The growth of groups, with application to Thompson's group F.

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Infinite finitely generated groups

- For example: The group $\mathbb{Z}^2$, with the operation $+$ is generated by $(1, 0)$ and $(0, 1)$.
- The free group on two generators F_2.
- Non-example: The group of rational numbers with operation $+$ is not finitely generated.

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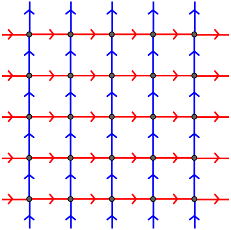

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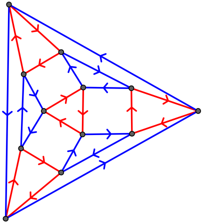

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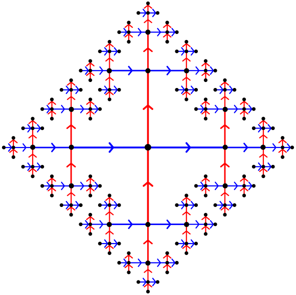

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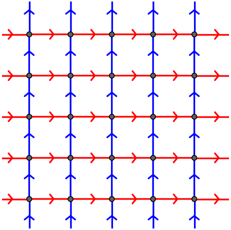

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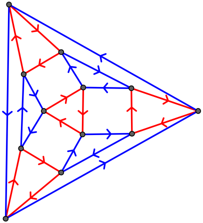

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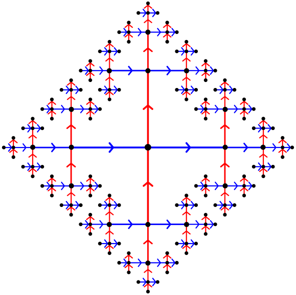

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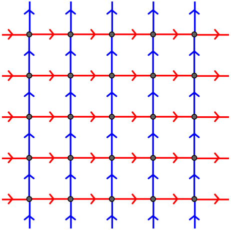

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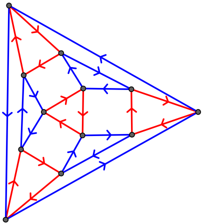

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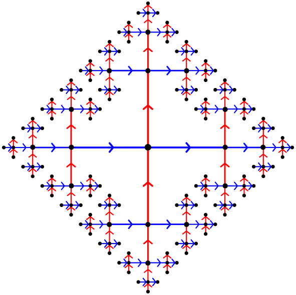

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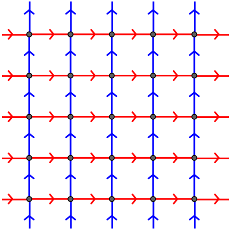

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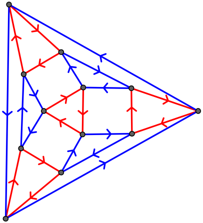

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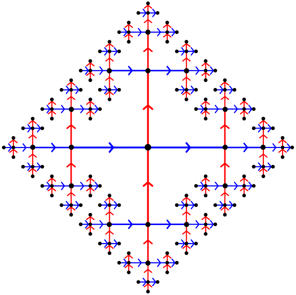

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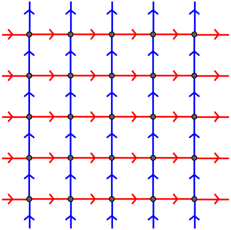

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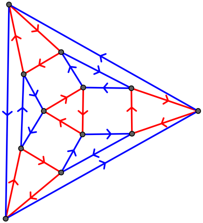

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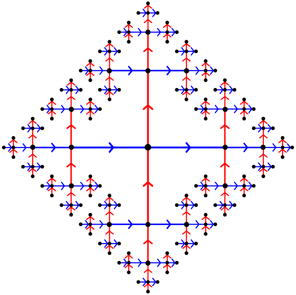

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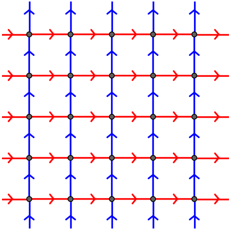

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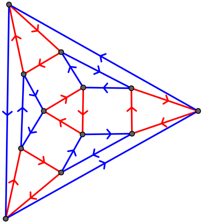

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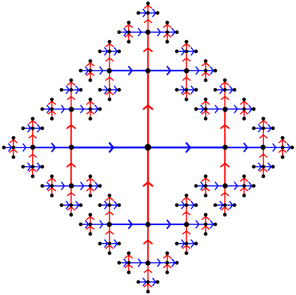

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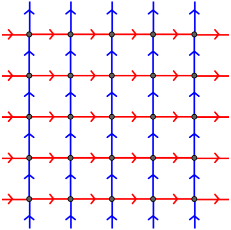

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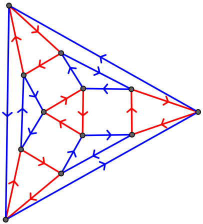

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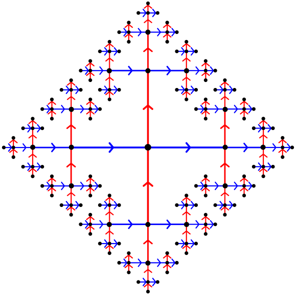

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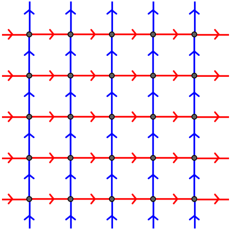

