The free boundary Schur process and applications

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Outline

1. The free boundary Schur process
   - A motivation: plane partitions
   - Free boundary conditions
   - General definition
2. Main results
   - Partition function
   - Correlation functions
3. Tidbits from the proof
   - Transfer matrix method
   - Boson-fermion correspondence
   - Handling free boundary states

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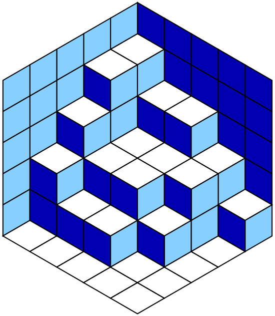
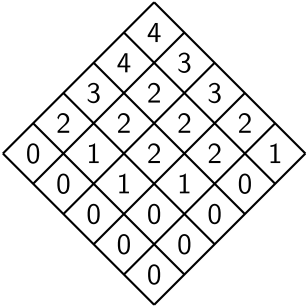
Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

Lozenge tiling



Plane partition



Sequence of interlaced integer partitions

$$\emptyset \prec 2 \prec \begin{matrix}3\\1\end{matrix} \prec \begin{matrix}4\\2\\1\end{matrix} \prec \begin{matrix}4\\2\\2\end{matrix} \succ \begin{matrix}3\\2\\1\end{matrix} \succ \begin{matrix}3\\2\end{matrix} \succ 2 \succ 1$$

Partitions, interlacing, Schur functions

An (integer) partition λ is a nonincreasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

that vanishes eventually. Its size is $|\lambda| := \sum \lambda_i$.

Two partitions λ, μ are said interlaced, which we write $\lambda \succ \mu$, iff

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

Such constraint can be implemented via skew Schur functions of a single variable:

$$s_{\lambda/\mu}(q) = q^{|\lambda|-|\mu|} \mathbb{1}_{\lambda \succ \mu}.$$

Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

Consider a sequence $\underline{\lambda} = \cdots, \lambda^{(-2)}, \lambda^{(-1)}, \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots$ of integer partitions with finite support, and set

$$W(\underline{\lambda}) = \cdots s_{\lambda^{(-1)}/\lambda^{(-2)}}(q^{3/2}) s_{\lambda^{(0)}/\lambda^{(-1)}}(q^{1/2}) s_{\lambda^{(0)}/\lambda^{(1)}}(q^{1/2}) s_{\lambda^{(1)}/\lambda^{(2)}}(q^{3/2}) \cdots$$

Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a plane partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)}$.
Plane partitions whose shape fits in a $N \times N$ square correspond to sequences vanishing outside the interval $[-N, N]$.

Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

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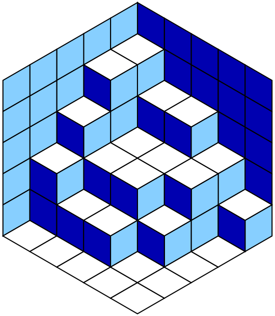

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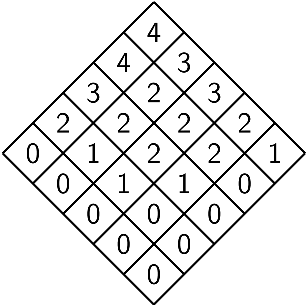

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Proposition

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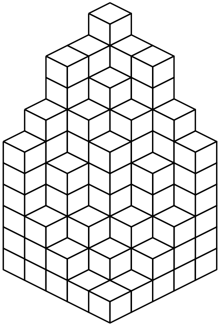
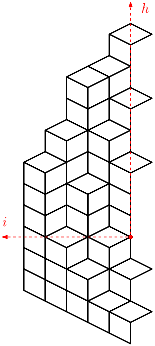
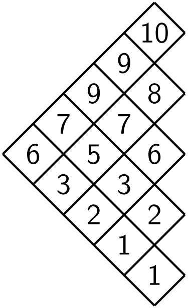
Plane partitions whose shape fits in a $N \times N$ square correspond to sequences vanishing outside the interval $[-N, N]$.

- By changing the order of some interlacings in $W(\underline{\lambda})$, one can treat "skew plane partitions" [Okounkov-Reshetikhin 2007].
- The form of $W(\underline{\lambda})$ is suitable for the transfer matrix method.

