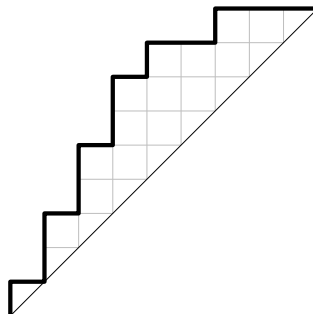


Tamari-like intervals and planar maps

Wenjie Fang
TU Graz

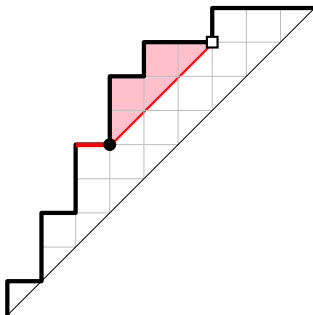
Workshop on Enumerative Combinatorics, 19 October 2017
Erwin Schrödinger Institute

Dyck paths and Tamari lattice, ...



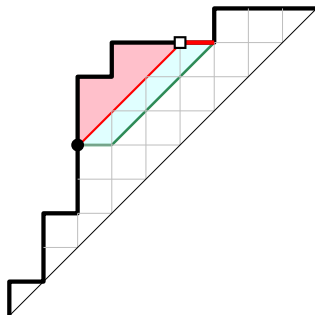
Dyck path: n north(N) and n east(E) steps, always above the diagonal
 Counted by the n -th Catalan numbers $\text{Cat}(n) = \frac{1}{2n+1} \binom{2n+1}{n}$

Dyck paths and Tamari lattice, ...



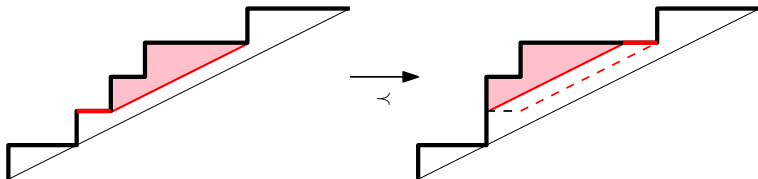
Covering relation: take a valley point ●, find the next point □ with the same distance to the diagonal ...

Dyck paths and Tamari lattice, ...



... and push the segment to the left. This gives the **Tamari lattice** (Huang-Tamari 1972).

..., m -Tamari lattice, ...



m -ballot paths: n north steps, mn east steps, above the " m -diagonal".

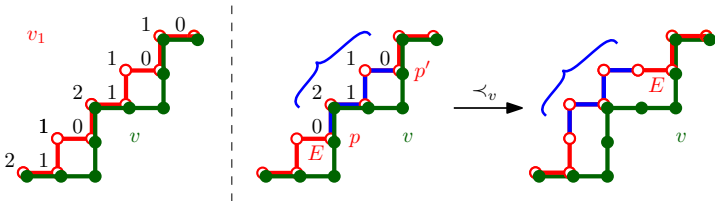
Counted by Fuss-Catalan numbers $\text{Cat}_m(n) = \frac{1}{mn+1} \binom{mn+1}{n}$.

A similar covering relation gives the **m -Tamari lattice** (Bergeron 2010).

... and beyond.

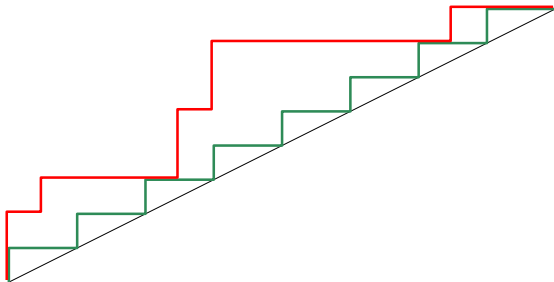
But we can use an arbitrary path v as "diagonal"!

Horizontal distance = # steps one can go without crossing v



Generalized Tamari lattice (Préville-Ratelle and Viennot 2014):
 $TAM(v)$ over arbitrary v (called the **canopy**) with N, E steps.

