Apparent Singularities of Differential Systems with Rational Function Coefficients.

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Introduction - Preliminaries

Apparent Singularities

$\mathbf{K} = \mathbb{C}(z)$, $' = \frac{d}{dz}$. For $A \in Mat_n(\mathbf{K})$, we denote by $[A]$ the system:

$$[A] \qquad \frac{dX}{dz} = A(z)X,$$

- The (finite) singularities of system $[A]$ are the poles of the entries of $A(z)$.
- Singularities of solutions of the system $[A]$ are among the singularities of $[A]$, but the converse is not always true.

Def. An apparent singularity of $[A]$ is a singular point where the general solution of $[A]$ is holomorphic.

Question:
How to detect and remove the apparent singularities of a given system $[A]$?

Example

$$[A] \qquad \frac{dX}{dz} = A(z)X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1+\frac{2}{z} \end{bmatrix}.$$

- A fundamental matrix solution of $[A]$ is

$$\begin{bmatrix} e^z & 1+z+\frac{z^2}{2} \\ e^z & 1+z \end{bmatrix}.$$

- Hence $z = 0$ is an apparent singularity for $[A]$.
- The polynomial gauge transformation

$$X = T(z)\ Y, \quad T(z) = \begin{bmatrix} 1 & 0 \\ 1 & z^2 \end{bmatrix}$$

  takes $[A]$ into the equivalent system

$$[B] \quad \frac{d}{dz} Y = B\ Y$$

  where

$$B = T[A] := T^{-1}\left(AT - \frac{dT}{dz}\right) = \begin{bmatrix} 1 & z^2 \\ 0 & 0 \end{bmatrix}.$$

What this talk is about?

- A general fact : Any system $[A]$ with rational coefficients can be reduced to a *gauge equivalent* system $[B]$ with rational coefficients, such that the finite singularities of $[B]$ coincide with the non-apparent singularities of $[A]$.
- Such a system $[B]$ will be called a *complete desingularization* of $[A]$
- We present an algorithm which outputs a desingularization for any input system $[A]$.
- More generally, given $[A]$, to find a polynomial matrix $T \in \mathrm{GL}_n(\mathcal{C}(z))$ such that $B = T[A]$ satisfies $\mathrm{den}(B) \mid \mathrm{den}(A)$, and $\mathrm{den}(B)$ is as "small" as possible.

Previous and related works

- Desingularization of linear difference/ differential (and more generally Ore) operators, e.g.
  - Abramov, van Hoeij 1999
  - Tsai 2000
  - Abramov-B.-van Hoeij'2006,
  - Chen-Jaroschek-Kauers-Singer'2013, Chen-Kauers-Singer'2015
  - Yi Zhang, ISSAC'2016
- Desingularization of linear differential systems:
  - B.'2010,
  - B.-Maddah ISSAC'2015
- Desingularization of linear difference systems Maximilian Jaroschek' talk

Classification of Singularities

Consider a System of first order linear differential equations:

$$[A] \qquad \frac{dX}{dz} = A(z)X, \;\; A(z) \in Mat_n(\mathbb{C}(z))$$

- A pole z_0 of $A(z)$ is a regular singular point for $[A]$ if there is a fundamental solution W of $[A]$ has the form:

  $$W(z) = \Phi(z)(z - z_0)^{\Lambda}$$

  where $\Phi(z)$ is holomorphic and Λ is a constant matrix.
- Otherwise z_0 is called an irregular singular point.
- A system [A] has regular singularity at z_0 iff it is gauge equivalent to a system [B] with a simple pole at z_0.
- We shall refer to simple poles of $A(z)$ as simple singularities of $[A]$.
- if z_0 is an apparent singularity then z_0 is a regular singularity and thus can be reduced to a simple one.

