Tilings and plane partitions

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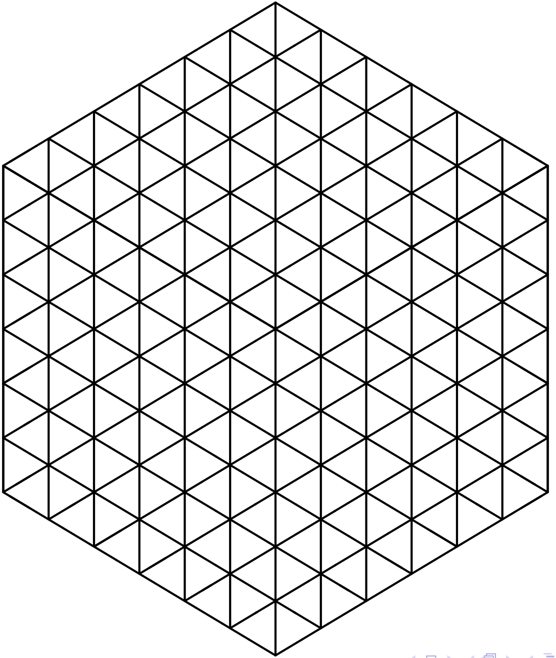

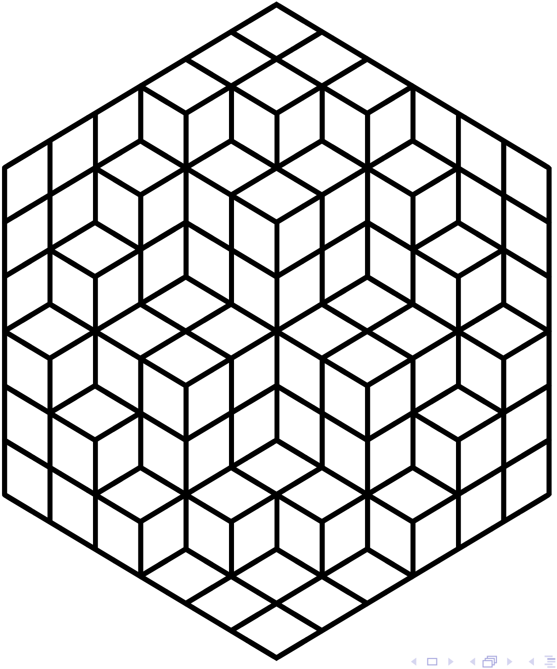

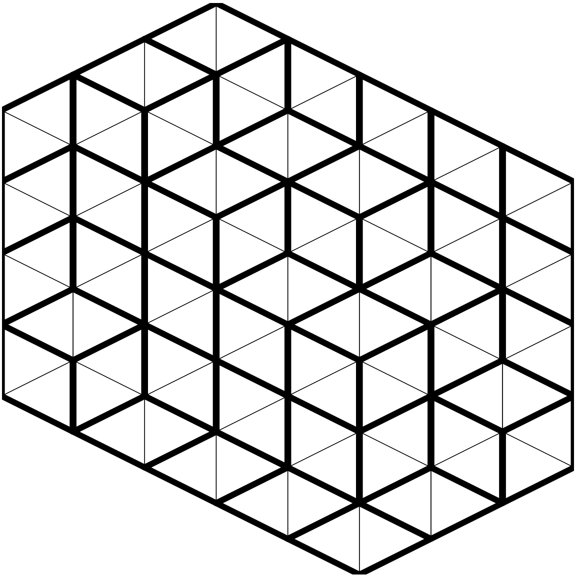

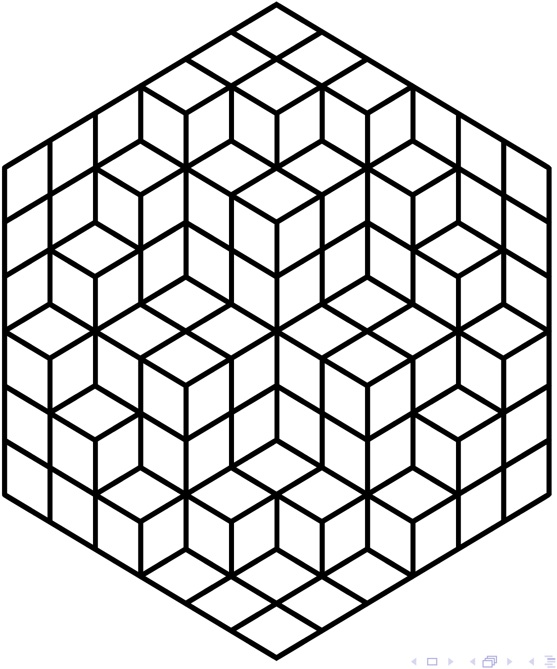

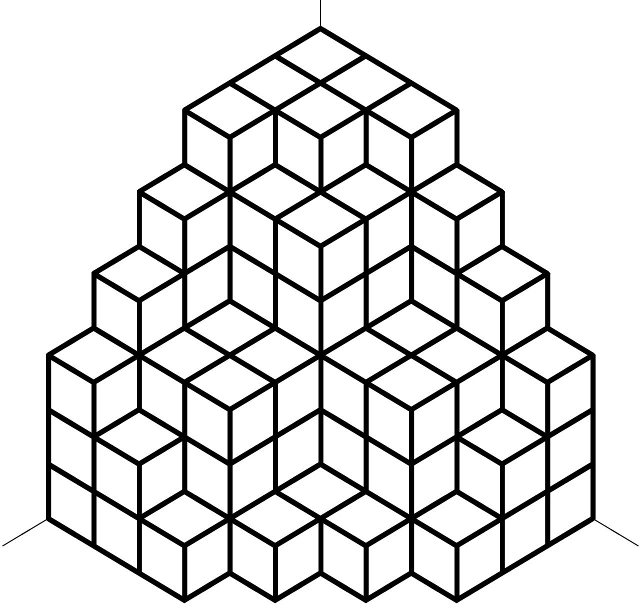

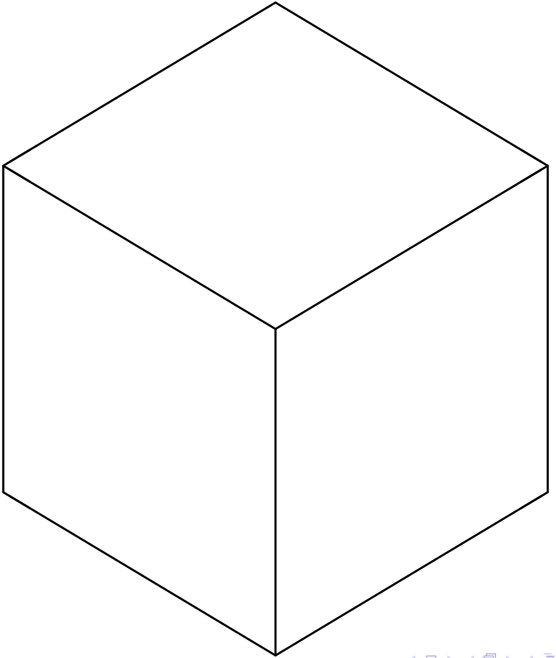

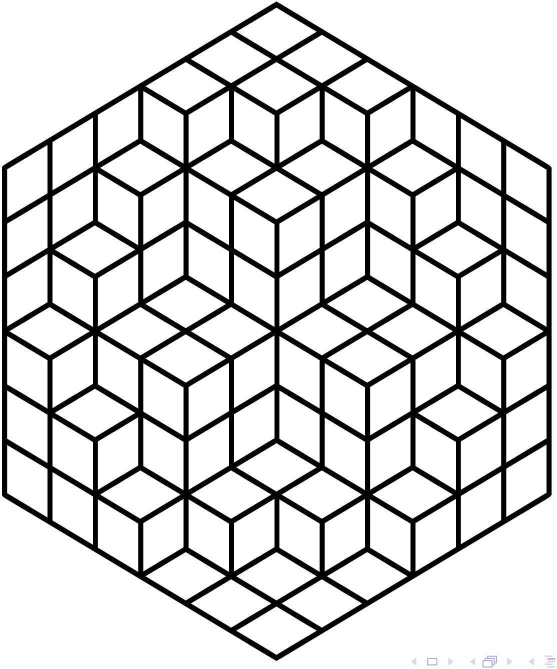

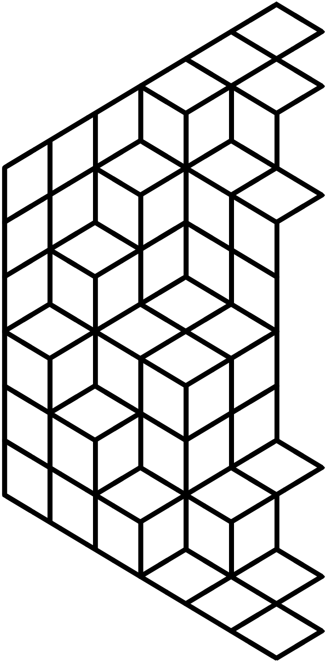

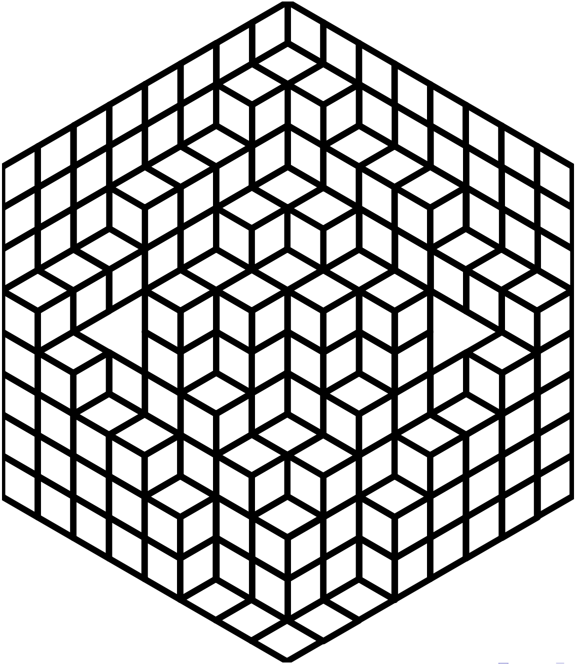

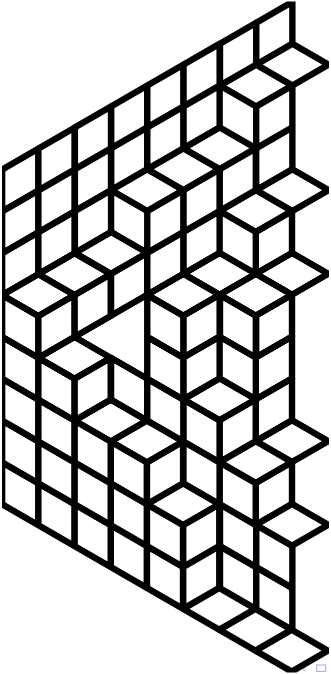

