A (lattice) path formula for birational rowmotion on a product of two chains

Tom Roby (UConn)

Describing joint research with
Gregg Musiker (University of Minnesota)

Workshop on Computer Algebra in Combinatorics
Programme on Algorithmic and Enumerative Combinatorics
Erwin Schödinger Institute for Mathematics and Physics
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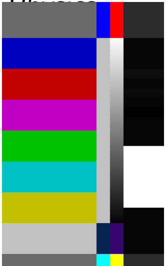

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Abstract:

We give a formula in terms of families of non-intersecting lattice paths for iterated actions of the birational rowmotion map on a product of two chains. Birational rowmotion is an action on the space of assignments of rational functions to the elements of a poset. It is lifted from the well-studied *rowmotion* (aka "Panyushev Complementation") map on order ideals (equivariantly on antichains) of a partially ordered set P, which when iterated on special posets has unexpectedly nice properties in terms of periodicity, cyclic sieving, and homomesy (constant averages for each orbit). Darij Grinberg has contributed an implementation of this map to SageMath, which the authors found invaluable for numerical experiments and making conjectures.

Acknowledgments

This seminar talk discusses joint work with Gregg Musiker (UMN), as well as some earlier joint work with Darij Grinberg (UMN) and James Propp (UML). We are particularly grateful to Darij for coding birational rowmotion in SageMath. We also acknowledge the hospitality of the American Institute of Mathematics (San Jose, CA USA), where this collaboration began at a workshop on Dynamical Algebraic Combinatorics in 2015.

Please feel free to interrupt with questions or comments.

Outline & Setting

1. Classical (combinatorial) rowmotion;
2. Birational rowmotion;
3. Lattice Path Formula for birational rowmotion on rectangular posets;
4. Consequences: periodicity, reciprocity, and homomesy.
5. Key Lemma: Colorful bijections on pairs of families of NILPs

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [2].
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [1, EiPr13, EiPr14], though that is not our focus here.
- Periodicity of these systems is generally nontrivial to prove.

Classical rowmotion is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

Let P be a finite poset. **Classical rowmotion** is the map
$\mathbf{r} : J(P) \longrightarrow J(P)$

sending every order ideal S to a new order ideal $\mathbf{r}(S)$ generated by the minimal elements of $P \setminus S$.

Example: Let S be the following order ideal

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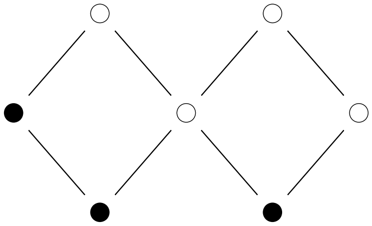

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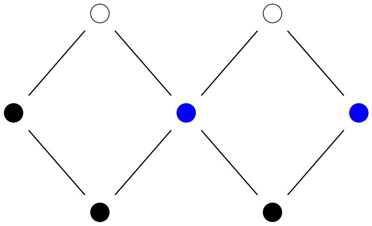
Mark M (the minimal elements of the complement) in blue.



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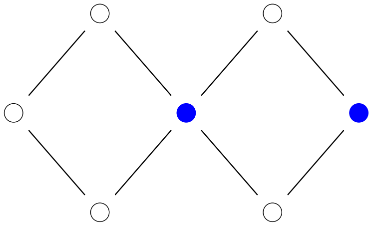
Remove the old order ideal:



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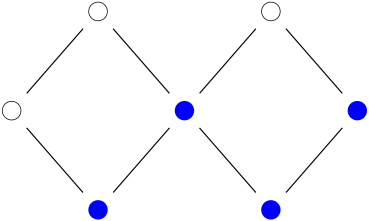
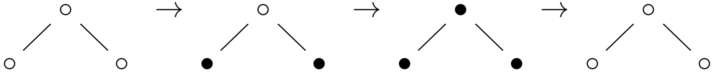
Example: Let S be the following order ideal

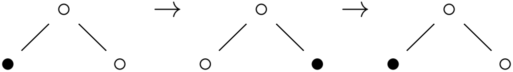
$\mathbf{r}(S)$ is the order ideal generated by M ("everything below M"):



