

Positive m -divisible non-crossing partitions and their cyclic sieving

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Prologue

24th International Conference on Formal Power Series and
Algebraic Combinatorics, Nagoya, Hotel Lobby, 2012

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Christian to Christian: *“Willst Du ein weiteres
,Zyklisches-Sieben-Phänomen‘ beweisen?”*¹

¹ *“Do you want to prove another cyclic sieving phenomenon?”*

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Christian to Christian: *“Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen‘ beweisen?”*¹

Christian to Christian: *“Sicher. Warum nicht?”*²

¹ *“Do you want to prove another cyclic sieving phenomenon?”*

² *“Sure. Why not?”*

m -divisible non-crossing partitions associated with reflection groups

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Let W be a finite real reflection group.

The *absolute length* (*reflection length*) $\ell_T(w)$ of an element $w \in W$ is defined by the smallest k such that

$$w = t_1 t_2 \cdots t_k,$$

where all t_i are reflections.

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The *absolute order* (*reflection order*) \leq_T is defined by

$$u \leq_T w \quad \text{if and only if} \quad \ell_T(u) + \ell_T(u^{-1}w) = \ell_T(w).$$

m -divisible non-crossing partitions associated with reflection groups

Definition (ARMSTRONG)

The m -divisible non-crossing partitions for a reflection group W are defined by

$$NC^{(m)}(W) = \{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and} \\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\},$$

where c is a Coxeter element in W .

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where c is a Coxeter element in W .

In particular,

$$NC^{(1)}(W) \cong NC(W),$$

the “ordinary” non-crossing partitions for W .

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Combinatorial realisation in type A (Armstrong)

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EXAMPLE FOR $m = 3$, $W = A_6 (= S_7)$:

$w_0 = (4, 5, 6)$, $w_1 = (3, 6)$, $w_2 = (1, 7)$, and $w_3 = (1, 2, 6)$. Indeed,

$$(4, 5, 6) (3, 6) (1, 7) (1, 2, 6) = (1, 2, 3, 4, 5, 6, 7) = c$$

and

$$\ell_T((4, 5, 6)) + \ell_T((3, 6)) + \ell_T((1, 7)) + \ell_T((1, 2, 6)) \\ = 2 + 1 + 1 + 2 = 6 = \ell_T((1, 2, 3, 4, 5, 6, 7)).$$

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Now “blow-up” w_1, w_2, w_3 :

$$(1, 2, \dots, 21) (7, 16)^{-1} (2, 20)^{-1} (3, 6, 18)^{-1}$$

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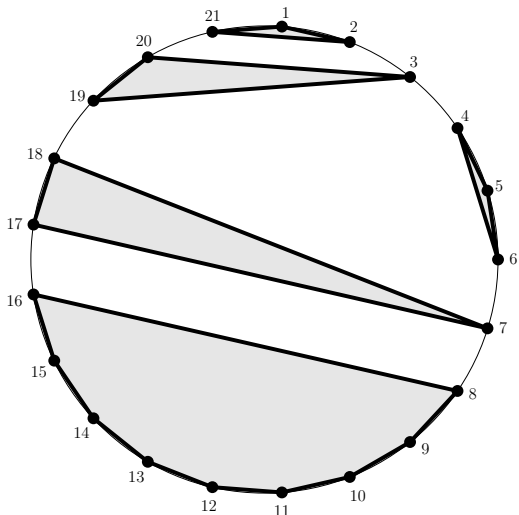
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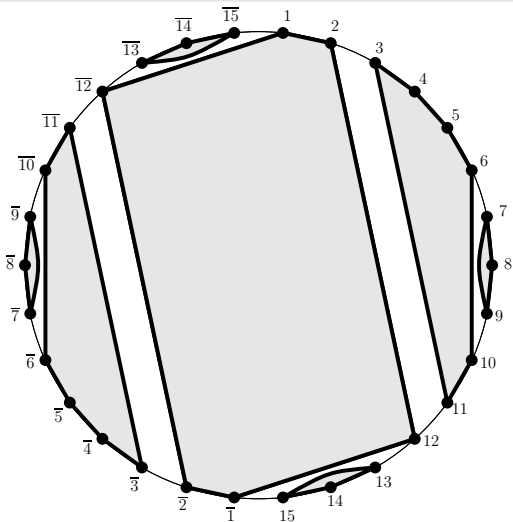
$$(1, 2, \dots, 21)(7, 16)^{-1}(2, 20)^{-1}(3, 6, 18)^{-1} \\ = (1, 2, 21)(3, 19, 20)(4, 5, 6)(7, 17, 18)(8, 9, \dots, 16).$$

