

A Riccati differential equation and free subgroup numbers for $PSL_2(\mathbb{Z})$ modulo powers of primes

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Ramanujan?

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- congruences

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- congruences
- hypergeometric series

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- congruences
- hypergeometric series
- computer algebra

Counting subgroups

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$$f_n = 6nf_{n-1} + \sum_{m=1}^{n-2} f_m f_{n-m-1}, \quad f_1 = 5.$$


```

f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[f[n], {n, 1, 20}]
{5, 60, 1105, 27120, 828250, 30220800, 1282031525,
 61999046400, 3366961243750, 202903221120000,
 13437880555850250, 970217083619328000,
 75849500508999712500, 6383483988812390400000,
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 5649301494178851172304968750,
 610768380520654474629120000000,
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 8370071726919812448859648819200000}

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The recurrence is equivalent to the **differential equation**

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0$$

for the generating function $F(z) = 1 + \sum_{n=1}^{\infty} f_n z^n$.

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We are going to analyse the sequence $(f_n)_{n \geq 1}$ modulo prime powers p^α .

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For the primes $p = 2$ and $p = 3$ there exists a method based on generating function calculus which is able to solve the problem modulo any power of 2 and of 3, respectively.

f_n modulo 2 and 3

Theorem

The number f_n is odd if and only if n is of the form $n = 2^k - 1$ for some positive integer k .

Theorem

We have:

- ① $f_n \equiv -1 \pmod{3}$ if, and only if, the 3-adic expansion of n is an element of $\{0, 2\}^*1$;
- ② $f_n \equiv 1 \pmod{3}$ if, and only if, the 3-adic expansion of n is an element of $\{0, 2\}^*100^* \cup \{0, 2\}^*122^*$;
- ③ for all other n , we have $f_n \equiv 0 \pmod{3}$.

f_n modulo powers of 5

```

f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[f[n], {n, 1, 20}]
{5, 60, 1105, 27120, 828250, 30220800, 1282031525,
 61999046400, 3366961243750, 202903221120000,
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```

f_n modulo powers of 5

It is easy to convince oneself that, for fixed α , we have $f_n \equiv 0 \pmod{5^\alpha}$ for large enough n .

f_n modulo powers of 7

```

f[n_] := f[n] =
  6 n f[n - 1] + Sum[f[m] f[n - m - 1],
    {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 7], {n, 1, 40}]
{5, 4, 6, 2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4,
  6, 2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4, 6,
  2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4, 6, 2}
(1 + 5 z + 4 z^2 + 6 z^3 + 2 z^4 + 3 z^5) /
(1 - z^6)
1 + 5 z + 4 z^2 + 6 z^3 + 2 z^4 + 3 z^5
      1 - z^6
Factor[%, Modulus -> 7]
1
2 (4 + z)

```

```
f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 49], {n, 1, 100}]
{5, 11, 27, 23, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13,
  37, 45, 15, 5, 18, 6, 2, 17, 22, 40, 46, 48, 16, 38,
  29, 26, 25, 41, 30, 10, 36, 12, 4, 34, 44, 31, 43, 47,
  32, 27, 9, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13, 37,
  45, 15, 5, 18, 6, 2, 17, 22, 40, 46, 48, 16, 38, 29,
  26, 25, 41, 30, 10, 36, 12, 4, 34, 44, 31, 43, 47, 32,
  27, 9, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13, 37}
```

```
1 + 5 z + 11 z ^ 2 + 27 z ^ 3 + 23 z ^ 4 + z ^ 4
(3 z + z ^ 2 + 33 z ^ 3 + 11 z ^ 4 + 20 z ^ 5 + 23 z ^ 6 + 24 z ^ 7 + 8 z ^ 8 + 19 z ^ 9 +
  39 z ^ 10 + 13 z ^ 11 + 37 z ^ 12 + 45 z ^ 13 + 15 z ^ 14 + 5 z ^ 15 + 18 z ^ 16 +
  6 z ^ 17 + 2 z ^ 18 + 17 z ^ 19 + 22 z ^ 20 + 40 z ^ 21 + 46 z ^ 22 + 48 z ^ 23 +
  16 z ^ 24 + 38 z ^ 25 + 29 z ^ 26 + 26 z ^ 27 + 25 z ^ 28 + 41 z ^ 29 + 30 z ^ 30 +
  10 z ^ 31 + 36 z ^ 32 + 12 z ^ 33 + 4 z ^ 34 + 34 z ^ 35 + 44 z ^ 36 + 31 z ^ 37 +
```

$$\begin{aligned}
 & 39 z^{10} + 13 z^{11} + 37 z^{12} + 45 z^{13} + 15 z^{14} + 5 z^{15} + 18 z^{16} + \\
 & 6 z^{17} + 2 z^{18} + 17 z^{19} + 22 z^{20} + 40 z^{21} + 46 z^{22} + 48 z^{23} + \\
 & 16 z^{24} + 38 z^{25} + 29 z^{26} + 26 z^{27} + 25 z^{28} + 41 z^{29} + 30 z^{30} + \\
 & 10 z^{31} + 36 z^{32} + 12 z^{33} + 4 z^{34} + 34 z^{35} + 44 z^{36} + 31 z^{37} + \\
 & 43 z^{38} + 47 z^{39} + 32 z^{40} + 27 z^{41} + 9 z^{42} \Big/ (1 - z^{42})
 \end{aligned}$$

$$1 + 5 z + 11 z^2 + 27 z^3 + 23 z^4 +$$

$$\frac{1}{1 - z^{42}} z^4 \left(3 z + z^2 + 33 z^3 + 11 z^4 + 20 z^5 + 23 z^6 + 24 z^7 + 8 z^8 +$$

$$\begin{aligned}
 & 19 z^9 + 39 z^{10} + 13 z^{11} + 37 z^{12} + 45 z^{13} + 15 z^{14} + 5 z^{15} + \\
 & 18 z^{16} + 6 z^{17} + 2 z^{18} + 17 z^{19} + 22 z^{20} + 40 z^{21} + 46 z^{22} + \\
 & 48 z^{23} + 16 z^{24} + 38 z^{25} + 29 z^{26} + 26 z^{27} + 25 z^{28} + \\
 & 41 z^{29} + 30 z^{30} + 10 z^{31} + 36 z^{32} + 12 z^{33} + 4 z^{34} + 34 z^{35} + \\
 & 44 z^{36} + 31 z^{37} + 43 z^{38} + 47 z^{39} + 32 z^{40} + 27 z^{41} + 9 z^{42} \Big)
 \end{aligned}$$

$$\left(1 + 9 z + 35 z^2 + 42 z^3 + 28 z^4 + 7 z^5 + 7 z^6 \right) \Big/ (1 + 2 z)^2$$

f_n modulo powers of 7

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n \geq 1}$, considered modulo 7^α , is eventually periodic, with period length $6 \cdot 7^{\alpha-1}$.

Moreover, the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo 7^α , equals

$$\frac{P_\alpha(z)}{(1 + 2z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 11

```
f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 11], {n, 1, 40}]
{5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5,
  5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5}
Table[Mod[f[n], 11 ^ 2], {n, 1, 40}]
{5, 60, 16, 16, 5, 82, 104, 27, 49, 82, 115, 27, 60, 93,
  5, 38, 71, 104, 16, 49, 82, 115, 27, 60, 93, 5, 38,
  71, 104, 16, 49, 82, 115, 27, 60, 93, 5, 38, 71, 104}
```

$$7 + 99 z + 44 z^3 + 11 z^4 + 88 z^5 +$$

$$11 z^6 + 11 z^8 + \frac{99}{(-1 + 12 z)^2} + \frac{105}{-1 + 12 z}$$

f_n modulo powers of 11

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n \geq 1}$, considered modulo 11^α , is eventually periodic, with period length $11^{\alpha-1}$.

