PLANE PARTITIONS IN THE WORK OF RICHARD STANLEY AND HIS SCHOOL

C. KRATTENTHALER

ABSTRACT. These notes provide a survey of the theory of plane partitions, seen through the glasses of the work of Richard Stanley and his school.

1. INTRODUCTION

Plane partitions were introduced to (combinatorial) mathematics by Major Percy Alexander MacMahon [71] around 1900. What he had in mind was a planar analogue of a(n integer) partition.[1]

In order to explain this, let us start with the definition of a(n integer) partition. A *partition of a positive integer* n is a way to represent n as a sum of positive integers, where the order of the summands does not play any role. So, a partition of n is

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \tag{1.1}$$

for some k, where all summands λ_i are positive integers and, since the order of summands is irrelevant, we may assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. For example, there are 7 partitions of 5, namely

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

If we want to represent a partition as in (1.1) *very* succinctly, then we just write

$$\lambda_1\, \lambda_2\, \ldots\, \lambda_k.$$

A plane partition is a planar analogue of this. A *plane partition of a positive integer* n is a planar array π of non-negative integers of the form

$$\begin{array}{llll} \pi_{1,1} & \cdots\cdots\cdots\cdots\cdots & \pi_{1,\lambda_1} \\ \pi_{2,1} & \cdots\cdots\cdots\cdots & \pi_{2,\lambda_2} \\ \vdots & & \iddots \\ \pi_{r,1} & \ldots & \pi_{r,\lambda_r} \end{array} \tag{1.2}$$

such that entries along rows *and* along columns are weakly decreasing, i.e.,

$$\pi_{i,j} \geq \pi_{i,j+1} \quad \text{and} \quad \pi_{i,j} \geq \pi_{i+1,j}, \tag{1.3}$$

and such that the sum $\sum \pi_{i,j}$ of all entries $\pi_{i,j}$ equals n. Here, the sequence of row-lengths, $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, is assumed to form a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, and this partition is called the *shape* of the plane partition π. The individual entries $\pi_{i,j}$ are called

[1] In particular, a plane partition is *not* a partition of the plane.

parts of π. The sum $\sum \pi_{i,j}$ of all parts of the plane partition π — so-to-speak the "size" of π — i.e., the number that π partitions, will be denoted by $|\pi|$. For example, Figure 1 shows a plane partition of 24. That is, if π_0 denotes this plane partition, then $|\pi_0| = 24$, and its shape is $(4, 3, 2)$.

$$\begin{array}{cccc} 5 & 3 & 3 & 2 \\ 5 & 1 & 1 & \\ 3 & 1 & & \end{array}$$

FIGURE 1. A plane partition

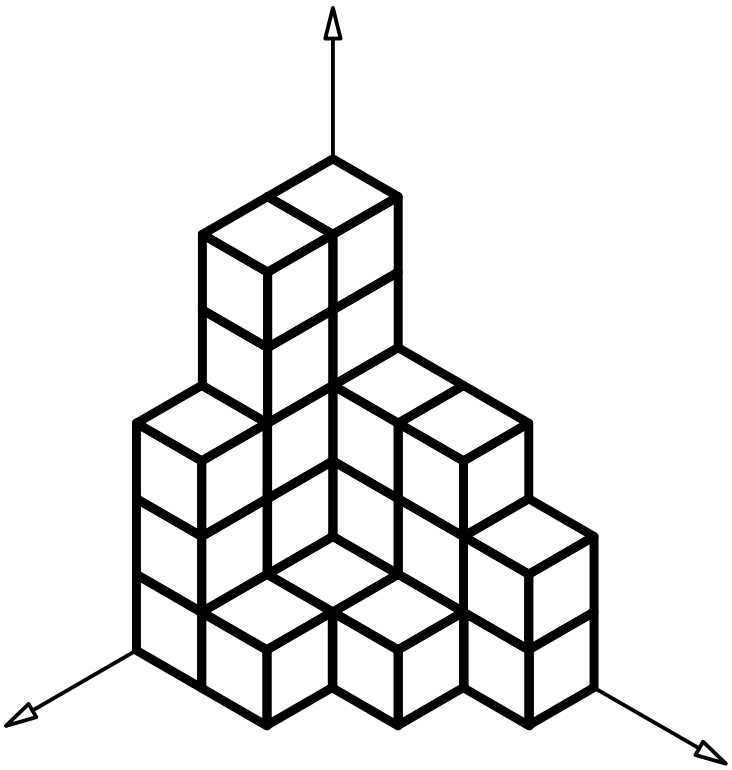
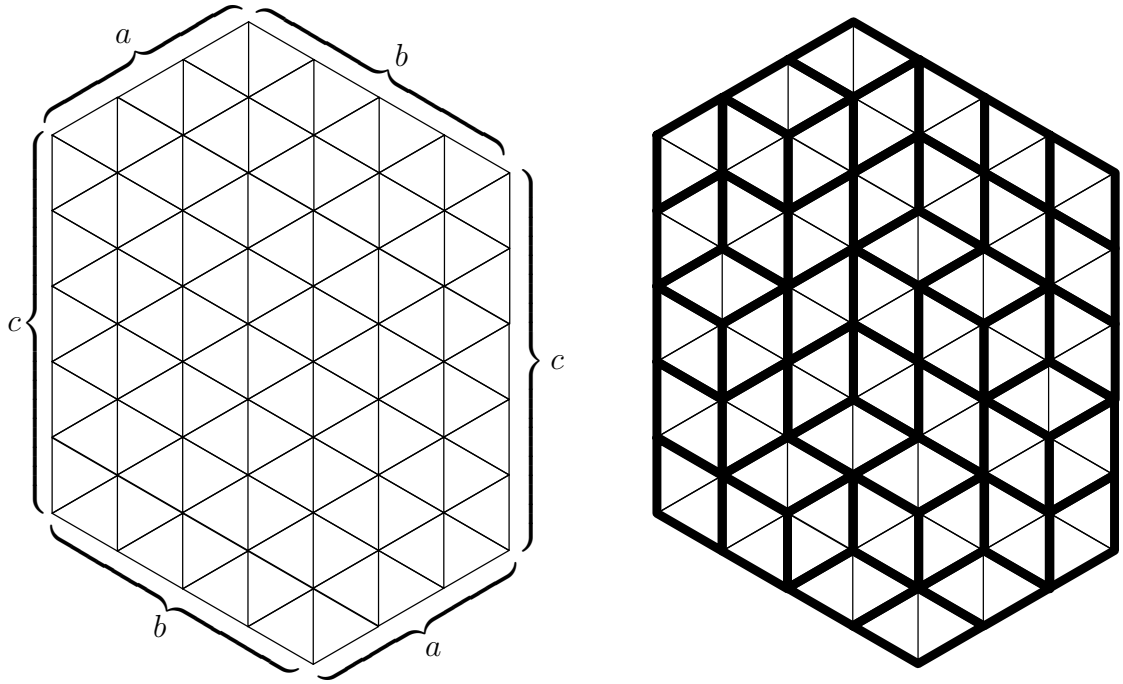


FIGURE 2. A pile of cubes

There is a nice way to represent a plane partition as a three-dimensional object: this is done by replacing each part k of the plane partition by a stack of k unit cubes. Thus we obtain a pile of unit cubes. The pile of cubes corresponding to the plane partition in Figure 1 is shown in Figure 2. It is placed into a coordinate system (the coordinate axes being indicated by arrows in the figure) so that its "back corner" resides in the origin of the coordinate system and it aligns with the coordinate axes as shown in the figure. These piles are not arbitrary piles; clearly they inherit the monotonicity condition on the parts of the original plane partition.

If we forget that Figure 2 is meant to show a 3-dimensional object, but rather consider it as graphic object in the plane (which it really is in this printed form), then we realise that this object consists entirely of rhombi with side length 1 (say) and angles of 60° and 120°, respectively, which fit together perfectly, in the sense that there are no "holes" in the figure which are not such rhombi. By adding a few more rhombi, we may enlarge this figure to a hexagon, see Figure 3.b. (At this point, the thin lines should be ignored). This hexagon has the property that opposite sides have the same length, and its angles are all 120°. The upshot of all this is that plane partitions with a rows and b columns, and with parts bounded above by c (recall that parts are non-negative integers) are in bijection with tilings of a hexagon with side lengths a, b, c, a, b, c into unit rhombi, cf. again Figures 1–3, with Figure 3.a showing the assignments of side lengths of the hexagon. This is an important point of view which has only been observed relatively recently [22]. In particular, it was not known at the time when Richard Stanley studied plane partitions. Nevertheless, I shall use this point of view in the exposition here, since it facilitates many explanations and arguments enormously. I would even say that, had it been known earlier, much of the plane partition literature could have been presented much more elegantly ...



a. A hexagon with sides a, b, c, a, b, c, where $a = 3$, $b = 4$, $c = 5$

b. A rhombus tiling of a hexagon with sides a, b, c, a, b, c

FIGURE 3

2. Percy Alexander MacMahon

The problem that MacMahon posed to himself was:

Given positive integers a, b, c, compute the generating function $\sum_{\pi} q^{|\pi|}$,

where the sum is over all plane partitions π contained in an $a \times b \times c$ box!

As explained in the previous section, "contained in an $a \times b \times c$ box" has to be understood in the sense of the "pile of cubes"-interpretation (see Figure 2) of plane partitions. In terms of rhombus tilings, we are considering rhombus tilings of a hexagon with side lengths a, b, c, a, b, c (see Figure 3), while in terms of the original definition (1.2)/(1.3), we are considering plane partitions of shape $(b, b, \dots, b)$ (with a occurrences of b) with parts at most c.

Why were plane partitions so fascinating for MacMahon, and for legions of followers? From his writings, it is clear that MacMahon did not have any external motivation to consider these objects, nor did he have any second thoughts. For him it was obvious that these plane partitions are very natural, as two-dimensional analogues of (linear) partitions (for which at the time already a well established theory was available), and as such of intrinsic interest. Moreover, this intuition was "confirmed" by the extremely elegant product formula in Theorem 1 below. He himself — conjecturally — found another intriguing product formula for so-called "symmetric" plane partitions contained in a given box (see (6.2)). Later many more such formulae were found (again, first conjecturally, and some of them still quite mysterious); see Section 6 below. Moreover, over time it turned out that plane partitions (and rhombus tilings) are related to many other areas of mathematics, most notably to the theory of symmetric functions and representation theory of classical groups (as Richard Stanley pointed out; see Sections 4 and 7), representation theory of quantum groups (cf. [64]), enumeration of integer points in polytopes and commutative algebra (cf. [12, 79, 17] and the references therein), enumeration of matchings in graphs (cf. [62, 66]), and to statistical physics (cf. [39, 61]). In brief, the theory of plane partitions offered challenging conjectures (most of them solved now, but see Section 8), connections to many other areas, and therefore many different views and approaches to attack problems are possible, making this a very rich research field.

Returning to MacMahon, the surprisingly simple formula [73, Sec. 429] he found for the generating function for all plane partitions contained in a given box was the following.

Theorem 1. *The generating function $\sum_{\pi} q^{|\pi|}$ for all plane partitions π contained in an $a \times b \times c$ box is given by*[2]

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}. \tag{2.1}$$

[2]MacMahon did not write the result in this form, for which there are many ways to express it. The particular product (2.1) is due to Macdonald [70, Eq. (2) on p. 81].

We note that, if we let $a, b, c \to \infty$, then we obtain

$$\sum_{\pi} q^{|\pi|} = \prod_{i \geq 1} \frac{1}{(1-q^i)^i}, \tag{2.2}$$

an elegant product formula for the generating function for *all* plane partitions.

MacMahon developed two very interesting methods to prove (2.2) and Theorem 1. First, MacMahon developed a whole theory, which he called *"partition analysis"*, and which today runs under the name of *"omega calculus."* However, he realised finally that it would not do what it was supposed to do (namely prove (2.2)). So he abandoned this approach. It is interesting to note though that recently there has been a revival of MacMahon's partition analysis (see [12]), and Andrews and Paule [6] finally "made MacMahon's dream true."

