Noncrossing partitions, Bruhat order, and the cluster complex

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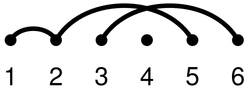
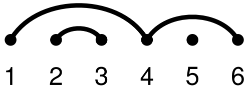
ESI Enumerative combinatorics workshop

Set partitions

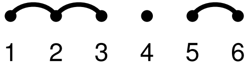
Set partition: 125|36|4 (={{1,2,5},{3,6},{4}})



Noncrossing partition: 146|23|5



Interval partition : 123|4|56.



Posets of set partitions

Set partitions, endowed with the reverse refinement order, form a lattice. Minimal element: $1|2|3|\dots|n$. Maximal element: $123\dots n$.

The cover relations are: $\pi \lessdot \rho$ if π is obtained from ρ by splitting a block $B \in \rho$ into two blocks of π.

There is also a lattice of noncrossing partitions,
with same cover relations, minimal element, maximal element.

The lattice of interval partitions is the boolean lattice.

Noncrossing partitions

Properties of NC_n, lattice of noncrossing partitions on $\{1, \ldots, n\}$:

$$\# \mathrm{NC}_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{(Catalan number)}$$

$$\#\{\pi \in \mathrm{NC}_n \ : \ \#\pi = k\} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \qquad \text{(Narayana numbers)}$$

They are related with free cumulants in free probability, factorizations of permutations, can be extended to finite Coxeter groups... [Kreweras, Reiner, Biane, Brady & Watt, Bessis, Speicher, Armstrong, ...]

Cover relations

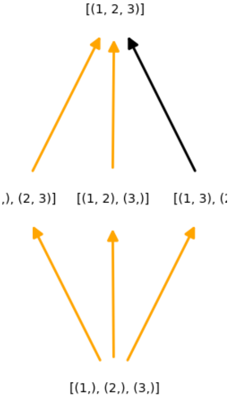
Distinguish two kinds of cover relations in NC_n. Let $\pi \lessdot \rho$, obtained by splitting a block $B \in \rho$ into $B_1, B_2 \in \pi$:

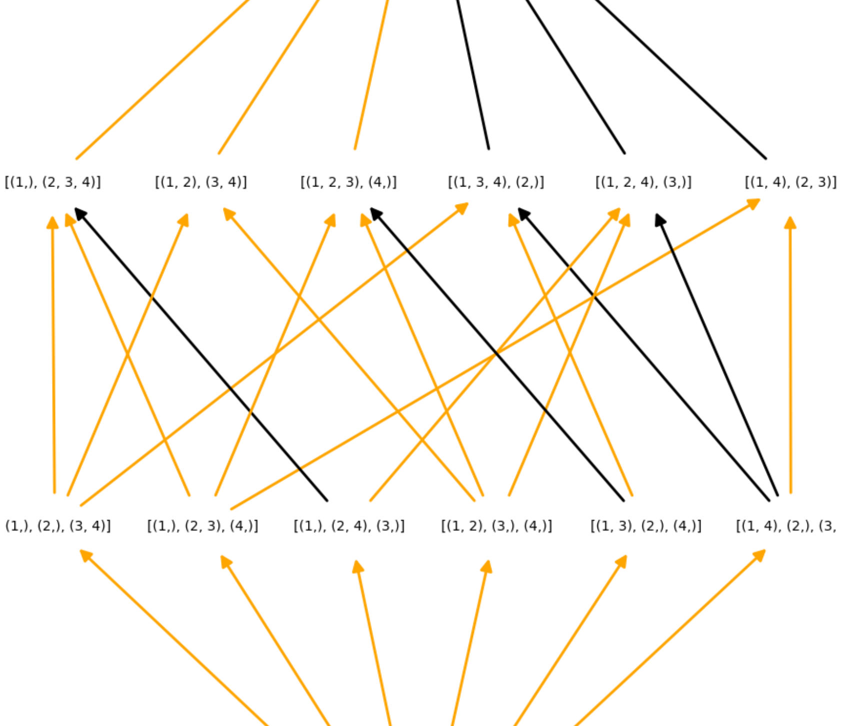
- we denote $\pi \ll\!\!\cdot\ \rho$ if $\max B, \min B$ both in B_1, or B_2.
- $\pi \sqsubset\!\!\cdot\ \rho$ otherwise.

Their transitive closure are two order relations denoted $\sqsubset$ and $\ll$.

Color code: $\sqsubset\!\!\cdot$, $\ll\!\!\cdot$

More direct characterization:
$\pi \sqsubset \rho$ if all "arcs" of π are also "arcs" of π.




