1			
		3	
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-avoiding permutations

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Australian Government
Australian Research Council

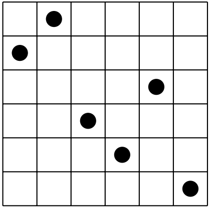
Collaboration with A. Conway, A. Guttmann / A. Sportiello.

Overview

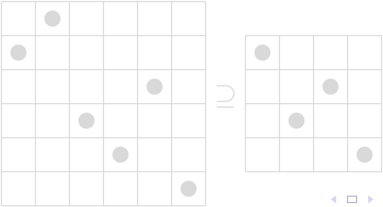
Permutation containment

We use the following representation of permutations:

$215346 \quad \mapsto$



A permutation β contains another permutation α if α can be obtained from β by removing rows/columns:



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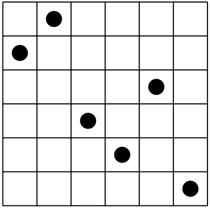
$215346 \quad \mapsto$ 

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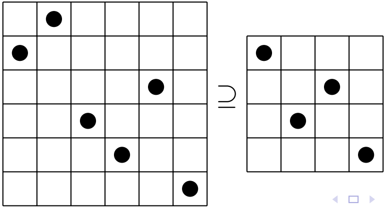

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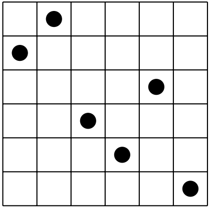

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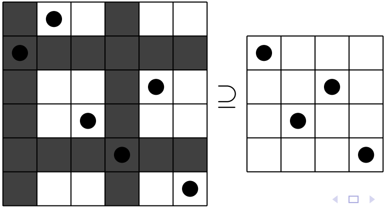

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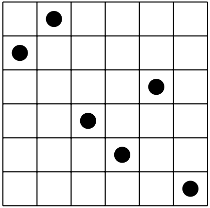
215346 $\mapsto$ 

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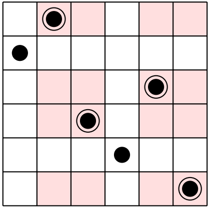
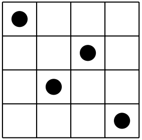
 $\supseteq$ 

Pattern avoidance

Given a fixed permutation α (the "pattern"), define

$$Av_n(\alpha) = \{\beta \in \mathcal{S}_n : \ \alpha \not\subseteq \beta\}$$

We are interested in exact and asymptotic enumeration of $Av_n(\alpha)$ for various α:

$$a_n(\alpha) = |Av_n(\alpha)|$$

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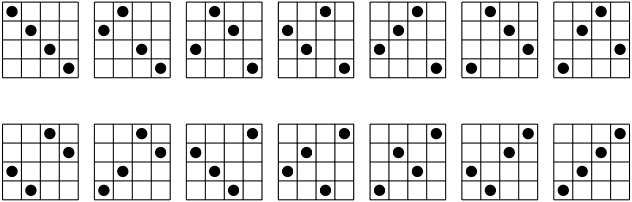
Example

$Av_4(132) = \{$  $\}$

$a_4(132) = 14$

Randomly selected history

- MacMahon (1915) considered $Av_n(123)$. Enumerated by Catalan numbers.
- More generally, the Robinson–Schensted correspondence allows to characterize $Av_n(12\cdots n)$ and $Av_n(n\cdots 21)$.
- Knuth (1968) considered permutations that can be stack-sorted; these are $Av_n(231)$, also enumerated by Catalan numbers.
- Pratt (1973) began a systematic study of $Av_n(\alpha)$.
- For *any* α of length 3, $a_n(\alpha)$ is the Catalan number.
- Patterns with the same a_n are said to be in the same Wilf class.

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Permutations of length 4

Permutations of length 4 fall into 3 Wilf classes:

- Gessel (1990) proved

$$a_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

- Bóna (1997) proved

$$a_n(1342) = (-1)^{n-1}\frac{7n^2-3n-2}{2} + 3\sum_{k=0}^{n}(-1)^{n-i}2^{i+1}\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}.$$

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Asymptotic enumeration

- Stanley and Wilf conjectured, and Marcus and Tardos proved (2004), that $\lim_{n\to\infty} \sqrt[n]{a_n(\alpha)}$ exists and is finite for all α. Denote

$$\mu(\alpha) = \lim_{n\to\infty} \sqrt[n]{a_n(\alpha)}$$

- Madras and Liu showed that for α of length less than 6,

$$\lim_{n\to\infty} \frac{a_{n+1}(\alpha)}{a_n(\alpha)} = \mu(\alpha)$$

- In lengths 3 and 4,

$$\mu(123) = 4$$

$$\mu(1234) = 9 \qquad \mu(1342) = 8 \qquad 9.81 < \mu(1324) < 13.002$$

The lower bound (Bevan, 2015) can be improved, as we shall see.