The second method consists in translating the problem from enumerating plane partitions to the enumeration of — what MacMahon called — *lattice permutations*. This idea led him finally to a proof of Theorem 1 in [73, Sec. 494]. The "correct" general framework for this second method was brought to light by Richard Stanley in his thesis (published in revised form as [88]), by developing his theory of *poset partitions*. See the next section, and the article by Ira Gessel in this volume for an extensive account.

3. The revival of plane partitions and Richard Stanley

As mentioned in the previous section, MacMahon left behind a very intriguing conjecture on "symmetric" plane partitions (these are plane partitions which are invariant under reflection in the main diagonal; see Section 6, Class 2). However, it seems that, at the time, there were not many others who shared MacMahon's excitement for plane partitions. (In particular, nobody seemed to care about his conjecture — or perhaps it was too difficult at the time . . .) In any case, after MacMahon plane partitions were more or less forgotten, except that Wright [100] calculated the asymptotics of the number of plane partitions of n as n tends to infinity. It was Leonard Carlitz [15, 16], and Basil Gordon and Lorne Houten [35, 36] who, in the 1960s, relaunched the interest in plane partitions.

However, the "rebirth" of intensive investigation of plane partitions was instigated by Stanley's two-part survey article [89] *"Theory and applications of plane partitions"*, together with Bender and Knuth's article [10]. Already in his thesis (published in revised form as [88]) Stanley had dealt with plane partitions. More precisely, he introduced a vast generalisation of the notion of plane partitions, *poset partitions*, and built a sophisticated theory around it. In particular, this theory generalised the earlier mentioned key idea of MacMahon in his proof of Theorem 1 to a correspondence between poset partitions and linear extensions of the underlying poset (again, see the article by Ira Gessel). In [89], he laid out the state of affairs in the theory of plane partitions as of 1971, and he presented *his* view of the subject matter. This meant, among others, to emphasise the intimate connection with the theory of symmetric functions. Most importantly, the article [89] offered a truly fascinating reading, in particular pointing out that there were several intriguing open problems, some of them waiting for a solution already for a very long time. I am saying all this frankly admitting that it was this article of Stanley which "sparked the fire"

for plane partitions in me (which is still "burning" in the form of my fascination for the enumeration of rhombus tilings).

One of the original contributions one finds in [89] is an extremely elegant bijective proof of MacMahon's formula (2.2), reported from [87] (but which was published later than [89]). It is an adaption of an idea of Bender and Knuth [10, proof of Theorem 2]. It had the advantage of allowing for an additional parameter to be put in, the *trace* of a plane partition, which by definition is the sum of the parts along the diagonal of the plane partition. More on this is to be found in Section 5.

With the increased interest in the enumeration of plane partition, several new authors entered the subject, the most prominent being George Andrews and Ian Macdonald. This led on the one hand to a proof of MacMahon's conjecture by Andrews [1], a proof of a conjecture of Bender and Knuth in the aforementioned article [10, p. 50] by Gordon [37] and later by Andrews [2], and alternative proofs of both by Macdonald [70, Ex. 16, 17, 19, pp. 83–86]. Moreover, Macdonald introduced another symmetry operation on plane partitions, *cyclic symmetry* (see Section 6, Class 3), and observed that, again, the generating function for plane partitions contained in a given box which are invariant under this operation seems to be given by an elegant closed form product formula (see (6.4)).

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

FIGURE 4. An alternating sign matrix

Then, William Mills, David Robbins and Howard Rumsey "entered the scene," and Stanley had again a part in this development. Robbins and Rumsey [83] had played with a parametric generalisation of the determinant which they called *λ-determinant*,[3] and in that context were led to new combinatorial objects which they called *alternating sign matrices*. An alternating sign matrix is a square matrix consisting of 0's, 1's and (-1)'s such that, ignoring 0's, along each row and each column one reads $1, -1, 1, \dots, -1, 1$ (that is, 1's and (-1)'s alternate, and at the beginning and at the end there stands a 1). Figure 4 shows a 6×6 alternating sign matrix. At some point, they became interested in how many such matrices there were. More precisely, they asked the question: how many $n \times n$ alternating sign matrices are there? By computer experiments combined with human insight, they observed that, apparently, that number was given by the amazingly simple product formula in (6.12). As Robbins writes in [84], after having in vain tried for some time to prove their

[3]It is interesting to note that this — at the time, isolated, and maybe somewhat obscure — object has now become part of a fascinating theory, namely the theory of discrete integral systems (cf. e.g. [23]), and is maybe the first (non-trivial) example of the so-called *Laurent phenomenon* of Fomin and Zelevinsky [29].

conjecture, they began to suspect that this may have something to do with the theory of plane partitions. So they called *the* expert on plane partitions, Richard Stanley. After a few days, Stanley replied that, while he was not able to prove the conjecture, he had seen these numbers before in [3], namely as the numbers of "size n" *descending plane partitions*, a certain variation of cyclically symmetric plane partitions, which Andrews had introduced in his attempts to prove Macdonald's conjectured formula for the latter plane partitions. (Andrews "only" succeeded to prove the $q = 1$ case of Macdonald's conjecture.)

This led to further remarkable developments in the theory of plane partitions. Andrews and Robbins independently came up with a conjectured formula for a certain generating function for plane partitions which are invariant under both reflective and cyclic symmetry (see Section 6, Class 4). Moreover, Mills, Robbins and Rumsey now started to look at alternating sign matrices and plane partitions in parallel, trying to see connections between these seemingly so different objects. One of the directions they were investigating was the study of operations on these objects and whether there existed "analogous" operations on the "other side." One of the outcomes was the discovery of a new (but, in retrospective, "obvious") symmetry operation on plane partitions: *complementation* (see the more detailed account of this discovery in [84]). If combined with the other two symmetry operations — reflective symmetry, cyclic symmetry —, this gave rise to six further symmetry classes, beyond the already "existing" four (see Section 6, Classes 5–10). Amazingly, also for the six new classes, it seemed that the number of plane partitions in each of these six symmetry classes, the plane partitions being contained in a given box, was again given by a nice closed form product formula. Mills, Robbins and Rumsey found striking, much finer enumerative coincidences between plane partitions and alternating sign matrices (see Section 8), and they initiated a programme of enumerating symmetry classes of alternating sign matrices (in analogy with the programme for plane partitions summarised here in Section 6). Ironically, they never succeeded to prove any of their conjectures on alternating sign matrices, but their investigations did enable them to provide a full proof [74] of Macdonald's conjecture on cyclically symmetric plane partitions and a proof of the conjectured formula for another symmetry class of plane partitions (see Section 6, Class 8).

In 1986, Stanley summarised the statements, conjectures, and results (at the time) for symmetry classes of plane partitions in [90]. In another survey article [91] he also included the statements, conjectures, and results for symmetry classes of alternating sign matrices. So-to-speak, he "used the opportunity" in [90] to resolve one of the (then) conjectures for symmetry classes of plane partitions, namely the conjecture on the number of self-complementary plane partitions contained in a given box (see Section 6, Class 6). Stanley's insight that led him to the solution was that the problem could be formulated as the problem of calculating a certain complicated sum, which he identified as a sum of specialised Schur functions. As Stanley demonstrates, "by coincidence," this sum of Schur functions is precisely the expansion of a product of two special Schur functions, as an application of the Littlewood–Richardson rule shows; see Section 7.

So, in brief, here is a summary of Stanley's contributions to the theory of plane partitions:

- *poset partitions* [88];

- *trace generating functions* for plane partitions [87];
- survey of the theory of plane partitions as of 1971, [89];
- application of *symmetric functions* to the theory of plane partitions [89, 90];
- surveys of the state of the art concerning *symmetry classes* of plane partitions and alternating sign matrices as of 1986 [90, 91];
- proof of the conjecture on *self-complementary plane partitions* [90].

However, Stanley's contributions extend beyond this through the work of his students, particularly the work of Emden Gansner (who continued Stanley's work on traces of plane partitions), Ira Gessel (who, together with Viennot, developed the powerful method of non-intersecting lattice paths), Bob Proctor (who proved the enumeration formulae for Classes 6 and 7 and provided new proofs of several others), and John Stembridge (who, among other things, paved the way for Andrews' proof of the enumeration formula for Class 10, and who proved the $q = 1$ case of the enumeration formula for Class 4). Their work will also be discussed below.

In the next section, I give a quick survey of methods that have been used to enumerate plane partitions, covering in particular the non-intersecting path method. Then, in Section 5, I shall describe Stanley's beautiful bijective proof of the formula (2.2) for the generating function for all plane partitions, leading to a refined formula which features the statistic "trace." This section also contains a brief exposition of the subsequent work of Gansner on trace generating functions. Section 6 presents the project of enumeration of symmetry classes of plane partitions, listing all cases and those who proved the corresponding formulae. Pointers to further work are also included. This is followed by a detailed discussion of one of the classes in Section 7, namely the class of self-complementary plane partitions. The central piece of that section is Stanley's proof of the corresponding enumeration formulae. The final section addresses the role of plane partitions in research work of today.

4. Methods for the enumeration of plane partitions

It was said in Section 2 that plane partitions can be approached from many different angles. Which are the methods which have (so far) been applied for the enumeration of plane partitions?

As already explained, MacMahon himself already introduced two methods: first, his *partition analysis* (nowadays often called *"omega calculus"*; cf. [12]), which is a generating function method that, in its utmost generality, can be applied for the enumeration of integer points in d-dimensional space obeying linear inequalities and equalities.

Second, MacMahon introduced a translation of counting problems for plane partitions to counting problems of so-called "lattice permutations," which Stanley [88] generalised to his theory of *poset partitions* (see again Gessel's article in the same volume).

The standard method nowadays for the enumeration of plane partitions is to use *non-intersecting lattice paths* to "reduce" the counting problem to the problem of evaluating

a certain determinant.[4] I want to illustrate this by explaining the non-intersecting paths approach to MacMahon's formula in Theorem 1. For the sake of simplicity, I concentrate on the $q = 1$ case, that is, on the plain enumeration of plane partitions contained in a given box. The case of generic q is by no means more complicated or difficult, it would only require more notation, which I want to avoid here.

Consider the plane partition in Figure 3, which is copied in Figure 5.a. We mark the midpoints of the edges along the south-west side of the hexagon and we start paths there, where the individual steps of the paths always connect midpoints of opposite sides of rhombi. See Figure 5.a,b for the result in our running example. We obtain a collection of paths which connect the midpoints of the edges along the south-west side with the midpoints of edges along the north-east side. Clearly, the paths are *non-intersecting*, meaning that no two paths have any points in common. By slightly deforming the obtained paths, we may place them into the plane integer lattice, see Figure 5.c. What we have obtained is a bijection between plane partitions in an $a \times b \times c$ box and families $(P_1, P_2, \dots, P_b)$ of non-intersecting lattice paths consisting of unit horizontal and vertical steps in the positive direction, where P_i connects $(-i, i)$ with $(a - i, c + i)$, $i = 1, 2, \dots, b$.

The (first) main theorem on non-intersecting lattice paths implies that the number of the families of non-intersecting lattice paths that we have obtained above from our plane partitions can be written in terms of a determinant, see (4.2) below.

