Generating functions for shifted plane partitions

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Abstract. With the help of a tableaux method, determinant formulas for trace generating functions for various classes of shifted plane partitions are derived. Proceeding from these determinants, generalizations of Gansner's (J. Combin. Theory A 30 (1981), 71–89; Illinois J. Math. 25 (1985), 533–554) hook formulas for shifted plane partitions and alternative proofs of recent results of Proctor about alternating trace generating functions are given.

1. Introduction. Generating functions for certain families of plane partitions with a given shape and given row bounds can be written down in determinantal form. This has been shown by various methods ([Chu (1989), Gessel and Viennot (1989), Krattenthaler (1990a), Wachs (1985)]), the most prominent being the Gessel–Viennot [(1985, 1989)] method of nonintersecting lattice paths. In special cases these determinants can be evaluated to result into closed forms ([Gessel and Viennot (1989), Krattenthaler (1990a, 1990b)]). Among them are the famous hook-content formulas of Stanley [(1971)].

With the help of a tableaux method, which is related to the nonintersecting path method, and a simple determinant evaluation, the author ([Krattenthaler (1990a, 1990b)]) provided a uniform proof method to establish essentially all known results concerning generating functions for plane partitions of a given shape (without symmetries) and, moreover, some new results. In this paper we continue this work. We transfer the method of [Krattenthaler (1990a)] to the case of shifted plane partitions. These arise naturally when considering generating function problems for plane partitions with symmetries ([Andrews (1979), Mills et al. (1982, 1983, 1986), Okada (1989)]). Using our tableaux method we derive determinant formulas for generating functions for shifted plane partitions of a given shape with prescribed entries on the main diagonal (Theorems 1,2 below).

Again there are special cases in which the determinants may be evaluated. With the help of a summation formula for Schur functions we are able to generalize Gansner's [(1981a, 1981b)] hook formulas for trace generating functions for two particular families of shifted plane partitions.

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A particular specialization of trace generating functions leads to the alternating trace generating functions of Proctor [(1990)]. In the case of trapezoidal shapes, there are closed forms for the alternating trace generating function of shifted plane partitions with prescribed entries on the main diagonal. These formulas were first proved by Proctor [private communication] using group representation theory and bideterminants. With the help of a simple determinant evaluation (Lemma 7) we are able to provide alternative proofs which start with the determinantal formula of Theorem 1.

We should remark that recently Stembridge [(1990a)] discovered a powerful approach to the computation of generating functions for shifted plane partitions, which involves Pfaffians. For example, Gansner's hook formula ([Gansner (1981a), Theorem 7.1]) for the generating function of shifted reverse plane partitions could be easily proved that way, too. (In [Stembridge (1990a)] only the case $x_i = q$ is considered.) Other results about shifted plane partitions, which we were not able to prove using our methods, occur in [Proctor (1983, 1990), and Mills et al. (1986)].

The next section contains all relevant definitions. In section 3 we derive the determinant formulas for three types of generating functions of shifted plane partitions. The generalizations of Gansner's results are given in section 4. Finally, in section 5 the computations concerning the alternating trace generating functions are discussed.

2. Definitions. We use the usual multidimensional notations. For r-tuples of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_r)$, we write $\lambda + \mu$ for $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_r + \mu_r)$. If σ is an element of the symmetric group S_r of order r we define λ_σ to be the r-tuple $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(r)})$. An r-tuple λ of integers is called a *partition* if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0$.

Let α, β be arbitrary integers and λ a partition with $\lambda_r \ge r$. We call an array π of integers of the form

$$
\begin{array}{cccccccc}
\pi_{11} & \pi_{12} & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\
 & \pi_{22} & \cdots & \cdots & \cdots & \cdots & \pi_{2,\lambda_2} & \\
 & & \ddots & & & \iddots & & \\
 & & & \pi_{rr} & \dots & \pi_{r,\lambda_r} & &
\end{array}
\tag{2.1}
$$

a shifted (α, β)–*plane partition of shape* λ if

$$\pi_{ij} \ge \pi_{i,j+1} + \alpha \qquad \text{for} \quad 1 \le i \le r,\ i \le j < \lambda_i \tag{2.2)(a}$$

and

$$\pi_{ij} \ge \pi_{i+1,j} + \beta \qquad \text{for} \quad 1 \le i \le r-1,\ i \le j \le \lambda_{i+1}\ . \tag{2.2)(b}$$

It should be noted that in our definition of the shifted shape there are $\lambda_i - i + 1$ integers in the i-th row, while usually there are λ_i integers in the i-th row.