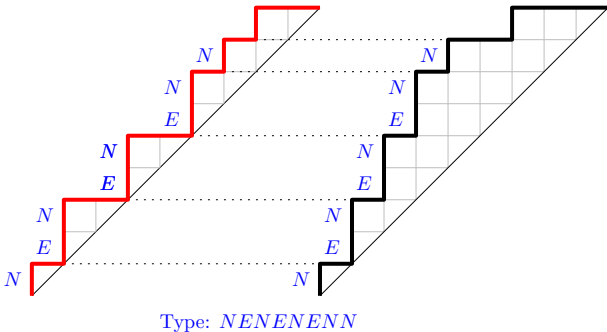
... and beyond.



$$\text{TAM}((NE^m)^n) \simeq m\text{-Tamari lattice}$$

Type of a Dyck path

North step: followed by an east step $\rightarrow N$, by a north step $\rightarrow E$.
 Mind the change!



The two paths have the same type, therefore **synchronized**.

Tamari restricted to type $v \simeq \text{TAM}(v)$

How exactly?

The next level: intervals

Interval in a lattice: $[a, b]$ with comparable $a \leq b$

Motivation: conjecturally related to the dimension of diagonal coinvariant spaces

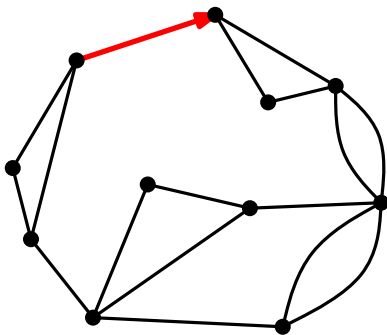
For generalized Tamari intervals:

Interval in $\text{TAM}(v)$ with v of length $n - 1 \Leftrightarrow$ **synchronized interval** of length $2n$, i.e., Tamari interval $[D, E]$ with D and E of the same type.

For Tamari and m -Tamari intervals:

- Counting: Bousquet-Mélou, Chapoton, Chapuy, Fusy, Préville-Ratelle, Viennot, ...
- Interval poset: Chapoton, Châtel, Pons, ...
- λ -terms: N. Zeilberger, ...
- **Planar maps**

What is a planar map?



Planar map: embedding of a connected multigraph on the plane (loops and multiple edges allowed), defined up to homeomorphism, cutting the plane into **faces**

Planar maps are **rooted** at an edge on the infinite outer face.

Intervals that count like planar maps

Chapoton 2006: # intervals in Tamari lattice of size $n =$

$$\frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

= # 3-connected planar triangulations with $n+3$ vertices (Tutte 1963)

= # bridgeless planar maps with n edges (Walsh and Lehman 1975)

Bousquet-Mélou, Fusy and Préville-Ratelle 2011:

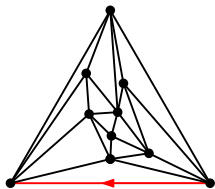
intervals in m -Tamari lattice of size $n =$

$$\frac{m+1}{n(mn+1)} \binom{n(m+1)^2+m}{n-1},$$

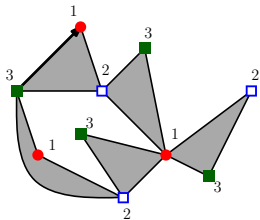
and it also looks like an enumeration of **planar maps**!

Deeper connections

For **Tamari intervals** and **3-connected planar triangulations**: bijective proof using orientations (Bernardi and Bonichon 2009)

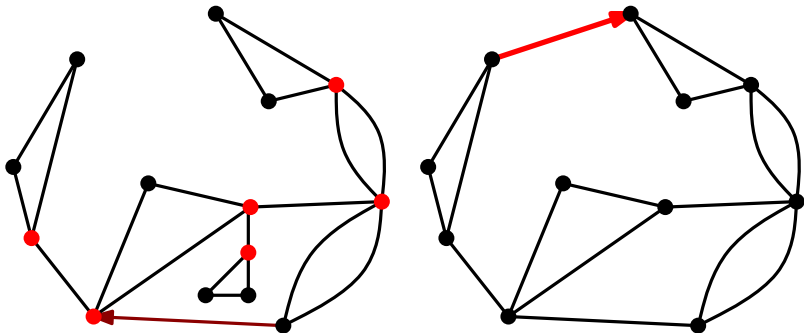


For **m -Tamari intervals**, the formal method used to solve for its generating function (the “differential-catalytic” method) can also be used on **planar m -constellations**.



