# Music AND Mathematics?

## Personal views on a difficult relationship

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Computer presentation designed by Theresia Eisenkölbl

# Music AND Mathematics?

Robert SCHUMANN (1810 – 1856)
„Aveu“ from Carnaval op. 9

*"Mathematics and Music, they are so close to each other!"*

*"Mathematics and Music, they are so close to each other!"*

"Is that really so?"

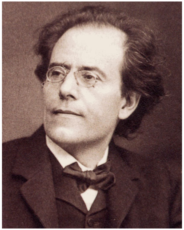
A prominent visitor to the Dean of the Fakultät für Mathematik of the University of Vienna:

*"I hear that you are chairing a department of pianists!"*

**Why are there so many mathematicians who also have a strong affinity for music, and why are there so many musicians who also have a strong affinity for mathematics?**

**How do we imagine the typical mathematician?**

## How do we imagine the typical mathematician?

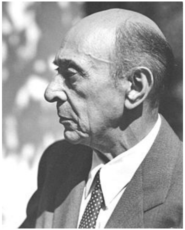


Gustav
Mahler

**How do we imagine the typical mathematician?**

Dmitry
Shostakovich

## How do we imagine the typical mathematician?



Arnold Schönberg

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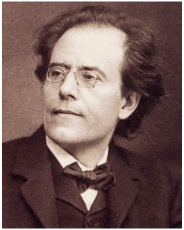

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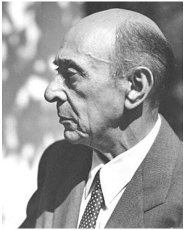

Arnold
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## How do we imagine the typical musician?

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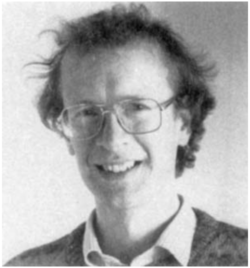
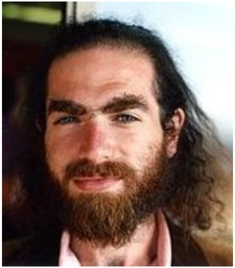


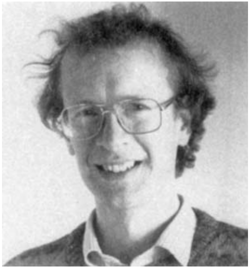
Andrew
Wiles

## How do we imagine the typical musician?

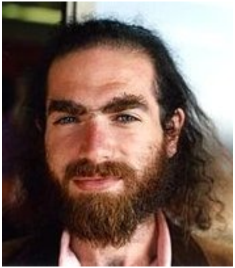


Grigori
Perelman

## How do we imagine the typical musician?



Andrew
Wiles



Grigori
Perelman

Music ... $\longrightarrow$ art form

Music $\quad \ldots \quad \longrightarrow$ art form

Mathematics $\quad \ldots \quad \longrightarrow$ science

$1 \quad \sqrt{2} \quad 12 \quad 8 \quad \oint$

$5 \sum k^2 \quad 16\% \quad 0.123456789\ldots$

$\frac{3}{14}$

$8 \quad \pi \quad \frac{1+\sqrt{5}}{2}$

$27 \quad 175560 \quad \frac{2}{7} \quad 52954$

$20.5 \quad e \quad 14 \quad 99.999\ldots$

**Why are there so many mathematicians who also have a strong affinity for music, and why are there so many musicians who also have a strong affinity for mathematics?**

**Both Mathematics AND Music**

**are food for the soul AND the brain.**

# Soul in music

**Good cheer!**

Scott JOPLIN (1867/1868? – 1917)
Maple Leaf Rag (beginning)

# Soul in music

**Bad mood!**

Robert Schumann (1810 – 1856)
Pantalon et Colombine (beginning) from Carnaval op. 9

# Soul in music

**heartbreakingly sad**

Franz Schubert (1797 – 1828)
Andantino (beginning) from the Sonata in A major, D 959

# Soul in music

# Soul in music

**transcendently joyful**

Franz SCHUBERT (1797 – 1828)
Impromptu A flat major, D 899, Nr. 4 (end)

