A Galoisian Approach to Counting Walks

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Erwin Schrödinger International Institute for Mathematics and Physics

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- Talk 1: An Introduction to the Galois theory of difference equations
- Talk 2: Walks, Difference Equations and Elliptic Curves

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Galois theory give us *computable* conditions to determine when generating functions satisfy or do not satisfy differential equations.

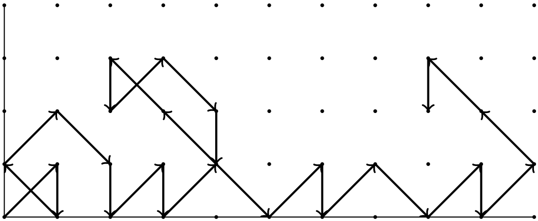
Walks

Consider the walks in the quarter plane starting from $(0,0)$ with steps in a fixed set

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}.$$

Example with possible directions

$$\mathcal{D} \subset \{\nwarrow, \nearrow, \searrow, \downarrow\}.$$



256 possible choices for $\mathcal{D}$. Triviality, Symmetries $\Rightarrow$ 79 interesting ones.

Walks

$q_{\mathcal{D},i,j,k}$ = the number of walks in $\mathbb{N}^2$ starting from $(0,0)$ ending at (i,j) using k steps from $\mathcal{D}$.

Generating series: $Q_{\mathcal{D}}(x,y,t) := \sum_{i,j,k} q_{\mathcal{D},i,j,k} x^i y^j t^k$.

Classification problem: when is $Q_{\mathcal{D}}(x,y,t)$

- Algebraic over $\mathbb{C}(x,y,t)$?
- Holonomic over $\mathbb{C}(x,y,t)$?(x-, y-, and t-holonomic)
- Differentially Algebraic over $\mathbb{C}(x,y,t)$? (x-,y-, and t-diff. algebraic)

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$f(x,y,t)$ is <u>x-holonomic</u> if for some n and $a_i \in \mathbb{C}(x,y,t)$,

$$a_n(x,y,t)\frac{\partial^n f}{\partial x^n} + \ldots + a_1(x,y,t)\frac{\partial f}{\partial x} + a_0(x,y,z)f = 0$$

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$f(x,y,t)$ is x-differentially algebraic if for some n and polynomial $P \neq 0$,

$$P(x,y,t,f,\frac{\partial f}{\partial x},\ldots,\frac{\partial^n f}{\partial x^n}) = 0$$

Walks

Fayolle, Iasnorodski, Malyshev (1999), Bousquet-Mélou, Mishna (2010) - associate to a set of steps $\mathcal{D}$,

- an algebraic curve $E_{\mathcal{D}}$ of genus 0 or 1, and
- a group $G_{\mathcal{D}}$, finite or infinite.

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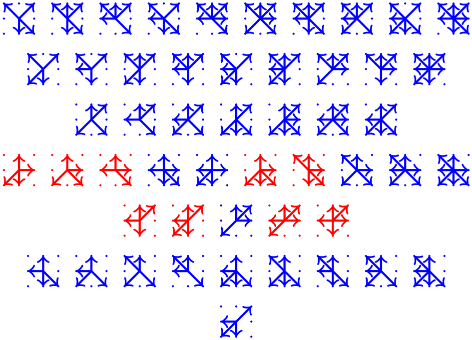
- a group $G_{\mathcal{D}}$, finite or infinite.

Results: For the **79** walks

- $|G_{\mathcal{D}}| < \infty$ for **23** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ algebraic or holonomic.
  $\rightarrow$ A. Bostan, M. Bousquet-Mélou, M. van Hoeij, M. Kauers, M. Mishna, ...
- $|G_{\mathcal{D}}| = \infty$ for **56** walks $\Rightarrow Q_{\mathcal{D}}(x, y, t)$ **not** holonomic.
  - 5 walks with $\text{genus}(E_{\mathcal{D}}) = 0$ $\rightarrow$ S. Melzcer, M. Mishna, A. Rechnitzer, ...
  - 51 walks with $\text{genus}(E_{\mathcal{D}}) = 1$ $\rightarrow$A. Bostan, I. Kurkova, K. Raschel, B. Salvy, ...
- Differentially Algebraic???