Apparent singularities are removable

Prop.0 If $z = z_0$ is a finite apparent singularity of $[A]$ then there exists a polynomial matrix $T(z)$ with

$$\det T(z) = c(z - z_0)^{\alpha}, \quad c \in \mathbb{C}^*, \alpha \in \mathbb{N}$$

such that $[B] := T[A]$ has no pole at $z = z_0$.

Proof.

- Every fundamental solution F of $[A]$ is holomorphic (in a neighborhood of z_0);
- There exists matrices $P(z) \in GL_n(\mathbb{C}[z])$, and $Q(z) \in GL_n(\mathbb{C}[[z - z_0]])$ such that

$$P(z)F(z)Q(z) = Diag((z - z_0)^{\alpha_1}, \ldots, (z - z_0)^{\alpha_n})$$

  where $\alpha_1, \ldots \alpha_n \in \mathbb{N}$
- Take

$$T(z) \; = \; P^{-1}(z) \; Diag((z - z_0)^{\alpha_1}, \ldots, (z - z_0)^{\alpha_n})$$

Characterization of Regular Singularities

How to recognize regular singularities?

Problem 1: Given a system $[A]$ and a pole z_0 of order $p_{z_0}(A) > 1$ to decide whether z_0 is regular singular or not.

In other words, to decide if the order of a given singularity can be reduced to 1 or not?

- There is no analogue of the Fuchs' Criterion.

Problem 2: Given a system $[A]$ and a pole z_0 of order $p_{z_0}(A) > 1$, to decide whether there exists $T \in \mathrm{GL}(n, \mathbb{C}((z - z_0)))$ such that $p_{z_0}(T[A]) < p_{z_0}(A)$.

- There is a method due to Moser (1960) which solve these two problems.

- Rational Moser -Algorithm Barkatou'1995: It transforms a given system over $\mathbb{C}(z)$ into an equivalent one for which the orders of the finite poles are reduced to their minimal values.

- Other methods for reducing the rank of a singularity (to its minimal value) do exist: Levelt (1992), Wagenfurer (1989), . . . , B.-Pfluegel (2007, 2008), B.-El Bacha (2012).

An example

$$A = \begin{bmatrix} \frac{-1}{z} & \frac{-1+z^{17}-8z^{14}+24z^{11}-32z^{8}+16z^{5}}{z^3\left(z^3-2\right)^2} & 0 & \frac{1}{z^3\left(z^3-2\right)^2} \\ 0 & 0 & \frac{1}{z^3\left(z^3-2\right)^2} & 0 \\ \frac{1}{\left(z^3-2\right)^3 z^4} & 0 & \frac{4}{z} & 0 \\ 0 & -\frac{2}{z} & \frac{1}{z^3\left(z^3-2\right)^2} & \frac{2}{z} \end{bmatrix} \in M_4(\mathbb{Q}(z))$$

- When applied to $[A]$ and the roots of the irreducible polynomial $z^3 - 2$, Algorithm [Bar'95] produces the equivalent matrix

$$B = T[A] = \begin{bmatrix} -\frac{10z^3+4}{z(z^3-2)} & 0 & 0 & 0 \\ \frac{1}{z^3(z^3-2)} & -\frac{8z^3+2}{z(z^3-2)} & 0 & \frac{z^2}{z^3-2} \\ 0 & \frac{1}{z^4(z^3-2)} & \frac{z^3-8}{z(z^3-2)} & 0 \\ 0 & 0 & \frac{1}{z^3(z^3-2)} & 0 \end{bmatrix}$$

and the gauge transformation

$$T = \begin{bmatrix} 0 & (z^3-2)^3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & z^3-2 & 0 \\ (z^3-2)^4 & 0 & 0 & 1 \end{bmatrix}$$

- The denominator of the matrix B is $z^4(z^3-2)$.
- Hence the system $[A]$ has regular singularities at the zeros of $q = z^3 - 2$.

