Hopf algebras and factorial divergent power series: Algebraic tools for graphical enumeration

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Renormalized asymptotic enumeration of Feynman diagrams, Annals of Physics, October 2017 *arXiv:1703.00840*

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Overview

Motivation

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- Each integral maybe represented as a graph.
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- Divergence is believed to be caused by proliferation of graphs.

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- For instance, the partition function in φ^3 theory is formally,

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- 'Formal' $\Rightarrow$ expand under the integral sign and integrate over Gaussian term by term,

$$Z^{\varphi^3}(\hbar) := \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}]e^{\frac{x^3}{3!\hbar}}$$

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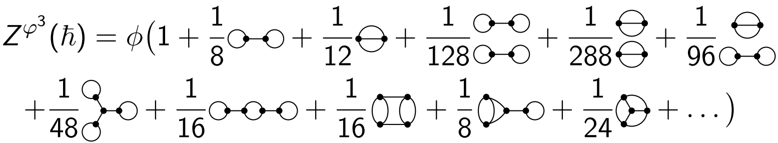

$$= 1 + \left(\frac{1}{8} + \frac{1}{12}\right)\hbar + \frac{385}{1152}\hbar^2 + \dots$$

where $\phi(\Gamma) = \hbar^{|E(\Gamma)|-|V(\Gamma)|}$ maps a graph with excess n to $\hbar^n$.

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- More general with arbitrary degree distribution,

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- This corresponds to the pairing model of multigraphs [Bender and Canfield, 1978].

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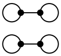
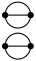
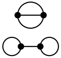
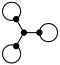
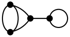
$$\sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 3} \frac{\lambda_k}{k!} x^k}{\hbar}} = \phi\Big(1 + \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \frac{1}{8}$$ 

$$+ \frac{1}{128}$$  $$+ \frac{1}{288}$$  $$+ \frac{1}{96}$$  $$+ \frac{1}{48}$$ 

$$+ \frac{1}{16}$$  $$+ \frac{1}{16}$$  $$+ \frac{1}{8}$$  $$+ \frac{1}{24}$$  $$+ \cdots \Big)$$

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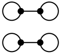
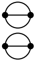
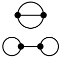
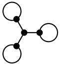
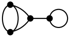
$+ \frac{1}{128}$  $+ \frac{1}{288}$  $+ \frac{1}{96}$  $+ \frac{1}{48}$ 

$+ \frac{1}{16}$  $+ \frac{1}{16}$  $+ \frac{1}{8}$  $+ \frac{1}{24}$  $+ \cdots \Big)$

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$$= 1 + \left(\left(\frac{1}{8} + \frac{1}{12} \right) \lambda_3^2 + \frac{1}{8} \lambda_4 \right) \hbar$$

$$+ \left(\frac{385}{1152} \lambda_3^4 + \frac{35}{64} \lambda_3^2 \lambda_4 + \frac{35}{384} \lambda_4^2 + \frac{7}{48} \lambda_3 \lambda_5 + \frac{1}{48} \lambda_6 \right) \hbar^2 + \cdots$$

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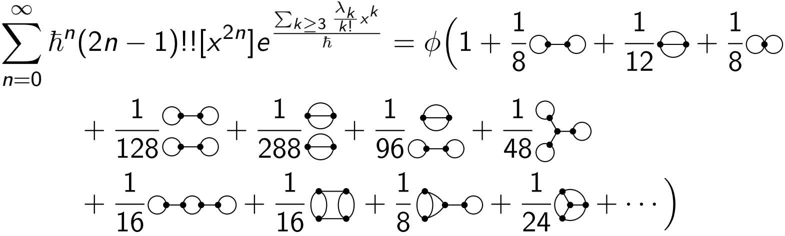

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&= \sum_{n=0}^{\infty} P_n(\lambda_3, \lambda_4, \ldots)\hbar^n
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Example

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$\Rightarrow$ Stirling series as a signed sum over multigraphs.

- Define a functional $\mathcal{F} : \mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$

$$\mathcal{F} : \mathcal{S}(x) \mapsto \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\frac{x^2}{2} + \mathcal{S}(x)}{\hbar}}$$

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- $\mathcal{F}$ maps a degree sequence to the corresponding generating function of multigraphs.