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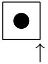
A naive transfer matrix

Basic idea: build the permutation row by row (recording only the *relative* horizontal locations of dots).



At each stage remains only a 132-avoiding permutation.

Remark: the length of the permutation is the distance of the forbidden zone to the diagonal.

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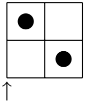

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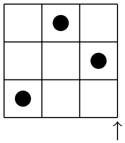

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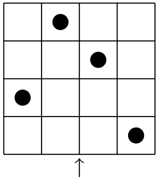

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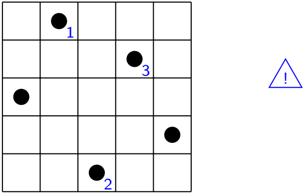

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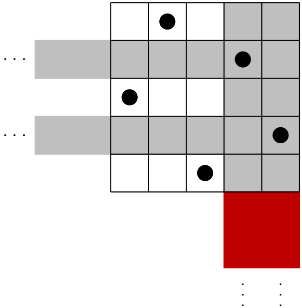

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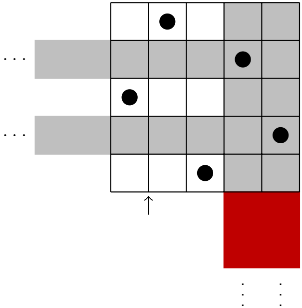

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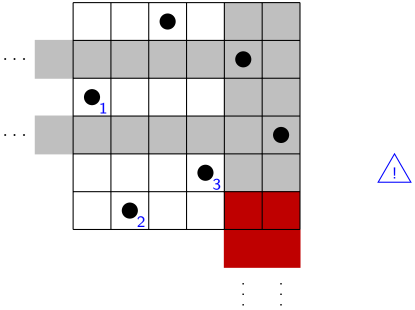

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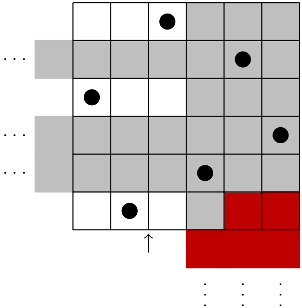

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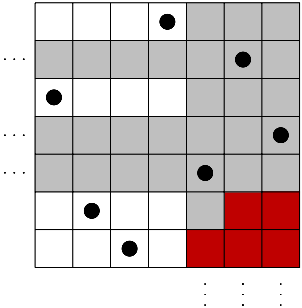

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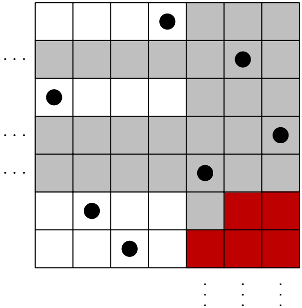

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A naive transfer matrix cont'd

The algorithm can be formulated as follows: given a 132-avoiding permutation, insert one more dot on the next bottom row, and erase all dots to the right of any 132 (including the 3).

Equivalently, define a matrix

$$T_{\alpha,\beta} = \begin{cases} 1 & \text{if the algorithm produces } \alpha \text{ from } \beta \\ 0 & \text{else} \end{cases}$$

where α, β are arbitrary length 132-avoiding permutations.

Theorem

$$a_n(1324) = \sum_{\alpha} (T^n)_{\alpha,\varnothing} = \langle \mathbb{1} | \, T^n \, | \varnothing \rangle$$

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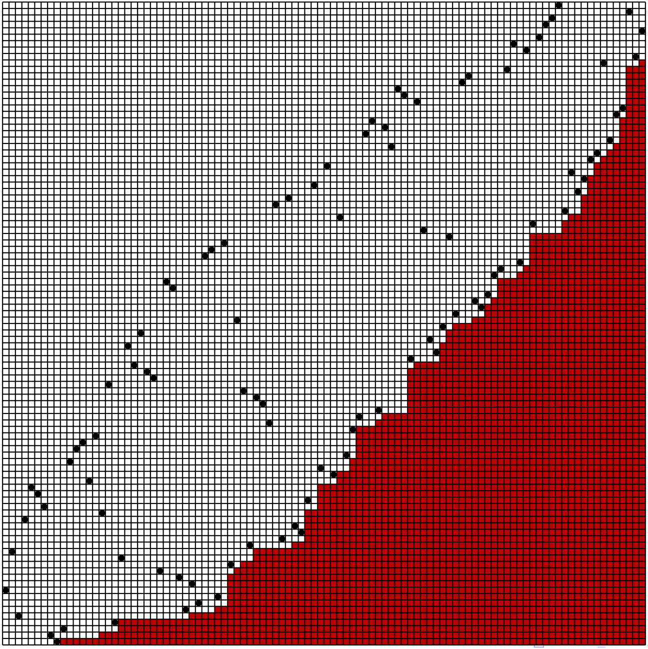
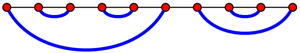


Link patterns

A link pattern is a planar pairing in the half-plane of $2n$ points on its boundary.