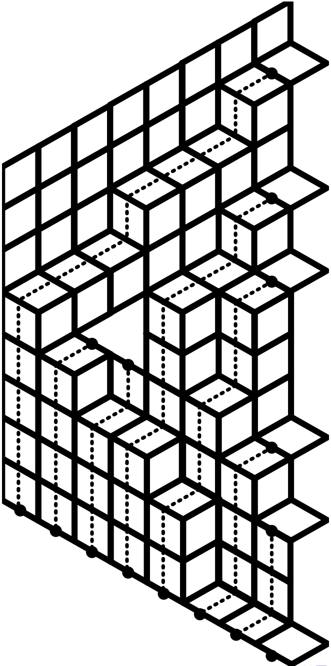

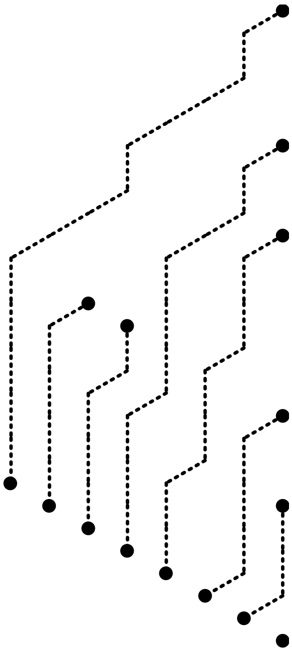

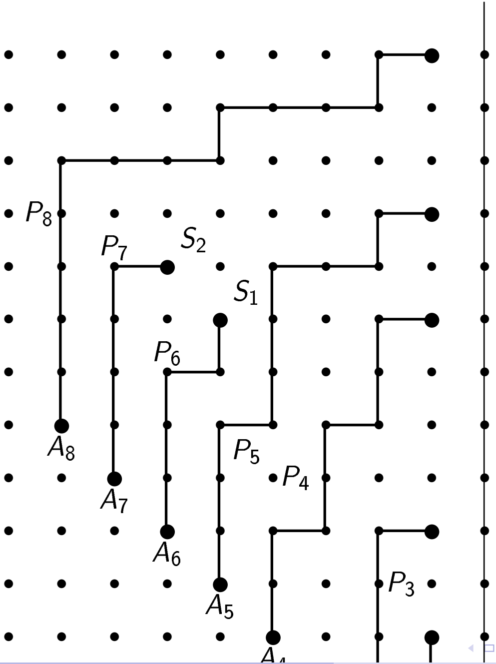
P_8
P_7
S_2
S_1
P_6
P_5
P_4
P_3
A_8
A_7
A_6
A_5
A_4

```
In[1]:= V[2]
Out[1]= {0, c[1]}
In[2]:= V[3]
Out[2]= {0, c[1], c[1]}
In[3]:= V[4]
Out[3]= {0, c[1], 2 c[1], c[1]}
In[4]:= V[5]
Out[4]= {0, c[1], 3 c[1], c[3], c[1]}
In[5]:= V[6]
Out[5]= {0, c[1], 4 c[1], 2 c[1] + c[4], c[4], c[1]}
In[6]:= V[7]
Out[6]= {0, c[1], 5 c[1], c[3], -10 c[1] + 2 c[3],
