Applications of integer relation algorithms

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This talk shows ways in which **integer relation** algorithms have empowered quantum field theorists to turn numerical results into conjecturally exact evaluations of **Feynman periods**. Ideas on **quasi-periods** are fermenting.

1. **1985**: Periods in the **Dark Ages**
2. **1995**: Renaissance by **PSLQ**
3. **1999**: Improvements and **parallelization**
4. **2009**: Work on the Broadhurst–**Kreimer** conjecture
5. **2015**: Periods from **Panzer** and **Schnetz**
6. **2017**: Periods from **Laporta** in electrodynamics
7. **Heute**: Quasi-periods from **Brown** and **Zhou**

1 1985: Periods in the Dark Ages

Problem: Given numerical **approximations** to $n > 2$ real numbers, x_k, is there is at least one **probable** relation

$$\sum_{k=1}^{n} z_k x_k = 0$$

with integers z_k, at least two of which are non-zero? If so, produce one.

Examples: I studied periods from 6-loop Feynman diagrams in 1985:

$$P_{6,1} = 168\zeta_9, \quad P_{6,2} = \frac{1063}{9}\zeta_9 + 8\zeta_3^3, \quad 16P_{6,3} + P_{6,4} = 1440\zeta_3\zeta_5$$

with Riemann zeta values $\zeta_a := \sum_{n>0} n^{-a}$. I had a strong intuition that $P_{6,3}$ and $P_{6,4}$ would involve ζ_8 and the **multiple zeta value** (MZV)

$$\zeta_{5,3} := \sum_{m>n>0} \frac{1}{m^5 n^3} = 0.037707672984847544011304782\ldots$$

but did not have enough digits for the **periods** to test this.

2 1995: Renaissance by PSLQ

In response to a request from **Dirk Kreimer**, I obtained
$P_{6,3} = 256N_{3,5} + 72\zeta_3\zeta_5$ and $P_{6,4} = -4096N_{3,5} + 288\zeta_3\zeta_5$, with

$$N_{3,5} := \frac{27}{80}\zeta_{5,3} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8$$

found by PSLQ, after more digits were obtained for the periods.

We found $\zeta_{3,5,3}$, with weight 11 and depth 3, in some 7-loop periods.

Much experimenting with PSLQ led to the Broadhurst-Kreimer (BK) conjecture that the number $N(w,d)$ of independent **primitive** MZVs of **weight** w and **depth** d is generated by

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{N(w,d)} = 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

with a final term inferred by relating MZVs to **alternating** sums.

2.1 PSLQ: Partial Sums, Lower triangular, orthogonal Quotient

PSLQ came from work by **Helaman Ferguson** and **Rodney Forcade** in 1977, was implemented in multiple-precision ForTran by **David Bailey** in 1992, improved and parallelized in 1999. See David H. Bailey and David J. Broadhurst, *Parallel Integer Relation Detection: Techniques and Applications*, Math. Comp. 70 (2001), 1719–1736. **Initialization:**

1. For $j := 1$ to n: for $i := 1$ to n: if $i = j$ then set $A_{ij} := 1$ and $B_{ij} := 1$ else set $A_{ij} := 0$ and $B_{ij} := 0$; endfor; endfor.

2. For $k := 1$ to n: set $s_k :=$ **sqrt** $\left(\sum_{j=k}^{n} x_j^2\right)$; endfor. Set $t = 1/s_1$. For $k := 1$ to n: set $y_k := tx_k$; $s_k := ts_k$; endfor.

3. For $j := 1$ to $n - 1$: for $i := 1$ to $j - 1$: set $H_{ij} := 0$; endfor; set $H_{jj} := s_{j+1}/s_j$; for $i := j + 1$ to n: set $H_{ij} := -y_i y_j/(s_j s_{j+1})$; endfor; endfor.

4. For $i := 2$ to n: for $j := i - 1$ to 1 step -1: set $t :=$ **round**(H_{ij}/H_{jj}); and $y_j := y_j + ty_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - tH_{jk}$; endfor; for $k := 1$ to n: set $A_{ik} := A_{ik} - tA_{jk}$ and $B_{kj} := B_{kj} + tB_{ki}$; endfor; endfor; endfor.