Walks: 51 walks with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{D}}) = 1$



Theorem (D-H-R-S, 2017a): For $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$

1. In 42 cases, $x \mapsto Q_{\mathcal{D}}(x, 0, t)$ is not x-DA, $y \mapsto Q_{\mathcal{D}}(0, y, t)$ is not y-DA.
2. In 9 cases, $x \mapsto Q_{\mathcal{D}}(x, 0, t)$ is x-DA, $y \mapsto Q_{\mathcal{D}}(0, y, t)$ is y-DA but neither is holon.

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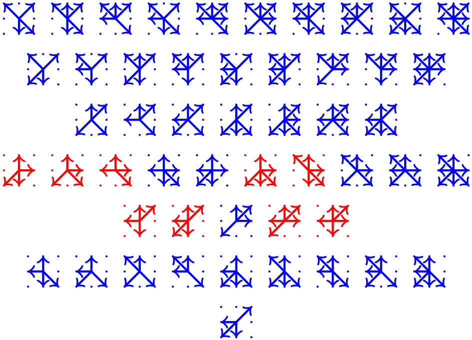

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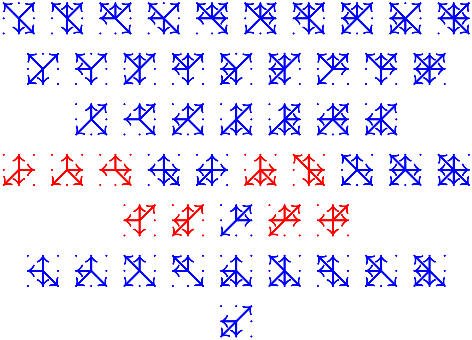

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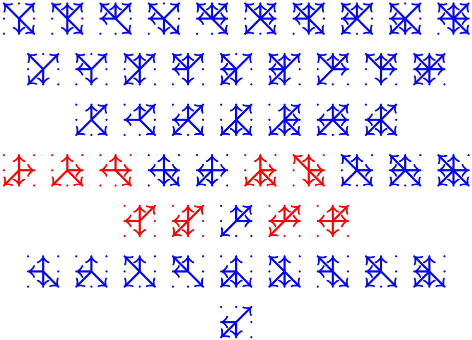

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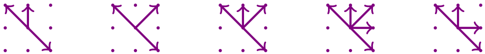
- 1. true for weighted cases as well. See recent paper of Dreyfus/Raschel.

Walks: 5 walks with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{D}}) = 0$



Theorem (D-H-R-S, 2017b): For $t \in \mathbb{R} \backslash \overline{\mathbb{Q}}$

In all cases, $x \mapsto Q_{\mathcal{D}}(x, 0, t)$, is not x-DA and $y \mapsto Q_{\mathcal{D}}(0, y, t)$ is not y-DA.

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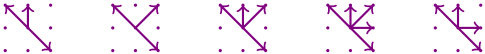

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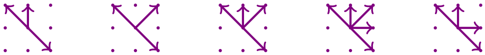

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- This implies $Q_{\mathcal{D}}(x, y, t)$ is not DA (and so not holon.) in these cases.
- True for weighted cases as well.

- Generalities about Walks

- Differential Transcendence of the 42 walks, $|G_{\mathcal{D}}| = \infty, \text{genus}(E_{\mathcal{D}}) = 1$.

- Differential Algebraicity of the 9 walks, $|G_{\mathcal{D}}| = \infty, \text{genus}(E_{\mathcal{D}}) = 1$.

- Differential Transcendence of the 5 walks, $|G_{\mathcal{D}}| = \infty, \text{genus}(E_{\mathcal{D}}) = 0$.

