AN IDENTITY OF ANDREWS, MULTIPLE INTEGRALS, AND VERY-WELL-POISED HYPERGEOMETRIC SERIES

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Dedicated to Richard Askey

Abstract. We give a new proof of a theorem of Zudilin that equates a very-well-poised hypergeometric series and a particular multiple integral. This integral generalizes integrals of Vasilenko and Vasilyev which were proposed as tools in the study of the arithmetic behaviour of values of the Riemann zeta function at integers. Our proof is based on limiting cases of a basic hypergeometric identity of Andrews.

1. Introduction

After Apéry's 1978 proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ (see [3]), $\zeta(s)$ denoting the Riemann zeta function, Beukers [6] gave another proof with the help of his famous integrals

$$\int_{[0,1]^2} \frac{x^n(1-x)^n y^n(1-y)^n}{(1-(1-x)y)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y = \alpha_n\zeta(2) - \beta_n$$

and

$$\int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 2a_n\zeta(3) - b_n.$$

Here, $n \geq 0$ is an integer, and α_n, β_n, a_n, b_n are rational numbers. More precisely, α_n, a_n, $\mathrm{d}_n^2\beta_n$ and $\mathrm{d}_n^3 b_n$ are integers, with $\mathrm{d}_n = \mathrm{lcm}\{1, 2, \ldots, n\}$.

Extending Vasilenko's method [17], Vasilyev [18, 19] considered a family of integrals generalizing Beukers' pattern:

$$J_{E,n} = \int_{[0,1]^E} \frac{\prod_{i=1}^E x_i^n(1-x_i)^n}{Q_E(x_1, x_2, \ldots, x_E)^{n+1}}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_E, \tag{1.1}$$

where

$$Q_E(x_1, x_2, \ldots, x_E) = 1 - (\cdots(1-(1-x_E)x_{E-1})\cdots)x_1,$$

and where E is an integer ≥ 2.

He then formulated the following conjecture, which he proved for $E = 4, 5$ and which is also true for $E = 2, 3$ because of Beukers' work.

2000 *Mathematics Subject Classification.* Primary 33C20; Secondary 11J72.

Key words and phrases. Riemann zeta function, hypergeometric series.

†Research partially supported by the programme "Improving the Human Research Potential" of the European Commission, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe".

Conjecture 1 (Vasilyev)**.**
(i) *For all integers $E \ge 2$ and $n \ge 0$, there exist rational numbers $p_{m,E,n}$ such that*

$$J_{E,n} = p_{0,E,n} + \sum_{\substack{m=2,\dots,E \\ m\equiv E \,(\mathrm{mod}\,2)}} p_{m,E,n}\zeta(m). \tag{1.2}$$

(ii) *Furthermore, $\mathrm{d}_n^E p_{m,E,n}$ is an integer for all $m = 0, 2, 3, \dots, E$.*

Part (i) of this conjecture has been proved by Zudilin in [23, Sec. 8], thanks to an unexpected identity between a certain multiple integral J_m (generalizing Vasilyev's ones) and a non-terminating very-well-poised hypergeometric series (see Theorem 1 below). Part (ii) was also proved by Zudilin in [23], except for the coefficient $p_{0,E,n}$. A sharper version of Part (ii) has been established by the authors in [10] (for $m = 0$ only up to multiplication by a factor of 2).

The aim of this note is to give a new proof of Zudilin's identity, which is the content of Theorem 1 in the next section. In fact, our proof shows that, modulo a more or less evident expansion of the Vasilyev-type integral J_m as a multiple sum (see Proposition 2 in Section 4), Zudilin's identity is a limiting case of a thirty year old identity between a terminating multiple sum and a terminating very-well-poised hypergeometric series due to Andrews (see (3.1) below). We believe that this is an interesting observation because attempts to prove Zudilin's identity by manipulating hypergeometric series directly failed because of convergence problems. Zudilin circumvents these problems by having recourse to a Barnes-type (i.e., contour) integral in place of the very-well-poised hypergeometric series. Our proof shows that there is indeed a "purely hypergeometric" proof (i.e., a proof just using summation and transformation formulas for hypergeometric series), but to be able to accomplish it, one has to go "one level higher in hierarchy," meaning that one finds a *terminating* identity "above," of which the identity which one actually wants to prove is a limiting case.[1] This identity "above" is Andrews' identity, and it does indeed have a purely hypergeometric proof (see Section 3 for more information).

As an aside, we mention that, in the recent paper [22], Zlobin shows that the multiple integral J_m is also equal to an integral of the same type as those of Sorokin in [15, 16] (see also Fischler [7] for similar results). The latter integral can also be expanded as a multisum, in a manner completely analogous to the way we derive the multisum expansion for J_m in the proof of Proposition 2. As a result, one obtains again exactly the right-hand sides of (4.1) and (4.2), respectively. Thus, this provides an alternative proof of Zlobin's

[1] This point is also of interest "philosophically." There are several proposers (of whom Koornwinder [9] seems to have been the first; see [13, Remark 3.2] and [20, paragraph after the second Eq. (*Apery*)] for printed versions) of the "conjecture" that above every identity for non-terminating hypergeometric series (which are very often difficult to prove; in particular, the automatic tools described in [11] do not apply) there sits a more general identity for terminating series (which, at least in principle, can be proved automatically), of which the non-terminating identity is a limiting case. Of course, the analyst would object that above every terminating identity there exists an even more general non-terminating identity, of which the terminating one is a special case. Clearly, this dispute is as easy to settle as the dispute about the question of which was first, the hen or the egg ...

result. As a matter of fact, when this work originated, we went the other way, that is, our starting point was the multisum expansion of Zlobin's integral, until we realized that, actually, the integral J_m admits the same treatment.

Zudilin's identity is recalled in Theorem 1 in the next section. The limiting cases of Andrews' identity which we need are stated and proved in Proposition 1 in Section 3. One of the lemmas which we need for carrying out these limits generalizes a lemma of Zhao [21] on the convergence of multizeta functions, see the remark after the proof of Lemma 3. The purpose of Section 4 is to relate these identities to the Vasilyev-type integral J_m, see Proposition 2. We finally prove Theorem 1 in Section 5.