Theorem 2. *Let G by a directed, acyclic graph, and let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be vertices in G with the property that any path from A_i to E_l and any path from A_j to E_k with $i < j$ and $k < l$ have a common vertex. Then the number of all families $(P_1, P_2, \dots, P_n)$ of non-intersecting paths, where P_i runs from A_i to E_i, $i = 1, 2, \dots, n$, is given by*

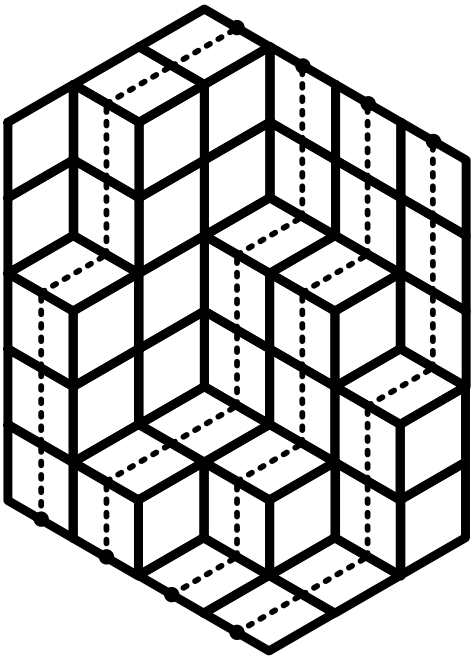
$$\det_{1 \le i,j \le n} \big(L_G(A_j \to E_i) \big), \tag{4.1}$$

where $L_G(A \to E)$ denotes the number of all paths starting in A and ending in E.

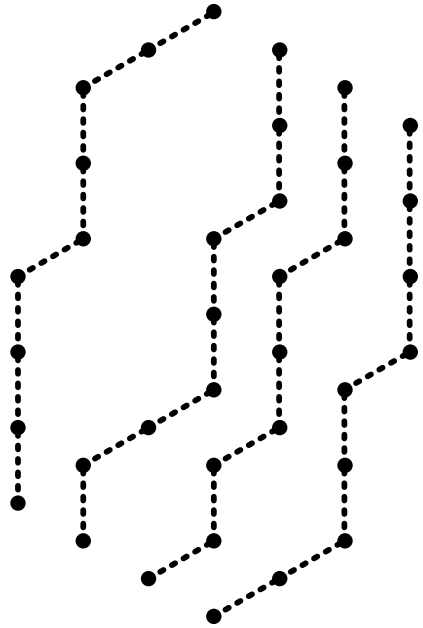
This theorem was discovered and independently rediscovered several times. It is originally due to Lindström [69, Lemma 1] (who, in fact, proved a more general theorem, and which also includes weights in a straightforward way; for a proof of Theorem 1 with generic q, we would need the weighted version), who discovered and used it in the context of matroid representations. It was Gessel and Viennot [32, 33] who realised its enormous significance for the enumeration of plane partitions.[5]

[4] Here, "reduce" is in quotes since, in the harder cases, the most difficult part then is the evaluation of the determinant.

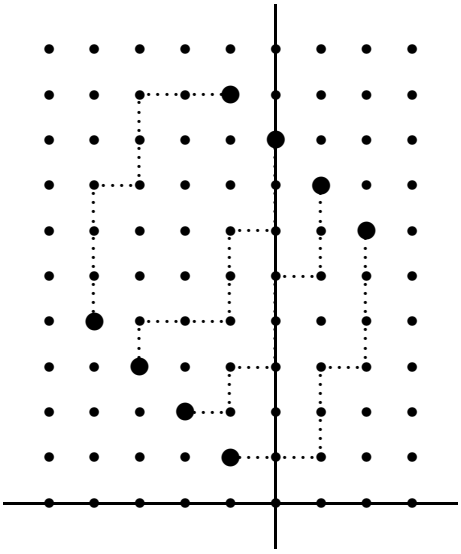
[5] Lindström's theorem was rediscovered (not always in its most general form) in the 1980s at about the same time in three different communities, not knowing of each other at that time: in enumerative combinatorics by Gessel and Viennot [32, 33] in order to count tableaux and plane partitions, in statistical physics by Fisher [25, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, and in combinatorial chemistry by John and Sachs [45] and Gronau, Just, Schade, Scheffler and Wojciechowski [38] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules. It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [47, 46] in a probabilistic framework, as well as that the so-called "Slater determinant" in



a. A plane partition



b. Non-intersecting lattice paths



c. Non-intersecting lattice paths made orthogonal

FIGURE 5

quantum mechanics (cf. [85] and [86, Ch. 11]) may qualify as an "ancestor" of the determinantal formula of Lindström.

If we apply Theorem 2 with the graph consisting of the integer points in the plane as vertices and edges being horizontal and vertical unit vectors in the positive direction, with $A_i = (-i, i)$ and $E_i = (a - i, c + i)$, then we obtain that the number of plane partitions contained in an $a \times b \times c$ box equals the determinant

$$\det_{1\le i,j\le b}\left(\binom{a+c}{a-i+j}\right). \tag{4.2}$$

There are many ways to evaluate this determinant, see e.g. [60, Secs. 2.2, 2.3, 2.5], and the result can be written in the form

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}, \tag{4.3}$$

which is the $q \to 1$ limit of (2.1).

Theorem 2 solves the problem of counting non-intersecting lattice paths for which starting *and* end points are fixed. If one tries the non-intersecting lattice path approach for symmetry classes of plane partitions, then one often encounters the problem that starting or/and end points are not fixed, but rather may vary in given sets. Inspired by work of Okada [78] on summations of minors of given matrices (which was later generalised to the minor summation theorem of Ishikawa and Wakayama [43]), Stembridge developed the corresponding theory, which provides Pfaffian formulae (and thus again determinantal formulae, given the fact that the Pfaffian is the square root of a determinant) for the number of non-intersecting lattice paths where either starting or end points, or both, vary in given sets. With the exception of only very few, it is the non-intersecting lattice path method which constitutes the first step in proofs of enumeration formulae for symmetry classes of plane partitions, may it be explicitly or implicitly. We refer the reader to Section 7 for an "implicit" application of non-intersecting lattice paths (in the sense that Stanley's original proof did not mention non-intersecting lattice paths; they are still there).

Symmetric functions and representation theory come into play because plane partitions are very close to the tableau-like objects that index representations of classical groups. As said earlier, Stanley [89] was the first to emphasise and exploit this connection. It was picked up and deepened by Macdonald [70, Ch. I, Sec. 5, Ex. 13–19], Proctor [80, 81, 82], Stembridge [93, 94, 96], Okounkov, Reshetikhin, and Vuletić (see [99] and the references therein), and Kuperberg [64]. To give a flavour, let us again consider the enumeration of all plane partitions in an $a \times b \times c$ box. In terms of the original definition (1.2), these are plane partitions of shape $(b, b, \dots, b)$ (with a occurrences of b) with entries at most c. An example with $a = 3$, $b = 4$, $c = 6$ is shown in Figure 4.a. (The corresponding "graphical" representations are the ones in Figures 2 and 3.) We rotate the plane partition by 180° and add i to each entry in row i, for $i = 1, 2, \dots, c$. Thereby we obtain an array of integers of the same shape as the original plane partition, however with the property that entries along rows are weakly increasing and entries along columns are strictly increasing, all of them between 1 and $a + c$. Arrays satisfying the above two monotonicity properties are

called *semistandard tableaux*. See Figure 4.b for the semistandard tableau which arises in this way from the plane partition in Figure 4.a.

5	3	3	2
5	1	1	0
3	1	0	0

a. A plane partition

1	1	2	4
2	3	3	7
5	6	6	8

b. The corresponding semistandard tableau

FIGURE 6

It is a central fact of the representation theory of $SL_n(\mathbb{C})$ that semistandard tableaux of shape λ (where shape is defined in the same way as for a plane partition) with entries between 1 and n index a basis of an irreducible representation of $SL_n(\mathbb{C})$. On the character level, this is reflected by the *Schur function* $s_\lambda(x_1, x_2, \dots, x_n)$ (cf. [70]), which is defined by

$$s_\lambda(x_1, x_2, \dots, x_n) = \sum_T \prod_{i=1}^{n} x_i^{\#\text{entries } i \text{ in T}}, \tag{4.4}$$

where the sum is over all semistandard tableaux T of shape λ with entries at most n. Thus, we see that MacMahon's generating function for plane partitions essentially equals a specialised Schur function, namely we have

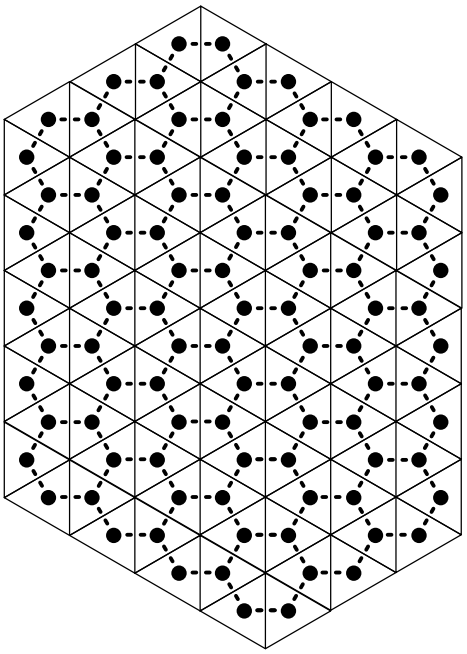
$$\sum_\pi q^{|\pi|} = q^{-b\binom{a+1}{2}} s_{(b,b,\dots,b)}(q, q^2, \dots, q^{a+c}), \tag{4.5}$$

the sum on the left-hand side being over all plane partitions π contained in an $a \times b \times c$ box. If this "principal specialisation" of the variables, i.e., $x_i = q^i$, $i = 1, 2, \dots, a + c$, is done in the Weyl character formula for Schur functions (cf. [70, p. 40, Eq. (3.1)]),

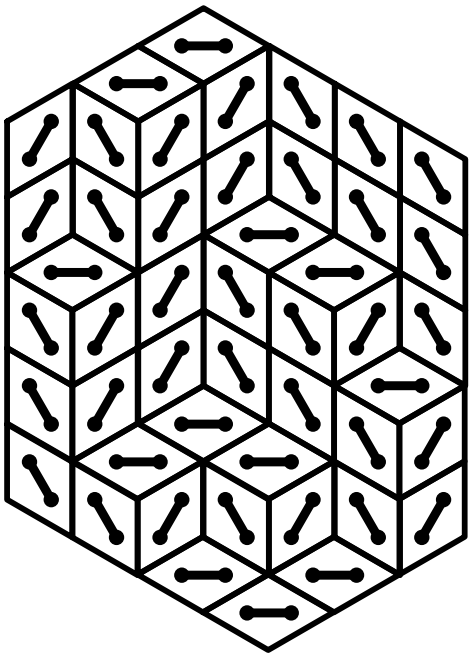
$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det_{1\le i,j\le n}(x_i^{\lambda_j+n-j})}{\det_{1\le i,j\le n}(x_i^{n-j})},$$

then both determinants are Vandermonde determinants and can therefore be evaluated, thus establishing (2.1).