Generalities about Walks

Functional Equation of the Walk

$q_{\mathcal{D},i,j,k}$ = the number of walks in $\mathbb{N}^2$ starting from $(0,0)$ ending at (i,j) using k steps from $\mathcal{D}$.

Generating series: $Q_{\mathcal{D}}(x,y,t) := \sum_{i,j,k} q_{\mathcal{D},i,j,k} x^i y^j t^k$.

Step Inventory: $\mathcal{S}_{\mathcal{D}}(x,y) = \sum_{(i,j)\in\mathcal{D}} x^i y^j$
Kernel of the Walk: $K_{\mathcal{D}}(x,y,t) = xy(1 - t\mathcal{S}_{\mathcal{D}}(x,y))$
Functional Equation:

$$\begin{aligned} K_{\mathcal{D}}(x,y,t)Q_{\mathcal{D}}(x,y,t) = \\ & xy - K_{\mathcal{D}}(x,0,t)Q_{\mathcal{D}}(x,0,t) - K_{\mathcal{D}}(0,y,t)Q_{\mathcal{D}}(0,y,t) \\ & + K_{\mathcal{D}}(0,0,t)Q_{\mathcal{D}}(0,0,t). \end{aligned}$$

Curve of the Walk

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The Curve of the Walk is the curve

$$E_{\mathcal{D}} = \overline{\{(x,y) \mid K_{\mathcal{D}}(x,y,t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

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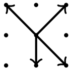
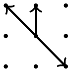
Ex: 1) $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 1$

2) $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + xy^2 + x^2) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 0$

for $t \in \mathbb{C}\backslash\overline{\mathbb{Q}}$

Group of the Walk

$$E_{\mathcal{D}} = \overline{\{(x, y) \mid K_{\mathcal{D}}(x, y, t) = 0\}}^{\textit{Zariski}} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

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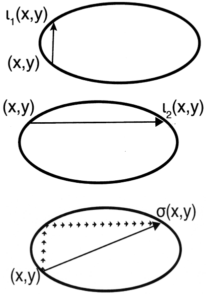
We define two involutions of $E_{\mathcal{D}}$ and an automorphism:

$$\iota_1(x,y) = \left(x, \frac{1}{y}\frac{\sum_{(i,-1)\in\mathcal{D}} x^i}{\sum_{(i,+1)\in\mathcal{D}} x^i}\right)$$

$$\iota_2(x,y) = \left(\frac{1}{x}\frac{\sum_{(-1,j)\in\mathcal{D}} y^j}{\sum_{(+1,j)\in\mathcal{D}} y^j}, y\right)$$

$$\sigma_{\mathcal{D}} = \iota_2 \circ \iota_1$$



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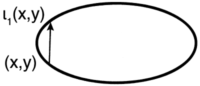
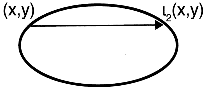


$$\iota_2(x,y) = \left(\frac{1}{x} \frac{\sum_{(-1,j)\in\mathcal{D}} y^j}{\sum_{(+1,j)\in\mathcal{D}} y^j}, y\right)$$



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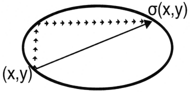

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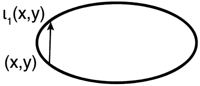

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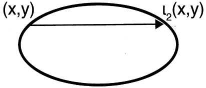

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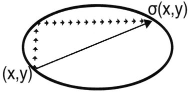

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Facts: 1) $G_{\mathcal{D}}$ is infinite iff $\sigma_{\mathcal{D}}$ is infinite.

2) $g(E_{\mathcal{D}}) = 1 \Rightarrow \exists P \in E_{\mathcal{D}}$, s.t. $\sigma_{\mathcal{D}}(Q) = Q \oplus P$. $\sigma_{\mathcal{D}}$ is infinite iff P nontorsion.

3) Of the **79** interesting walks, $|G_{\mathcal{D}}| = \infty$ for **56** walks, 5 with $g=0$ and 51 with $g=1$ when $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$ (Bousquet-Mélou/Mishna).