Moreover, the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo 11^α , equals

$$\frac{P_\alpha(z)}{(1-z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 13

```
f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 13], {n, 1, 50}]
{5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8, 0, 2, 7,
  12, 8, 5, 0, 11, 6, 1, 5, 8, 0, 2, 7, 12, 8, 5, 0,
  11, 6, 1, 5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8}
```

```
Factor[
  1 +  $\frac{1}{1 - z^{12}}$  (5 z + 8 z2 + 2 z4 + 7 z5 + 12 z6 + 8 z7 + 5 z8 + 11 z10 +
  6 z11 + z12), Modulus -> 13]

$$\frac{7(5+z)}{(6+z)(8+z)}$$

```

f_n modulo powers of 13

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n \geq 1}$, considered modulo 13^α , is eventually periodic, with period length $12 \cdot 13^{\alpha-1}$.

Moreover, the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo 13^α , equals

$$\frac{P_\alpha(z)}{((1-2z)(1+5z))^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 17

```

f[n_] := f[n] = 6 n f[n - 1] +
  Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 17], {n, 1, 200}]
{5, 9, 0, 5, 10, 2, 2, 7, 0, 2, 4, 11, 11, 13, 0, 11,
  5, 1, 1, 12, 0, 1, 2, 14, 14, 15, 0, 14, 11, 9, 9,
  6, 0, 9, 1, 7, 7, 16, 0, 7, 14, 13, 13, 3, 0, 13, 9,
  12, 12, 8, 0, 12, 7, 15, 15, 10, 0, 15, 13, 6, 6, 4,
  0, 6, 12, 16, 16, 5, 0, 16, 15, 3, 3, 2, 0, 3, 6, 8,
  8, 11, 0, 8, 16, 10, 10, 1, 0, 10, 3, 4, 4, 14, 0,
  4, 8, 5, 5, 9, 0, 5, 10, 2, 2, 7, 0, 2, 4, 11, 11,
  13, 0, 11, 5, 1, 1, 12, 0, 1, 2, 14, 14, 15, 0, 14,
  11, 9, 9, 6, 0, 9, 1, 7, 7, 16, 0, 7, 14, 13, 13, 3,
  0, 13, 9, 12, 12, 8, 0, 12, 7, 15, 15, 10, 0, 15,
  13, 6, 6, 4, 0, 6, 12, 16, 16, 5, 0, 16, 15, 3, 3,
  2, 0, 3, 6, 8, 8, 11, 0, 8, 16, 10, 10, 1, 0, 10,
  3, 4, 4, 14, 0, 4, 8, 5, 5, 9, 0, 5, 10, 2, 2, 7}

```


f_n modulo powers of 17

In terms of generating functions, this is equivalent to

$$F(z) = 13 + \frac{5 + 12z}{1 + 15z + 7z^2} \quad \text{modulo } 17.$$

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More generally it seems:

Conjecture

Let α be a positive integer. The generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo 17^α , equals

$$\frac{P_\alpha(z)}{(1 + 15z + 7z^2)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

How to prove these conjectures?

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Conjecture

The generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$ satisfies

$$F(z) = \frac{P_\alpha(z)}{(1+2z)^\alpha} \quad \text{modulo } 7^\alpha,$$

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where $P_\alpha(z)$ is a polynomial in z over the integers.

Maybe we make the **Ansatz** $F(z) = P_\alpha(z)/(1+2z)^\alpha$, substitute in the differential equation

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

and then it becomes clear that $P_\alpha(z)$ must be a polynomial, and which polynomial it is.

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If this does not work . . . , we should maybe try a **recursive approach**: let us suppose that $F_\alpha(z) = P_\alpha(z)/(1+2z)^\alpha$ solves

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modulo 7^α . Then make the Ansatz $F_{\alpha+1}(z) = F_\alpha(z) + 7^\alpha G_{\alpha+1}(z)$, substitute in the differential equation, and solve for $G_{\alpha+1}(z)$.

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modulo 7^α . Then make the Ansatz $F_{\alpha+1}(z) = F_\alpha(z) + 7^\alpha G_{\alpha+1}(z)$, substitute in the differential equation, and solve for $G_{\alpha+1}(z)$.

This comes very close: in this manner one can prove that $F(z) = \bar{P}_\alpha(z)/(1+2z)^{e_\alpha}$ modulo 7^α , for some integer e_α .

How to prove these conjectures?

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New approach: Padé approximants!!

How to prove these conjectures?

New approach: **Padé approximants**!!

Apparently, the generating function $F(z)$ is always rational when reduced modulo 7, 11, 13, 17, ...

On the other hand, over \mathbb{Z} , the solution to

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0$$

is certainly *not* rational. But we may **approximate** $F(z)$ by a rational function, say

$$F(z) = \frac{P_n(z)}{Q_n(z)} + O(z^{2n+1}),$$

where $P_n(z)$ and $Q_n(z)$ are polynomials of degree at most n .

How to prove these conjectures?

So:

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So:

substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

which, after clearing denominators, becomes

$$(1 - 4z)P_n(z)Q_n(z) - 6z^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\ - zP_n^2(z) - Q_n^2(z) = O(z^{2n+1}),$$

How to prove these conjectures?

So:

substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

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How to prove these conjectures?

So:

substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

which, after clearing denominators, becomes

$$(1 - 4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) - zP_n^2(z) - Q_n^2(z) = \text{const}(n) \times z^{2n+1},$$

and compute the polynomials $P_n(z)$ and $Q_n(z)$ for $n = 1, 2, 3, \dots$

Then stare at these polynomials and try to come up with a guess for $\text{const}(n)$, $P_n(z)$, $Q_n(z)$.


```

SolPQ[n_] := Module[{Erg},
  P = (Sum[p[i] z ^ i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z ^ i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
    6 z ^ 2 (D[P, z] Q - P D[Q, z]) - z P ^ 2 - Q ^ 2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
    Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
]

```

SolPQ[1]

$$\left\{ \frac{1 - 7 z}{1 - 12 z}, -385 \right\}$$

SolPQ[2]

```

SolPQ[n_] := Module[{Erg},
  P = (Sum[p[i] z ^ i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z ^ i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
    6 z ^ 2 (D[P, z] Q - P D[Q, z]) - z P ^ 2 - Q ^ 2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
    Table[q[i], {i, 1, n}]]];
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]

```

SolPQ[1]

$$\left\{ \frac{1 - 7 z}{1 - 12 z}, -385 \right\}$$

SolPQ[2]

$$\left[1 - 31 z + 91 z^2 \right]$$

```

P = (Sum[p[i] z ^ i, {i, 1, n}] + 1);
Q = (Sum[q[i] z ^ i, {i, 1, n}] + 1);
Erg = Expand[(1 - 4 z) P Q -
  6 z ^ 2 (D[P, z] Q - P D[Q, z]) - z P ^ 2 - Q ^ 2];
Var = Coefficient[Erg, z, 2 n + 1];
Erg =
  Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
  Table[q[i], {i, 1, n}]]];
{(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
]