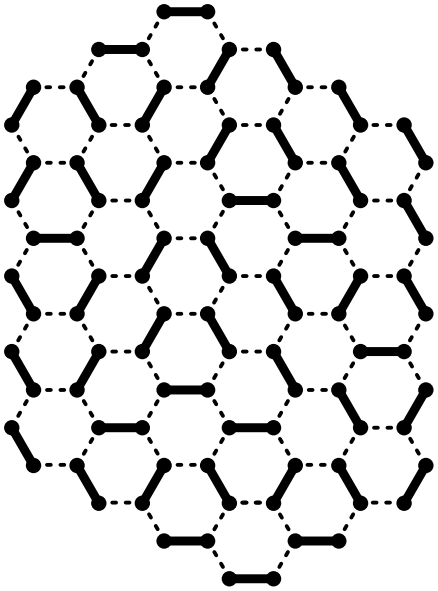
The next important method to enumerate plane partitions is actually a more general method to enumerate *perfect matchings of bipartite graphs*. A perfect matching of a graph is a collection of pairwise non-incident edges which cover all vertices of the graph. To see how plane partitions can indeed be seen as perfect matchings of certain graphs, in the triangular grid inside the bounding hexagon (see Figure 3.a) place a vertex into the centre of each triangle and connect vertices in adjacent triangles by an edge. In this way, we obtain a hexagonal graph; see Figure 7.a. A unit rhombus is the union of two adjacent triangles. Consequently, a rhombus tiling of a given hexagon corresponds to a collection of edges in the hexagonal graph which are pairwise non-incident and cover all vertices of the graph; in other words: a perfect matching of the hexagonal graph. See Figure 7.b,c for an example.



a. The triangular grid with its "dual" graph



b. A rhombus tiling and corresponding edges in the "dual" graph



c. The corresponding perfect matching of the "dual" graph

FIGURE 7

Let G be a given graph. If we consider the defining expansion of the Pfaffian (cf. [94, p. 102]) of the adjacency matrix of G, then each term in this expansion corresponds to a perfect matching of G. Kasteleyn [48, 49] (see [98] for an excellent exposition) showed that for planar graphs there is a consistent way to introduce signs to the 1's in the adjacency matrix so that the Pfaffian of this *modified* adjacency matrix (often called *Kasteleyn–Percus matrix*) has the property that all terms in its expansion have the same sign, and hence gives the number of perfect matchings of G. Since in the case of a bipartite graph, this Pfaffian reduces to a determinant (namely the determinant of the suitably modified bipartite adjacency matrix of the graph), this approach has been called *"permanent-determinant method"* by Kuperberg [62, 66]. He has been the first and only one to effectively apply this approach for exact enumeration of plane partitions, providing in particular the first proof for the enumeration of cyclically symmetric self-complementary plane partitions (Class 9 in Section 6). As a matter of fact, since the permanent-determinant method produces determinants (and permanents and Pfaffians) of very large matrices, this approach is best suited for conceptual insight in the enumeration of plane partitions (and, more generally, perfect matchings of bipartite graphs; see e.g. the last paragraph in Section 8), whereas less so for proving explicit exact enumeration formulae. It is interesting to note, though, that Kuperberg has shown that, for planar graphs, the non-intersecting lattice path method and the permanent-determinant method are equivalent, in a sense explained in [67, Sec. 3.3]. This does not include the situations where the starting or/and end points of the non-intersecting lattice paths to be counted vary in given sets (cf. the paragraph below (4.3)); Kuperberg has however also presented workarounds in [62] to simulate certain such situations within the permanent-determinant method, and he speculates in [67, end of Sec. 3.3] that this is always possible (in the planar case).

Extremely useful tools for the enumeration of plane partitions (and, more generally, perfect matchings) are moreover Kuo's condensation method [54], which allows for inductive proofs of (conjectured) enumeration formulae, Ciucu's matchings factorisation theorem [18] for enumerating perfect matchings of graphs with a reflective symmetry, and Jockusch's theorem [44] for graphs with a rotational symmetry. As Fulmek [28] (for the former) and Kuperberg [66] (for the latter two) have shown, all three are consequences of the permanent-determinant method.

Finally, among the purely combinatorially methods that have been applied to the enumeration of plane partition, Robinson–Schensted–Knuth-like correspondences must be mentioned; see [30, 31, 87] and the next section.

5. Trace generating functions

In this section, I present one of the main results from Stanley's first article [87] on plane partitions.[6] There, Stanley introduces the notion of "trace" of a plane partition.

[6]It is indeed the chronologically first article, see Stanley's publication list `http://www-math.mit.edu/~rstan/pubs/` on his website. The order of his papers in the References section at the end of this article follows this order.

As already said above, the *trace* of a plane partition π as in (1.2) is the sum of its parts along the main diagonal, that is, $\sum_{i\geq 1}\pi_{i,i}$. He is led to this notion by observing that a combination of a construction of Frobenius [27, p. 523] for partitions with a variation of the Robinson–Schensted–Knuth correspondence [52] leads to an elegant and effortless proof of (2.2), and that in this proof a certain statistic, namely the trace, is preserved, leading to a refinement of (2.2) (see (5.5) below).

More precisely, let us consider the plane partition

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 4 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & & \\ 1 & 1 & & \end{array} \tag{5.1}$$

Each row by itself is a(n ordinary) partition. For example, the first row represents the partition $4+3+2+2$. A partition $\lambda_1+\lambda_2+\cdots+\lambda_k$ may be represented as a *Ferrers diagram*, that is, as a left-justified diagram of dots, with λ_i dots in the i-th row. Thus, the Ferrers diagram of the above partition is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array} \tag{5.2}$$

The partition *conjugate* to λ is the partition λ' corresponding to the transpose of the Ferrers diagram of λ. Thus, the partition conjugate to $4+3+2+2$ is $4+4+2+1$.

By conjugating each row of (5.1), we obtain

$$\begin{array}{cccc} 4 & 4 & 2 & 1 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & & \\ 2 & & & \\ 2 & & & \end{array} \tag{5.3}$$

It should be noted that the trace of the original plane partition — in our example in (5.1) this is $4+3+1=8$ — equals the number of parts $\pi_{i,j}$ with $\pi_{i,j}\geq i$ in the plane partition one obtains under this mapping. Moreover, the total sum of the parts of the plane partition is preserved under this operation.

Obviously, each *column* of the plane partition obtained is a(n ordinary) partition. We now write each column in (essentially) its *Frobenius notation*. More precisely, out of a partition $\lambda_1+\lambda_2+\cdots+\lambda_k$, we form the pair

$$(\lambda_1, \lambda_2-1, \lambda_3-2, \dots \mid \lambda'_1, \lambda'_2-1, \lambda'_3-2, \dots),$$

with λ' denoting the conjugate of λ, and only recording positive numbers. (In the Ferrers diagram picture, this corresponds to splitting the partition λ along its main diagonal, "counting" the main diagonal "twice". In Frobenius' original notation, the main diagonal is *not* accounted for.) For example, from the partition $4+3+2+2$ in our running example

we obtain the pair $(4,2 \mid 4,3)$. We apply this modified Frobenius notation to each column of the plane partition we have obtained so far, and we collect the first components and the second components separately. Thus, for our running example (5.3), we obtain

$$\begin{array}{cccc} 4 & 4 & 2 & 1 \\ 3 & 1 & 1 & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} 5 & 3 & 2 & 2 \\ 4 & 2 & 1 & \\ 1 & & & \end{array} \tag{5.4}$$

(Here, the pair $(4,3,2 \mid 5,4,1)$ corresponds to the first column in (5.3), etc.) It is easy to see that in this way we obtain a pair (C_1, C_2) of so-called *column-strict* plane partitions, which are plane partitions with the additional condition that parts along columns are *strictly* decreasing. Moreover, C_1 and C_2 have the same shape. It should be observed that the trace of the original plane partition equals the number of parts of C_1 (or of C_2). It is not true that the total sum of the parts of the original plane partition equals the sum of all the parts of C_1 and C_2. However, what is true is that the total sum of the original plane partition equals the sum of all parts of C_1 and C_2 *minus* the *number* of parts of C_1.

By a variant of the Robinson–Schensted–Knuth correspondence (see [92, p. 368]), the pairs (C_1, C_2) correspond to matrices $(m_{i,j})_{i,j\geq 1}$ with a finite number of positive integer entries, all other entries being zero. In this correspondence, the total sum of the parts of C_1 and C_2 equals $\sum_{i,j\geq 1}(i+j)m_{i,j}$ (actually, something much finer is true), and the number of parts of C_1 equals $\sum_{i,j\geq 1} m_{i,j}$. If one puts everything together, this shows (see [87, Sec. 3 in combination with Theorem 2.2])

$$\sum_{\pi} t^{\mathrm{trace}(\pi)} q^{|\pi|} = \prod_{i,j\geq 1} \frac{1}{1-tq^{i+j-1}}. \tag{5.5}$$

In [87], we also find the notion of *"conjugate trace"*, together with further generating function results for plane partitions featuring that variation of "trace."

Almost ten years later, Stanley's first student, Emden Gansner, showed that there is much more to Stanley's concept of "trace." He generalised it and defined the *i-trace* of a plane partition π, denoted by $t_i(\pi)$, as the sum of the parts of the i-diagonal of π, that is, as $\sum_{\ell\geq 1} \pi_{\ell,\ell+i}$. Thus, in this more general context, Stanley's original trace is the 0-trace.

Gansner's result on trace generating functions in [31] actually concerns *reverse plane partitions*, which are arrays of non-negative integers as in (1.2) with the property that entries along rows and along columns are weakly *increasing*. For the formulation of the result, we need to define the monomial $\mathbf{x}(\rho;\lambda)$ for a "cell" ρ of a partition λ (a cell corresponds to a bullet in the Ferrers diagram of λ; see (5.2)). If ρ is located in the i-th row and the j-th column of λ, then

$$\mathbf{x}(\rho) := x_{j-\lambda'_j} \cdots x_{\lambda_i - i - 1} x_{\lambda_i - i},$$

where λ' is again the partition conjugate to λ (see the paragraph after (5.2)). Using this notation, Gansner's elegant result [31, Theorem 5.1] is

$$\sum_{\pi} \prod_{i=1-\ell(\lambda)}^{\lambda_1-1} x_i^{t_i(\pi)} = \prod_{\rho\in\lambda} \frac{1}{1-\mathbf{x}(\rho)}, \tag{5.6}$$

where the sum is over all reverse plane partitions π of shape λ. The proof is bijective; it exploits hidden properties of a correspondence of Hillman and Grassl [42] that Gansner lays open. He continued that work in [31], where he considers trace generating functions for (ordinary) plane partitions. As it turns out, the trace generating function (with all traces present) for plane partitions of a given shape does not evaluate to an equally elegant product formula as for reverse plane partitions (except for rectangular shapes, which however follows trivially from the result (5.6) for reverse plane partitions by a 180° degree rotation), one has to be content with formulae given in terms of certain sums. In order to derive these results, Gansner uses Robinson–Schensted–Knuth-like correspondences due to Burge [13]. It is shown in [55, Theorems 7.6, 7.7] that, for generating functions for plane partitions which only take into account the size and 0-trace of plane partitions, there are closed form products available for certain shapes.

Finally, we mention that Proctor [82] has derived several results for *alternating trace* generating functions by relating these to characters of symplectic groups.

6. The 10 symmetry classes of plane partitions

The purpose of this section is to summarise the programme of enumeration of symmetry classes of plane partitions, the history of which was reviewed in Section 3. I start by defining the symmetry operations which give rise to 10 symmetry classes of plane partitions. Then, for each symmetry class, I state the corresponding enumeration formula, and report who had proved that formula, sometimes with pointers to further work.

The first operation is *reflection*. To reflect a plane partition π as in (1.2) means to reflect it along the main diagonal, that is, to map $\pi = (\pi_{i,j})_{i,j}$ to $\pi = (\pi_{j,i})_{i,j}$. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, reflection means reflection along the vertical symmetry axis of the hexagon (if there is one).

Rotation is the second operation. Given a plane partition π, viewed as a pile of cubes as in Figure 2, to rotate it means to rotate it by 120° with rotation axis $\{(t,t,t) : -\infty < t < \infty\}$ (this is the rotation which leaves the origin of the coordinate system invariant and maps the coordinate axes to each other; for our purposes it is irrelevant whether we consider a "left" or "right" rotation). In the representation of a plane partition as a rhombus tiling as in Figure 3.b, this rotation corresponds to a rotation of the hexagon (together with the tiling) by 120°.

