Continuous TASEP on a ring

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Outline of talk

I will discuss a probability distribution Ξ of labeled particles on a ring that can be described in several ways and seems to have interesting combinatorial properties.

- Process of last row, the simplest definition

- Other definition, multiline queues

- Density function for a fixed permutation

- Correlations

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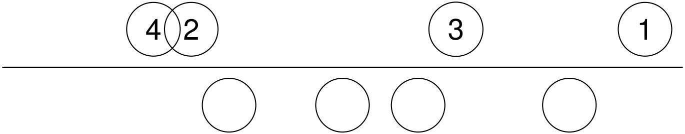

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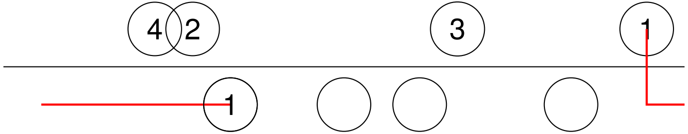

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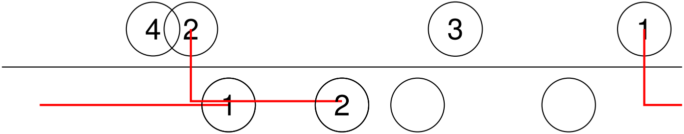

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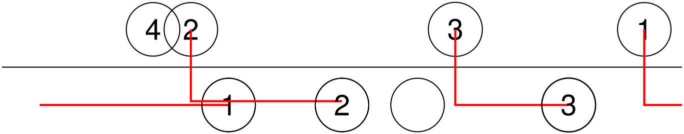

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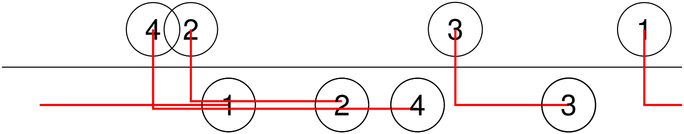

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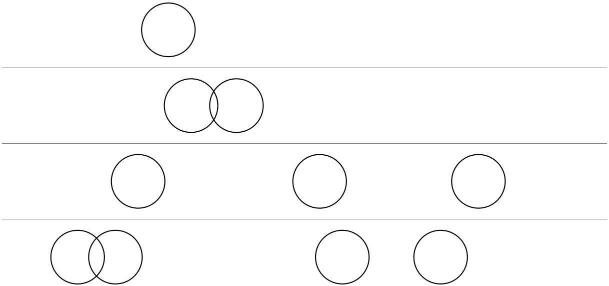
Note that the positions are uniformly distributed by definition. The difficulty is with the labelling.

Continuous multi-line queues (mlq)

One other way to define Ξ is the following.

A continuous multi-line queue (for permutations) has n (continuous) rows and on row i we choose uniformly at random i positions.



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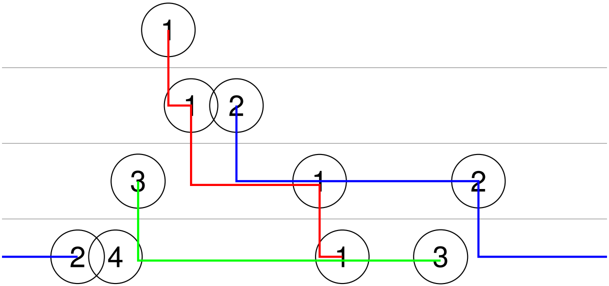

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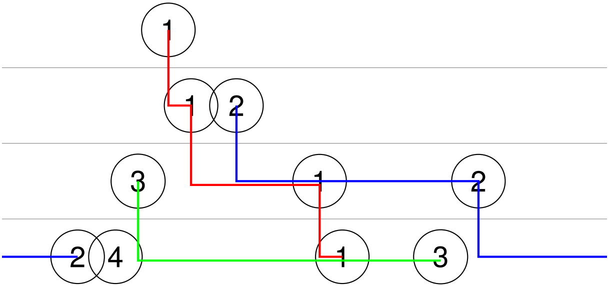

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From a paper by Ferrari-Martin it follows that Ξ_n is the distribution of the labels on the bottom row for a uniformly chosen continuous multiline queue.