⟩ -5 c[1] + c[3], c[1]}
In[7]:= V[8]
Out[7]= {0, c[1], 6 c[1], c[3], -25 c[1] + 3 c[3],
⟩ c[5], -9 c[1] + c[3], c[1]}
In[8]:= V[9]
Out[8]= {0, c[1], 7 c[1], c[3], -49 c[1] + 4 c[3],
⟩ -28 c[1] + 2 c[3] + c[6], c[6], -14 c[1] + c[3],
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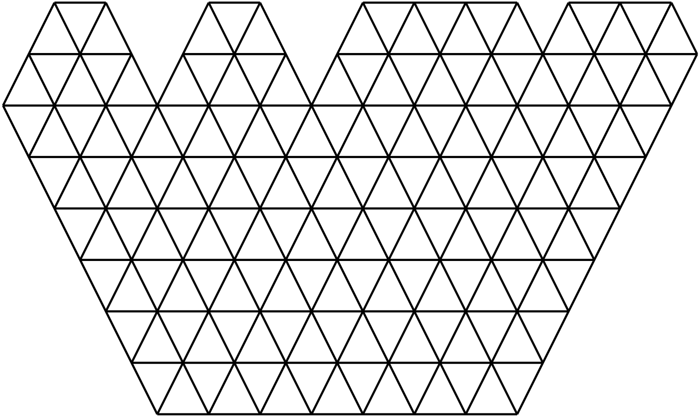

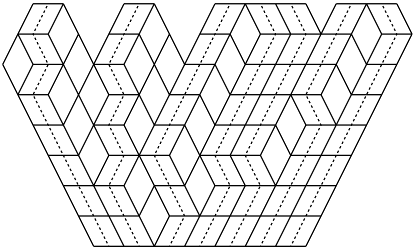

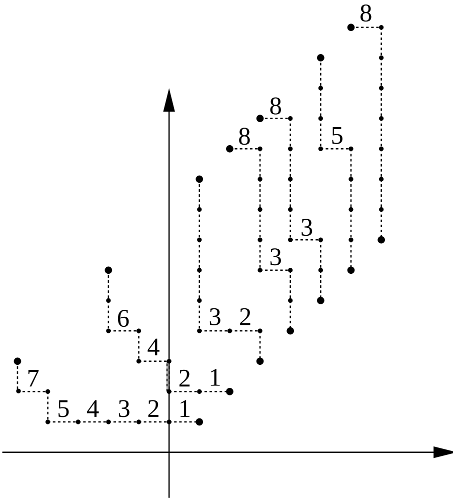
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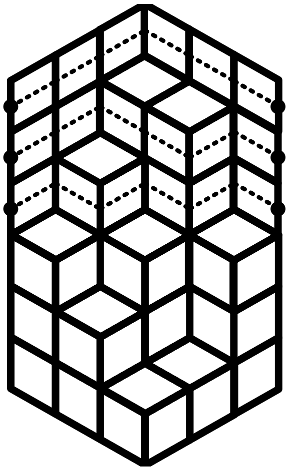