Applied to $[B]$ and $z_0 = 0$, Algorithm [Bar'95] produces the equivalent matrix

$$C = S[B] = \begin{bmatrix} 0 & 0 & \frac{1}{z(z^3-2)} & \frac{15\,z^3-6}{z(z^3-2)} \\ \frac{1}{z^2(z^3-2)} & -\frac{12\,z^3-6}{z(z^3-2)} & 0 & 0 \\ 0 & \frac{1}{z^2(z^3-2)} & -\frac{4+z^3}{z(z^3-2)} & 0 \\ 0 & 0 & 0 & -\frac{15\,z^3-6}{z(z^3-2)} \end{bmatrix}$$

and the transformation

$$S = \begin{bmatrix} 0 & 0 & 0 & -z^5 \\ 0 & z^4 & 0 & 0 \\ 0 & 0 & z^2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- $z = 0$ is still a pole of order > 1, hence the point $z = 0$ is an irregular singular point of the original system $[A]$.

Moser's Method

▶We assume here $z_0 = 0$

$$A(z) = \frac{1}{z^p} \sum_{k=0}^{\infty} A_k z^k, \quad A_k \in M_n(\mathbb{C}), \quad A_0 \neq 0.$$

▶ A necessary condition that there exist a gauge transformation $T \in GL(n, \mathbb{C}((z)))$ such that $T[A] = \frac{1}{x^{p'+1}}(B_0 + B_1 z + \cdots)$ with $p' < p$ $(B_0 \neq 0)$, is that A_0 is nilpotent.

Moser rank: $m(A) = p - 1 + \frac{rank(A_0)}{n}$ if $p > 1$, otherwise $m(A) = 1$.

Moser invariant: $\mu(A) = \min\{m(T[A]) \mid T \in GL(n, \mathbb{C}((z)))\}$

Definition. $[A]$ is said to be Moser-reducible if $m(A) > \mu(A)$.

- $[A]$ is Moser-reducible $\iff \exists T \in GL(n, \mathbb{C}((z)))$ such that $m(T[A]) < m(A)$.

- $z = 0$ is regular singular for $[A] \iff \mu(A) = 1$.

A Criterion for Moser-reducibility

Theorem. [Moser 1960]

1. If $p > 1$ then A is Moser-reducible iff the polynomial

$$\mathcal{B}_A(\lambda) := z^{rank(A_0)} \det\left(\lambda I - A_0/z - A_1\right)_{|_{z=0}} \equiv 0.$$

2. If A is Moser-reducible then the reduction can be carried out with a transformation of the form
$T = (P_0 + zP_1)diag(1, \ldots, 1, z, \ldots, z), \quad P_i \in \mathbb{C}^{n\times n}, det P_0 \neq 0.$

- Applying Moser's Theorem several times, if necessary, $\mu(A)$ can be determined.
- Further, a polynomial matrix T such that $m(T[A]) = \mu(A)$ can be computed in this way

Review of Moser-reduction Algorithms

- There are various algorithms to compute T such that $T[A]$ is Moser-reduced.
- Moser's paper: no constructive algorithm given.
- Dietrich (1978), Hilali/Wazner (1987): first efficient algorithms,
- Bar'1995: version for rational function coefficients, implemented in ISOLDE
- B.-Pflügel (2007): New reduction algorithm + complexity analysis.

Description of Moser Algorithm

- By a constant gauge transformation we can put A_0 in the form:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & 0 \end{pmatrix}, \quad A_0^{11} \in \mathbb{C}^{r \times r} \quad r = rank(A_0).$$

- Let A_1 be partitioned so that A_1^{11} is a square matrix of order r:

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix},$$

- Consider

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{pmatrix}.$$

- Then $\det G_\lambda(A)) = \mathcal{B}_A(\lambda)$.
- A is Moser-reducible $\iff \det G_\lambda(A) \equiv 0$.