# Soul in music

**Elegance**

Frédéric Chopin (1810 – 1849)
Grande Valse Brillante in E flat major, op. 18 (beginning)

# Soul in music

## Humour in music

**Humour in music**

Max Reger (1873 – 1916)
Humoreske D major, op. 20/1

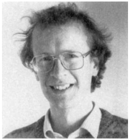
# Soul in music

**Tour de force!**

Franz Liszt (1811 – 1886)
Sonata b minor (excerpt)

# Soul in mathematics

# Soul in mathematics



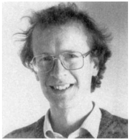
Andrew
Wiles

**Theorem (Wiles, Taylor 1995)**

(*Fermat's Last Theorem*)

*Let $n$ be a natural number which is at least 3. Then there are no natural numbers $x, y, z$ such that*

$$x^n + y^n = z^n.$$



Andrew Wiles

# Soul in mathematics

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Excerpt from a reviewer's report on a mathematical article:

"This is a very nice paper."

# Soul in mathematics

Excerpt from a reviewer's report on a mathematical article:

"This is a very nice paper."

"This is a very elegant proof!"

# Elegance

**Theorem**

*There are infinitely many prime numbers.*

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*There are infinitely many prime numbers.*

**Proof:** Suppose that there are only finitely prime numbers, say $2, 3, 5, 7, 11, 13, \dots, 1031$.
Now we consider

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \cdots \cdot 1031 + 1.$$

This (big) number can be decomposed into prime factors. However, 2 does not divide this number, 3 also does not divide it, nor does $5, \dots, 1031$. Consequently, these were not yet all prime numbers.

# Elegance

**Theorem**

*There are infinitely many prime numbers.*

**Proof:** Suppose that there are only finitely prime numbers, say $p_1, p_2, p_3, p_4, p_5, p_6, \ldots, p_n$.
Now we consider

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot \cdots \cdot p_n + 1.$$

This (big) number can be decomposed into prime factors. However, $p_1$ does not divide this number, $p_2$ also does not divide it, nor does $p_3$, ..., $p_n$. Consequently, these were not yet all prime numbers.

# Soul in mathematics

**Humour in mathematics**

**Humour in mathematics**

Excerpt from a reviewer's report on a mathematical article:
"This is a funny construction!"

# Humour in mathematics

# Humour in mathematics

Srinivasa Ramanujan

## Humour in mathematics

A *partition* of a number $n$ is a representation of this number as a sum of positive integers such that the summands are arranged in (weakly) increasing order.

Srinivasa Ramanujan

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Srinivasa Ramanujan

$n = 1$: $1$
$n = 2$: $2,\ 1+1$
$n = 3$: $3,\ 1+2,\ 1+1+1$
$n = 4$: $4,\ 1+3,\ 2+2,\ 1+1+2,\ 1+1+1+1$
$n = 5$: $5,\ 1+4,\ 2+3,\ 1+1+3,\ 1+2+2,\ 1+1+1+2,$
$1+1+1+1+1$

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$n = 3$: $3,\ 1+2,\ 1+1+1$

$n = 4$: $4,\ 1+3,\ 2+2,\ 1+1+2,\ 1+1+1+1$

$n = 5$: $5,\ 1+4,\ 2+3,\ 1+1+3,\ 1+2+2,\ 1+1+1+2,$
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## Humour in mathematics

Let $p(n)$ denote the number of all partitions of $n$.

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$n = 1$: 1
$n = 2$: 2, $1 + 1$
$n = 3$: 3, $1 + 2$, $1 + 1 + 1$
$n = 4$: 4, $1 + 3$, $2 + 2$, $1 + 1 + 2$, $1 + 1 + 1 + 1$
$n = 5$: 5, $1 + 4$, $2 + 3$, $1 + 1 + 3$, $1 + 2 + 2$, $1 + 1 + 1 + 2$, $1 + 1 + 1 + 1 + 1$

$$\begin{array}{ccccc}
p(1) = 1 & p(2) = 2, & p(3) = 3, & p(4) = 5, & p(5) = 7 \\
p(6) = 11 & p(7) = 15, & p(8) = 22, & p(9) = 30, & p(10) = 42 \\
p(11) = 56 & p(12) = 77, & p(13) = 101, & p(14) = 135, & p(15) = 176 \\
p(16) = 231 & p(17) = 297, & p(18) = 385, & p(19) = 490, & p(20) = 627
\end{array}$$

## Humour in mathematics

Let $p(n)$ denote the number of all partitions of $n$.