Expressions as

$$\mathcal{F}[\mathcal{S}(x)] = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\frac{x^2}{2}+\mathcal{S}(x)}{\hbar}}$$

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where $x(y)$ is the (power series) solution of

$$\frac{y^2}{2} = -\mathcal{S}(x(y)).$$

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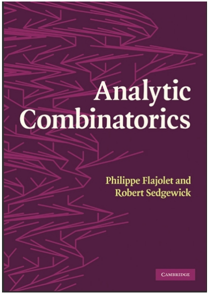

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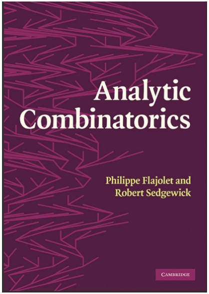

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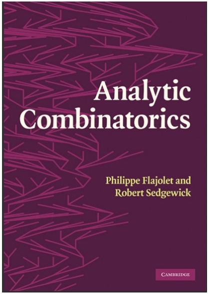
$\Rightarrow$ Locate the dominant singularity of $x(y)$ by analysis of the (generalized) hyperelliptic curve,

$$\frac{y^2}{2} = -\mathcal{S}(x).$$



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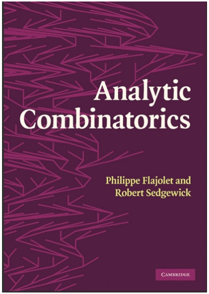


- The dominant singularity of $x(y)$ coincides with a branch-cut of the local parametrization of the curve.

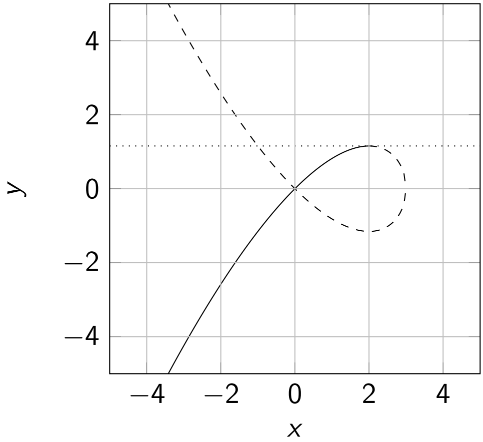
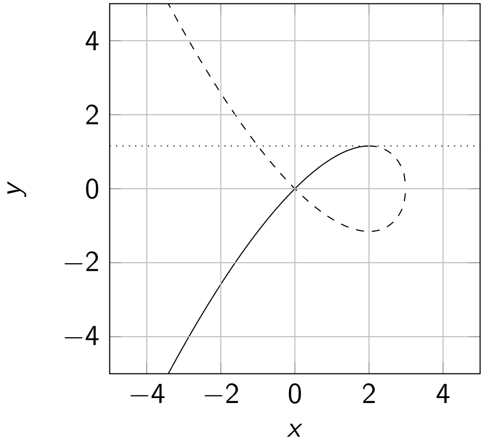


Figure: Example: The curve $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^3}{3!}$ associated to Z^{φ^3}.



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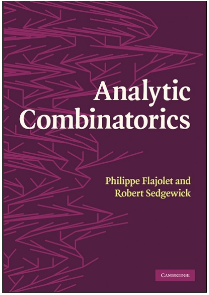
$\Rightarrow x(y)$ has a (dominant) branch-cut singularity at $y = \rho = \frac{2}{\sqrt{3}}$, where $x(\rho) = \tau = 2$.