Case 1: $rank(A_0^{11} \;\; A_1^{12}) < r$ *(I)*

A is Moser-reducible $\iff \begin{vmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{vmatrix} = 0.$

Proposition 1 If $m(A) > 1$ and $rank(A_0^{11} \;\; A_1^{12}) < r$, then A is M-reducible and the reduction can be carried out with the gauge transformation

$$T = diag(zI_r, I_{n-r}).$$

Proof: Let $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dz}$.

$$B = z^{-p}[B_0 + zB_1 + \cdots] + z^{-1}diag(I_r, 0)$$

where

$$B_0 = \begin{pmatrix} A_0^{11} & A_1^{12} \\ 0 & 0 \end{pmatrix},$$

Since $p > 1$, then $m(B) = p - 1 + rank(B_0)/n < m(A) = p - 1 + r/n$.

Case 1: $rank(A_0^{11} \;\; A_1^{12}) < r$ *An example*

$$A = z^{-2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + z^{-1}\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

- Here $p = 2$, $r = 1 \Rightarrow m(A) = 1 + 1/2 = 3/2 > 1$.
- $\det G_\lambda(A) = \begin{vmatrix} 0 & 0 \\ 2 & -3+\lambda \end{vmatrix} = 0 \Rightarrow A$ is Moser-reducible.
- Take

$$T = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

$$B := T[A] = T^{-1}AT - T^{-1}T' = \frac{1}{z}\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}.$$

- The system $Z' = BZ$ has a singularity of first kind at $z = 0$.
- Hence $Y' = AY$ has a regular singularity at $z = 0$.

Case 2: $rank(A_0^{11} \;\; A_1^{12}) = r$ *(I)*

Proposition 2 If A is M-reducible and $rank(A_0^{11} \;\; A_1^{12}) = r$, then there exists a constant matrix Q such that the matrix $G_\lambda(Q[A])$ has the form has the following particular form:

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & U_1 & U_2 \\ V_1 & W_1 + \lambda I_{n-r-h} & W2 \\ 0 & 0 & W_3 + \lambda I_h \end{pmatrix}, \tag{1}$$

where $1 \leq h \leq n - r$, $W1, \; W3$ are square matrices of order $(n - r - h)$ and h respectively, W_3 is upper triangular with zero diagonal with the condition

$$rank(A_0^{11} \;\; U_1) < r \tag{2}$$

Case 2: $rank(A_0^{11} \;\; A_1^{12}) = r$ *(II)*

Proposition 3 If $m(A) > 1$ and $G_\lambda(A)$ has the form (1) with the condition (2), then A is reducible and the reduction can be carried out with the transformation

$$T = diag(zI_r, I_{n-r-h}, zI_h)$$

Proof: Put $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dz}$. One has

$$B = z^{-p}[B_0 + zB_1 + \cdots] + z^{-1}\mathrm{diag}(I_r, 0, I_h)$$

where

$$B_0 = \begin{pmatrix} A_0^{11} & U_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and then $rank(B_0) = rank(A_0^{11} \;\; U_1) < r = rank(A_0)$. On the other hand since $p > 1$, then $m(B) = p - 1 + rank(B_0)/n$. Hence $m(B) < m(A)$.

Moser-reduction (End)

If A is Moser-reducible and $m(A) > 1$ then one can construct a matrix polynomial S of the form :

$$S = U\mathrm{diag}(z, z, \cdots, z, 1, 1, \cdots, 1)$$

where $U \in \mathrm{GL}(n, \mathbb{C})$, such that $m(S[A]) < m(A)$.

- Moser's Theorem allows us to check whether A is Moser-reducible.
- If A is Moser-reducible then by the above theorem we can find a matrix S such that $m(S[A]) < m(A)$.
- After this reduction has been carried out we can apply Moser's Theorem to check whether further reduction is possible and so on.
- After at most $n(p-1)$ steps we obtain an equivalent matrix B such that $m(B) = \mu(A)$.
- The nature of the singularity depends on the first $n(p-1)$ coefficients in the series expansion of A

Removal of Apparent Singularities

How to detect and remove an apparent singularity? (I)

Prop.1: If $z = z_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^{\alpha}$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $T[A]$ has at worst a simple pole at $z = z_0$.