$n=1$: $1$
$n=2$: $2,\ 1+1$
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$n=4$: $4,\ 1+3,\ 2+2,\ 1+1+2,\ 1+1+1+1$
$n=5$: $5,\ 1+4,\ 2+3,\ 1+1+3,\ 1+2+2,\ 1+1+1+2,$
$1+1+1+1+1$

$$
\begin{array}{ccccc}
p(1)=1 & p(2)=2, & p(3)=3, & {\color{red}p(4)=5,} & p(5)=7 \\
p(6)=11 & p(7)=15, & p(8)=22, & {\color{red}p(9)=30,} & p(10)=42 \\
p(11)=56 & p(12)=77, & p(13)=101, & {\color{red}p(14)=135,} & p(15)=176 \\
p(16)=231 & p(17)=297, & p(18)=385, & {\color{red}p(19)=490,} & p(20)=627
\end{array}
$$

# Humour in mathematics

## Humour in mathematics

Srinivasa Ramanujan

## Humour in mathematics

**Theorem ("Ramanujan's most beautiful theorem" 1919)**

$p(5n+4)$ *is always divisible by 5.*

Srinivasa Ramanujan

## Theorem (Euler)

$$1 + p(1)q + p(2)q^2 + p(3)q^3 + p(4)q^4 + \cdots$$
$$= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}$$

## Ramanujan's proof:

The product $(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots$ will be abbreviated by $(q;q)_\infty$.

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### Lemma

*Let $\omega^5 = 1$, $\omega \neq 1$. Then*

$$(q;q)_\infty(\omega q;\omega q)_\infty(\omega^2 q;\omega^2 q)_\infty(\omega^3 q;\omega^3 q)_\infty(\omega^4 q;\omega^4 q)_\infty = \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty}.$$

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### Lemma

$$\frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty} = q^{-1}R - 1 - qR^{-1},$$

*where* $R$ *is a power series in* $q^5$.

## Ramanujan's proof:

**Lemma**

$$q^{-5}R^5 - 11 - q^5R^{-5} = \frac{(q^5;q^5)_\infty^6}{q^5(q^{25};q^{25})_\infty^6}.$$

## Ramanujan's proof:

$$
\begin{aligned}
1 &+ p(1)q + p(2)q^2 + p(3)q^3 + p(4)q^4 + p(5)q^5 \\
&\quad + p(6)q^6 + p(7)q^7 + p(8)q^8 + p(9)q^9 + p(10)q^{10} \\
&\quad + p(11)q^{11} + p(12)q^{12} + p(13)q^{13} + p(14)q^{14} + \dots \\
&= q^4 \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \cdot (q^{-4}R^4 + q^{-3}R^3 + 2q^{-2}R^2 + 3q^{-1}R + 5 \\
&\qquad - 3qR^{-1} + 2q^2R^{-2} - q^3R^{-3} + q^4R^{-4})
\end{aligned}
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$$p(4)q^4 + p(9)q^9 + p(14)q^{14} + \dots$$

$$= q^4 \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \cdot 5$$

## Ramanujan's proof:

$$p(4)q^4 + p(9)q^9 + p(14)q^{14} + \cdots = q^4 \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \times 5$$

# Soul in mathematics

**Tour de force!**

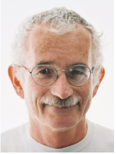
## Doron Zeilberger's theorem on alternating sign matrices

Doron
Zeilberger

## Doron Zeilberger's theorem on alternating sign matrices

An alternating sign matrix is a quadratic arrangement of 0's, 1's, and $(-1)$'s such that

- in each row and in each column 1's and $(-1)$'s alternate (if one ignores the 0's) and
- in each row and in each column, the first and last entry different from 0 is a 1.