Our interest here: symmetric/free boundary tilings







$$6 \prec \begin{matrix} 7 \\ 3 \end{matrix} \prec \begin{matrix} 9 \\ 5 \\ 2 \end{matrix} \prec \begin{matrix} 9 \\ 7 \\ 3 \\ 1 \end{matrix} \prec \begin{matrix} 10 \\ 8 \\ 6 \\ 2 \\ 1 \end{matrix}$$

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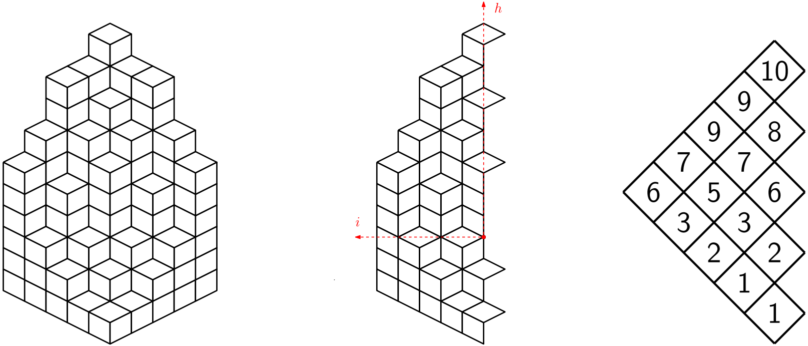

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One free boundary (or pfaffian) Schur process

Consider a sequence $\underline{\lambda} = \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots$ of integer partitions with finite support, and set

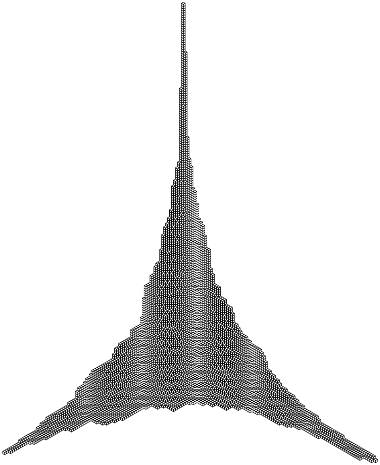
$$W(\underline{\lambda}) = s_{\lambda^{(0)}/\lambda^{(1)}}(q^{1/2}) s_{\lambda^{(1)}/\lambda^{(2)}}(q^{3/2}) \cdots$$

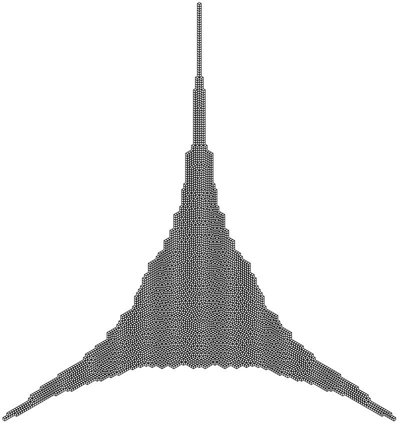
Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a vertically symmetric plane partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)/2}$.

- The "boxed" case is treated in [Panova 2015] by representation-theoretic methods.
- The pfaffian Schur process was introduced in [Borodin-Rains 2005], appears implicitly in [Sasamoto-Imamura 2003].

Large objects ($q \to 1$): limit shape





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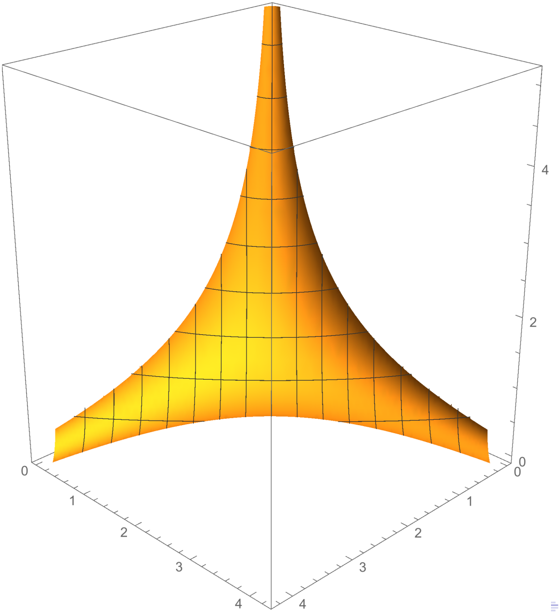
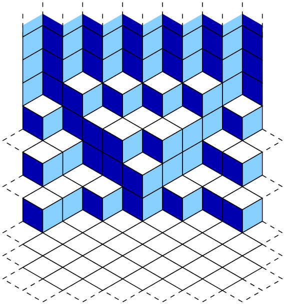


Cylindric partitions and periodic Schur process

[Gessel-Krattenthaler 1997, Borodin 2007]

Lozenge tiling



Cylindric partition

$$\begin{array}{ccccccccccccc}
 & 3 & & 4 & & 4 & & 4 & & 4 & & 3 & \\
\color{red}{2} & & 2 & & 3 & & 3 & & 3 & & 2 & & \color{red}{2} \\
 & 2 & & 1 & & 3 & & 3 & & 1 & & 2 & \\
\color{red}{1} & & 1 & & 1 & & 2 & & 1 & & 1 & & \color{red}{1} \\
 & 1 & & & & & & & & & & 1 &
\end{array}$$