m -divisible non-crossing partitions associated with reflection groups

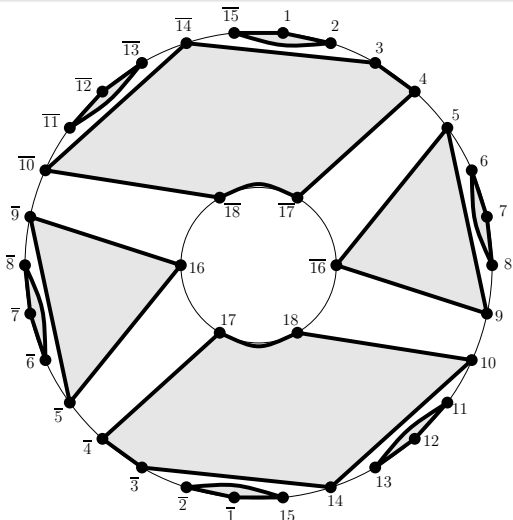


A 3-divisible non-crossing partition of type A_6

m -divisible non-crossing partitions associated with reflection groups



m -divisible non-crossing partitions associated with reflection groups



A 3-divisible non-crossing partition of type D_6

Positive m -divisible non-crossing partitions

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These were defined by Buan, Reiten and Thomas, as an aside in “ *m -noncrossing partitions and m -clusters.*” There, they constructed a bijection between the facets of the m -cluster complex of Fomin and Reading and the m -divisible non-crossing partitions of Armstrong.

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The positive m -clusters are those which do not contain any negative roots. They are enumerated by the *positive Fuß–Catalan numbers*

$$\text{Cat}_+^{(m)}(W) := \prod_{i=1}^n \frac{mh + d_i - 2}{d_i}.$$

Here, $d_1 | d_2 \leq \dots \leq d_n$ are the *degrees* of W , and $h = d_n$ is the *Coxeter number* of W .

Positive m -divisible non-crossing partitions

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One can give an intrinsic definition:

Definition

An m -divisible non-crossing partition $(w_0; w_1, \dots, w_m)$ in $NC^{(m)}(W)$ is **positive**, if and only if $w_0 w_1 \cdots w_{m-1}$ is not contained in any proper standard parabolic subgroup of W .

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Let $NC_+^{(m)}(W)$ denote the set of all positive m -divisible non-crossing partitions for W .

Trivial corollary:

$$|NC_+^{(m)}(W)| = \text{Cat}_+^{(m)}(W) = \prod_{i=1}^n \frac{mh + d_i - 2}{d_i}.$$

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Buan, Reiten and Thomas then write:

“Other than that, there do not seem to be enumerative results known for these families.”

Enumeration of positive m -divisible non-crossing partitions

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For “ordinary” m -divisible non-crossing partitions, closed-form enumeration results are known for:

- total number;
- number of those of given rank;
- number of those with given block sizes (in types A , B , D);
- number of chains;
- number of chains with elements at given ranks;
- number of chains with elements at given ranks and bottom element with given block sizes (in types A , B , D).

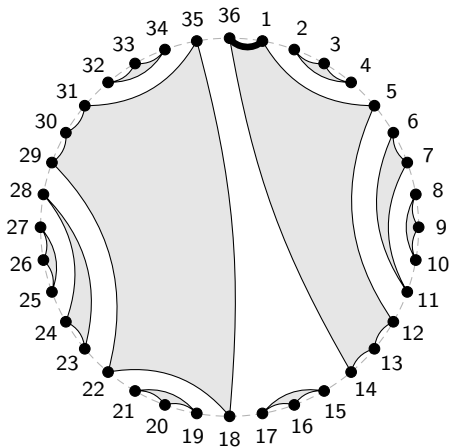
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Fact: Under Armstrong's map, the elements of $NC_+^{(m)}(A_{n-1})$ correspond to those m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ in which mn and 1 are in the same block.

How do elements of $NC_+^{(m)}(A_{n-1})$ look like?



A **positive** 3-divisible non-crossing partition of type A_{11}

Theorem

Let m, n be positive integers, The total number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ is given by

$$\frac{1}{n} \binom{(m+1)n-2}{n-1}.$$

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Theorem

Let m, n, l be positive integers, The number of multi-chains $\pi_1 \leq \pi_2 \leq \dots \leq \pi_{l-1}$ in the poset of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ is given by

$$\frac{1 + (l-1)(m-1)}{n-1} \binom{n-1 + (l-1)(mn-1)}{n-2}.$$

Theorem

Let m and n be positive integers, For non-negative integers b_1, b_2, \dots, b_n , the number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which have exactly b_i blocks of size mi , $i = 1, 2, \dots, n$, is given by

$$\frac{1}{mn-1} \binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \binom{mn-1}{b_1 + b_2 + \dots + b_n}$$

if $b_1 + 2b_2 + \dots + nb_n = n$, and 0 otherwise.

Theorem

Let m, n, l be positive integers, and let s_1, s_2, \dots, s_l be non-negative integers with $s_1 + s_2 + \dots + s_l = n - 1$. The number of multi-chains $\pi_1 \leq \pi_2 \leq \dots \leq \pi_{l-1}$ in the poset of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\text{rk}(\pi_i) = s_1 + s_2 + \dots + s_i$, $i = 1, 2, \dots, l - 1$, is given by

$$\frac{mn - s_2 - s_3 - \dots - s_l - 1}{(mn - 1)n} \binom{n}{s_1} \binom{mn - 1}{s_2} \dots \binom{mn - 1}{s_l}.$$

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$$\frac{mn - b_1 - b_2 - \dots - b_n}{(mn - 1)(b_1 + b_2 + \dots + b_n)} \binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \times \binom{(l-1)(mn-1)}{b_1 + b_2 + \dots + b_n - 1}$$

if $b_1 + 2b_2 + \dots + nb_n = n$, and 0 otherwise.