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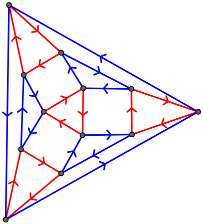

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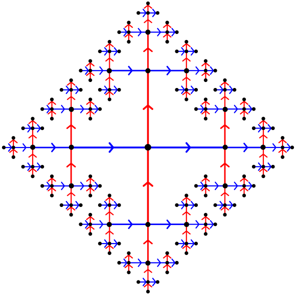

COGROWTH

Let G be a group with generating set S and Cayley graph $\Gamma(G, S)$. The cogrowth sequence $l_0, l_1, \ldots$ of G is defined as follows:

- For each $n \in \mathbb{Z}_{\geq 0}$, let l_n be the number of walks of length $2n$ in Γ starting and ending at the root vertex.
- Equivalently, l_n is the number of words w of length $2n$ over the alphabet $S \cup S^{-1}$ which are equal to the identity in G.

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AMENABILITY

A group G, with Cayley graph Γ is amenable if any of the following (equivalent) conditions hold:

- There exists a left invariant, finitely additive probability measure on G.
- Something about Folner sets.
- The cogrowth sequence $l_0, l_1, \ldots$ satisfies

$$\lim_{n\to\infty} \sqrt[n]{l_n} = 4|S|^2 = \lim_{n\to\infty} \frac{l_n}{l_{n-1}}.$$

Surprisingly, this does not depend on the choice of Cayley graph.

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- The Cayley graph for $\mathbb{Z}^2$ is the square lattice.
- The cogrowth sequence $l_0, l_1, \ldots$ is given by

$$l_n = \binom{2n}{n}^2,$$

the number of loops of length $2n$ on the square lattice.

- $\mathbb{Z}^2$ is amenable because

$$\lim_{n\to\infty} \sqrt[n]{\binom{2n}{n}^2} = 16.$$

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- The Cayley graph for F_2 (the free group on 2 generators) is the infinite 4-regular tree.
- The cogrowth sequence $l_0, l_1, \ldots$ is given by the generating function

$$\sum_{n=0}^{\infty} l_n t^n = \frac{3}{1 + 2\sqrt{1 - 12t}},$$

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Von Neumann conjecture

- The Von Neumann conjecture states that a finitely generated group is non-amenable if and only if it contains F_2 as a subgroup.
- The if direction is easy to prove.
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Big question: Is Thompson's group F amenable?

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- (1980) Ol'shanskii disproves the Von Neumann conjecture anyway.
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- (1980) Ol'shanskii disproves the Von Neumann conjecture anyway.
- (2009) Akhmedov announces a proof that Thompson's group is not amenable.
- (2009) Akhmedov withdraws his claim.
- (2009) Moore proves that if Thompson's group is amenable, the Folner sets have to grow extremely quickly.

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- (2010) Shavgulidze announces a proof that Thompson's group is amenable.
- (2011) Moore points out that Shavgulidze's method is hopeless, because the Folner sets wouldn't grow extremely quickly.
- (2012) Moore announces a proof the Thompson's group is amenable.
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Plan to determine whether F is amenable

- We know that Thompson's group F is amenable if and only if its cogrowth sequence $t_0, t_1, t_2, \ldots$ has exponential growth rate 16.
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Outline of algorithm

- For each group element $g \in F$ and $n \in \mathbb{Z}_{\geq 0}$, let $p_{g,n}$ be the number of paths of length n from the identity vertex v_ϵ to v_g.
- Then $p_{g,n}^2$ is the number of loops of length $2n$ whose midpoint is v_g. So,

$$t_n = \sum_{g \in F} p_{g,n}^2.$$

- We can calculate $p_{g,n}$ recursively by

$$p_{g,n} = p_{ga,n-1} + p_{ga^{-1},n-1} + p_{gb,n-1} + p_{gb^{-1},n-1}.$$

but the number of group elements g with $p_{g,n} > 0$ is about 2.618^n, so this algorithm takes a lot of memory.

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Hausdorff Moment problem

- Let the moments c_n of a given measure $\phi(x)$ be given by

$$c_n = \int_a^b x^n d\phi(x) = \int_a^b x^n \mu(x) dx, \;\; n = 1, 2, \ldots .$$

- The measure is unique.
- The Stieltjes transform of ϕ is

$$S(z, \mu) = \int_a^b \frac{\mu dx}{z - x} = \frac{1}{z} \sum_{k \geq 0} \frac{c_k}{z^k} = \sum_{i \geq 1} \frac{\lambda_i}{z - x_i}.$$

- $[N-1, N]$ Padé approximant converges to the Stieltjes transform.

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- The denominator zeros of the PAs provide rigorous bounds on the support $[a, b]$.
- Such a measure exists if the Hankel determinants are all non-negative.
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- Haagerup, Haagerup and Ramirez-Solano proved that the cogrowth sequence for F *is* the moment sequence of a probability measure.
- In fact their proof applies to the cogrowth sequence of any (locally finite) Cayley graph.
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GROUPS TO STUDY

- $\mathbb{Z}^2$. Then $c_n = \binom{2n}{n}^2$, and $C(x) = 2\mathbf{K}\left(\frac{4\sqrt{x}}{\pi}\right)$.
- Discrete Heisenberg group. $c_n \sim 16^n/(2n^2)$, and $C_H(x) \sim \frac{1}{2}(1-16x)\log(1-16x)$.
- Lamplighter group. $c_n \sim c \cdot 9^n \cdot \kappa^{n^{1/3}} \cdot n^{1/6}$. Stretched exponential.
- $\mathbb{Z} \wr \mathbb{Z}$. $c_n \sim const \cdot 16^n \cdot \kappa^{n^{1/3} \log^{2/3} n} \cdot n^g$.
- Navas-Brin group. An amenable sub-group of Thompson F. $c_n \sim c.16^n$, and more slowly than $c_n \sim c.16^n \cdot \kappa^{n^\sigma} \cdot n^g$.