Iteration:

1. Select m such that $(\mathbf{4/3})^{i/2}|H_{ii}|$ is maximal when $i = m$. **Swap** the entries of y indexed m and $m+1$, the corresponding rows of A and H, and the corresponding columns of B.

2. If $m \le n-2$ then set $t_0 := \mathbf{sqrt}(H_{mm}^2 + H_{m,m+1}^2)$, $t_1 := H_{mm}/t_0$ and $t_2 := H_{m,m+1}/t_0$; for $i := m$ to n: set $t_3 := H_{im}$, $t_4 := H_{i,m+1}$, $H_{im} := t_1 t_3 + t_2 t_4$ and $H_{i,m+1} := -t_2 t_3 + t_1 t_4$; endfor; endif.

3. For $i := m+1$ to n: for $j := \min(i-1, m+1)$ to 1 step -1: set $t := \mathbf{round}(H_{ij}/H_{jj})$ and $y_j := y_j + t y_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - t H_{jk}$; endfor; for $k := 1$ to n: set $A_{ik} := A_{ik} - t A_{jk}$ and $B_{kj} := B_{kj} + t B_{ki}$; endfor; endfor; endfor.

4. If the largest entry of A exceeds the precision, then **fail**, else if a component of the y vector is very small, then output the **relation** from the corresponding column of B, else go back to Step 1.

For big problems, the **parallelization** of PSLQ has been vital, especially for the magnetic moment of the electron. For smaller problems, there is now a handy alternative.

2.2 LLL

In 1982, Arjen Lenstra, Hendrik Lenstra and László Lovász gave the LLL algorithm for lattice reduction to a basis with short and almost orthogonal components. An extension of this underlies `lindep` in Pari-GP.

```
$ Z53=0.037707672984847544011304782;
$ P63=107.71102484102;
$ V=[P63,Z53,zeta(3)*zeta(5),zeta(8)];
$ for(d=10,16,U=lindep(V,d);U*=sign(U[1]);print([d,U~]));
[10, [12, 44, -936, -127]]
[11, [4, -827, -460, 173]]
[12, [4, -827, -460, 173]]
[13, [4, -827, -460, 173]]
[14, [5, -432, -1260, 1044]]
[15, [5, -432, -1260, 1044]]
[16, [196, 1652, -9701, -9045]]
```

3 1999: Improvements and parallelization

Multi-level improvement: perform most operations at 64-bit precision, some at intermediate precision (we chose 125 digits) and only the bare **minimum** of the most delicate operations at **full** precision (more than 10000 digits, for some big problems).

Multi-pair improvement: swap up to $0.4n$ disjoint **pairs** of the n indices at each iteration. In this case, it is not proven that the algorithm will succeed, but it ain't yet been found to fail.

Parallelization: distribute the disjoint-pair jobs; for each pair, distribute the full-precision matrix multiplication in the outermost loop.

3.1 Fourth bifurcation of the logistic map

Working at **10000** digits, we found that the constant associated with the fourth bifurcation is the root of a polynomial of degree **240**.

3.2 Alternating sums

We tested my conjecture on alternating sums defined by

$$\zeta\left(\begin{array}{cccc} s_1, & s_2 & \cdots & s_r \\ \sigma_1, & \sigma_2 & \cdots & \sigma_r \end{array}\right) := \sum_{k_1>k_2>\cdots>k_r>0} \frac{\sigma_1^{k_1}}{k_1^{s_1}} \frac{\sigma_2^{k_2}}{k_2^{s_2}} \cdots \frac{\sigma_r^{k_r}}{k_r^{s_r}}$$

where $\sigma_j = \pm 1$ are signs and $s_j > 0$ are integers, namely that at weight $w = \sum_j s_j$ every alternating sum is a rational linear combination of elements of a basis of size $F_{w+1} = F_w + F_{w-1}$, i.e. the **Fibonacci** number with index $w + 1$. At $w = 11$, integer relations of size $n = F_{12} + 1 =$ **145** were readily found, working at **5000**-digit precision.

3.3 Inverse binomial sums

Noting that $S(4) = \frac{17}{36}\zeta_4$, I conjectured that

$$S(w) := \sum_{n=1}^{\infty} \frac{1}{n^w \binom{2n}{n}}$$

is reducible to weigth w nested sums that involve **sixth roots of unity**, i.e. with $\sigma_j^6 = 1$, above. This was confirmed for all weights $w \leq 20$, with 525990827847624469523748125835264000$S(20)$ given by **106** terms.