Any other links? Especially for **generalized Tamari intervals**...

Non-separable planar maps



A **cut vertex** cuts the map into two sets of edges.

A **non-separable planar map** is a planar map without cut vertex.

Another type of intervals that counts like map

Theorem (W.F. and Louis-François Prévaille-Ratelle 2016)

There is a natural bijection between *intervals in* $\mathsf{TAM}(v)$ for all possible v of length n and *non-separable planar maps* with $n + 2$ edges.

Intermediate object: **decorated trees**

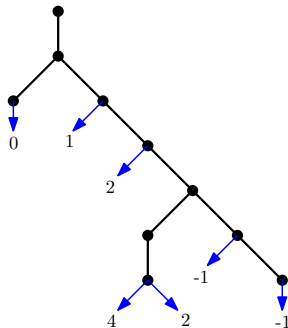
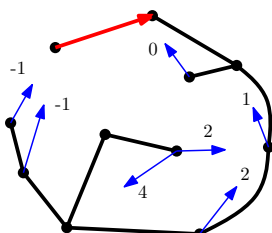
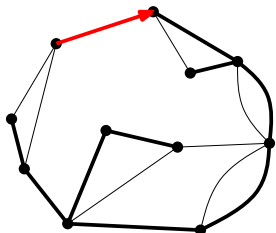
Corollary

The total number of *intervals in* $\mathsf{TAM}(v)$ for all possible v of length n is

$$\sum_{v \in (N, E)^n} \text{Int}(\mathsf{TAM}(v)) = \frac{2}{(n+1)(n+2)} \binom{3n+3}{n}.$$

This formula was first obtained in (Tutte 1963).

What are decorated trees?



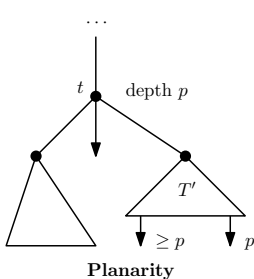
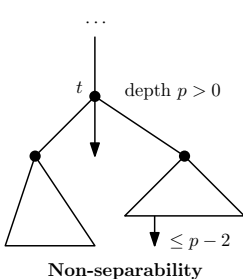
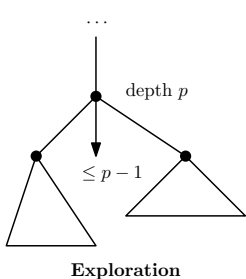
Property

If the exploration of an edge e adjacent to a vertex u reaches an already visited vertex w , then w is an ancestor of u .

Characterizing decorated trees

A **decorated tree** is a rooted plane tree with labels ≥ -1 on leaves such that (depth of the root is 0):

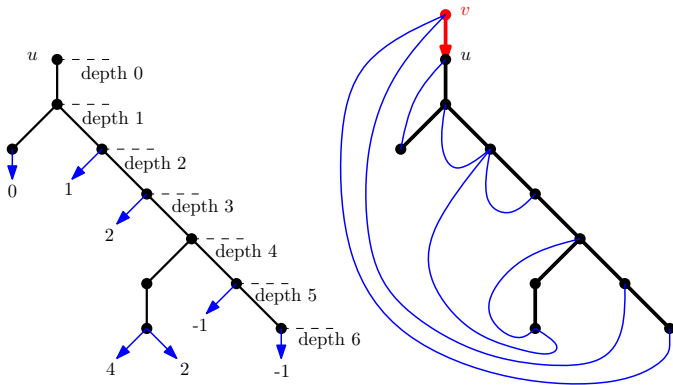
- 1 (Exploration) For a leaf ℓ of a node of depth p , the label of ℓ is $< p$;
- 2 (Non-separability) For a non-root node t of depth p , there is at least one descendant leaf with label $\leq p - 2$ (the first such leaf is the **certificate** of t);
- 3 (Planarity) For t a node of depth p and T' a direct subtree of t , if a leaf ℓ in T' is labeled p , every leaf in T' before ℓ has a label $\geq p$.