Theorem

Let m, n, l be positive integers, and let $s_1, s_2, \dots, s_l, b_1, b_2, \dots, b_n$ be non-negative integers with $s_1 + s_2 + \dots + s_l = n - 1$. The number of multi-chains $\pi_1 \leq \pi_2 \leq \dots \leq \pi_{l-1}$ in the poset of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ with the property that $\text{rk}(\pi_i) = s_1 + s_2 + \dots + s_i$, $i = 1, 2, \dots, l - 1$, and that the number of blocks of size mi of π_1 is b_i , $i = 1, 2, \dots, n$, is given by

$$\frac{mn - b_1 - b_2 - \dots - b_n}{(mn - 1)(b_1 + b_2 + \dots + b_n)} \binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \times \binom{mn - 1}{s_2} \dots \binom{mn - 1}{s_l}$$

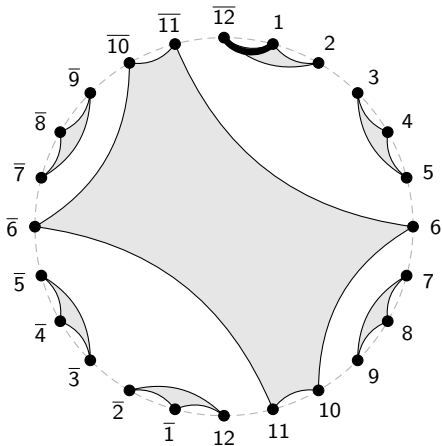
if $b_1 + 2b_2 + \dots + nb_n = n$ and $s_1 + b_1 + b_2 + \dots + b_n = n$, and 0 otherwise.

How do elements of $NC_+^{(m)}(B_n)$ look like?

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Fact: Under Armstrong's map, the elements of $NC_+^{(m)}(B_n)$ correspond to those m -divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ which are invariant under rotation by 180° , and in which **the block of 1 contains a negative element.**

How do elements of $NC_+^{(m)}(B_n)$ look like?



A **positive** 3-divisible non-crossing partition of type B_4

Enumeration in $NC_+^{(m)}(B_n)$

Theorem

Let m, n be positive integers, The total number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type B is given by

$$\binom{(m+1)n-1}{n}.$$

Theorem

Let m, n be positive integers, The total number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type B which have a zero block of size $2ma$ is given by

$$\binom{(m+1)n-a-2}{n-a}.$$

Theorem

Let m, n be positive integers. The number of positive m -divisible non-crossing partitions in $NC^{(m)}(B_n)$ with the property that the number of non-zero blocks of size mi of π_1 is $2b_i$, $i = 1, 2, \dots, n$, is given by

$$\binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \binom{mn - 1}{b_1 + b_2 + \dots + b_n}.$$

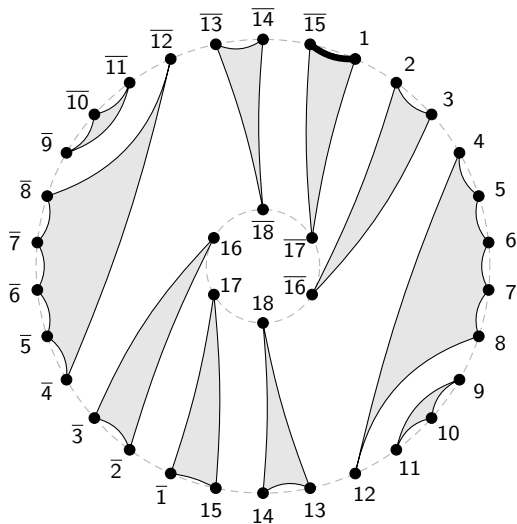
REMARK. We do not have results on chain enumeration.

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Fact: Under CK's map, the elements of $NC_+^{(m)}(D_n)$ correspond to those m -divisible non-crossing partitions on the annulus with $\{1, 2, \dots, m(n-1), -1, -2, \dots, -m(n-1)\}$ on the outer circle and $\{m(n-1) + 1, \dots, mn, -m(n-1) - 1, \dots, -mn\}$ on the inner circle which are invariant under rotation by 180° , satisfy the earlier mentioned and non-defined technical constraint, and in which the predecessor of 1 in its block is a negative element on the outer circle.

How do elements of $NC_+^{(m)}(D_n)$ look like?



A **positive** 3-divisible non-crossing partition of type D_6

Theorem

Let m and n be positive integers. The number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type D equals

$$\frac{2m(n-1) + n - 2}{n} \binom{(m+1)(n-1) - 1}{n-1},$$

while the number of these partitions of which all blocks have size m equals

$$\frac{2m(n-1) - n}{n} \binom{m(n-1) - 1}{n-1}.$$

REMARK. We did not go to finer levels although this may be doable.