```

SolPQ[1]

$$\left\{ \frac{1 - 7 z}{1 - 12 z}, -385 \right\}$$

SolPQ[2]

$$\left\{ \frac{1 - 31 z + 91 z^2}{1 - 36 z + 211 z^2}, -85085 \right\}$$

```

P = (Sum[p[i] z^i, {i, 1, n}] + 1);
Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
Erg = Expand[(1 - 4 z) P Q -
  6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
Var = Coefficient[Erg, z, 2 n + 1];
Erg =
  Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
  Table[q[i], {i, 1, n}]]];
{(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
]

```

SolPQ[1]

$$\left\{ \frac{1 - 7 z}{1 - 12 z}, -385 \right\}$$

SolPQ[2]

$$\left\{ \frac{1 - 31 z + 91 z^2}{1 - 36 z + 211 z^2}, -85085 \right\}$$

SolPQ[3]

```

Erg = Expand[(1 - 4 z) P Q -
  6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
Var = Coefficient[Erg, z, 2 n + 1];
Erg =
  Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
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Erg = Expand[(1 - 12 z) P - Q];
6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
Var = Coefficient[Erg, z, 2 n + 1];
Erg =
Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
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SolPQ[3]

$$\left\{ 1 - 67z + 986z^2 - 1729z^3 \right.$$

37182145}



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{{5, 1}, {7, 1}, {11, 1},

{13, 1}, {17, 1}, {19, 1}, {23, 1}}

```
Table[q[i], {i, 1, n}]]];  
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{{5, 1}, {7, 1}, {11, 1},  
{13, 1}, {17, 1}, {19, 1}, {23, 1}}
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So, apparently, there exist polynomials $P_n(z)$ and $Q_n(z)$ of degree n such that

$$(1 - 4z)P_n(z)Q_n(z) - 6z^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\ - zP_n^2(z) - Q_n^2(z) = -5z^{2n+1} \prod_{\ell=1}^n (6\ell + 1)(6\ell + 5),$$

or, equivalently, $R_n(z) = P_n(z)/Q_n(z)$ satisfies

$$(1 - 4z)R_n(z) - 6z^2R'_n(z) - zR_n^2(z) - 1 \\ = -\frac{5z^{2n+1}}{Q_n^2(z)} \prod_{\ell=1}^n (6\ell + 1)(6\ell + 5).$$

Let us pause here for a moment!

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This would **prove immediately** that, **modulo any prime power** p^α with $p \geq 5$, our generating function $F(z)$ is **rational**. In particular, the sequence $(f_n)_{n \geq 1}$ is **eventually periodic**, if considered modulo a prime power p^α with $p \geq 5$!

So, we know what we have to do:

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After having guessed $\text{const}(n)$, we must now guess polynomials $P_n(z)$ and $Q_n(z)$ of degree n such that

$$(1 - 4z)P_n(z)Q_n(z) - 6z^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\ - zP_n^2(z) - Q_n^2(z) = -5z^{2n+1} \prod_{\ell=1}^n (6\ell + 1)(6\ell + 5).$$

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Well: „*Knapp daneben ist auch vorbei!*“
(“*A miss is as good as a mile!*”)

Generalise!

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$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0.$$

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Let us consider

$$(1 - Az)F(z) - Bz^2F'(z) - CzF^2(z) - 1 - Dz = 0.$$

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Let us consider

$$(1 - Az)F(z) - Bz^2F'(z) - CzF^2(z) - 1 - Dz = 0.$$

For the Padé approximant $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$, we then have

$$(1 - Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - CzP_n^2(z) - (1 + Dz)Q_n^2(z) = \text{const}(n) \times z^{2n+1}.$$

```

SolABCD[n_] := SolABCD[n] = Module[{Erg},
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - A z) P Q -
    B z^2 (D[P, z] Q - P D[Q, z]) -
    CC z P^2 - Q^2 - DD z Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
    Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
]

```

SolABCD[1]

$$\left\{ \frac{1 + (-B - CC + DD) z}{1 + (-A - B - 2 CC) z}, \right.$$


```

SolABCD[n_] := SolABCD[n] = Module[{Erg},
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
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  Erg = Expand[(1 - A z) P Q -
    B z^2 (D[P, z] Q - P D[Q, z]) -
    CC z P^2 - Q^2 - DD z Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
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```

```
SolABCD[1]
```

$$\left\{ \frac{1 + (-B - CC + DD) z}{1 + (-A - B - 2 CC) z}, \right.$$

$$\left. - (A + CC + DD) (A B + B^2 + A CC + 2 B CC + CC^2 + CC DD) \right\}$$

```

Solve[1] := Module[{Erg},
  P = (Sum[p[i] z ^ i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z ^ i, {i, 1, n}] + 1);
  Erg = Expand[(1 - A z) P Q -
    B z ^ 2 (D[P, z] Q - P D[Q, z]) -
    CC z P ^ 2 - Q ^ 2 - DD z Q ^ 2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
    Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
]

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$$\left\{ \frac{1 + (-B - CC + DD) z}{1 + (-A - B - 2 CC) z}, \right. \\ \left. - (A + CC + DD) (A B + B^2 + A CC + 2 B CC + CC^2 + CC DD) \right\}$$

```

P = (Sum[p[i] z^i, {i, 1, n}] + 1);
Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
Erg = Expand[(1 - A z) P Q -
  B z^2 (D[P, z] Q - P D[Q, z]) -
  CC z P^2 - Q^2 - DD z Q^2];
Var = Coefficient[Erg, z, 2 n + 1];
Erg =
  Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
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```

Q = (Sum[q[[1]] z^i, {i, 1, n}] + 1);
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]

```

SolABCD[1]

$$\left\{ \frac{1 + (-B - CC + DD) z}{1 + (-A - B - 2 CC) z}, \right. \\
 \left. - (A + CC + DD) (A B + B^2 + A CC + 2 B CC + CC^2 + CC DD) \right\}$$

SolABCD[2]

$$\left\{ \left(1 + (-A - 4B - 3CC + DD) z + \right. \right. \\ \left. \left. (2B^2 + 3BCC + CC^2 - ADD - 3BDD - 3CCDD) z^2 \right) / \right. \\ \left. (1 - 2(A + 2B + 2CC) z + \right. \\ \left. (A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD) z^2 \right), \\ - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right\}$$

SolABCD[3]

$$\left\{ \left(1 + (-2A - 9B - 5CC + DD) z + (A^2 + 7AB + 18B^2 + 4ACC + \right. \right. \\ \left. \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD) z^2 + \right. \right. \\ \left. \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \right. \\ \left. \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2) z^3 \right) / \right. \\ \left. (1 - 3(A + 3B + 2CC) z + (3A^2 + 15AB + 18B^2 + \right.$$

SolABCD[2]

$$\left\{ \left(1 + (-A - 4B - 3CC + DD) z + \right. \right. \\ \left. \left(2B^2 + 3BCC + CC^2 - ADD - 3BDD - 3CCDD \right) z^2 \right) / \\ \left(1 - 2(A + 2B + 2CC) z + \right. \\ \left. (A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD) z^2 \right), \\ - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right\}$$

