# The Alternating Group and Noncrossing Partitions

PHILIPPE NADEAU (CNRS & Université Lyon 1)
Joint work with HENRI MÜHLE (TU Dresden)

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# Permutations and generators

Let $\mathfrak{S}_n$ be the set of permutations of $\{1, \dots, n\}$.

$\mathfrak{S}_n$ has many standard sets of generators : elementary transpositions $(i, i+1)$, star transpositions $(1, i)$, the set of transpositions $C_2 = \{(i,j), 1 \leq i < j \leq n\}$, the $2$-element set $\{(1,2), (1, \dots, n)\}$,. . .

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More generally, for $k \geq 2$ we can consider the set $C_k$ of $k$-cycles in $\mathfrak{S}_n$. These are closed under conjugation.

**Remark** If $k$ is even, $C_k$ generates $\mathfrak{S}_n$ while if $k$ is odd, $C_k$ generates the alternating group $\mathfrak{A}_n$.

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In this talk we will be interested in $\mathfrak{A}_n$ with the set $C_3$ of $3$-cycles as generators.

We recall that $x \in \mathfrak{A}_n$ iff $x$ has an even number of even cycles in its cycle decomposition.

# Generated Group

Let $G$ be a group with a given set of generators $\mathcal{T}$. The pair $(G, \mathcal{T})$ forms a generated group.

**Length**: given $g \in G$, define $\ell_{\mathcal{T}}(g)$ to be the smallest $k$ such that $g = t_1 \cdots t_k$ for certain $t_i \in \mathcal{T}$.

**Order**: given $g, h \in G$, write $g \leq_{\mathcal{T}} h$ if

$$\ell_{\mathcal{T}}(g) + \ell_{\mathcal{T}}(g^{-1}h) = \ell_{\mathcal{T}}(h).$$

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$\rightarrow$ These notions have a natural interpretation inside the (right) Cayley graph of $(G, \mathcal{T})$: the length of $x$ is its distance to the identity $e$, while $x$ is smaller than $y$ if it lies one a geodesic from $e$ to $y$.

For each generated group we get a graded poset $(G, \leq_{\mathcal{T}})$.

# In this talk

We will study the combinatorics of the poset $(\mathfrak{A}_n, \leq_{C_3})$ :

- Rank function
- Covering relations
- Interval structure
- Enumeration of chains
- ....

We will compare our results with the well-known corresponding ones for $(\mathfrak{S}_n, C_2)$, which is where noncrossing partitions occur naturally.

We use the subscripts $\mathbf{2}, \mathbf{3}$ instead of $C_2, C_3$, so that our object of study is $(\mathfrak{A}_n, \leq_{\mathbf{3}})$.

# Rank function

Given $x \in \mathfrak{S}_n$, one has $\ell_{\mathbf{2}}(x) = n - \mathrm{cyc}(x)$ where $\mathrm{cyc}(x)$ denotes the number of cycles of $x$ (including fixed points).

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To prove this well-known result, notice that multiplying by a transposition either cuts a cycle or joins two cycles.

**Theorem**[Herzog-Reid '76][Mühle-N] For any $x \in \mathfrak{A}_n$,

$$\ell_{\mathbf{3}}(x) = \frac{n - \mathrm{oddcyc}(x)}{2}.$$

The proof follows the same line as for $\ell_{\mathbf{2}}$. The key is to figure out the effect on $\mathrm{oddcyc}$ of the multiplication by a $3$-cycle (remark that $(ijk) = (ij)(jk)$.)

# Cover relations

We write $x \lessdot y$ if $x < y$ and there is no $z$ such that $x < z < y$ ($\Leftrightarrow \ell(y) = \ell(x) + 1$ in a graded poset).

One has $x \lessdot_{\mathbf{2}} y$ in $\mathfrak{S}_n$ if and only if $x$ is obtained by cutting a cycle of $y$ in two.

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The case of $\leq_{\mathbf{3}}$ is more involved:

**Proposition**[Mühle-N] $x \lessdot_{\mathbf{3}} y$ in $\mathfrak{A}_n$ if and only if $x$ is obtained by one of the following operations on $y$:

1. Cut an odd cycle of $y$ into three odd cycles.
2. Cut an even cycle of $y$ into two odd cycles and one even cycle.
3. Cut an even cycle of $y$ into two odd cycles, and join one of these with another even cycle of $y$.

# Decomposition of intervals

It is easily shown that when $\mathcal{T}$ is closed under conjugation, $[x, y]_{\mathcal{T}}$ is isomorphic to $[e, x^{-1}y]_{\mathcal{T}}$, so we can focus on intervals of this form.