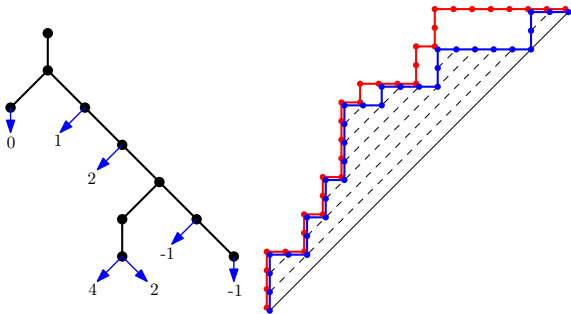
From maps to trees

Just glue leaves with label d to their ancestor of depth d .

Only one way to glue back to a planar map.

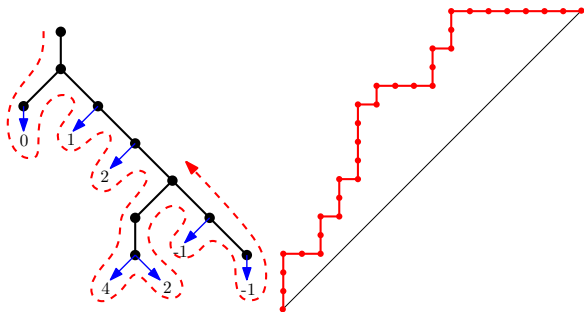


From trees to intervals



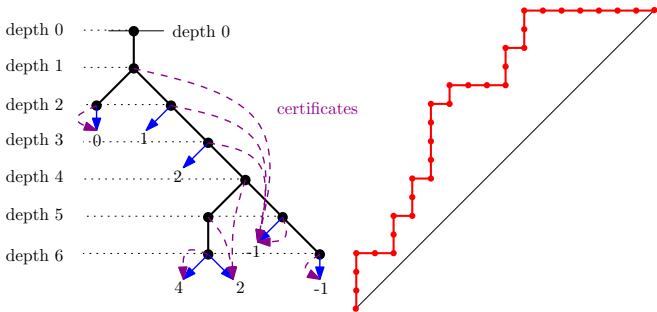
From a decorated tree T to a synchronized interval $[P(T), Q(T)]$

From trees to intervals



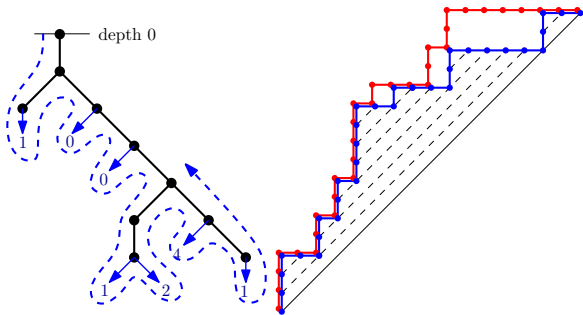
Path Q : a traversal

From trees to intervals



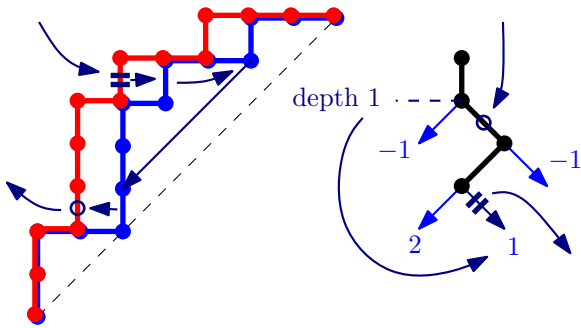
Function c : for a leaf ℓ , $c(\ell) = \# \text{nodes with } \ell \text{ as certificate}$