An array π of the form (2.1) is called a *shifted* (α, β)–*reverse plane partition of shape* λ, if it satisfies

$$\pi_{ij} \le \pi_{i,j+1} + \alpha \qquad \text{for} \quad 1 \le i \le r,\ i \le j < \lambda_i \tag{2.3)(a}$$

and

$$\text{(2.3)(b)} \qquad \pi_{ij} \le \pi_{i+1,j} + \beta \qquad \text{for} \quad 1 \le i \le r-1,\ i \le j \le \lambda_{i+1}\ .$$

To give an example, the array in Figure 1 below is a shifted $(0,2)$–plane partition of shape (8,6,6,4).

$$\begin{matrix} 11 & 9 & 7 & 7 & 6 & 6 & 3 & 1 \\ & 6 & 5 & 5 & 4 & 3 & & \\ & & 3 & 2 & 1 & 1 & & \\ & & & -1 & & & & \end{matrix}$$

Figure 1

The entries of π are called *parts* of π. The sum of all parts of π, $\sum \pi_{ij}$, is called the *norm* of π, for which we write $n(\pi)$. The *l-diagonal* of an array π of the form (2.1) is the sequence of parts π_{ij} with $j-i=l$. The *l-trace* of π, denoted by $t_l(\pi)$, is the sum of the parts of the l-diagonal of π.

We shall consider three types of generating functions for shifted (α,β)–plane partitions subject to certain restrictions. The *part generating function* for a certain class C of shifted (α,β)–plane partitions is $\sum_\pi \prod x_{\pi_{ij}}$, where the sum is over all π belonging to C and the product is over all parts of π. The *norm generating function* for C is $\sum_\pi x^{n(\pi)}$, while the *trace generating function* for C is $\sum_\pi \prod_{l\in\mathbb{Z}} x_l^{t_l(\pi)}$, where in both cases the sum is over all π being an element of that class C. ($\mathbb{Z}$ is the set of integers.) Of course, the norm generating function comes out of both the part and the trace generating function by setting $x_i = x^i$, respectively $x_i = x$.

For x being an indeterminate, we introduce the "x-notations" $[\beta] = 1 - x^\beta$, $[n]! = [1][2]\cdots[n]$, $[0]! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}\ ,$$

$$(a;x)_\infty = \prod_{j=0}^{\infty}(1-ax^j)\ ,$$

and

$$(a;x)_\beta = (a;x)_\infty / (ax^\beta;x)_\infty\ .$$

Finally, for the integers m,n, $m \le n$, we set

$$x(m,n) = \prod_{j=m+1}^{n} x_j\ ,$$

with the empty product being equal to 1 by convention.

3. The determinant formulas. In this section we are going to prove the following theorems.

Theorem 1. *Let α, β be arbitrary integers, λ be a partition with $\lambda_r \geq r$, and let a, b be r-tuples of integers satisfying*

$$(3.1) \qquad a_i - \alpha - \beta \geq a_{i+1} \quad \text{and} \quad b_i + \alpha(\lambda_i - \lambda_{i+1}) + (1-\beta) \geq b_{i+1}$$

for $i = 1, 2, \ldots, r-1$. Then the trace generating function for shifted (α, β)-plane partitions of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least b_i is

$$(3.2) \qquad x_0^{\sum_{i=1}^r a_i} \prod_{i=1}^{r} \left(x(0, \lambda_i - i)^{b_i} \prod_{l=0}^{\lambda_i - i - 1} x(0,l)^{\alpha} \right) \times \det_{1 \leq s,t \leq r} \Big(t(\lambda_s - s; a_t + (1-\alpha-\beta)(s-t) - \alpha(\lambda_s - s) - b_s) \Big) \,,$$

where $t(p;m)$ is the coefficient of z^m in $1 \Big/ \prod_{l=0}^{p}(1 - x(0,l)z)$. The norm generating function for the same class of shifted (α, β)-plane partitions is

$$(3.3) \qquad x^{N_1} \det_{1 \leq s,t \leq r} \left(\begin{bmatrix} (\lambda_s - s)(1-\alpha) + (1-\alpha-\beta)(s-t) + a_t - b_s \\ \lambda_s - s \end{bmatrix} \right) \,,$$

where $N_1 = \sum_{i=1}^r \Big(b_i(\lambda_i - i) + a_i + \alpha \binom{\lambda_i - i}{2} \Big)$. If $\beta = 1 - \alpha$, the part generating function for this class is given by

$$(3.4) \qquad \prod_{i=1}^{r} x_{a_i} \det_{1 \leq s,t \leq r} \Big(h^{(\alpha)}_{\lambda_s - s}(\mathbf{x}; a_t, b_s) \Big) \,,$$

where $h_n^{(\alpha)}(\mathbf{x}; A, B) = \sum x_{i_1} \cdots x_{i_n}$, the sum being over all $i_1, \ldots, i_n$ satisfying $i_1 \leq A$, $i_n \geq B$, and $i_j \geq i_{j+1} + \alpha$ for $j = 1, 2, \ldots, n-1$.

