\lfloor\underline{-x}}{\lfloor\underline{-x+1/2}}. \tag{4}$$

If we compare the right-hand side of this equation with (2), which, as we outlined, is equal to the left-hand side (1) of (XII, 43, Ex. 4), it is clear that Ramanujan missed a multiplicative factor x on the left-hand side of (XII, 43, Ex. 4), while his Entry (X, 10, Ex. 4) is correct. As we shall see, the same applies to the Entry (XII, 43, Ex. 7). To be precise, the factor (1) on the left-hand side of that entry must also be replaced by the correct value (4).

The series on the right-hand side of (XII, 43, Ex. 7) is, in hypergeometric notation:

$$ {}_3F_2\left[\begin{matrix} x, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}; 1\right]. $$

There are two factors on the left-hand side of (XII, 43, Ex. 7). Besides the factor (1), the other factor in (XII, 43, Ex. 7) is:

$$ \sum \frac{1}{2x} - \frac{1}{2} \sum \frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2} \tan(\pi x). \tag{5} $$

Ramanujan used the notation $\sum \frac{1}{x}$ to indicate the extension of the function

$$ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, $$

representing the harmonic numbers, from positive integers n to real x. In other words, $\sum \frac{1}{x}$ is Ramanujan's notation for the *digamma function* $\psi(x) := \Gamma'(x)/\Gamma(x)$, the logarithmic derivative of the gamma function, or, more precisely, for $\psi(x+1) + \gamma$, where γ is the Euler–Mascheroni constant, that is

$$ \psi(x+1) = \sum \frac{1}{x} - \gamma. \tag{6} $$

In fact, in Ramanujan's very first research paper [10], the digamma function occurs as:

$$ \frac{d}{dn} \log \Gamma(n+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \gamma = \sum \frac{1}{n} - \gamma. $$

Thus, the expression (5) becomes:

$$ \psi(2x+1) - \frac{1}{2}\psi(x+1) + \frac{1}{1-2x} - \frac{\pi}{2} \tan(\pi x) + \frac{\gamma}{2}\,. \tag{7} $$

The digamma function satisfies the recurrence relation [7, 1.7.1(8)]:

$$ \psi(z+1) = \psi(z) + \frac{1}{z}, \tag{8} $$

the reflection formula [7, 1.7.1(11)]:

$$ \psi(-z) = \psi(z+1) + \pi \cot(\pi z) \tag{9} $$

and the duplication formula [7, 1.7.1(12)]:

$$ 2\psi(2z) = \psi(z) + \psi\Big(z + \frac{1}{2}\Big) + 2\log 2. \tag{10} $$

We remark that the duplication formula appears implicitly in Ramanujan's first Notebook. Namely, a comparison of Entry (XII, 43, Ex. 6) with $n \to x-1$ in Entry (X, 10, Ex. 6) shows that Ramanujan obtained:

$$ \sum \frac{1}{x+\frac{1}{2}} - \sum \frac{1}{x} = 2\sum \frac{1}{2x} - 2\sum \frac{1}{x} + \frac{1}{x} - 2\log 2, $$

which by (6) and (8) is equivalent to (10). Furthermore, the digamma function has the special values

$$\psi\Big(\frac{1}{2}\Big) = -\gamma - 2\log 2 \qquad \text{and} \qquad \psi(1) = -\gamma. \tag{11}$$

We now use the duplication formula (10) to convert (7) into

$$\frac{1}{2}\psi\Big(x+\frac{1}{2}\Big) + \log 2 + \frac{1}{1-2x} - \frac{\pi}{2}\tan(\pi x) + \frac{\gamma}{2}.$$

The recurrence relation (8) implies that $\psi(x+\frac{1}{2}) = \psi(x-\frac{1}{2}) + \frac{2}{2x-1}$, while we know from (11) that $\log 2 = -\frac{1}{2}\psi(\frac{1}{2}) - \frac{1}{2}\gamma$. If this is substituted in the last equation, and if we then apply the reflection formula (9), we obtain

$$\frac{1}{2}\left(\psi\Big(\frac{3}{2}-x\Big) - \psi\Big(\frac{1}{2}\Big)\right) \tag{12}$$

for the expression (5). Therefore, if we recall that the factor (1) on the left-hand side of Ramanujan's Entry (XII, 43, Ex. 7) must be replaced by (4), this entry can be rewritten in contemporary notation as:

$$\begin{aligned} {}_3F_2\left[\begin{matrix} x, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}; 1\right] &= \frac{\sqrt{\pi}}{4}\frac{\Gamma(1-x)}{\Gamma(\frac{3}{2}-x)}\left\{\psi\Big(\frac{3}{2}-x\Big) - \psi\Big(\frac{1}{2}\Big)\right\} \\ &= \frac{1}{2}\left\{\psi\Big(\frac{3}{2}-x\Big) - \psi\Big(\frac{1}{2}\Big)\right\}\ {}_2F_1\left[\begin{matrix} x, \frac{1}{2} \\ \frac{3}{2} \end{matrix}; 1\right], \end{aligned} \tag{13}$$

for $\Re x < 2$. In view of (cf. [1, (1.2.13)])

$$\psi(x) - \psi(y) = \sum_{k=0}^{\infty}\left(\frac{1}{k+y} - \frac{1}{k+x}\right), \tag{14}$$

the first line of (13) is completely equivalent with [4, second displayed line on p. 410].

In the next section we state and prove a theorem which is a generalization of (13), that is, of Ramanujan's Entry (XII, 43, Ex. 7).

3. The theorem

Theorem 1. *Let N be a non-negative integer and a be a complex number which is not a negative integer. If $\Re x < N+2$, then*

$$\begin{aligned} {}_3F_2&\left[\begin{matrix} a, a, x \\ a+1, a+N+1 \end{matrix}; 1\right] \\ &= \frac{a\,\Gamma(a+N+1)\,\Gamma(1-x)}{N!\,\Gamma(a-x+1)}\left(\psi(a-x+1) - \psi(a) - \psi(N+1) - \gamma\right) \\ &\qquad - \frac{a\,\Gamma(a+N+1)\,\Gamma(1-x)}{N!\,\Gamma(a-x+1)}\sum_{k=1}^{N}\frac{(a)_k\,(-N)_k}{k\cdot k!\,(a-x+1)_k}. \end{aligned} \tag{15}$$

Proof. To evaluate the ${}_3F_2(1)$ series on the left-hand side of (15), let us introduce a parameter ε, and consider the series

$${}_3F_2\left[\begin{matrix} a, a-\varepsilon, x \\ 1+a-\varepsilon, 1+a-\varepsilon+N \end{matrix}; 1\right].$$

First we apply the (non-terminating) transformation formula (see [8, Ex. 3.6, $q \to 1$, reversed]):

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}A, B, C\\ D, E\end{matrix}; 1\right]\\
&= \frac{\Gamma(A-B)\,\Gamma(D)\,\Gamma(E)\,\Gamma(D+E-A-B-C)}{\Gamma(A)\,\Gamma(D-B)\,\Gamma(E-B)\,\Gamma(D+E-A-C)}\;{}_3F_2\left[\begin{matrix}B, D-A, E-A\\ 1-A+B, D+E-A-C\end{matrix}; 1\right]\\
&+\frac{\Gamma(B-A)\,\Gamma(D)\,\Gamma(E)\,\Gamma(D+E-A-B-C)}{\Gamma(B)\,\Gamma(D-A)\,\Gamma(E-A)\,\Gamma(D+E-B-C)}\;{}_3F_2\left[\begin{matrix}A, D-B, E-B\\ D+E-B-C, 1+A-B\end{matrix}; 1\right].
\end{aligned}
$$

and obtain

$$
\begin{aligned}
&\frac{\Gamma(1+a-\varepsilon)\,\Gamma(\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(2-\varepsilon+N-x)}{\Gamma(a)\,\Gamma(1+N)\,\Gamma(2+a-2\varepsilon+N-x)}\;{}_2F_1\left[\begin{matrix}a-\varepsilon, 1-\varepsilon+N\\ 2+a-2\varepsilon+N-x\end{matrix}; 1\right]\\
&\qquad+\frac{\Gamma(1+a-\varepsilon)\,\Gamma(-\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(2-\varepsilon+N-x)}{\Gamma(1-\varepsilon)\,\Gamma(a-\varepsilon)\,\Gamma(1-\varepsilon+N)\,\Gamma(2+a-\varepsilon+N-x)}\\
&\qquad\qquad\times\;{}_3F_2\left[\begin{matrix}a, 1, 1+N\\ 1+\varepsilon, 2+a-\varepsilon+N-x\end{matrix}; 1\right].
\end{aligned}
$$

Clearly, the convergence of the hypergeometric series on the right-hand side will only be guaranteed if $\Re x < 1$. Therefore. for the moment we suppose $\Re x < 1$.