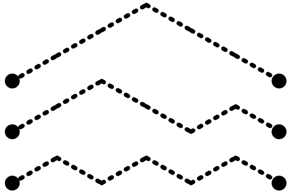
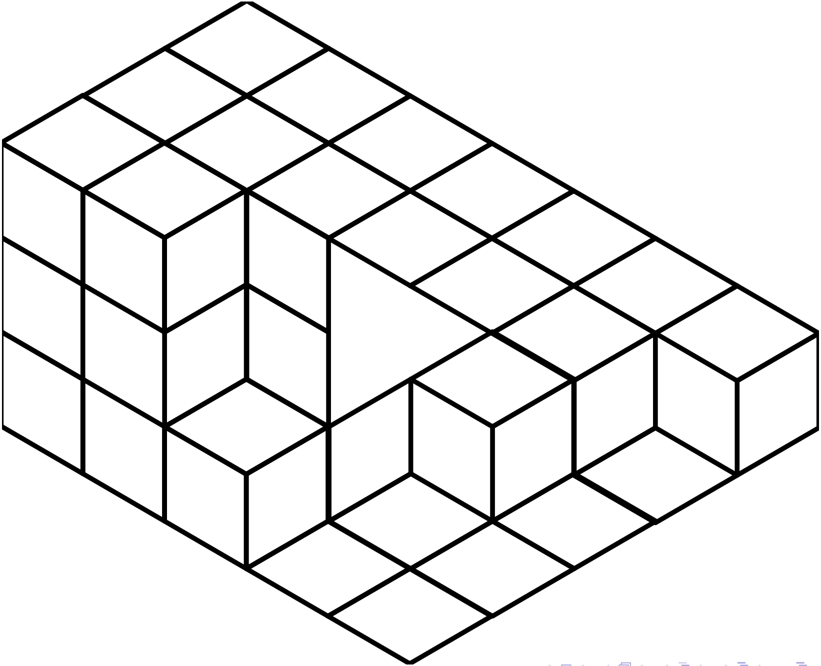




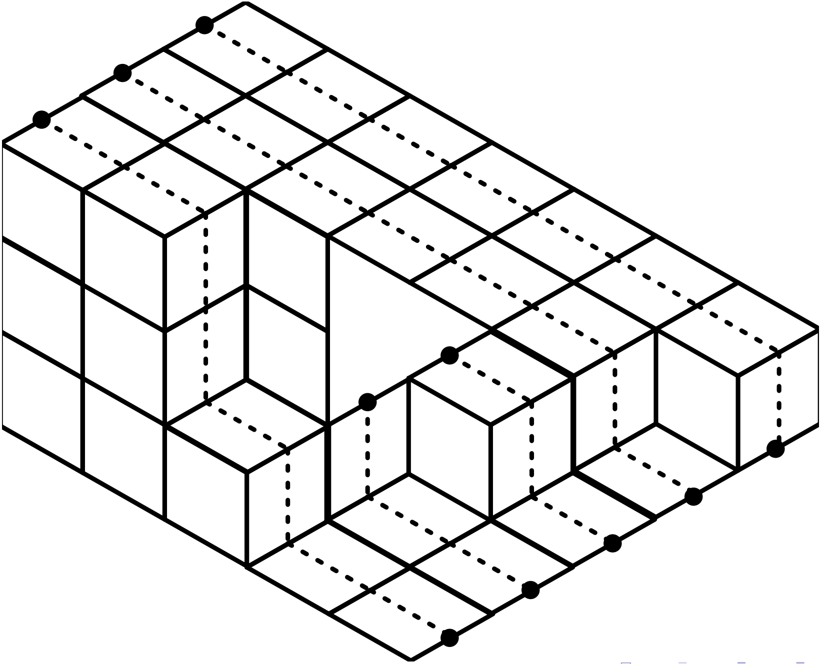


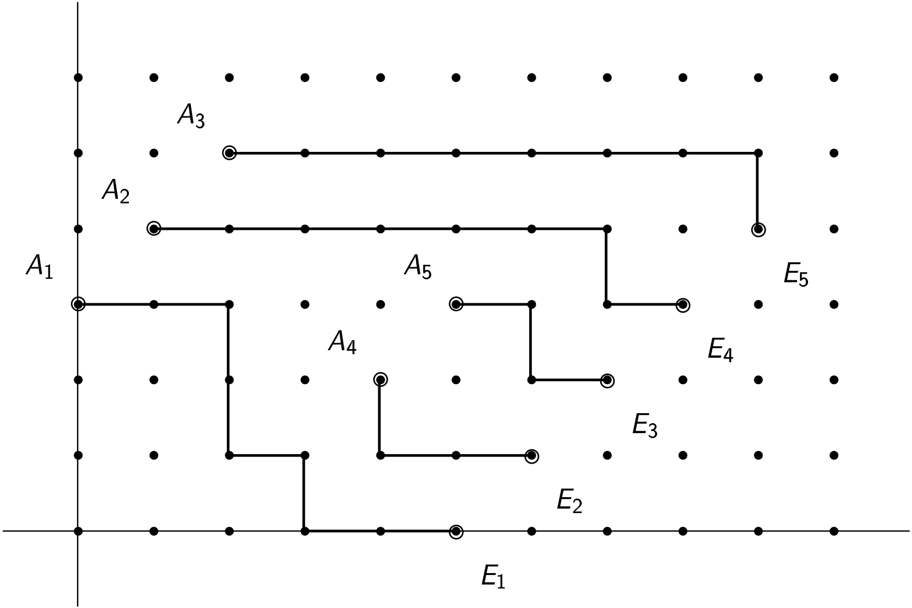
Theorem

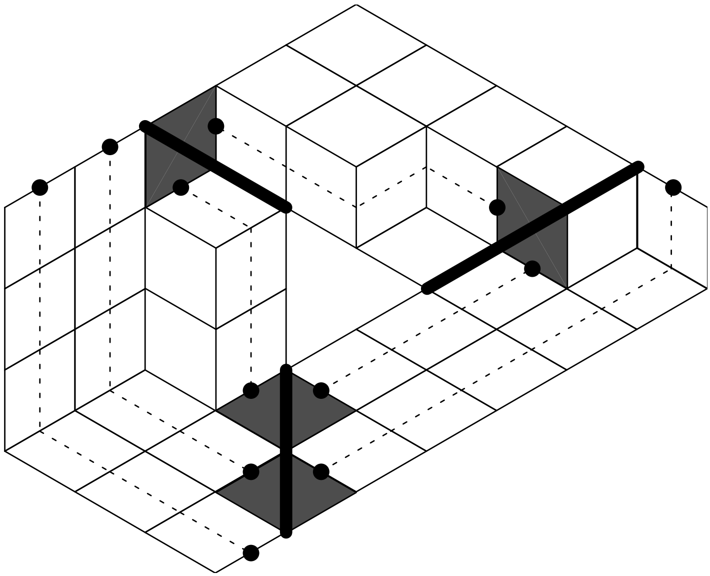
The number of rhombus tilings of the hexagon minus triangle is given by

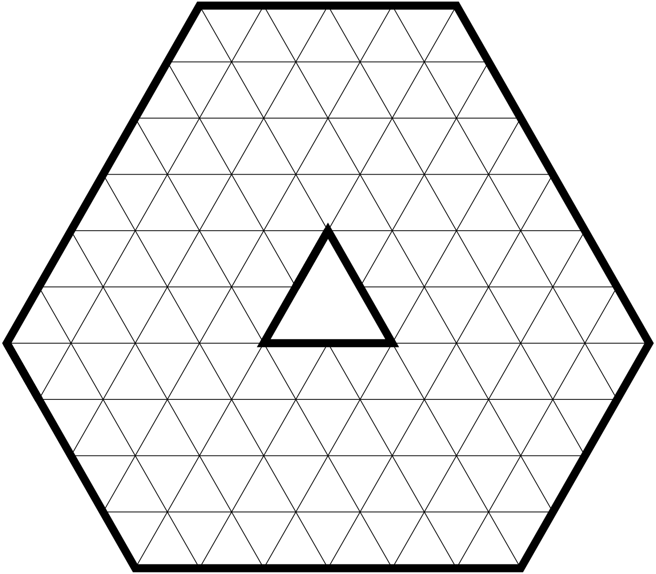
$$\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)}$$
$$\times\frac{\mathrm{H}(m+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(m+\lfloor\frac{a+b+c}{2}\rfloor)}{\mathrm{H}(\frac{a+b}{2}+m)\,\mathrm{H}(\frac{a+c}{2}+m)\,\mathrm{H}(\frac{b+c}{2}+m)}$$
$$\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\;\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\;\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\;\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\;\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\;\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\;\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\;\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{a+c+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2})\,\mathrm{H}(\frac{a+c}{2})\,\mathrm{H}(\frac{b+c}{2})},$$

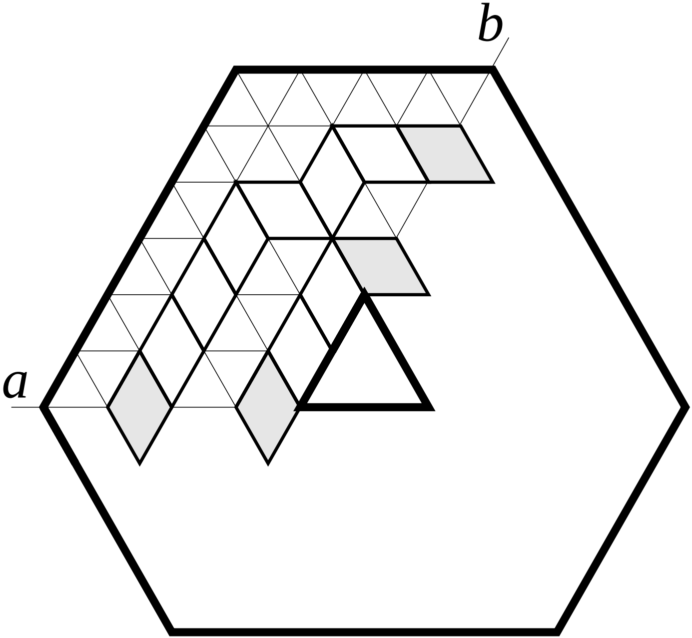
where $\mathrm{H}(n) := \begin{cases}\prod_{k=0}^{n-1}\Gamma(k+1) & \textit{for } n \textit{ an integer,}\\ \prod_{k=0}^{n-\frac{1}{2}}\Gamma(k+\frac{1}{2}) & \textit{for } n \textit{ a half-integer.}\end{cases}$



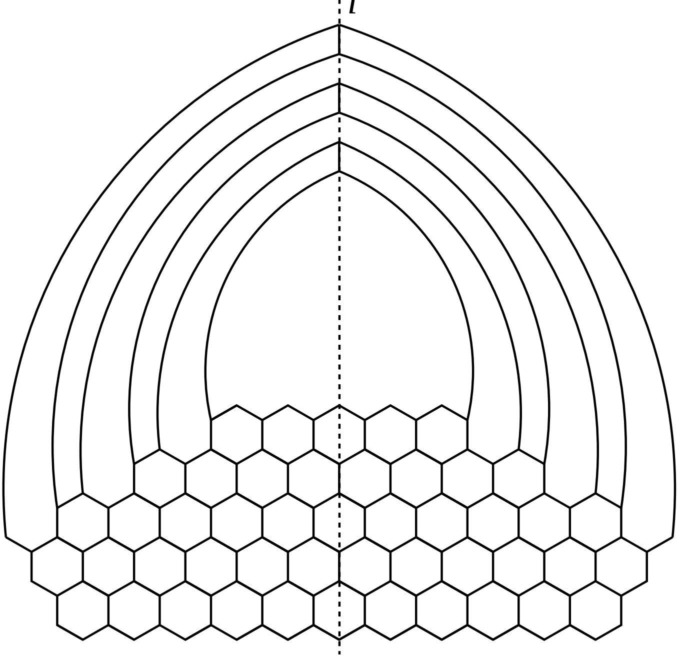