4 2009: Work on the BK conjecture

The BK conjecture was a rash leap based on a PSLQ dicovery:

$$
\begin{aligned}
&2^5 \cdot 3^3 \zeta_{4,4,2,2} - 2^{14} \sum_{m>n>0} \frac{(-\mathbf{1})^{m+n}}{(m^3 n)^3} = \\
&\quad 2^5 \cdot 3^2\, \zeta_3^4 + 2^6 \cdot 3^3 \cdot 5 \cdot 13\, \zeta_9\, \zeta_3 + 2^6 \cdot 3^3 \cdot 7 \cdot 13\, \zeta_7\, \zeta_5 \\
&+ 2^7 \cdot 3^5\, \zeta_7\, \zeta_3\, \zeta_2 + 2^6 \cdot 3^5\, \zeta_5^2\, \zeta_2 - 2^6 \cdot 3^3 \cdot 5 \cdot 7\, \zeta_5\, \zeta_4\, \zeta_3 \\
&- 2^8 \cdot 3^2\, \zeta_6\, \zeta_3^2 - \frac{13177 \cdot 15991}{691}\, \zeta_{12} \\
&+ 2^4 \cdot 3^3 \cdot 5 \cdot 7\, \zeta_{6,2}\, \zeta_4 - 2^7 \cdot 3^3\, \zeta_{8,2}\, \zeta_2 - 2^6 \cdot 3^2 \cdot 11^2\, \zeta_{10,2}
\end{aligned}
$$

is reducible to MZVs of depth $d \leq 2$ and their products. It means that $\zeta_{4,4,2,2}$ is **pushed down** to depth $d = 2$, if we allow **alternating** sums in the MZV basis. When constructing the MZV datamine, **Johannes Blümlein** and **Jos Vermaseren** and I were able to **prove** this by massive use of computer algebra. There seems little hope of proving my discovery of pushdown at weight 21 and depth 7, where

$$
81 \zeta_{6,2,3,3,5,1,1} + 326 \sum_{j>k>l>m>n>0} \frac{(-\mathbf{1})^{k+m}}{(jk^2 lm^2 n)^3}
$$

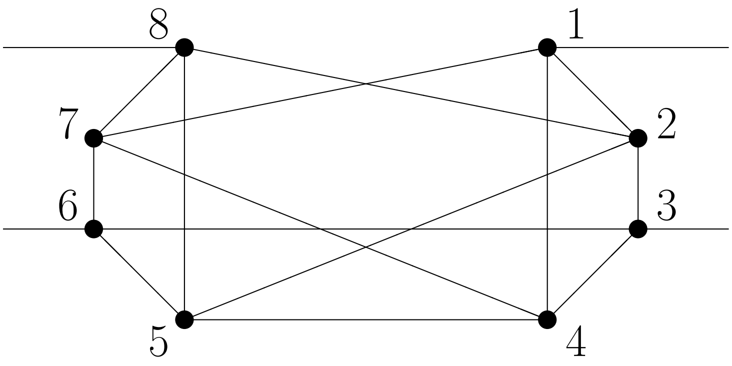
is empirically reducible to **150** terms containing MZVs of depths $d \leq 5$.

5 2015: Periods from Panzer and Schnetz

I found empirical reductions to MZVs for a pair of 7-loop periods

$$\begin{aligned}
P_{7,8} &= \frac{22383}{20}\zeta_{11} + \frac{4572}{5}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - 700\zeta_3^2\zeta_5 \\
&\quad + 1792\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right) \\
P_{7,9} &= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 \\
&\quad + 896\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right)
\end{aligned}$$

that had been expected to involve alternating sums. These results were later proven, one by **Erik Panzer** and the other by **Oliver Schnetz**. They obtained complicated combinations of **alternating** sums which then gave my MZV formulas by use of proven results in the datamine.



The period from this 7-loop diagram is called $P_{7,11}$ in the census of Schnetz. All other periods up to 7 loops reduce to MZVs; only $P_{7,11}$ requires nested sums with **sixth roots of unity**. Panzer evaluated $\sqrt{3}P_{7,11}$ in terms of 4589 such sums, each of which he evaluated to 5000 digits. Then he found an empirical reduction to a 72-dimensional basis. The rational coefficient of π^{11} in his result was

$$C_{11} = -\frac{964259961464176555529722140887}{2733669078108291387021448260000}$$

whose **denominator** contains 8 primes greater than 11, namely 19, 31, 37, 43, 71, 73, **50909** and **121577**.