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Series analysis 101.

- Given $f(z) = \sum c_n z^n$, the Cauchy-Hadamard theorem tells us

$$r = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}.$$

- Alternatively, the ratio test tells us that

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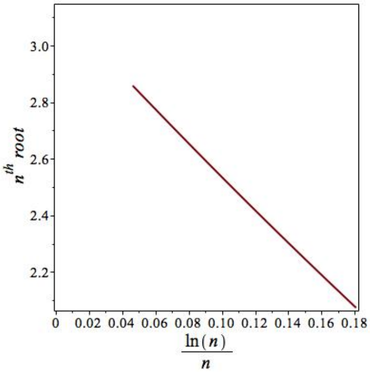
Test series $f(z) = \exp(-z) \cdot (1 - \pi \cdot z)^{2/3}$.



Ratios are better!

Note: If $c_n \sim C \cdot \mu^n \cdot n^g$, ratios eliminate the leading constant.
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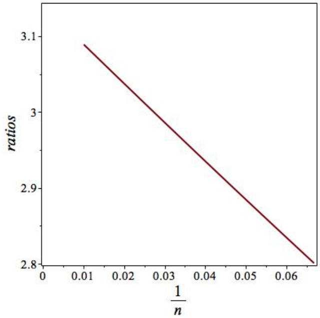

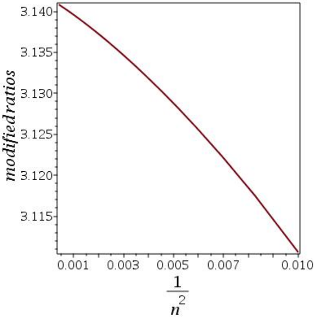

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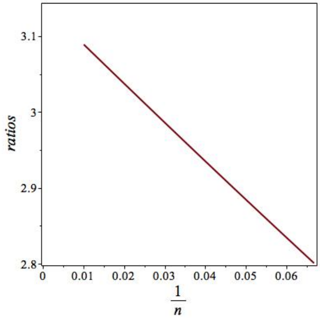

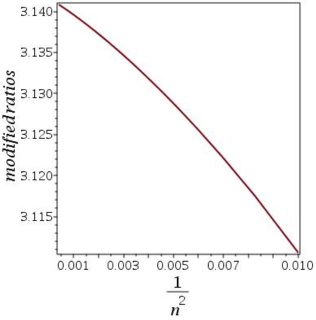

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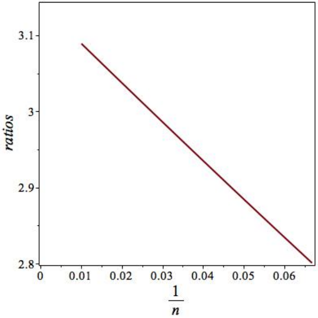

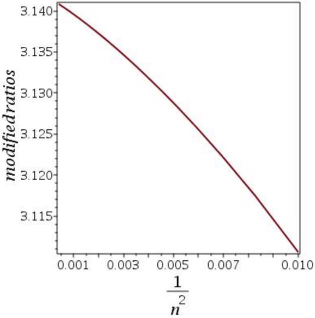

Series analysis 101. Differential approximants.

- $\sum_{k=0}^{M} Q_k(z)(z\frac{\mathrm{d}}{\mathrm{d}z})^k \tilde{F}(z) = P(z)$
- The singularities of $\tilde{F}(z)$ are approximated by the zeros $z_i, \; i = 1, \ldots, N_M$ of $Q_M(z)$.
- Exponents γ_i from the indicial equation. If only a single root at z_i,

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Example of differential approximants.

Critical point and exponent estimates for self-avoiding polygons. Numbers in parentheses give the uncertainty in the last quoted digits.

L	Second order DA		Third order DA	
	x_c^2	$2-\alpha$	x_c^2	$2-\alpha$
0	0.29289321854(19)	1.50000065(41)	0.29289321865(12)	1.50000040(28)
5	0.29289321875(21)	1.50000010(59)	0.29289321852(48)	1.50000041(99)
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Getting more ratios – the method of Series Extension.

In many cases, for example Thompson F, we need more terms.

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

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The first approximate coefficient will be the most accurate, with accuracy declining with increasing order of predicted coefficients.

In practice we construct many DAs. We then calculate the average of the predicted coefficients (or ratios) across all constructed DAs, as well as their standard deviation.

We have experimentally found the true error to be between 1 and 2 standard deviations.

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Cogrowth series of infinite, finitely-generated groups.

- We wish to determine if Thompsons group F is amenable.
- To that end, we study the asymptotics of some amenable groups.
- We develop techniques to deal with all known behaviour.
- Recall that if Thompsons group F is amenable, it is only just amenable.
- If the growth constant of the cogrowth series is 16, the group is amenable, otherwise not.
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The group $\mathbb{Z}^2$.