Theorem 2. *Let α, β be arbitrary integers, λ be a partition with $\lambda_r \geq r$, and let a, b be r-tuples of integers satisfying*

$$(3.5) \qquad a_i + \alpha(\lambda_{i-1} - \lambda_i) + (1-\beta) \geq a_{i-1} \quad \text{and} \quad b_i - \alpha - \beta \geq b_{i-1}$$

for $i = 1, 2, \ldots, r-1$. Then the trace generating function for shifted (α, β)-reverse plane partitions of shape λ in which the first part in row i is equal to b_i and the last part in row i is at most a_i is

$$(3.6) \qquad \prod_{i=1}^{r} \prod_{l=0}^{\lambda_i - i - 1} x(l, \lambda_i - i)^{\alpha} \det_{1 \leq s,t \leq r} \Big(x(-1, \lambda_s - s)^{b_t - (s-t)(1-\alpha-\beta)} \cdot t'(\lambda_s - s; a_s + (1-\alpha-\beta)(s-t) - \alpha(\lambda_s - s) - b_t) \Big) \,,$$

where $t'(p;m)$ is the coefficient of z^m in $1\big/\prod_{l=0}^{p}(1-x(l,p)z)$. The norm generating function for the same class of shifted (α,β)-plane partitions is

(3.7)
$$x^{N_2}\det_{1\le s,t\le r}\left(x^{(\lambda_s-s+1)(b_t+t(1-\alpha-\beta))}\begin{bmatrix}(\lambda_s-s)(1-\alpha)+(1-\alpha-\beta)(s-t)+a_s-b_t\\ \lambda_s-s\end{bmatrix}\right),$$

where $N_2=\sum_{i=1}^{r}\left(\alpha\binom{\lambda_i-i+1}{2}-i(\lambda_i-i+1)(1-\alpha-\beta)\right)$. If $\beta=1-\alpha$, the part generating function for this class is given by

$$\prod_{i=1}^{r}x_{b_i}\det_{1\le r,s\le t}\left(h^{(\alpha)}_{\lambda_s-s}(\mathbf{x};a_s,b_t)\right)\ . \tag{3.8}$$

In order to prove these theorems, we shall apply Lemma 2.1 of [Krattenthaler (1990a)], which we restate here without proof.

Lemma 3. *Let $\{M_\sigma:\sigma\in S_r\}$ be a collection of pairwise disjoint sets with a weight function w acting on $\bigcup_{\sigma\in S_r}M_\sigma$. Let M_{id} be partitioned into two disjoint subsets, M^+_{id} and M^-_{id}, $M_{\mathrm{id}}=M^+_{\mathrm{id}}\cup M^-_{\mathrm{id}}$. Assume a weight-preserving involution φ acting on $M^-_{\mathrm{id}}\cup\bigcup_{\sigma\ne\mathrm{id}}M_\sigma$ satisfying the following property, holding for all $\sigma\in S_r$: For every π on which φ is applicable, $\pi\in M_\sigma$, there exists a transposition (i,j), $1\le i,j\le r$, for which $\varphi(\pi)\in M_{\sigma(i,j)}$. Then*

$$F(M^+_{\mathrm{id}};w)=\sum_{\sigma\in S_r}\operatorname{sgn}\sigma\,F(M_\sigma;w)\ , \tag{3.9}$$

where $F(M;w)$ denotes the generating function $\sum_{\pi\in M}w(\pi)$. $\square$

Proof of Theorems. In analogy with [Krattenthaler (1990a)] we first define a shifted α-tableaux of shape λ to be an array of the form (2.1) such that (2.2)(a) holds. Thus, for shifted α-tableaux there are only restrictions on the rows but none on the columns. All the notions of section 2, parts, norm, trace, etc., are extended to shifted α-tableaux in the obvious way.

For the set of all shifted α-tableaux of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least b_i we write $ST_\alpha(\lambda;a,b)$. The subset of $ST_\alpha(\lambda;a,b)$ containing all shifted (α,β)-plane partitions of $ST_\alpha(\lambda;a,b)$ is denoted by $ST^+_\alpha(\lambda;a,b)$, its complement by $ST^-_\alpha(\lambda;a,b)$.