Differential Transcendence of the 42 walks,

$|G_{\mathcal{D}}| = \infty, \text{genus}(E_{\mathcal{D}}) = 1.$

Proving Differential Transcendence: The Gamma Function

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$$f(x+1) - f(x) = \frac{1}{x}.$$
- Galois Theory: If $f(x)$ is DA then for some n and complex numbers a_i
$$\frac{d^n}{dx^n}(\frac{1}{x}) + a_{n-1}\frac{d^{n-1}}{dx^{n-1}}(\frac{1}{x}) + \ldots + a_0(\frac{1}{x}) = h(x+1) - h(x)$$
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for some rational function $h(x)$
- Computation: LHS has only one pole and RHS has at least two poles $\Rightarrow$ CONTRADICTION.

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  $\sigma(x) = x + 1$ or qx or ... and $g(x)$ a rational function.
- Galois Theory implies that if f is DA then for some n and complex numbers a_i

$$\frac{d^n g}{dx^n} + a_{n-1}\frac{d^{n-1} g}{dx^{n-1}} + \ldots + a_0 g = h(\sigma(x)) - h(x)$$

  for some rational function $h(x)$

Proving Differential Transcendence of Function $F(x)$

- Analysis is used to find that a related function $f(x)$ s.t.
  - $F(x)$ DA $\Rightarrow$ $f(x)$ DA, and
  - $f(x)$ satisfies a functional equation

$$f(\sigma(x)) - f(x) = g(x).$$

  $\sigma(x) = x + 1$ or qx or ... and $g(x)$ a rational function.

- Galois Theory implies that if f is DA then for some n and complex numbers a_i

$$\frac{d^n g}{dx^n} + a_{n-1}\frac{d^{n-1} g}{dx^{n-1}} + \ldots + a_0 g = h(\sigma(x)) - h(x)$$

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- Computation of poles shows that this Telescoper Equation cannot happen.

Differential Transcendence: $|G_{\mathcal{D}}| = \infty, g(E_{\mathcal{D}}) = 1,$ $(t \in \mathbb{C}\backslash\overline{\mathbb{Q}})$

Generating Series: $Q_{\mathcal{D}}(x,y,t) := \sum_{i,j,k} q_{\mathcal{D},i,j,k} x^i y^j t^k$ satisfies

$$K_{\mathcal{D}}(x,y,t)Q_{\mathcal{D}}(x,y,t) = xy - K_{\mathcal{D}}(x,0,t)Q_{\mathcal{D}}(x,0,t) - K_{\mathcal{D}}(0,y,t)Q_{\mathcal{D}}(0,y,t) + K_{\mathcal{D}}(0,0,t)Q_{\mathcal{D}}(0,0,t)$$

Curve: $E_{\mathcal{D}} := \overline{\{(x,y) \mid K_{\mathcal{D}}(x,y,t) = 0\}}^{Zariski} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

Group: $G_{\mathcal{D}} := \langle \iota_1, \iota_2 \rangle, \sigma_{\mathcal{D}} = \iota_2 \circ \iota_1 \quad \sigma_{\mathcal{D}}(Q) = Q \oplus P.$

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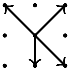
3) Each $F^i_{\mathcal{D}}(X)$ satisfies

$$F^i_{\mathcal{D}}(\sigma_{\mathcal{D}}(X)) - F^i_{\mathcal{D}}(X) = g^i_{\mathcal{D}}(X)$$

on $E_{\mathcal{D}}$ for some $g^i_{\mathcal{D}}(X) \in \mathbb{C}(E_{\mathcal{D}}) = \mathbb{C}(x, y)$.