- For this group the coefficients of the cogrowth series are just $c_n = \binom{2n}{n}^2 \sim \frac{16^n}{n\pi}$.
- So the ratio of successive terms is

$$r_n = \frac{c_n}{c_{n-1}} = 16\left(1 - \frac{1}{n} + \frac{1}{4n^2}\right).$$

- A ratio plot, based on the first 50 coefficients is clearly going to the expected limit of 16. The exponent should be -1, corresponding to a logarithmic singularity of the generating function, $C_{\mathbb{Z}^2}(x) \sim c \cdot \log(1 - 16x)$.
- For this simple example from the first 20 or so coefficients one immediately obtains

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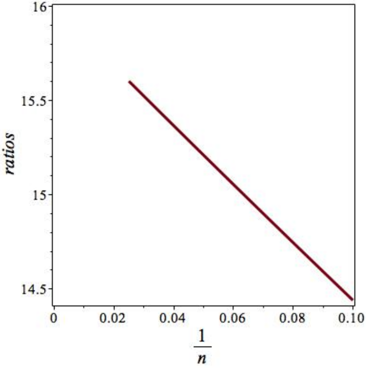
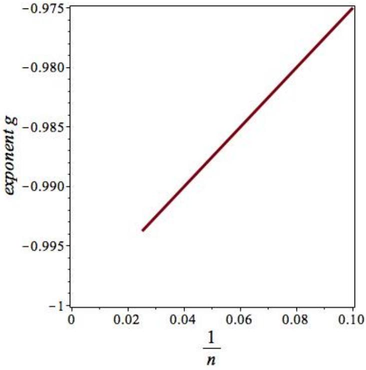


Plot of $\mathbb{Z}^2$ ratios against $1/n$.



Estimators of exponent g for $\mathbb{Z}^2$ vs. $1/n$.

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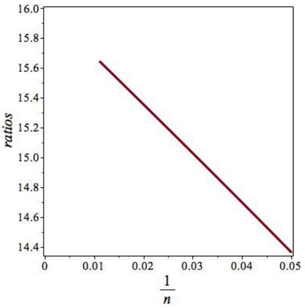
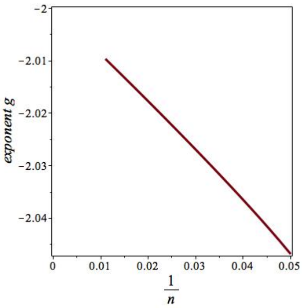
- The coefficient asymptotics are $c_n \sim 16^n/(2n^2)$, from the OGF

$$C_{Heisenberg} \sim \frac{1}{2}(1-16x)\log(1-16x).$$

- A ratio plot, based on the first 90 coefficients is clearly going to the expected limit of 16. The exponent should be -2.





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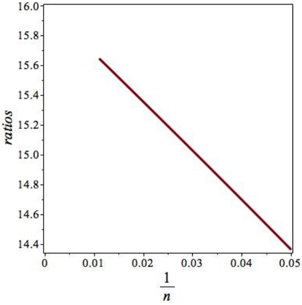

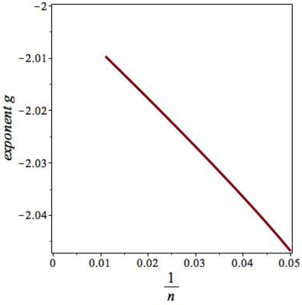

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- We can estimate higher-order asymptotic terms by subtracting the leading-order term from the coefficients.
- Further ratio analysis suggests asymptotics involve increasing powers of $1/n$.
- Accordingly, we fit to the assumed form $c_n/16^n = 1/(2n^2) + k_1/n^3 + k_2/n^4 + k_3/n^5$.
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$$c_n = 16^n \left(\frac{1}{2n^2} + \frac{0.93341}{n^3} + \frac{1.530}{n^4} + \frac{3.30}{n^5} + O\left(\frac{1}{n^6}\right)\right).$$

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The lamplighter group.

- The lamplighter group L is the wreath product of the group of order two with the integers, $L = \mathbb{Z}_2 \wr \mathbb{Z}$.
- For this group,

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- If the series coefficients of a series include a stretched-exponential term, so that

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with $0 < \sigma,\ \kappa < 1$, then the ratios behave as

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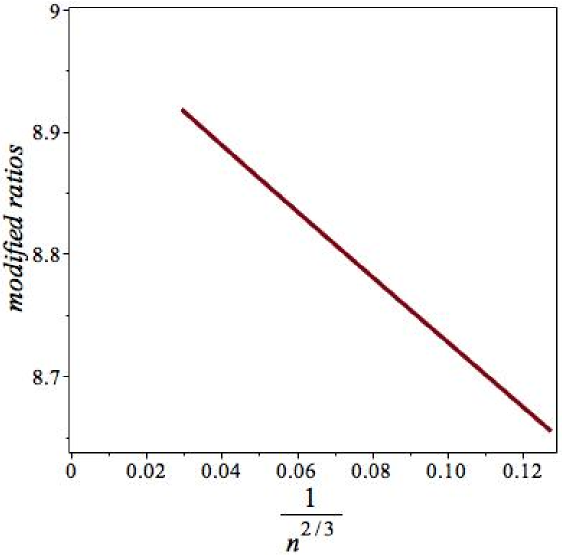