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$$\left\{ \left(1 + (-2A - 9B - 5CC + DD) z + (A^2 + 7AB + 18B^2 + 4ACC + \right. \right. \\ \left. \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD) z^2 + \right. \right. \\ \left. \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \right. \\ \left. \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2) z^3 \right) / \\ \left(1 - 3(A + 3B + 2CC) z + (3A^2 + 15AB + 18B^2 + \right. \right. \\ \left. \left. 10ACC + 30BCC + 10CC^2 - 2CCDD) z^2 + \right. \right.$$

$$\left\{ \left(1 + (-A - 4B - 3CC + DD) z + \right. \right. \\ \left. \left(2B^2 + 3BCC + CC^2 - ADD - 3BDD - 3CCDD \right) z^2 \right) / \\ \left(1 - 2(A + 2B + 2CC) z + \right. \\ \left. (A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD) z^2 \right), \\ - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right\}$$

solABCD [3]

$$\left\{ \left(1 + (-2A - 9B - 5CC + DD) z + (A^2 + 7AB + 18B^2 + 4ACC + \right. \right. \\ \left. \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD) z^2 + \right. \right. \\ \left. \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \right. \\ \left. \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2) z^3 \right) / \\ \left(1 - 3(A + 3B + 2CC) z + (3A^2 + 15AB + 18B^2 + \right. \right. \\ \left. \left. 10ACC + 30BCC + 10CC^2 - 2CCDD) z^2 + \right. \right. \\ \left. \left. (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \right. \right. \\ \left. \left. 18B^2CC - 6CC^3 + A^2DD + 6ABDD + 11B^2DD + 4ACCDD + \right. \right. \\ \left. \left. 18BCCDD + 6CC^2DD - CCDD^2) z^3 \right) \right\}$$

$$\left\{ \left(1 + (-A - 4B - 3CC + DD)z + \right. \right. \\ \left. \left. (2B^2 + 3BCC + CC^2 - ADD - 3BDD - 3CCDD)z^2 \right) / \right. \\ \left. (1 - 2(A + 2B + 2CC)z + \right. \\ \left. (A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD)z^2 \right), \\ - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right\}$$

SolABCD[3]

$$\left\{ \left(1 + (-2A - 9B - 5CC + DD)z + \left(A^2 + 7AB + 18B^2 + 4ACC + \right. \right. \right. \\ \left. \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD \right)z^2 + \right. \\ \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \\ \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2)z^3 \right) / \\ \left(1 - 3(A + 3B + 2CC)z + \left(3A^2 + 15AB + 18B^2 + \right. \right. \\ \left. \left. 10ACC + 30BCC + 10CC^2 - 2CCDD \right)z^2 + \right. \\ \left. (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \right.$$

$$\begin{aligned} & \left(2 B^2 + 3 B C C + C C^2 - A D D - 3 B D D - 3 C C D D \right) z^2) / \\ & \left(1 - 2 (A + 2 B + 2 C C) z + \right. \\ & \quad \left. (A^2 + 3 A B + 2 B^2 + 3 A C C + 6 B C C + 3 C C^2 - C C D D) z^2 \right), \\ & - (A + C C + D D) (A B + B^2 + A C C + 2 B C C + C C^2 + C C D D) \\ & \left. (2 A B + 4 B^2 + A C C + 4 B C C + C C^2 + C C D D) \right\} \end{aligned}$$

SolABCD[3]

$$\begin{aligned} & \left\{ \left(1 + (-2 A - 9 B - 5 C C + D D) z + (A^2 + 7 A B + 18 B^2 + 4 A C C + \right. \right. \\ & \quad \left. \left. 22 B C C + 6 C C^2 - 2 A D D - 8 B D D - 6 C C D D) z^2 + \right. \right. \\ & \quad \left. \left. (-6 B^3 - 11 B^2 C C - 6 B C C^2 - C C^3 + A^2 D D + 6 A B D D + 11 B^2 \right. \right. \\ & \quad \left. \left. D D + 4 A C C D D + 18 B C C D D + 6 C C^2 D D - C C D D^2) z^3 \right) / \right. \\ & \left. \left(1 - 3 (A + 3 B + 2 C C) z + (3 A^2 + 15 A B + 18 B^2 + \right. \right. \\ & \quad \left. \left. 10 A C C + 30 B C C + 10 C C^2 - 2 C C D D) z^2 + \right. \right. \\ & \quad \left. \left. (-A^3 - 6 A^2 B - 11 A B^2 - 6 B^3 - 4 A^2 C C - 18 A B C C - \right. \right. \\ & \quad \left. \left. 22 B^2 C C - 6 A C C^2 - 18 B C C^2 - 4 C C^3 + \right. \right. \end{aligned}$$

$$\begin{aligned} & (1 - 2(A + 2B + 2CC)z + \\ & (A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD)z^2), \\ & - (A + CC + DD)(AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ & (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \} \end{aligned}$$

Sol ABCD [3]

$$\begin{aligned} & \{ (1 + (-2A - 9B - 5CC + DD)z + (A^2 + 7AB + 18B^2 + 4ACC + \\ & 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD)z^2 + \\ & (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \\ & DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2)z^3) / \\ & (1 - 3(A + 3B + 2CC)z + (3A^2 + 15AB + 18B^2 + \\ & 10ACC + 30BCC + 10CC^2 - 2CCDD)z^2 + \\ & (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \\ & 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + \\ & 2ACCDD + 6BCCDD + 4CC^2DD)z^3) , \end{aligned}$$

$$\begin{aligned} & \left(A^2 + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD \right) z^2, \\ & - (A + CC + DD) \left(AB + B^2 + ACC + 2BCC + CC^2 + CCDD \right) \\ & \left(2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD \right) \} \end{aligned}$$

SolABCD[3]

$$\begin{aligned} & \left\{ \left(1 + (-2A - 9B - 5CC + DD) z + \left(A^2 + 7AB + 18B^2 + 4ACC + \right. \right. \right. \\ & \quad \left. \left. \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD \right) z^2 + \right. \right. \\ & \quad \left. \left(-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \right. \\ & \quad \left. \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2 \right) z^3 \right\} / \\ & \left(1 - 3(A + 3B + 2CC) z + \left(3A^2 + 15AB + 18B^2 + \right. \right. \\ & \quad \left. \left. 10ACC + 30BCC + 10CC^2 - 2CCDD \right) z^2 + \right. \\ & \quad \left. \left(-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \right. \right. \\ & \quad \left. \left. 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + \right. \right. \\ & \quad \left. \left. 2ACCDD + 6BCCDD + 4CC^2DD \right) z^3 \right), \\ & - (A + CC + DD) \left(AB + B^2 + ACC + 2BCC + CC^2 + CCDD \right) \end{aligned}$$

$$\begin{aligned}
 & (A + 3AB + 2B^2 + 3ACC + 6BCC + 3CC^2 - CCDD) z^2, \\
 & - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\
 & (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \}
 \end{aligned}$$