From trees to intervals

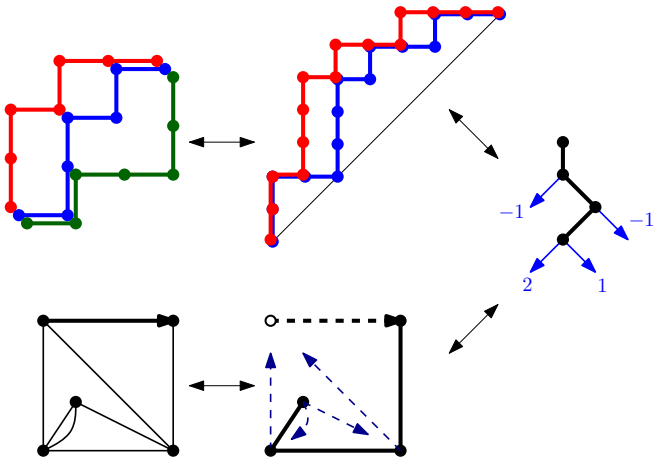


Path P : an altered traversal where descents are $c(\ell) + 1$

The other direction



The whole bijection



Structural result

Our bijections are **canonical** w.r.t. appropriate recursive decompositions of related objects.

Theorem (W.F. 2017)

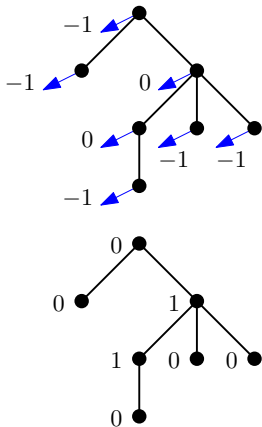
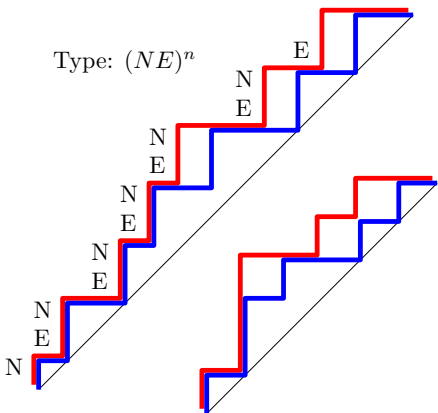
*Under our bijections, the **involution** from intervals in $\text{TAM}(v)$ to those in $\text{TAM}(\overleftarrow{v})$ is equivalent to **map duality**.*

Also connection with β -(1,0) trees (Cori, Schaeffer, Jacquard, Kitaev, de Mier, Steingrímsson, ...), leading to a bijective proof of a result in Kitaev–de Mier(2013).

Also equi-distribution results on various statistics

Restriction to the original Tamari intervals...

Tamari lattice = $TAM((NE)^n)$

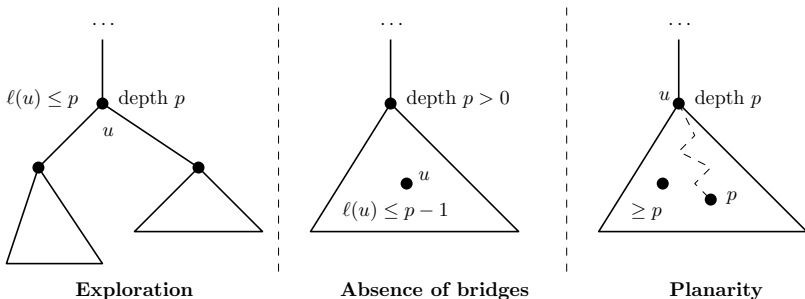


Restriction to type $(NE)^n$: decorated trees where each leaf is the first child of each internal node.

Sticky tree

Decorated trees restricted in $\text{TAM}((NE)^n) \rightsquigarrow$ **sticky trees**

A **sticky tree** is a plane tree with a label $\ell(u) \geq 0$ on each node u such that:



Essentially adapted from the condition of decorated trees! Now every non-root node has a certificate, which is a node (and can be itself).