[(1,), (2, 3, 4)]
[(1, 2), (3, 4)]
[(1, 2, 3), (4,)]
[(1, 3, 4), (2,)]
[(1, 2, 4), (3,)]
[(1, 4), (2, 3)]
(1,), (2,), (3, 4)]
[(1,), (2, 3), (4,)]
[(1,), (2, 4), (3,)]
[(1, 2), (3,), (4,)]
[(1, 3), (2,), (4,)]
[(1, 4), (2,), (3,

The Belinschi-Nica order

- $\sqsubset$ was introduced by the speaker, in the context of noncrossing partitions associated to a finite Coxeter group W, to give a refined enumeration of maximal chains. It also gives a more general refinement of minimal cycle factorizations of permutations (joint work with P. Biane).
- $\ll$ was introduced by Belinschi and Nica in the context of noncommutative probability theories, more precisely, interactions between free and boolean probability. Also introduced by Senato and Petrullo to give formulas for Kerov polynomials.

Theorem (Nica, Senato-Petrullo)

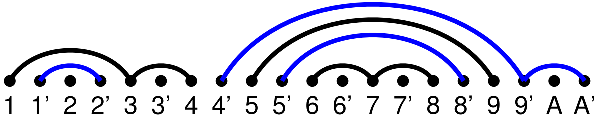
Every upper ideal $\{\tau \in \mathrm{NC}_n \mid \tau \gg \sigma\}$ *is boolean.*
The number of intervals in $(\mathrm{NC}_n, \ll)$ *(or pairs* $\alpha \ll \beta$*) is* S_n *with*

$$\sum_{n\geq 0} S_n z^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \qquad \textit{(Schröder numbers).}$$

On the other side, it is easy to see that lower ideals $\{\tau \in \mathrm{NC}_n \mid \tau \sqsubset \sigma\}$ are boolean. For example, $\{\tau \in \mathrm{NC}_n \mid \tau \sqsubset 123\ldots n\}$ contains exactly the interval partitions.

Kreweras complement

The Kreweras complement of a noncrossing partition π :
put a i' between i and $i+1$ (and n' at the far right).
$K(\pi)$ is the coarsest noncrossing partitions of $1', \ldots, n'$
that can be drawn without crossing arcs of π.



$$K(134|2|59|678|A) = 12|3|49A|58|6|7.$$

$\pi \mapsto K(\pi)$ is a poset anti-automorphism: $u \leq v$ iff $K(v) \leq K(u)$.

Kreweras complement

Proposition

If $\pi \lessdot \rho$ and π is <u>full</u> (it is not below an interval partition other than the maximal one), we have:

$$\pi \ll\!\cdot\ \rho \quad \textit{if and only if} \quad K(\rho) \sqsubset\!\!\cdot\ K(\pi).$$

So K gives an anti-isomorphism of poset:

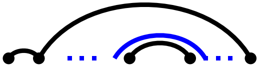
$$\{\alpha \in \mathrm{NC}_n \ : \ \alpha \gg \pi\} \longrightarrow \{\alpha \in \mathrm{NC}_n \ : \ \alpha \sqsubset K(\pi)\},$$

and they are both boolean.

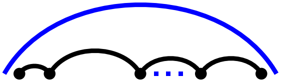
Proof

If $\pi \ll \rho$:

$\rho =$ 

$\pi =$ 

If $\pi \sqsubset \rho$ (the blue arc exists because π is full) :

$\rho =$ 

$\pi =$ 

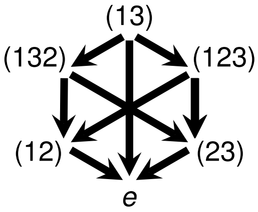
Orienting the Cayley graph

Definition
The *Cayley graph* has vertex set $\mathfrak{S}_n$,
and there is an edge $\sigma - \tau$ if $\sigma = \tau(ij)$.

Remark
Transpositions are a generating set, so the graph is connected.
They are conjugation-closed so one might as well take $\sigma = (ij)\tau$.

Definition
To get the *Bruhat graph*, orient the edges:
$\sigma \to \tau$ if σ has more inversions than τ.

The *Bruhat order* is its transitive closure.

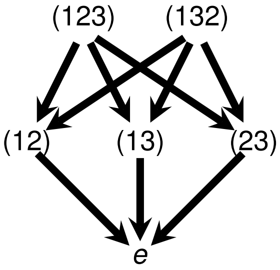


Orienting the Cayley graph

Definition

Another graph: orient the edges
$\sigma \to \tau$ if σ has less cycles than τ.

The transitive closure of this graph
is the *absolute order*.



Remark

This graph is the Hasse diagram of the absolute order.

