The second life of George Andrews' multidimensional extension of Watson's transformation formula

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The first life . . .

The Rogers–Ramanujan identities

The Rogers–Ramanujan identities (1894–1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}$$

Watson's tranformation formula (1929)

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})\,(a,b,c,d,e,q^{-N};q)_k}{(1-a)\,(q,aq/b,aq/c,aq/d,aq/e,aq^{N+1};q)_k} \left(\frac{a^2q^{N+2}}{bcde}\right)^k$$
$$= \frac{(aq,aq/de;q)_N}{(aq/d,aq/e;q)_N} \sum_{k=0}^{N} \frac{(aq/bc,d,e,q^{-N};q)_k}{(q,aq/b,aq/c,de/aq^N;q)_k} q^k,$$

where N is a nonnegative integer.

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$$= \frac{(aq,aq/de;q)_N}{(aq/d,aq/e;q)_N} \sum_{k=0}^{N} \frac{(aq/bc,d,e,q^{-N};q)_k}{(q,aq/b,aq/c,de/aq^N;q)_k} q^k,$$

where N is a nonnegative integer.
Letting $b,c,d,e,N\to\infty$, one obtains

$$\sum_{k=0}^{\infty}(-1)^k a^{2k} q^{k(5k-1)/2} \frac{1-aq^{2k}}{1-a} \frac{(a;q)_k}{(q;q)_k} = (aq;q)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q;q)_k}.$$

Now, specialisation of $a=1$ and $a=q$ yield the two Rogers–Ramanujan identities upon invoking Jacobi's triple product identity.

Generalising Watson's transformation formula

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})\,(a, b_1, c_1, \dots, b_m, c_m, q^{-N}; q)_k}{(1-a)\,(q, aq/b_1, aq/c_1, \dots, aq/b_m, aq/c_m, aq^{N+1}; q)_k} q^{?}\, ?^k$$

$$= \frac{(?; q)_N}{(?; q)_N} \sum_{k=0}^{N} ??.$$

Andrews' extension of Watson's transformation formula (1975)

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})\,(a,b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(1-a)\,(q,aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k} \cdot \left(\frac{a^m q^{N+m}}{b_1c_1\cdots b_mc_m}\right)^k$$

$$= \frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N} \sum_{N\ge i_{m-1}\ge\dots\ge i_1\ge 0} \frac{(q^{-N};q)_{i_{m-1}}}{(b_mc_mq^{-N}/a;q)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{q^{i_j}a^{i_{j-1}}}{(b_jc_j)^{i_{j-1}}} \frac{(aq/b_jc_j;q)_{i_j-i_{j-1}}\,(b_{j+1};q)_{i_j}\,(c_{j+1};q)_{i_j}}{(q;q)_{i_j-i_{j-1}}\,(aq/b_j;q)_{i_j}\,(aq/c_j)_{i_j}},$$

where i_0 is interpreted as 0.

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$$= \frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N} \sum_{N\ge i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{(q^{-N};q)_{i_{m-1}}}{(b_mc_mq^{-N}/a;q)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{q^{i_j}a^{i_{j-1}}}{(b_jc_j)^{i_{j-1}}} \frac{(aq/b_jc_j;q)_{i_j-i_{j-1}}\,(b_{j+1};q)_{i_j}\,(c_{j+1};q)_{i_j}}{(q;q)_{i_j-i_{j-1}}\,(aq/b_j;q)_{i_j}\,(aq/c_j)_{i_j}},$$

where i_0 is interpreted as 0.

This identity can be proved by an m-fold iteration along the Bailey chain.

Rogers–Ramanujan-type identities

In the previous identity let $b_1, c_1, \dots, b_m, c_m, N \to \infty$. This leads to

$$\sum_{k=0}^{\infty} (-1)^k q^{mk^2+\binom{k}{2}} a^{mk} \frac{(1-aq^{2k})\,(a;q)_k}{(1-a)\,(q;q)_k}$$

$$= (aq;q)_\infty \sum_{i_{m-1}\ge\dots\ge i_1\ge 0} \frac{q^{i_1^2+\dots+i_{m-1}^2} a^{i_1+\dots+i_{m-1}}}{(q;q)_{i_{m-1}-i_{m-2}} \cdots (q;q)_{i_2-i_1}\,(q;q)_{i_1}}.$$

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$$= (aq;q)_\infty \sum_{i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{q^{i_1^2+\cdots+i_{m-1}^2}\, a^{i_1+\cdots+i_{m-1}}}{(q;q)_{i_{m-1}-i_{m-2}}\cdots(q;q)_{i_2-i_1}\,(q;q)_{i_1}}.$$

Again, we specialise of $a=1$ and $a=q$.

Rogers–Ramanujan-type identities

Specialisation of $a = 1$ and $a = q$ together with Jacobi's triple product identity yields

$$\sum_{i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{q^{i_1^2+\cdots+i_{m-1}^2}}{(q;q)_{i_{m-1}-i_{m-2}}\cdots(q;q)_{i_2-i_1}(q;q)_{i_1}} = \frac{(q^m,q^{m+1},q^{2m+1};q^{2m+1})_\infty}{(q;q)_\infty}$$

and

$$\sum_{i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{q^{i_1^2+\cdots+i_{m-1}^2+i_1+\cdots+i_{m-1}}}{(q;q)_{i_{m-1}-i_{m-2}}\cdots(q;q)_{i_2-i_1}(q;q)_{i_1}} = \frac{(q,q^{2m},q^{2m+1};q^{2m+1})_\infty}{(q;q)_\infty}.$$

Intermission . . .