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- Let's also estimate σ without assuming the growth constant μ by taking the ratio of modified ratios.
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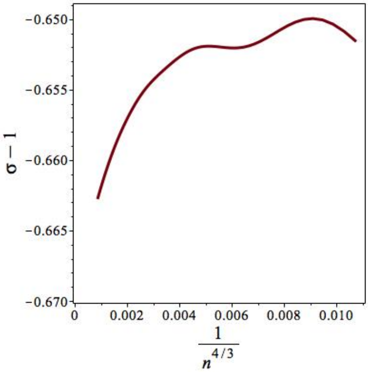
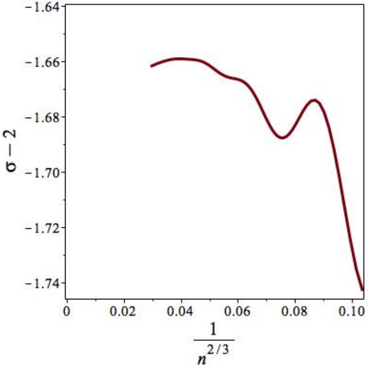


Estimates of $\sigma - 1$ vs. $n^{-4/3}$.



Estimates of $\sigma - 2$ vs. $n^{-2/3}$.

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$$\log c_n - n \cdot \log 9 \sim n^{1/3} \cdot \log \kappa + g \cdot \log n + \log c.$$

We fit successive triples of coefficients to estimate the three unknowns, $\log \kappa$, g and $\log c$.
We estimate $\log \kappa \approx -2.78$, $g \approx 0.17$, and $\log c \approx -0.6$.
Using the known value $g = 1/6$, we can get $\log \kappa \approx -2.775$, and $\log c \approx -0.5$.
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Analysis of group $\mathbb{Z} \wr \mathbb{Z}$

- This group has coefficients that behave as

$$a_n \sim const \cdot 16^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g, \text{ with } \sigma = 1/3 \text{ and } \delta = 2/3.$$

- The ratios behave as

$$r_n = \frac{a_n}{a_{n-1}} \sim 16 \left(1 + \frac{\sigma \cdot \log \kappa \cdot \log^\delta n}{n^{1-\sigma}} + \frac{\delta \cdot \log \kappa \cdot \log^{\delta-1} n}{n^{1-\sigma}} + \frac{g}{n} \cdots\right)$$

- We have generated series to order x^{276} for this group.
- A ratio plot against $1/n$ is strongly concave. Plotting against $1/n^{2/3}$ is much closer to linearity, but is still slightly concave.
- Again we eliminate the $O(1/n)$ term using modified ratios $r_n^{(1)} = n \cdot r_n - (n-1) \cdot r_{n-1}$:

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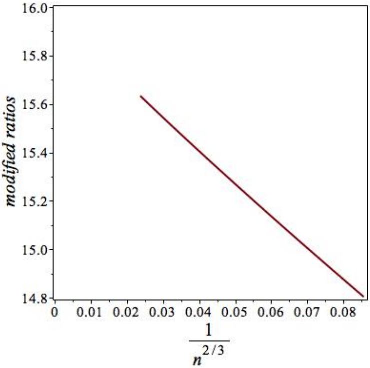
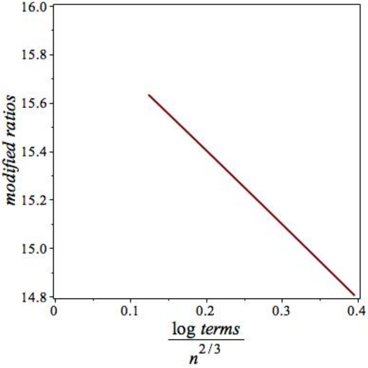


Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $n^{-2/3}$.



Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $\left(\log^{2/3} n + 4\log^{-1/3} n - 2\log^{-4/3} n\right) n^{-2/3}$

What about estimating μ? One way is from the modified ratio plots. All are seen to be tracking towards a value very close to 16. To estimate σ without assuming μ we use ratios of ratios:

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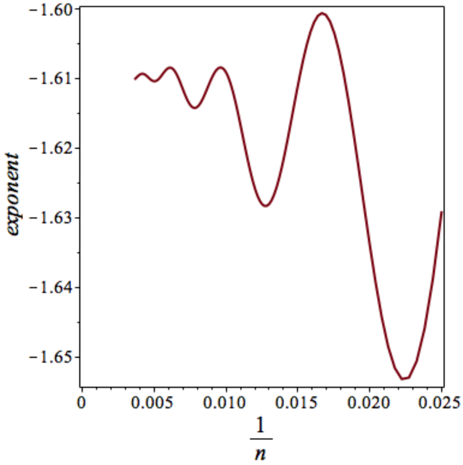
We plot the local gradient against $1/n$.

It appears to be going to -1.62 to -1.61, implying $\sigma \approx 0.38$ or 0.39, rather than the known value of $1/3$.



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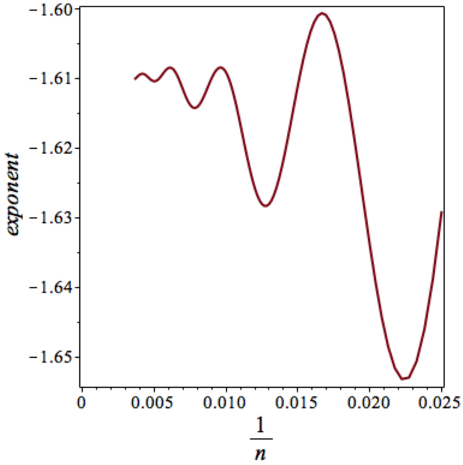
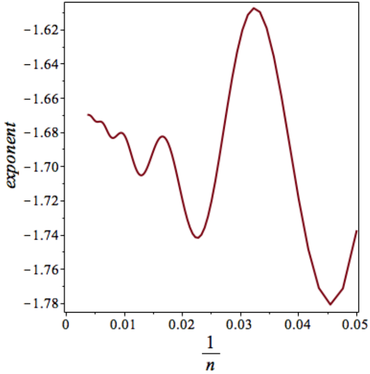


If we include the confluent logarithmic term $\log^{2/3} n$ in the exponent of the stretched-exponential term, plotting instead $\log\left(\frac{r_n^{(2)}-1}{\log^{2/3} n}\right)$ against $\log n$, the corresponding plot of the local gradient is clearly going to a limit around $-5/3$, consistent with the known value $\sigma = 1/3$.