Sol ABCD [3]

$$\begin{aligned}
 & \{ (1 + (-2A - 9B - 5CC + DD) z + (A^2 + 7AB + 18B^2 + 4ACC + \\
 & \quad 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD) z^2 + \\
 & \quad (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \\
 & \quad \quad DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2) z^3) / \\
 & (1 - 3(A + 3B + 2CC) z + (3A^2 + 15AB + 18B^2 + \\
 & \quad 10ACC + 30BCC + 10CC^2 - 2CCDD) z^2 + \\
 & \quad (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \\
 & \quad 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + \\
 & \quad 2ACCDD + 6BCCDD + 4CC^2DD) z^3), \\
 & - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\
 & (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD)
 \end{aligned}$$

$$- (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \}$$

Sol ABCD [3]

$$\left\{ (1 + (-2A - 9B - 5CC + DD)z + (A^2 + 7AB + 18B^2 + 4ACC + 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD)z^2 + \right. \\ \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2)z^3) / \right. \\ \left. (1 - 3(A + 3B + 2CC)z + (3A^2 + 15AB + 18B^2 + 10ACC + 30BCC + 10CC^2 - 2CCDD)z^2 + \right. \\ \left. (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + 2ACCDD + 6BCCDD + 4CC^2DD)z^3) , \right. \\ \left. - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \right. \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right\}$$

$$(2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \}$$

SolABCD[3]

$$\left\{ \left(1 + (-2A - 9B - 5CC + DD)z + (A^2 + 7AB + 18B^2 + 4ACC + 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD)z^2 + (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2)z^3 \right) / \left(1 - 3(A + 3B + 2CC)z + (3A^2 + 15AB + 18B^2 + 10ACC + 30BCC + 10CC^2 - 2CCDD)z^2 + (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + 2ACCDD + 6BCCDD + 4CC^2DD)z^3 \right), \right. \\ \left. - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \right. \\ \left. (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \right. \\ \left. (3AB + 9B^2 + ACC + 6BCC + CC^2 + CCDD) \right\}$$

Sol ABCD [3]

$$\left\{ \begin{aligned} & \left(1 + (-2A - 9B - 5CC + DD)z + (A^2 + 7AB + 18B^2 + 4ACC + \right. \\ & \quad \left. 22BCC + 6CC^2 - 2ADD - 8BDD - 6CCDD)z^2 + \right. \\ & \quad \left. (-6B^3 - 11B^2CC - 6BCC^2 - CC^3 + A^2DD + 6ABDD + 11B^2 \right. \\ & \quad \left. DD + 4ACCDD + 18BCCDD + 6CC^2DD - CCDD^2)z^3 \right) / \\ & \left(1 - 3(A + 3B + 2CC)z + (3A^2 + 15AB + 18B^2 + \right. \\ & \quad \left. 10ACC + 30BCC + 10CC^2 - 2CCDD)z^2 + \right. \\ & \quad \left. (-A^3 - 6A^2B - 11AB^2 - 6B^3 - 4A^2CC - 18ABCC - \right. \\ & \quad \left. 22B^2CC - 6ACC^2 - 18BCC^2 - 4CC^3 + \right. \\ & \quad \left. 2ACCDD + 6BCCDD + 4CC^2DD)z^3 \right), \\ & - (A + CC + DD) (AB + B^2 + ACC + 2BCC + CC^2 + CCDD) \\ & \quad (2AB + 4B^2 + ACC + 4BCC + CC^2 + CCDD) \\ & \quad (3AB + 9B^2 + ACC + 6BCC + CC^2 + CCDD) \end{aligned} \right\}$$

So, apparently,

So, apparently, for every positive integer n , there exist polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$, such that

$$\begin{aligned}
 & (1 - Az)P_n(z)Q_n(z) - Bz^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\
 & \quad - CzP_n^2(z) - (1 + Dz)Q_n^2(z) \\
 = & -z^{2n+1}(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2).
 \end{aligned}$$

Let $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$.

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By staring at the first few polynomials, one is led to conjecture:

Let $P_n(z) = 1 + \sum_{k=1}^n p_{n,k}z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k}z^k$.

By staring at the first few polynomials, one is led to conjecture:

$$p_{n,1} = -(n-1)A - n^2B - (2n-1)C + D,$$

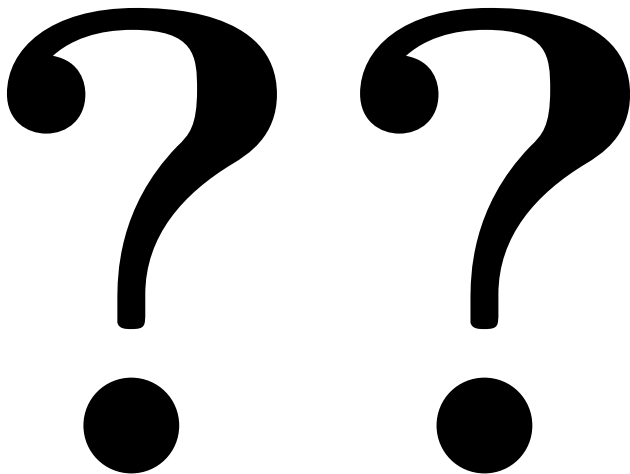
$$q_{n,1} = -nA - n^2B - 2nC,$$

$$\begin{aligned} p_{n,2} = & \frac{1}{2}(n-2)(n-1)A^2 + \frac{1}{2}(n-2)(n-1)(2n+1)AB \\ & + 2(n-2)(n-1)AC - (n-1)AD + \frac{1}{2}(n-1)^2n^2B^2 \\ & + (n-1)(2n^2 - 2n - 1)BC \\ & - (n-1)(n+1)BD + (n-1)(2n-3)C^2 - 3(n-1)CD, \end{aligned}$$

$$\begin{aligned} q_{n,2} = & \frac{1}{2}(n-1)nA^2 + \frac{1}{2}(n-1)n(2n-1)AB + (n-1)(2n-1)AC \\ & + \frac{1}{2}(n-1)^2n^2B^2 + (n-1)n(2n-1)BC \\ & + (n-1)(2n-1)C^2 - (n-1)CD, \end{aligned}$$

$$\begin{aligned}
p_{n,3} = & -\frac{1}{6}(n-3)(n-2)(n-1)A^3 - \frac{1}{2}(n-3)(n-2)(n^2-n-1)A^2B \\
& - \frac{1}{2}(n-3)(n-2)(2n-3)A^2C + \frac{1}{2}(n-2)(n-1)A^2D \\
& - \frac{1}{6}(n-3)(n-2)(3n^3-3n^2-n-2)AB^2 \\
& - \frac{1}{2}(n-3)(n-2)(2n-3)(2n+1)ABC \\
& + \frac{1}{2}(n-2)(n+1)(2n-3)ABD \\
& - (n-3)(n-2)(2n-3)AC^2 + (n-2)(3n-5)ACD \\
& - \frac{1}{6}(n-2)^2(n-1)^2n^2B^3 \\
& - \frac{1}{6}(n-2)(2n-3)(3n^3-6n^2-n-2)B^2C \\
& + \frac{1}{2}(n-2)(n^3-n-2)B^2D \\
& - (n-2)(2n-3)(n^2-2n-1)BC^2 \\
& + (n-2)(3n^2-2n-3)BCD - \frac{1}{3}(n-2)(2n-5)(2n-3)C^3 \\
& + 2(n-2)(2n-3)DC^2 - (n-2)CD^2,
\end{aligned}$$

$$\begin{aligned}
q_{n,3} = & -\frac{1}{6}(n-2)(n-1)nA^3 - \frac{1}{2}(n-2)(n-1)^2nA^2B \\
& - (n-2)(n-1)^2A^2C \\
& - \frac{1}{6}(n-2)(n-1)n(3n^2 - 6n + 2)AB^2 \\
& - (n-2)(n-1)n(2n-3)ABC \\
& - (n-2)(n-1)(2n-3)AC^2 + (n-2)(n-1)ACD \\
& - \frac{1}{6}(n-2)^2(n-1)^2n^2B^3 \\
& - \frac{1}{3}(n-2)(n-1)n(3n^2 - 6n + 2)B^2C \\
& - (n-2)(n-1)n(2n-3)BC^2 + (n-2)(n-1)nBCD \\
& - \frac{2}{3}(n-2)(n-1)(2n-3)C^3 + 2(n-2)(n-1)C^2D.
\end{aligned}$$



An interim thought

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If we should be able to come up with a guess for $P_n(z)$ and $Q_n(z)$, how would we ever be able to prove that this guess is correct?