So much for the first life of Andrews' identity.

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Very subtle number-theoretic properties of Andrews' identity lay dormant for about 30 years . . .

I. Linear forms in zeta values

In order to prove irrationality results for values of the zeta function at positive integers, Rivoal investigated the following construction of linear forms in zeta values :

$$\mathbf{S}_{n,A,B,C,r} = n!^{A-2Br} \sum_{k=1}^{\infty} \frac{1}{C!} \frac{\partial^C}{\partial k^C} \left(\left(k + \frac{n}{2}\right) \frac{(k-rn)_{rn}^B (k+n+1)_{rn}^B}{(k)_{n+1}^A} \right),$$

where A, B, C, r are non-negative integers with A even and $0 \le 2Br < A$.

One can then expand

$$\mathbf{S}_{n,A,B,C,r} = \mathbf{p}_{0,C,n} + (-1)^C \sum_{l=1}^{A} \binom{C+l-1}{l-1} \mathbf{p}_{l,n}\, \zeta(C+l),$$

where, for $l \geq 1$,

$$\mathbf{p}_{l,n} = n!^{A-2Br} \sum_{j=0}^{n} \frac{1}{(A-l)!} \frac{\partial^{A-l}}{\partial k^{A-l}} \cdot \left(\left(k+\frac{n}{2}\right) \frac{(k-rn)_{rn}^{B}(k+n+1)_{rn}^{B}}{(k)_{n+1}^{A}} (k+j)^A \right) \Bigg|_{k=-j},$$

and a similar (but more complicated) formula for $\mathbf{p}_{0,n}$.

Yuri Nesterenko's criterion

Let $\xi_0, \xi_1, \dots, \xi_r$ be real numbers, $0 < \alpha < 1$, and $\beta > 1$. Furthermore, for all positive integers n let $\ell_{0,n}, \ell_{1,n}, \dots \ell_{r,n}$ be integers such that

(i) $\displaystyle \lim_{n\to\infty} \left| \sum_{i=0}^{r} \ell_{i,n}\xi_i \right|^{1/n} = \alpha\,;$

(ii) $\displaystyle \limsup_{n\to\infty} |\ell_{i,n}|^{1/n} \le \beta$ for all i.

Then

$$\dim \operatorname{Span}_{\mathbb{Q}}(\xi_0, \xi_1, \dots, \xi_r) \ge 1 - \frac{\log \alpha}{\log \beta}.$$

Let $\mathrm{d}_n := \operatorname{lcm}(1, 2, \ldots, n)$.
It is relatively straightforward to show that

$$\mathrm{d}_n^{A+C}\mathbf{p}_{0,C,n} \in \mathbb{Z}, \quad \mathrm{d}_n^{A-l}\mathbf{p}_{l,n} \in \mathbb{Z}, \ l \geq 1.$$

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However, computer experiments led Tanguy Rivoal to :

Denominator Conjecture

$$\mathrm{d}_n^{A+C-1} \mathbf{p}_{0,C,n} \in \mathbb{Z}, \quad \mathrm{d}_n^{A-l-1} \mathbf{p}_{l,n} \in \mathbb{Z},\ l \geq 1.$$

For example, if $r = 1$, $A = 4$, $B = 1$, $C = 0$, we have

$$\mathbf{p}_{3,n} = (-1)^n \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_j - 5H_{n-j} + H_{2n-j} - H_{n+j} + \frac{1}{\frac{n}{2} - j}\right).$$

where $H_m := \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers. According to the conjecture, this should be an integer. Really ?

$$\mathbf{p}_{3,n} = (-1)^n \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_j - 5H_{n-j} + H_{2n-j} - H_{n+j} + \frac{1}{\frac{n}{2} - j}\right).$$

$$\mathbf{p}_{3,n} = (-1)^n \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_j - 5H_{n-j} + H_{2n-j} - H_{n+j} + \frac{1}{\frac{n}{2} - j}\right).$$

So, wouldn't it be great if

$$(-1)^n \sum_{j=0}^{n} \left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{n+j}{n} \binom{2n-j}{n} \cdot \left(5H_j - 5H_{n-j} + H_{2n-j} - H_{n+j} + \frac{1}{\frac{n}{2} - j}\right) = - \sum_{0 \le i \le j \le n} (-1)^j \binom{n}{j} \binom{n}{i}^2 \binom{n+j}{n} \binom{n+j-i}{n} ?$$

By (guided) guessing, Rivoal produced more of these strange identities. Finally, we were able to produce a proof. This proof consisted of an application of a balanced ${}_4F_3$-transformation formula and a repeated application of the Pfaff–Saalschütz formula.

Subsequently, we put as many parameters as possible into this proof. The resulting identity is coming up on the next page.