2. Zudilin's identity

In order to be able to state Zudilin's identity, we need to recall the standard notation for (generalized) hypergeometric series,

$$ {}_{p+1}F_p\left[\begin{matrix}\alpha_0,\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_p\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_0)_k\,(\alpha_1)_k\cdots(\alpha_p)_k}{k!\,(\beta_1)_k\cdots(\beta_p)_k}\,z^k, $$

where $p\ge 1$, $\alpha_j\in\mathbb{C}$, $\beta_j\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}$ and, by definition, $(x)_0=1$ and $(x)_\ell=x(x+1)\cdots(x+\ell-1)$ for $\ell\ge 1$. The series is absolutely convergent for all $z\in\mathbb{C}$ such that $|z|<1$, and also for $|z|=1$ provided $\mathfrak{Re}(\beta_1+\dots+\beta_p)>\mathfrak{Re}(\alpha_0+\alpha_1+\dots+\alpha_p)$. Furthermore, it is said to be *balanced* if $\alpha_0+\dots+\alpha_p+1=\beta_1+\dots+\beta_p$ and *very-well-poised* if $\alpha_0+1=\alpha_1+\beta_1=\dots=\alpha_p+\beta_p$ and $\alpha_1=\frac{1}{2}\,\alpha_0+1$. See the books [2, 4, 8, 14] for more information on hypergeometric series.

Let z, $a_0,a_1,\dots,a_m$, and $b_1,\dots,b_m$ be complex numbers such that $|z|<1$, $\mathfrak{Re}(b_i)>\mathfrak{Re}(a_i)>0$ for all $i=1,2,\dots,m$, and let us define the Vasilyev-type integral

$$ J_m\left[\begin{matrix}a_0,a_1,\dots,a_m\\ b_1,\dots,b_m\end{matrix};z\right]=\int_{[0,1]^m}\frac{\prod_{i=1}^m x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{(1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1z)^{a_0}}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_m, \tag{2.1} $$

which is absolutely convergent under the above conditions. (We will sometimes use the short notation J_m for this integral if there is no ambiguity about the parameters.) It is also absolutely convergent for $z=1$, provided that we also assume that $\mathfrak{Re}(b_1-a_1)>\mathfrak{Re}(a_0)$ if $m=1$, respectively $\mathfrak{Re}(b_1-a_1)\ge\mathfrak{Re}(a_0)$ if $m>1$. Since previous authors assume more restrictive conditions in the case $z=1$ (in particular, restrictions that are not satisfied by Vasilyev's integrals (1.1)), we sketch the verification of the convergence here for the sake of completeness.

If $m=1$, then $J_m=J_1$ is a beta integral. If $m\ge 2$, then, because of

$$ 1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1\ge 1-x_1 $$

and $\mathfrak{Re}(b_1 - a_1) \geq \mathfrak{Re}(a_0)$, we have

$$\begin{aligned}
&\int_{\varepsilon_m}^{1-\varepsilon_m} \cdots \int_{\varepsilon_2}^{1-\varepsilon_2} \int_{\varepsilon_1}^{1-\varepsilon_1} \left| \frac{\prod_{i=1}^m x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{(1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1)^{a_0}} \right| \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_m \\
&\quad \leq \int_{\varepsilon_m}^{1-\varepsilon_m} \cdots \int_{\varepsilon_2}^{1-\varepsilon_2} \int_{\varepsilon_1}^{1-\varepsilon_1} \frac{\prod_{i=2}^m x_i^{\mathfrak{Re}(a_i-1)}(1-x_i)^{\mathfrak{Re}(b_i-a_i-1)}}{(1-x_1+Xx_2x_1)} \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_m \\
&\quad \leq \int_{\varepsilon_m}^{1-\varepsilon_m} \cdots \int_{\varepsilon_2}^{1-\varepsilon_2} \left(\prod_{i=2}^m x_i^{\mathfrak{Re}(a_i-1)}(1-x_i)^{\mathfrak{Re}(b_i-a_i-1)} \right) \\
&\qquad\qquad \cdot \left(-\frac{\log(1-x_1+Xx_2x_1)}{1-Xx_2} \right) \Bigg|_{x_1=\varepsilon_1}^{1-\varepsilon_1} \mathrm{d}x_2 \cdots \mathrm{d}x_m,
\end{aligned}$$

for any small $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m > 0$, where we wrote X for $1-(\cdots(1-x_m)x_{m-1})\cdots)x_3$. (In case that $m=2$, X has to be interpreted as 1.) If we perform the limit $\varepsilon_1 \to 0$, then the right-hand side of this inequality becomes the integral

$$\int_{\varepsilon_m}^{1-\varepsilon_m} \cdots \int_{\varepsilon_2}^{1-\varepsilon_2} \left(\prod_{i=2}^m x_i^{\mathfrak{Re}(a_i-1)}(1-x_i)^{\mathfrak{Re}(b_i-a_i-1)} \right) \left(-\frac{\log(Xx_2)}{1-Xx_2} \right) \mathrm{d}x_2 \cdots \mathrm{d}x_m. \tag{2.2}$$

In the integrand, there is no problem as $Xx_2 \to 1$, since the function $\log(Xx_2)/(1-Xx_2)$ is continuous at $Xx_2 = 1$. On the other hand, if we fix $\eta > 0$, then for Xx_2 sufficiently close to 0, we have

$$|\log(Xx_2)| < (Xx_2)^{-\eta} \leq (1-x_3)^{-\eta} x_2^{-\eta}.$$

Thus, choosing $\eta = \frac{1}{2}\min\{\mathfrak{Re}(a_2), \mathfrak{Re}(b_3 - a_3)\}$, we see that the integral in (2.2), and thus the original integral J_m, exists.