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If we should be able to prove anything, then this proof must come from **hypergeometrics!**

An interim thought

Hypergeometrics?

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Hypergeometrics?

We are looking at the differential equation:

$$\begin{aligned} & (1 - Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ & \quad - CzP_n^2(z) - (1 + Dz)Q_n^2(z) \\ = & -z^{2n+1}(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2). \end{aligned}$$

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If this wants to be part of the realm of hypergeometrics, then we should better factor the product on the right-hand side into **linear** factors.

An interim thought

Here is this product:

$$(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2).$$

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If we make the substitution $E^2 = A^2 - 4CD$, then this becomes

$$\begin{aligned} (A + C + D) \prod_{\ell=1}^n \left(\ell B + \frac{A+2C+E}{2} \right) \left(\ell B + \frac{A+2C-E}{2} \right) \\ = \frac{1}{C} \prod_{\ell=0}^n \left(\ell B + \frac{A+2C+E}{2} \right) \left(\ell B + \frac{A+2C-E}{2} \right). \end{aligned}$$

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Let us write

$$\Pi_+ = \prod_{\ell=0}^n \left(\ell B + \frac{A+2C+E}{2} \right) \quad \text{and} \quad \Pi_- = \prod_{\ell=0}^n \left(\ell B + \frac{A+2C-E}{2} \right).$$

Our “new” problem: find polynomials $P_n(z)$ and $Q_n(z)$ of degree n with

$$(1 - Az)P_n(z)Q_n(z) - Bz^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\ - CzP_n^2(z) - (1 + Dz)Q_n^2(z) = -\frac{1}{C}\Pi_+\Pi_- z^{2n+1}.$$

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Maybe we should “work from the other end”?!

In other words: let us try to come up with guesses for $p_{n,n}$, $q_{n,n}$, $p_{n,n-1}$, $q_{n,n-1}$, etc.

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In other words: let us try to come up with guesses for $p_{n,n}$, $q_{n,n}$, $p_{n,n-1}$, $q_{n,n-1}$, etc.

Comparing coefficients of z^{2n+1} in our differential equation above, we obtain

$$-Ap_{n,n}q_{n,n} - Cp_{n,n}^2 - Dq_{n,n}^2 = -\frac{1}{C}\Pi_+\Pi_-.$$

The equation again:

$$-Ap_{n,n}q_{n,n} - Cp_{n,n}^2 - Dq_{n,n}^2 = -\frac{1}{C}\Pi_+\Pi_-.$$

Performing the substitution $E^2 = A^2 - 4CD$, the left-hand side factors:

$$(Cp_{n,n} + \frac{1}{2}(A - E)q_{n,n}) (Cp_{n,n} + \frac{1}{2}(A + E)q_{n,n}) = \Pi_+\Pi_-.$$

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$$(Cp_{n,n} + \frac{1}{2}(A - E)q_{n,n}) (Cp_{n,n} + \frac{1}{2}(A + E)q_{n,n}) = \Pi_+\Pi_-.$$

Indeed, the computer says:

$$\begin{aligned} Cp_{n,n} + \frac{1}{2}(A - E)q_{n,n} &= \Pi_-, \\ Cp_{n,n} + \frac{1}{2}(A + E)q_{n,n} &= \Pi_+. \end{aligned}$$

Indeed, the computer says:

$$Cp_{n,n} + \frac{1}{2}(A - E)q_{n,n} = \Pi_{-},$$

$$Cp_{n,n} + \frac{1}{2}(A + E)q_{n,n} = \Pi_{+}.$$

Indeed, the computer says:

$$Cp_{n,n} + \frac{1}{2}(A - E)q_{n,n} = \Pi_-,$$

$$Cp_{n,n} + \frac{1}{2}(A + E)q_{n,n} = \Pi_+.$$

Solve for $p_{n,n}$ and $q_{n,n}$:

$$p_{n,n} = \frac{(-1)^{n+1}}{2CE} ((A - E)\Pi_+ - (A + E)\Pi_-),$$

$$q_{n,n} = \frac{(-1)^n}{E} (\Pi_+ - \Pi_-),$$

where $\Pi_+ = B^{n+1} \left(\frac{A+2C+E}{2B} \right)_{n+1}$ and $\Pi_- = B^{n+1} \left(\frac{A+2C-E}{2B} \right)_{n+1}$.

We continue:

Comparing coefficients of z^{2n} in our differential equation, we obtain

$$\begin{aligned} p_{n,n}q_{n,n} - Ap_{n,n}q_{n,n-1} - Ap_{n,n-1}q_{n,n} - Bp_{n,n}q_{n,n-1} + Bp_{n,n-1}q_{n,n} \\ - 2Cp_{n,n-1}p_{n,n} - q_{n,n}^2 - 2Dq_{n,n-1}q_{n,n} = 0. \end{aligned}$$

We consider this equation **modulo $\Pi_+ - \Pi_-$** , and **modulo $\Pi_+ + \Pi_-$** . This yields two **linear congruences**:

$$C_1p_{n,n-1} + C_2q_{n,n-1} + C_3 \equiv 0 \pmod{\Pi_+ - \Pi_-},$$

$$C_4p_{n,n-1} + C_5q_{n,n-1} + C_6 \equiv 0 \pmod{\Pi_+ + \Pi_-},$$

with explicitly known $C_1, C_2, C_3, C_4, C_5, C_6$. In other words,

$$C_1p_{n,n-1} + C_2q_{n,n-1} + C_3 = D_1(\Pi_+ - \Pi_-),$$

$$C_4p_{n,n-1} + C_5q_{n,n-1} + C_6 = D_2(\Pi_+ + \Pi_-),$$

for some (unknown) D_1 and D_2 .