Estimators of exponent $\sigma - 2$ vs. $1/n$, assuming a confluent logarithmic term

Assuming $\mu = 16$, $\sigma = 1/3$ and $\kappa = 2/3$, we again estimate the remaining parameters by fitting to the log of the coefficients.

From $c_n \sim c \cdot 16^n \cdot \kappa^{n^{1/3} \log^{2/3} n} \cdot n^g$, we get

$$\log c_n - n \cdot \log 16 \sim n^{1/3} \cdot \log^{2/3} n \cdot \log \kappa + g \cdot \log n + \log c.$$

Proceeding as before, we estimate $\log \kappa \approx -1.64$. It is difficult to estimate the other parameters with any certainty.

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Anticipating our analysis of Thompson's group F, we attempt to estimate both the exponents σ and δ without knowing the value of μ.

We first form the ratio of ratios $rr_n^{(1)}$ to eliminate μ.

If we now form the sequence

$$t_n = \frac{rr_n^{(1)} - 1}{rr_{n-1}^{(1)} - 1} \quad (1)$$

this eliminates the base κ of the stretched-exponential term, since

$$n(t_n - 1) \sim \sigma - 2 + \frac{\delta}{n \log n}.$$

So plot $n(t_n - 1)$ against $1/(n \log n)$ to estimate of $\sigma - 2$.

We estimate δ, from

$$n \log^2 n(n(t_n - 1) - (n-1)(t_{n-1} - 1)) \sim -\delta + O(1/\log n).$$

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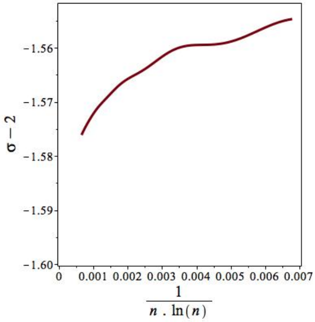
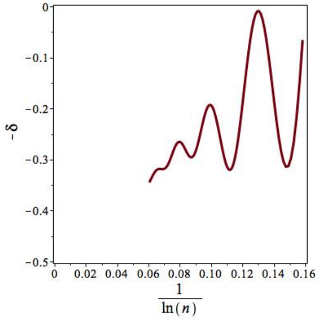
The estimate of $\sigma - 2$ appears to be going to a limit of around -1.6 or below, c.f. the known exact value of $-5/3$, while the estimate of δ is harder to estimate, but the plot is certainly consistent with the known value $2/3$. This exponent is difficult to estimate without many more terms than we currently have.



Estimates of $\sigma - 2$ vs. $1/(n \log n)$



Estimates of exponent $-\delta$ vs. $1/\log n$.

THE NAVAS-BRIN GROUP B.

- This amenable group was introduced independently by Navas and Brin, and is a subgroup of Thompson's group F.
- It is an infinite wreath product, with an extra generator conjugating each generator of the wreath product to the next one.
- Two generators: Growth rate of the cogrowth sequence is 16.
- It also has a sub-exponential growth term that is very close to exponential, and so makes the growth rate difficult to estimate.
- We have generated 128 terms exactly, and used the method of series extension to predict the next 590 ratios, the last of which we expect to be accurate to 1 part in 5×10^7.

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- The asymptotic form of the coefficients is not known.
- They must grow more slowly than

$$c_n \sim c \cdot 16^n \cdot \kappa^{n^\sigma} \cdot n^g,$$

where $0 < \sigma < 1$, and $0 < \kappa < 1$.

- Possible behaviour might be

$$c_n \sim c \cdot 16^n \cdot \kappa^{n/\log n} \cdot n^g,$$

- In that case the ratios will be

$$r_n = \frac{c_n}{c_{n-1}} \sim 16 \left(1 + \frac{constant}{\log n} + \frac{g}{n} + \cdots \right).$$

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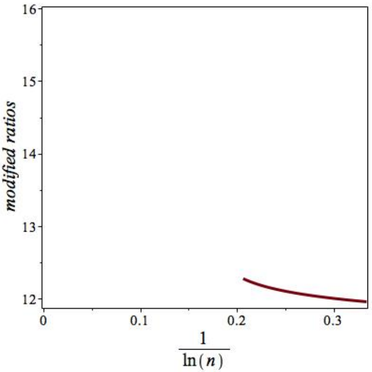
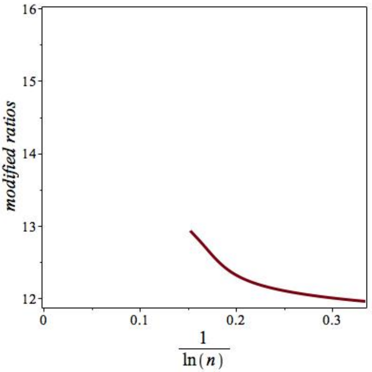
Modified ratio plots, exact and extrapolated terms.