Density functions

Let $q_1, \ldots, q_n$ be the positions of the labeled particles.
Restricting Ξ to a permutation π the density functions $g_\pi(q_1, \ldots, q_n)$ for the positions of the particles behave nicely for some π.

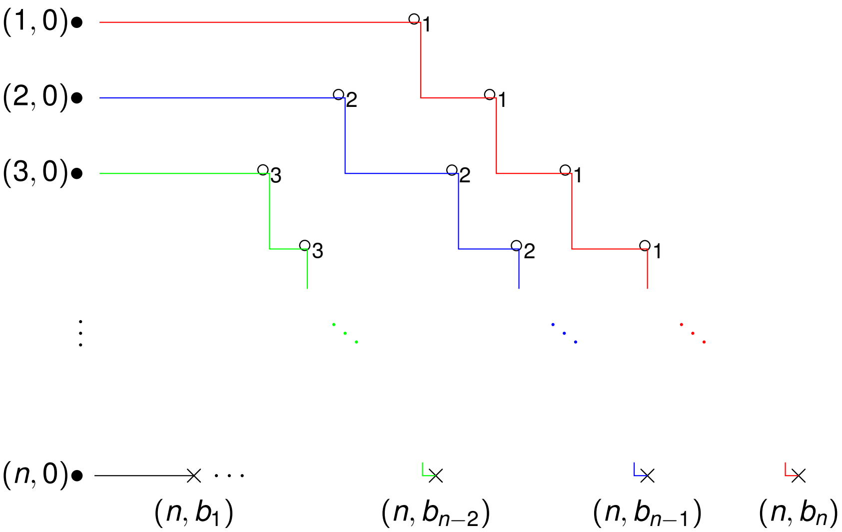
Theorem (Aas-L.)

For any $n \geq 2$ and $0 \leq q_1 < \cdots < q_n < 1$ we have

$$g_{w_0}(q_1, \ldots, q_n) = n! \prod_{1 \leq k < l \leq n} (q_l - q_k).$$

Proof uses that the continuous mlq:s are limits of discrete mlq:s with the number of empty positions going to infinity. For the discrete case we can use the determinantal formula.

Density functions



Density functions

Conjecture (Aas-L.)

For any $n > k \geq 1$ and any $0 \leq q_1 < \cdots < q_n < 1$ we have

$$g_{s_k w_0} = \left(\frac{1}{k!} \frac{\partial^k}{\partial q_{n-k+1} \ldots \partial q_n} - 1 \right) g_{w_0}.$$

A similar conjecture for many commuting s_k:s.

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NEWSFLASH: Proof of conjecture claimed by Aas-Grinberg-Scrimshaw, to be submitted to FPSAC.

Density functions

Examples.

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Does not hold for all longer permutations.

Question: Is there a better operator than the derivative?

Probabilities

Let p_π be the probability that the particles form a certain permutation π.

Theorem

The probability that the particles form the reverse permutation w_0 is

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We have no formula for general π. For $n = 2, 3$ we have

π	12	21	123	231	312	132	213	321
p_π	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{25}{60}$	$\frac{13}{60}$	$\frac{10}{60}$	$\frac{5}{60}$	$\frac{5}{60}$	$\frac{2}{60}$

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Correlations

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Conjecture (Aas-L.)

For $n \geq 2$, we have the following two-point correlations at stationarity

$$c_{i,j}(n) = \begin{cases} \frac{n}{\binom{n+j}{2}}, & \text{if } i+1 < j \leq n, \\ \frac{n}{\binom{n+j}{2}} + \frac{ni}{\binom{n+i}{2}}, & \text{if } i+1 = j \leq n, \\ \frac{n}{\binom{n+j}{2}} - \frac{n}{\binom{n+i}{2}}, & \text{if } j < i < n, \\ \frac{n(j+1)}{\binom{n+j}{2}} - \frac{n(j-1)}{\binom{n+j-1}{2}} - \frac{n}{\binom{2n}{2}}, & \text{if } j < i = n. \end{cases}$$

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It would be very interesting if anyone could shed some light on this.