This follows from the fact that:

- if z_0 is an apparent singularity then z_0 is a regular singularity,
- and that a system with a regular singularity at z_0 is equivalent to a system with a simple pole at z_0.

• The transformation T can be constructed using the *rational Moser algorithm* (developed in Bar'1995).

How to detect and remove an apparent singularity? (II)

Prop.2 Suppose that $A(z)$ has simple pole at $z = z_0$ and let

$$A(z) = \frac{A_0}{(z - z_0)} + \sum_{k \geq 1} A_k (z - z_0)^{k-1}, \quad A_k \in \mathbb{C}^{n \times n}.$$

If z_0 is an apparent singularity then the eigenvalues of A_0 are nonnegative integers and A_0 is diagonalizable.

This follows from the fact that:

- A system having a simple singularity at $z = z_0$ with residue matrix A_0 possesses a local fundamental solution of the form:

  $$\Phi(z)(z - z_0)^{\Lambda}$$

  where $\Phi(z)$ is holomorphic at $z = z_0$ and Λ is a constant matrix with

  $$spec(\Lambda) \subset spec(A_0) - \mathbb{N}$$

- When A_0 is not diagonalizable, the local solutions at z_0 involve logarithmic terms.

How to detect and remove an apparent singularity? (III)

Prop.3: Suppose that $z = z_0$ is a simple pole of $A(z)$ and that its residue matrix A_0 has only nonnegative integer eigenvalues. Then one can construct a polynomial matrix $T(z)$ with

$$\det T(z) = c(z - z_0)^{\alpha}$$

for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that

$$B := T[A] = B_0(z - z_0)^{-1} + \cdots$$

has at worst a simple pole at $z = z_0$ with

$$B_0 = mI_n + N$$

where $m \in \mathbb{N}$ and N nilpotent.

- Moreover z_0 is an apparent singularity iff $N = 0$.
- In this case the gauge transformation $Y = (z - z_0)^m \tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.

How to detect and remove an apparent singularity? (IV)

Skech of the proof of Prop. 3:

- The eigenvalues of A_0 of are nonnegative integers:

$$m_1 > m_2 > \ldots > m_s, \quad m_i - m_{i+1} = \ell_i \in \mathbb{N}^*, \quad i = 1, \ldots, s-1.$$

- By applying a constant gauge transformation we can assume that:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix},$$

where A_0^{11} is an ν_1 by ν_1 matrix having one single eigenvalue m_1

$$A_0^{11} = m_1 I_{\nu_1} + N_1$$

N_1 being a nilpotent matrix.

• Apply the gauge transformation $U = diag((z - z_0)I_{\nu_1}, I_{n-\nu_1})$ yields the new system:

$$Z' = (z - z_0)^{-1}\tilde{A}(z)Z, \quad \tilde{A}(z) = (z - z_0)U^{-1}A(z)U - (z - z_0)U^{-1}U'$$

with the leading matrix:

$$\tilde{A}(z_0) = \left(A_0 + (z - z_0)U^{-1}A_1 U - (z - z_0)U^{-1}U'\right)_{|z=z_0}.$$

• Let A_1 be partitioned as A_0 :

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \quad A_1^{11} \in \mathbb{C}^{\nu_1 \times \nu_1}$$

Then

$$\tilde{A}(z_0) = \begin{pmatrix} A_0^{11} - I_{\nu_1} & A_1^{12} \\ 0 & A_0^{22} \end{pmatrix}.$$

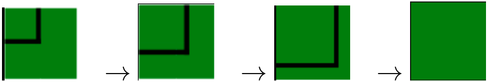
Hence the eigenvalues of $\tilde{A}(0)$ are: $m_1 - 1, m_2, \ldots, m_s$, each with the same initial multiplicity ν_i.