There is a bijection between 132-avoiding permutations of length n and link patterns of size $2n$:



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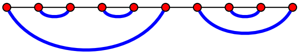

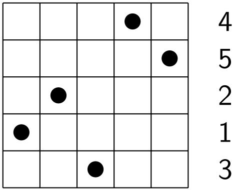

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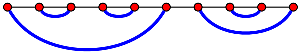

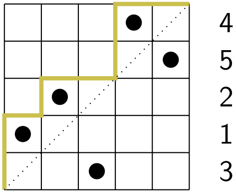

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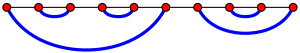

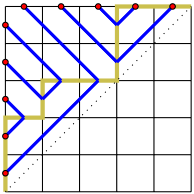

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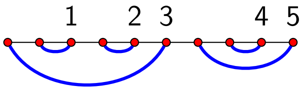
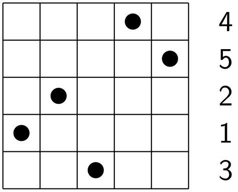


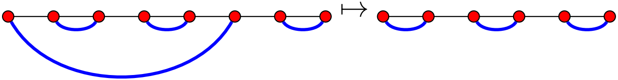


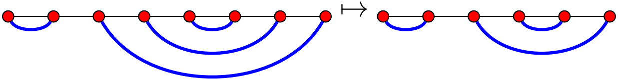
A modified naive transfer matrix

Define matrices L and R:

$$L/R_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha \text{ is obtained from } \beta \\ & \text{by removing the leftmost/rightmost arc} \\ 0 & \text{else} \end{cases}$$

where α, β are arbitrary size link patterns, e.g.,

$L :$ 

$R :$ 

A modified naive transfer matrix cont'd

Combining the 132-avoiding/link pattern bijection and a simple upper triangular transformation, we arrive at the following transfer matrix:

Theorem

$$a_n(1324) = \langle\varnothing|\, \tilde T^n \,|\varnothing\rangle\,, \qquad \tilde T = L^t + 1/(1-R) = L^t + \sum_{i=0}^{\infty} R^i$$

Remark. Other classes have similar expressions:

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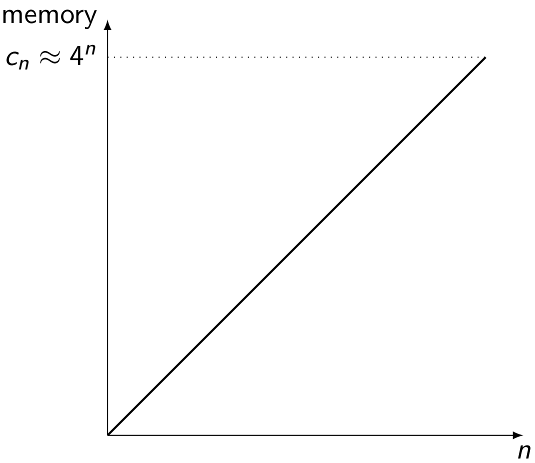
Memory usage

Simplistic log plot of memory usage vs n:



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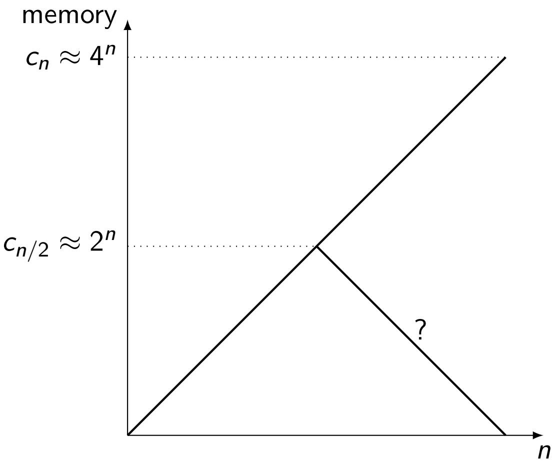