Analytic
Combinatorics
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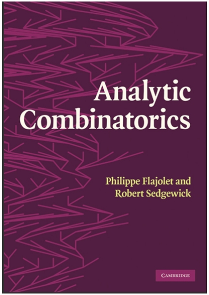
⇒ Near the dominant singularity:

$$x(y) - \tau = -d_1 \sqrt{1 - \frac{y}{\rho}} + \sum_{j=2}^{\infty} d_j \left(1 - \frac{y}{\rho}\right)^{\frac{j}{2}}$$



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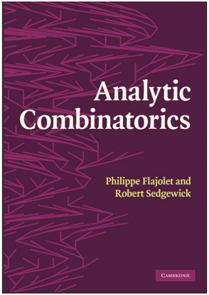

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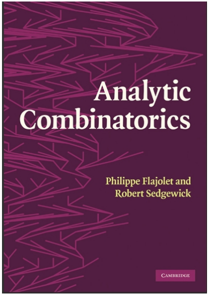

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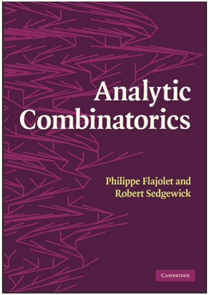

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$\Rightarrow$ Near the dominant singularity:

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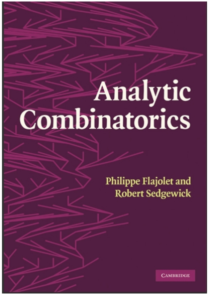

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$$\begin{aligned}[\hbar^n]\mathcal{F}[\mathcal{S}(x)] &= (2n+1)!![y^{2n+1}]x(y) \\ &= \frac{2^n \Gamma(n + \frac{3}{2})}{\sqrt{\pi}} [y^{2n+1}]x(y) \\ &\sim C' 2^n \rho^{-2n} \Gamma(n) \left(1 + \sum_{k=1}^{\infty} \frac{e'_k}{n^k}\right)\end{aligned}$$

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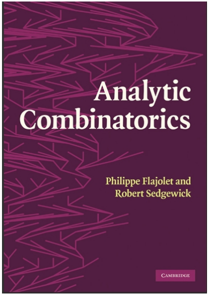
$$x(y) - \tau = -d_1 \sqrt{1 - \frac{y}{\rho}} + \sum_{j=2}^{\infty} d_j \left(1 - \frac{y}{\rho}\right)^{\frac{j}{2}}$$

$$[y^n]x(y) \sim C \rho^{-n} n^{-\frac{3}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right)$$



- Therefore,

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- What is the value of C' and the e'_k?

Theorem ([MB, 2017])

Let $(x, y) = (\tau, \rho)$ *be the location of the dominant branch-cut singularity of* $\frac{y^2}{2} = -\mathcal{S}(x)$*. Then*

$$[\hbar^n]\mathcal{F}[\mathcal{S}(x)](\hbar) = \sum_{k=0}^{R-1} c_k A^{-(n-k)}\Gamma(n-k) + \mathcal{O}\left(A^{-n}\Gamma(n-R)\right),$$

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$\Rightarrow$ The asymptotic expansion can be expressed as a generating function of graphs.

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- For cubic graphs or equivalently φ^3 theory, we are interested in $\mathcal{S}(x) = -\frac{x^2}{2} + \frac{x^3}{3!}$,

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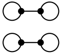
$\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \phi\Big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{128}$  $+ \ldots\Big)$

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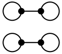
$\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \phi\Big(1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{128}$  $+ \ldots\Big)$

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- We find $\tau = 2$, $A = \frac{2}{3}$ and the coefficients of the asymptotic expansion

$$\sum_{k=0}^{\infty} c_k \hbar^k = \frac{1}{2\pi}\mathcal{F}[\mathcal{S}(\tau) - \mathcal{S}(\tau + x)](-\hbar) = \frac{1}{2\pi}\mathcal{F}[-\frac{x^2}{2} + \frac{x^3}{3!}](-\hbar)$$

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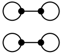
$$\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \phi\big(1 + \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \frac{1}{128}$$  $$+ \dots\big)$$

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$\Rightarrow$ The asymptotic expansion is $[\hbar^n]\mathcal{F}\left[\mathcal{S}(x)\right](\hbar) = \sum_{k=0}^{R-1} c_k A^{-n+k}\Gamma(n-k) + \mathcal{O}(A^{-n+R}\Gamma(n-R))$.