$$p_{n,n} = \frac{(-1)^{n+1}}{2CE} ((A - E)\Pi_+ - (A + E)\Pi_-),$$

$$q_{n,n} = \frac{(-1)^n}{E} (\Pi_+ - \Pi_-),$$

$$p_{n,n-1} = \frac{(-1)^{n+1}}{2C(E - B)E(E + B)} \\ \times \left(\Pi_+ (n(-E + B)(A - E) + (A + 2C - E)(A + B)) \right. \\ \left. - \Pi_- ((E + B)(A + E) + (A + 2C + E)(A + B)) \right),$$

$$q_{n,n-1} = \frac{(-1)^n}{(E - B)E(E + B)} \\ \times \left(\Pi_+ (n(-E + B) + (A + 2C - E)) \right. \\ \left. - \Pi_- (n(E + B) + (A + 2C + E)) \right),$$

$$\begin{aligned}
p_{n,n-2} = & \frac{(-1)^{n+1}}{2C(E-2B)(E-B)E(E+B)(E+2B)} \\
& \times \left(\Pi_+ \left(\frac{1}{2}n(n-1)(-E+B)(-E+2B)(A-E) \right. \right. \\
& \quad \left. \left. + \frac{3}{2}(n-1)(-E+2B)(A+2C-E)\left(A+\frac{4}{3}B-\frac{1}{3}E\right) \right. \right. \\
& \quad \left. \left. + \frac{3}{2}(A+2C-E)(A+2C-E+2B)(A+2B) \right) \right) \\
& - \Pi_- \left(\frac{1}{2}n(n-1)(E+B)(E+2B)(A+E) \right. \\
& \quad \left. + \frac{3}{2}(n-1)(E+2B)(A+2C+E)\left(A+\frac{4}{3}B+\frac{1}{3}E\right) \right. \\
& \quad \left. + \frac{3}{2}(A+2C+E)(A+2C+E+2B)(A+2B) \right) \Bigg),
\end{aligned}$$

$$\begin{aligned}
q_{n,n-2} = & \frac{(-1)^n}{(E-2B)(E-B)E(E+B)(E+2B)} \\
& \times \left(\Pi_+ \left(\frac{1}{2}n(n-1)(-E+B)(-E+2B) \right. \right. \\
& \qquad \qquad \qquad \frac{3}{2}(n-1)(-E+2B)(A+2C-E) \\
& \qquad \qquad \qquad \left. \left. \frac{3}{2}(A+2C-E)(A+2C-E+2B) \right) \right) \\
& - \Pi_- \left(\frac{1}{2}n(n-1)(E+B)(E+2B) \right. \\
& \qquad \qquad \qquad \frac{3}{2}(n-1)(E+2B)(A+2C+E) \\
& \qquad \qquad \qquad \left. \left. \frac{3}{2}(A+2C+E)(A+2C+E+2B) \right) \right),
\end{aligned}$$

$$\begin{aligned}
 q_{n,n-k} &= \frac{(-1)^n B^{n-k}}{\left(\frac{E}{B} - k\right)_{2k+1}} \\
 &\times \left(\left(\frac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^k q_{n,k,j} (n-k+1)_{k-j} \left(-\frac{E}{B} + j + 1\right)_{k-j} \left(\frac{A+2C-E}{2B}\right)_j \right. \\
 &\quad \left. - \left(\frac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^k q_{n,k,j} (n-k+1)_{k-j} \left(\frac{E}{B} + j + 1\right)_{k-j} \left(\frac{A+2C+E}{2B}\right)_j \right).
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 &\quad \left. - \left(\frac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^k q_{n,k,j} (n-k+1)_{k-j} \left(\frac{E}{B} + j + 1\right)_{k-j} \left(\frac{A+2C+E}{2B}\right)_j \right).
 \end{aligned}$$

How can we guess the coefficients $q_{n,k,j}$?

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Let us assume the form of $q_{n,k}$ from the previous page, and the earlier observed **polynomiality of $q_{n,k}$** in A, B, C, D , and, hence, also in A, B, C, E .

$(\frac{E}{B} - k)_{2k+1}$ must divide the expression between parentheses, that is, this expression must vanish for $E = Bs$, $s = -k, -k + 1, \dots, k$. Let us do the substitution $E = Bs$ in this expression, and let us suppose that $s > 0$.

How can we guess the coefficients $q_{n,k,j}$?

Let us assume the form of $q_{n,k}$ from the previous page, and the earlier observed **polynomiality of $q_{n,k}$** in A, B, C, D , and, hence, also in A, B, C, E .

$(\frac{E}{B} - k)_{2k+1}$ must divide the expression between parentheses, that is, this expression must vanish for $E = Bs$, $s = -k, -k + 1, \dots, k$. Let us do the substitution $E = Bs$ in this expression, and let us suppose that $s > 0$.

Then we have

$$\left(\frac{A+2C+E}{2B}\right)_{n+1} = \left(\frac{A+2C+Bs}{2B}\right)_{n+1} = \left(\frac{A+2C+Bs}{2B}\right)_{n+1-s} \left(\frac{A+2C-Bs}{2B} + n + 1\right)_s$$

Comparing with

$$\left(\frac{A+2C-E}{2B}\right)_{n+1} = \left(\frac{A+2C-Bs}{2B}\right)_{n+1},$$

we infer that $\left(\frac{A+2C-Bs}{2B} + n + 1\right)_s$ must divide the second sum over j as a *polynomial in $n!$* . This provides many non-trivial vanishing conditions for this sum over j , viewed as polynomial in n , and this suffices to compute the coefficients $q_{n,k,j}$ for a large range of k 's and corresponding j 's.

Voilà! Here is our **guess** for $q_{n,n-k}$:

$$q_{n,n-k} = \frac{(-1)^n B^{n-k}}{\left(\frac{E}{B} - k\right)_{2k+1}} \times \left(\left(\frac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} \left(-\frac{E}{B} + j + 1\right)_{k-j} \left(\frac{A+2C-E}{2B}\right)_j - \left(\frac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} \left(\frac{E}{B} + j + 1\right)_{k-j} \left(\frac{A+2C+E}{2B}\right)_j \right).$$

Similarly, here is our **guess** for $p_{n,n-k}$:

$$\begin{aligned}
 p_{n,n-k} = & \frac{(-1)^{n+1} B^{n-k}}{2C \left(\frac{E}{B} - k\right)_{2k+1}} \\
 & \times \left(\left(\frac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} \left(-\frac{E}{B} + j + 1\right)_{k-j} \right. \\
 & \quad \cdot \left(\frac{A+2C-E}{2B}\right)_j \left(A + \frac{2kj}{k+j} B - \frac{k-j}{k+j} E\right) \\
 & \quad - \left(\frac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} \left(\frac{E}{B} + j + 1\right)_{k-j} \\
 & \quad \left. \cdot \left(\frac{A+2C+E}{2B}\right)_j \left(A + \frac{2kj}{k+j} B + \frac{k-j}{k+j} E\right) \right),
 \end{aligned}$$

We have a full-fledged guess:

Conjecture

For every positive integer n , there exist uniquely determined polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$, where $p_{n,k}$ and $q_{n,k}$ are homogeneous polynomials in A, B, C, D of degree k over the integers such that the rational function $R_n(z) = P_n(z)/Q_n(z)$ satisfies

$$\begin{aligned} & (1 - Az)R_n(z) - Bz^2R'_n(z) - CzR_n^2(z) - 1 - Dz \\ &= -\frac{z^{2n+1}}{Q_n^2(z)}(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2). \end{aligned}$$

Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

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Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

So far, we did not prove anything!!

Here is what we would like to prove:

Theorem?

For every positive integer n , there exist uniquely determined polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$, where $p_{n,k}$ and $q_{n,k}$ are homogeneous polynomials in A, B, C, D of degree k over the integers such that the rational function $R_n(z) = P_n(z)/Q_n(z)$ satisfies

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Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

So far, we did not prove anything!!

However, in principle, as soon as the guess for the polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$ is written down, it is already proved!

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However, **in principle**, as soon as the guess for the polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$ is written down, it is already proved!

Sketch of proof that $P_n(z)/Q_n(z)$ satisfies the differential equation

The differential equation:

$$\begin{aligned} & (1 - Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ & \quad - CzP_n^2(z) - (1 + Dz)Q_n^2(z) \\ &= -z^{2n+1}(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2). \end{aligned}$$

We actually already verified that the coefficients of z^{2n+1} match.
(This is how we found the formulae for $p_{n,n}$ and $q_{n,n}$.)