Doron Zeilberger

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Doron Zeilberger

Example:

$$\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

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One-row alternating sign matrices: 1

Two-row alternating sign matrices: $\begin{matrix}1&0\\0&1\end{matrix} \qquad \begin{matrix}0&1\\1&0\end{matrix}$

Three-row alternating sign matrices:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \qquad \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} \qquad \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} \qquad \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix}$$

$$\begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix} \qquad \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix} \qquad \begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix}$$

## Doron Zeilberger's theorem on alternating sign matrices

One-row alternating sign matrices: $1$

Two-row alternating sign matrices: $\begin{matrix}1&0\\0&1\end{matrix} \qquad \begin{matrix}0&1\\1&0\end{matrix}$

Three-row alternating sign matrices:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \quad \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} \quad \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} \quad \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} \quad \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix} \quad \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix} \quad \begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix}$$

Let $A(n)$ denote the number of $n$-row alternating sign matrices.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $A(n)$ | 1 | 2 | 7 | 42 | 429 | 7436 |

Conjecture (Mills, Robbins, Rumsey $\sim$ 1980)

$$A(n) = \frac{1! \cdot 4! \cdot 7! \cdot \cdots \cdot (3n-2)!}{n! \cdot (n+1)! \cdot (n+2)! \cdot \cdots \cdot (2n-1)!},$$

*where* $m! = m \cdot (m-1) \cdot (m-2) \cdot \cdots \cdot 2 \cdot 1.$

**PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE** [1]

*Doron ZEILBERGER*[2]

Submitted: May 1, 1995; Accepted: July 25, 1995

**Checked by**[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

**Dedicated to my Friend, Mentor, and Guru, Dominique Foata.**

**Lemma 1:** For $n \geq k \geq 1$, the number of $n \times k$-Gog trapezoids equals the number of $n \times k$-Magog trapezoids.

[ The number of n by k Magog trapezoids, for specific n and k, is obtained by typing `b(k,n);` while the number of n by k Gog trapezoids is given by `m(k,n);`. To verify lemma 1, type `S1(k,n):`.]

This would imply, by setting $n = k$, that,

**Corollary 1':** For $n \geq 1$, the number of $n$-Gog triangles equals the number of $n$-Magog triangles.

Since $n$-Gog triangles are equi-numerous with $n \times n$ alternating sign matrices, and $n$-Magog triangles are equi-numerous with TSSCPPs bounded in $[0, 2n]^3$, this would imply, together with Andrews's[A2] affirmative resolution of the TSCCPP conjecture, the following result, that was conjectured in [MRR1].

**The Alternating Sign Matrix Theorem**: The number of $n \times n$ alternating sign matrices, for $n \geq 1$, is:

$$\frac{1!4!\dots(3n-2)!}{n!(n+1)!\dots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \quad .$$

We now need the following $(sub)^6$ lemma:

**Subsubsubsubsubsublemma 1.2.1.2.1.1.1:** Let $U_j$, $j = 1, \dots, l$, be quantities in an associative algebra, then:

$$1 - \prod_{j=1}^{l} U_j = \sum_{j=1}^{l} \left\{ \prod_{h=1}^{j-1} U_h \right\} (1 - U_j) \quad .$$

Doron Zeilberger in a talk 1991

*"Extreme UGLINESS is new BEAUTY!"*

## Brains in mathematics

**Brains in mathematics** ✓

**Brains in mathematics** ✓

**Brains in music?**

Franz SCHUBERT (1797 – 1828) Sonata in A major, D 959

1. *Allegro.*
2. *Andantino.*
3. *Scherzo. Allegro vivace.*
4. *Rondo. Allegretto.*

Franz Schubert (1797 – 1828) Sonata in A major, D 959

1. *Allegro.*
2. *Andantino.*
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4. *Rondo. Allegretto.*

## Differences between music and mathematics

# Differences between music and mathematics

Ludwig van BEETHOVEN (1770 – 1827)
Sonata f minor, op. 57, „Appassionata"

# Differences between music and mathematics

# Differences between music and mathematics

7. $\operatorname{Sec} x = E_1 + \frac{x^2}{\underline{|2}} E_3 + \frac{x^4}{\underline{|4}} E_5 + \frac{x^6}{\underline{|6}} E_7 + \&c$ 32

Cor. $\frac{B_{2n}}{2n} 2^{2n}(2^{2n}-1) = 2 E_1 E_{2n-1} + 2 E_3 E_{2n-3} \frac{(2n-2)(2n-3)}{\underline{|2}} + \&c$

the last term being $2 E_{n-1} E_{n+1} \frac{\underline{|2n-2}}{\underline{|n-2}\,\underline{|n}}$ or $(E_n)^2 \frac{\underline{|2n-2}}{(\underline{|n-1})^2}$

according as $n$ is even or odd.