To the ${}_3F_2$-series we apply the transformation formula (see [2, Ex. 7, p. 98]):

$$
{}_3F_2\left[\begin{matrix}A, B, C\\ D, E\end{matrix}; 1\right] = \frac{\Gamma(E)\,\Gamma(D+E-A-B-C)}{\Gamma(E-A)\,\Gamma(D+E-B-C)}\;{}_3F_2\left[\begin{matrix}A, D-B, D-C\\ D, D+E-B-C\end{matrix}; 1\right],
$$

to get the expression

$$
\begin{aligned}
&\frac{\Gamma(1+a-\varepsilon)\,\Gamma(\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(2-\varepsilon+N-x)}{\Gamma(1)\,\Gamma(a)\,\Gamma(1+N)\,\Gamma(2+a-2\varepsilon+N-x)}\;{}_2F_1\left[\begin{matrix}a-\varepsilon, 1-\varepsilon+N\\ 2+a-2\varepsilon+N-x\end{matrix}; 1\right]\\
&\qquad+\frac{\Gamma(1+a-\varepsilon)\,\Gamma(-\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(1-x)}{\Gamma(1-\varepsilon)\,\Gamma(a-\varepsilon)\,\Gamma(1-\varepsilon+N)\,\Gamma(1+a-x)}\;{}_3F_2\left[\begin{matrix}a, \varepsilon, \varepsilon-N\\ 1+\varepsilon, 1+a-x\end{matrix}; 1\right].
\end{aligned}
$$

The ${}_2F_1$-series is summed by means of the Gauß summation theorem (3), while the ${}_3F_2$-series is written as a sum over k, and subsequently split into the ranges $k=0$, $k=1,2,\dots,N$, and $k=N+1,N+2,\dots$. This yields the expression

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Bigg(\frac{\Gamma(1+a-\varepsilon)\,\Gamma(1+\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(1-x)}{\Gamma(a)\,\Gamma(1+N)\,\Gamma(1+a-\varepsilon-x)}\\
&\qquad\qquad-\frac{\Gamma(1+a-\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(1-x)}{\Gamma(a-\varepsilon)\,\Gamma(1-\varepsilon+N)\,\Gamma(1+a-x)}\Bigg)\\
&\quad-\frac{\Gamma(1+a-\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(1-x)}{\Gamma(a-\varepsilon)\,\Gamma(1-\varepsilon+N)\,\Gamma(1+a-x)}\sum_{k=1}^{N}\frac{(a)_k\,(\varepsilon-N)_k}{k!\,(k+\varepsilon)\,(1+a-x)_k}\\
&\quad-\frac{\Gamma(1+a-\varepsilon)\,\Gamma(1+a-\varepsilon+N)\,\Gamma(1-x)}{\Gamma(a-\varepsilon)\,\Gamma(1-\varepsilon+N)\,\Gamma(1+a-x)}\sum_{k=N+1}^{\infty}\frac{(a)_k\,(\varepsilon-N)_k}{k!\,(k+\varepsilon)\,(1+a-x)_k}.
\end{aligned}
$$

Now we perform the limit $\varepsilon \to 0$. Thus, our original ${}_3F_2$-series becomes

$$ {}_3F_2\left[\begin{matrix} a, a, x \\ 1+a, 1+a+N \end{matrix}; 1\right]. $$

On the other hand, the four-line expression which we obtained for this ${}_3F_2$-series simplifies significantly. The last term simply vanishes because of the occurrence of the factor $(\varepsilon - N)_k$, which is equal to $(\varepsilon - N)_N\, \varepsilon\, (1+\varepsilon)_{k-N-1}$ for $k \geq N+1$, which becomes zero as $\varepsilon \to 0$ for N a non-negative integer. On the other hand, the limit of $\varepsilon \to 0$ of the first two lines can be easily calculated by means of l'Hôpital's rule to obtain the final result (15).

As it stands, the assertion is only demonstrated for $\Re x < 1$. However, by analytic continuation, Equation (15) is true for any value of x for which the ${}_3F_2$-series on the left-hand side converges, i.e., for $\Re x < N+1$. □

4. Some remarks

The following observations can be made:

(i) Clearly, the theorem (15) reduces to Ramanujan's entry (XII, 43, Ex. 7), in our notation (13), if $a = 1/2$ and $N = 0$.