Periodic sequence of interlaced integer partitions

$$\begin{matrix}\color{red}{2}\\ \color{red}{1}\end{matrix} \prec \begin{matrix}3\\2\\1\end{matrix} \succ \begin{matrix}2\\1\end{matrix} \prec \begin{matrix}4\\1\end{matrix} \succ \begin{matrix}3\\1\end{matrix} \prec \begin{matrix}4\\3\end{matrix} \succ \begin{matrix}3\\2\end{matrix} \prec \begin{matrix}4\\3\end{matrix} \succ \begin{matrix}3\\1\end{matrix} \prec \begin{matrix}4\\1\end{matrix} \succ \begin{matrix}2\\1\end{matrix} \prec \begin{matrix}3\\2\\1\end{matrix} \succ \begin{matrix}\color{red}{2}\\ \color{red}{1}\end{matrix}$$

Periodic Schur process [Borodin 2007]

Consider a sequence $\underline{\lambda} = \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(2N)} = \lambda^{(0)}$ of integer partitions, and set

$$W(\underline{\lambda}) = s_{\lambda^{(0)}/\lambda^{(1)}}(q^{\cdots}) s_{\lambda^{(2)}/\lambda^{(1)}}(q^{\cdots}) \cdots s_{\lambda^{(2N)}/\lambda^{(2N-1)}}(q^{\cdots}) \times q^{2N|\lambda^{(0)}|}$$

Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a cylindric partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)}$.

- The extra factor $q^{2N|\lambda^{(0)}|}$ is needed, as otherwise constant sequences would all have weight 1.

Different type of boundary conditions

We have encountered instances of Schur process with various types of "boundary conditions":

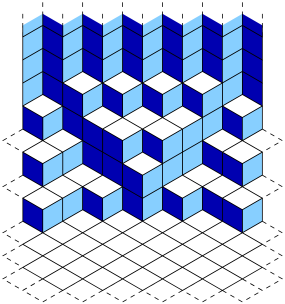
- empty/empty [Okounkov-Reshetikhin 2003],
- free/empty [Borodin-Rains 2005],
- periodic [Borodin 2007].

Missing case: free/free (equivalent to periodic with reflection symmetry).

Cylindric partitions and periodic Schur process

with reflection symmetry

Lozenge tiling



Cylindric partition

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\color{red}{2} & & 2 & & 3 & & 3 & & 3 & & 2 & & \color{red}{2} \\
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Cylindric partitions and periodic Schur process

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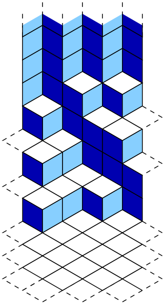

partition

$$\begin{array}{ccccccc}
 & 3 & & 4 & & 4 & \\
2 & & 2 & & 3 & & 3 \\
 & 2 & & 1 & & 3 & \\
1 & & 1 & & 1 & & 2 \\
 & 1 & & & & &
\end{array}$$

sequence of interlaced integer partitions

$$\begin{matrix}2\\1\end{matrix} \prec \begin{matrix}3\\2\\1\end{matrix} \succ \begin{matrix}2\\1\end{matrix} \prec \begin{matrix}4\\1\end{matrix} \succ \begin{matrix}3\\1\end{matrix} \prec \begin{matrix}4\\3\end{matrix} \succ \begin{matrix}3\\2\end{matrix}$$

Schur point processes

We consider the point process:

$$\mathfrak{S}(\underline{\lambda}) = \left\{ (i, \lambda_j^{(i)} - j + 1/2), i \in \mathbb{Z}, j \geq 0 \right\}$$

(related to the position of horizontal lozenges in the tiling picture).

What is the “nature” of this point process ?

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What is the "nature" of this point process ?

- empty/empty: determinantal [Okounkov-Reshetikhin 2003],
- free/empty: pfaffian [Borodin-Rains 2005],
- periodic: determinantal after a "shift-mixing" [Borodin 2007],
- free/free: pfaffian after mixing [BBNV17].

Determinantal and pfaffian point processes

A simple point process ξ in a discrete space X is said:

- determinantal if

$$\mathrm{Prob}(\{x_1, \ldots, x_n\} \subset \xi) = \det_{1 \leq i,j \leq n} k(x_i, x_j)$$

for some $k : X \times X \to \mathbb{C}$,

- pfaffian if

$$\mathrm{Prob}(\{x_1, \ldots, x_n\} \subset \xi) = \mathrm{pf}[K(x_i, x_j)]_{1 \leq i,j \leq n}$$

for some $K : X \times X \to M_2(\mathbb{C})$ with $K(x,y) = -K(y,x)^T$,

for any finite set $\{x_1, \ldots, x_n\} \subset X$.

Determinantal is a subcase of pfaffian, when taking

$$K(x,y) = \begin{pmatrix} 0 & k(x,y) \\ -k(y,x) & 0 \end{pmatrix}.$$

General definition

The free boundary Schur process is a random sequence of partitions

$$\mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \mu^{(N)}$$

such that

$$\mathrm{Prob}(\underline{\lambda}, \underline{\mu}) = \frac{1}{Z} u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right).$$

Here:

- u, v are nonnegative real parameters (recover empty boundary conditions by taking them zero),
- the $\rho_k^\pm$ are specializations (e.g. single variables for plane partitions),
- $Z = Z(u, v, \ldots)$ is the partition function (normalizing constant).