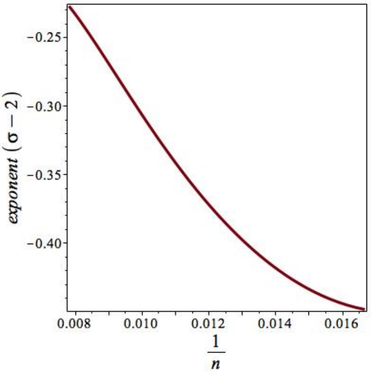


The first 128 modified ratios for the Navas-Brin group B vs. $1/\log n$. Limit 16 far from obvious.

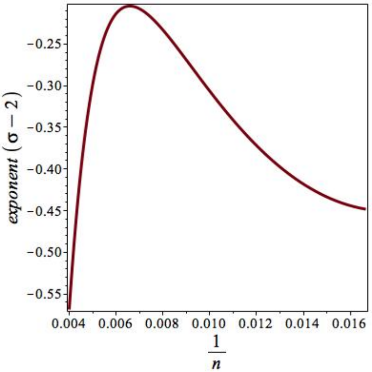
The first 718 modified ratios for the Navas-Brin group B vs. $1/\log n$. Limit 16 more plausible.

Estimating σ, exact and extrapolated terms.

We next try and estimate the exponent σ, which should be 1, without assuming $\mu = 16$.



Estimates of $\sigma - 2$ from 128 terms of the Navas-Brin group B.



Estimates of $\sigma - 2$ from 256 terms of the Navas-Brin group B.

Thompson's group F.

- It is known that the series grows exponentially like μ^n. If $\mu = 16$, the group is amenable.
- **Theorem:** Let c_n be the number of loops of length $2n$ in the standard Cayley graph for Thompson's group. Then for any real numbers $0 < a < 1$ and $0 < \kappa < 1$, the inequality

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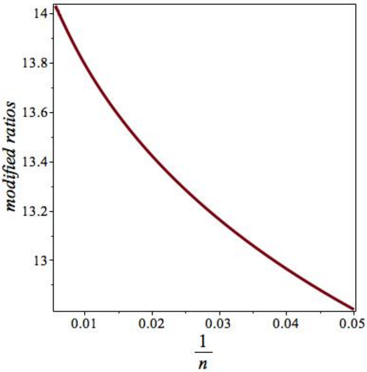
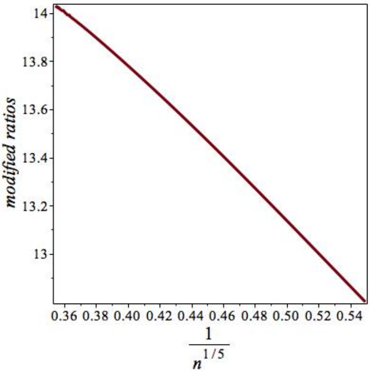
The modified ratios plotted against $1/n$ display curvature. The same data plotted against $n^{-1/5}$ shows curvature in the opposite direction.



Modified ratios vs. $1/n$ for Thompson's group F.



Modified ratios vs. $n^{-1/5}$ for Thompson's group F.

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- This is strong evidence for the presence of a conventional stretched-exponential term.
- The presence of such a term is incompatible with amenability. This is our first piece of evidence that the group is not amenable.
- This is quite different to the behaviour observed for the coefficients of the Navas-Brin group B.

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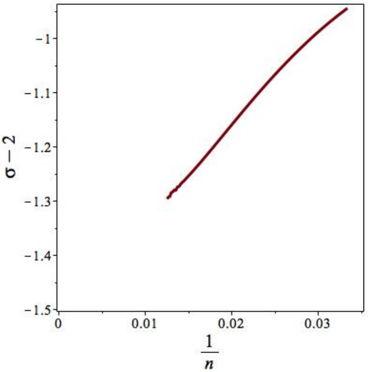
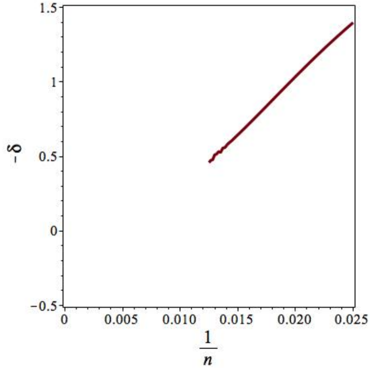
We estimate the exponents in the stretched-exponential term as for $\mathbb{Z} \wr \mathbb{Z}$. This allows for a stretched-exponential term $\kappa^{n^\sigma \log^\delta n}$.



Estimators of $\sigma - 2$ for Thompson's group F vs. $1/n$.



Estimators of $-\delta$ for Thompson's group F vs. $1/n$.

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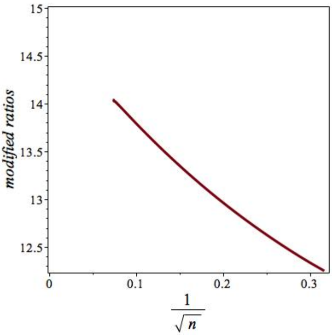
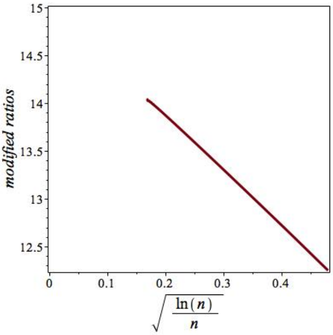
We plot the modified ratios against $1/\sqrt{n}$. A little curvature is seen. But a plot against $\sqrt{\log n/n}$, is essentially linear.