I built an empirical datamine to enable substantial simplification.

Let $A = \mathrm{d}\log(x)$, $B = -\mathrm{d}\log(1-x)$ and $D = -\mathrm{d}\log(1-\exp(2\pi\mathrm{i}/6)x)$ be **letters**, forming **words** W that define **iterated integrals** $Z(W)$. Let

$$W_{m,n} \equiv \sum_{k=0}^{n-1} \frac{\zeta_3^k}{k!} A^{m-2k} D^{n-k}$$

$P_n \equiv (\pi/3)^n/n!$, $I_n \equiv \mathrm{Cl}_n(2\pi/3)$ and $I_{a,b} \equiv \Im Z(A^{b-a-1}DA^{2a-1}B)$. Then

$$\begin{aligned}
\sqrt{3}P_{7,11} &= -10080\Im Z(W_{7,4} + W_{7,2}P_2) + 50400\zeta_3\zeta_5 P_3 \\
&+ \left(35280\Re Z(W_{8,2}) + \frac{46130}{9}\zeta_3\zeta_7 + 17640\zeta_5^2\right) P_1 \\
&- 13277952 T_{2,9} - 7799049 T_{3,8} + \frac{6765337}{2} I_{4,7} - \frac{583765}{6} I_{5,6} \\
&- \frac{121905}{4}\zeta_3 I_8 - 93555\zeta_5 I_6 - 102060\zeta_7 I_4 - 141120\zeta_9 I_2 \\
&+ \frac{42452687872649}{6} P_{11}
\end{aligned}$$

with the datamine transformations

$$\begin{aligned}
I_{2,9} &= 91(11T_{2,9}) - 898T_{3,8} + 11I_{4,7} - 292P_{11} \\
I_{3,8} &= 24(11T_{2,9}) + 841T_{3,8} - 190I_{4,7} - 255P_{11}
\end{aligned}$$

removing denominator primes greater than 3.

6 2017: Periods from Laporta in electrodynamics

The **magnetic moment** of the electron, in Bohr magnetons, has electrodynamic contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L=4$ loops by

$$
\begin{aligned}
a_0 &= 1 \quad [\textbf{Dirac}, 1928] \\
a_1 &= 0.5 \quad [\textbf{Schwinger}, 1947] \\
a_2 &= -0.328478965579193784582172816964892392411111929867962\ldots \\
a_3 &= 1.181241456587200006274753982212877853368789390932 13\ldots \\
a_4 &= -1.912245764926445574152647167439830054060873390658 72\ldots
\end{aligned}
$$

In 1957, corrections by **Petermann** and **Sommerfield** resulted in

$$a_2 = \frac{197}{144} + \frac{\zeta_2}{2} + \frac{3\zeta_3 - 2\pi^2 \log 2}{4}\,.$$

In 1996, **Laporta** and **Remiddi** [hep-ph/9602417] gave us

$$
\begin{aligned}
a_3 = {} & \frac{28259}{5184} + \frac{17101\zeta_2}{135} + \frac{139\zeta_3 - 596\pi^2 \log 2}{18} \\
& - \frac{39\zeta_4 + 400U_{3,1}}{24} - \frac{215\zeta_5 - 166\zeta_3\zeta_2}{24}\,.
\end{aligned}
$$

The 3-loop contribution contains a weight-4 depth-2 **polylogarithm**

$$U_{3,1} := \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{\zeta_4}{2} + \frac{(\pi^2 - \log^2 2)\log^2 2}{12} - 2\sum_{n>0} \frac{1}{2^n n^4}$$

encountered in my study of **alternating** sums [arXiv:hep-th/9611004].

Equally fascinating is the **Bessel** moment B, at weight 4, in the breath-taking evaluation by **Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals. U comes from 6 difficult integrals, still under investigation.