Outline

The partition function

There is a general product formula for the partition function. To state it in full generality we need some notations:

$$H(\rho; \rho') := \sum_{\lambda} s_\lambda(\rho) s_\lambda(\rho')$$

$$\tilde{H}(\rho) := \sum_{\lambda} s_\lambda(\rho)$$

The partition function

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$$H(\rho;\rho') := \sum_\lambda s_\lambda(\rho) s_\lambda(\rho') = \prod_{i,j} \frac{1}{1 - x_i y_j} \qquad \text{(Cauchy identity)}$$

$$\tilde{H}(\rho) := \sum_\lambda s_\lambda(\rho) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} \qquad \text{(Littlewood identity)}$$

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Theorem [B.-Chapuy-Corteel 2014, BBNV17]

$$Z = \prod_{1 \le k \le \ell \le N} H(\rho_k^+; \rho_\ell^-) \prod_{n \ge 1} \frac{\tilde{H}(u^{n-1} v^n \rho^+) \tilde{H}(u^n v^{n-1} \rho^-) H(u^{2n}\rho^+; v^{2n}\rho^-)}{1 - u^n v^n}$$

where $\rho^\pm = \rho_1^\pm \cup \rho_2^\pm \cup \cdots \cup \rho_N^\pm$.

Particular cases

- The original Schur process $u = v = 0$:

$$Z = \prod_{1 \le k \le \ell \le N} H(\rho_k^+; \rho_\ell^-)$$

Plane partitions:

$$Z = H(q^{1/2}, q^{3/2}, \dots; q^{1/2}, q^{3/2}, \dots) = \prod_{j \ge 1} \frac{1}{(1-q^j)^j}.$$

- The pfaffian Schur process $u = 0$, $v = 1$:

$$Z = \prod_{1 \le k \le \ell \le N} H(\rho_k^+; \rho_\ell^-) \times \tilde{H}(\rho^+)$$

Symmetric plane partitions:

$$Z = \tilde{H}(q, q^3, \dots) = \prod_{j \ge 1} \frac{1}{(1-q^{2j-1}) \times (1-q^{2j})^{2j-2}}.$$

Correlation functions

Recall that the point configuration is

$$\mathfrak{S}(\underline{\lambda}) := \left\{ \left(i, \lambda_j - j + \frac{1}{2} \right), 1 \leq i \leq N, j \geq 1 \right\} \subset \mathbb{Z} \times \left(\mathbb{Z} + \frac{1}{2} \right).$$

We will encounter pfaffian correlation kernels with the following "universal" form

$$K_{1,1}(i,k;i',k') = \left[z^k w^{k'} \right] F(i,z) F(i',w) \kappa_{1,1}(z,w)$$

$$K_{1,2}(i,k;i',k') = \left[z^k w^{-k'} \right] \frac{F(i,z)}{F(i',w)} \kappa_{1,2}(z,w)$$

$$K_{2,2}(i,k;i',k') = \left[z^{-k} w^{-k'} \right] \frac{1}{F(i,z) F(i',w)} \kappa_{2,2}(z,w)$$

where F and κ are Laurent series in z and w (obtained as expansions of meromorphic functions in certain compatible annuli). Only F depends on the specializations $\rho_k^{\pm}$, κ just depends on the boundary conditions.

Correlation functions: one free boundary

Theorem [Borodin-Raines 2005, Ghosal 2017, BBNV 2017]

For $u = 0$, the point process $\mathfrak{S}(\underline{\lambda})$ is pfaffian, and its correlation kernel takes the universal form with

$$F(i,z) = \frac{\prod_{1 \le \ell \le i} H(\rho_\ell^+; z)}{H(v^2\rho^+; z^{-1}) \prod_{i \le \ell \le N} H(\rho_\ell^-; z^{-1})}$$

$$\kappa_{1,1}(z,w) = \frac{v^2(z-w)\sqrt{zw}}{(z+v)(w+v)(zw-v^2)}$$

$$\kappa_{1,2}(z,w) = \frac{(zw-v^2)\sqrt{zw}}{(z+v)(w-v)(z-w)}$$

$$\kappa_{2,2}(z,w) = \frac{v^2(z-w)\sqrt{zw}}{(z-v)(w-v)(zw-v^2)}.$$

We shall expand the κ's in the annuli $|z|, |w| > v$, with $|z| > |w|$ for $i \le i'$ and vice versa otherwise.