A Fundamental Principle of Combinatorial Enumeration (2004ff)

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Every family of combinatorial objects satisfies the
cyclic sieving phenomenon!

Cyclic sieving (Reiner, Stanton, White)

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Ingredients:

- a set M of *combinatorial objects*,
- a *cyclic group* $C = \langle g \rangle$ acting on M ,
- a *polynomial* $P(q)$ in q with non-negative integer coefficients.

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- a set M of *combinatorial objects*,
- a *cyclic group* $C = \langle g \rangle$ acting on M ,
- a *polynomial* $P(q)$ in q with non-negative integer coefficients.

Definition

The triple (M, C, P) exhibits the *cyclic sieving phenomenon* if

$$|\text{Fix}_M(g^p)| = P\left(e^{2\pi ip/|C|}\right).$$

Cyclic sieving (Reiner, Stanton, White)

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$$M = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{1, 3\}, \{2, 4\}\}$$

$$g : j \mapsto j + 1 \pmod{4}$$

$$P(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

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$$|\text{Fix}_M(g^0)| = 6 = P(1) = P\left(e^{2\pi i \cdot 0/4}\right),$$

Cyclic sieving (Reiner, Stanton, White)

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$$|\text{Fix}_M(g^0)| = 6 = P(1) = P\left(e^{2\pi i \cdot 0/4}\right),$$

$$|\text{Fix}_M(g^1)| = 0 = P(i) = P\left(e^{2\pi i \cdot 1/4}\right),$$

Cyclic sieving (Reiner, Stanton, White)

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$$P(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

$$|\text{Fix}_M(g^0)| = 6 = P(1) = P\left(e^{2\pi i \cdot 0/4}\right),$$

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Cyclic sieving (Reiner, Stanton, White)

EXAMPLE:

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A Fundamental Principle of Combinatorial Enumeration (2004ff)

Every family of combinatorial objects satisfies the
cyclic sieving phenomenon!

A Fundamental Principle of Combinatorial Enumeration (2004ff)

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cyclic sieving phenomenon!

Corollary

The positive m -divisible non-crossing partitions satisfy the cyclic sieving phenomenon.

A cyclic action for m -divisible non-crossing partitions

A cyclic action for m -divisible non-crossing partitions

Let $K : NC^{(m)}(W) \rightarrow NC^{(m)}(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \\ \mapsto ((c w_m c^{-1}) w_0 (c w_m c^{-1})^{-1}; c w_m c^{-1}, w_1, w_2, \dots, w_{m-1}).$$

It generates a cyclic group of order mh .

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Furthermore, let

$$\text{Cat}^{(m)}(W; q) := \prod_{i=1}^n \frac{[mh + d_i]_q}{[d_i]_q},$$

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Theorem (with T. W. MÜLLER)

The triple $(NC^{(m)}(W), \langle K \rangle, \text{Cat}^{(m)}(W; q))$ exhibits the cyclic sieving phenomenon.

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Theorem (with T. W. MÜLLER)

Let $NC^{(m;0)}(W)$ denote the subset of $NC^{(m)}(W)$ consisting of those elements for which $w_0 = id$. Then the triple $(NC^{(m;0)}(W), \langle K \rangle, \text{Cat}^{(m-1)}(W; q))$ exhibits the cyclic sieving phenomenon.

A cyclic action for **positive** m -divisible non-crossing partitions?

A cyclic action for **positive** m -divisible non-crossing partitions?

Bad news:

The map $K : NC^{(m)}(W) \rightarrow NC^{(m)}(W)$ defined by

$$(w_0; w_1, \dots, w_m) \\ \mapsto ((cw_m c^{-1})w_0(cw_m c^{-1})^{-1}; cw_m c^{-1}, w_1, w_2, \dots, w_{m-1})$$

does not necessarily map positive m -divisible non-crossing partitions to positive ones!

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Bad news:

The map $K : NC^{(m)}(W) \rightarrow NC^{(m)}(W)$ defined by

$$(w_0; w_1, \dots, w_m) \\ \mapsto ((cw_m c^{-1})w_0(cw_m c^{-1})^{-1}; cw_m c^{-1}, w_1, w_2, \dots, w_{m-1})$$

does not necessarily map positive m -divisible non-crossing partitions to positive ones!

Consequently: we have to modify the above action.

A cyclic action for **positive** m -divisible non-crossing partitions?

A cyclic action for positive m -divisible non-crossing partitions?

Let $K_+ : NC^{(m)}(W) \rightarrow NC^{(m)}(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \\ \mapsto ((c w_{m-1}^R w_m c^{-1}) w_0 (c w_{m-1}^R w_m c^{-1})^{-1}; \\ c w_{m-1}^R w_m c^{-1}, w_1, \dots, w_{m-1}^L),$$

where $w_{m-1} = w_{m-1}^L w_{m-1}^R$ is the factorisation of w_{m-1} into its “left” and its “right” part.

A cyclic action for **positive** m -divisible non-crossing partitions?

Factorisation into “left” and “right” part

A cyclic action for **positive** m -divisible non-crossing partitions?