Intuitively, the *complement* of a plane partition in a given $a\times b\times c$ box is what the name says, namely the pile of cubes which remains (suitably rotated and reflected) after the plane partition is removed from the box. To be precise, if the cubes of a plane partition π contained in an $a\times b\times c$ box are coordinatised in the obvious way by (i,j,k) with $1\le i\le a$,

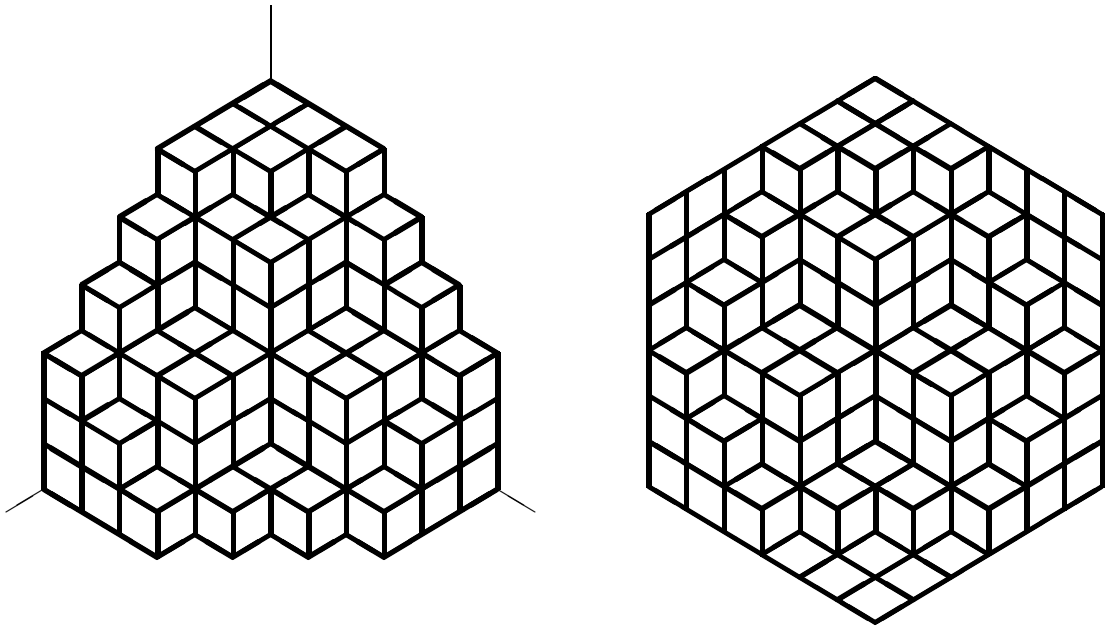


FIGURE 8. A totally symmetric self-complementary plane partition — as a pile of cubes, and as a rhombus tiling of a hexagon

$1 \le j \le b$, and $1 \le k \le c$, then the complement π^c of π is defined by

$$\pi^c = \{(a+1-i, b+1-j, c+1-k) : (i,j,k) \notin \pi\}.$$

In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to take the complement of a plane partition means to rotate the hexagon (together with the tiling) by 180°.

If one combines the three operations — reflection, rotation, complementation — in all possible ways, then there result ten *symmetry classes* of plane partitions. Remarkably, in all ten cases, there exist closed form product formulae for the number of plane partitions contained in a given box, respectively for certain generating functions. The first complete presentation of the corresponding results and conjectures was given by Stanley in [90], together with the state of the art at the time, and the proof of the conjectured formulae for Class 5 (see also Section 7).

While reading the descriptions of the various symmetry classes, Figure 8 may be helpful. It shows a plane partition which has *all* possible symmetries. Thus, it belongs to all ten classes.

In order to have a convenient notation to write down some of the formulae, we use the symbol $N_d(a,b,c)$ for the number of all plane partitions in symmetry class d which are contained in an $a \times b \times c$ box.

CLASS 1: *Unrestricted Plane Partitions.* As we already said, MacMahon proved (see Theorem 1) that the generating function for all plane partitions contained in an $a \times b \times c$

box is given by

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}, \tag{6.1}$$

where the sum is over all plane partitions π contained in an $a \times b \times c$ box.

Class 2: *Symmetric Plane Partitions.* A plane partition π as in (1.2) is called *symmetric* if it is invariant under reflection along the main diagonal. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be symmetric means to be invariant under reflection along the vertical symmetry axis of the hexagon (two of the side lengths of the hexagon must be equal).

For this class, two different weights lead to generating functions which have closed form product formulae. The first weight is the usual "size" of a plane partition, and it leads to MacMahon's conjecture [72] that

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \frac{1-q^{c+2i-1}}{1-q^{2i-1}} \prod_{1\le i<j\le a} \frac{1-q^{2(c+i+j-1)}}{1-q^{2(i+j-1)}}, \tag{6.2}$$

where the sum is over all symmetric plane partitions π that are contained in an $a \times a \times c$ box. MacMahon's conjecture was first proved by Andrews [1] and Macdonald [70, Ex. 16 and 17, pp. 83–85]. Since then, several other proofs and refinements were given, see [81, Prop. 7.3], [82, Theorem 1, Case BYI], [63, Sec. 5], [56, Cor. 12], [57, Theorem 5].

The second weight is roughly "half" the size of a plane partition, and it leads to a conjecture of Bender and Knuth [10, p. 50]. Given a plane partition π as in (1.2), the weight $|\pi|_0$ of π is defined as the sum $\sum_{1\le j\le i} \pi_{i,j}$. Then

$$\sum_{\pi} q^{|\pi|_0} = \prod_{1\le i\le j\le a} \frac{1-q^{c+i+j-1}}{1-q^{i+j-1}}, \tag{6.3}$$

where the sum is over all symmetric plane partitions π that are contained in an $a \times a \times c$ box. Bender and Knuth's conjecture was first proved by Gordon [37] (as reported in [89, Prop. 16.1]), but published only much later. The first published proofs are due to Andrews [2] and Macdonald [70, Ex. 19, p. 86]. Since then, several other proofs and refinements were given, see [81, Prop. 7.2], [82, Theorem 1, Case BYH], [56, Theorem 21], [57, Theorem 6], [24].

Class 3: *Cyclically Symmetric Plane Partitions.* A plane partition π is called *cyclically symmetric* if, viewed as a pile of cubes as in Figure 2, it is invariant under rotation by 120° as described above. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be cyclically symmetric means to be invariant under rotation of the hexagon (together with the tiling) by 120° (all sides of the hexagon must have the same length).

For this class, Macdonald [70, Ex. 18, p.85] conjectured

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{a} \frac{1-q^{3i-1}}{1-q^{3i-2}} \prod_{1\le i<j\le a} \frac{1-q^{3(2i+j-1)}}{1-q^{3(2i+j-2)}} \prod_{1\le i<j,k\le a} \frac{1-q^{3(i+j+k-1)}}{1-q^{3(i+j+k-2)}}, \tag{6.4}$$

where the sum is over all cyclically symmetric plane partitions π that are contained in an $a \times a \times a$ box.

Andrews [3, Theorem 4] had found a determinant for the generating function in question; in retrospective this determinant can be explained by non-intersecting lattice paths (which have been discussed in Section 4). The conjecture was proved by Mills, Robbins and Rumsey [74] by evaluating this determinant using some beautiful linear algebra, and it is still the only proof.

Class 4: *Totally Symmetric Plane Partitions.* A plane partition is called *totally symmetric* if it is at the same time symmetric and cyclically symmetric.

For this class, Andrews and Robbins (see [4]) conjectured

$$\sum_{\pi} q^{|\pi|_0} = \prod_{1\le i\le j\le k\le a} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}, \tag{6.5}$$

where the sum is over all totally symmetric plane partitions π that are contained in an $a \times a \times a$ box.

Okada [78, Theorem 5] found a determinant for the generating function in question. However, for a long time nobody knew how to evaluate this determinant. In the $q = 1$ special case, Stembridge [97] was able to relate a slightly different determinant to the enumeration of cyclically symmetric plane partitions, thereby solving the problem of plain enumeration of totally symmetric plane partitions. Unfortunately, it seems that this very conceptual approach does not extend to the q-case. This was the longest standing conjecture of all plane partition conjectures. Finally, Koutschan, Kauers and Zeilberger [53] succeeded to evaluate Okada's determinant, by a (heavily) computer-assisted argument.

Class 5: *Self-Complementary Plane Partitions.* A plane partition is called *self-complementary* if it is equal to its complement. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be self-complementary means to be invariant under rotation of the hexagon (together with the tiling) by 180°. In other words, to be self-complementary in the rhombus tiling picture means to be centrally symmetric.

For this class, Robbins and Stanley independently conjectured that

$$N_5(2a, 2b, 2c) = N_1(a, b, c)^2, \tag{6.6a}$$

$$N_5(2a+1, 2b, 2c) = N_1(a, b, c)N_1(a+1, b, c), \tag{6.6b}$$

$$N_5(2a+1, 2b+1, 2c) = N_1(a+1, b, c)N_1(a, b+1, c). \tag{6.6c}$$

As mentioned above, Stanley proved this conjecture in [90, Sec. 3], see Section 7. For a different proof see [63, Sec. 4] and [96, Ex. 4.2].

Class 6: *Transpose-Complementary Plane Partitions.* A plane partition is called *transpose-complementary* if it is equal to the reflection of its complement. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be transpose-complementary means to be invariant under reflection of the hexagon (together with the tiling) in its horizontal symmetry axis (two of the side lengths of the hexagon must be equal).

The number of all transpose-complementary plane partitions contained in an $a \times a \times c$ box is equal to

$$\binom{c+a-1}{a-1} \prod_{1 \le i \le j \le a-2} \frac{2c+i+j+1}{i+j+1}. \tag{6.7}$$

This was proved by Proctor [82, paragraph above Cor. 1, Cases 'CG'] by (in essence) relating these plane partitions to the symplectic tableaux of King and El-Sharkaway [51]. For a different proof see [63, Sec. 6].

Class 7: *Symmetric Self-Complementary Plane Partitions.* A plane partition is called *symmetric self-complementary* if it is equal to its complement and invariant under reflection in the main diagonal. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be symmetric self-complementary means to be invariant under reflection of the hexagon (together with the tiling) in its vertical symmetry axis and under reflection in its horizontal symmetry axis (all side lengths of the hexagon must be equal). For this class, we have

$$N_7(2a, 2a, 2c) = N_1(a, a, c), \tag{6.8a}$$

$$N_7(2a+1, 2a+1, 2c) = N_1(a+1, a, c). \tag{6.8b}$$

This was proved by Proctor [80] by again using the representation theory of symplectic groups. For a different proof see [96, Ex. 4.3].

Class 8: *Cyclically Symmetric Transpose-Complementary Plane Partitions.* A plane partition is called *cyclically symmetric transpose-complementary* if it is equal to the transpose of its complement and invariant under rotation by 120°. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be cyclically symmetric transpose-complementary means to be invariant under rotation of the hexagon (together with the tiling) by 120° and under reflection in its horizontal symmetry axis (all side lengths of the hexagon must be equal).

The number of all cyclically symmetric transpose-complementary plane partitions contained in a $2a \times 2a \times 2a$ box is equal to

$$\prod_{i=0}^{a-1} \frac{(3i+1)!\,(6i)!\,(2i)!}{(4i+1)!\,(4i)!}. \tag{6.9}$$

This was proved by Mills, Robbins and Rumsey [77] by an interesting determinant evaluation. For a different proof, using Ciucu's matchings factorisation theorem [18], see [20, Sec. 4].

Class 9: *Cyclically Symmetric Self-Complementary Plane Partitions.* A plane partition is called *cyclically symmetric self-complementary* if it is equal to its complement and invariant under rotation by 120°. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be cyclically symmetric self-complementary means to be invariant under rotation of the hexagon (together with the tiling) by 60° (all side lengths of the hexagon must be equal).

The number of all cyclically symmetric self-complementary plane partitions contained in a $2a \times 2a \times 2a$ box is equal to

$$\prod_{i=0}^{a-1} \frac{(3i+1)!^2}{(a+i)!^2}. \tag{6.10}$$

This was first proved by Kuperberg [62] by using Kasteleyn's method (cf. [48, 49]). Ciucu [19] found a combinatorial explanation of the factorisation

$$N_9(2a, 2a, 2a) = N_{10}(2a, 2a, 2a)^2 \tag{6.11}$$

that results from a comparison of (6.10) and (6.12), based on his matching factorisation theorem [18]. A direct proof of (6.10), using Ciucu's matchings factorisation theorem [18], can be found in [20, Sec. 5].

Class 10: *Totally Symmetric Self-Complementary Plane Partitions.* A plane partition is called *totally symmetric self-complementary* if it is equal to its complement, is invariant under rotation by 120°, and also under reflection in the main diagonal. In the representation of a plane partition as a rhombus tiling as in Figure 3.b, to be totally symmetric self-complementary means to be invariant under all symmetry operations of the hexagon, that is, under reflection of the hexagon (together with the tiling) in its horizontal symmetry axis and under rotation by 60° (all side lengths of the hexagon must be equal).