Correlation functions: one free boundary

Remarks:

- in [Borodin-Raines 2005], the expressions appear slightly different because the first partition is assumed to have even columns,
- for $v = 0$, the diagonal entries $K_{1,1}$ and $K_{2,2}$ vanish, and we recover the result from [Okounkov-Reshetikhin 2003] that $\mathfrak{S}(\underline{\lambda})$ is determinantal with kernel

$$k(i,k;i',k') = \left[z^k w^{-k'}\right] \frac{F(i,z)}{F(i',w)} \frac{\sqrt{zw}}{(z-w)}$$

where

$$F(i,z) := \frac{\prod_{1 \le \ell \le i} H(\rho_\ell^+; z)}{\prod_{i \le \ell \le N} H(\rho_\ell^-; z^{-1})}.$$

Correlation function: two free boundaries

For $uv > 0$ (the general free boundary case), the process is not Pfaffian anymore, but a closely related process is: define the shifted point configuration

$$\mathfrak{S}_d(\underline{\lambda}) = \left\{ \left(i, \lambda_j - j + \frac{1}{2} + 2d \right), 1 \leq i \leq N, j \geq 1 \right\} \subset \mathbb{Z} \times \left(\mathbb{Z} + \frac{1}{2} \right).$$

and take d independant of $\underline{\lambda}$ with law

$$\mathrm{Prob}(d) = \frac{t^{2d}(uv)^{2d^2}}{\theta_3(t^2;(uv)^4)},$$

t being an arbitrary parameter (θ_3: Jacobi theta function).

Correlation function: two free boundaries

We find that $\mathfrak{S}_d(\underline{\lambda})$ is pfaffian, and its correlation kernel takes the universal form with

$$F(i,z) = \frac{\prod_{1\leq \ell \leq i} H(\rho_\ell^+; z)}{\prod_{i \leq \ell \leq N} H(\rho_\ell^-; z^{-1})} \prod_{n\geq 1} \frac{H(u^{2n}v^{2n-2}\rho^-; z) H(u^{2n}v^{2n}\rho^+; z)}{H(u^{2n-2}v^{2n}\rho^+; z^{-1}) H(u^{2n}v^{2n}\rho^-; z^{-1})}$$

$$\kappa_{1,1}(z,w) = \frac{v^2}{tz^{1/2}w^{3/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(uz, uw, -\frac{v}{z}, -\frac{v}{w}; uv)_\infty} \frac{\theta_{u^2v^2}(\frac{w}{z})}{\theta_{u^2v^2}(u^2zw)} \frac{\theta_3\left((\frac{tzw}{v^2})^2; u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

$$\kappa_{1,2}(z,w) = \frac{w^{1/2}}{z^{1/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(uz, -uw, -\frac{v}{z}, \frac{v}{w}; uv)_\infty} \frac{\theta_{u^2v^2}(u^2zw)}{\theta_{u^2v^2}(\frac{w}{z})} \frac{\theta_3\left((\frac{tz}{w})^2; u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

$$\kappa_{2,2}(z,w) = \frac{tv^2}{z^{1/2}w^{3/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(-uz, -uw, \frac{v}{z}, \frac{v}{w}; uv)_\infty} \frac{\theta_{u^2v^2}(\frac{w}{z})}{\theta_{u^2v^2}(u^2zw)} \frac{\theta_3\left((\frac{tv^2}{zw})^2; u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

where $(a_1,\ldots,a_m;q)_\infty := \prod_{k=0}^\infty (1-a_1q^k)\cdots(1-a_mq^k)$ and $\theta_q(z) := (z;q)_\infty(q/z;q)_\infty$. Laurent expansion is for $u^{-1} > |z|, |w| > v$.

Correlation function: two free boundaries

- For $u = 0$ we recover the previous case of one free boundary (exercise !).
- It is actually possible to evaluate the pfaffians and express the general n-point correlation for both $\mathfrak{S}(\underline{\lambda})$ and $\mathfrak{S}_d(\underline{\lambda})$ as a coefficient in a Laurent series in $2n$ variables. Behind this, there is an elliptic pfaffian identity which can be rewritten as a particular case of an identify due to Okada (2006). For $u = 0$, we recover Schur's pfaffian identity

$$\operatorname*{pf}_{1\leq i<j\leq 2n} \frac{x_i - x_j}{x_i + x_j} = \prod_{1\leq i<j\leq 2n} \frac{x_i - x_j}{x_i + x_j}$$

by a simple change of variables.

An elliptic pfaffian identity

$$\mathrm{pf}\begin{bmatrix} \kappa_{1,1}(z_i,z_j) & \kappa_{1,2}(z_i,w_j) \\ -\kappa_{1,2}(z_j,w_i) & \kappa_{2,2}(w_i,w_j) \end{bmatrix}_{1\le i,j\le n} = \sqrt{\frac{w_1\cdots w_n}{z_1\cdots z_n}}\times$$

$$\frac{((uv)^2;(uv)^2)_\infty^{2n}}{\prod_{i=1}^n (uz_i,-uw_i,-vz_i^{-1},vw_i^{-1};uv)_\infty}\frac{\theta_3\left(\left(t\frac{z_1\cdots z_n}{w_1\cdots w_n}\right)^2;(uv)^4\right)}{\theta_3(t^2;(uv)^4)}\times$$

$$\frac{\prod_{i,j=1}^n \theta_{(uv)^2}(u^2 z_i w_j)}{\prod_{1\le i\le j\le n}\theta_{(uv)^2}(w_j/z_i)\prod_{1\le i<j\le n}\theta_{(uv)^2}(z_j/w_i)}\times$$

$$\prod_{1\le i<j\le n}\frac{\theta_{(uv)^2}(z_j/z_i)\theta_{(uv)^2}(w_j/w_i)}{\theta_{(uv)^2}(u^2 z_i z_j)\theta_{(uv)^2}(u^2 w_i w_j)}.$$

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Outline

Tidbits from the proof

Our proof uses the formalism of free fermions (aka semi-infinite wedge space), as already used in [OR 2003]. Our main contribution is a treatment of free boundary states.