Given a permutation $\sigma\in S_r$, let the r-tuple e_σ be defined by $(1-\sigma(1),2-\sigma(2),\dots,r-\sigma(r))$. Next we construct an involution $\varphi_{\alpha,\beta}$ acting on $\bigcup_{\sigma\in S_r}ST^-_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)$ which satisfies the properties required in Lemma 3. For a shifted α-tableaux τ of shape λ this involution is introduced in two steps:

(A) Consider all pairs $(\tau_{ij},\tau_{i+1,j})$ of parts of τ with

$$\tau_{ij}<\tau_{i+1,j}+\beta\ . \tag{3.10}$$

Let J be the largest integer with $\tau_{iJ}<\tau_{i+1,J}+\beta$ for some i. Let I be the smallest integer with $\tau_{IJ}<\tau_{I+1,J}+\beta$. (The idea is to search for the right-most pairs and, among them, for the highest pair satisfying (3.10) in the tableaux τ.)

(B) Let K be the smallest integer with $\tau_{IK} < \tau_{I+1,K} + \beta$. The image of τ under application of $\varphi_{\alpha,\beta}$ is the tableaux τ', which is obtained by leaving all rows of τ unchanged, with the exception of the I'th and $(I+1)$'th row. Here we set

$$\tau'_{Ij} = \begin{cases} \tau_{Ij} & j \geq K \\ \tau_{I+1,j+1} - (1-\alpha-\beta) & j < K \end{cases}$$

and

$$\tau'_{I+1,j} = \begin{cases} \tau_{I+1,j} & j > K \\ \tau_{I,j-1} + (1-\alpha-\beta) & j \leq K \end{cases} .$$

(For the geometrical interpretation of the action of $\varphi_{\alpha,\beta}$ confer [Krattenthaler (1990a), Def. 4.1].)

Let $\tau \in ST^-_\alpha(\lambda; a_\sigma + (1-\alpha-\beta)e_\sigma, b)$ and let I, K be the integers being determined in (A). It is clear from the definition of $\varphi_{\alpha,\beta}$ that $\varphi_{\alpha,\beta}(\tau)$ again is a shifted (α,β)-plane partition of shape λ with diagonal entries being given by the vector $a_{\sigma(I,I+1)} + (1-\alpha-\beta)e_{\sigma(I,I+1)}$. Besides, the last part in row i of $\varphi_{\alpha,\beta}(\tau)$ is again at least b_i: For $i \neq I+1$ this is obvious, since the last parts in each row of τ remain unchanged under application of $\varphi_{\alpha,\beta}$, with the exception of the $(I+1)$-th row. Here the last part might be changed, namely in case $K = \lambda_{I+1}$, where the last part of row $(I+1)$ of $\varphi_{\alpha,\beta}(\tau)$ is given by $\tau_{I,\lambda_{I+1}-1} + (1-\alpha-\beta)$. Use of (2.2)(a) yields

$$\begin{aligned} \tau_{I,\lambda_{I+1}-1} &\geq \tau_{I,\lambda_I} + \alpha(\lambda_I - \lambda_{I+1} + 1) \\ &\geq b_i + \alpha(\lambda_I - \lambda_{I+1} + 1) \ . \end{aligned}$$

Hence, by the second part of (3.1), we obtain

$$\tau_{I,\lambda_{I+1}-1} + (1-\alpha-\beta) \geq b_{i+1} \ ,$$

just as being claimed.

Another fact is that, if $\sigma \neq \mathrm{id}$, we have

$$ST_\alpha(\lambda; a_\sigma + (1-\alpha-\beta)e_\sigma, b) = ST^-_\alpha(\lambda; a_\sigma + (1-\alpha-\beta)e_\sigma, b) \ .$$

Since $\sigma \neq \mathrm{id}$, there must exist an integer i, $1 \leq i \leq r-1$, such that $\sigma(i) > \sigma(i+1)$. Using the inequality

$$a_R < a_S + (R-S)(1-\alpha-\beta), \quad \text{for } r \geq R > S \geq 1, \tag{3.11}$$

which is equivalent with (3.1), it can be shown that

$$a_{\sigma(i)} + (1-\alpha-\beta)(i-\sigma(i)) < a_{\sigma(i+1)} + (1-\alpha-\beta)(i+1-\sigma(i+1)) + \alpha + \beta \ .$$

This implies that for any $\tau \in ST_\alpha(\lambda; a_\sigma + (1-\alpha-\beta)e_\sigma, b)$ the inequality $\tau_{ii} < \tau_{i+1,i+1} + \alpha + \beta$ is true, hence $\tau_{i,i+1} < \tau_{i+1,i+1} + \beta$, and thus $\tau \in ST^-_\alpha(\lambda; a_\sigma + (1-\alpha-\beta)e_\sigma, b)$.