The number of all totally symmetric self-complementary plane partitions contained in a $2a \times 2a \times 2a$ box is equal to

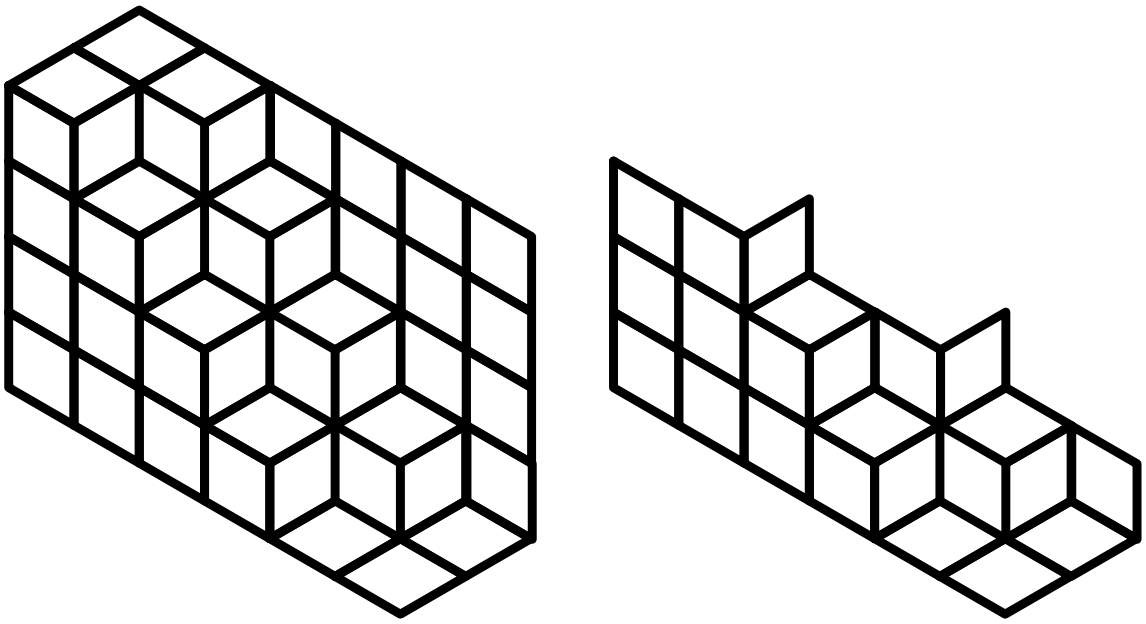
$$\prod_{i=0}^{a-1} \frac{(3i+1)!}{(a+i)!}. \tag{6.12}$$

Stembridge [94, Theorem 8.3] found a determinant for this number using non-intersecting lattice paths, and Andrews [5] succeeded to evaluate this determinant. See [7] for a much simpler proof of that determinant evaluation, and [59] for a generalisation containing an additional parameter.

7. Stanley's proof of the formula for self-complementary plane partitions

Here is a "modern" version of Stanley's proof of (6.6), "modern" in the sense that we argue using the rhombus tiling point of view of plane partitions. We want to count all self-complementary plane partitions contained in an $a \times b \times c$ box, or, equivalently, all rhombus tilings of a hexagon with side lengths a, b, c, a, b, c which are invariant under rotation by 180°. There are in fact three cases to consider: all three of a, b, c are even, two of a, b, c are even, or only one of a, b, c is even. Clearly, if all of a, b, c are odd, then there does not exist such a self-complementary plane partition (the plane partition together with its complement have to fill the complete $a \times b \times c$ box, which consequently must have even volume).

We focus on the case where all of a, b, c are even, say $a = 2a_1$, $b = 2b_1$, $c = 2c_1$, all other cases being similar. Figure 9.a shows an example of a rhombus tiling contained in a $2 \times 6 \times 4$ box which is invariant under rotation by 180°. Obviously, one half of the tiling determines

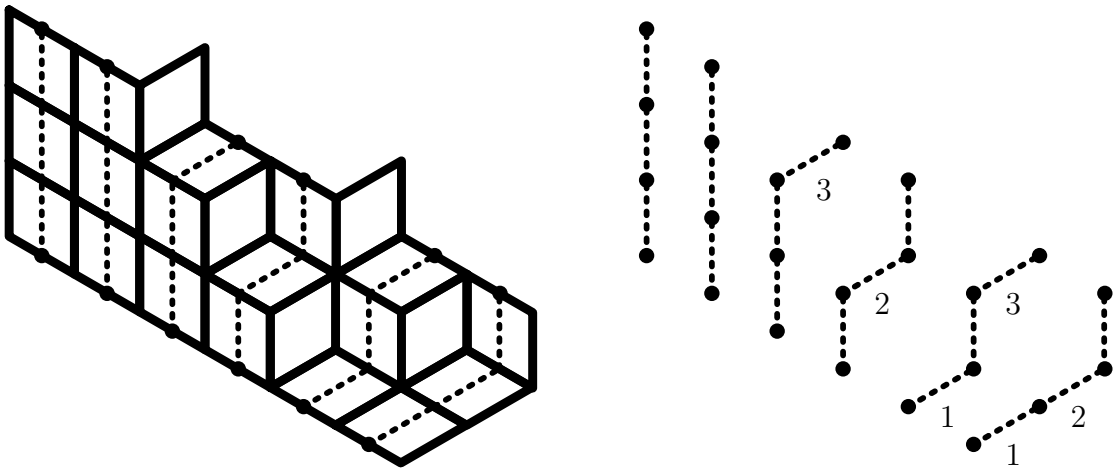


a. A self-complementary plane partition b. Its lower "half"

FIGURE 9

the rest. Therefore we may concentrate on just the lower half, see Figure 9.b. It should be noted that there are some rhombi of the original tiling which stick out of the half-hexagon along the cut. In fact, there are exactly $\min\{a, c\}$ such rhombi. Moreover, since the original tiling was centrally symmetric, these rhombi must be arranged symmetrically along the cut. Now, as in Section 4 (see Figure 5), we mark the midpoints of the edges along the south-west side of the half-hexagon, and we start paths there, where the individual steps of the paths always connect midpoints of opposite sides of rhombi. See Figure 10 for the result in our running example. We obtain a collection of paths which connect the midpoints of the edges along the south-west side with the midpoints of parallel edges along the cut. Clearly, the paths are *non-intersecting*, meaning that no two paths have any points in common. Each path consists of $\frac{1}{2}(a + c)$ steps.

The next step is to translate these sets of paths into so-called semistandard tableaux. In order to do this, we label "right steps" of paths by their distance to the south-west side of the half-hexagon. More precisely, a right step in a path is labelled by i if it is the i-th step of this path. Up-steps remain unlabelled. See Figure 10.b for the labelling in our running example. Now we read the labels of the paths in order, and put the labels of the bottom-most path into the first column, the labels of the path next to it into the second column, etc., of a tableau. Figure 11 shows the tableau which we obtain for the tiling in Figure 9.a. By construction, the entries of the tableau are strictly increasing along columns. Furthermore, it is not difficult to see that entries along rows are weakly increasing. As we said in Section 4, these two properties define a *semistandard tableau*.



a. Half of a self-complementary plane partition b. Non-intersecting lattice paths

FIGURE 10

1	1	2	3
2	3		

FIGURE 11. The corresponding semistandard tableau

The shape of the tableau we obtain (where shape is defined in the same way as for a plane partition) is determined by the position of the rhombi which stick out of the cut line of the half-hexagon. Thus, it is not unique, but on the other hand it cannot be arbitrary since it inherits the property of these rhombi to be arranged symmetrically along the cut. To be precise, the only shapes which are possible are the ones of the form

$$(b_1+\delta_1, b_1+\delta_2, \dots, b_1+\delta_{a_1}, b_1-\delta_{a_1}, \dots, b_1-\delta_2, b_1-\delta_1), \tag{7.1}$$

where $(\delta_1, \delta_2, \dots, \delta_{a_1})$ ranges over all sequences with $b_1 \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{a_1} \geq 0$.

If we summarise the discussion so far, we see that the number of self-complementary plane partitions contained in a $2a_1 \times 2b_1 \times 2c_1$ box is equal to

$$\sum_{\lambda} \left| \mathcal{T}_\lambda \left(\tfrac{1}{2}(a+c) \right) \right|, \tag{7.2}$$

where $\mathcal{T}_\lambda(m)$ denotes the set of semistandard tableaux of shape λ with entries between 1 and m, and the sum is over all partitions λ in (7.1). By (4.4), we see that $|\mathcal{T}_\lambda(m)|$ equals a specialised Schur function. More precisely, we have

$$|\mathcal{T}_\lambda(m)| = s_\lambda(1, 1, \dots, 1), \tag{7.3}$$

with m occurrences of 1. Thus, the sum in (7.2) is a sum of specialised Schur functions. The crucial observation of Stanley is that in fact

$$\sum_{\lambda} s_\lambda(x_1, x_2, \dots) = s_{(b_1, b_1, \dots, b_1)}(x_1, x_2, \dots)^2, \tag{7.4}$$

with a_1 repetitions of b_1, where the sum is over all partitions λ in (7.1). Once this is observed, it is readily verified by means of the Littlewood–Richardson rule (cf. [70, Ch. 1, Sec. 9]). Thus, in view of (7.2)–(7.4), all we have to do is to set $x_i = 1$ for $i = 1, 2, \dots, \frac{1}{2}(a+c)$ in (7.4) and evaluate the specialised Schur function on the right-hand side. How the latter is done, was reviewed in the paragraph following (4.5). (In the result, one has to perform the limit $q \to 1$.) Thus we have proved (6.6a).

8. All problems solved?

Is this the "end of the story"? Or is there still something left? Yes, as reported in Section 6, all conjectures on the enumeration of symmetry classes of plane partitions have now (finally) been established. However, although significant advances have been made since Richard Stanley studied plane partitions, one cannot claim that we have a good understanding of the formulae. Namely, an immediate question which poses itself is whether there are insightful explanations why, for all ten symmetry classes, there are closed form product formulae for the number or the generating function of plane partitions in the class which are contained in a given box? Representation theory, in the form of the observation that certain plane partitions index bases of representations of classical groups and the fact that there are closed form product formulae for the dimensions of such representations provide explanations for several symmetry classes, namely for Classes 1, 2, 3 (only for $q = 1$), 5, 6, 7, 8 (see [63, 64, 82] and the references therein). Not only is it not one idea that works for all these classes but several, each of which working for a subset, for the remaining cases, no such explanation is available. The "worst" case is certainly the class of totally symmetric plane partitions, where the proof consists of a heavily computer-assisted verification of a (complicated) determinant evaluation.

A combinatorialist would dream of a combinatorial (in the best case: bijective) argument which would explain the product formulae. However, we are very far off a realisation of that dream. Only in the simplest case, namely the case of MacMahon's theorem for the generating function for *all* plane partitions in a given box — presented here in Theorem 1 —, there exists a bijective proof (see [58]). In all other cases, to find a bijective argument is a wide open problem. As mentioned earlier, there exists at least a combinatorial explanation of the factorisation (6.10) occurring for cyclically symmetric self-complementary plane partitions, see [19].

However, the greatest, still unsolved, mystery concerns the question what plane partitions have to do with *alternating sign matrices* (see Section 3 for their definition). This question was first raised in [91], and there is still no answer to it. Mills, Robbins and Rumsey [75, Conj. 1] conjectured and Zeilberger [101] (see also [66]) proved that the number of $n \times n$ alternating sign matrices is given by (6.12) (with a replaced by n), which also

counts totally symmetric self-complementary plane partitions contained in a $2n \times 2n \times 2n$ box. Moreover, as Andrews [3] had shown, the same numbers also arise as the numbers of "size n" *descending plane partitions* (which I will not define here). Can these be coincidences? Certainly not. However, so far nobody has found any conceptual explanation.