- These were introduced in [B.-Chapuy-Corteel 2014], for the computation of the partition function.
- It took us some time to understand how to compute correlation functions, as “Wick's lemma” does not generalize straightforwardly.
- The key idea was to introduce “extended” free boundary states, which are not eigenvalues of the charge operator, but correspond to “rotations” (Bogoliubov transformations) of the empty boundary state. This explains the need to consider a shifted point process.

Transfer matrix method

We consider an infinite-dimensional vector space with basis $|\lambda\rangle$ indexed by partitions, and the matrices $\Gamma_{\pm}(\rho)$ such that

$$\langle\lambda|\Gamma_+(\rho)|\mu\rangle = \langle\mu|\Gamma_-(\rho)|\lambda\rangle = s_{\lambda/\mu}(\rho).$$

For empty boundary conditions ($u = v = 0$), the partition function reads

$$Z = \langle\emptyset|\Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-)\cdots\Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-)|\emptyset\rangle.$$

For free boundaries, we introduce free boundary states

$$|\underline{v}\rangle := \sum_\lambda v^{|\lambda|}|\lambda\rangle, \qquad \langle\underline{u}| := \sum_\lambda u^{|\lambda|}\langle\lambda|$$

and then

$$Z = \langle\underline{u}|\Gamma_+(\rho_1^+)\Gamma_-(\rho_1^-)\cdots\Gamma_+(\rho_N^+)\Gamma_-(\rho_N^-)|\underline{v}\rangle.$$

Transfer matrix method: evaluating the partition function

In the empty boundary case, we need the Cauchy “commutation” identity

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho)$$

and the observation that $\Gamma_+(\rho)|\emptyset\rangle = |\emptyset\rangle$, $\langle\emptyset|\Gamma_-(\rho') = \langle\emptyset|$.

With free boundaries, we also need the Littlewood “reflection” identity

$$\Gamma_+(\rho)|\underline{v}\rangle = \tilde{H}(v\rho)\Gamma_-(v^2\rho)|\underline{v}\rangle, \qquad \langle\underline{u}|\Gamma_-(\rho) = \tilde{H}(u\rho)\Gamma_+(u^2\rho)\langle\underline{u}|.$$

See details

Transfer matrix method: correlation functions

To compute correlation, we need to insert "observables". Let N_k be the diagonal matrix such that

$$N_k|\lambda\rangle = \begin{cases} |\lambda\rangle & \text{if } k = \lambda_i - i + 1/2 \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

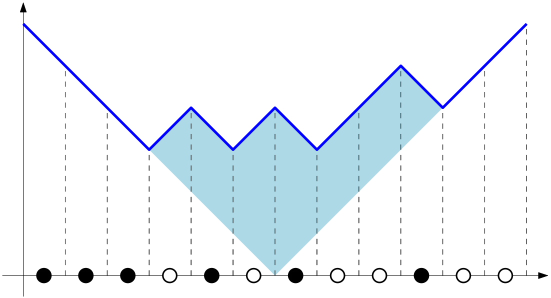
Then, for instance,

$$Z' = \langle \emptyset | \Gamma_+(\rho_1^+) N_k \Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+) \Gamma_-(\rho_N^-) | \emptyset \rangle.$$

counts sequences $\underline{\lambda}$ such that $(1,k) \in \mathfrak{S}(\underline{\lambda})$. Insert several N's at appropriate places and normalize for general correlation functions.

To evaluate such expressions, we represent the observables N_k in terms of "fermionic" operators.

Combinatorial boson-fermion correspondence



There is a bijection between:

- charged partitions (λ, c) with λ a partition and c an integer "charge"
- admissible sets, i.e. subsets S of $\mathbb{Z}' := \mathbb{Z} + 1/2$ such that S has a largest element and $\mathbb{Z}' \setminus S$ a smallest element.

This mapping reads explicitly $(\lambda, c) \mapsto \{\lambda_i - i + c + 1/2, i \geq 1\}$ hence is closely related to point configurations.

Semi-infinite wedge space

We consider an infinite-dimensional vector space with basis $|S\rangle$ indexed by admissible sets. Denoting $s_1 > s_2 > \cdots$ the elements of S, we may identify

$$|S\rangle := \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

as a semi-infinite wedge (exterior) product. Partitions form the subspace of charge 0.