Summarizing we have the following properties of $\varphi_{\alpha,\beta}$:

(1) $\varphi_{\alpha,\beta}$ is an involution on

$$ST_\alpha^-(\lambda;a,b)\cup\bigcup_{\substack{\sigma\in S_r\\ \sigma\neq\mathrm{id}}}ST_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)\ .$$

If $\tau\in ST_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)$ and τ is an element of the domain of $\varphi_{\alpha,\beta}$, then there exists a transposition $(I,I+1)$ with

$$\varphi_{\alpha,\beta}(\tau)\in ST_\alpha(\lambda,a_{\sigma(I,I+1)}+(1-\alpha-\beta)e_{\sigma(I,I+1)},b)\ .$$

(2) The l-trace and norm are invariant under application of $\varphi_{\alpha,\beta}$. If $\alpha+\beta=1$ the set of parts remains unchanged under application of $\varphi_{\alpha,\beta}$, too.

Since, because of (3.11), the map $\sigma\to a_\sigma+(1-\alpha-\beta)e_\sigma$ is injective, the sets $ST_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)$ are pairwise disjoint. Hence, all assumptions of Lemma 3 are satisfied with $M_\sigma=ST_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)$, etc., and the trace weight $w(\tau)=\prod_{l\in\mathbb{Z}}x_l^{t_l(\tau)}$. The generating functions $F\big(ST_\alpha(\lambda;a_\sigma+(1-\alpha-\beta)e_\sigma,b)\big)$ are equal to the product of the corresponding generating functions of the rows. Therefore, for this choice of M_σ, the sum on the right-hand side of (3.9) can be written in the form of a determinant, which finally, after taking some factors out of this determinant, results into the right-hand side of (3.2).

Formula (3.3) is obtained by setting $x_i=x$ for all i in (3.2) and using the well-known expansion for $1/(z;x)_{p+1}$ ([Andrews (1976), (3.3.7)]). Formula (3.4) is proved in analogy with (3.2) by observing that if $\alpha+\beta=1$, all assumptions of Lemma 3 are satisfied with $M_\sigma=ST_\alpha(\lambda;a_\sigma,b)$, etc., and $w(\tau)=\prod x_{\tau_{ij}}$.

Theorem 2 can be proved by constructing a similar involution, which acts on shifted (α,β)-reverse tableaux, or by using the transformation $\pi_{ij}\to-\pi_{ij}$, which maps shifted (α,β)-plane partitions onto shifted (α,β)-reverse plane partitions, and transforming the results of Theorem 1 into the "reverse" case. $\square$

The special case $\alpha=0$, $\beta=1$, $b_i=0$ of (3.3) was used to obtain generating functions for symmetry classes of plane partitions ([Andrews (1979), Mills et al. (1982, 1983, 1986), Okada (1989)]). It is also interesting to note that a slight modification of φ_{01} serves to provide an alternative proof of Okada's [(1990)] Theorem about partially strict shifted plane partitions.

4. Trace generating functions with no upper part bounds. The main tool which is used in this section to obtain closed forms for generating functions for shifted (α,β)-(reverse) plane partitions with no upper bound on the parts in each row, is the following summation theorem, which is proved by inserting the bideterminantal form of Schur functions ([Macdonald (1979), p. 24, (3.1)]) into the summation formula ([Mcdonald (1979), p. 45, Ex. 4]) for Schur functions.

Lemma 4. *Let $X_1,X_2,\dots,X_r$ be indeterminates. Then*

$$\sum_{a_1>a_2>\dots>a_r\geq0}\det_{1\leq s,t\leq r}\left(X_s^{a_t}\right)=\frac{\prod_{1\leq i<j\leq r}(X_i-X_j)}{\prod_{i=1}^r(1-X_i)\prod_{1\leq i<j\leq r}(1-X_iX_j)}\ .\qquad\square$$

Since it is more straight-forward to deal with, we first consider the case of shifted (α,β)-reverse plane partitions.