Sol. $\frac{d \tan x}{dx} = \operatorname{Sec}^2 x$. Equate the coeff$^{ts}$ of $x^{2n-2}$

$E_1 = 1,\ E_3 = 1,\ E_5 = 5,\ E_7 = 61,\ E_9 = 1385,\ E_{11} = 50521,$

$E_{13} = 2702765,\ E_{15} = 199360981,$ &c.

8. i. $\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots \&c$

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c

and $\frac{\sin x}{x} = 1$ when $x = 0$.

ii. In a similar manner

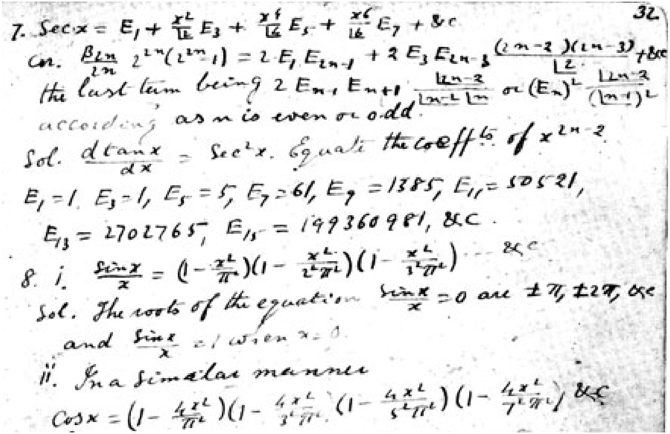
$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\left(1 - \frac{4x^2}{7^2\pi^2}\right) \&c$

Srinivasa Ramanujan (1887 – 1920)
Notebook I

## Differences between music and mathematics

Ludwig van BEETHOVEN (1770 – 1827)
Sonata f minor, op. 57, „Appassionata"

# Differences between music and mathematics


$7.\ \sec x = E_1 + \frac{x^2}{\lfloor\underline{2}} E_3 + \frac{x^4}{\lfloor\underline{4}} E_5 + \frac{x^6}{\lfloor\underline{6}} E_7 + \&c$ 32

Cor. $\frac{B_{2n}}{2n} 2^{2n}(2^{2n}-1) = 2E_1E_{2n-1} + 2E_3E_{2n-3}\frac{(2n-2)(2n-3)}{\lfloor\underline{2}} + \&c$

the last term being $2E_{n-1}E_{n+1}\frac{\lfloor\underline{2n-2}}{\lfloor\underline{n-2}\,\lfloor\underline{n}}$ or $(E_n)^2\frac{\lfloor\underline{2n-2}}{(\lfloor\underline{n-1})^2}$

according as $n$ is even or odd.

Sol. $\frac{d\tan x}{dx} = \sec^2 x$. Equate the coeff^ts of $x^{2n-2}$

$E_1 = 1,\ E_3 = 1,\ E_5 = 5,\ E_7 = 61,\ E_9 = 1385,\ E_{11} = 50521,$

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Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c

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Srinivasa RAMANUJAN (1887 – 1920)
Notebook I

$$\mathcal{E}$$

Alban BERG (1885 – 1935)
Sonata op. 1


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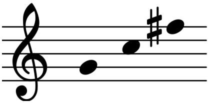
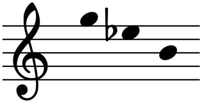


ascent figure



thirds figure

sigh motive

Alban Berg (1885 – 1935)
Sonata op. 1

Alban Berg (1885 – 1935)
Sonata op. 1

Johannes Brahms (1833 – 1897)
Intermezzo in b minor, op. 119/1