If $\sigma$ has cycles $(\zeta_i)_i$, then one has a simple decomposition

$$[e, \sigma]_{\mathbf{2}} \simeq \prod_i [e, \zeta_i]_{\mathbf{2}}$$

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**Proposition**[Mühle-N] Let $x \in \mathfrak{A}_N$. Let $(\zeta_i)_i$ be the odd cycles of $x$ and let $\xi$ be the product of its even cycles. Then

$$[e, x]_{\mathbf{3}} \simeq \prod_i [e, \zeta_i]_{\mathbf{3}} \times [e, \xi]_{\mathbf{3}}$$

# Odd cycles and Noncrossing Partitions

Assume $y$ only has odd cycles. Then the cover relations imply that any permutation $x \leq_{\mathbf{3}} y$ only has odd cycles also.

It follows that each such interval $[x, y]_{\mathbf{3}}$ is isomorphic to $\prod_i \mathcal{ENC}_{2k_i+1}$ where $2k_i + 1$ are the cycle sizes of $x^{-1}y$ and

$$\mathcal{ENC}_{2k+1} := [e, (1, \ldots, 2k+1)]_{\mathbf{3}}.$$

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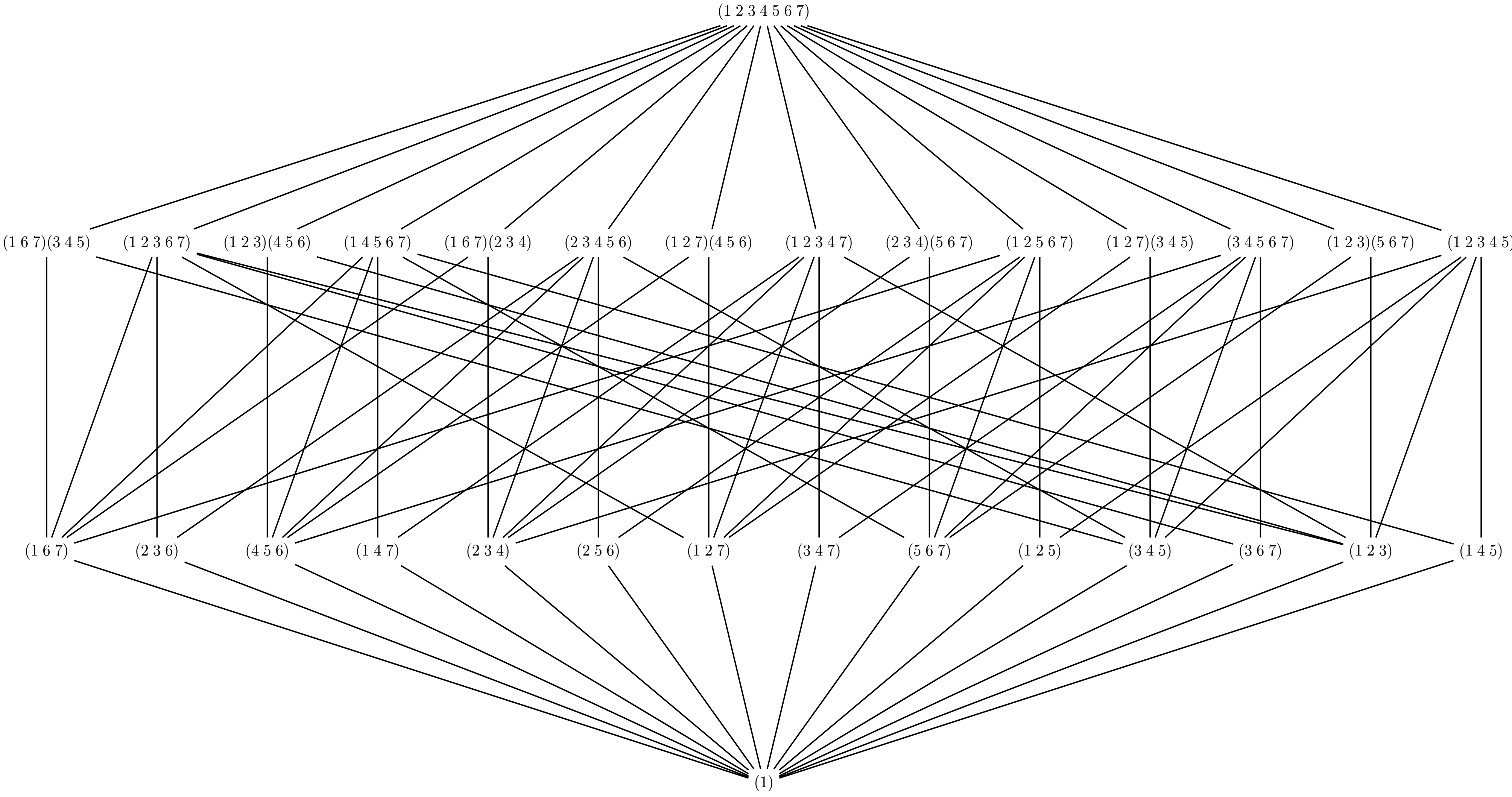
**Theorem**[Mühle-N] $x \in \mathcal{ENC}_{2n+1}$ if and only if $x \in \mathcal{NC}_{2n+1}$ and each cycle $(a_1 < \cdots < a_{2p+1})$ of $x$ satisfies that $a_{i+1} - a_i$ is odd for all $i < 2p + 1$.
This realizes $\mathcal{ENC}_{2n+1}$ as an (induced) subposet of $\mathcal{NC}_{2n+1}$.