Galois Theory

Ex. $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 1$

$F^2_{\mathcal{D}}(X)$ satisfies $F^2_{\mathcal{D}}(\sigma_{\mathcal{D}}(X)) - F^2_{\mathcal{D}}(X) = g^2_{\mathcal{D}}(X) := x(\dfrac{x^2 + x}{y(x^2+1)} - y)$

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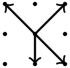
<u>Ex.</u> $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 1$

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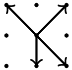
Ex. $\mathcal{D} =$  $E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 1$

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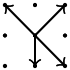
<u>Ex.</u> $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\Rightarrow g(E_{\mathcal{D}}) = 1$

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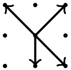
- $F^2_{\mathcal{D}}$ is DA wrt $\delta_{\mathcal{D}}$ iff $Q_{\mathcal{D}}(0, y, t)$ is y-DA over $\mathbb{C}(x, y, t)$.

Galois Theory

<u>Ex.</u> $\mathcal{D} =$  $E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \ \Rightarrow g(E_{\mathcal{D}}) = 1$

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How does one decide if such a telescoper equation exists?

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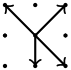
<u>Ex.</u> $\mathcal{D} =$  $\quad E_{\mathcal{D}} : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \quad \Rightarrow g(E_{\mathcal{D}}) = 1$

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How does one decide if such a telescoper equation exists?

Computation of poles shows when this happens.

Telescoper Equations

$$k = \mathbb{C}(x), \sigma(x) = x+1, \delta = \frac{d}{dx} \qquad y(x+1) - y(x) = g(x) \quad g(x) \in k$$

When does g satisfy a telescoper equation

$$\frac{d^n g}{dx^n} + a_{n-1}\frac{d^{n-1}g}{dx^{n-1}} + \ldots + a_0 g = h(x+1) - h(x)?$$

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Definition Let $g \in \mathbb{C}(x)$, $\alpha \in \mathbb{C}$ and c_α^i be the coefficient of $(x-\alpha)^{-i}$ in the partial fraction expansion of g. The **$\mathbf{i^{th}}$ orbit residue** of g at α is

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Existence of Telescopers. $k = \mathbb{C}(x), \sigma(x) = x + 1, \delta = \frac{d}{dx}$ and $g \in k$. The following are equivalent:

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- $g = h(x+1) - h(x)$ for some $h \in k$.

Corollary. If for some $\alpha \in \mathbb{C}$, g has a unique pole in $\{\alpha + n\}_{n \in \mathbb{Z}}$, then g satisfies no telescoper eqn.

Telescopers in $\mathbb{C}(E)$, E an Elliptic Curve

E elliptic curve, P nontorsion point, $k = \mathbb{C}(E)$, $\sigma(f(Y)) = f(Y \oplus P)$, δ deriv $\delta\sigma = \sigma\delta$

When does an $g \in \mathbb{C}(E)$ satisfy a telescoper equation $L(g) = \sigma h - h$?

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Def. 1) $\{u_Q \mid Q \in E\}$ local param. are **coherent** if $u_{Q \ominus P} = \sigma(u_Q)$.

2) $f \in \mathbb{C}(E)$, $Q \in E$, and $c_Q^i =$ coeff. of $\frac{1}{u_Q^i}$ in u-adic expansion of g. The **$\mathbf{i}^{\mathbf{th}}$ orbit residue** of g at Q is

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- g satisfies a telescoper equation.
- For each $i \in \mathbb{N}_{>0}, Q \in E$, $\mathrm{ores}_Q^i(g) = 0$.
- There exists $Q \in E, h \in k$ and $e \in \mathcal{L}(Q + (Q \oplus P))$ s.t. $g = \sigma h - h + e$.

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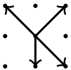
Corollary. If for some $Q \in E$, g has a unique pole in $\{Q \oplus nP\}_{n \in \mathbb{Z}}$, then no telescoper for g.

An Example

$\mathcal{D} =$ 

$$E_{\mathcal{D}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : xy - t(y^2 + x^2y^2 + x^2 + x) = 0$$

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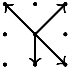
$\mathcal{D} =$ 

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$K_{\mathcal{D}}(0, y, t)Q_{\mathcal{D}}(0, y, t)$ is y-DA $\Rightarrow$ $F^2_{\mathcal{D}}(x)$ is DA $\Rightarrow$

$$g_{\mathcal{D}} = x\left(\frac{x^2 + x}{y(x^2 + 1)} - y\right)$$

would satisfy a telescoper equation. This cannot happen because g has a pole unique in its orbit.