Theorem 1 (Zudilin)**.** *For every integer $m \geq 1$, the following identity holds:*

$$\begin{aligned}
&J_m \left[\begin{matrix} h_1, h_2, h_3, \ldots, h_{m+1} \\ 1+h_0-h_3, 1+h_0-h_4, \ldots, 1+h_0-h_{m+2} \end{matrix}; 1 \right] \\
&\quad = \frac{\Gamma(1+h_0)\prod_{j=3}^{m+1}\Gamma(h_j)}{\prod_{j=1}^{m+2}\Gamma(1+h_0-h_j)} \cdot \left(\prod_{j=1}^{m+1} \Gamma(1+h_0-h_j-h_{j+1}) \right) \\
&\qquad\qquad \times {}_{m+4}F_{m+3} \left[\begin{matrix} h_0, \frac{1}{2}h_0+1, h_1, \ldots, h_{m+2} \\ \frac{1}{2}h_0, 1+h_0-h_1, \ldots, 1+h_0-h_{m+2} \end{matrix}; (-1)^{m+1} \right],
\end{aligned} \tag{2.3}$$

provided that $1 + \mathfrak{Re}(h_0) > \frac{2}{m+1}\sum_{j=1}^{m+2}\mathfrak{Re}(h_j)$, $\mathfrak{Re}(1+h_0-h_{j+1}) > \mathfrak{Re}(h_j) > 0$ for $j = 2, 3, \ldots, m+1$, and $\mathfrak{Re}(1+h_0-h_3-h_2) \geq \mathfrak{Re}(h_1)$, these conditions ensuring that both sides of (2.3) are well-defined.

In the case of the original integrals $J_{E,n}$ of Vasilyev, the identity in Theorem 1 reads as follows: for any integers $n \geq 0$ and $E \geq 2$,

$$J_{E,n} = \frac{n!^{2E+1}(3n+2)!}{(2n+1)!^{E+2}} {}_{E+4}F_{E+3} \left[\begin{matrix} 3n+2, \frac{3}{2}n+2, n+1, \ldots, n+1 \\ \frac{3}{2}n+1, 2n+2, \ldots, 2n+2 \end{matrix}; (-1)^{E+1} \right].$$

From [5, 12], it follows that such a very-well-poised hypergeometric series gives rise to a decomposition of the shape (1.2).

3. Limiting cases of Andrews' hypergeometric identity

Let N and s be positive integers, and a, $b_1,\dots,b_{s+1}$, $c_1,\dots,c_{s+1}$ be complex numbers such that none of $1+a-b_j$, $1+a-c_j$, $j=1,2,\dots,s+1$, and $1+a+N$ are non-positive integers.

Andrews' identity [1, Theorem 4] relates a terminating very-well-poised basic hypergeometric series to a terminating multiple basic hypergeometric series. We shall need here the limiting case of this identity when $q\to 1$, so that the series there reduce to "ordinary" hypergeometric series. That is, we replace a by q^a, b_i by q^{b_i}, c_i by q^{c_i}, there, and then let q tend to 1. The result can be compactly written in the form

$$
\begin{aligned}
&{}_{2s+5}F_{2s+4}\left[\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \dots, b_{s+1}, c_{s+1}, -N \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \dots, 1+a-b_{s+1}, 1+a-c_{s+1}, 1+a+N \end{matrix}; 1\right] \\
&\quad = \frac{(1+a)_N\,(1+a-b_{s+1}-c_{s+1})_N}{(1+a-b_{s+1})_N\,(1+a-c_{s+1})_N} \sum_{k_1,k_2,\dots,k_s\ge 0} \frac{(-N)_{k_1+\dots+k_s}}{(b_{s+1}+c_{s+1}-a-N)_{k_1+\dots+k_s}} \\
&\qquad \cdot \prod_{j=1}^{s} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\dots+k_j}\,(c_{j+1})_{k_1+\dots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\dots+k_j}\,(1+a-c_j)_{k_1+\dots+k_j}}.
\end{aligned}
\tag{3.1}
$$

The proof in [1] uses Whipple's transformation between a balanced ${}_4F_3$-series and a very-well-poised ${}_7F_6$-series,

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, b, c, -N \\ e, f, 1+a+b+c-e-f-N \end{matrix}; 1\right] = \frac{(-a-b+e+f)_N\,(-a-c+e+f)_N}{(-a+e+f)_N\,(-a-b-c+e+f)_N} \\
&\quad \cdot {}_7F_6\left[\begin{matrix} -1-a+e+f, \frac{1}{2}-\frac{a}{2}+\frac{e}{2}+\frac{f}{2}, -a+f, -a+e, b, c, -N \\ -\frac{1}{2}-\frac{a}{2}+\frac{e}{2}+\frac{f}{2}, e, f, -a-b+e+f, -a-c+e+f, -a+e+f+N \end{matrix}; 1\right],
\end{aligned}
$$

and the Pfaff–Saalschütz summation in an iterative fashion. In particular, the identity (3.1) reduces to Whipple's transformation for $s=1$.

We prove that the same kind of identity holds for *non-terminating* hypergeometric series provided the parameters a, b_j, and c_j, $j=1,2,\dots,s+1$, satisfy some further conditions.

Proposition 1. (i) *Let $s\ge 1$ be an integer, and let a, $b_1,\dots,b_{s+1}$, $c_1,\dots,c_{s+1}$ be complex numbers such that none of $1+a-b_j$, $1+a-c_j$, $j=1,2,\dots,s+1$, is a non-positive integer. Furthermore, we assume that*

$$
\mathfrak{Re}\left((2s+1)(a+1)-2\sum_{j=1}^{s+1}(b_j+c_j)\right)>0 \tag{3.2}
$$

and

$$
\mathfrak{Re}\left((1+a-b_{s+1}-c_{s+1})+\sum_{j=r}^{s}A_j(1+a-b_j-c_j)\right)>0 \tag{3.3}
$$

for all $r = 2, 3, \ldots, s+1$ (*in the case that* $r = s+1$, *the empty sum* $\sum_{j=r}^{s}$ *has to be interpreted as* 0), *for all possible choices of* $A_j = 1$ *or* 2, *for* $j = 2, 3, \ldots, s$. *Then*

$$
\begin{aligned}
&{}_{2s+4}F_{2s+3}\left[\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \ldots, b_{s+1}, c_{s+1} \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \ldots, 1+a-b_{s+1}, 1+a-c_{s+1} \end{matrix}; -1\right] \\
&\qquad = \frac{\Gamma(1+a-b_{s+1})\,\Gamma(1+a-c_{s+1})}{\Gamma(1+a)\,\Gamma(1+a-b_{s+1}-c_{s+1})} \\
&\qquad\qquad \times \sum_{k_1,k_2,\ldots,k_s \ge 0} \prod_{j=1}^{s} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\cdots+k_j}\,(c_{j+1})_{k_1+\cdots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\cdots+k_j}\,(1+a-c_j)_{k_1+\cdots+k_j}}. \qquad (3.4)
\end{aligned}
$$