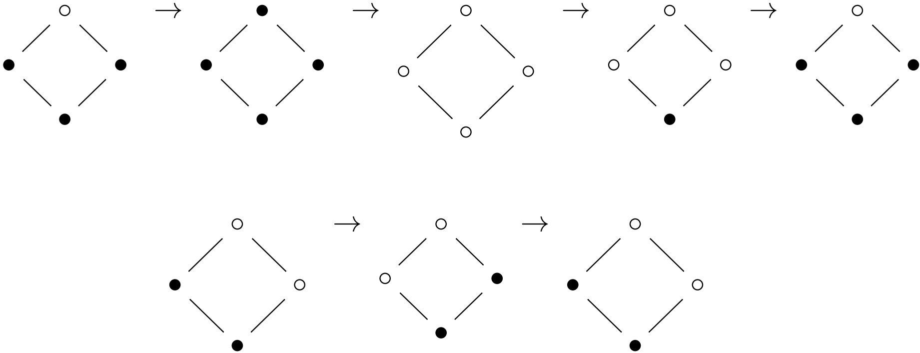
Earlier Examples Revisited

We can think of these orbits also as a dynamic on order ideals.





Rowmotion orbits



Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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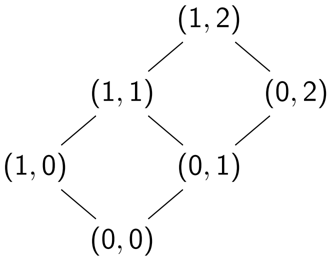
However, **for some types of** P, the order can be explicitly computed or bounded from above.

See Striker-Williams for an exposition of known results.

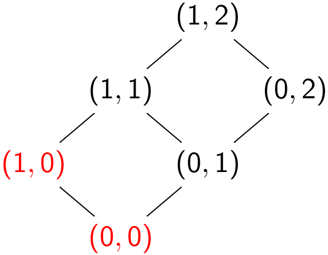
- If P is a $p \times q$-rectangle:



(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

Example:

Let S be the order ideal of the 2×3-rectangle $[0,1] \times [0,2]$ given by:



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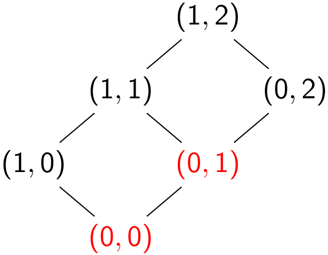

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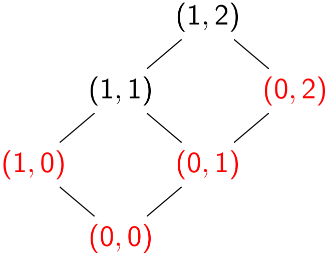
$\mathbf{r}^2(S)$ is



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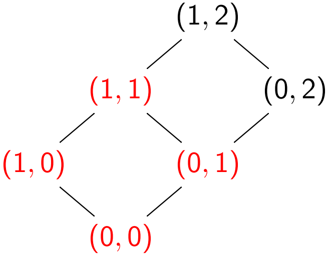

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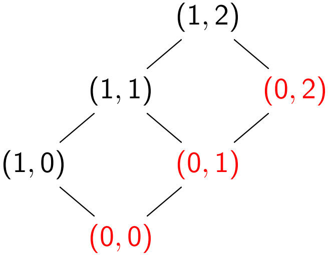
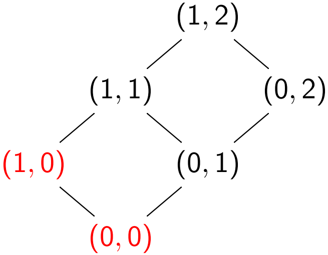


Classical rowmotion: properties

Example:
$\mathbf{r}^5(S)$ is



which is precisely the S we started with.

$\mathrm{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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Rowmotion: the toggling definitions

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- More formally, if P is a poset and $v \in P$, then the v-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal S to:
  - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
  - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S;
  - S otherwise.
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.