Theorem 5. *Let K, B be integers, $K \ge 1$. The trace generating function for shifted (α,β)-reverse plane partitions π of shape λ in which the first part in row i is $\equiv -i(1-\alpha-\beta) \pmod K$ and $\pi_{11} \ge (KB-(1-\alpha-\beta))$ is*

$$\prod_{i=1}^{r} x(-1,\lambda_i-i)^{-i(1-\alpha-\beta)+K(B+i-1)} \prod_{i=1}^{r}\prod_{l=0}^{\lambda_i-i-1} \frac{x(l,\lambda_i-i)^{\alpha}}{(1-x(l,\lambda_i-i))}$$
$$\times \frac{\prod_{1\le j<i\le r}(1-x(\lambda_i-i,\lambda_j-j)^K)}{\prod_{i=1}^{r}(1-x(-1,\lambda_i-i)^K)\prod_{1\le i<j\le r}(1-x(-1,\lambda_i-i)^K x(-1,\lambda_j-j)^K)}.$$

Proof. In (3.6) set $b_t = KB_t - t(1-\alpha-\beta)$ and let $a_s \to \infty$. Using the fact that, if $\lim_{i\to\infty} c_i = c$ then also $\lim_{z\to 1^-}(1-z)\sum_j c_j z^j = c$, we obtain, after taking some factors out of the determinant, for the trace generating function for shifted (α,β)-reverse plane partitions of shape λ in which the first part in row i is equal to $KB_i - i(1-\alpha-\beta)$ the expression

$$\frac{\prod_{i=1}^{r}\prod_{l=0}^{\lambda_i-i-1} x(l,\lambda_i-i)^{\alpha}\prod_{i=1}^{r} x(-1,\lambda_i-i)^{-i(1-\alpha-\beta)}}{\prod_{i=1}^{r}\prod_{l=0}^{\lambda_i-i-1}(1-x(l,\lambda_i-i))} \det\left(x(-1,\lambda_s-s)^{KB_t}\right).$$

When taking into account (3.5), one observes that, to obtain the desired generating function, these expressions have to be summed up over all $B_1, B_2, \dots, B_r$ with $B \le B_1 < \cdots < B_r$. Application of Lemma 4 after some manipulation yields the claimed expression. □

Gansner's [(1981a)] Theorem 7.1 is the special case $\alpha=\beta=0$, $K=B=1$ of Theorem 5.

In the case of ordinary shifted plane partitions there is only a closed form result analogous to the previous one in case of a triangular shape (r^r), the shape λ with $\lambda_i = r$ for all i.

Theorem 6. *Let K, B be integers, $K \ge 1$. The trace generating function for shifted (α,β)-plane partitions π of shape (r^r) in which the first part in row i is $\equiv i(1-\alpha-\beta) \pmod K$ and $\pi_{rr} \ge (KB + r(1-\alpha-\beta))$ is*

$$\prod_{i=0}^{r-1} x_i^{N_3(i)} \prod_{0\le i<j\le r-1} \frac{(1-x(i,j)^K)}{(1-x(i,j))(1-x(-1,i)^K x(-1,j)^K)} \prod_{i=0}^{r-1}\frac{1}{(1-x(-1,i)^K)},$$

where $N_3(i) = (r-i)(r+i+1)/2 + (KB-r\alpha)(r-i) + (\alpha+K)\binom{r-i}{2} - \beta\binom{r-i+1}{2}$.

Proof. Let $\hat{B} = KB - r\alpha$. First we consider the generating function of shifted (α,β)-plane partitions of shape (r^r) where the first part in row i is equal to a_i and

the last part in row i is at least $\hat{B}+i(1-\beta)$. If these choices are inserted into (3.2), we get
(4.1)
$$x_0^{\sum a_i}\prod_{i=1}^{r} x(0,r-i)^{\hat{B}+i(1-\beta)}\prod_{l=0}^{r-1} x(0,l)^{\alpha(r-l-1)}\det\Big(t(r-s;a_t-t(1-\alpha-\beta)-\alpha r-\hat{B})\Big)\ ,$$

where $t(p;m)$ is the coefficient of z^m in the power series expansion given in Theorem 1. Using the partial fraction expansion of this power series, it turns out that the determinant in (4.1) by simple row manipulations may be simplified to

$$\det\left(x(0,r-s)^{a_t-t(1-\alpha-\beta)-\alpha r-\hat{B}}\prod_{l=0}^{r-s-1}\frac{(-1)^{r-s}x(l,r-s)}{(1-x(l,r-s))}\right)\ . \tag{4.2}$$

Because of (2.2)(b), any shifted (α,β)-plane partition of shape (r^r) with $\pi_{rr}\ge KB+r(1-\alpha-\beta)$ automatically possesses the property that the last part in row i is greater than $\hat{B}+i(1-\beta)$. Therefore, by setting $a_i=KA_i+i(1-\alpha-\beta)$ in (4.1) and summing up over all $A_1,\dots,A_r$ with $A_1>\dots>A_r\ge KB+r(1-\alpha-\beta)$, the desired generating function will be obtained. Writing the determinant in (4.1) in the form (4.2), it is seen that again Lemma 4 may be applied. Some tedious manipulations finally result into the claimed expression. □

The case $K=2$ of Theorem 6 is somewhat related to a result of Proctor [(1990) ("CYI" case of Theorem 1)] (cf. also [Stembridge (1990b)]), although there is no overlap. Gansner's [(1981b) Theorem 4.2] is obtained from Theorem 6 by setting $\alpha=\beta=0$, $K=1$, and $B=-r$.