(ii) *Let* $s \ge 1$ *be an integer, and let* a, c_0, $b_1, \ldots, b_s$, $c_1, \ldots, c_s$ *be complex numbers such that none of* $1+a-b_j$, $1+a-c_j$, $j = 0, 1, \ldots, s$, *is a non-positive integer. Furthermore, we assume that*

$$
\mathfrak{Re}\left(2s(a+1) - 2c_0 - 2\sum_{j=1}^{s}(b_j + c_j)\right) > 0, \qquad (3.5)
$$

$$
\mathfrak{Re}\left((1+a-b_s-c_s) + \sum_{j=r}^{s-1} A_j(1+a-b_j-c_j)\right) > 0 \qquad (3.6)
$$

for all $r = 2, 3, \ldots, s$ (*in the case that* $r = s$, *the empty sum* $\sum_{j=r}^{s-1}$ *has to be interpreted as* 0), *and*

$$
\mathfrak{Re}\left((1+a-c_0-b_1-c_1) + \sum_{j=2}^{s-1} A_j(1+a-b_j-c_j)\right) > 0, \qquad (3.7)
$$

for all possible choices of $A_j = 1$ *or* 2, *for* $j = 2, 3, \ldots, s-1$. *Then*

$$
\begin{aligned}
&{}_{2s+3}F_{2s+2}\left[\begin{matrix} a, \frac{a}{2}+1, c_0, b_1, c_1, \ldots, b_s, c_s \\ \frac{a}{2}, 1+a-c_0, 1+a-b_1, 1+a-b_1, \ldots, 1+a-b_s, 1+a-c_s \end{matrix}; 1\right] \\
&\qquad = \frac{\Gamma(1+a-b_s)\,\Gamma(1+a-c_s)}{\Gamma(1+a)\,\Gamma(1+a-b_s-c_s)} \sum_{k_1,k_2,\ldots,k_s \ge 0} \frac{(b_1)_{k_1}\,(c_1)_{k_1}}{k_1!\,(1+a-c_0)_{k_1}} \\
&\qquad\qquad \cdot \prod_{j=2}^{s} \frac{(1+a-b_{j-1}-c_{j-1})_{k_j}\,(b_j)_{k_1+\cdots+k_j}\,(c_j)_{k_1+\cdots+k_j}}{k_j!\,(1+a-b_{j-1})_{k_1+\cdots+k_j}\,(1+a-c_{j-1})_{k_1+\cdots+k_j}}. \qquad (3.8)
\end{aligned}
$$

Our proof of this proposition is based on three lemmas, which we state and prove first.

Lemma 1. *Let* α *and* β *be complex numbers such that* β *is not a non-positive integer. Then for any non-negative integer* k *we have*

$$
\left|\frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)}\right| \le D_1 \cdot (k+1)^{\mathfrak{Re}(\alpha-\beta)},
$$

where D_1 *is a constant which does not depend on* k.

Proof. By Stirling's formula, we have

$$\frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} \sim (k+1)^{\alpha-\beta}$$

as $k \to \infty$. Hence, the claim follows immediately. □

Lemma 2. *Let A and B be real numbers such that $A+B+1<0$, and let C be a non-negative integer. Then, for any non-negative integer h, we have*

$$\sum_{k=0}^{\infty}(k+1)^A(h+k+1)^B\left(\log(h+k+2)\right)^C \le D_2(h+1)^{\max\{B,A+B+1\}}\left(\log(h+2)\right)^{C+1},$$

where D_2 is a constant independent of h.

Proof. We split the summation range into the ranges $R_0=\{0,1,\dots,2^{\lceil\log_2(h+1)\rceil+1}-h-1\}$ and

$$R_s=\{2^s-h,2^s-h+1,\dots,2^{s+1}-h-1\}, \qquad s=\lceil\log_2(h+1)\rceil+1,\lceil\log_2(h+1)\rceil+2,\dots,$$

where $\lceil x\rceil$ denotes the least integer $\ge x$. Since, depending on whether A and B are positive or not, for $k\in R_s$, $s>0$, we have

$$(k+1)^A(h+k+1)^B\left(\log(h+k+2)\right)^C \le \left(\log 2^{s+2}\right)^C\cdot\begin{cases}2^{(s+1)(A+B)} & \text{if } A,B\ge 0,\\ 2^{(s-1)A+(s+1)B} & \text{if } A<0,\ B\ge 0,\\ 2^{(s+1)A+sB} & \text{if } A\ge 0,\ B<0,\\ 2^{(s-1)A+sB} & \text{if } A,B<0,\end{cases}$$

we infer that

$$(k+1)^A(h+k+1)^B\left(\log(h+k+2)\right)^C \le D_3\cdot(s+2)^C\cdot 2^{s(A+B)},$$

with a constant D_3 which is independent of h. Thus, for the sum over the range $\{k\ge 2^{\lceil\log_2(h+1)\rceil+1}-h\}$ we have

$$\begin{aligned}\sum_{k=2^{\lceil\log_2(h+1)\rceil+1}-h}^{\infty}&(k+1)^A(h+k+1)^B\left(\log(h+k+2)\right)^C\\ &=\sum_{s=\lceil\log_2(h+1)\rceil+1}^{\infty}\sum_{k\in R_s}(k+1)^A(h+k+1)^B\left(\log(h+k+2)\right)^C\\ &\le\sum_{s=\lceil\log_2(h+1)\rceil+1}^{\infty}D_3\cdot 2^s\cdot(s+1)_{C+1}\cdot 2^{s(A+B)}\\ &\le D_3\cdot 2^{(\lceil\log_2(h+1)\rceil+1)(A+B+1)}\frac{(C+1)!}{\left(1-2^{A+B+1}\right)^{C+2}}\\ &\le D_4\cdot(h+1)^{A+B+1},\end{aligned}$$

for a constant D_4 independent of h.