Classical rowmotion: the toggling definition

- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

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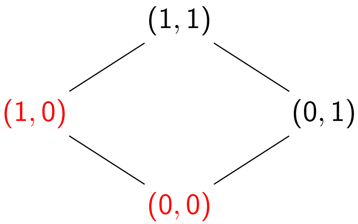

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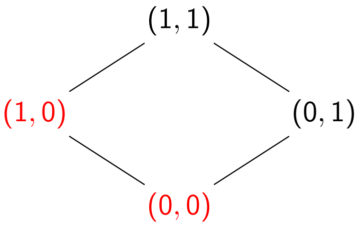

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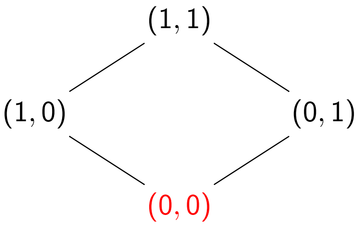

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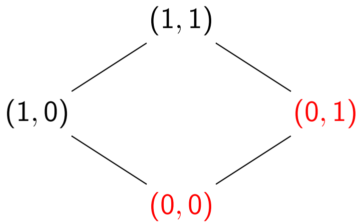

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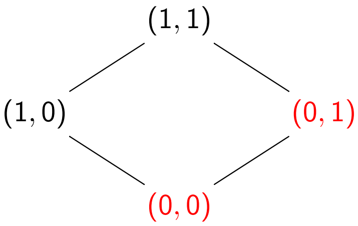

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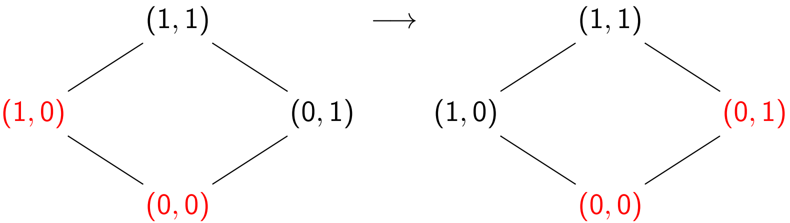

Generalizing to the piece-wise linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\widehat{0}) = 0$, $f(\widehat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

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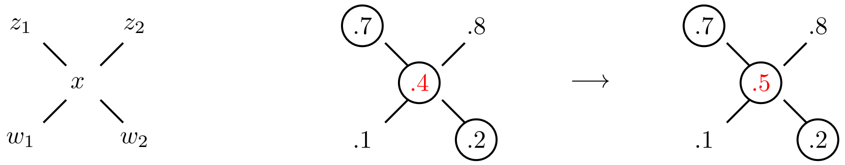
Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

if $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \text{ to } \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints.

Example of flipping at a node

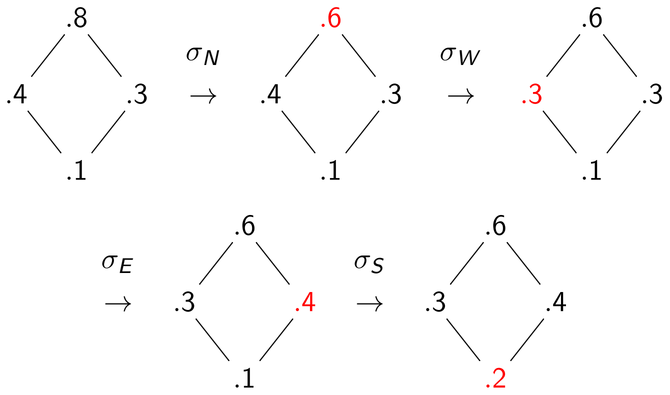


$$\min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

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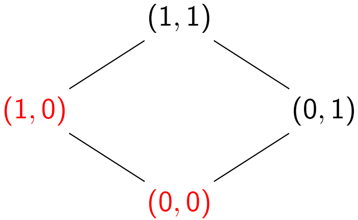

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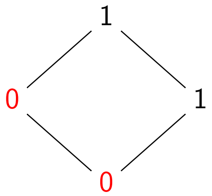
Example:

Translated to the PL setting:



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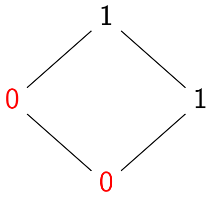

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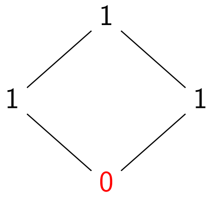

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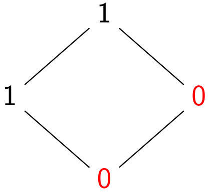

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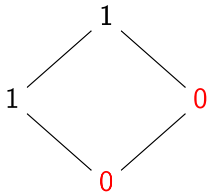