As a matter of fact, already Mills, Robbins and Rumsey looked deeper into this mysterious triple occurrence of the same fascinating number sequence. They attempted to find parametric refinements and variations of these "coincidences." Their idea was that, if the numerical equality took place even at a finer level, then this may be the guide to the sought-for conceptual explanation(s) and, in the best case, to bijections between these objects. The result of their search for refinements was several conjectures, one [75, Conj. 3] connecting refined enumerations of descending plane partitions and of alternating sign matrices with each other, and several [76, Conjectures 2–7'] connecting refined enumerations of totally symmetric self-complementary plane partitions and of alternating sign matrices with each other. The first conjecture has been recently demonstrated by Behrend, Di Francesco and Zinn–Justin [8] (and even further refined in [9]) in a remarkable combination of the new ideas developed by the second and third author in their (so far, unsuccessful) attempt to prove the so-called Razumov–Stroganov conjecture (cf. [14]) with a skillful determinant calculation. As this description indicates, this proof is again far from being bijective or illuminating as to why the numerical equality occurs.

The second group of conjectures is still wide open, although Zeilberger's magnum opus [101] actually proves a weak version of [76, Conj. 7']. Again, the proof is highly non-bijective and non-illuminating. In 1996, the author of these notes found a generalisation of this second conjecture of Mills, Robbins and Rumsey. It is also still open, and in particular it has so far not led to the sought-for bijection between alternating sign matrices and totally symmetric self-complementary plane partitions. Nevertheless, since this conjecture has not yet been published anywhere, I use the opportunity to present it here.

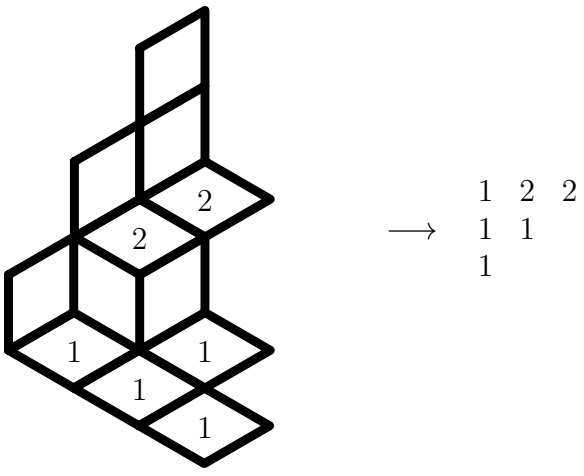


FIGURE 12. A totally symmetric self-complementary plane partition turned into a triangular array

The starting point for [76, Conjecture 7'] and its announced generalisation is the fact that both totally symmetric self-complementary plane partitions and alternating sign matrices can be encoded in terms of certain triangular arrays. Namely, a totally symmetric self-complementary plane partition (see Figure 8) contained in a $2n \times 2n \times 2n$ box is determined by what we see in just one twelveth of the hexagon, everything else is forced by symmetry. See Figure 12 for one twelveth of the plane partition in Figure 8. The top-most point in Figure 12 corresponds to the centre in Figure 8. Although this is just one twelveth, we still view this in three dimensions as earlier. For the "flat" rhombi, we record their heights in relation to the "floor" (in the three-dimensional picture) which we assume at height 1; see the numbers in the left half of Figure 12. These numbers are then arranged in a triangular array as in the right half of Figure 12. The entries in the triangular array are positive integers. they have the property that they are weakly increasing along rows and weakly decreasing along columns, and the entries in column i are bounded above by i, $i = 1, 2, \ldots, n$. It is easy to see that the above defines a bijection.

The "Magog trapezoids" (this is terminology stolen from [101]) below generalise these triangular arrays. In view of the above correspondence, totally symmetric self-complementary plane partitions correspond to $(0, n, n)$-Magog trapezoids.

Definition 3. An (m, n, k)-*Magog trapezoid* is an array of positive integers consisting of the first k rows of an array

$$\begin{array}{llll} b_{11} & b_{12} & \ldots\ldots\ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2,n-1} \\ \ldots\ldots\ldots\ldots & & & \\ b_{n1} & & & \end{array}$$

such that entries along rows are weakly increasing, entries along columns are weakly decreasing, and such that the entries in the first row are bounded by $b_{11} \leq m+1$, $b_{12} \leq m+2$, $\ldots$, $b_{1n} \leq m+n$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & \\ 2 & 3 & 5 & 6 & & \\ 2 & 4 & 5 & & & \\ 3 & 5 & & & & \\ 4 & & & & & \end{array}$$

FIGURE 13. An alternating sign matrix turned into a monotone triangle

Likewise, we may transform an $n \times n$ alternating sign matrix into a triangular array. This is done by replacing the i-th row of the matrix by the sum of rows $i, i+1, \ldots, n$, for $i = 1, 2, \ldots, n$. The result is a matrix with only 0's and 1's, see the middle of Figure 13. To obtain the corresponding triangular array, for each row we record the positions of the 1's in that row; see again Figure 13. The entries in the triangular array are positive integers. they have the property that they are strictly increasing along rows, weakly increasing along

columns, weakly increasing along diagonals in direction north-east, and the entries in the first row are $1, 2, \dots, n$. Again, it is easy to see that the above defines a bijection. The triangular arrays which one obtains here are most often called *monotone triangles*.

The "Gog trapezoids" (terminology again stolen from [101]) below generalise these monotone triangles. In view of the above correspondence, $n \times n$ alternating sign matrices correspond to $(0, n, n)$-Gog trapezoids.

Definition 4. An (m, n, k)-*Gog trapezoid* is an array of positive integers consisting of the first k columns of an array

$$\begin{array}{llcl} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-1} \\ \dots\dots\dots\dots & & & \\ a_{n1} & & & \end{array}$$

such that entries along rows are strictly increasing, entries along columns are weakly increasing, and entries along diagonals from lower-left to upper-right are weakly increasing, and such that the entries in the right-most column are bounded by $a_{1k} \le m + k$, $a_{2k} \le m + k + 1$, $\dots$, $a_{n+1-k,k} \le m + n$.

Here is now the announced conjecture.

Conjecture 5. *The number of (m, n, k)-Magog trapezoids with s Maxima in the first row and t Minima in the last row equals the number of (m, n, k)-Gog trapezoids with t Maxima in the right-most column and s Minima in the left-most column. Here, a Maximum is an entry that is equal to its upper bound, whereas a Minimum is an entry that is* 1*.*

I am convinced that there must exist a jeu-de-taquin-like procedure to turn the objects in Definition 3 into the ones in Definition 4, but I am not the only one to have failed to find such a procedure. For a different attempt to construct a bijection between totally symmetric self-complementary plane partitions and alternating sign matrices (and new conjectures), see [11] and the references contained therein.

What is known about Conjecture 5?

(1) If $m = 0$ and $k = n$, and if we forget about Maxima and Minima, then the conjecture reduces to the equality of the number of $n \times n$ alternating sign matrices and the number of totally symmetric self-complementary plane partitions contained in a $(2n) \times (2n) \times (2n)$ box. As was said earlier, this is a theorem, thanks to Andrews [5] and Zeilberger [101], but no bijection is known.

(2) If $m = 0$ and if we forget about Maxima and Minima, then this is Zeilberger's main theorem in [101] (namely Lemma 1), but, again, his proof is highly non-bijective.

(3) If $m = 0$, then the above conjecture reduces to Conjecture 7' in [76].

(4) If $k = 1$ then we want to show that the number of arrays of positive integers

$$a_1 \quad a_2 \quad \dots \quad a_n$$

with $a_k \le m + k$ and with s Maxima and t Minima is exactly the same as the number of arrays of the same type but with t Maxima and s Minima. It is easy to show that the

number in question is

$$\binom{m+2n-s-t-2}{m+n-2}-\binom{m+2n-s-t-2}{m+n-1},$$

which is indeed symmetric in s and t. For this special case, it is not very difficult to construct a bijection.

I refer the reader to [84, 40, 41] for further conjectures around totally symmetric self-complementary plane partitions (some of them having been proved in [26]).

In order to (finally) answer the question in the section header: interest in plane partitions is still very much alive, if though in different forms than at the time of MacMahon or when Stanley studied plane partitions. Aside from the notorious problem of connecting plane partitions and alternating sign matrices in a conceptual way, it is rhombus tilings and, more generally, perfect matchings of bipartite graphs which are primarily in the focus of current research. On the one hand, unexpected closed form product formulae for rhombus tilings of particular regions are constantly discovered, see e.g. [34, 68, 21] and the references therein. On the other hand, a deep theory of asymptotic properties of perfect matchings (and thus, in particular, of rhombus tilings and plane partitions) has been developed in the recent past, see the excellent survey [50]. Kasteleyn–Percus matrices (see Section 4) are the starting point of this theory. To go into more detail of the subsequent asymptotic analysis would however definitely go beyond the scope of these notes, and therefore this is the place to stop.

References

[1] G. E. Andrews, *Plane partitions I: The MacMahon conjecture*, Studies in foundations and combinatorics, G.-C. Rota (ed.), Adv. in Math. Suppl. Studies, vol. 1, 1978, pp. 131–150.

[2] G. E. Andrews, *Plane partitions II: The equivalence of the Bender–Knuth and MacMahon conjectures*, Pacific J. Math. **72** (1977), 283–291.

[3] G. E. Andrews, *Macdonald's conjecture and descending plane partitions*, in: Combinatorics, representation theory and statistical methods in groups, Young Day Proceedings, T. V. Narayana, R. M. Mathsen, J. G. Williams (eds.), Lecture Notes in Pure Math., vol. 57, Marcel Dekker, New York, Basel, 1980, pp. 91–106.

[4] G. E. Andrews, *Totally symmetric plane partitions*, Abstracts Amer. Math. Soc. **1** (1980), 415.

[5] G. E. Andrews, *Plane partitions (V): The t.s.s.c.p.p. conjecture*, J. Combin. Theory Ser. A **66** (1994), 28–39.

[6] G. E. Andrews and P. Paule, *MacMahon's partition analysis, XII: Plane partitions*, J. London Math. Soc. (2) **76** (2007), 647—666.

[7] G. E. Andrews and W. H. Burge, *Determinant identities*, Pacific J. Math. **158** (1993), 1–14.

[8] R. E. Behrend, P. Di Francesco and P. Zinn-Justin, *On the weighted enumeration of alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **119** (2012), 331–363.

[9] R. E. Behrend, P. Di Francesco and P. Zinn-Justin, *A doubly-refined enumeration of alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **120** (2013), 409–432.

[10] E. A. Bender and D. E. Knuth, *Enumeration of plane partitions*, J. Combin. Theory Ser. A **13** (1972), 40–54.

[11] P. Biane and H. Cheballah, *Gog and GOGAm pentagons*, J. Combin. Theory Ser. A **138** (2016), 133–154.