A monstrosity

$$\frac{2\pi}{\sin(\pi a_1)} \cdot \frac{\Gamma(n+b_1+b_2-a_1+1)\,\Gamma(n+b_1+1)}{n!\,\Gamma(n-a_2+b_1+b_2+1)\,\Gamma(a_2-b_2+1)}$$
$$\times \sum_{j=0}^{n} \left(\left(\frac{n}{2}+\frac{a_1}{2}-j\right) \binom{n}{j} \cdot \frac{\Gamma(a_3+1)}{\Gamma(-a_1+j+1)\,\Gamma(n+a_1-j+1)} \right.$$
$$\cdot \frac{\Gamma(-a_1+a_2+j+1)}{\Gamma(-n-a_1+a_3+j+1)\,\Gamma(n+a_2-a_3+1)} \frac{\Gamma(n+a_2-j+1)}{\Gamma(a_3-j+1)\,\Gamma(a_2-a_1+1)}$$
$$\left. \cdot \left(\prod_{k=1}^{2m} \frac{\Gamma(n-a_1+b_k+b_{k+2}+1)}{\Gamma(n+b_k-j+1)\,\Gamma(-a_1+b_k+j+1)} \right) \right)$$

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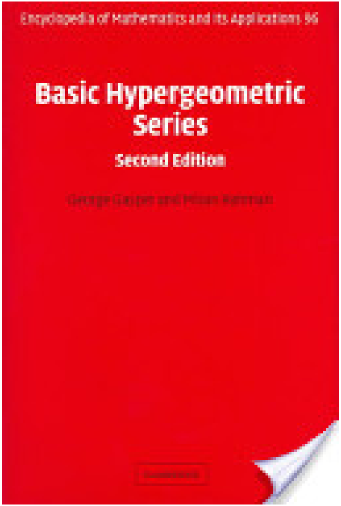
$$\times \sum_{j=0}^{n} \Bigg(\left(\frac{n}{2}+\frac{a_1}{2}-j\right) \binom{n}{j} \cdot \frac{\Gamma(a_3+1)}{\Gamma(-a_1+j+1)\,\Gamma(n+a_1-j+1)}$$
$$\cdot \frac{\Gamma(-a_1+a_2+j+1)}{\Gamma(-n-a_1+a_3+j+1)\,\Gamma(n+a_2-a_3+1)} \frac{\Gamma(n+a_2-j+1)}{\Gamma(a_3-j+1)\,\Gamma(a_2-a_1+1)}$$
$$\cdot \left(\prod_{k=1}^{2m} \frac{\Gamma(n-a_1+b_k+b_{k+2}+1)}{\Gamma(n+b_k-j+1)\,\Gamma(-a_1+b_k+j+1)} \right) \Bigg)$$

(actually not quite ...)

A monstrosity

$$= (-1)^n \sum_{0\le i_1\le i_2\le\cdots\le i_m\le n} \binom{a_2-b_1}{n-i_m}\binom{-a_1+a_2+i_m-i_{m-1}}{i_m-i_{m-1}}$$
$$\cdot \frac{\Gamma(n+b_1+b_2-a_1+1)}{\Gamma(a_2-a_3+i_m+1)\,\Gamma(n+a_3-a_1-a_2+b_1+b_2-i_m+1)}$$
$$\cdot \frac{\Gamma(a_2+i_m+1)}{\Gamma(a_2-b_2-n+i_m+1)\,\Gamma(n+b_2+1)}$$
$$\cdot \frac{\Gamma(n+a_3-a_1-a_2+b_1+b_2-i_{m-1}+1)}{\Gamma(-a_1+a_3-i_{m-1}+1)\,\Gamma(n-a_2+b_1+b_2+1)}$$
$$\cdot \Bigg(\prod_{k=1}^{m-1}\binom{n-a_1+b_{2m-2k+1}+b_{2m-2k+2}+i_k-i_{k-1}}{i_k-i_{k-1}}$$
$$\cdot \frac{\Gamma(n-a_1+b_{2m-2k-1}+b_{2m-2k+1}+1)}{\Gamma(b_{2m-2k+1}+i_k+1)\,\Gamma(n-a_1+b_{2m-2k-1}-i_k+1)}$$
$$\cdot \frac{\Gamma(n-a_1+b_{2m-2k}+b_{2m-2k+2}+1)}{\Gamma(b_{2m-2k+2}+i_k+1)\,\Gamma(n-a_1+b_{2m-2k}-i_k+1)}\Bigg).$$

The bible


Encyclopedia of Mathematics and its Applications 96
Basic Hypergeometric Series
Second Edition

Other inspiring reading

George E. Andrews, *Problems and prospects for basic hypergeometric functions*, Theory and application of special functions, R. A. Askey, ed., Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, pp. 191–224, 1975.