The weight-4 **non-polylogarithm** at 4 loops has $N = 6$ Bessel functions:

$$B = -\int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t) K_0^5(t) \mathrm{d}t.$$

6.1 Bessel moments and modular forms

Gauss noted on 30 May 1799 that the **lemniscate** constant

$$A := \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \frac{(\Gamma(1/4))^2}{4\sqrt{2\pi}} = \frac{\pi/2}{\mathbf{agm}(1,\sqrt{2})}$$

is given by the reciprocal of an **arithmetic-geometric mean**. This is an example of the Chowla-Selberg formula (1949) at the **first** singular value, seen in the talk by Dan Romick. In Bruno Salvy's talk, we encountered the **sixth** singular value, where an integral evaluated by **Watson** in 1939 in terms of $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{6}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{24})$ with $k=1,5,7,11$, as observed by Glasser and Zucker in 1977. In 2007, I identified a **Feynman** period at the **fifteenth** singular value, where $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{15})$ with $k=1,2,4,8$.

With $N=a+b$ **Bessel** functions and $c\geq 0$, I define **moments**

$$M(a,b,c) \equiv \int_0^\infty I_0^a(t)K_0^b(t)t^c\mathrm{d}t$$

that converge for $b>a>0$. Then the 5-Bessel matrix is

$$\begin{bmatrix} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{bmatrix} = \begin{bmatrix} \pi^2 C & \pi^2\left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2}C & \frac{\sqrt{15}\pi}{2}\left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right) \end{bmatrix}.$$

The **determinant** $2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C \equiv \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{\mathbf{15}}\right)$$

The **L-series** for $N=5$ Bessel functions comes from a **modular form** of weight **3** and level **15** [arXiv:1604.03057]:

$$\begin{aligned}
\eta_n &\equiv q^{n/24}\prod_{k>0}(1-q^{nk})\\
f_{3,15} &\equiv (\eta_3\eta_5)^3+(\eta_1\eta_{15})^3=\sum_{n>0}A_5(n)q^n\\
L_5(s) &\equiv \sum_{n>0}\frac{A_5(n)}{n^s} \quad \text{for } s>2\\
L_5(1) &= \sum_{n>0}\frac{A_5(n)}{n}\left(2+\frac{\sqrt{\mathbf{15}}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{\mathbf{15}}}\right)\\
&= 5C = \frac{5}{\pi^2}\int_0^\infty I_0(t)K_0^4(t)t\mathrm{d}t\,.
\end{aligned}$$

Laporta's work engages the first row of the **6-Bessel determinant**

$$\det\begin{bmatrix} M(1,5,1) & M(1,5,3)\\ M(2,4,1) & M(2,4,3)\end{bmatrix}=\frac{5\zeta_4}{32}$$

associated to $f_{4,6}=(\eta_1\eta_2\eta_3\eta_6)^2$ with weight **4** and level **6**.

7 Heute: Quasi-periods from Brown and Zhou

7.1 Proofs of conjectures on determinants

A few days ago, **Yajun Zhou** posted impressive proofs [arXiv:1711.01829] of conjectures that **Anton Mellit** and I had made about determinants of matrices of Feynman integrals. Let $\mathbf{M}_k$ be the $k\times k$ matrix with elements $M(a,2k+1-a,2b-1)$, for a and b running from 1 to k. Then I discovered that with $N=2k+1=$ **31 Bessel** functions

$$\mathrm{det}\mathbf{M}_{15}=\frac{2^{182}\pi^{120}}{3^{33}\,5^{20}\,7^{5}\sqrt{11^{3}\,13^{9}\,17^{17}\,19^{19}\,23^{23}\,29^{29}\,31^{31}}}$$

after seeking an **integer relation** between **logs** of the determinant, small primes and π. Then I inferred a general formula which Zhou has **proven**. My result for **even** numbers of Bessel functions is also proven and hence the **6 Bessel determinant** is secure, in quantum electrodynamics.