How to detect and remove an apparent singularity? (V)

- By repeating this process ℓ_1 times, the eigenvalues become:

$$m_1 - \ell_1 = m_2, m_2, \ldots, m_s.$$

- Thus, after $\ell_1 + \ldots + \ell_{s-1}$ steps, the eigenvalues $m_1, \ldots, m_s$ are reduced to one single eigenvalue m_s of multiplicity $\nu_1 + \ldots + \nu_s = n$.

$A_0 =$  $= m_s I_n + N$

- z_0 is an apparent singularity iff $N = 0$.

- In this case the gauge transformation $Y = (z - z_0)^{m_s} \tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.

Complete Desingularization

• The matrix T in Prop3 is obtained as a product of invertible constant matrices or diagonal matrices of the form $U = diag((z - z_0)I_\nu, I_{n-\nu})$.

Hence T is a polynomial matrix with $\det T(z) = c(z - z_0)^\alpha$ for some $c \in \mathbb{C}$ and $\alpha \in \mathbb{N}$.

• Due to the form of its determinant, the gauge transformation $T(z)$ in the above proposition does not affect the other finite singularities of $[A]$.

Theorem One can construct a polynomial matrix $T(z)$ which is invertible in $\mathbb{C}(z)$ such that the finite poles of $B := T[A]$ are exactly the true singularities for $[A]$.

Algorithm of Desingularization

Step 1 Reduce the rank of each singularity to its minimal value :

Compute a polynomial matrix $T(z)$ such that

- the zeros of $\det T(z)$ are in $\mathcal{P}(A)$
- $T[A]$ has the same poles as A with minimal orders.

Step 2 Select simple singularities with nonnegative exponents:

For each simple pole z_0 compute $A_{0,z_0} := res_{z=z_0} A(z)$ and its spectrum.

$\mathcal{A}pp(A) := \{z_0 \textit{ simple singularities such that } spec(A_{0,z_0}) \subset \mathbb{N}\}$

Step 3 Make all exponents equal:

For $z_0 \in \mathcal{A}pp(A)$ compute a polynomial matrix $T_{z_0}(z)$ with $\det T_{z_0}(z) = c(z - z_0)^{\alpha}$ such that $T_{z_0}[A]$ has at worst a simple pole at $z = z_0$ with residue matrix of the form $R_{z_0} = m_{z_0} I_n + N_{z_0}$ where $m_{z_0} \in \mathbb{N}$ and N_{z_0} nilpotent.

Step 4 Determine the apparent singularities of $[A]$:

Keep in $\mathcal{A}pp(A)$ only the points z_0 for which $N_{z_0} = 0$.

Step 5 Shift the exponent to 0 :

Apply the scalar transformation $T = \prod_{z_0 \in \mathcal{A}pp(A)} (z - z_0)^{m_{z_0}} I_n$.

Application to Desingularization of Scalar Differential Equations

Desingularization of Scalar differential Equations

• Removing apparent singularities of $L \in \mathbb{C}(z)[\partial]$:

$\rightarrow$ to construct another operator $\tilde{L} \in \mathbb{C}(z)[\partial]$ such that:

(i) any solution of $L(y) = 0$ is a solution of $\tilde{L}(y) = 0$,
i.e. $\tilde{L} = R \circ L$ for some $R \in \mathbb{C}(z)[\partial]$

(ii) and the singularities of $\tilde{L}$ are exactly the singularities of L that are not apparent.

- Such an operator $\tilde{L}$ is called a desingularization of L.

Example: $L = \partial - \frac{\mu}{z}, \qquad \mu \in \mathbb{N}.$

The operator $\tilde{L} = \partial^{\mu+1}$ is a desingularization of L.

• Several algorithms have been developed for linear differential (and more generally Ore) operators, e.g.

