Cyclic Sieving for Generalized Non-Crossing Partitions Associated with Complex Reflection Groups

Christian Krattenthaler

Universität Wien

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Ingredients:

— a set M of *combinatorial objects*,

— a *cyclic group* $C = \langle g \rangle$ acting on M,

— a *polynomial* $P(q)$ in q with non-negative integer coefficients.

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Definition

The triple (M, C, P) exhibits the *cyclic sieving phenomenon* if

$$|\operatorname{Fix}_M(g^p)| = P\left(e^{2\pi i p/|C|}\right).$$

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$$M = \big\{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,3\},\{2,4\}\big\}$$

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Cyclic sieving: equivalent characterisations

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Fact

The triple (M, C, P) *exhibits the* cyclic sieving phenomenon *if and only if*

$$P(q) \equiv \sum_{j=0}^{|C|-1} a_j q^j \mod q^{|C|} - 1,$$

where a_j *is the number of* C*-orbits for which the stabilizer order divides* j.

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Fact

Let g *be a generator of the cyclic group* C*, and let* $V^{(j)}$ *denote the (one-dimensional) irreducible representation of* C *given by* $g \cdot v = e^{2\pi ij/|C|} v$*. Furthermore, let* $P(q) = \sum_{j \ge 0} p_j q^j$*. Then the triple* (M, C, P) *exhibits the* cyclic sieving phenomenon *if and only if* $\mathbb{C}M$ *is isomorphic to* $\bigoplus_{j \ge 0} p_j V^{(j)}$*.*

History of Cyclic sieving

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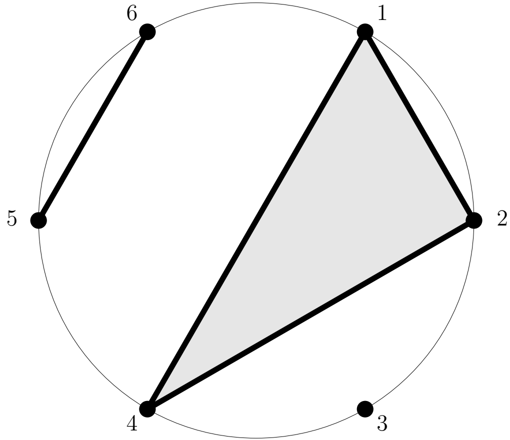
- early 1990s: "(-1)-phenomenon" for plane partitions (John Stembridge)
- 2004: "The cyclic sieving phenomenon" (Vic Reiner, Dennis Stanton, Dennis White)
- Instances of cylic sieving were discovered for permutations, for tableaux, for non-crossing matchings, for non-crossing partitions, for triangulations, for dissections of polygons, for clusters, for faces in the cluster complex, ...

Non-crossing partitions (Kreweras)



A non-crossing partition of $\{1, 2, 3, 4, 5, 6\}$

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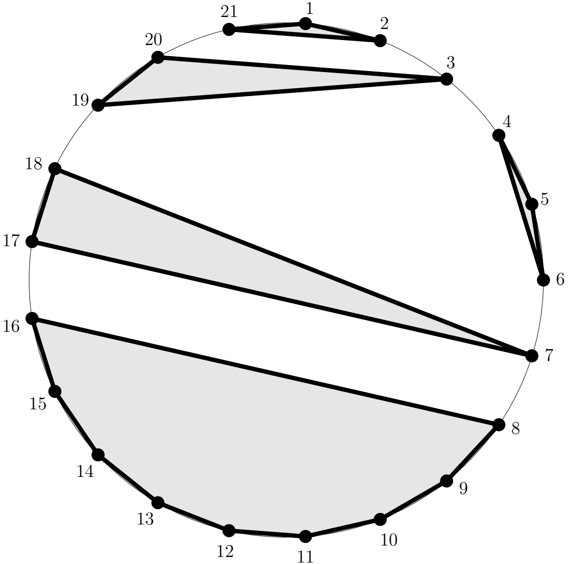
The non-crossing partitions of $\{1, 2, \ldots, n\}$, say $NC(n)$, can be (partially) ordered by refinement.