5. Alternating trace generating functions. Recently, [Proctor (1990)] considered unusual weightings for shifted plane partitions π, namely $w_1(\pi)=x^{\sum_{i\ge0}(-1)^{i-1}it_i(\pi)}$ and $w_2(\pi)=x^{\sum_{i\ge0}(-1)^i(i+1)t_i(\pi)}$, which he called "alternating trace" weightings. He also observed [private communication] that with these weightings there are closed expressions for trapezoidal shapes, the shapes λ, where $\lambda_i=\Lambda-i$ for some Λ. His proofs base on the representation theory of the symplectic group. We give more elementary proofs below, which use the determinants in Theorems 1,2 and the following computational lemma. Special cases of these results earlier appeared in [Proctor (1988), Prop. 4.1, and Proctor (1990), Theorem 1].

Lemma 7. *Let $A_1,A_2,\dots,A_r,B_2,\dots,B_r$ be indeterminates. Then we have*

$$\det\left((C/A_t+B_r)\cdots(C/A_t+B_{s+1})(A_t+B_r)\cdots(A_t+B_{s+1})\right) \tag{5.1}$$
$$=\prod_{i=2}^{r}B_i^{i-1}\prod_{1\le i<j\le r}(A_i-A_j)(1-C/A_iA_j)\ .$$

PROOF. The determinant on the left-hand side of (5.1) is a polynomial in $A_1,\dots,A_r$ of degree $3\binom{r}{2}$ divided by $\prod_i A_i^{r-1}$. Since, for $1\le t_1<t_2\le r$, the determinant is zero

for $A_{t_1}=A_{t_2}$ or $A_{t_1}=C/A_{t_2}$, respectively, the left-hand side is equal to

$$p(A_1,\dots,A_r)\prod_{i<j}(A_i-A_j)(A_iA_j-C)\Big/\prod_i A_i^{r-1}\ ,$$

where $p(A_1,\dots,A_r)$ is some polynomial in the A_i's. Comparing the degrees yields that p is a constant, which is computed by comparing coefficients of $\prod_i A_i^{r-i}$ on both sides of (5.1). $\square$

Theorem 8. *Let λ be a trapezoidal shape, $\lambda_i=\Lambda-i$, with $\Lambda\ge 2r$. Given an integer B and an r-tuple a satisfying (3.1), the generating function with respect to the alternating trace weight w_1 for shifted (α,β)-plane partitions of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least $B-\beta i+\alpha i$ is*

$$x^{N_4}\prod_{i=1}^{r}\frac{[\Lambda-r+a_i-i(1-\alpha-\beta)-\Lambda\alpha-B]!}{[\Lambda-2i]!\,[r+a_i-i(1-\alpha-\beta)-\Lambda\alpha-B]!}\times$$

$$\prod_{1\le i<j\le r}[a_i-a_j-(i-j)(1-\alpha-\beta)][a_i+a_j-(i+j)(1-\alpha-\beta)-\Lambda(2\alpha-1)-2B+1]\ ,$$

where

$$N_4=\sum_{i=1}^r a_i(i-\lfloor\tfrac{\Lambda}{2}\rfloor)+\alpha\left(r\Lambda\lfloor\tfrac{\Lambda}{2}\rfloor-\tfrac{\Lambda}{2}r(2r+1)+\tfrac{r(4r^2+3r-1)}{6}\right)+\chi(\Lambda\text{ odd})$$
$$\cdot\left(\alpha(\Lambda\tfrac{r^2}{2}-4\tbinom{r+1}{3})+\beta(\tfrac{r(4r^2+3r-1)}{6}-r(r+1)\lfloor\tfrac{\Lambda}{2}\rfloor)+B(\Lambda r-r(r+1))\right)\ .$$

(χ is the usual truth function.)