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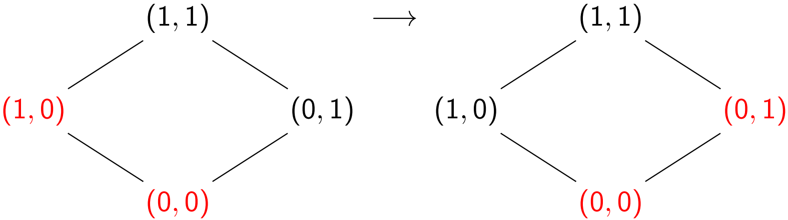
Example:

So this is $S \longrightarrow \mathbf{r}(S)$:



De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get

$$f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

- The rowmotion map $\mathbf{r}$ is a map of 0-1 labelings of P. It has a natural generalization to labelings of P by real numbers in $[0, 1]$, i.e., the *order polytope* of P. Toggles get replaced by piecewise-linear toggling operations that involve max, min, and $+$.
- *Detropicalizing* these toggles leads to the definition below of birational toggling. Results at the birational level imply those at the order polytope and combinatorial level.
- This is originally due to Einstein and Propp [EiPr13, EiPr14]. Another exposition of these ideas can be found in [Rob16], from the IMA volume *Recent Trends in Combinatorics*.

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A $\mathbb{K}$-**labelling of** P will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

Birational rowmotion

- For any $v \in P$, define the **birational** v**-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

  for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at v,
  - **multiply** by the **sum** of the labels at vertices **covered by** v,
  - **multiply** by the **parallel sum** of the labels at vertices **covering** v.

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for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at v; all other labels stay the same.
- We have $T_v^2 = \mathrm{id}$ (on the range of T_v), and T_v is a birational map.

- We define **birational rowmotion** as the rational map

$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because:

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- This is indeed independent of the linear extension, because:
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

poset	labelling

$$\begin{tikzcd}[column sep=small] & \widehat{1} \arrow[d, dash] & \\ & (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (1,0) \arrow[dr, dash] & & (0,1) \arrow[dl, dash] \\ & (0,0) \arrow[d, dash] & \\ & \widehat{0} & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & 1 & \end{tikzcd}$$

Example:

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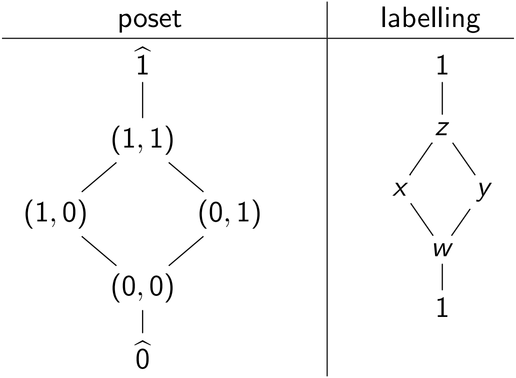

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension

$((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

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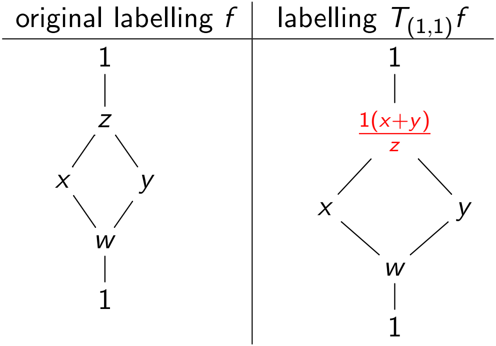

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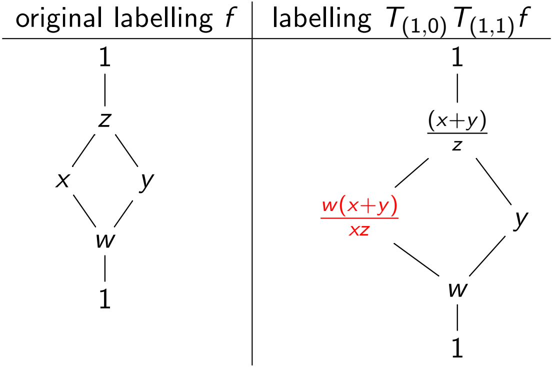

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Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,1)} T_{(1,0)} T_{(1,1)} f$