A_3
A_2
A_1
A_5
E_5
A_4
E_4
E_3
E_2
E_1

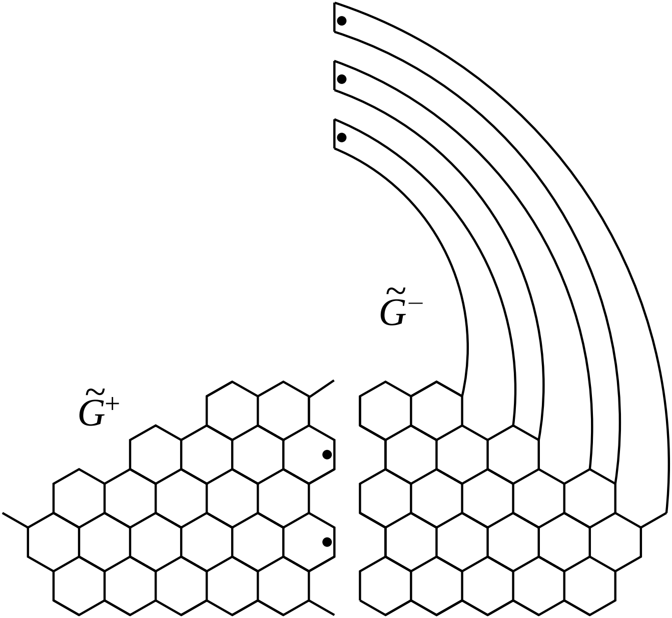


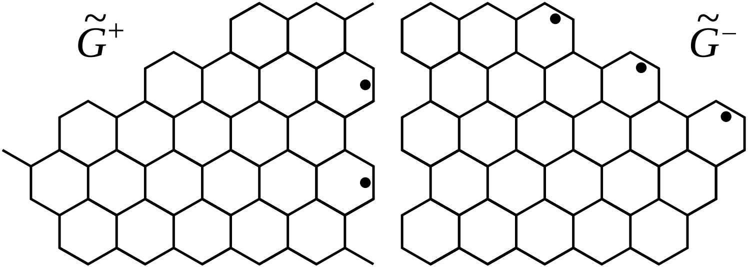


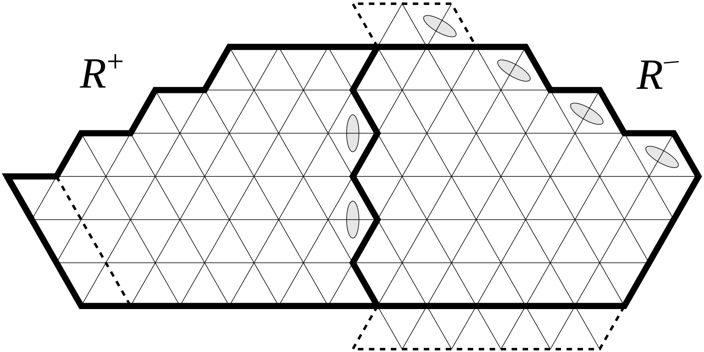


b
a

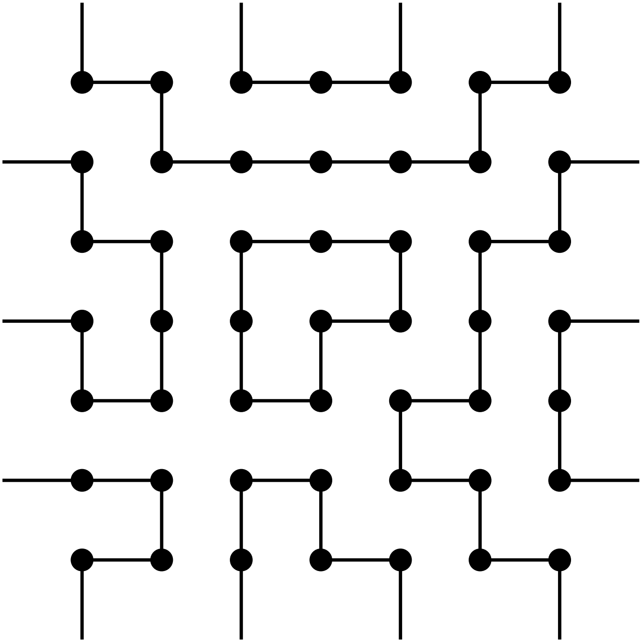

l

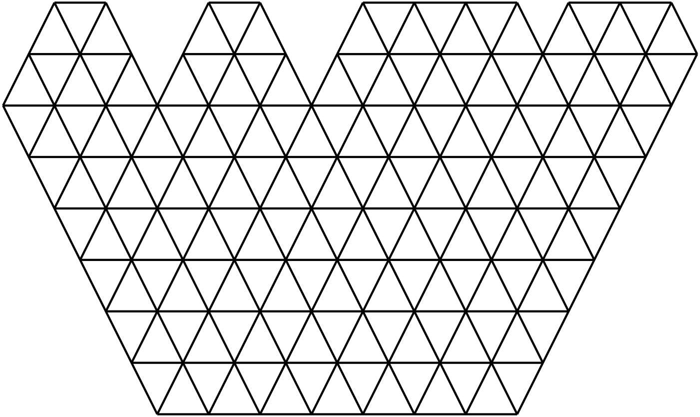

$\tilde{G}^-$
$\tilde{G}^+$

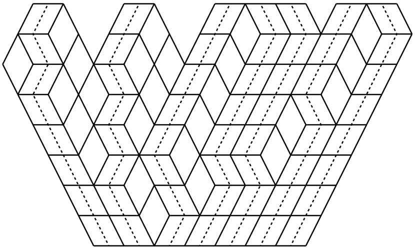

$\widetilde{G}^{+}$
$\widetilde{G}^{-}$

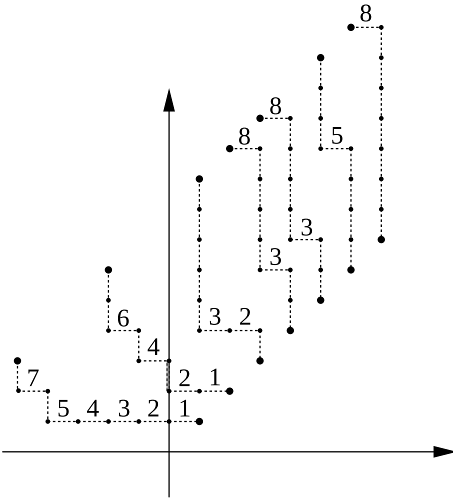


R^+
R^-





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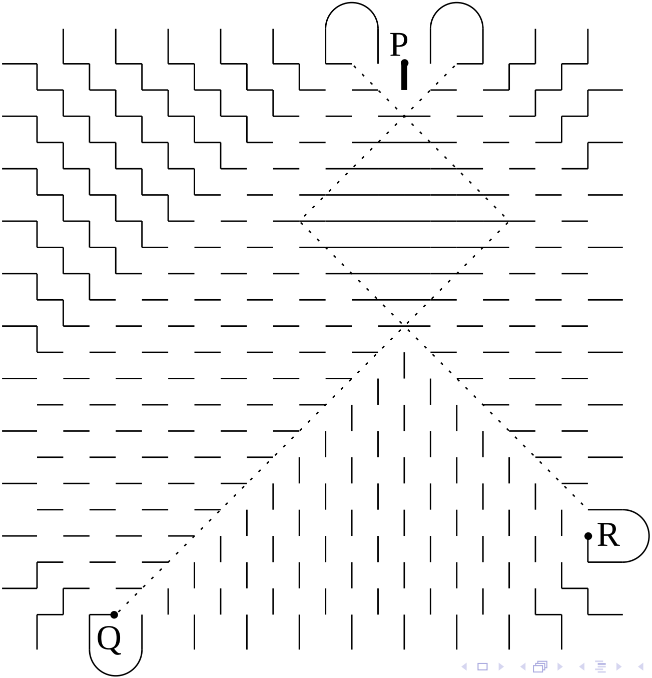