- Abramov-B.-van Hoeij'2006,
- Chen-Jaroschek-Kauers-Singer'2013, Chen-Kauers-Singer'2015

- We developed, in [ABH 2006][1] an algorithm that, given an operator L of order n, produces a desingularization $\tilde{L}$ with minimal order $m \geq n+1$.
- This algorithm has been implemented in Maple.
- I will refer to this algorithm as ABH method.

[1] S. ABRAMOV, M. BARKATOU and M. van HOEIJ AAECC 2006

Example 1

Consider the second order operator

$$L := \partial^2 - \frac{(z+2)}{z}\partial + \frac{2}{z}.$$

- $z = 0$ is a singularity of L.
- The general solution of $L(y) = 0$ is given by

$$c_1 e^z + c_2 \left(1 + z + \frac{z^2}{2}\right) \qquad c_1, c_2 \in \mathbb{C}.$$

- L has an apparent singularity at $z = 0$.
- The desingularization computed by ABH method is of order 4

$$\tilde{L} = \partial^4 + \left(-1 + \frac{z}{4}\right)\partial^3 + \left(-\frac{1}{4} - \frac{3\,z}{8}\right)\partial^2 + \left(\frac{1}{2} + \frac{z}{8}\right)\partial - \frac{1}{4}$$

- The apparent singularity of L at $z = 0$ can be removed by computing a *gauge equivalent* first-order differential system with coefficient in $\mathbb{C}(z)$ of size $\mathrm{ord}(L) = 2$.
- Consider the first-order differential system associated with L

$$[A] \qquad \frac{d}{dz}X = A(z)X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z} \end{bmatrix}.$$

- Set

$$X = T(z)\ Y, \quad \text{where} \quad T(z) = \begin{bmatrix} 1 & 0 \\ 1 & z^2 \end{bmatrix}.$$

- The new variable Y satisfies the gauge equivalent first-order differential system of the same dimension given by

$$[B] \quad \frac{d}{dz}Y = B\ Y$$

where

$$B := T^{-1}AT - T^{-1}\frac{d}{dz}T = \begin{bmatrix} 1 & z^2 \\ 0 & 0 \end{bmatrix}.$$

Example 2

- Consider

$$L = \partial^2 + \frac{\left(3\,z^2 - 4\right)}{z\,(z^2+2)}\,\partial - 2\,\frac{-1+2\,z^2}{z^2+2}$$

- L has an apparent singularity at $z = 0$ with local exponents 0 and 3.

- The desingularization computed by ABH method is of order 4

$$\tilde{L} = \partial^4 + 1/2\,\frac{z\left(24+7\,z^2\right)}{z^2+2}\,\partial^3 + 1/2\,\frac{\left(58\,z^2+88+27\,z^4\right)}{\left(z^2+2\right)^2}\,\partial^2$$

$$-1/2\,\frac{z\left(-4\,z^2+4+93\,z^4+28\,z^6\right)}{\left(z^2+2\right)^3}\,\partial - 4\,\frac{44\,z^2+16+42\,z^4+7\,z^6}{\left(z^2+2\right)^3}.$$

- The companion matrix of L is

$$A = \begin{bmatrix} 0 & 1 \\ 2\,\frac{-1+2\,z^2}{z^2+2} & -\frac{3\,z^2-4}{z(z^2+2)} \end{bmatrix}$$

- It has a simple pole at $z = 0$ with a residue matrix $A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

- Our algorithm computes the following gauge transformation T

$$T = \begin{bmatrix} 1 & 0 \\ z & -z^2 \end{bmatrix}$$

- The matrix of the new equivalent system is

$$B = T^{-1}(AT - T') = \begin{bmatrix} z & -z^2 \\ 1 & -\frac{z\left(z^2+7\right)}{z^2+2} \end{bmatrix}$$

- It has $z = 0$ as ordinary point.

Example 3

- Let $\partial = \frac{d}{dz}$ and consider

$$L = \partial^2 - \frac{(z^2-3)\,(z^2-2\,z+2)}{(z-1)\,(z^2-3\,z+3)\,z}\partial + \frac{(z-2)\,(2\,z^2-3\,z+3)}{(z-1)(z^2-3\,z+3)z}.$$

- L has apparent singularities at $z = 0$ and the roots of $z^2 - 3\,z + 3 = 0$.