Factorisation into “left” and “right” part

Fix a reduced word $c = c_1 \cdots c_n$ for the Coxeter element c . Define the c -*sorting word* $w(c)$ for $w \in W$ to be the lexicographically first reduced word for w when written as a subword of c^∞ .

Let $w_\circ(c) = s_{k_1} \cdots s_{k_N}$ with $N = nh/2$ be the c -sorting word of the longest element $w_\circ \in W$.

The word $w_\circ(c)$ induces a *reflection ordering* given by

$$T = \left\{ s_{k_1} <_c s_{k_1} s_{k_2} s_{k_1} <_c s_{k_1} s_{k_2} s_{k_3} s_{k_2} s_{k_1} <_c \cdots \right. \\ \left. <_c s_{k_1} \cdots s_{k_{N-1}} s_{k_N} s_{k_{N-1}} \cdots s_{k_1} \right\}.$$

Associate to every element $w \in NC(W)$ a reduced T -word $\mathcal{T}_c(w)$ given by the lexicographically first subword of T that is a reduced T -word for w .

We decompose w as $w = w^L w^R$ where w^R is the part of $\mathcal{T}_c(w)$ within the last n reflections in T .

Cyclic sieving for **positive** m -divisible non-crossing partitions

Cyclic sieving for **positive** m -divisible non-crossing partitions

Let $K_+ : NC^{(m)}(W) \rightarrow NC^{(m)}(W)$ be the earlier defined map.

Furthermore, let

$$\text{Cat}_+^{(m)}(W; q) := \prod_{i=1}^n \frac{[mh + d_i - 2]_q}{[d_i]_q},$$

Cyclic sieving for **positive** m -divisible non-crossing partitions

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$$\text{Cat}_+^{(m)}(W; q) := \prod_{i=1}^n \frac{[mh + d_i - 2]_q}{[d_i]_q},$$

Conjecture

The triple $(NC_+^{(m)}(W), \langle K_+ \rangle, \text{Cat}_+^{(m)}(W; q))$ exhibits the cyclic sieving phenomenon.

Conjecture

Let $NC_+^{(m;0)}(W)$ denote the subset of $NC_+^{(m)}(W)$ consisting of those elements for which $w_0 = \text{id}$. Then the triple $(NC_+^{(m;0)}(W), \langle K_+ \rangle, \text{Cat}_+^{(m-1)}(W; q))$ exhibits the cyclic sieving phenomenon.

Cyclic sieving for **positive** m -divisible non-crossing partitions

24th International Conference on Formal Power Series and Algebraic Combinatorics, Nagoya, Hotel Lobby, **2012**

Christian to Christian: *“Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen‘ beweisen?”*¹

Christian to Christian: *“Sicher. Warum nicht?”*²

Cyclic sieving for **positive** m -divisible non-crossing partitions

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Christian to Christian: *“Willst Du ein weiteres ‚Zyklisches-Sieben-Phänomen‘ beweisen?”*¹

Christian to Christian: *“Sicher. Warum nicht?”*²

To actually carry this out turned out to be slightly more involved than originally anticipated by Christian.

Cyclic sieving for **positive** m -divisible non-crossing partitions

Theorem

The triple $(NC_+^{(m)}(W), \langle K_+ \rangle, \text{Cat}_+^{(m)}(W; q))$ exhibits the cyclic sieving phenomenon.

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Let $NC_+^{(m;0)}(W)$ denote the subset of $NC_+^{(m)}(W)$ consisting of those elements for which $w_0 = \text{id}$. Then the triple $(NC_+^{(m;0)}(W), \langle K_+ \rangle, \text{Cat}_+^{(m-1)}(W; q))$ exhibits the cyclic sieving phenomenon.

Cyclic sieving for **positive** m -divisible non-crossing partitions

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Our proof is by a careful case-by-case verification. Along the way, we also prove some **finer** cyclic sieving phenomena.

Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

Realisation of the cyclic action in type A_{n-1}

Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

Realisation of the cyclic action in type A_{n-1}

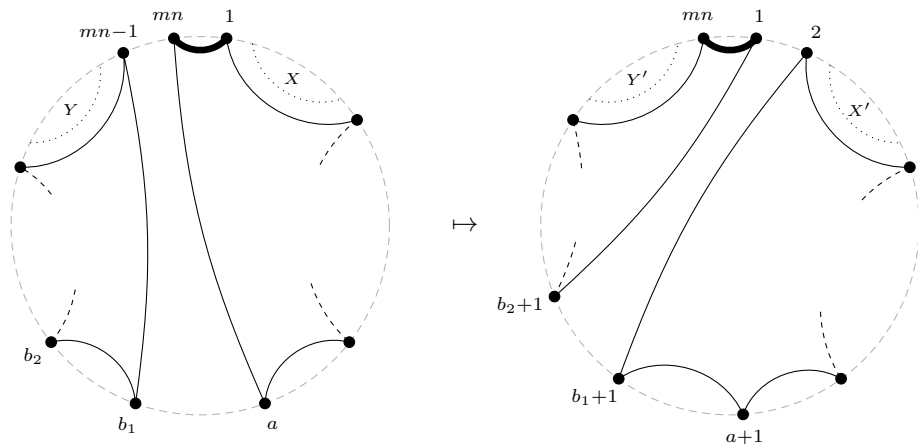
“In principle,” under Armstrong’s combinatorial realisation, the map K_+ becomes rotation by one unit, unless this would produce a non-positive m -divisible partition.