The $q = 1$ special case of Andrews' identity :

$$\sum_{k=0}^{N} \frac{(a+2k)\,(a)_k\,(b_1)_k\,(c_1)_k \cdots}{a \cdot k!\,(1+a-b_1)_k\,(1+a-c_1)_k \cdots} \cdot \frac{\cdots (b_m)_k\,(c_m)_k\,(-N)_k}{\cdots (1+a-b_m)_k\,(1+a-c_m)_k\,(1+a+N)_k}$$
$$= \frac{(1+a)_N\,(1+a-b_m-c_m)_N}{(1+a-b_m)_N\,(1+a-c_m)_N} \times \sum_{N \ge i_{m-1} \ge \cdots \ge i_1 \ge 0} \frac{(-N)_{i_{m-1}}}{(b_m+c_m-a-N)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{(1+a-b_j-c_j)_{i_j-i_{j-1}}\,(b_{j+1})_{i_j}\,(c_{j+1})_{i_j}}{(i_j-i_{j-1})!\,(1+a-b_j)_{i_j}\,(1+a-c_j)_{i_j}}.$$

I. Linear forms in zeta values

In the end, it turned out that, after application of a balanced ${}_4F_3$-transformation formula to the innermost sum, the "monstrosity" is just equivalent to (the $q = 1$ special case of) Andrews' identity.

Denominator Theorem (K., Rivoal (2003))

$$\mathrm{d}_n^{A+C-1}\mathbf{p}_{0,C,n} \in \mathbb{Z}, \quad \mathrm{d}_n^{A-l-1}\mathbf{p}_{l,n} \in \mathbb{Z}, \; l \geq 1.$$

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$$\mathrm{d}_n^{A+C-1}\mathbf{p}_{0,C,n} \in \mathbb{Z}, \quad \mathrm{d}_n^{A-l-1}\mathbf{p}_{l,n} \in \mathbb{Z},\ l \geq 1.$$

Although this proved several open conjectures at the time, the consequences for irrationality questions remained somewhat modest.

For example, Rivoal's original result that one out of $\zeta(5), \zeta(7), \ldots, \zeta(21)$ is irrational can be improved to asserting that one out of $\zeta(5), \zeta(7), \ldots, \zeta(19)$ is irrational. However, in the meantime Zudilin had already shown that one out of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, by using a more general hypergeometric construction of linear forms.

In combination with a refinement of Nesterenko's criterion, due to Fischler and Zudilin (2007), some new irrationality results could be obtained by these authors.

But what about q?

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A q-zeta function has been introduced by

$$\zeta_q(s) = \sum_{k=1}^{\infty} \sigma_{s-1}(k)\, q^k = \sum_{m=1}^{\infty} m^{s-1} \frac{q^m}{1-q^m},$$

where $\sigma_{s-1}(k) = \sum_{d|k} d^{s-1}$.

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where $\sigma_{s-1}(k) = \sum_{d|k} d^{s-1}$.

Don't miss Frédéric Jouhet's talk for the strongest irrationality results for these q-zeta values!

In the late 1990s, D. V. Vasilyev considered the integrals

$$J_{E,n}=\int_{[0,1]^E}\frac{\prod_{i=1}^{E}x_i^n(1-x_i)^n}{1-(\cdots(1-(1-x_E)x_{E-1})\cdots)x_1)^{n+1}}\cdot dx_1\,dx_2\cdots dx_E.$$

Conjecture (Vasilyev)

(i) *For all integers $E\ge 2$ and $n\ge 0$, there exist rational numbers $p_{m,E,n}$ such that*

$$J_{E,n}=p_{0,E,n}+\sum_{\substack{m=2,\dots,E\\ m\equiv E\ (\mathrm{mod}\ 2)}}p_{m,E,n}\,\zeta(m).$$

(ii) *Furthermore, $d_n^E\,p_{m,E,n}$ is an integer for all $m=0,2,3,\dots,E$.*

II. An integral identity

$$J_{E,n} = \int_{[0,1]^E} \frac{\prod_{i=1}^{E} x_i^n (1-x_i)^n}{1-(\cdots(1-(1-x_E)x_{E-1})\cdots)x_1)^{n+1}} \cdot dx_1\, dx_2 \cdots dx_E .$$

$$J_{E,n} = \int_{[0,1]^E} \frac{\prod_{i=1}^{E} x_i^n (1-x_i)^n}{1-(\cdots(1-(1-x_E)x_{E-1})\cdots)x_1)^{n+1}} \cdot dx_1\, dx_2 \cdots dx_E .$$

Zudilin proved Part (i) of the conjecture, by establishing a (more general) integral identity. Let us define

$$J_m \left[\begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix} ; z \right] = \int_{[0,1]^m} \frac{\prod_{i=1}^{m} x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{(1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1 z)^{a_0}} \, dx_1\, dx_2 \cdots dx_m .$$

Theorem (Zudilin (2002))

For every integer $m \ge 1$, we have

$$J_m\left[\begin{matrix} h_1, h_2, h_3, \dots, h_{m+1} \\ 1+h_0-h_3, 1+h_0-h_4, \dots, 1+h_0-h_{m+2} \end{matrix}; 1\right]$$

$$= \frac{\Gamma(1+h_0)\prod_{j=3}^{m+1}\Gamma(h_j)}{\prod_{j=1}^{m+2}\Gamma(1+h_0-h_j)} \cdot \left(\prod_{j=1}^{m+1}\Gamma(1+h_0-h_j-h_{j+1})\right)$$

$$\times \sum_{k=0}^{\infty} \frac{(h_0+2k)\,(h_0)_k\,(h_1)_k \cdots (h_{m+2})_k}{h_0 \cdot k!\,(1+h_0-h_1)_k \cdots (1+h_0-h_{m+2})_k}(-1)^{(m+1)k}.$$