Now we consider the remaining range, $R_0 = \{0, 1, \ldots, 2^{\lceil \log_2(h+1) \rceil + 1} - h - 1\}$. For any $k \in R_0$ we have $k \le 3h + 3$, and therefore

$$(h+k+1)^B \left(\log(h+k+2)\right)^C \le \left(\log(4h+5)\right)^C \cdot \begin{cases} (h+1)^B & \text{if } B \le 0, \\ (4h+4)^B & \text{if } B > 0. \end{cases}$$

In particular, there is a constant D_5 independent of k such that

$$(h+k+1)^B \left(\log(h+k+2)\right)^C \le D_5 \left(\log(h+2)\right)^C (h+1)^B$$

for all $k \in R_0$. Using this fact, we are able to conclude that

$$\sum_{k \in R_0} (k+1)^A (h+k+1)^B \left(\log(h+k+2)\right)^C \le D_5 \cdot \left(\log(h+2)\right)^C (h+1)^B \sum_{k \in R_0} (k+1)^A.$$

Now, if $A < -1$, then $\sum_{k \in R_0} (k+1)^A < \zeta(-A)$. If $A = -1$, then $\sum_{k \in R_0} (k+1)^{-1} < \log(4h+4)$. Finally, if $A > -1$, then

$$\sum_{k \in R_0} (k+1)^A < \int_0^{4h+4} x^A \, \mathrm{d}x = \frac{1}{A+1} (4h+4)^{A+1}.$$

In all cases, we obtain that

$$\sum_{k \in R_0} (k+1)^A (h+k+1)^B \left(\log(h+k+2)\right)^C < D_6 \left(\log(h+2)\right)^{C+1} (h+1)^{\max\{B, A+B+1\}},$$

where D_6 is a constant independent of h.

To conclude the proof of the lemma, the two estimates for the two ranges are combined, and the claimed result follows. □

In the statement of the next lemma, we use the following notation: given two sets S and T, we write $S + T$ for the sum-set $\{x + y : x \in S \text{ and } y \in T\}$.

Lemma 3. *Let E_j and F_j be real numbers and let Z_j denote the set $\{F_j, E_j + F_j + 1\}$, $j = 1, 2, \ldots, s$. If*

$$E_r + F_r + 1 + \max(Z_{r+1} + Z_{r+2} + \cdots + Z_s) < 0 \tag{3.9}$$

for $r = 1, 2, \ldots, s$, then the multiple series

$$\sum_{k_1, \ldots, k_s \ge 0} \prod_{j=1}^{s} (k_j + 1)^{E_j} (k_1 + \cdots + k_j + 1)^{F_j} \tag{3.10}$$

converges.

Proof. By applying Lemma 2 iteratively, we have

$$\begin{aligned}
&\sum_{k_1,\dots,k_s\ge 0}\prod_{j=1}^{s}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\\
&\qquad\le D_7\sum_{k_1,\dots,k_{s-1}\ge 0}\left(\prod_{j=1}^{s-1}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\right)\\
&\qquad\qquad\cdot(k_1+\cdots+k_{s-1}+1)^{\max(Z_s)}\log(k_1+\cdots+k_{s-1}+2)\\
&\qquad\le D_8\sum_{k_1,\dots,k_{s-2}\ge 0}\left(\prod_{j=1}^{s-2}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\right)\\
&\qquad\qquad\cdot(k_1+\cdots+k_{s-2}+1)^{\max(Z_{s-1}+Z_s)}\left(\log(k_1+\cdots+k_{s-2}+2)\right)^2,
\end{aligned}$$

and, after the t-th iteration, $1\le t\le s-1$,

$$\begin{aligned}
&\sum_{k_1,\dots,k_s\ge 0}\prod_{j=1}^{s}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\\
&\qquad\le D_9\sum_{k_1,\dots,k_{s-t}\ge 0}\left(\prod_{j=1}^{s-t}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\right)\\
&\qquad\qquad\cdot(k_1+\cdots+k_{s-t}+1)^{\max(Z_{s-t+1}+\cdots+Z_s)}\left(\log(k_1+\cdots+k_{s-t}+2)\right)^t.
\end{aligned}$$

To justify these steps, we have to verify that the condition $A+B+1<0$ in Lemma 2 is satisfied in each iteration. However, this is exactly the condition (3.9) with r replaced by $s-t$.

Thus, for $t=s-1$ we arrive at the estimate

$$\begin{aligned}
&\sum_{k_1,\dots,k_s\ge 0}\prod_{j=1}^{s}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}\\
&\qquad\le D_{10}\sum_{k_1\ge 0}(k_1+1)^{E_1+F_1+\max(Z_2+\cdots+Z_s)}\left(\log(k_1+2)\right)^{s-1}.
\end{aligned}$$

Since the sum over k_1 at the right-hand side converges because of (3.9) with $r=1$, the claim follows. □

Remark. A careful check of our arguments reveals that, in fact, the conditions in Lemma 3 are optimal, meaning that they describe *exactly* the domain of convergence of the multiple sum (3.10). This can be seen by verifying that, if condition (3.9) is violated for a particular r, then the subsum

$$\sum_{k_r,\dots,k_s\ge 0}\prod_{j=1}^{s}(k_j+1)^{E_j}\,(k_1+\cdots+k_j+1)^{F_j}$$

of (3.10) does not converge. Thus, this lemma generalizes Proposition 1 in [21]. It does at the same time *correct* that proposition, and it answers the question raised after the (incomplete) proof of the proposition. The question, which is asked there, is to determine the domain of absolute convergence of the multizeta function

$$\zeta(s_d, s_{d-1}, \dots, s_1) = \sum_{0<n_1<\dots<n_d} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_d^{s_d}}. \tag{3.11}$$

Proposition 1 in [21] states that, for all d-tuples $(s_1, s_2, \dots, s_d)$ with $\mathfrak{Re}(s_d) > 1$ and $\sum_{i=1}^{d} \mathfrak{Re}(s_i) > d$, the series $\zeta(s_d, s_{d-1}, \dots, s_1)$ converges absolutely. (As the case $d = 3$, $s_1 = 3$, $s_2 = -1$, $s_3 = 2$ shows, these conditions are not sufficient.)