- $NC(n)$ is a *ranked poset*.
- $NC(n)$ is in fact a *lattice*.
- $NC(n)$ is *self-dual* ($\rightarrow$ Kreweras complement).
- $|NC(n)| = \dfrac{1}{n+1}\dbinom{2n}{n}$.
- There exist nice formulae for *Möbius function*, *zeta polynomial*, ...

m-divisible non-crossing partitions (Edelman)



A 3-divisible non-crossing partition of $\{1, 2, \ldots, 21\}$

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- $NC^m(n)$ is a *ranked poset*.
- $NC^m(n)$ is a *join-semilattice*.
- $|NC^m(n)| = \frac{1}{n}\binom{(m+1)n}{n-1}$.
- There exist nice formulae for *Möbius function*, *zeta polynomial*, ...
- In particular, the number of elements of $NC^m(n)$ all block sizes of which are *equal* to m is

$$\frac{1}{n}\binom{mn}{n-1}.$$

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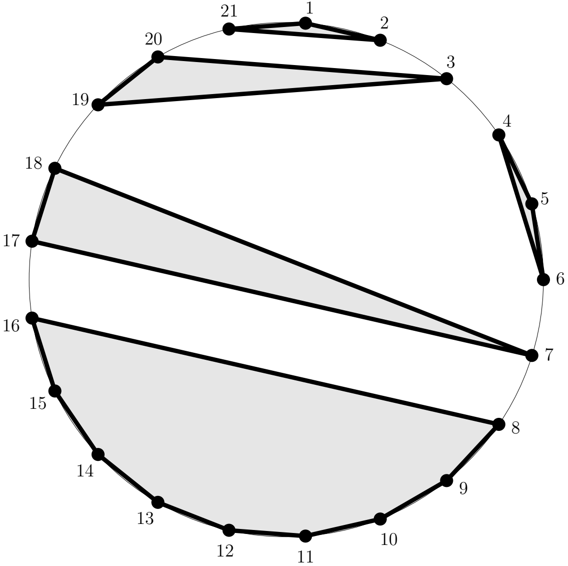

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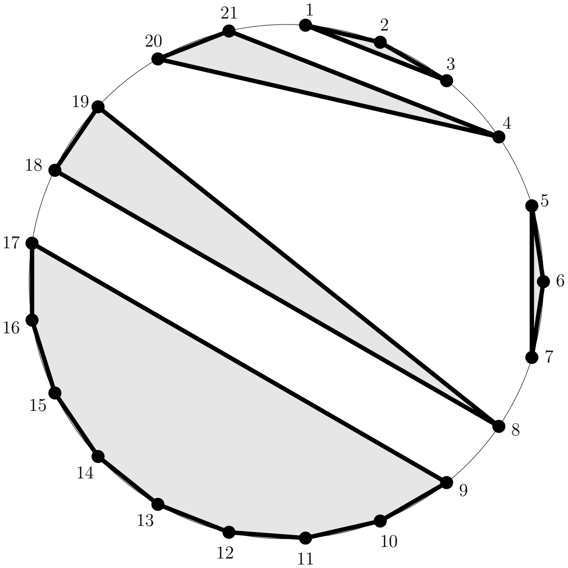

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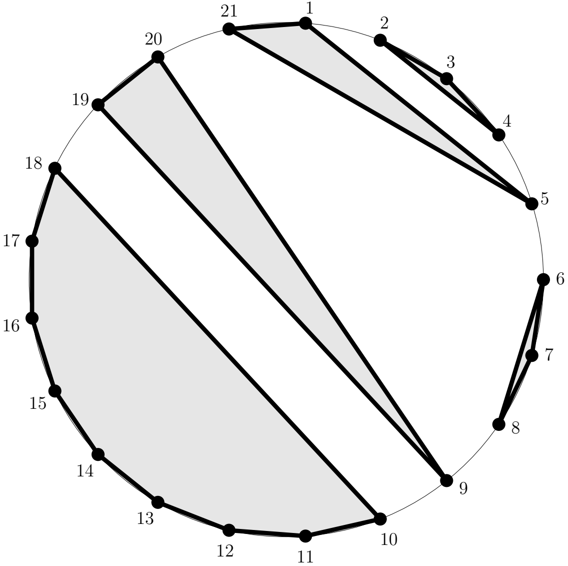