Bijections to classical objects

Theorem (W.F. 2017+)

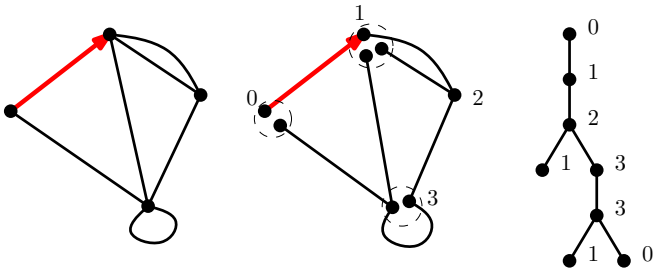
Sticky trees with n edges are in natural bijection with

- 1 *Tamari intervals with n up steps;*
- 2 *bridgeless planar maps with n edges;*
- 3 *3-connected triangulations with $n + 3$ vertices.*

A new bijective proof of (1) = (3), different from (Bernardi–Bonichon 2009).

Also a new bijective (and direct!) proof of (2) = (3), different from the recursive ones in (Wormald 1980) and (Fusy 2010).

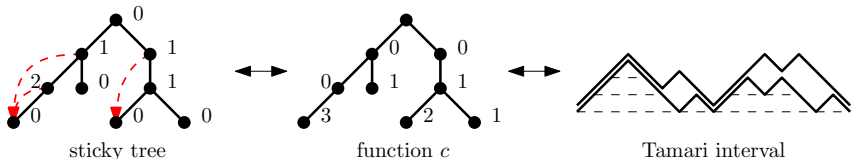
Bijection with bridgeless planar maps



An exploration on **edges**

There is also a bijection between sticky trees and 3-connected planar triangulations (with a different exploration process)

Bijection with Tamari intervals

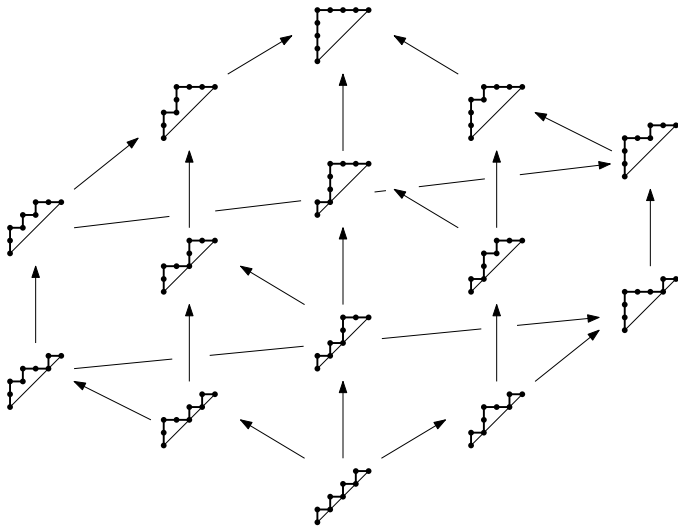


Also with closed flows of plane forests (Chapoton–Châtel–Pons 2014), recovering a result therein.