Applying Lemma 3 to

$$\sum_{0<n_1\le\dots\le n_d} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_d^{s_d}} = \sum_{k_1,\dots,k_d\ge 0} \prod_{j=1}^{d} \frac{1}{(k_1 + k_2 + \dots + k_j + 1)^{s_j}}$$

(that is, one chooses $s = d$, $E_i = 0$ and $F_i = -\mathfrak{Re}(s_i)$, $i = 1, 2, \dots, d$, there), it is seen that the domain of absolute convergence of this latter multisum is the set of all d-tuples $(s_1, s_2, \dots, s_d)$ such that

$$\sum_{i=r}^{d} \mathfrak{Re}(s_i) > d - r + 1, \qquad r = 1, 2, \dots, d. \tag{3.12}$$

Moreover, it is not difficult to see that, for the domain of absolute convergence, it does not matter whether we sum the summand on the right-hand side of (3.11) over $0 < n_1 < \dots < n_d$ or over $0 < n_1 \le \dots \le n_d$. Therefore, the domain described by the inequalities (3.12) is at the same time the domain of absolute convergence of $\zeta(s_d, s_{d-1}, \dots, s_1)$. That is, one has to add the conditions (3.12) for $i = 2, \dots, d-1$ to Zhao's two conditions to obtain a complete description of the domain of absolute convergence. As a matter of fact, all the arguments given in the proof of Proposition 1 in [21] are correct. However, it is only the case $d = 2$ which is carried out in detail (in which case there are no missing conditions), and therefore the additional $d - 2$ conditions are overlooked.

Proof of Proposition 1. (i) We consider first the left-hand side of Andrews' identity (3.1). We write the hypergeometric series as a sum over k. Let S_k denote the k-th summand. Since for $N \ge k > |a|$ we have

$$\left| \frac{(-N)_k}{(1+a+N)_k} \right| \le \frac{(N-k+1)\cdots(N-1)N}{(N+1-|a|)(N+2-|a|)\cdots(N+k-|a|)} \le 1,$$

and since for $k > N$ we have $(-N)_k = 0$, the modulus of $(-N)_k/(1+a+N)_k$ is bounded above by a constant for *all* $k = 0, 1, \dots$. Hence, using Lemma 1, we obtain that

$$|S_k| \le D_{11} \cdot (k+1)^{-E-1},$$

where D_{11} is some constant independent of k, and where E is the left-hand side of (3.2). Since, by (3.2), we have $E > 0$, the absolutely convergent series $\sum_{k=0}^{\infty} D_{11} \cdot (k+1)^{-E-1}$ dominates the hypergeometric series on the left-hand side of (3.1) term-wise. Thus, by

Lebesgue's dominated convergence theorem, we may perform its limit as $N \to \infty$ term-wise. This term-wise limit is exactly the left-hand side of (3.4).

Now we consider the right-hand side of (3.1). We need to temporarily assume that

$$\mathfrak{Re}(a - b_{s+1} - c_{s+1}) > 0. \tag{3.13}$$

(This is slightly stronger than (3.3) with $r = s+1$.) Writing A for $a - b_{s+1} - c_{s+1}$, for any non-negative integer $K \le N$ we have

$$\left| \frac{(-N)_K}{(b_{s+1} + c_{s+1} - a - N)_K} \right| \le \frac{N(N-1)\cdots(N-K+1)}{(N + \mathfrak{Re}(A))(N + \mathfrak{Re}(A) - 1)\cdots(N + \mathfrak{Re}(A) - K + 1)} \le 1,$$

and since for $K > N$ we have $(-N)_K = 0$, the modulus of $(-N)_K/(b_{s+1} + c_{s+1} - a - N)_K$ is bounded above by a constant for *all* $K = 0, 1, \dots$. Thus, again using Lemma 1, the modulus of the summand indexed by $k_1, k_2, \dots, k_s$ on the right-hand side of (3.1) is bounded above by

$$D_{12} \prod_{j=1}^{s} (k_j + 1)^{\mathfrak{Re}(a - b_j - c_j)} \, (k_1 + \cdots + k_j + 1)^{\mathfrak{Re}(b_j + c_j + b_{j+1} + c_{j+1} - 2(a+1))}, \tag{3.14}$$

for some constant D_{12} independent of the summation indices. Now we apply Lemma 3 with $E_j = \mathfrak{Re}(a - b_j - c_j)$ and $F_j = \mathfrak{Re}(b_j + c_j + b_{j+1} + c_{j+1} - 2(a+1))$. This is indeed justified since, for this choice of parameters, the set of conditions (3.9) is exactly the set (3.3). Hence, the sum of the expression (3.14) over all $k_1, \dots, k_s \ge 0$ converges. Another application of Lebesgue's dominated convergence theorem then implies that we may perform the limit of the multiple sum on the right-hand side of (3.1) as $N \to \infty$ term-wise. Together with the fact that

$$\lim_{N \to +\infty} \frac{(1+a)_N \, (1 + a - b_{s+1} - c_{s+1})_N}{(1 + a - b_{s+1})_N \, (1 + a - c_{s+1})_N} = \frac{\Gamma(1 + a - b_{s+1}) \, \Gamma(1 + a - c_{s+1})}{\Gamma(1+a) \, \Gamma(1 + a - b_{s+1} - c_{s+1})},$$

this establishes the identity (3.4), provided (3.13) holds in addition to the conditions of the statement of the proposition.

We can finally get rid of the restriction (3.13) by analytic continuation. Indeed, by using arguments very similar to those above, one can show that both sides of (3.4) are analytic in the parameters $a, b_1, \dots, b_{s+1}, c_1, \dots, c_{s+1}$ as long as (3.2) and (3.3) are satisfied. In particular, in variation of Lemma 1, one would use the fact that, for fixed complex numbers α and β, there are constants D_{13} and D_{14} such that

$$D_{13} \cdot (k+1)^{\mathfrak{Re}(\alpha - \beta)} \log(k+2) \le \left| \frac{\Gamma(x+k) \, \psi(x+k)}{\Gamma(\beta + k)} \right| \le D_{14} \cdot (k+1)^{\mathfrak{Re}(\alpha - \beta)} \log(k+2)$$

for all non-negative integers k and all complex numbers x in a sufficiently small neighbourhood of α, say for $|x - \alpha| < 1$. Here, $\psi(x)$ denotes the logarithmic derivative of $\Gamma(x)$.

