Product formulae for trace generating functions for semistandard tableaux of unusual shapes†

C. Krattenthaler

Fakultät für Mathematik der Universität Wien,
Nordbergstraße 15, A-1090 Wien, Austria.
WWW: http://www.mat.univie.ac.at/~kratt

Abstract. Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we show that the trace generating functions for semistandard tableaux of shapes $\lambda/(0,0,1,1,\dots)$ and $\lambda/(0,1,2,2,2,\dots)$ are both given by closed product formulae.

An r-tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r \ge 0$ is called a *partition*. The components λ_i are called *parts* of the partition. Let λ be a partition and $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ an arbitrary r-tuple of non-negative integers with $\mu_i \le \lambda_i$ for $i = 1, 2, \dots, r$. A (*semistandard*) *tableau* τ *of shape* λ/μ is an array

$$\begin{matrix} \tau_{1,\mu_1+1} & \tau_{1,\mu_1+2} & \cdots\cdots\cdots\cdots\cdots\cdots & & & \tau_{1,\lambda_1} \\ & & \tau_{2,\mu_2+1} & \tau_{2,\mu_2+2} & \cdots\cdots & \tau_{2\lambda_2} \\ & & \vdots & \cdots\cdots\cdots & \iddots & \\ & & \tau_{r,\mu_r+1} & \cdots & \tau_{r\lambda_r} & \end{matrix} \tag{1}$$

of non-negative integers τ_{ij}, $1 \le i \le r$, $\mu_i < j \le \lambda_i$, such that the rows are weakly and the columns are strictly increasing.[1]

Given a (semistandard) tableau τ we consider the trace weight (cf. [1, 2, 3])

$$w(\tau) = \prod_{(i,j)} x_{j-i}$$

where the product is over all cells (i,j) of the shape of τ. In order to facilitate notation, we write $x(-\infty, n) := \prod_{i=-\infty}^{n} x_i$, and $x(m,n) := x(-\infty,n)/x(-\infty,m)$.

The purpose of this note is to present the following two product formulae for trace generating functions for semistandard tableaux of unusual shapes.

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[1]We emphasise that, in this note, the r-tuple μ will *not* be a partition, so that the tableaux which we are considering here do not *not* have skew shapes in the classical sense.

Theorem 1. *Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ be a partition. The trace generating function $\sum_\tau w(\tau)$, where the sum is over all tableaux τ of shape $\lambda/(0,0,1,1,\dots)$, is given by*

$$\frac{x_{-1}\prod_{i=1}^{r} x(-i+1,\lambda_i-i)^{i-1}\prod_{1\le i<j\le r}(1-x(0,\lambda_j-j,\lambda_i-i))}{\prod_{i=1}^{r}\prod_{\ell=-r+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))} \\ \times\frac{\prod_{i=2}^{r-1}\Big(1-x_{-1}\prod_{j=1}^{r}x(-j,\lambda_j-j)\Big/x(-r,-i)\Big)}{\prod_{i=1}^{r}\Big(1-x_{-1}\prod_{j=1}^{r}x(-j,\lambda_j-j)\Big/x(-r,\lambda_i-i)\Big)}. \tag{2}$$

Theorem 2. *Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ be a partition. The trace generating function $\sum_\tau w(\tau)$, where the sum is over all semistandard tableaux τ of shape $\lambda/(0,1,2,2,2,\dots)$, is given by*

$$\frac{x_{0}\prod_{i=1}^{r} x(-i+2,\lambda_i-i)^{i-1}\prod_{1\le i<j\le r}(1-x(0,\lambda_j-j,\lambda_i-i))}{\prod_{i=1}^{r}\prod_{\ell=-r+2}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))} \\ \times\frac{\prod_{i=2}^{r-2}\Big(1-x_{0}\prod_{j=1}^{r}x(-j+1,\lambda_j-j)\Big/x(-r+1,-i)\Big)}{\prod_{i=1}^{r}\Big(1-x_{0}\prod_{j=1}^{r}x(-j+1,\lambda_j-j)\Big/x(-r+1,\lambda_i-i)\Big)}. \tag{3}$$

Remark. Both expressions should have a nicer formulation when using Gansner's [2] "trace hook lengths."

We provide the details for the proof of Theorem 1, the proof of Theorem 2 being similar.

Proof of Theorem 1. From [3, Theorem 7.2], we know that the generating function for tableaux of shape $\lambda/(0,0,\dots)$ with a_1 being the first entry in the first row and a_2 being the first entry in rows $2,3,\dots,r$ is given by

$$\det_{1\le i,j\le r}\left(\frac{x(-1,\lambda_i-i)^{a_1}}{\prod_{\ell=0}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\ \vdots\ \frac{x(-j,\lambda_i-i)^{a_2}}{\prod_{\ell=-j+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\right).$$

In the determinant the row index is i and the column index is j. The notation of the determinant means that the first expression gives the first column and the second expression gives the remaining columns, $j\ge 2$.