Expanding the integral

$$J_m\begin{bmatrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{bmatrix}; z\end{bmatrix}$$

$$= \int_{[0,1]^m} \frac{\prod_{i=1}^m x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{(1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1 z)^{a_0}}\, dx_1\, dx_2 \cdots dx_m.$$

Expanding the integral

$$J_m\left[\begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix}; z\right]$$
$$= \int_{[0,1]^m} \frac{\prod_{i=1}^m x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{(1-(1-(\cdots(1-x_m)x_{m-1})\cdots)x_1 z)^{a_0}}\, dx_1\, dx_2 \cdots dx_m.$$

Consider the case where $m = 2s$ even. Let

$$Q_{2s}(x_1, x_2, \dots, x_{2s}) = 1 - (\cdots(1-(1-x_{2s})x_{2s-1})\cdots)x_1 z$$

denote the denominator term. We expand

$$Q_{2s}(x_1, \dots, x_{2s}; z) = Q_{2s-2}(x_1, \dots, x_{2s-2}; z) - z x_1 \cdots x_{2s-1}(1-x_{2s})$$
$$= Q_{2s-2}(x_1, \dots, x_{2s-2}; z)\left(1 - \frac{z x_1 \cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1, \dots, x_{2s-2}; z)}\right).$$

Expanding the integral

We apply the binomial theorem :

$$\left(1-\frac{zx_1\cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)}\right)^{-a_0} \\ =\sum_{k_1=0}^{\infty} z^{k_1}\,\frac{\Gamma(a_0+k_1)}{\Gamma(a_0)\Gamma(k_1+1)}\left(\frac{x_1\cdots x_{2s-1}(1-x_{2s})}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)}\right)^{k_1}.$$

Hence

$$J_{2s}\left[\begin{matrix}a_0,a_1,\dots,a_{2s}\\ b_1,\dots,b_{2s}\end{matrix};z\right]=\int_{[0,1]^{2s}}\sum_{k_1=0}^{\infty}\frac{z^{k_1}\,\Gamma(a_0+k_1)}{\Gamma(a_0)\Gamma(k_1+1)}x_{2s-1}^{a_{2s-1}+k_1-1}(1-x_{2s-1})^{b_{2s-1}-a_{2s-1}-1} \\ \cdot\, x_{2s}^{a_{2s}-1}(1-x_{2s})^{b_{2s}-a_{2s}+k_1-1}\frac{\prod_{j=1}^{2s-2}x_j^{a_j+k_1-1}(1-x_j)^{b_j-a_j-1}}{Q_{2s-2}(x_1,\dots,x_{2s-2};z)^{k_1+a_0}}\,dx_1\cdots dx_{2s}.$$

Expanding the integral

$$J_{2s}\left[\begin{matrix} a_0, a_1, \dots, a_{2s} \\ b_1, \dots, b_{2s} \end{matrix}; z\right] = \frac{\Gamma(a_{2s})\,\Gamma(a_{2s-1})\,\Gamma(b_{2s}-a_{2s})\,\Gamma(b_{2s-1}-a_{2s-1})}{\Gamma(b_{2s})\,\Gamma(b_{2s-1})}$$

$$\cdot \sum_{k_1=0}^{\infty} z^{k_1} \frac{(b_{2s}-a_{2s})_{k_1}\,(a_{2s-1})_{k_1}\,(a_0)_{k_1}}{k_1!\,(b_{2s})_{k_1}\,(b_{2s-1})_{k_1}} J_{2s-2}\left[\begin{matrix} a_0+k_1, a_1+k_1, \dots, a_{2s-2}+k_1 \\ b_1+k_1, \dots, b_{2s-2}+k_1 \end{matrix}; z\right].$$

Iteration then leads to the following result.

Expanding the integral

Proposition

We have

$$J_{2s}\left[\begin{matrix} a_0, a_1, \dots, a_{2s} \\ b_1, \dots, b_{2s} \end{matrix}; 1\right] = \prod_{j=1}^{2s} \frac{\Gamma(a_j)\,\Gamma(b_j - a_j)}{\Gamma(b_j)}$$

$$\times \sum_{k_1,k_2,\dots,k_s \ge 0} \prod_{j=1}^{s} \frac{(b_{2s-2j+2} - a_{2s-2j+2})_{k_j}}{k_j!} \frac{(a_{2s-2j+1})_{k_1+\dots+k_j}}{(b_{2s-2j+1})_{k_1+\dots+k_j}} \cdot \frac{(a_{2s-2j})_{k_1+\dots+k_j}}{(b_{2s-2j+2})_{k_1+\dots+k_j}}.$$

There holds a similar identity if m is odd.

What would we have to show to reprove Zudilin's identity ?