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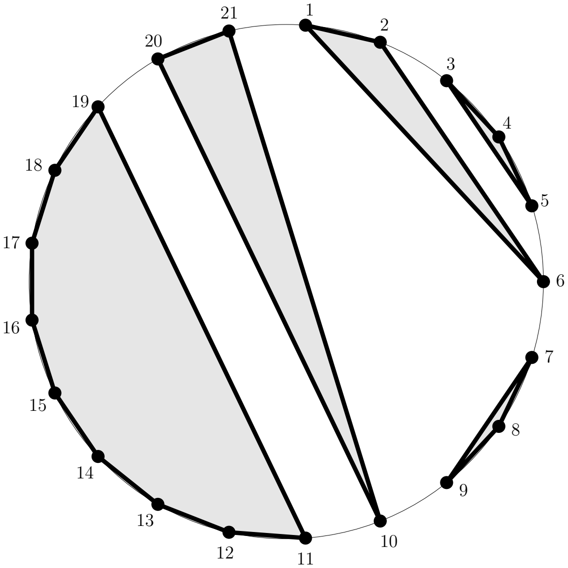

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- M = m-divisible non-crossing partitions of $\{1, 2, \ldots, mn\}$,
- $C = \langle \text{rotation} \rangle$,
- $P(q) = \frac{1}{[n]_q} \begin{bmatrix} (m+1)n \\ n-1 \end{bmatrix}_q$.

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m**-divisible non-crossing partitions for complex reflection groups!**

(Armstrong, Brady, Watt, Bessis)

An algebraic point of view

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Define the *absolute length* $\ell_T(\sigma)$ of a permutation $\sigma \in S_n$ by the smallest k such that

$$\sigma = t_1 t_2 \cdots t_k,$$

where all t_i are transpositions.

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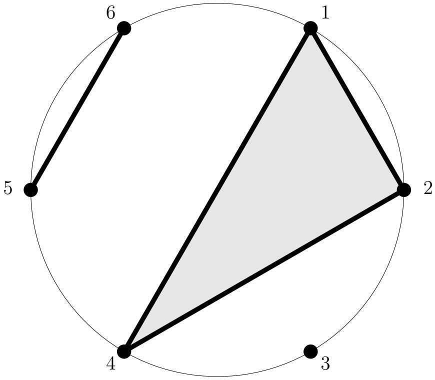

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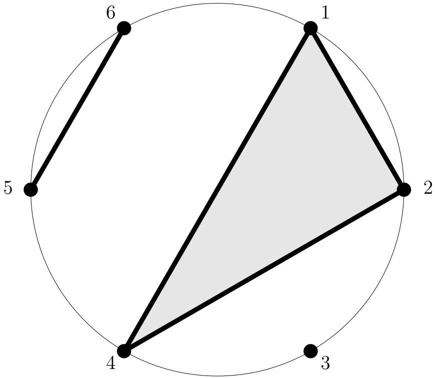

Indeed, one can show that the non-crossing partitions of $\{1,2,\dots,n\}$ are in bijection with

$$\{\sigma \in S_n : \sigma \leq_T (1,2,\dots,n)\}.$$

Complex reflection groups

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A *complex reflection* is a linear transformation on $\mathbb{C}^n$ which fixes a hyperplane pointwise, and which has finite order. In other words, a complex reflection is a diagonalisable linear transformation on $\mathbb{C}^n$ whose eigenvalues are 1 with multiplicity $n-1$, and whose remaining eigenvalue is a root of unity.

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A *complex reflection group* W is a group generated by (complex) reflections. Here, we consider always *finite* complex reflection groups.

The classification of all finite complex reflection groups

(Shephard and Todd)

All finite complex reflection groups are known!

All *irreducible* finite complex reflection groups are:

— the infinite family $G(d, e, n)$, where d, e, n are positive integers such that $e \mid d$,

— the exceptional groups $G_4, G_5, \ldots, G_{37}$.

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Any finite complex reflection group is a direct product of irreducible ones.

The groups $G(d, e, n)$

Let d, e, n be positive integers such that $e \mid d$. The group $G(d, e, n)$ consists of all $n \times n$ matrices, in which:

- *exactly* one entry in each row and in each column is non-zero;
- this non-zero entry is always some d-th root of unity;
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- $G(2,2,n) = D_n$.

Well-generated complex reflection groups

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A complex reflection group W of rank n is called *well-generated*, if it is generated by n (complex) reflections.

The classification of all well-generated complex reflection groups

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All *irreducible* well-generated complex reflection groups are:

— the two infinite families $G(d, 1, n)$ and $G(e, e, n)$, where d, e, n are positive integers,

— the exceptional groups
$G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$, $G_{23} = H_3, G_{24}, G_{25}, G_{26}, G_{27},\ G_{28} = F_4, G_{29}, G_{30} = H_4, G_{32}$, $G_{33},\ G_{34}, G_{35} = E_6,\ G_{36} = E_7,\ G_{37} = E_8$.