Extrapolating this we estimate the growth constant, to be $14.8 - 15.1$. This is well away from 16, which would be required for amenability.





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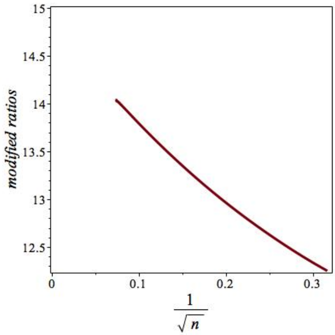

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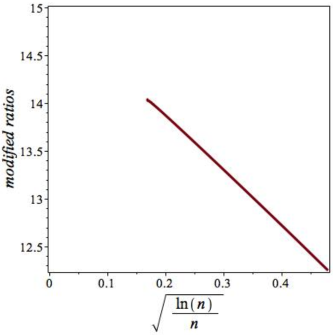

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- A simple test for amenability is that the ratios of successive coefficients asymptote to the growth constant $\mu = 4|S|^2$.
- For the lamplighter group, this ratio behaves as $r_n^{(L)} = 9\left(1 + \frac{c}{n^{2/3}} + o\left(\frac{1}{n^{2/3}}\right)\right)$.
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If Thompson's group F is amenable, these quotients should all go to 1.

THOMPSON'S GROUP F.

- A simple test for amenability is that the ratios of successive coefficients asymptote to the growth constant $\mu = 4|S|^2$.
- For the lamplighter group, this ratio behaves as $r_n^{(L)} = 9\left(1 + \frac{c}{n^{2/3}} + o\left(\frac{1}{n^{2/3}}\right)\right)$.
- For $\mathbb{Z} \wr \mathbb{Z}$ one has $r_n^{(2)} = 16\left(1 + \frac{c \cdot \log^{2/3} n}{n^{2/3}} + o\left(\frac{\log^{2/3} n}{n^{2/3}}\right)\right)$.
- For the triple wreath product the corresponding result is $r_n^{(3)} = 36\left(1 + \frac{c\sqrt{\log n}}{n^{1/2}} + o\left(\frac{\sqrt{\log n}}{n^{1/2}}\right)\right)$,
- For Thompson's group F all we know is $r_n = \mu\left(1 + \text{lower order terms}\right)$, and amenable iff $\mu = 16$.
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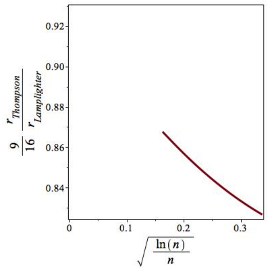
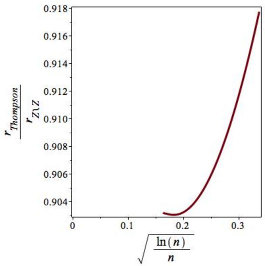
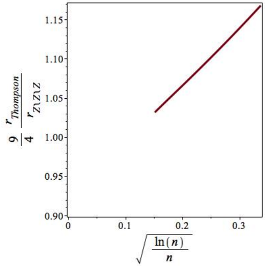


Quotient of Thompson group and lamplighter group ratios using 200 terms.



Quotient of Thompson group and $\mathbb{Z} \wr \mathbb{Z}$ ratios using 200 terms.

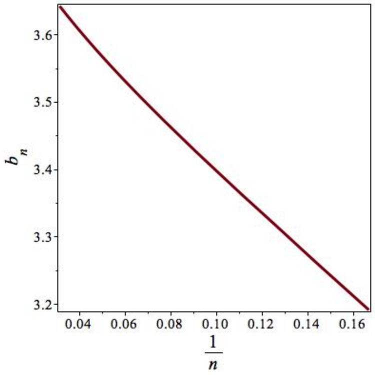


Quotient of Thompson group and $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ ratios using 200 terms.

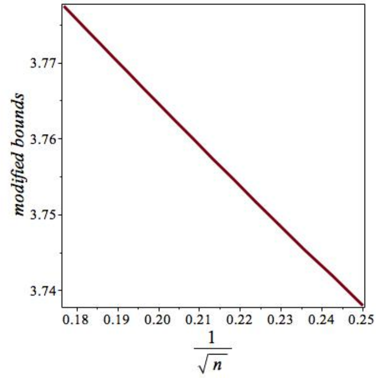
All are consistent with a limit around 0.93 ± 0.02, corresponding to $\mu = 14.9 \pm 0.3$.

EXTRAPOLATING BOUNDS.

We can extrapolate the lower bounds $\{b_n\}$, which are bounds on $\sqrt{\mu}$.
Extrapolates to 3.875, so that $\mu \approx 15.0$.



Plot of bounds b_n for Thompson's group F against $1/n$.



Plot of modified bounds $b_n^{(1)}$ for Thompson's group F against $1/\sqrt{n}$.

CONCLUSION.

The growth constant for Thompson's group F is estimated to be close to 15 by a variety of methods. As a consequence, we conjecture that Thompson's group F is not amenable.

With lower confidence, we suggest that the coefficients behave as

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g,$$

where $\mu \approx 15$, $\kappa \approx 1/e$, $\sigma \approx 1/2$, $\delta \approx 1/2$, and $g \approx -1$.

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