Theorem (Zudilin (2002))

For every integer $m \ge 1$, we have

$$J_m\left[\begin{matrix} h_1, h_2, h_3, \dots, h_{m+1} \\ 1+h_0-h_3, 1+h_0-h_4, \dots, 1+h_0-h_{m+2} \end{matrix}; 1\right]$$

$$= \frac{\Gamma(1+h_0)\prod_{j=3}^{m+1}\Gamma(h_j)}{\prod_{j=1}^{m+2}\Gamma(1+h_0-h_j)} \cdot \left(\prod_{j=1}^{m+1}\Gamma(1+h_0-h_j-h_{j+1})\right)$$

$$\times \sum_{k=0}^{\infty} \frac{(h_0+2k)\,(h_0)_k\,(h_1)_k\cdots(h_{m+2})_k}{h_0\cdot k!\,(1+h_0-h_1)_k\cdots(1+h_0-h_{m+2})_k}(-1)^{(m+1)k}.$$

The $q = 1$ special case of Andrews' identity again :

$$\sum_{k=0}^{N} \frac{(a+2k)\,(a)_k\,(b_1)_k\,(c_1)_k\cdots}{a\cdot k!\,(1+a-b_1)_k\,(1+a-c_1)_k\cdots} \cdot \frac{\cdots(b_m)_k\,(c_m)_k\,(-N)_k}{\cdots(1+a-b_m)_k\,(1+a-c_m)_k\,(1+a+N)_k}$$
$$= \frac{(1+a)_N\,(1+a-b_m-c_m)_N}{(1+a-b_m)_N\,(1+a-c_m)_N} \times \sum_{N\ge i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{(-N)_{i_{m-1}}}{(b_m+c_m-a-N)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{(1+a-b_j-c_j)_{i_j-i_{j-1}}\,(b_{j+1})_{i_j}\,(c_{j+1})_{i_j}}{(i_j-i_{j-1})!\,(1+a-b_j)_{i_j}\,(1+a-c_j)_{i_j}}.$$

The $q = 1$ special case of Andrews' identity again :

$$\sum_{k=0}^{N} \frac{(a+2k)\,(a)_k\,(b_1)_k\,(c_1)_k \cdots}{a \cdot k!\,(1+a-b_1)_k\,(1+a-c_1)_k \cdots} \cdot \frac{\cdots (b_m)_k\,(c_m)_k\,(-N)_k}{\cdots (1+a-b_m)_k\,(1+a-c_m)_k\,(1+a+N)_k}$$

$$= \frac{(1+a)_N\,(1+a-b_m-c_m)_N}{(1+a-b_m)_N\,(1+a-c_m)_N} \times \sum_{N \ge i_{m-1} \ge \cdots \ge i_1 \ge 0} \frac{(-N)_{i_{m-1}}}{(b_m+c_m-a-N)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{(1+a-b_j-c_j)_{i_j-i_{j-1}}\,(b_{j+1})_{i_j}\,(c_{j+1})_{i_j}}{(i_j-i_{j-1})!\,(1+a-b_j)_{i_j}\,(1+a-c_j)_{i_j}}.$$

By reindexing $i_j = k_1 + \cdots + k_j$, one obtains an equivalent form.

Equivalent form of the $q = 1$ case of Andrews' identity

$$\sum_{k=0}^{N} \frac{(a+2k)\,(a)_k\,(b_1)_k\,(c_1)_k \cdots}{a \cdot k!\,(1+a-b_1)_k\,(1+a-c_1)_k \cdots} \cdot \frac{\cdots (b_m)_k\,(c_m)_k\,(-N)_k}{\cdots (1+a-b_m)_k\,(1+a-c_m)_k\,(1+a+N)_k}$$
$$= \frac{(1+a)_N\,(1+a-b_m-c_m)_N}{(1+a-b_m)_N\,(1+a-c_m)_N} \times \sum_{k_1,k_2,\dots,k_{m-1}\ge 0} \frac{(-N)_{k_1+\dots+k_{m-1}}}{(b_m+c_m-a-N)_{k_1+\dots+k_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\dots+k_j}\,(c_{j+1})_{k_1+\dots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\dots+k_j}\,(1+a-c_j)_{k_1+\dots+k_j}}.$$

II. An integral identity

Equivalent form of the $q=1$ case of Andrews' identity

$$\sum_{k=0}^{N} \frac{(a+2k)\,(a)_k\,(b_1)_k\,(c_1)_k\cdots}{a\cdot k!\,(1+a-b_1)_k\,(1+a-c_1)_k\cdots} \cdot \frac{\cdots(b_m)_k\,(c_m)_k\,(-N)_k}{\cdots(1+a-b_m)_k\,(1+a-c_m)_k\,(1+a+N)_k}$$
$$= \frac{(1+a)_N\,(1+a-b_m-c_m)_N}{(1+a-b_m)_N\,(1+a-c_m)_N} \times \sum_{k_1,k_2,\dots,k_{m-1}\ge 0} \frac{(-N)_{k_1+\dots+k_{m-1}}}{(b_m+c_m-a-N)_{k_1+\dots+k_{m-1}}}$$
$$\cdot \prod_{j=1}^{m-1} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\dots+k_j}\,(c_{j+1})_{k_1+\dots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\dots+k_j}\,(1+a-c_j)_{k_1+\dots+k_j}}.$$

We would have to derive an "infinite" form of Andrews' identity!
This is easy : just let $N\to\infty$.

The Legendre transform

Given a sequence $(c_k)_{k\geq 0}$, transform it to the sequence $(a_n)_{n\geq 0}$ by

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k .$$

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Conjecture (A. Schmidt (1992))

For each positive integer r, the sequence

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r$$

is the Legendre transform of a sequence of integers.