Theorem (Biane)

There is an isomorphism between the interval $[e, (1, 2, 3, \ldots, n)]$ *in the absolute order, and the poset of noncrossing partitions.*

Orienting the Cayley graph

Theorem (Biane)

There is an isomorphism between the interval $[e, (1, 2, 3, \dots, n)]$ *in the absolute order and the noncrossing partitions poset.*

Proof

Send a permutation to its cycle decomposition.
c gives the maximal partition. Then:

$$(1, 2, 3, \dots, n)(i, j) = (1, 2, 3, \dots, i, j + 1, \dots, n)(i + 1, \dots, j)$$

gives a two-block noncrossing partition.

Then go on inductively:
if the cycle decomposition of $\sigma \leq_{\text{abs}} c$ gives a noncrossing partition, the same is true for $\sigma(ij)$ if it has one more cycle.

Orienting the Cayley graph

For each edge $\sigma - \tau$ in the Cayley graph, the endpoints are comparable for <u>both</u> the Bruhat order and the absolute order. Define:

- $\sigma \mathrel{\sqsubset\!\!\cdot}\ \tau$ if and only if $\sigma \lessdot_{\text{abs}} \tau$ and $\sigma \leq_{\text{Bruhat}} \tau$,
- $\sigma \mathrel{\ll\!\!\cdot}\ \tau$ if and only if $\sigma \lessdot_{\text{abs}} \tau$ and $\sigma \geq_{\text{Bruhat}} \tau$,

and $\sqsubset$, $\ll$ their transitive closures.

Theorem

Restricted to the interval $[e, (1, 2, 3, \dots, n)]$, *this coincides with the previous definition on noncrossing partitions.*

Remark

It might happen that $\sigma \not\sqsubset \tau$, and still $\sigma \leq_{\text{abs}} \tau$, $\sigma \leq_{\text{Bruhat}} \tau$. Smallest example :

$$\sigma = (2, 4), \qquad \tau = (1, 5)(2, 3, 4).$$

Orienting the Cayley graph

Theorem
Restricted to the interval $[e, (1, 2, 3, \ldots, n)]$*, this coincides with the previous definition on noncrossing partitions.*

Proof.
Let $\tau = (123\ldots n) = s_1 \cdots s_{n-1}$ with $s_i = (i, i+1)$.
If $\sigma \lessdot_{\text{abs}} \tau$ and $\sigma \leq_{\text{Bruhat}} \tau$, it is obtained from $s_1 \cdots s_{n-1}$ by removing a factor and you get a two block interval partition.

If τ is not the maximal element, use the same argument in a parabolic subgroup (where τ <u>is</u> the maximal element, and the Bruhat graph is the induced subgraph).

The cluster complex

The cluster complex

Computing both sides in terms of Narayana numbers shows that

$$\sum_{\sigma \ll \tau} (x+1)^{\mathrm{rank}(\sigma)} y^{\mathrm{rank}(\tau)} = \sum_{\sigma \leq \tau} \mu(\sigma,\tau)(-x)^{\mathrm{rank}(\sigma)}(-y)^{\mathrm{rank}(\sigma)}$$

where μ is the Möbius function of noncrossing partitions.

The right hand side is related to the enumeration of faces in the cluster complex by Chapoton's $F = M$ identity (proved by Athanasiadis).

A consequence of the identity: there are as many

- pairs of noncrossing partitions $\sigma \ll \tau$, with $\mathrm{rank}(\tau) = k$,
- positive faces of cardinality k in the cluster complex.

The cluster complex

Cluster complex:

- Introduced by [Fomin-Zelevinski] in the crystallographic case. Related with cluster algebras and generalized associahedra.
- Extended by [Fomin-Reading] to the noncrystallographic case.
- Enumerative properties: Chapoton's $F = M = H$ identities.
- Alternative definition by [Brady-Watt] using the absolute order.
- Extented to non-bipartite Coxeter elements by [Reading].

The (positive part of the) cluster complex

Let c be the long cycle $(1, 2, \ldots, n)$.
(We could take the product of $(1, 2), \ldots, (n-1, n)$ in any order.)

Definition

A *positive c-cluster* is a factorization $c = t_1 \cdots t_{n-1}$
where t_i are transpositions, and:

- t_i and t_j are compatible in the sense that:
  either $t_i t_j = t_j t_i$,
  or they are $(ij), (ik)$ or $(ik), (jk)$ with $i < j < k$.
  (so the forbidden pairs are $(ij), (jk)$ with $i < j < k$)
- Two factorizations are considered as equal
  if they only differ by commutations.

Example

$$\begin{aligned}(1,2,3,4) &= (1,4)(1,3)(1,2) = (3,4)(2,4)(1,4)\\ &= (3,4)(1,4)(1,2) = (2,4)(2,3)(1,4)\\ &= (1,4)(2,3)(1,3)\end{aligned}$$

The (positive part of the) cluster complex

The number of c-clusters is the $(n-1)$st Catalan number.