(ii) In (3.4), we first shift the parameters to $b_j \to b_{j-1}$ and $c_j \to c_{j-1}$, and then we let $b_0 \to +\infty$. The same kind of argument as above then yields (3.8). $\square$

4. Multisum expansions of the Vasilyev-type integral J_m

The link between Andrews' identity and the Vasilyev-type integrals J_m becomes apparent in the next proposition.

Proposition 2. *Let z, $a_0, a_1, \dots, a_m$, and $b_1, \dots, b_m$ be complex numbers such that $|z| < 1$, $\mathfrak{Re}(a_0) > 0$, $\mathfrak{Re}(b_i) > \mathfrak{Re}(a_i) > 0$ for all $i = 1, 2, \dots, m$.*

(i) *If $m = 2s \geq 2$ is even, then*

$$J_m\left[\begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; z\right] = \prod_{j=1}^{2s} \frac{\Gamma(a_j)\,\Gamma(b_j - a_j)}{\Gamma(b_j)}$$
$$\times \sum_{k_1,k_2,\dots,k_s \geq 0} z^{k_1+\cdots+k_s} \prod_{j=1}^{s} \frac{(b_{2s-2j+2} - a_{2s-2j+2})_{k_j}}{k_j!} \frac{(a_{2s-2j+1})_{k_1+\cdots+k_j}}{(b_{2s-2j+1})_{k_1+\cdots+k_j}} \frac{(a_{2s-2j})_{k_1+\cdots+k_j}}{(b_{2s-2j+2})_{k_1+\cdots+k_j}}. \tag{4.1}$$

This identity holds also for $z = 1$ provided $\mathfrak{Re}(b_1 - a_1) \geq \mathfrak{Re}(a_0)$, and provided (3.9) *holds with $E_j = \mathfrak{Re}(b_{2s-2j+2} - a_{2s-2j+2} - 1)$ and $F_j = \mathfrak{Re}(a_{2s-2j} + a_{2s-2j+1} - b_{2s-2j+1} - b_{2s-2j+2})$, $j = 1, 2, \dots, s$.*

(ii) *If $m = 2s + 1 \geq 3$ is odd, then*

$$J_m\left[\begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; z\right] = \prod_{j=1}^{2s+1} \frac{\Gamma(a_j)\,\Gamma(b_j - a_j)}{\Gamma(b_j)} \cdot \sum_{k_1,\dots,k_{s+1} \geq 0} z^{k_1+\cdots+k_s} \frac{(a_{2s+1})_{k_1}}{k_1!} \frac{(a_{2s})_{k_1}}{(b_{2s+1})_{k_1}}$$
$$\times \prod_{j=2}^{s+1} \frac{(b_{2s-2j+4} - a_{2s-2j+4})_{k_j}}{k_j!} \frac{(a_{2s-2j+3})_{k_1+\cdots+k_j}}{(b_{2s-2j+3})_{k_1+\cdots+k_j}} \frac{(a_{2s-2j+2})_{k_1+\cdots+k_j}}{(b_{2s-2j+4})_{k_1+\cdots+k_j}}. \tag{4.2}$$

This identity holds also for $z = 1$ provided $\mathfrak{Re}(b_1 - a_1) \geq \mathfrak{Re}(a_0)$, and provided (3.9) *holds with $E_1 = \mathfrak{Re}(a_{2s+1} - 1)$, $F_1 = \mathfrak{Re}(a_{2s} - b_{2s+1})$, $E_j = \mathfrak{Re}(b_{2s-2j+4} - a_{2s-2j+4} - 1)$, and $F_j = \mathfrak{Re}(a_{2s-2j+2} + a_{2s-2j+3} - b_{2s-2j+3} - b_{2s-2j+4})$, $j = 2, 3, \dots, s + 1$.*

Proof. For $m \geq 2$, we denote by $Q_m(x_1, \dots, x_m; z)$ the nested expression in the denominator of the integrand in (2.1), that is

$$Q_m(x_1, \dots, x_m; z) = 1 - (1 - (\cdots(1 - x_m)x_{m-1})\cdots)x_1 z.$$

(i) We prove the claim by induction on m. For $m = 0$, the (empty) integral $J_0\left[\begin{matrix} a_0 \\ \ \end{matrix}; z\right]$ can be consistently interpreted as 1. In order to do the induction step, we fix $m = 2s \geq 2$ and z such that $|z| < 1$. Then, trivially,

$$\begin{aligned} Q_{2s}(x_1, \dots, x_{2s}; z) &= Q_{2s-2}(x_1, \dots, x_{2s-2}; z) - z x_1 \cdots x_{2s-1}(1 - x_{2s}) \\ &= Q_{2s-2}(x_1, \dots, x_{2s-2}; z)\left(1 - \frac{z x_1 \cdots x_{2s-1}(1 - x_{2s})}{Q_{2s-2}(x_1, \dots, x_{2s-2}; z)}\right), \end{aligned}$$

where for $s=1$ the term $Q_0(-;z)$ has to be interpreted as 1. Since for $x_j \in [0,1]$, we have

$$\left| \frac{z x_1 \cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)} \right| \le |z| < 1,$$

we may apply the binomial theorem to obtain

$$\left(1 - \frac{z x_1 \cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)}\right)^{-a_0} = \sum_{k_1=0}^{\infty} z^{k_1} \frac{\Gamma(a_0+k_1)}{\Gamma(a_0)\Gamma(k_1+1)} \left(\frac{x_1 \cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)}\right)^{k_1}.$$

Hence

$$\begin{aligned} J_{2s}\begin{bmatrix} a_0, a_1, \dots, a_{2s} \\ b_1, \dots, b_{2s} \end{bmatrix}; z\Big] &= \int_{[0,1]^{2s}} \sum_{k_1=0}^{\infty} z^{k_1} \frac{\Gamma(a_0+k_1)}{\Gamma(a_0)\Gamma(k_1+1)} x_{2s-1}^{a_{2s-1}+k_1-1}(1-x_{2s-1})^{b_{2s-1}-a_{2s-1}-1} \\ &\quad \cdot x_{2s}^{a_{2s}-1}(1-x_{2s})^{b_{2s}-a_{2s}+k_1-1} \frac{\prod_{j=1}^{2s-2} x_j^{a_j+k_1-1}(1-x_j)^{b_j-a_j-1}}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)^{k_1+a_0}} \, dx_1 \cdots dx_{2s}. \end{aligned}$$