The inverse of the Legendre transform

We have

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k$$

if and only if

$$c_n = \binom{2n}{n}^{-1} \sum_{k=0}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} a_k.$$

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Conjecture (A. Schmidt (1992) ; rephrased)

For each positive integer r, the sequence

$$c_n^{(r)} := \binom{2n}{n}^{-1} \sum_{k=0}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} \sum_{j=0} k \binom{k}{j}^{r} \binom{k+j}{j}^{r}$$

is a sequence of integers.

Partial results

Conjecture (A. Schmidt (1992) ; rephrased)

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$$= \sum_{j=0}^{n} \binom{2n}{n}^{-1} \binom{2j}{j}^r \sum_{k=j}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} \binom{k+j}{k-j}^r$$

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$$= \sum_{j=0}^{n} \binom{2n}{n}^{-1} \binom{2j}{j}^r \sum_{k=j}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} \binom{k+j}{k-j}^r$$

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The case $r = 1$ is trivial.
Asmus Schmidt and Volker Strehl were able to resolve the cases $r = 2$ and $r = 3$.

Wadim Zudilin's solution (2003)

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We want to show that

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$$t_{n,j}^{(r)} = \binom{2n}{n}^{-1} \binom{2j}{j} \sum_{k=j}^{n} (-1)^{n-k} \frac{2k+1}{n+k+1} \binom{2n}{n-k} \binom{k+j}{k-j}^{r}.$$

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If $r = 2m+1$ is odd, then choose $a = -2n-1$, $b_1 = c_1 = \cdots = b_m = c_m = -N = -n+j$ in Andrews' identity. With this choice, everything condenses to

$$t_{n,j}^{(r)} = \binom{n}{j}^{2} \sum_{i_{m-1} \ge \cdots \ge i_1 \ge 0} \binom{2j}{n - i_{m-1} - j} \cdot \prod_{k=1}^{m-1} \binom{2j}{i_k - i_{k-1}} \binom{n - i_k + j}{n - i_k - j}^{2}.$$

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This is manifestly an integer.

There is a similar specialisation which proves the case where r is even.

IV. A polynomiality result

In 1998, Neil Calkin proved that

$$\binom{2n}{n}^{-1} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k}^r$$

is an integer for all positive integers n and r.

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Theorem (Guo, Jouhet, Zeng (2005))

For all positive integers $n_1, \dots, n_m, n_{m+1} = n_1$, the alternating sum

$$\begin{bmatrix} n_1 + n_m \\ n_1 \end{bmatrix}_q^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{jk^2 + \binom{k}{2}} \prod_{j=1}^{m} \begin{bmatrix} n_j + n_{j+1} \\ n_j + k \end{bmatrix}_q$$

is a polynomial in q with non-negative integral coefficients for $0 \le j \le m-1$.

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is a polynomial in q with non-negative integral coefficients for $0 \le j \le m-1$.

They prove this by first observing that it is sufficient to establish this for $j = m-1$. Then they prove

$$\begin{bmatrix} n_1 + n_m \\ n_1 \end{bmatrix}_q^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{(m-1)k^2 + \binom{k}{2}} \prod_{j=1}^{m} \begin{bmatrix} n_j + n_{j+1} \\ n_j + k \end{bmatrix}_q$$

$$= \sum_{n_1 \ge i_{m-2} \ge \dots \ge i_1 \ge 0} \prod_{j=1}^{m-2} q^{i_j^2} \begin{bmatrix} i_{j+1} \\ i_j \end{bmatrix}_q \begin{bmatrix} n_j + n_{j+1} \\ n_j + i_j \end{bmatrix}_q ,$$

where $i_{m-1} = n_1 = n_{m+1}$.

The identity again :

$$\begin{bmatrix} n_1+n_m \\ n_1 \end{bmatrix}_q^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{(m-1)k^2+\binom{k}{2}} \prod_{j=1}^{m} \begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix}_q$$

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Andrews' identity

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})\,(a,b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(1-a)\,(q,aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k} \cdot \left(\frac{a^m q^{N+m}}{b_1c_1\cdots b_mc_m}\right)^k$$

$$= \frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N} \sum_{N\ge i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{(q^{-N};q)_{i_{m-1}}}{(b_mc_mq^{-N}/a;q)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{q^{i_j}a^{i_{j-1}}}{(b_jc_j)^{i_{j-1}}} \frac{(aq/b_jc_j;q)_{i_j-i_{j-1}}\,(b_{j+1};q)_{i_j}\,(c_{j+1};q)_{i_j}}{(q;q)_{i_j-i_{j-1}}\,(aq/b_j;q)_{i_j}\,(aq/c_j)_{i_j}}.$$

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Guo, Jouhet and Zeng replace m by $m-1$ in Andrews' identity, and then choose $a \to 1$ (!), $c_1, \dots, c_{m-1} \to \infty$, $b_j = q^{-n_j}$, $j = 1, 2, \dots, m-1$.