When $c = (1, 2, \ldots, n)$, a Catalan decomposition is as follows: the c-cluster must contain $(1, n)$, then separate what is on the left and on the right.

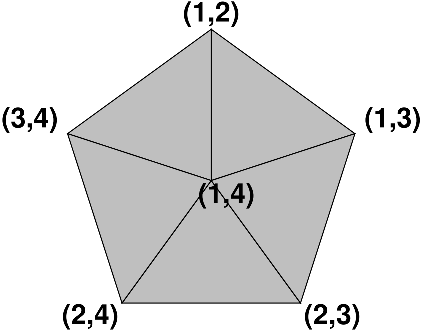
Definition

The *positive faces of the cluster complex* are the factorizations that are subwords of some c-cluster. (Equivalently, they are factorizations of some $w \leq_{\text{abs}} c$ with pairwise compatible factors.)

They are partially ordered by inclusion, and are counted by big Schröder numbers (2,6,22,90,...)

The (positive part of the) cluster complex

$$\begin{aligned}(1,2,3,4) &= (1,4)(1,3)(1,2) = (3,4)(2,4)(1,4)\\ &= (3,4)(1,4)(1,2) = (2,4)(2,3)(1,4)\\ &= (1,4)(2,3)(1,3)\end{aligned}$$



5 triangles + 10 edges + 6 vertices + 1 empty face = 22 faces

The (positive part of the) cluster complex

Remark

The (full) cluster complex is the disjoint union of positive parts of the 2^n complexes associated with standard parabolic subgroups (= Young subgroups).

Topologically, they glue nicely to form a sphere.

As for enumeration, we can just focus on the positive part.
In particular, having a bijection: clusters $\to$ noncrossing partitions, is equivalent to having a bijection: positive clusters $\to$ full noncrossing partitions.

From clusters to noncrossing partitions

Each positive c-cluster $c = t_1 \cdots t_{n-1}$ gives
(nondeterministically, because of possible commutations)
a path in the absolute order: with $u_i = t_1 \cdots t_i$,

$$e = u_0 \lessdot_{\mathrm{abs}} u_1 \lessdot_{\mathrm{abs}} \cdots \lessdot_{\mathrm{abs}} u_{n-1} = c.$$

Theorem

Up to commutations among $t_1, \ldots, t_{n-1}$,
we can arrange the factorization so that:

$$e = u_0 \sqsubset\!\!\cdot \ \ldots \sqsubset\!\!\cdot \ u_k \lll\!\cdot \ \ldots \lll\!\cdot \ u_{n-1} = c.$$

Then $t_1 \ldots t_{n-1} \mapsto u_k$ is a bijection from positive c-clusters to noncrossing partitions w such that $w \ll c$.

We recover a bijection of
[Athanasiadis, Brady, MacCammond, Watt].

From clusters to noncrossing partitions

The inverse bijection: write the noncrossing partition w as a product of the simple generators of its parabolic subgroup

$$w = \alpha_1 \cdots \alpha_k.$$

(It means that the forbidden pairs among the α_i are $(i,j), (i,k)$ and $(i,k)(j,k)$ with $i < j < k$).
Note: it is far from obvious that we can do that in general, but follows from results by [Reading], [Brady-Watt].
Do the same for the <u>Kreweras complement</u>:

$$w^{-1}c = \beta_1 \cdots \beta_{n-1-k}.$$

Then the positive cluster is:

$$\left\{\alpha_k \cdots \alpha_i \cdots \alpha_k \ : \ 1 \leq i \leq k\right\} \bigcup \left\{\beta_1 \cdots \beta_i \cdots \beta_1 \ : \ 1 \leq i \leq n-1-k\right\}.$$

From clusters to noncrossing partitions

In general, a positive face of the cluster complex
is given by a factorization $w = t_1 \cdots t_k$ where $w \leq_{\mathrm{abs}} c$.

Theorem

Up to commutations among $t_1, \ldots, t_k$,
we can arrange the factorization so that (with $u_i = t_1 \cdots t_i$):

$$e = u_0 \sqsubset\!\!\cdot \;\ldots \sqsubset\!\!\cdot\; u_j \ll\!\!\cdot \;\ldots \ll\!\!\cdot\; u_k .$$

Then $t_1 \ldots t_k \mapsto (u_j, u_k)$ is a bijection from positive faces
to pair of noncrossing partitions (v, w) such that $v \ll w$.

From clusters to noncrossing partitions

The bijection from positive faces to intervals is an immediate extension of the bijection between positive clusters and noncrossing partitions.

To be able do this, we need to show that the complex behaves well with respect to (possibly non-standard) parabolic subgroups. This is done by extending some results of Brady-Watt to Reading's c-clusters.

thanks for your attention