(ii) For $x = 1$, in (13), the ${}_3F_2(1)$ is a special case of Dixon's theorem (see e.g. (III.8) of [12], for $a = 1, b = c = 1/2$), and it has the value $\frac{\pi^2}{8}$.

(iii) For $x = \frac{3}{2}$, the left-hand side of (13) is:

$$ {}_3F_2\left[\begin{matrix} \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}; 1\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix}; 1\right] = \frac{\pi}{2}, $$

by the Gauß summation theorem (3), which is the result for the right-hand side evaluated by l'Hôpital's rule.

(iv) For $x = -k$, a negative integer, in (13), we get:

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, -k \\ \frac{3}{2}, \frac{3}{2} \end{matrix}; 1\right] = \frac{\sqrt{\pi}}{2} \frac{\Gamma(k+1)}{\Gamma(k+3/2)} \sum_{j=1}^{k+1} \frac{1}{2j-1}\,. \tag{16} $$

(v) For $N = 0$, the identity (13) reduces to

$$ {}_3F_2\left[\begin{matrix} a, a, x \\ 1+a, 1+a \end{matrix}; 1\right] = \frac{a\,\Gamma(a+1)\,\Gamma(1-x)}{\Gamma(a-x+1)} \left(\psi(a-x+1) - \psi(a)\right). \tag{17} $$

In view of (14), this is equivalent with Example 5 of Section 10 of Chapter 10 of Ramanujan's second Notebook (see [3, p. 26]).

(vi) The ${}_3F_2(a, a, x; 1+a, 1+a+N; 1)$ series can be related to the 3-j coefficient $\begin{pmatrix} -\frac{x}{2} & -\frac{x}{2} & 0 \\ a-\frac{x}{2} & -a+\frac{x}{2} & 0 \end{pmatrix}$ (cf. [13]) provided x is a negative integer and $-x \leq a \leq 0$. It also corresponds to the dual Hahn polynomial [1] $S_n(0; a, b = 1, c = 1+N)$, for $x = -n$.

Finally, as we already announced in Section 2, it has to be noted that as in the case of Entry (XII, 43, Ex. 4), Ramanujan has missed a multiplicative factor x on the left-hand side of his Entry (XII, 43, Ex. 7).

Acknowledgments : Two of the authors (KSR and GVB) thank Prof. Dr. Niceas Schamp and the Koninklijke Vlaamse Academie van Belgie voor Wetenschappen en Kunsten for excellent hospitality and one of us (KSR) thanks Prof. Dr. Walter Van Assche for an interesting discussion on a visit to the Katholieke Universiteit Leuven for a lecture on the Life and Work of Ramanujan.

References

[1] G. E. Andrews, R. A. Askey and R. Roy, Special Functions, *Encycl. Math. and Its Applns.*, **71**, Cambridge University Press, Cambridge (1999).
[2] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, Cambridge, 1935.
[3] B.C. Berndt, *Ramanujan Notebooks*, Part II, Springer-Verlag, New-York (1987), Chapter X, 7–47.
[4] B.C. Berndt, *Ramanujan Notebooks*, Part IV, Springer-Verlag, New-York (1994).
[5] D. Bradley, On a claim of Ramanujan about certain hypergeometric series, *Proc. Amer. Math. Soc.* **121** (1994), 1145–1149.
[6] J. Dougall, On Vandermonde's theorem and some more general expansions, *Proc. Edinburgh Math. Soc.* **25** (1907), 114–132.
[7] A. Erdélyi, V. Magnus, F. Oberhettinger and F. Tricomi, *Higher transcendental functions*, McGraw–Hill, New York, 1953.
[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.
[9] G. H. Hardy, A chapter from Ramanujan's notebook, *Proc. Cambridge Philos. Soc.* **21** (1923), 492–503 and Chapter VII in *Ramanujan: Twelve lecture on subjects suggested by his life and work*, G. H. Hardy, Chelsea, New York (1944).
[10] S. Ramanujan, Some properties of Bernoulli's numbers, *J. Indian Math. Soc.* **3** (1911), 219–234.
[11] S. Ramanujan's historic first letter to G.H. Hardy, quoted in *Ramanujan: Letters and Commentary*, Bruce C Berndt and Robert A. Rankin, (Eds.), AMS – LMS publication (1995).
[12] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.
[13] K. Srinivasa Rao, A note on the symmetries of the $3j$-coefficient, *J. Phys. A: Math. Gen.* **11** (1978), L69–L73.