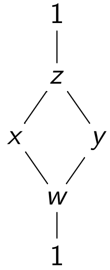
$$\begin{tikzcd}[column sep=small]
& 1 \arrow[d, dash] & \\
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 1 \arrow[d, dash] & \\
& \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{w(x+y)}{xz} \arrow[dr, dash] & & {\color{red}\frac{w(x+y)}{yz}} \arrow[dl, dash] \\
& w \arrow[d, dash] & \\
& 1 &
\end{tikzcd}$$

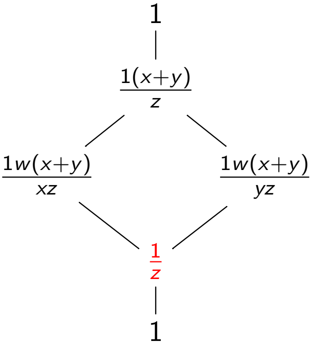
We are using $\rho_B = T_{(0,0)} \circ {\color{red}T_{(0,1)}} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$





We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Birational rowmotion orbit on a product of chains

Example: Iterating this procedure we get:

$$\rho_B f = \begin{tikzcd}[column sep=small] & \frac{(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{(x+y)w}{xz} \arrow[dr, dash] & & \frac{(x+y)w}{yz} \arrow[dl, dash] \\ & \frac{1}{z} & \end{tikzcd}, \qquad \rho_B^2 f = \begin{tikzcd}[column sep=small] & \frac{(x+y)w}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{1}{y} \arrow[dr, dash] & & \frac{1}{x} \arrow[dl, dash] \\ & \frac{z}{x+y} & \end{tikzcd},$$

$$\rho_B^3 f = \begin{tikzcd}[column sep=small] & \frac{1}{w} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{yz}{(x+y)w} \arrow[dr, dash] & & \frac{xz}{(x+y)w} \arrow[dl, dash] \\ & \frac{xy}{(x+y)w} & \end{tikzcd}, \qquad \rho_B^4 f = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}.$$

Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also “antipodal reciprocity”.

Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

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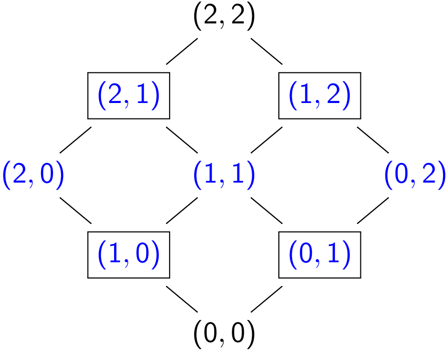
1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \geq (m,n)\}$ be the *principal order filter at* (m,n), $\hexagon^{k}_{(m,n)}$be the *rank-selected subposet*, of elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $k-1$ and whose corank is at most $k-1$.



2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$.

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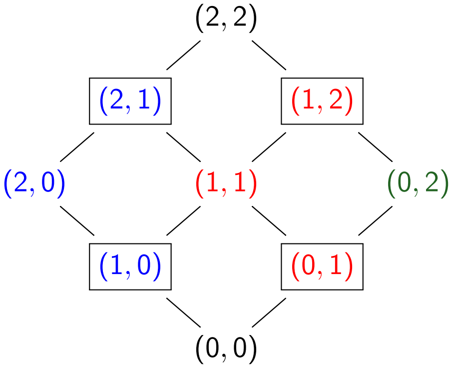
Let $A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij}'s as follows.

Birational Rowmotion on the Rectangular Poset

We define a **lattice path of length k** within $P = [0, r] \times [0, s]$ to be a sequence $v_1, v_2, \ldots, v_k$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.



Birational Rowmotion on the Rectangular Poset

3) Let $S_k(m,n)$ be the set of non-intersecting lattice paths in $\hexagon^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = (L_1, L_2, \ldots L_k) \in S^k_k(m,n)$ denote a k-tuple of such lattice paths.