- A desingularization computed by the classical algorithm[2] is given by:

$$\begin{aligned} \tilde{L}_{Classical} \quad = \quad & (z-1)\,(z^4 - z^3 + 3\,z^2 - 6\,z + 6)\partial^4 \\ & -(z^5 - 2\,z^4 + z^3 - 12\,z^2 + 24\,z - 24)\,\partial^3 \\ & -(3\,z^3 + 9\,z^2)\,\partial^2 + (6\,z^2 + 18\,z)\,\partial - (6\,z + 18). \end{aligned}$$

[2] Exm 1, Chen-Kauers-Singer'14

- A desingularization computed by the probabilistic method of CKS14[3] is given by:

$$\begin{aligned}\tilde{L}_{CKS} &= (z-1)\,(z^6-3\,z^5+3\,z^4-z^3+6)\,\partial^4\\ &\quad -(2\,z^6-9\,z^5+15\,z^4-11\,z^3+3\,z^2-24)\,\partial^3\\ &\quad -(z^7-4\,z^6+6\,z^5-4\,z^4+z^3+6\,z-6)\,\partial\\ &\quad +(2\,z^6-9\,z^5+15\,z^4-11\,z^3+3\,z^2-24).\end{aligned}$$

- The removal of one apparent singularity introduces new singularities. The latter can then be removed by using a trick introduced in ABH algorithm.

[3] Exm 7(1), Chen-Kauers-Singer'14

- The desingularization computed by ABH method is:

$$\begin{aligned}\tilde{L}_{ABH} \quad = \quad & \partial^4 + \frac{(16\,z^4 - 55\,z^3 + 63\,z^2 - 42\,z + 36)}{9\,(z-1)}\,\partial^3 \\ & - \frac{(64\,z^5 - 316\,z^4 + 591\,z^3 - 468\,z^2 + 123\,z + 42)}{9\,(z-1)^2}\,\partial^2 \\ & - \frac{96\,z^5 - 570\,z^4 + 1333\,z^3 - 1597\,z^2 + 993\,z - 219}{9\,(z-1)^3} \\ & + \frac{\beta}{9\,(z-1)^3}\,\partial,\end{aligned}$$

where

$$\beta = (48\,z^6 - 197\,z^5 + 148\,z^4 + 488\,z^3 - 1162\,z^2 + 999\,z - 288).$$

- The companion matrix of L is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{(z-2)\,(2\,z^2-3\,z+3)}{(z-1)\,(z^2-3\,z+3)\,z} & \frac{(z^2-3)\,(z^2-2\,z+2)}{(z-1)\,(z^2-3\,z+3)\,z} \end{bmatrix}$$

- Our new algorithm computes the following gauge transformation T

$$T = \begin{bmatrix} 1 & 0 \\ 1 & (-z^2 + 3\,z - 3)\,z^2 \end{bmatrix}$$

- The matrix of the new equivalent system is

$$B = T^{-1}(AT - T') = \begin{bmatrix} 1 & -z^2\,(z^2 - 3\,z + 3) \\ 0 & \frac{2}{1-z} \end{bmatrix}$$

- It has $z = 0$ and roots of $z^2 - 3\,z + 3 = 0$ as ordinary points.
- No new apparent singularities are introduced.

Comments

- The desingularization algorithms developed specifically for scalar equations are based on computing a least common left multiple of the operator in question and an appropriately chosen operator.
- This outputs an equation whose solution space contains strictly the solution space of the input equation.
- The new algorithm is based on an adequate choice of a gauge transformation.
- The desingularized output system is always equivalent to the input system and the dimension of the solution space is preserved.
- The transformations and the equivalent systems computed by our algorithm, have rational function coefficients.