Proof. The alternating trace weight w_1 comes out of the general trace weighting $\prod_{l\in\mathbb{Z}}x_l^{t_l(\pi)}$ by setting $x_i=x^{(-1)^{i-1}i}$. Since for this choice of the variables x_i, we have

$$t(p;m)=x^{-m\lfloor\frac{p}{2}\rfloor}\begin{bmatrix}p+m\\ p\end{bmatrix}\ ,$$

which is again shown using [Andrews (1976), (3.3.7)], by (3.2) the desired generating function allows the form

$$x^{N_5}\det\left(x^{(-\lfloor\frac{\Lambda}{2}\rfloor+s)(a_t-t(1-\alpha-\beta))}\begin{bmatrix}\Lambda(1-\alpha)-s+a_t-t(1-\alpha-\beta)-B\\ \Lambda-2s\end{bmatrix}\right),\tag{5.2}$$

where we suppress the exact expression for N_5 for brevity. Taking some factors out of the determinant, this expression may be written

$$x^{N_6}\prod_{i=1}^{r}\frac{[\Lambda-r+\hat A_i]!}{[\Lambda-2i]!\,[r+\hat A_i]}\det\Big((x^{-\Lambda-1-\hat A_t}-x^{-r})\cdots(x^{-\Lambda-1-\hat A_t}-x^{-s-1})$$
$$\cdot(x^{\hat A_t}-x^{-r})\cdots(x^{\hat A_t}-x^{-s-1})\Big)\ ,$$

where $\hat A_t=a_t-t(1-\alpha-\beta)-\Lambda\alpha-B$. The determinant can be evaluated using Lemma 7 with $A_i=x^{\hat A_i}$, $B_i=-x^{-i}$, and $C=x^{-\Lambda-1}$. Simplifications finally lead to the claimed expression. $\square$

For the weighting w_2 we only obtained a closed form expression for the first row length of the shifted shape being even. In case of odd first row length, it does not seem to be possible to evaluate the determinant.

Theorem 9. *Let λ be a trapezoidal shape, $\lambda_i = 2\Lambda - 1 - i$, with $\Lambda > r$. Given an integer B and an r-tuple a satisfying (3.1), the generating function with respect to the alternating trace weight w_2 for shifted (α,β)-plane partitions of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least $B-\beta i+\alpha i$ is*

$$x^{N_7}\prod_{i=1}^{r}\frac{(1+x^{a_i-i(1-\alpha-\beta)-\alpha(2\Lambda-1)-B+\Lambda})}{(1+x^{\Lambda-i})}\\ \times\frac{[2\Lambda-1-r+a_i-i(1-\alpha-\beta)-(2\Lambda-1)\alpha-B]!}{[2\Lambda-1-2i]!\,[r+a_i-i(1-\alpha-\beta)-(2\Lambda-1)\alpha-B]!}\\ \times\prod_{1\le i<j\le r}[a_i-a_j-(i-j)(1-\alpha-\beta)][a_i+a_j-(i+j)(1-\alpha-\beta)-(2\Lambda-1)(2\alpha-1)-2B+1]\,,$$

where $N_7=\sum_{i=1}^{r}a_i(i-\Lambda)+\alpha\left(2\Lambda^2r-\Lambda r(3r+2)+\frac{r(2r^2+3r-2)}{3}\right)+\Lambda r^2+r$.

Proof. The alternating trace weight w_2 comes out of the general trace weighting by setting $x_i = x^{(-1)^i(i+1)}$. For this choice of the variables x_i, we get, another time using [Andrews (1976), (3.3.7)],

$$\begin{aligned}t'(p;m)&=x^{-m\lfloor\frac{p+3}{2}\rfloor}\begin{bmatrix}p+m+1\\p+1\end{bmatrix}-x^{-1-(m-1)\lfloor\frac{p+3}{2}\rfloor}\begin{bmatrix}p+m\\p+1\end{bmatrix}\\&=x^{-m\lfloor\frac{p+3}{2}\rfloor}\begin{bmatrix}p+m\\p\end{bmatrix}\frac{[\lfloor\frac{p+1}{2}\rfloor]+x^{m+\lfloor\frac{p+1}{2}\rfloor}[\lfloor\frac{p+2}{2}\rfloor]}{[p+1]}\,.\end{aligned}$$

Using this identity, by (3.2) we obtain, after taking some factors out of the determinant, for the desired generating function

$$x^{N_7}\prod_{i=1}^{r}\frac{(1+x^{a_i-i(1-\alpha-\beta)-\alpha(2\Lambda-1)-B+\Lambda})}{(1+x^{\Lambda-i})}\det\Bigg(x^{(-\Lambda+1+s)(a_t-t(1-\alpha-\beta))}\\ \cdot\begin{bmatrix}(2\Lambda-1)(1-\alpha)-s+a_t-t(1-\alpha-\beta)-B\\2\Lambda-1-2s\end{bmatrix}\Bigg)\,.$$

The determinant in this expression is a special case of the determinant in (5.2) (replace Λ by $2\Lambda-1$ in (5.2)). $\square$

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