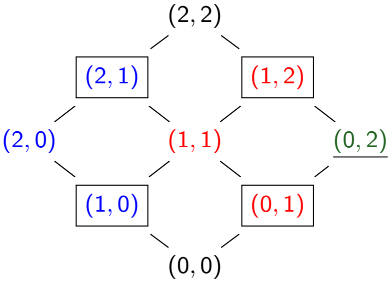
4) Define $\varphi_k(m,n) :=$
$$\sum_{\mathcal{L} \in S^k_k(m,n)} \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}.$$

$\approx$**Thm:**

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

EG: $\rho_B^2(1,1) = \dfrac{\varphi_1(0,0)}{\varphi_2(0,0)}.$

$$= \frac{\text{sum of 6 quartic terms in } A_{ij}}{A_{20} + A_{11} + A_{02}}$$



Main Theorem (Musiker-R 2017+)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

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(a1) When $M = 0$, i.e., $(i-k, j-k)$ still lies in the poset $[0,r] \times [0,s]$:

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

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(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\mu^{(a,b)}$ is the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term: $A_{(u,v)} \mapsto A_{(u-a,v-b)}$.

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(a) When $0 \le M \le k$:

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where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined in 4) and 5) above.

(b) When $M \geq k$: $\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity: $\rho_B^{r+s+2+d} = \rho_B^d$; thus we get a formula for **all** iterations of the birational rowmotion map.

Corollary

For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.

Corollary ([GrRo15, Thm. 30])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is periodic, with period* $r+s+2$.

Corollary ([GrRo15, Thm. 32])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *satisfies the following reciprocity:*
$\rho_B^{i+j+1} = 1/\rho_B^0(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Example in Further Depth

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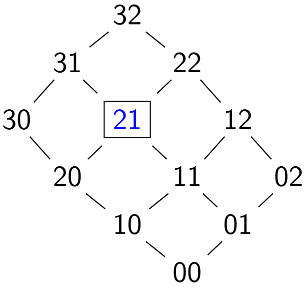
Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.



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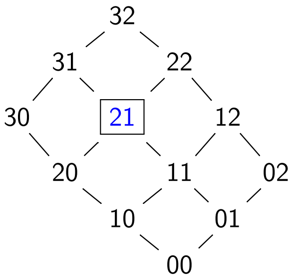
When $\mathbf{k} = \mathbf{0}$, $M = 0$ and we get

$$\rho_B^1(2,1) = \frac{\varphi_0(2,1)}{\varphi_1(2,1)} = \frac{A_{21}A_{22}A_{31}A_{32}}{A_{22} + A_{31}}.$$



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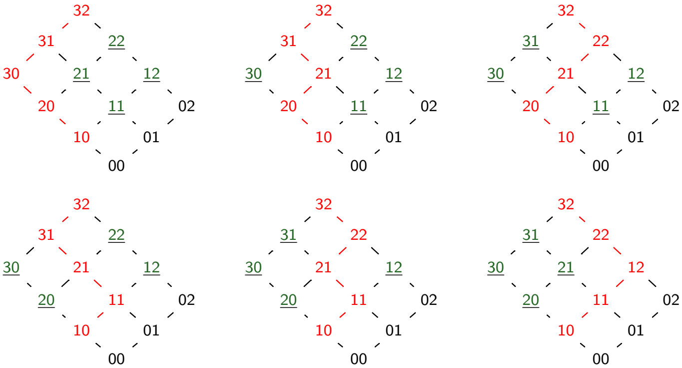
When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$



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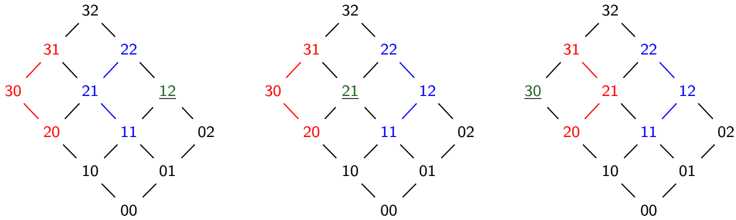

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Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

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When k $=$ **2**, we get $M = [2-2]_+ + [2-1]_+ = 1 \leq 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2,1) = \mu^{(1,0)}\left[\frac{\varphi_1(1,0)}{\varphi_2(1,0)}\right] = \text{(just shifting indices in the } k = 1 \text{ formula)}$$

$$\frac{A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}}{A_{02} + A_{11} + A_{20}}.$$

Example in Further Depth

Corollary: For $k \leq \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0,1,2,3,4,5,6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k} = \mathbf{3}$, we get $M = [3-2]_+ + [3-1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2,1) = \mu^{(2,1)}\left[\frac{\varphi_0(2,1)}{\varphi_1(2,1)}\right] = \mu^{(2,1)}\left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}\right] = \frac{A_{00}A_{01}A_{10}A_{11}}{A_{01}+A_{10}}.$$

In this situation, we can also use part (b) of the main theorem to get

$$\rho_B^4(2,1) = 1/\rho_B^{3-2-1}(3-2,2-1) = 1/\rho_B^0(1,1) = \frac{1}{x_{11}}.$$

The equality between these two expressions is easily checked.

Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 4, we get $M = [4-2]_+ + [4-1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2, 1) = 1/\rho_B^{4-2-1}(3-2, 2-1) = 1/\rho_B^1(1, 1) = \frac{\varphi_1(1,1)}{\varphi_0(1,1)} = \frac{A_{12}A_{22} + A_{12}A_{31} + A}{A_{11}A_{12}A_{21}A_{22}A_{31}}$$

Each term in the numerator is associated with one of the three lattice paths from $(1, 1)$ to $(3, 2)$ in P, while the denominator is just the product of all A-variables in the principal order filter $\bigvee(1, 1)$.

Example in Further Depth

Corollary: For $k \leq \min\{i, j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When $\mathbf{k = 5}$, we get $M = [5-2]_+ + [5-1]_+ = 7 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(3-2, 2-1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(0,0)}{\varphi_1(0,0)}$$

$$= \frac{A_{02}A_{12} + A_{02}A_{21} + A_{11}A_{21} + A_{30}A_{02} + A_{30}A_{11} + A_{30}A_{20}}{\text{A sum of 10 degree-6 monomials in } A_{ij}}.$$

The numerator here represents the empty product, since the unique (unordered) pair of lattice paths from $s_1 = (2,1)$ and $s_2 = (1,2)$ to $t_1 = (3,1)$ and $t_2 = (2,2)$ covers **all** elements of $\hexagon^2_{(1,1)}$. The denominator here is the same as the numerator of the previous case.

Corollary: For $k \le \min\{i, j\}$, $\rho_B^{k+1}(i, j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$

Example 3: We use our main theorem to compute $\rho_B^{k+1}(2, 1)$ for $P = [0, 3] \times [0, 2]$ for $k = 0, 1, 2, 3, 4, 5, 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

When k = 6, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right] =$$

$$\mu^{(1,1)}\left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x_{21}$$

When k $=$ **6**, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2, 2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right] =$$

$$\mu^{(1,1)}\left[\frac{A_{12}A_{22} + A_{12}A_{31} + A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11} + A_{01}A_{20} + A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x_{21}$$

The lattice paths involved here are the same as for the $k = 4$ computation.

We can deduce this by $A_{00} = 1/x_{00}, A_{10} = x_{00}/x_{10}, A_{01} = x_{00}/x_{01}$, $A_{11} = (x_{10} + x_{01})/x_{11}, A_{20} = x_{10}/x_{20}$, and $A_{21} = (x_{20} + x_{11})/x_{21}$.

Periodicity also kicks in: $\rho_B^7(2,1) = \rho_B^0(2,1) = x_{21}$ using $(r + s + 2) = 7$.

Sketch of Proof

The proof is a complicated triple induction on (i, j, k). Start with $k = 0$ and work top down through the poset, repeat with $k = 1$, etc.

The key to making it work is the following lemma, and a variation on it which includes the shifting $\mu^{(i,j)}$'s.

Lemma

For $1 \leq k \leq \min\{i, j\}$ we have the Plücker-like relation

$$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1)$$

$$= \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1)$$

$$+\varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k).$$

The proof of this involves a colorful bijection between families of NILPs.

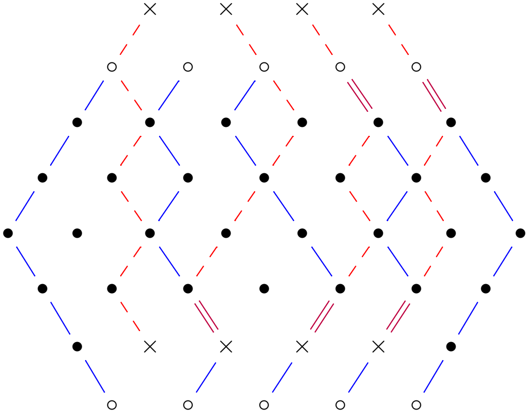
Sketch of Proof: Colorful Bjection

$$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) =$$
$$\varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1).$$

Sketch of Proof: Colorful Bjection

$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) =$
$\varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1)$.