The fermionic operator ψ_k is

$$\psi_k|S\rangle := \underline{k} \wedge \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

and ψ_k^* is its adjoint. We have $N_k = \psi_k \psi_k^*$.

- The ψ's and ψ^*'s obey the canonical anticommutation relations.
- The $\psi_k \psi_\ell^*$'s span the Lie algebra A_∞.
- The $\psi_k \psi_\ell^*$'s, $\psi_k \psi_\ell$'s and $\psi_k^* \psi_\ell^*$'s span the Lie algebra D'_∞.

Correlation functions

Adapting ideas from [OR 2003], we may rewrite the general correlation function in the form

$$\mathrm{Prob}\left(\{(i_1, k_1), \ldots, (i_n, k_n)\} \subset \mathfrak{S}(\underline{\lambda})\right) = \frac{\langle \underline{u} | \Psi_{k_1}(i_1) \Psi^*_{k_1}(i_1) \cdots \Psi_{k_n}(i_n) \Psi^*_{k_n}(i_n) | \underline{v} \rangle}{\langle \underline{u} | \underline{v} \rangle}$$

where $\Psi_k(i)$ is some linear combination of ψ's.

Wick's lemma

For empty boundary conditions, we may conclude that the process $\mathfrak{S}(\underline{\lambda})$ is determinantal from:

Wick's lemma

Let Ψ be the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$, we have

$$\langle \emptyset | \phi_1 \cdots \phi_{2n} | \emptyset \rangle = \mathrm{pf}\, A \tag{1}$$

where A is the antisymmetric matrix defined by $A_{ij} := \langle \emptyset | \phi_i \phi_j | \emptyset \rangle$ for $i < j$.

In our situation, half of the entries of A vanish and the pfaffian reduces to a determinant.

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But naive generalizations of Wick's lemma to free boundaries are false!

Extended free boundary states

To fix this problem, we need to replace free boundary states by some "nicer" states.

Proposition

There exists an operator $X(v,t) \in D'_\infty$ such that

$$|\underline{v}\rangle = \Pi_0 e^{X(v,t)} |\emptyset\rangle$$

with Π_0 the projector on the space of zero charge.

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The state $|\underline{v,t}\rangle := e^{X(v,t)}|\emptyset\rangle$ is called the extended free boundary state, and $\langle \underline{u,t}|$ is defined similarly. We have

$$|\underline{v,t}\rangle = \sum_{\lambda \text{ partition}} \sum_{c \in 2\mathbb{Z}} v^{|\lambda| + c^2/2} t^{c/2} |S(\lambda, c)\rangle.$$

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It satisfies the same Littlewood reflection identity as $|\underline{v}\rangle$, but also a "fermionic reflection identity".

Extended free boundary states

Wick's lemma for two free boundaries

Let Ψ be again the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$ and $uv < 1$, we have

$$\frac{\langle \underline{u,t} | \phi_1 \cdots \phi_{2n} | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle} = \operatorname{pf} A \tag{2}$$

where A is the antisymmetric matrix defined by
$A_{ij} = \langle \underline{u,t} | \phi_i \phi_j | \underline{v,t} \rangle / \langle \underline{u,t} | \underline{v,t} \rangle$ for $i < j$.

Extended free boundary states

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where A is the antisymmetric matrix defined by $A_{ij} = \langle \underline{u,t} | \phi_i \phi_j | \underline{v,t} \rangle / \langle \underline{u,t} | \underline{v,t} \rangle$ for $i < j$.

This lemma implies that the shift point process $\mathfrak{S}_d(\underline{\lambda})$ is pfaffian as its n-point correlation function reads

$$\mathrm{Prob}\left(\{(i_1, k_1), \ldots, (i_n, k_n)\} \subset \mathfrak{S}_d(\underline{\lambda})\right) = \frac{\langle \underline{u,t} | \Psi_{k_1}(i_1) \Psi^*_{k_1}(i_1) \cdots \Psi_{k_n}(i_n) \Psi^*_{k_n}(i_n) | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle}.$$

Pfaffian structure of $\mathfrak{S}_d(\underline{\lambda})$

The correlation kernel is given by

$$K_{1,1}(i,k;i',k') = \frac{\langle \underline{u,t} | \Psi_k(i)\Psi_{k'}(i') | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle},$$

$$K_{2,2}(i,k;i',k') = \frac{\langle \underline{u,t} | \Psi^*_k(i)\Psi^*_{k'}(i') | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle},$$

$$K_{1,2}(i,k;i',k') = -K_{2,1}(i',k';i,k) = \begin{cases} \frac{\langle \underline{u,t} | \Psi_k(i)\Psi^*_{k'}(i') | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle}, & \text{if } i \leq i', \\ -\frac{\langle \underline{u,t} | \Psi^*_{k'}(i')\Psi_k(i) | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle}, & \text{otherwise.} \end{cases}$$