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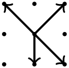
$\mathcal{D} =$ 

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$K_{\mathcal{D}}(0, y, t)Q_{\mathcal{D}}(0, y, t)$ is y-DA $\Rightarrow F^2_{\mathcal{D}}(x)$ is DA $\Rightarrow$

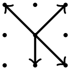
$$g_{\mathcal{D}} = x\left(\frac{x^2 + x}{y(x^2 + 1)} - y\right)$$

would satisfy a telescoper equation. This cannot happen because g has a pole unique in its orbit.

<u>Poles:</u> $\mathcal{P} = \{(\infty, \pm \mathrm{i}), (\pm \mathrm{i}, \infty), (\pm \mathrm{i}, \pm \mathrm{i}t + t)\}$

<u>Fact:</u> The autom. $\tau : \mathrm{i} \mapsto -\mathrm{i}$ of $\mathbb{Q}(\mathrm{i})$ commutes with $\sigma_{\mathcal{D}} : (\infty, i) \mapsto (\infty, i) \oplus P$.

An Example

$\mathcal{D} =$  $\qquad E_{\mathcal{D}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : xy - t(y^2 + x^2y^2 + x^2 + x) = 0$

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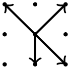
$\mathcal{D} =$  $\quad E_{\mathcal{D}} \subset \mathbb{P}^1 \times \mathbb{P}^1 : xy - t(y^2 + x^2y^2 + x^2 + x) = 0$

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Proof: If $(\infty, -\mathrm{i}) = \sigma^n_{\mathcal{D}}(\infty, \mathrm{i})$, then

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So $(\infty, \mathrm{i}) = (\infty, \mathrm{i}) \oplus 2nP \Rightarrow 0 = 2nP$, contradicting the fact that P is nontorsion. $\sigma^n(\infty, \mathrm{i}) \neq$ other poles similarly.

Differential Algebraicity of the 9 walks,

$$|G_{\mathcal{D}}| = \infty, \text{genus}(E_{\mathcal{D}}) = 1.$$

Showing Differential Transcendence

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For 9 cases $g(x)$ does satisfy these conditions.

DA for 9 cases

- For these walks, $g(x)$ satisfies a telescoper equation on $E_{\mathcal{D}}$

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- $\tilde{\mathcal{F}}(x)$ doubly periodic $\Rightarrow \tilde{\mathcal{F}}(x)$ DA $\Rightarrow Q_{\mathcal{D}}(0, y, t)$ y-DA.

Differential Transcendence of the 5 walks,

$$|G_{\mathcal{D}}| = \infty, \mathrm{genus}(E_{\mathcal{D}}) = 0.$$

5 walks with $|G_{\mathcal{D}}| = \infty$, genus$(E_{\mathcal{D}}) = 0$. $(t \in \mathbb{R}\backslash\overline{\mathbb{Q}})$

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Can select ϕ so that

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- Restrict $K_{\mathcal{D}}(0, y, t)Q(0, y, t)$ to a small open set in $E_{\mathcal{D}}$ and PULL-BACK to open set in $\mathbb{C}$.
- Analytically continue to get a function $f(z)$ on $\mathbb{C}$ that satisfies $f(qz) - f(z) = g(z)$ for some $g \in \mathbb{C}(x)$.
- f is DA $\Leftrightarrow Q(0, y, t)$ is y-DA.
- f is DA $\Rightarrow g(z) = h(qz) - h(z)$ for some $h \in \mathbb{C}(z)$. Conditions on poles give contradiction.

On the nature of the generating series of walks in the quarter plane

arXiv:1702.04696

Walks in the quarter plane, genus zero case

arXiv:1710.02848