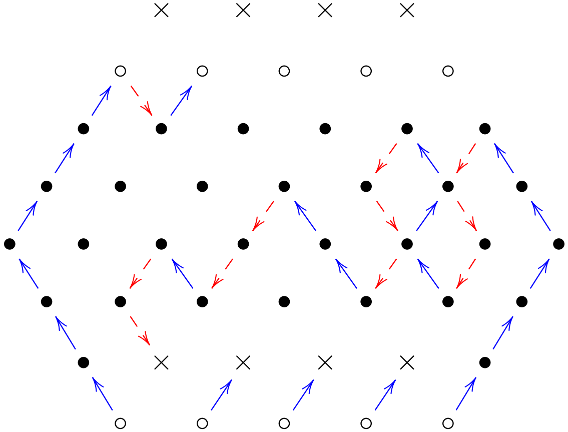
Example (k=5):



Sketch of Proof

We build **bounce paths** and **twigs** (paths of length one from $\circ$ to $\times$) starting from the bottom row of $\circ$'s.

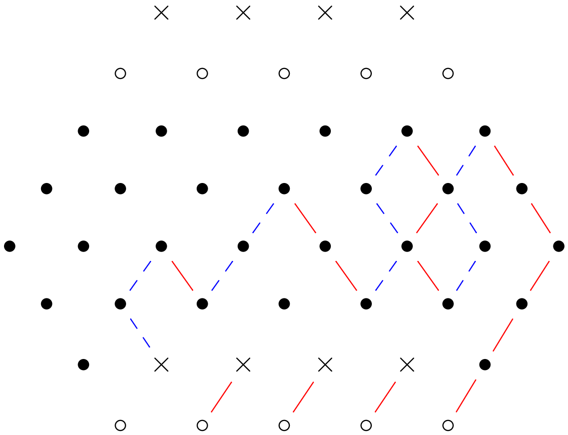
Example (k=5):



Sketch of Proof

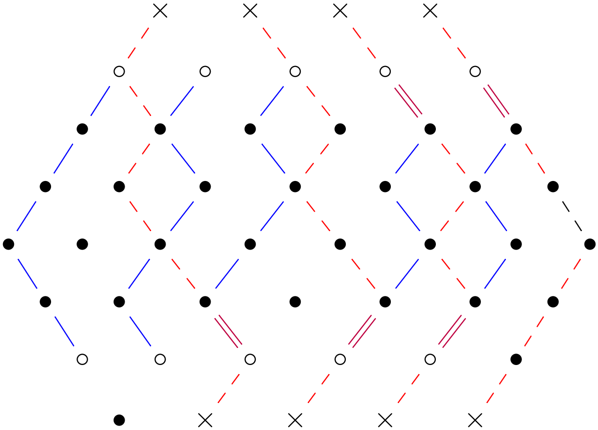
We then reverse the colors along the $(k-2)$ **twigs** and the **one bounce path from** $\circ$ **to** $\times$ (rather than $\circ$ to $\circ$).

Example (k=5):



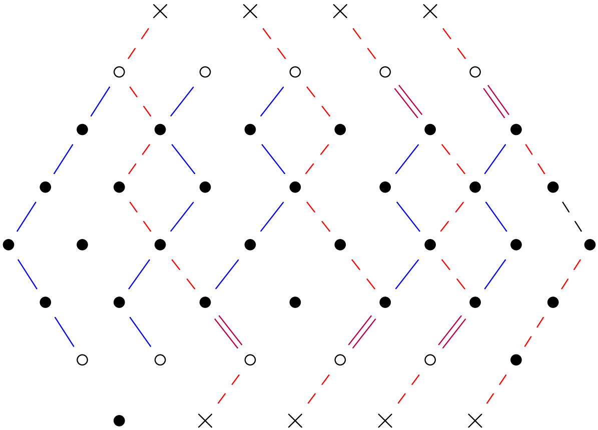
Swap in the new colors and shift the $\circ$'s and $\times$'s in the bottom two rows.

Example (k=5):



$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \\ \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) \\ +\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

Example (k=5):



The final slide of this talk (before the references)

We're happy to discuss this further with anyone who's interested.

Slides for this talk are available online (or will be soon) at

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Thanks very much for coming to this talk!

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