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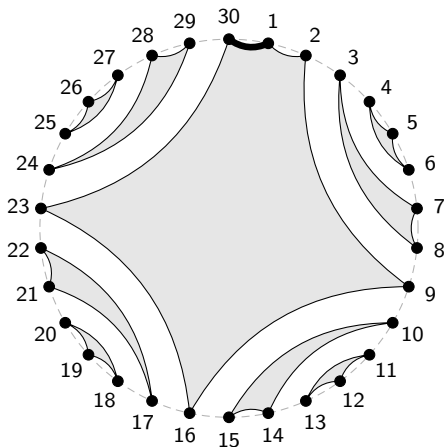


Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

How do “pseudo-rotationally” invariant elements look like?

Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

How do “pseudo-rotationally” invariant elements look like?



Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

Theorem

Let m, n, r be positive integers with $r \geq 2$ and $r \mid (mn - 2)$. Furthermore, let b_1, b_2, \dots, b_n be non-negative integers. The number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ which are invariant under the r -pseudo-rotation $K_+^{(mn-2)/r}$, the number of non-zero blocks of size m_i being rb_i , $i = 1, 2, \dots, n$, the zero block having size $ma = mn - mr \sum_{j=1}^n jb_j$, is given by

$$\binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \binom{(mn-2)/r}{b_1 + b_2 + \dots + b_n}$$

if $b_1 + 2b_2 + \dots + nb_n < n/r$, or if $r = 2$ and $b_1 + 2b_2 + \dots + nb_n = n/2$, and 0 otherwise.

REMARK. We do not have results on chain enumeration.

Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

Theorem

Let C be the cyclic group of pseudo-rotations of an mn -gon generated by K_+ .

Then the triple (M, C, P) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P :

- 1 $M = \widetilde{NC}_+^{(m)}(n)$, and $P(q) = \frac{1}{[n]_q} \left[\begin{matrix} (m+1)n-2 \\ n-1 \end{matrix} \right]_q$;
- 2 M consists of all elements of $\widetilde{NC}_+^{(m)}(n)$ the block sizes of which are all equal to m , and $P(q) = \frac{1}{[n]_q} \left[\begin{matrix} mn-2 \\ n-1 \end{matrix} \right]_q$;
- 3 M consists of all elements of $\widetilde{NC}_+^{(m)}(n)$ which have rank s (or, equivalently, their number of blocks is $n - s$), and

$$P(q) = \frac{1}{[n]_q} \left[\begin{matrix} n \\ s \end{matrix} \right]_q \left[\begin{matrix} mn-2 \\ n-s-1 \end{matrix} \right]_q;$$

Cyclic sieving for **positive** m -divisible non-crossing partitions for type A_{n-1}

- ④ M consists of all elements of $\widetilde{NC}_+^{(m)}(n)$ whose number of blocks of size mi is b_i , $i = 1, 2, \dots, n$, and

$$P(q) = \frac{1}{[b_1 + b_2 + \dots + b_n]_q} \begin{bmatrix} b_1 + b_2 + \dots + b_n \\ b_1, b_2, \dots, b_n \end{bmatrix}_q \times \begin{bmatrix} mn - 2 \\ b_1 + b_2 + \dots + b_n - 1 \end{bmatrix}_q.$$

Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

Realisation of the cyclic action in type B_n

Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

Realisation of the cyclic action in type B_n

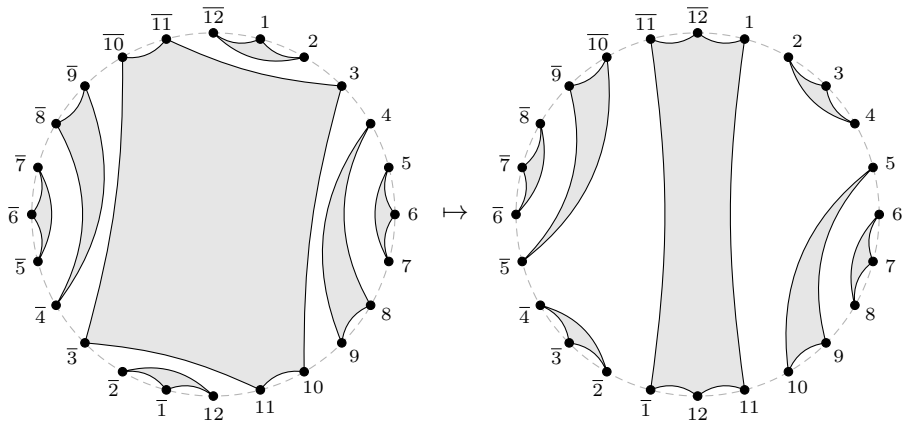
“In principle,” under Armstrong’s combinatorial realisation, the map K_+ becomes rotation by one unit (forward on the outer circle, backward on the inner circle), unless this would produce a non-positive m -divisible partition.

Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

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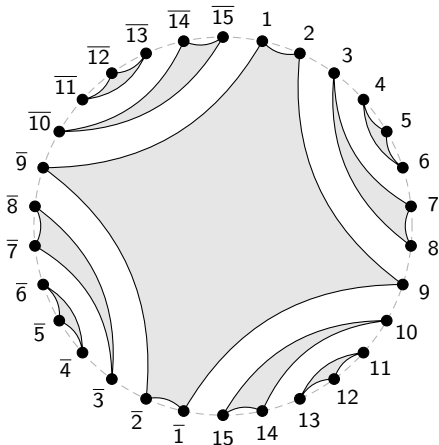


Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

How do “pseudo-rotationally” invariant elements look like?

Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

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Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

Theorem

Let m, n, a, r be positive integers with $r \mid (mn - 1)$. Furthermore, let b_1, b_2, \dots, b_n be non-negative integers. The number of positive m -divisible non-crossing partitions of $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ of type B which are invariant under the $2r$ -pseudo-rotation $K_+^{(mn-1)/r}$, where the number of non-zero blocks of size mi is $2rb_i$, $i = 1, 2, \dots, n$, the zero block having size $2ma = 2mn - 2mr \sum_{j=1}^n jb_j > 0$, is given by

$$\binom{b_1 + b_2 + \dots + b_n}{b_1, b_2, \dots, b_n} \binom{(mn-1)/r}{b_1 + b_2 + \dots + b_n}.$$

Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

Theorem

Let C be the cyclic group of pseudo-rotations of the $2mn$ -gon consisting of the elements $\{1, 2, \dots, mn, -1, -2, \dots, -mn\}$ generated by K_+ , viewed as a group of order $2mn - 2$.

Then the triple (M, P, C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P :

- 1 $M = \widetilde{NC}_+^{(m)}(B_n)$, and $P(q) = \left[\begin{matrix} (m+1)n-1 \\ n \end{matrix} \right]_{q^2}$;
- 2 M consists of the elements of $\widetilde{NC}_+^{(m)}(B_n)$ all of whose blocks have size m , and $P(q) = \left[\begin{matrix} mn-1 \\ n \end{matrix} \right]_{q^2}$;
- 3 M consists of all elements of $\widetilde{NC}_+^{(m)}(B_n)$ which have rank s (or, equivalently, their number of non-zero blocks is $2(n - s)$), and

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Cyclic sieving for **positive** m -divisible non-crossing partitions for type B_n

- ④ M consists of all elements of $\widetilde{NC}_+^{(m)}(B_n)$ whose number of non-zero blocks of size mi is $2b_i$, $i = 1, 2, \dots, n$, and

$$P(q) = \begin{bmatrix} b_1 + b_2 + \dots + b_n \\ b_1, b_2, \dots, b_n \end{bmatrix}_{q^2} \begin{bmatrix} mn - 1 \\ b_1 + b_2 + \dots + b_n \end{bmatrix}_{q^2}.$$

Cyclic sieving for **positive** m -divisible non-crossing partitions for type D_n

Realisation of the cyclic action in type D_n

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Realisation of the cyclic action in type D_n

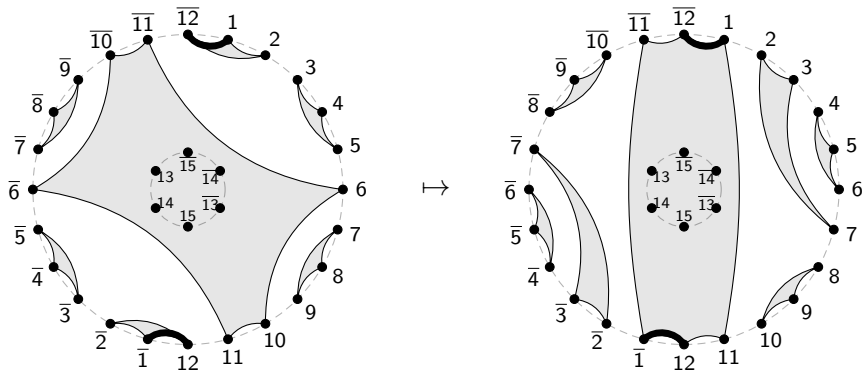
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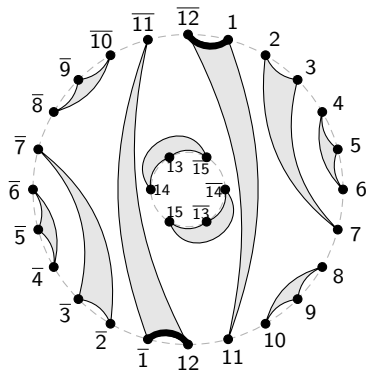
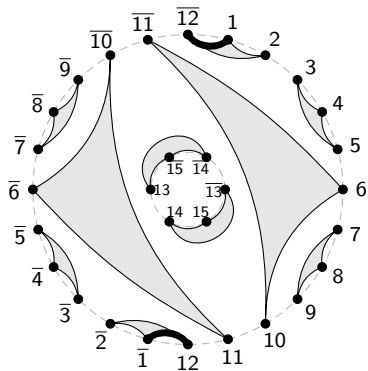
Cyclic sieving for **positive** m -divisible non-crossing partitions for type D_n