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  form a subring of $\mathbb{R}[[x]]$ which is closed under composition and inversion of power series [MB, 2016].
- For instance, this allows us to calculate the complete asymptotic expansion of constructions such as

$$\log \mathcal{F}\left[\mathcal{S}(x)\right](\hbar)$$

  in closed form.

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- which is compatible with our generating function and asymptotic techniques.

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- This is a Riemann-Hilbert problem.

Hopf algebra of graphs

- Pick a set of bridgeless graphs P, such that

$$\text{if } \gamma_1 \subset \gamma_2 \text{ then } \gamma_1, \gamma_2 \in P \text{ iff } \gamma_1, \gamma_2/\gamma_1 \in P \tag{1}$$

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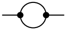
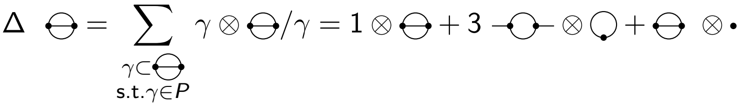
- Example: Suppose $\emptyset$, ,  $\in P$, then



where we had to consider the subgraphs

the complete and the empty subgraph.

- Gives us an action on algebra morphisms $\mathcal{G} \to \mathbb{R}[[\hbar]]$. For $\psi : \mathcal{H} \to \mathbb{R}[[\hbar]]$ and $\phi : \mathcal{G} \to \mathbb{R}[[\hbar]]$,

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- Coassociativity of Δ implies associativity of $\star$:

$$(\xi \star \psi) \star \phi = \xi \star (\psi \star \phi)$$

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- Simplifies on many (physical) cases to a (functional) inversion problem on power series.
- Solves the Riemann-Hilbert problem:
  Invert $\phi : \Gamma \mapsto \hbar^{|E(\Gamma)|-|V(\Gamma)|}$ restricted to $\mathcal{H}$. Then

$$(\phi|_{\mathcal{H}}^{\star-1} \star \phi)(\Gamma) = \begin{cases} \hbar^{|E(\Gamma)|-|V(\Gamma)|} \text{ if } \Gamma \in \mathcal{G}^+ \\ 0 \text{ else} \end{cases}$$

The identity van Suijlekom [2007], Yeats [2008],

$$\Delta X = \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} X_P^{(|v|)} \right) \otimes \frac{\Gamma}{|\operatorname{Aut} \Gamma|},$$

where $X = \sum_{\text{graphs } \Gamma} \frac{\Gamma}{|\operatorname{Aut} \Gamma|}$ and

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can be used to make this accessible for asymptotic analysis:

$$\phi|_{\mathcal{H}}^{\star -1} \star \phi(X) = m \circ \left(\phi|_{\mathcal{H}}^{\star -1} \otimes \phi\right) \circ \Delta X$$

$$= \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \phi|_{\mathcal{H}}^{\star -1} \left(X_P^{(|v|)} \right) \right) \frac{\phi(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

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- The generating function of all graphs with arbitrary weight for each vertex degree is

$$\sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 0} \frac{\lambda_k}{k!} x^k}{\hbar}}$$

$$= \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \lambda_{|v|} \right) \frac{\hbar^{|E(\Gamma)| - |V(\Gamma)|}}{|\operatorname{Aut}\Gamma|}$$

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$$= \sum_{\text{graphs } \Gamma} \left(\prod_{v \in V(\Gamma)} \lambda_{|v|} \right) \frac{\hbar^{|E(\Gamma)|-|V(\Gamma)|}}{|\operatorname{Aut}\Gamma|}$$

- Therefore,

$$\phi|_{\mathcal{H}}^{\star -1} \star \phi(X) = \sum_{n=0}^{\infty} \hbar^n (2n-1)!![x^{2n}] e^{\frac{\sum_{k\geq 0} \frac{x^k}{k!} \phi|_{\mathcal{H}}^{\star -1} \left(X_P^{(k)} \right)}{\hbar}}$$

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- We can obtain the full asymptotic expansions in these cases.

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- Arbitrary high order terms can be obtained by iteratively solving implicit equations.

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- Asymptotics are easily accessible: The asymptotic expansion also enumerates graphs.
- With Hopf algebra techniques restrictions on the set of enumerated graphs can be imposed.

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