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The differential equation:

$$\begin{aligned} & (1 - Az)P_n(z)Q_n(z) - Bz^2(P'_n(z)Q_n(z) - P_n(z)Q'_n(z)) \\ & \quad - CzP_n^2(z) - (1 + Dz)Q_n^2(z) \\ &= -z^{2n+1}(A + C + D) \prod_{\ell=1}^n (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2). \end{aligned}$$

Now we divide both sides by z^{2n+1} , then differentiate them with respect to z , and finally multiply both sides of the resulting equation by z^{2n+2} . In this way, we obtain

$$\begin{aligned} & P_n(z)(2CnzP_n(z) - 2Cz^2P'_n(z) - (2n + 1)Q_n(z) + zQ'_n(z) \\ & \quad - z^2(A + 2Bn - B)Q'_n(z) + 2AnzQ_n(z) + Bz^3Q''_n(z)) \\ & + Q_n(z)(zP'_n(z) - z^2(A - 2Bn + B)P'_n(z) - Bz^3P''_n(z) \\ & \quad + (2n + 1)Q_n(z) + 2DnzQ_n(z) - 2Dz^2Q'_n(z) - 2zQ'_n(z)) = 0. \end{aligned}$$

This is equivalent to the original equation!

We claim that, in fact,

$$2CnzP_n(z) - 2Cz^2P_n'(z) - (2n+1)Q_n(z) + zQ_n'(z) - z^2(A+2Bn-B)Q_n'(z) + 2AnzQ_n(z) + Bz^3Q_n''(z) = -(2n+1 - nAz + n^2Bz)Q_n(z)$$

and

$$zP_n'(z) - z^2(A-2Bn+B)P_n'(z) - Bz^3P_n''(z) + (2n+1)Q_n(z) + 2DnzQ_n(z) - 2Dz^2Q_n'(z) - 2zQ_n'(z) = (2n+1 - nAz + n^2Bz)P_n(z).$$

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This is not too difficult to prove by comparing coefficients of powers of z . The Gosper algorithm is used to prove one of the arising identities.

Sketch of proof of polynomiality of $p_{n,k}$ and $q_{n,k}$

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Recall:

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$\left(\frac{E}{B} - k\right)_{2k+1}$ must divide the term between parentheses. That is, if we put $E = Bs$ for $s \in \{-k, -k + 1, \dots, k\}$, then the term between parentheses must vanish.

So, what we have to establish is the identity

$$\begin{aligned} & \left(\frac{A+2C+Bs}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} (-s+j+1)_{k-j} \left(\frac{A+2C-Bs}{2B}\right)_j \\ & - \left(\frac{A+2C-Bs}{2B}\right)_{n+1} \sum_{j=0}^k \binom{k+j}{k} \binom{n-j}{k-j} (s+j+1)_{k-j} \left(\frac{A+2C+Bs}{2B}\right)_j = 0, \end{aligned}$$

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for $s \in \{-k, -k+1, \dots, k\}$.

The old hypergeometric transformation (THOMAE (1879))

$${}_3F_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix}; 1 \right] = \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-a) \Gamma(d+e-b-c)} {}_3F_2 \left[\begin{matrix} a, d-b, d-c \\ d, d+e-b-c \end{matrix}; 1 \right]$$

does the job.

Back to $p = 7$:

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n \geq 1}$, considered modulo 7^α , is eventually periodic, with period length $6 \cdot 7^{\alpha-1}$.

Moreover, the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo 7^α , equals

$$\frac{P_\alpha(z)}{(1 + 2z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

For the rest of the talk, we shall always talk about the special case $A = 4, B = 6, C = 1, D = 0, E = 4$, corresponding to the differential equation for the free subgroup numbers for $PSL_2(\mathbb{Z})$.

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Lemma

Let p be a prime with $p \equiv 1 \pmod{6}$. For $n \equiv \frac{p-1}{6} \pmod{p}$, we have

$$Q_n(z) = Q_{(p-1)/6}(z) \quad \text{modulo } p.$$

Furthermore, the polynomial $Q_{(p-1)/6}(z)$ has degree $(p-1)/6$ in z .

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SKETCH OF PROOF. One applies the same ${}_3F_2$ -transformation again to the sums over j in the definition of $q_{n,n-k}$, to obtain

$$q_{n,n-k} = (-1)^n 6^{n-k} \left(\sum_{j=0}^k \frac{(-1)^{k+j}}{k!} \binom{k}{j} \frac{\left(\frac{5}{6} - j\right)_{n+k+1}}{\left(\frac{2}{3} - j\right)_{k+1}} + \sum_{j=0}^k \frac{(-1)^{k+j}}{k!} \binom{k}{j} \frac{\left(\frac{1}{6} - j\right)_{n+k+1}}{\left(-\frac{2}{3} - j\right)_{k+1}} \right).$$

Now elementary (though tedious) p -adic analysis shows that each summand is divisible by p for $n \geq k + \frac{p-1}{6} + 1$.

Theorem

Let α be a positive integer. If p is a prime with $p \equiv 1 \pmod{6}$, then the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo p^α , equals $A_\alpha(z)/Q_{(p-1)/6}^\alpha(z)$, where $A_\alpha(z)$ is a polynomial in z over the integers.

There is a similar statement for $p \equiv 5 \pmod{6}$.

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Induction step: Choose $n \equiv \frac{p-1}{6} \pmod{p}$ large enough such that the product over ℓ on the right-hand side of the “Padé-approximated” differential equation vanishes modulo $p^{\alpha+1}$, and thus $P_n(z)/Q_n(z)$ solves the differential equation modulo $p^{\alpha+1}$. Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} \quad \text{modulo } p^\alpha.$$

PROOF (CONTINUED).

Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} \quad \text{modulo } p^\alpha.$$

PROOF (CONTINUED).

Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} \quad \text{modulo } p^\alpha.$$

Consequently, in the difference

$$\frac{P_n(z)}{Q_n(z)} - \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} = \frac{P_n(z)Q_{(p-1)/6}^\alpha(z) - A_\alpha(z)Q_n(z)}{Q_{(p-1)/6}^\alpha(z)Q_n(z)},$$

all coefficients of the integer polynomial in the numerator of the last fraction must be divisible by p^α . In other words, there is an integer polynomial $B_\alpha(z)$ such that

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} + p^\alpha \frac{B_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)Q_n(z)}.$$

By the previous lemma, this leads to

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q_{(p-1)/6}^\alpha(z)} + p^\alpha \frac{B_\alpha(z)}{Q_{(p-1)/6}^{\alpha+1}(z)} \quad \text{modulo } p^{\alpha+1}. \quad \square$$

Our differential equation for the generating function for the numbers of free subgroups in $PSL_2(\mathbb{Z})$:

$$(1 - 4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0.$$

Theorem

Let α be a positive integer. If p is a prime with $p \equiv 1 \pmod{6}$, then the generating function $F(z) = 1 + \sum_{n \geq 1} f_n z^n$, when coefficients are reduced modulo p^α , equals $A_\alpha(z)/Q_{(p-1)/6}^\alpha(z)$, where $A_\alpha(z)$ is a polynomial in z over the integers.

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A *Mathematica* implementation of the algorithm just described that proves the above theorem is available.