7.2 Brown's quasi-periods

Recently, **Francis Brown** posted impressive ideas [arXiv:1710.07912] on quasi-periods associated to modular forms. A definition of these has been strangely elusive at weights greater than 2. For the weight 12 level 1 modular form $\Delta(z) := \eta_1^{24}$ with $q := \exp(2\pi\mathrm{i}z)$, **periods** are defined via $L(\Delta, s)$ which has 11 critical values at integers $s \in [1, 11]$. At odd integers these are given, up to rational multiples of powers of π, by ω_+, while at even integers we use ω_-. Specifically, the **periods** are

$$\begin{aligned}\omega_+ &:= -70(2\pi)^{11}\int_0^\infty y^4\Delta(\mathrm{i}y)\mathrm{d}y\\ &= -68916772.809595194754310124655331030439069969 1\ldots\\ \omega_- &:= -6(2\pi)^{11}\int_0^\infty y^5\Delta(\mathrm{i}y)\mathrm{d}y\\ &= -5585015.3793104018668771392637962751296350334 3\ldots\end{aligned}$$

To define **quasi-periods**, Brown considers the **weakly** holomorphic modular form $\Delta'(z)$, defined in terms of Klein's j-invariant by

$$\begin{aligned}\Delta'(z) &:= (j^2 - 1464j + 142236)\Delta(z) = 1/q + O(q^2),\\ j &:= \frac{1}{\Delta(z)}\left(1 + 240\sum_{n>0}\frac{n^3q^n}{1-q^n}\right)^3.\end{aligned}$$

The quasi-periods are

$$\begin{aligned}\eta_+ &= 127202100647.17709477731716129861087749404598 8\ldots\\ \eta_- &= 10276732343.6491327508171930724009209088993990\ldots\end{aligned}$$

with numerical values obtainable from a determinant and permanent,

$$\begin{aligned}\omega_+\eta_- - \omega_-\eta_+ &= (2\pi)^{11}10!\\ \frac{\omega_+\eta_- + \omega_-\eta_+}{4\pi\omega_+\omega_-} &= -\sum_{c>0}\frac{I_{11}(4\pi/c)}{c}\sum_{r\in(\mathbf{Z}/\mathbf{Z}c)^*}\exp\left(\frac{2\pi\mathrm{i}(r-s)}{c}\right)\bigg|_{rs=1\bmod c}\end{aligned}$$

Brown is able to obtain these directly by Eichler-type integrals in the upper half plane, taking care to avoid the singularity at infinity in Δ'.

7.3 Quasi-periods from `lindep` and Zhou?

I conjectured and Zhou proved the determinant condition

$$\det\int_0^\infty\begin{bmatrix}K_0(t) & K_0(t)t^2\\ I_0(t) & I_0(t)t^2\end{bmatrix}I_0(t)K_0^4(t)t\mathrm{d}t = \frac{\pi^4}{2^6 3^2}$$

for the 6-Bessel problem encountered by Laporta in electrodynamics.

Using `lindep`, I discovered that this may be recast as

$$6\pi^3 \det \int_0^\infty \left[\begin{array}{cc} f(1/2+\mathrm{i}y) & g(1/2+\mathrm{i}y) \\ f(1/2+\mathrm{i}y)y & g(1/2+\mathrm{i}y)y \end{array} \right] \mathrm{d}y = 1$$

with the cuspform $f(z) = (\eta_1\eta_2\eta_3\eta_6)^2$,

$$\frac{g(z)}{f(z)} = w^4 - 6w^2 + c - 6w^{-2} + 9w^{-4}, \quad \frac{w}{3} = \left(\frac{\eta_3}{\eta_1}\right)^4 \left(\frac{\eta_2}{\eta_6}\right)^2.$$

Amusingly, w defines an external energy for the two-loop **sunrise** diagram that I evaluated in my first talk, using Domb's enumeration of returning walks on a **honeycomb**. Clearly the determinant alone cannot tell us the value of c. The Bessel moments choose $c = 2$ which makes $g(z)/f(z)$ vanish at the pseudo-threshold $w = 1$, where the Feynman integral is regular. This week, Zhou **proved** my empirical result, above.

It remains to be seen how, if at all, Francis Brown's definition of **quasi-periods** relates to the second column of the matrix above.

Summary

1. PSLQ and LLL have enlivened quests for analytical results.
2. PSLQ led to the Broadhurst-Kreimer conjecture.
3. PSLQ has been parallelized.
4. PSLQ and LLL have provided strong tests on conjectures.
5. PSLQ and LLL have condensed huge expressions.
6. PSLQ was of the essence in Laporta's work in electrodynamics.
7. PSLQ and LLL led to determinants that may relate to quasi-periods.
8. Yajun Zhou's remarkable proofs [arXiv:1711.01829; 1708.02857; 1706.08308; 1706.01068] continue to turn experimental findings into proven mathematics.

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