Hence, the desired trace generating function is given by

$$\sum_{0\le a_1<a_2} x(-r,-2)^{-a_2} \det_{1\le i,j\le r}\left(\frac{x(-1,\lambda_i-i)^{a_1}}{\prod_{\ell=0}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))} \vdots \frac{x(-j,\lambda_i-i)^{a_2}}{\prod_{\ell=-j+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\right).$$

If we sum over a_1, then we obtain

$$\sum_{0\le a_2\ \text{(sic!)}} x(-r,-2)^{-a_2} \det_{1\le i,j\le r}\left(\frac{1-x(-1,\lambda_i-i)^{a_2}}{\prod_{\ell=-1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))} \vdots \frac{x(-j,\lambda_i-i)^{a_2}}{\prod_{\ell=-j+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\right).$$

In the determinant add $x_{-1}^{-a_2}$ times the second column to the first column to obtain

$$\sum_{0\le a_2\ \text{(sic!)}} x(-r,-2)^{-a_2} \det_{1\le i,j\le r}\left(\frac{1}{\prod_{\ell=-1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))} \vdots \frac{x(-j,\lambda_i-i)^{a_2}}{\prod_{\ell=-j+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\right).$$

Now we expand the determinant along the first column,

$$\sum_{k=1}^{r} \frac{(-1)^{k-1}}{\prod_{\ell=-1}^{\lambda_k-k-1}(1-x(\ell,\lambda_k-k))} \\ \times \sum_{0\le a_2} x(-r,-2)^{-a_2} \det_{\substack{i=1,\dots,k-1,k+1,\dots,r\\ j=1,\dots,r-1}}\left(\frac{x(-j-1,\lambda_i-i)^{a_2}}{\prod_{\ell=-j}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}\right) \tag{4}$$

Next we compute the determinant in (4). By taking

$$\left(\frac{x_{-1}\prod_{i=1}^{r} x(-i,\lambda_i-i)}{x(-r,\lambda_k-k)}\right)^{a_2}$$

out of the determinant in (4), a determinant of the form

$$\det_{1\le i,j\le r-1}\left(\frac{1}{\prod_{\ell=-j}^{\mu_i-1}(1-x(\ell,\mu_i))}\right) \tag{5}$$

remains. This determinant equals

$$\frac{\prod_{i=1}^{r-1}\prod_{\ell=-r+1}^{-i-1} x(\ell,0)}{\prod_{i=1}^{r-1}\prod_{\ell=-r+1}^{\mu_i-1}(1-x(\ell,\mu_i))} \det_{1\le i,j\le r-1}\Big((x(0,-r+1)-x(0,\mu_i))\cdots(x(0,-j-1)-x(0,\mu_i))\Big).$$

The last determinant can be reduced to a Vandermonde determinant by elementary column operations, hence the determinant in (5) equals

$$\frac{\prod_{\ell=-r+1}^{-2} x(\ell,0)^{-\ell-1}}{\prod_{i=1}^{r-1}\prod_{\ell=-r+1}^{\mu_i-1}(1-x(\ell,\mu_i))}\times\prod_{1\le i,j\le r-1}(x(0,\mu_j)-x(0,\mu_i)). \tag{6}$$

If in (4) we use the equality of (5) and (6) and moreover sum over a_2, the expression (4) reduces to

$$\sum_{k=1}^{r}\frac{(-1)^{k-1}}{\prod_{\ell=-1}^{\lambda_k-k-1}(1-x(\ell,\lambda_k-k))}\frac{1}{(1-x_{-1}\prod_{i=1}^{r}x(-i,\lambda_i-i)\Big/x(-r,\lambda_k-k))}$$

$$\times\frac{\prod_{\ell=-r+1}^{-2}x(\ell,0)^{-\ell-1}}{\prod_{\substack{i=1\\i\ne k}}^{r}\prod_{\ell=-r+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}$$

$$\times\frac{\prod_{1\le i<j\le r}(x(0,\lambda_j-j)-x(0,\lambda_i-i))}{\prod_{i=1}^{j-1}(x(0,\lambda_j-j)-x(0,\lambda_i-i))\prod_{i=j+1}^{r}(x(0,\lambda_i-i)-x(0,\lambda_j-j))},$$

which equals

$$\frac{\prod_{\ell=-r+1}^{-2}x(\ell,0)^{-\ell-1}\prod_{1\le i<j\le r}(x(0,\lambda_j-j)-x(0,\lambda_i-i))}{\prod_{i=1}^{r}\prod_{\ell=-r+1}^{\lambda_i-i-1}(1-x(\ell,\lambda_i-i))}$$

$$\sum_{k=1}^{r}\frac{\prod_{i=2}^{r-1}(1-x(-i,\lambda_k-k))}{\prod_{\substack{i=1\\i\ne k}}^{r}(x(0,\lambda_i-i)-x(0,\lambda_j-j))}\frac{1}{(1-x_{-1}\prod_{i=1}^{r}x(-i,\lambda_i-i)\Big/x(-r,\lambda_k-k))}. \tag{7}$$

To evaluate the sum we apply the partial fraction expansion

$$\frac{z\,p(z)}{\prod_{i=1}^{r}(y_i-z)}=\sum_{k=1}^{r}\frac{p(y_k)}{\prod_{\substack{i=1\\i\ne k}}^{r}(y_i-y_k)}\frac{1}{1-z/y_k},$$

(here, p must be a polynomial of degree at most $r-2$), with $p(z)=\prod_{i=2}^{r-1}(1-x(-i,0)z)$, $y_i=x(0,\lambda_i-i)$, and $z=x_{-1}\prod_{i=1}^{r}x(-i,\lambda_i-i)\Big/x(-r,0)$. Application of this partial fraction expansion in (7) and some manipulation finally lead to the expression in (2). □

References

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