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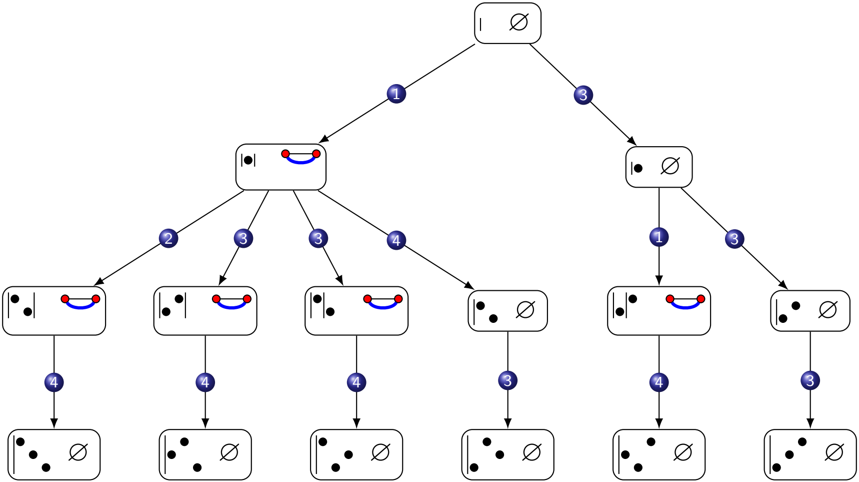

Transfer matrix formulation

Define $\mathcal{T}_{\alpha,\beta}$ to be the number of ways that α can be obtained from β by one of the following 4 moves:

1. Add an arc from a new leftmost vertex to a free spot.
2. (if not leftmost) Add this arc and erase the arc just left of it.
3. Extend arcs corresponding to the group of openings right of this spot all the way to the left.
4. (if not leftmost) Extend arcs corresponding to the group of openings right of this spot all the way to the left and erase the arc just left of it.

Theorem

$$a_n(1324) = \langle \varnothing | \mathcal{T}^n | \varnothing \rangle$$

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$$a_n(1324) = \langle \varnothing | \, \mathcal{T}^n \, | \varnothing \rangle$$

Since the memory usage only grows like $c_{n/2} \approx 2^n$, one can produce by computer the first 50 terms of the series $a_n(1324)$ (previously known up to $n = 36$ [Conway, Guttmann '15]).

One can then try to use various methods to analyze the asymptotic behavior as $n \to \infty$ (ask Tony!)

Our best guess:

$$a_n(1324) \overset{n\to\infty}{\sim} \mu^n \mu_1^{\sqrt{n}} n^g B$$

with $\mu = \mu(1324) \approx 11.60$, $\mu_1 \approx 0.040$, $g \approx -1$.

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Perron–Frobenius eigenvalues

Consider the truncated transfer matrix $\mathcal{T}_n$ which only allows link patterns up to size $2n$. Note that

$$a_k(1324) = \langle\varnothing|\, \mathcal{T}_n^k \,|\varnothing\rangle \qquad \forall k \leq 2n$$

Call its Perron–Frobenius eigenvalue μ_n. It is not hard to show that

$$\mu_n \leq \mu \quad \forall n, \qquad \text{and} \qquad \lim_{n\to\infty} \mu_n = \mu$$

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A lower bound

A few hours of computation on my laptop produce the eigenvalue μ_{18}:

Theorem

The growth constant $\mu = \lim_{n\to\infty} \sqrt[n]{a_n(1324)}$ *satisfies*

$$\mu > 9.9194195$$

Compared with existing methods for lower bounds (e.g., Bevan '15), this method only implicitly counts a particular subset of $Av_n(1324)$.

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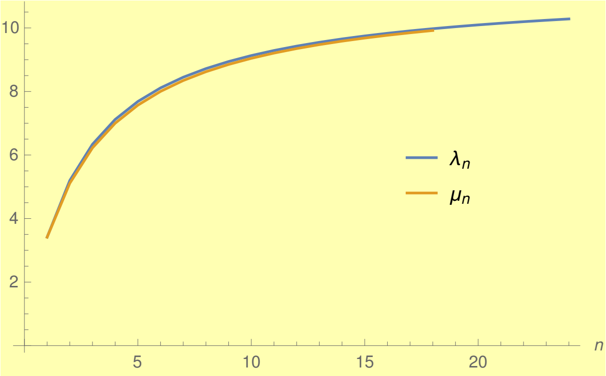
Further bounds

For comparison, define λ_n to be the inverse of the smallest zero of the denominator of the $(n, n+1)$ Padé approximant of $a_n(1324)$:



$\lambda_{24} = 10.28326\ldots$

It is likely that $\lambda_n - \mu_n$ tends monotonically to zero (for n large enough). Under certain technical hypotheses, λ_n itself is a lower bound for μ.