[12] F. Breuer and Z. Zafeirakopoulos, *Polyhedral omega: A new algorithm for solving linear diophantine systems*, preprint; arχiv:1501.07773.
[13] W. H. Burge, *Four correspondences between graphs and generalized Young tableaux*, J. Combin. Theory A **17** (1974), 12–30.
[14] L. Cantini and A. Sportiello, *Proof of the Razumov–Stroganov conjecture*, J. Combin. Theory Ser. A **118** (2011), 1549–1574.
[15] L. Carlitz, *Rectangular arrays and plane partitions*, Acta Arith. **13** (1967), 29–47.
[16] L. Carlitz, *Some determinants of q-binomial coefficients*, J. Reine Angew. Math. **226** (1967), 216–220.
[17] C. P. Chen, A. Guo, X. Jin and G. Liu, *Trivariate monomial complete intersections and plane partitions*, J. Commut. Algebra **3** (2011), 459–489.
[18] M. Ciucu, *Enumeration of perfect matchings in graphs with reflective symmetry*, J. Combin. Theory Ser. A **77** (1997), 67–97.
[19] M. Ciucu, *The equivalence between enumerating cyclically symmetric, self-complementary and totally symmetric, self-complementary plane partitions*, J. Combin. Theory Ser. A **86** (1999), 382–389.
[20] M. Ciucu and C. Krattenthaler, *Plane partitions II:* $5\frac{1}{2}$ *symmetry classes*, in: Combinatorial Methods in Representation Theory, M. Kashiwara, K. Koike, S. Okada, I. Terada, H. Yamada (eds.), Advanced Studies in Pure Mathematics, vol. 28, RIMS, Kyoto, 2000, pp. 83–103.
[21] D. Cook II and U. Nagel, *Enumerations of lozenge tilings, lattice paths, and perfect matchings and the weak Lefschetz property*, preprint; arχiv:1305.1314.
[22] G. David and C. Tomei, *The problem of the calissons*, Amer. Math. Monthly **96** (1989), 429–431.
[23] P. Di Francesco, *T-systems, networks and dimers*, Comm. Math. Phys. **331** (2014), 1237–1270.
[24] I. Fischer, *Another refinement of the Bender–Knuth (ex-)conjecture*, Europ. J. Combin. **27** (2006), 290–321.
[25] M. E. Fisher, *Walks, walls, wetting, and melting*, J. Statist. Phys. **34** (1984), 667–729.
[26] T. Fonseca and P. Zinn-Justin, *On the doubly refined enumeration of alternating sign matrices and totally symmetric self-complementary plane partitions*, Electron. J. Combin. **15** (2008), Article #R81, 35 pp.
[27] G. Frobenius, *Über die Charaktere der symmetrischen Gruppe*, Sitzber. Preuss. Akad. Berlin **417** (1900), 516–534.
[28] M. Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of matchings*, Electron. J. Combin. **17** (2010), Article #R83, 42 pp.
[29] S. Fomin and A. Zelevinsky, *The Laurent phenomenon*, Adv. Appl. Math. **28** (2002), 119–144.
[30] E. Gansner, *The enumeration of plane partitions via the Burge correspondence*, Illinois J. Math. **25** (1981), 533–554.
[31] E. Gansner, *The Hillman–Grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory Ser. A **30** (1981), 71–89.
[32] I. M. Gessel and X. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. **58** (1985), 300–321.
[33] I. M. Gessel and X. Viennot, *Determinants, paths, and plane partitions*, preprint, 1989; available at http://www.cs.brandeis.edu/~ira.
[34] T. Gilmore, *Three interactions of holes in two dimensional dimer systems*, preprint; arχiv:1501.05772.
[35] B. Gordon and L. Houten, *Notes on plane partitions I, II*, J. Combin. Theory Ser. A **4** (1968), 72–80; 81–99.
[36] B. Gordon and L. Houten, *Notes on plane partitions III*, Duke Math. J. **36** (1969), 801–824.
[37] B. Gordon, *A proof of the Bender–Knuth conjecture*, Pacific J. Math. **108** (1983), 99–113.
[38] H.-D. O. F. Gronau, W. Just, W. Schade, P. Scheffler, and J. Wojciechowski, *Path systems in acyclic directed graphs*, Zastos. Mat. **19** (1988), 399–411.

[39] A. J. Guttmann, A. L. Owczarek and X. G. Viennot, *Vicious walkers and Young tableaux I: Without walls*, J. Phys. A – Math. Gen. **3199** (1998), 8123–8135.
[40] M. Ishikawa, *On refined enumerations of totally symmetric self-complementary plane partitions, I*, preprint; `ar`χ`iv:math/0602068`.
[41] M. Ishikawa, *On refined enumerations of totally symmetric self-complementary plane partitions, II*, preprint; `ar`χ`iv:math/0606082`.
[42] A. P. Hillman and R. M. Grassl, *Reverse plane partitions and tableau hook numbers*, J. Combin. Theory Ser. A **21** (1976), 216–221.
[43] M. Ishikawa and M. Wakayama, *Minor summation formula for pfaffians*, Linear and Multilinear Algebra **39** (1995), 285–305.
[44] W. Jockusch, *Perfect matchings and perfect squares*, J. Combin. Theory Ser. A **67** (1994), 100–115.
[45] P. John and H. Sachs, *Wegesysteme und Linearfaktoren in hexagonalen und quadratischen Systemen*, in: Graphs in research and teaching (Kiel, 1985), Franzbecker, Bad Salzdetfurth, 1985, pp. 85–101.
[46] S. Karlin and J. L. McGregor. Coincidence probabilities. *Pacific J. Math.*, 9:1141–1164, 1959.
[47] S. Karlin and J. L. McGregor. Coincidence properties of birth-and-death processes. *Pacific J. Math.*, 9:1109–1140, 1959.
[48] P. W. Kasteleyn, *Dimer statistics and phase transitions*, J. Math. Phys. **4** (1963), 287–293.
[49] P. W. Kasteleyn, *Graph theory and crystal physics*, in: Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 43–110.
[50] R. Kenyon, *An introduction to the dimer model*, in: Statistical mechanics, IAS/Park City Math. Ser., vol. 16, Amer. Math. Soc., Providence, R.I., 2009, pp. 191–230.
[51] R. C. King and N. G. I. El-Sharkaway, *Standard Young tableaux and weight multiplicities of the classical Lie groups*, J. Physics A **16** (1983), 3153–3178.
[52] D. E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709—727.
[53] C. Koutschan, M. Kauers and D. Zeilberger, *A proof of George Andrews' and David Robbins' q-TSPP-conjecture*, Proc. Natl. Acad. Sci. USA **108** (2011), 2196–2199.
[54] E. H. Kuo, *Applications of graphical condensation for enumerating matchings and tilings*, Theoret. Comput. Sci. **319** (2004), 29–57.
[55] C. Krattenthaler, *Generating functions for plane partitions of a given shape*, Manuscripta Math. **69** (1990), 173–202.
[56] C. Krattenthaler, *The major counting of nonintersecting lattice paths and generating functions for tableaux*, Mem. Amer. Math. Soc. 115, no. 552, Providence, R. I, 1995.
[57] C. Krattenthaler, *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998), 1–64.
[58] C. Krattenthaler, *Another involution principle-free bijective proof of Stanley's hook-content formula*, J. Combin. Theory Ser. A **88** (1999), 66–92.
[59] C. Krattenthaler, *Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions*, Electron. J. Combin. **4**(1) (1997), Article #R27, 62 pp.
[60] C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** ("The Andrews Festschrift") (1999), Article B42q, 67 pp.
[61] C. Krattenthaler, A. J. Guttmann and X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux II: with a wall*, J. Phys. A: Math. Gen. **33** (2000), 8835–8866.
[62] G. Kuperberg, *Symmetries of plane partitions and the permanent determinant method*, J. Combin. Theory Ser. A **68** (1994), 115–151.
[63] G. Kuperberg, *Self-complementary plane partitions by Proctor's minuscule method*, Europ. J. Combin. **15** (1994), 545–553.
[64] G. Kuperberg, *Four symmetry classes of plane partitions under one roof*, J. Combin. Theory Ser. A **75** (1996), 295–315.

[65] G. Kuperberg, *Another proof of the alternating sign matrix conjecture*, Math. Research Lett. **3** (1996), 139–150.
[66] G. Kuperberg, *An exploration of the permanent-determinant method*, Electron. J. Combin. **5** (1998), Article #R46, 34 pp.
[67] G. Kuperberg, *Kasteleyn cokernels*, Electron. J. Combin. **9** (2002), Article #R29, 30 pp.
[68] T. Lai, *A q-enumeration of generalized plane partitions*, preprint; `arχiv:1502.01679`.
[69] B. Lindström, *On the vector representations of induced matroids*, Bull. London Math. Soc. **5** (1973), 85–90.
[70] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford University Press, New York/London, 1995.
[71] P.A. MacMahon, *Memoir on the theory of the partition of numbers, I*, Lond. Phil. Trans. (A) **187** (1897), 619–673.
[72] P. A. MacMahon, *Partitions of numbers whose graphs possess symmetry*, Trans. Cambridge Philos. Soc. **17** (1899), 149–170.
[73] P. A. MacMahon, *Combinatory Analysis*, vol. 2, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960.
[74] W. H. Mills, D. H. Robbins and H. Rumsey, *Proof of the Macdonald conjecture*, Invent. Math. **66** (1982), 73–87.
[75] W. H. Mills, D. H. Robbins and H. Rumsey, *Alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **34** (1983), 340–359.
[76] W. H. Mills, D. H. Robbins and H. Rumsey, *Self–complementary totally symmetric plane partitions*, J. Combin. Theory A **42** (1986), 277–292.
[77] W. H. Mills, D. H. Robbins and H. Rumsey, *Enumeration of a symmetry class of plane partitions*, Discrete Math. **67** (1987), 43–55.
[78] S. Okada, *On the generating functions for certain classes of plane partitions*, J. Combin. Theory Ser. A **51** (1989), 1–23.
[79] I. Pak, *Hook length formula and geometric combinatorics*, Sém. Lothar. Combin. **46** (2001/02), Article B46f, 13 pp.
[80] R. A. Proctor, *Shifted plane partitions of trapezoidal shape*, Proc. Amer. Math. Soc. **89** (1983), 553–559.
[81] R. A. Proctor, *Bruhat lattices, plane partition generating functions, and minuscule representations*, Europ. J. Combin. **5** (1984), 331–350.
[82] R. A. Proctor, *New symmetric plane partition identities from invariant theory work of De Concini and Procesi*, Europ. J. Combin. **11** (1990), 289–300.
[83] D. P. Robbins and H. Rumsey, *Determinants and alternating sign matrices*, Adv. in Math. **62** (1986), 169–184.
[84] D. P. Robbins, *The story of* 1, 2, 7, 42, 429, 7436, ..., Math. Intelligencer **13** (1991), 12–19.
[85] J. C. Slater. The theory of complex spectra. *Phys. Rev.*, 34:1293–1322, 1929.
[86] J. C. Slater, *Quantum Theory of Matter*, 2nd ed., McGraw–Hill, New York, 1968.
[87] R. P. Stanley, *The conjugate trace and trace of a plane partition*, J. Combin. Theory Ser. A **14** (1973), 53–65.
[88] R. P. Stanley, *Ordered Structures and Partitions*, Mem. Amer. Math. Soc., no. 119, Amer. Math. Soc., Providence, R. I., 1972.
[89] R. P. Stanley, *Theory and applications of plane partitions: Part 1,2*, Stud. Appl. Math **50** (1971), 167–188, 259–279.
[90] R. P. Stanley, *Symmetries of plane partitions*, J. Combin. Theory A **43** (1986), 103–113; Erratum **44** (1987), 310.
[91] R. P. Stanley, *A baker's dozen of conjectures concerning plane partitions*, in: Combinatoire énumérative, G. Labelle, P. Leroux (eds.), Lecture Notes in Math., vol. 1234, Springer–Verlag, Berlin–New York, 1986, pp. 285–293.

[92] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.
[93] J. R. Stembridge, *Hall–Littlewood functions, plane partitions and the Rogers–Ramanujan identities*, Trans. Amer. Math. Soc. **319** (1990), 469–498.
[94] J. R. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. Math. **83** (1990), 96–131.
[95] J. R. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser. A **68** (1994), 372–409.
[96] J. R. Stembridge, *On minuscule representations, plane partitions and involutions in complex Lie groups*, Duke Math. J. **73** (1994), 469–490.
[97] J. R. Stembridge, *The enumeration of totally symmetric plane partitions*, Adv. Math. **111** (1995), 227–245.
[98] G. Tesler, *Matchings in graphs on non-oriented surfaces*, J. Combin. Theory Ser. B **78** (2000), 198–231.
[99] S. Vuletić, *A generalization of MacMahon's formula*, Trans. Amer. Math. Soc. **361** (2009), 2789–2804.
[100] E. M. Wright, *Asymptotic partition formulae, I: Plane partitions*, Quart. J. Math. (Oxford) **2** (1931), 177–189.
[101] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. **3** (no. 2, "The Foata Festschrift") (1996), #R13, 84 pp.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria. WWW: http://www.mat.univie.ac.at/~kratt.