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The *degrees* $d_1 \le d_2 \le \cdots \le d_n$ of a (complex) reflection group W are the degrees of a system of homogeneous polynomial generators of the invariant ring of W. The largest degree, d_n, is called *Coxeter number*, and is denoted by h.

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A *regular element* (in the sense of Springer) is an element $w \in W$ which has an eigenvalue, ζ say, such that the corresponding eigenvector lies in no reflection hyperplane. If this eigenvalue ζ is a primitive h-th root of unity, then w is called a *Coxeter element*. We always write c for Coxeter elements.

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The non-crossing partitions for a well-generated complex reflection group W are defined by

$$NC(W) := \{w \in W : w \le_T c\},$$

where c is a Coxeter element in W.

Non-crossing partitions for reflection groups

Everything generalises to $NC(W)$:

— *order relation*: $\leq_T$

— $NC(W)$ is a *ranked* poset:

$$\text{rank of } w \;=\; \ell_T(w)$$

— $NC(W)$ is a *lattice*

— $NC(W)$ is *self-dual*:
"Kreweras-complement" is $w \mapsto cw^{-1}$

— *Catalan number* for W: if W is irreducible then

$$|NC(W)| = \prod_{i=1}^{n} \frac{h+d_i}{d_i}$$

m-divisible non-crossing partitions for reflection groups (Armstrong)

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The m-divisible non-crossing partitions for a complex reflection group W are defined by

$$NC^m(W) = \big\{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and } \\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\big\},$$

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In particular,

$$NC^1(W) \cong NC(W).$$

Combinatorial realisation in type A (Armstrong)

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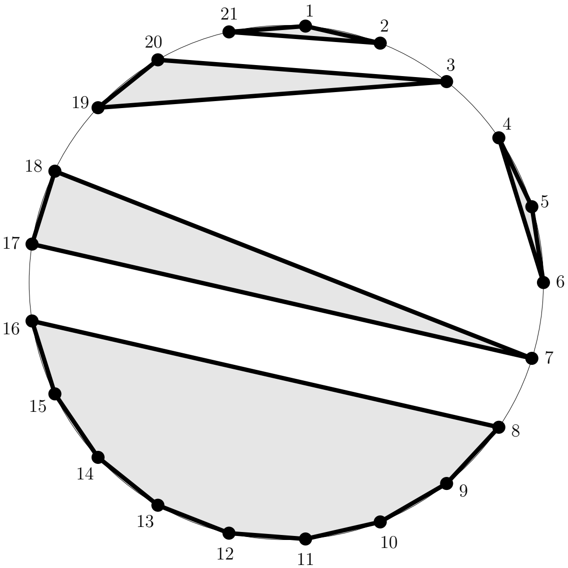
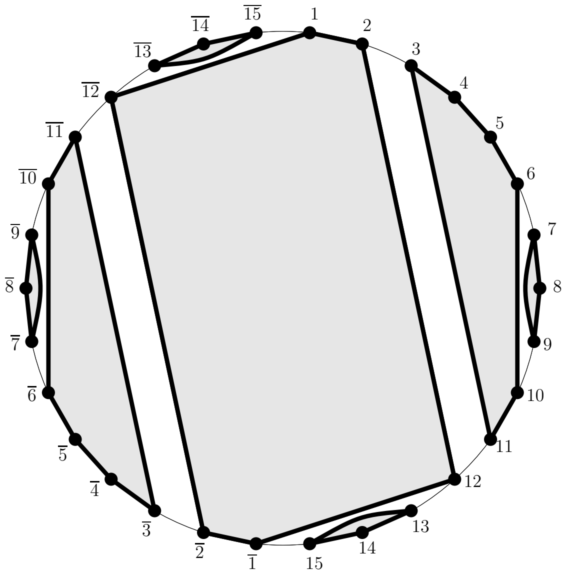
$$NC^m(W) = \big\{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and } \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\big\},$$

EXAMPLE FOR $m = 3$, $W = A_6 (= S_7)$:
$w_0 = (4,5,6)$, $w_1 = (3,6)$, $w_2 = (1,7)$, and $w_3 = (1,2,6)$.
Now "blow-up" w_1, w_2, w_3:

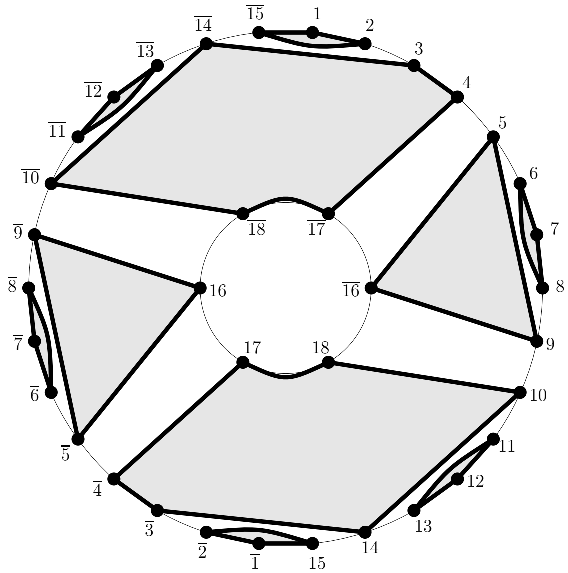
$$(1,2,\dots,21)\,(7,16)^{-1}\,(2,20)^{-1}\,(3,6,18)^{-1} = (1,2,21)\,(3,19,20)\,(4,5,6)\,(7,17,18)\,(8,9,\dots,16).$$



A 3-divisible non-crossing partition of type A_6



A 3-divisible non-crossing partition of type B_5



A 3-divisible non-crossing partition of type D_6

Properties of $NC^m(W)$

— *order relation*:

$$(u_0; u_1, \dots, u_m) \le (w_0; w_1, \dots, w_m)$$

if and only if $u_1 \ge w_1, \dots, u_m \ge w_m$;

— $NC^m(W)$ is a *join-semilattice*;

— $NC^m(W)$ is *ranked*:

$$\text{rank of } (w_0; w_1, \dots, w_m) = \ell_T(w_0)$$

The Fuß–Catalan numbers for reflection groups

Theorem (Athanasiadis, Bessis, Corran, Chapoton, Edelman, Reiner)

If W is irreducible then

$$|NC^m(W)| = \prod_{i=1}^{n} \frac{mh + d_i}{d_i}.$$

Let $\phi : NC^m(W) \to NC^m(W)$ be the map defined by

$$\begin{aligned}(w_0; w_1, \dots, w_m)\\ \mapsto \big((cw_mc^{-1})w_0(cw_mc^{-1})^{-1}; cw_mc^{-1}, w_1, w_2, \dots, w_{m-1}\big).\end{aligned}$$

It generates a cyclic group of order mh.

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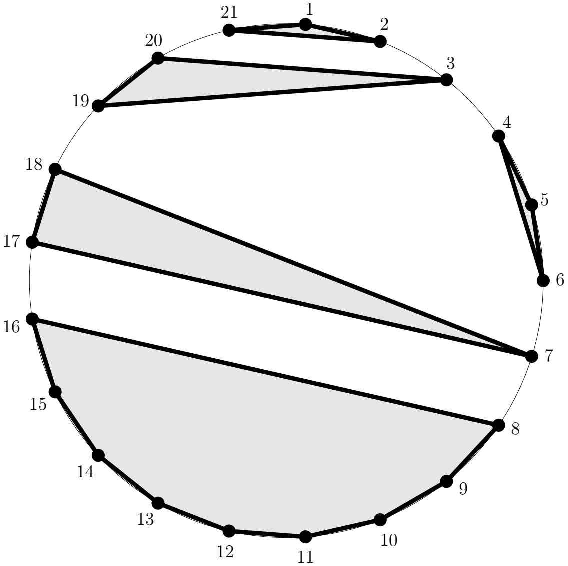

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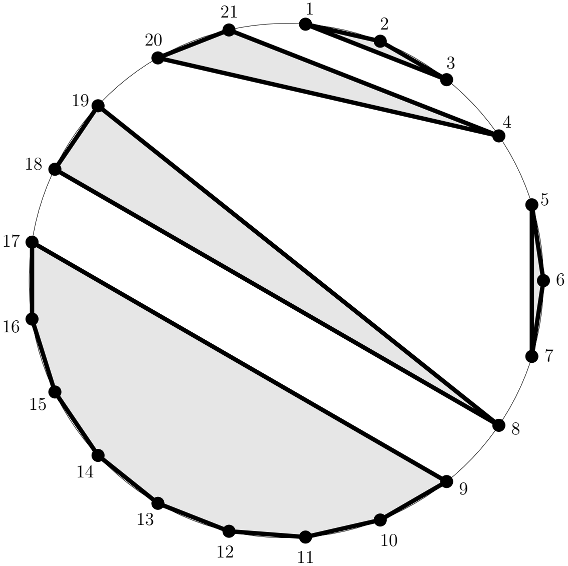