# The poset $\mathcal{ENC}7 = [e, (1234567)]_{\mathbf{3}}$




It is not a lattice, contrary to $\mathcal{NC}_n$.

# Enumeration

The zeta polynomial $Z(\mathcal{P}, q)$ of a finite poset $\mathcal{P}$ is the poynomial in $q$ such that $Z(\mathcal{P}, m)$ is the number of chains of $\mathcal{P}$ with $m-1$ elements $x_1 \leq x_2 \leq \cdots \leq x_{m-1}$.

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**Theorem**[Mühle-N] For $n \geq 1$,

$$Z\big(\mathcal{ENC}_{2n+1}, q\big) = \frac{q}{(2n+1)q-n}\binom{(2n+1)q-n}{n}.$$

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**Corollary** $\mathcal{ENC}_{2n+1}$ has cardinality $\frac{1}{n+1}\binom{3n+1}{n}$, Möbius number $(-1)^n \frac{1}{4n+1}\binom{4n+1}{n}$ and number of maximal chains equal to $(2n+1)^{n-1}$.

**Remark** Bijective proofs exist for some of these results.

# Hurwitz action

Consider a generated group $(G, \mathcal{T})$ and $g \in G$.
The expressions $t_1 \ldots t_k = g$ with $k = \ell_{\mathcal{T}}(g)$ form the set $\mathrm{Red}_{\mathcal{T}}(g)$ of reduced expressions of $g$ ($\Leftrightarrow$ the set of maximal chains in $[e, g]_{\mathcal{T}}$).

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Assume now $\mathcal{T}$ conjugation-invariant. Then for $i < k$ one can define a bijection $\sigma_i$ of $\mathrm{Red}_{\mathcal{T}}(g)$ by

$$\sigma_i \bullet t_1 \cdots t_i t_{i+1} \cdots t_k = t_1 \cdots t_{i+1} (t_{i+1}^{-1} t_i t_{i+1}) \cdots t_k$$

The $\sigma_i$ actually define an action of the braid group $B_k$ on $\mathrm{Red}_{\mathcal{T}}(g)$, the Hurwitz action.

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The $\sigma_i$ actually define an action of the braid group $B_k$ on $\mathrm{Red}_{\mathcal{T}}(g)$, the Hurwitz action.

**Remark** In the case of the factorization of permutations, the orbits of this action correspond to flexible equivalence classes of the corresponding coverings of the Riemann sphere, cf. [Lando-Zvonkin '04].

# Hurwitz action

**Theorem** Let $x \in \mathfrak{A}_N$ have $2k$ even cycles. The Hurwitz action on $\mathsf{Red}_{\mathbf{3}}(x)$ has $(2k)_k = (k+1)(k+2)\cdots(2k)$ orbits.

The important cases are $k = 0$ or $1$, which actually correspond to the case of transitive factorizations ($=$ connected coverings).

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**Example** $(12345)$ has 5 factorizations forming a single orbit

$$\{(1\ 2\ 3)(3\ 4\ 5), (3\ 4\ 5)(1\ 2\ 5), (1\ 2\ 5)(2\ 3\ 4), (2\ 3\ 4)(1\ 4\ 5), (1\ 4\ 5)(1\ 2\ 3)\}$$

while $(12)(34)$ has 8 factorizations falling into $2$ Hurwitz orbits

$\{$(1 2 3)(2 3 4),(2 3 4)(2 1 4),(2 1 4)(1 4 3), (1 4 3)(1 2 3)$\}$,
$\{$(1 2 4)(2 4 3),(2 4 3)(2 1 3),(2 1 3)(1 3 4), (1 3 4)(1 2 4)$\}$.

# Generalizations

- Extension to nonnesting partitions, $m$-divisible noncrossing partitions.
- Generation by $k$-cycles.
- Extension to other Coxeter groups.