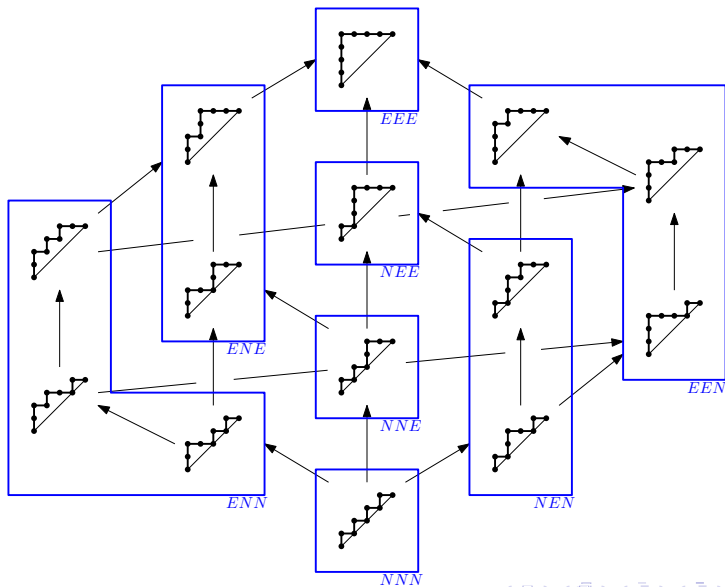
Discussion

- Other related lattices (Stanley, Kreweras, ...) and planar maps (bipartite, constellations)?
- Other structures (e.g. 2-stack-sortable permutations)?
- Asymptotic aspects of these objects (statistics, limit shape, ...)?
- Restricted bijections on m -Tamari lattice?
- Number of intervals in $\text{TAM}(w^n)$ for a fixed word w ?
Rotation of NE^m : **OK**.
For $w = NNEE$: 1, 20, 755, 37541, 2177653, ...
For $w = NEEN$: 6, 164, 7019, 373358, 22587911, ...
Are they all algebraic?

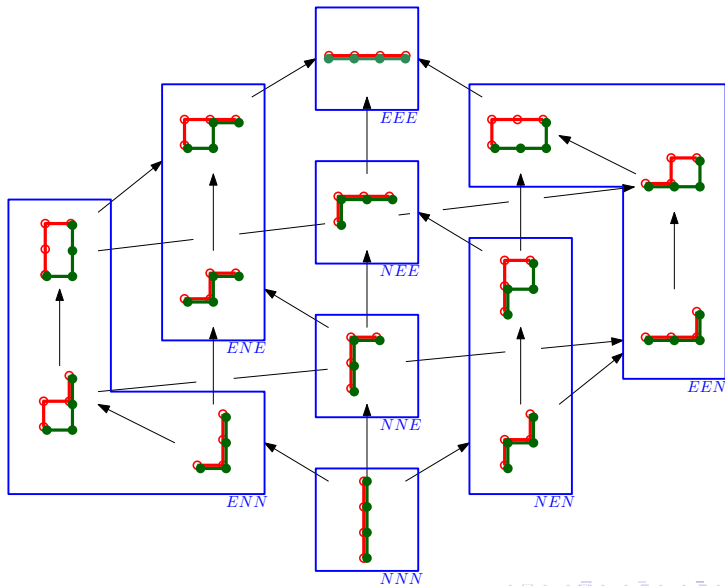
Partitioning the Tamari lattice by type



Partitioning the Tamari lattice by type



Partitioning the Tamari lattice by type



Partitioning the Tamari lattice by type

Delest and Viennot (1984): There is a bijection between Dyck path of length $2n$ and an element in $\text{TAM}(v)$ for some v of length $n - 1$.

Theorem (Préville-Ratelle and Viennot (2014))

The Tamari lattice of order n is partitioned by path types into 2^{n-1} sublattices, each isomorphic to the generalized Tamari lattice $\text{TAM}(v)$ with v the type (a word in N, E of length $n - 1$).

Theorem (Préville-Ratelle and Viennot (2014))

The lattice $\text{TAM}(v)$ is isomorphic to the order dual of $\text{TAM}(\overleftarrow{v})$, where \overleftarrow{v} is the word v read from right to left, with the substitution $N \leftrightarrow E$.

And back...

Some interesting sequences...

Number of intervals in $\text{TAM}(w^n)$ with w a word in $\{N, E\}$?

Observation

For $w = N^a EN^b$, the number of intervals in $\text{TAM}(w^n)$ is of the form

$$\frac{k_{a,b} + 1}{n(\ell_{a,b}n + 1)} \binom{(a + b + 1)^2 n + k_{a,b}}{n - 1},$$

where $k_{a,b}$ and $\ell_{a,b}$ are integers. *What are these constants?*

For $w = NNEE$: 1, 20, 755, 37541, 2177653, ...

For $w = NEEN$: 6, 164, 7019, 373358, 22587911, ...

What are these sequences?