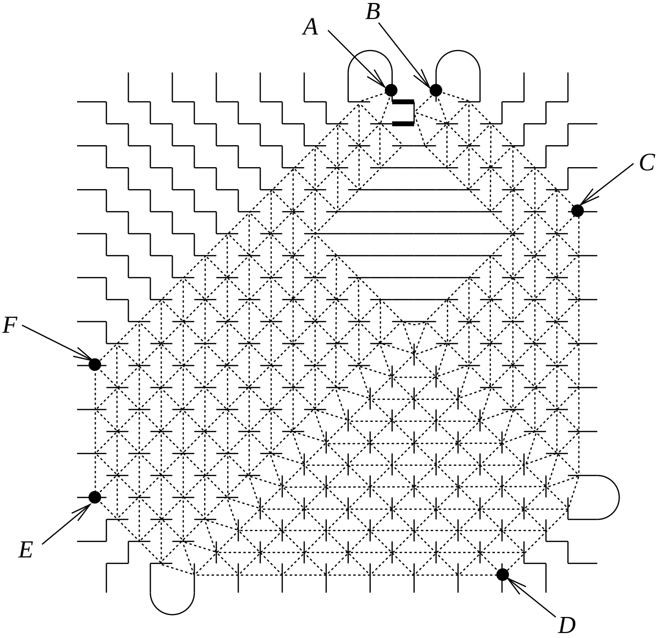
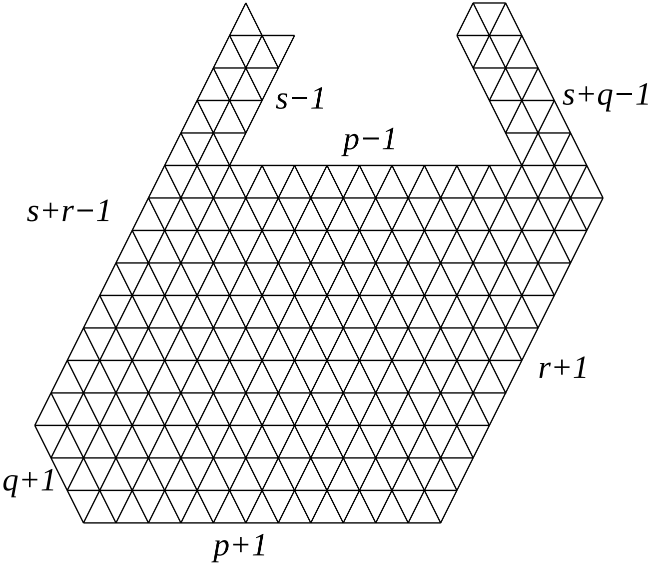
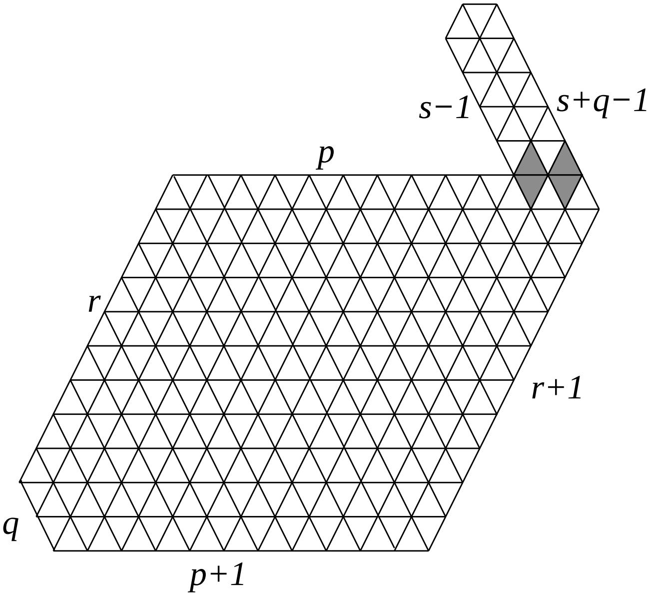

P
Q
R


A
B
C
D
E
F


$s-1$
$s+q-1$
$p-1$
$s+r-1$
$r+1$
$q+1$
$p+1$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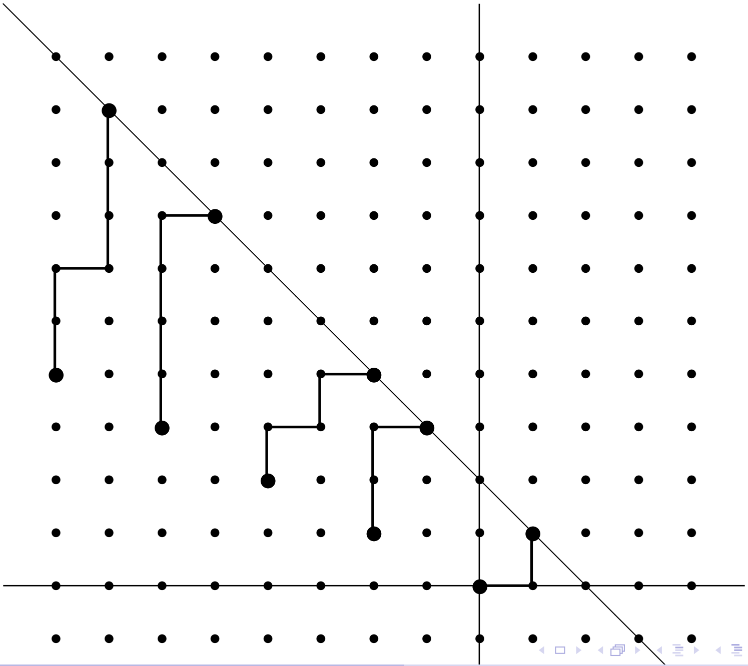
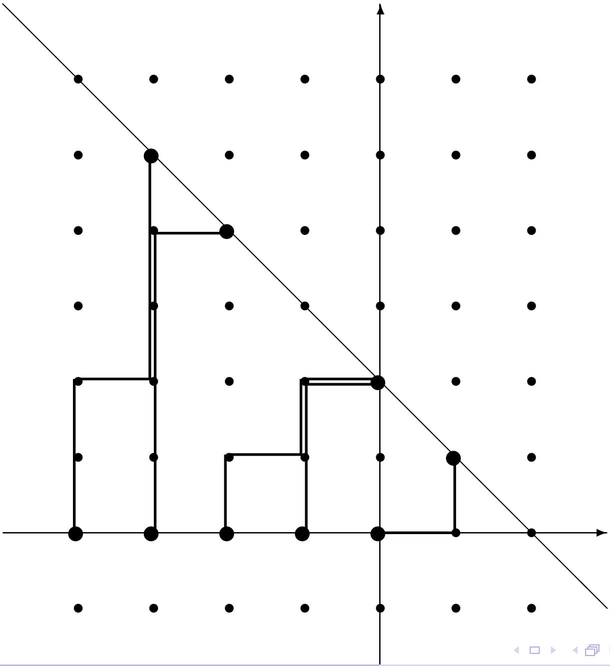
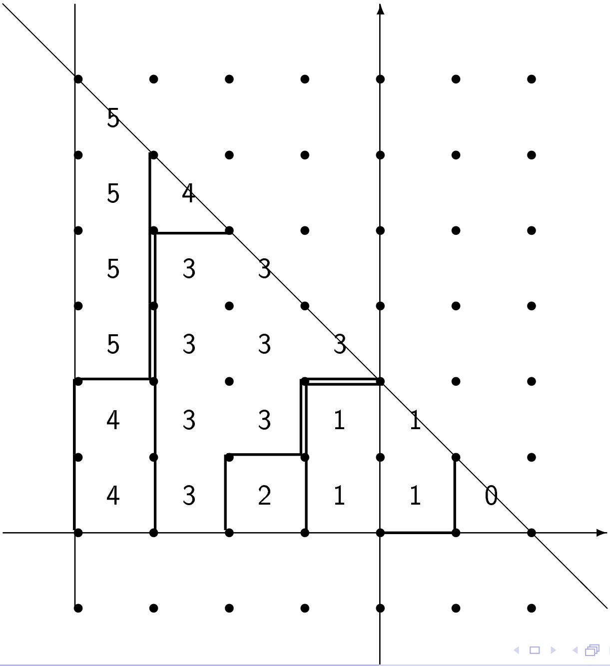
s+q−1
p
r
r+1
q
p+1






5
5 4
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