Pfaffian structure of $\mathfrak{S}_d(\underline{\lambda})$

Next step is standard, and consists in introducing fermion generating functions

$$\psi(z) := \sum_k \psi_k z^k, \qquad \psi^*(w) := \sum_k \psi_k^* w^{-k}.$$

By well-known commutation identities we find

$$\Psi_k(i) = [z^k] F(i,z)\psi(z), \qquad \Psi_k^*(i) = [w^{-k}] \frac{\psi^*(w)}{F(i,w)}. \tag{3}$$

with $F(i,z)$ given before. This explains the "universal form" for the correlation kernel, upon setting

$$\kappa_{1,1}(z,w) = \frac{\langle \underline{u},\underline{t} | \psi(z)\psi(w) | \underline{v},\underline{t} \rangle}{\langle \underline{u},\underline{t} | \underline{v},\underline{t} \rangle}, \quad \kappa_{2,2}(z,w) = \frac{\langle \underline{u},\underline{t} | \psi^*(z)\psi^*(w) | \underline{v},\underline{t} \rangle}{\langle \underline{u},\underline{t} | \underline{v},\underline{t} \rangle},$$

$$\kappa_{1,2}(z,w) = \begin{cases} \frac{\langle \underline{u},\underline{t} | \psi(z)\psi^*(w) | \underline{v},\underline{t} \rangle}{\langle \underline{u},\underline{t} | \underline{v},\underline{t} \rangle}, & \text{if } |z| > |w|, \\ -\frac{\langle \underline{u},\underline{t} | \psi^*(w)\psi(z) | \underline{v},\underline{t} \rangle}{\langle \underline{u},\underline{t} | \underline{v},\underline{t} \rangle}, & \text{if } |z| < |w|. \end{cases}$$

End of the proof

The final trick consists in using the identities

$$\psi(z) = z^{C-\frac{1}{2}} R\, \Gamma_-(z)\Gamma'_+\left(-z^{-1}\right),$$
$$\psi^*(w) = R^{-1} w^{-C+\frac{1}{2}} \Gamma'_-(-w)\Gamma_+\left(w^{-1}\right)$$

where C and R are respectively the charge and shift operators. This allows to compute the κ's by the same method as for the partition function.

Alternatively, we may use the representation

$$|\underline{v,t}\rangle = \sum_{\lambda \text{ partition}} \sum_{c \in 2\mathbb{Z}} v^{|\lambda| + c^2/2} t^{c/2} |S(\lambda, c)\rangle.$$

to obtain sum formulas.

An hypergeometric byproduct

By computing $\langle \psi\psi \rangle$ in two ways, get

$$s^2 u \sqrt{zw} \sum_{-\infty < L' < L < \infty} (z^L w^{L'} - z^{L'} w^L) u^{L+L'} \times \frac{(s^2\sqrt{uv}; uv)_L (-s^2\sqrt{uv}; uv)_{L'}}{(s^2 uv\sqrt{uv}; uv)_{L'} (-s^2 uv\sqrt{uv}; uv)_L} = \frac{((uv)^2; (uv)^2)_\infty^2 \theta_{(uv)^2}(\frac{w}{z})}{(uz, uw, -\frac{v}{z}, -\frac{v}{w}; uv)_\infty} \cdot \frac{v^2}{s^2 w\sqrt{zw}} \cdot \frac{\theta_3(\frac{s^2 zw}{v^2}; uv)}{\theta_3(s^2; uv)}$$

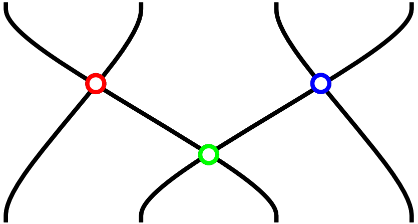
Another identity associated with $\langle \psi\psi^* \rangle$.

Conclusion

- We define the free boundary Schur process which generalizes the "original" process of Okounkov-Reshetikhin, and its "pfaffian" variant,
- partition function and correlations are explicitly computed,
- the proof uses the Fock space/semi-infinite wedge formalism, with some new tricks
- applications to symmetric plane partitions and also plane overpartitions/domino tilings and last passage percolation
- more work needed to analyze asymptotics in the case with two free boundaries ($uv > 0$). New universality classes?

Evaluation the partition function: empty bc

$$\langle\emptyset|\, \Gamma_+(\rho_1^+)\, \Gamma_-(\rho_1^-)\, \Gamma_+(\rho_2^+)\, \Gamma_-(\rho_2^-)\, |\emptyset\rangle \;=$$



$$\langle\emptyset|\, \Gamma_-(\rho_1^-)\, \Gamma_-(\rho_2^-)\, \Gamma_+(\rho_1^+)\, \Gamma_+(\rho_2^+)\, |\emptyset\rangle \;\times$$

$$H(\rho_1^+;\rho_1^-)H(\rho_1^+;\rho_2^-)H(\rho_2^+;\rho_2^-)$$

Evaluation the partition function: free bc