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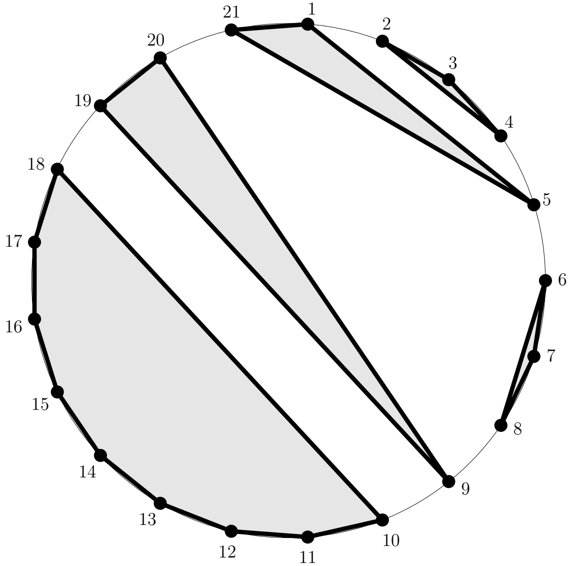

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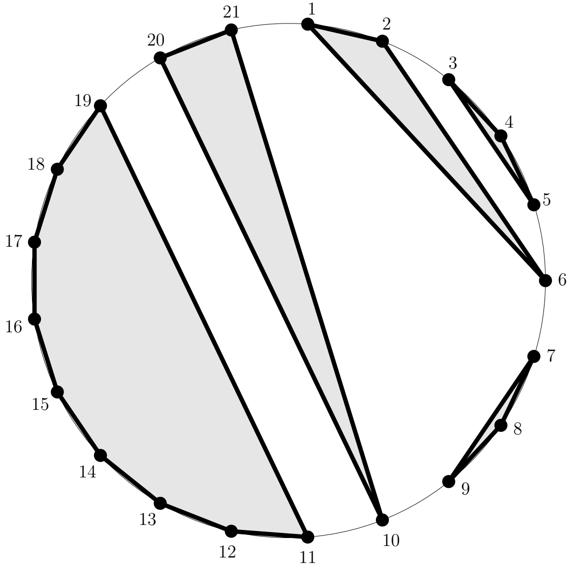

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Furthermore, let

$$\mathrm{Cat}^m(W;q) := \prod_{i=1}^{n} \frac{[mh+d_i]_q}{[d_i]_q},$$

where $[\alpha]_q := (1-q^\alpha)/(1-q)$.

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Theorem (with T. W. Müller)

The triple $(NC^m(W), \langle\phi\rangle, \mathrm{Cat}^m(W;q))$ *exhibits the cyclic sieving phenomenon.*

(Originally conjectured by Armstrong, Bessis and Reiner)

Let $\psi : NC^m(W) \to NC^m(W)$ be the map defined by

$$(w_0; w_1, \dots, w_m) \mapsto \big(c w_m c^{-1}; w_0, w_1, \dots, w_{m-1}\big).$$

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(If we embed

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The two cyclic sieving phenomena for $NC^m\big(G(d,1,n)\big)$ follow from the following result.

Theorem

Let m, n, r be positive integers such that $r \ge 2$ and $r \mid mn$. For non-negative integers $b_1, b_2, \dots, b_n$, the number of m-divisible non-crossing partitions of $\{1, 2, \dots, mn\}$ (in the sense of Edelman) which are invariant under the rotation $i \mapsto i + \frac{mn}{r}$ mod mn and have exactly rb_i non-zero blocks of size mi, $i = 1, 2, \dots, n$, is given by

$$\binom{b_1+b_2+\cdots+b_n}{b_1, b_2, \dots, b_n}\binom{mn/r}{b_1+b_2+\cdots+b_n}$$

if $b_1 + 2b_2 + \cdots + nb_n \le \lfloor n/r \rfloor$, and it is zero otherwise.

In order to establish the cyclic sieving phenomena for $NC^m\big(G(e, e, n)\big)$, one proves analogous enumeration results for m-divisible non-crossing partitions on an annulus.

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For the exceptional groups, we do a (lengthy) computer verification.