The conditions on the parameters ensure that the integral

$$\begin{aligned} \int_{[0,1]^{2s}} \sum_{k_1=0}^{\infty} \Bigg| z^{k_1} \frac{\Gamma(a_0+k_1)}{\Gamma(a_0)\Gamma(k_1+1)} x_{2s-1}^{a_{2s-1}+k_1-1}(1-x_{2s-1})^{b_{2s-1}-a_{2s-1}-1} \\ \cdot x_{2s}^{a_{2s}-1}(1-x_{2s})^{b_{2s}-a_{2s}+k_1-1} \frac{\prod_{j=1}^{2s-2} x_j^{a_j+k_1-1}(1-x_j)^{b_j-a_j-1}}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)^{a_0+k_1}} \Bigg| \, dx_1 \cdots dx_{2s} \end{aligned}$$

is convergent. Thus, we can exchange the integral and the summation, and, using the beta integral evaluation and some straightforward simplifications, we obtain

$$\begin{aligned} J_{2s}\left[\begin{matrix} a_0, a_1, \dots, a_{2s} \\ b_1, \dots, b_{2s} \end{matrix}; z\right] &= \frac{\Gamma(a_{2s})\,\Gamma(a_{2s-1})\,\Gamma(b_{2s}-a_{2s})\,\Gamma(b_{2s-1}-a_{2s-1})}{\Gamma(b_{2s})\,\Gamma(b_{2s-1})} \\ \cdot \sum_{k_1=0}^{\infty} z^{k_1} &\frac{(b_{2s}-a_{2s})_{k_1}\,(a_{2s-1})_{k_1}\,(a_0)_{k_1}}{k_1!\,(b_{2s})_{k_1}\,(b_{2s-1})_{k_1}} J_{2s-2}\left[\begin{matrix} a_0+k_1, a_1+k_1, \dots, a_{2s-2}+k_1 \\ b_1+k_1, \dots, b_{2s-2}+k_1 \end{matrix}; z\right]. \end{aligned} \tag{4.3}$$

If we substitute the induction hypothesis for J_{2s-2}, we arrive exactly at (4.1).

We now perform the limit $z \to 1$: since the conditions on the parameters guarantee that the integral J_{2s} is absolutely convergent for $z=1$, dominated convergence implies that one can interchange limit and integral. Similarly, if we put $z=1$ in the above multiple sum, then the conditions on the parameters allow us to apply Lemmas 1 and 3 and to conclude that it converges absolutely. Thus, again, dominated convergence implies that we may interchange limit and summation. As a result, Case (i) of Proposition 2 is now completely proved.

(ii) We do not provide all the details for the case where m is odd, $m = 2s+1 \ge 3$, since this case can be treated in a rather similar manner as the case where m is even. A main difference, however, is that, to get started, we use the alternative identity

$$Q_{2s+1}(x_1,\dots,x_{2s+1};z)=Q_{2s}(x_1,\dots,x_{2s};z)\left(1-\frac{zx_1\cdots x_{2s+1}}{Q_{2s}(x_1,\dots,x_{2s};z)}\right),$$

which implies the expansion

$$\begin{aligned}
&J_{2s+1}\begin{bmatrix}a_0,a_1,\dots,a_{2s+1}\\ b_1,\dots,b_{2s+1}\end{bmatrix};z\\
&\quad=\frac{\Gamma(a_{2s+1})\,\Gamma(b_{2s+1}-a_{2s+1})}{\Gamma(b_{2s+1})}\cdot\sum_{k=0}^{\infty}z^k\,\frac{(a_0)_k\,(a_{2s+1})_k}{k!\,(b_{2s+1})_k}J_{2s}\begin{bmatrix}a_0+k,a_1+k,\dots,a_{2s}+k\\ b_1+k,\dots,b_{2s}+k\end{bmatrix};z\Bigg].
\end{aligned}\tag{4.4}$$

At this point, we substitute the multiple series (4.1) for J_{2s}, and after some simple manipulations we arrive at (4.2). ☐

Remark. Both of the recursive formulas (4.3) and (4.4) appear already earlier in the article [22] of Zlobin which we mentioned in the Introduction. He used them to express the integrals J_m in terms of another family of integrals, like those considered by Sorokin in [15, 16].

5. Proof of Theorem 1

We are now in the position to prove Zudilin's theorem, by putting together the identities in Proposition 2 and 1. Because of the use of Proposition 2 when $z=1$, we shall need to temporarily impose stronger conditions on the parameters than required by the assertion of the theorem. We shall do this without mention. One gets rid of these restrictions at the end by analytic continuation.

Let first m be even, $m=2s$. We apply Proposition 2, Eq. (4.1), with $a_{j-1}=h_j$, $j=1,2,\dots,2s+1$, $b_j=1+h_0-h_{j+2}$, $j=1,2,\dots,2s$. Thus, using (4.1), we express the integral on the left-hand side of (2.3) in terms of a multiple sum. If we subsequently apply the identity (3.4) with $b_j=h_{2s-2j+4}$, $c_j=h_{2s-2j+3}$ for $j=1,2,\dots,s+1$, to the multiple sum, then we arrive at the very-well-poised hypergeometric series on the right-hand side of (2.3).

Similarly, if m is odd, $m=2s+1$, then we apply Proposition 2, Eq. (4.2), with $a_{j-1}=h_j$, $j=1,2,\dots,2s+2$, $b_j=1+h_0-h_{j+2}$, $j=1,2,\dots,2s+1$. Thus, using (4.2), we express the integral on the left-hand side of (2.3) in terms of a multiple sum. If we subsequently apply the identity (3.8) with s replaced by $s+1$, $b_j=h_{2s-2j+4}$, $j=1,2,\dots,s+1$, $c_j=h_{2s-2j+3}$ for $j=0,1,\dots,s+1$, to the multiple sum, then we arrive at the very-well-poised hypergeometric series on the right-hand side of (2.3).

Acknowledgement

We thank Wadim Zudilin for an attentive reading of an earlier version of the paper, and, in particular, for several useful suggestions for improvement of the exposition.

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