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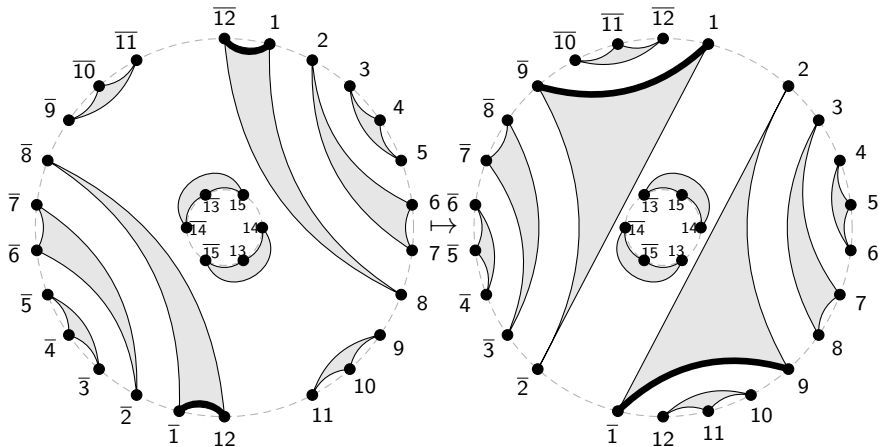
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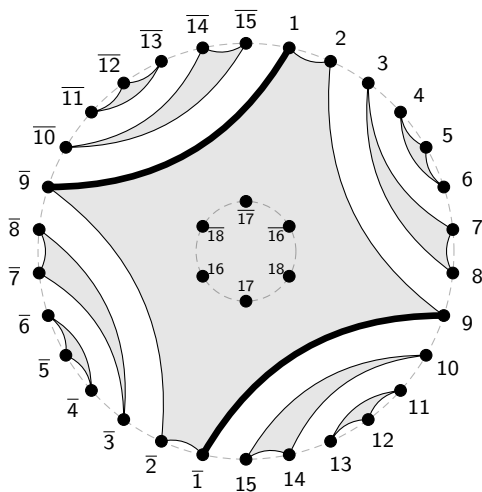


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How do “pseudo-rotationally” invariant elements look like?

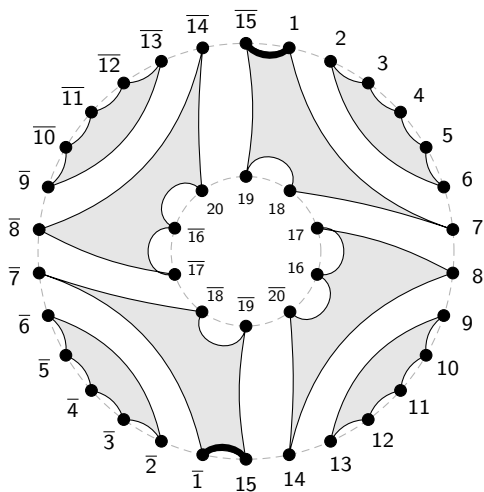
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Cyclic sieving for **positive** m -divisible non-crossing partitions for type D_n

In this case, we contented ourselves just proving the relevant enumeration formulae, since things get quite involved. Probably one can do more if one is braver . . .

Cyclic sieving for **positive** m -divisible non-crossing partitions for type D_n

Theorem

Let C be the cyclic group of pseudo-rotations of the annulus with $\{1, 2, \dots, m(n-1), -1, -2, \dots, -m(n-1)\}$ on the outer circle and $\{m(n-1)+1, \dots, mn, -(m(n-1)+1), \dots, -mn\}$ on the inner circle generated by K_+ , viewed as a group of order $2m(n-1) - 2$.

Then the triple (M, P, C) exhibits the cyclic sieving phenomenon for the following choices of sets M and polynomials P :

- 1 $M = \widetilde{NC}_+^{(m)}(D_n)$, and
$$P(q) = \frac{[2m(n-1)+n-2]_q}{[n]_q} \left[\begin{matrix} (m+1)(n-1)-1 \\ n-1 \end{matrix} \right]_{q^2};$$
- 2 M consists of the elements of $\widetilde{NC}_+^{(m)}(D_n)$ all of whose blocks have size m , and
$$P(q) = \frac{[2m(n-1)-n]_q}{[n]_q} \left[\begin{matrix} m(n-1)-1 \\ n-1 \end{matrix} \right]_{q^2}.$$

Proof method

- ① Careful combinatorial decomposition of the non-crossing objects;
- ② generating function calculus;
- ③ Lagrange inversion formula.

Cyclic sieving for **positive** m -divisible non-crossing partitions for the exceptional types

The (positive) m -divisible non-crossing partitions

$$(w_0; w_1, \dots, w_m)$$

for the exceptional types become “sparse” for large m .

This allows one to reduce the occurring enumeration problems to **finite** problems.

“Other than that, there do not seem to be enumerative results known for these families.”