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$$= \frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N} \sum_{N\ge i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{(q^{-N};q)_{i_{m-1}}}{(b_mc_mq^{-N}/a;q)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{q^{i_j}a^{i_{j-1}}}{(b_jc_j)^{i_{j-1}}} \frac{(aq/b_jc_j;q)_{i_j-i_{j-1}}\,(b_{j+1};q)_{i_j}\,(c_{j+1};q)_{i_j}}{(q;q)_{i_j-i_{j-1}}\,(aq/b_j;q)_{i_j}\,(aq/c_j)_{i_j}}.$$

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$$\begin{bmatrix} n_1 + n_m \\ n_1 \end{bmatrix}_q^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{(m-1)k^2 + \binom{k}{2}} \prod_{j=1}^{m} \begin{bmatrix} n_j + n_{j+1} \\ n_j + k \end{bmatrix}_q$$

$$= \sum_{n_1 \ge i_{m-2} \ge \cdots \ge i_1 \ge 0} \prod_{j=1}^{m-2} q^{i_j^2} \begin{bmatrix} i_{j+1} \\ i_j \end{bmatrix}_q \begin{bmatrix} n_j + n_{j+1} \\ n_j + i_j \end{bmatrix}_q .$$

Guo, Jouhet and Zeng replace m by $m-1$ in Andrews' identity, and then choose $a \to 1$ (!), $c_1, \dots, c_{m-1} \to \infty$, $b_j = q^{-n_j}$, $j = 1, 2, \dots, m-1$. After simplification, the result is the above identity !

By q-ifying Zudilin's approach, Guo, Jouhet and Zeng also succeed to define a natural q-analogue of the Legendre transform and to provide a q-analogue of the Asmus–Schmidt–problem.

V. Quantum invariants for 3-manifolds

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A typical result is :

Theorem (Anna Beliakova, Thang Lê (2007))

For every closed 3-manifold M and every root of unity ξ of odd order, the Witten–Reshetikhin–Turaev quantum $SO(3)$ invariant $\tau_M(\xi)$ is an element of $\mathbb{Z}[\xi]$.

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The integrality has many important applications, among them is the construction of an integral topological quantum field theory and representations of mapping class groups over $\mathbb{Z}$. The integrality is also a key property required for the categorification of quantum 3-manifold invariants.

A lot of q-calculus :

$$\{n\} := q^{n/2} - q^{-n/2} \qquad \{n\}! := \{1\}\{2\}\cdots\{n\}$$

$$[n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \qquad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{\{n\}!}{\{k\}!\,\{n-k\}!}$$

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In order to express this invariant $\tau_M(\xi)$, we need the quantity

$$Y_c(k,b) = (-1)^k \sum_{n=-\infty}^{\infty} \begin{bmatrix} 2k+1 \\ k+nc \end{bmatrix} q^{cbn^2}.$$

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We would like to prove the following divisibility result for $Y_c(b,k)$:

Theorem

For $b \in \mathbb{Z}$ and $k \le (r-3)/2$, $\frac{\gamma_d(\xi)}{\gamma_1(\xi)} Y_c(k,b)\big|_{q=\xi}$ is divisible by $\frac{\{2k+1\}!}{\{k\}!}\big|_{q=\xi}$, where $\gamma_d(\xi) = \sum_{j=0}^{r-1} \xi^{d(j^2+j)}$ and ξ is a primitive r-th root of unity.

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We write :

$$Y_c(k,b) = (-1)^k \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \left(1+\sum_{n=1}^{\infty} q^{cbn^2}(1+q^{cn}) \frac{q^{(k+1)cn}(q^{-k-1};q)_{cn}}{(q^{N+1};q)_{cn}}\right).$$

Let $s = (c+1)/2$, $k+1 \equiv t \pmod{c}$, $0 \le t \le c-1$. Now one applies Andrews' identity with $m = b+s$, $a \to 1$, $q \to q^c$, $b_j = q^{j+t-k-1}$, $c_j = q^{-j+t-k-1}$, $j = 1,2,\dots,s-1$, $b_s = q^{t-k-1}$, $c_s = q^{(k+2)c}$, $b_j, c_j \to \infty$, $j = s+1,\dots,m$. Under this specialisation, the left-hand side of Andrews' identity produces the expression in parentheses. After a careful analysis of the right-hand side, the result follows.

VI. George Andrews striking back again !

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In *"Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks"* [Invent. Math. **169** (2007), 37–73], George Andrews returned to his "old" (?) identity, and used it to derive generating function identities for "Durfee symbols."

Andrews' extension of Watson's transformation formula (1975)

$$\sum_{k=0}^{N} \frac{(1-aq^{2k})\,(a,b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(1-a)\,(q,aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k} \cdot \left(\frac{a^m q^{N+m}}{b_1c_1\cdots b_mc_m}\right)^k$$
$$= \frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N} \sum_{N\ge i_{m-1}\ge\cdots\ge i_1\ge 0} \frac{(q^{-N};q)_{i_{m-1}}}{(b_mc_mq^{-N}/a;q)_{i_{m-1}}} \cdot \prod_{j=1}^{m-1} \frac{q^{i_j}a^{i_{j-1}}}{(b_jc_j)^{i_{j-1}}} \frac{(aq/b_jc_j;q)_{i_j-i_{j-1}}\,(b_{j+1};q)_{i_j}\,(c_{j+1};q)_{i_j}}{(q;q)_{i_j-i_{j-1}}\,(aq/b_j;q)_{i_j}\,(aq/c_j)_{i_j}},$$

where i_0 is interpreted as 0.