Two applications of useful functions

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‘Special functions’ should be more appropriately called ‘useful functions’

(Pál Turán)
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The ‘useful functions’ of this talk will be hypergeometric series.
\[(\alpha; q)_m := (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \cdots (1 - \alpha q^{m-1})\]

\[8W_7 \left( a; b, c, d, e, f; q, \frac{a^2 q^2}{bcdef} \right) = \frac{(aq, aq/de, aq/df, aq/ef; q)_{\infty}}{(aq/d, aq/e, aq/f, aq/def; q)_{\infty}} \times 4\phi_3 \left[ \frac{aq/bc, d, e, f}{aq/b, aq/c, def/a; q, q} \right] \]
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The ‘useful functions’ of this talk will be hypergeometric series.

→ the Mathematica packages HYP and HYPQ
The first application: Counting standard Young tableaux

(joint work with Michael Schlosser)
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Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) be two \( n \)-tuples of non-negative integers which are in non-increasing order and satisfy \( \lambda_i \geq \mu_i \) for all \( i \).

A standard Young tableau of skew shape \( \lambda/\mu \) is an arrangement of the numbers 1, 2, \ldots, \( \sum_{i=1}^{n} (\lambda_i - \mu_i) \) of the form

\[
\begin{array}{cccc}
\pi_{1,\mu_1+1} & \cdots & \cdots & \pi_{1,\lambda_1} \\
\pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots & \pi_{2,\lambda_2} \\
\vdots & & \vdots & & \vdots \\
\pi_{n,\mu_n+1} & \cdots & \cdots & \cdots & \pi_{n,\lambda_n}
\end{array}
\]

such that numbers along rows and columns are increasing.
The first application: Counting standard Young tableaux

A standard Young tableau of skew shape \( \lambda/\mu \) is an arrangement of the numbers 1, 2, \ldots, \( \sum_{i=1}^{n} (\lambda_i - \mu_i) \) of the form

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\vdots & & \vdots & & \vdots \\
\pi_{n,\mu_n+1} & \cdots & \cdots & \cdots & \pi_{n,\lambda_n}
\end{array}
\]

such that numbers along rows and columns are increasing.
A standard Young tableau of skew shape $\lambda/\mu$ is an arrangement of the numbers $1, 2, \ldots, \sum_{i=1}^n (\lambda_i - \mu_i)$ of the form

\[
\begin{array}{ccccccc}
\pi_{1,\mu_1+1} & \ldots & \ldots & \pi_{1,\lambda_1} \\
\pi_{2,\mu_2+1} & \ldots & \pi_{2,\mu_1+1} & \ldots & \pi_{2,\lambda_2} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\pi_{n,\mu_n+1} & \ldots & \ldots & \pi_{n,\lambda_n}
\end{array}
\]

such that numbers along rows and columns are increasing.

A standard Young tableau of shape (6, 5, 4, 3, 2, 1)/(3, 3, 0, 0, 0, 0):

\[
\begin{array}{ccc}
2 & 5 & 13 \\
3 & 9 & \\
1 & 4 & 8 & 12 \\
6 & 11 & 15 \\
7 & 14 \\
10
\end{array}
\]
John Stembridge:

My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?
John Stembridge:

My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?
We shall do something more general than DeWitt here: we shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.
Our goal: Let $N, n, m, r$ be non-negative integers. Compute the number of all standard Young tableaux of shape $(N, N-1, \ldots, N-n+1)/(m^r)$, where $(m^r)$ stands for $(m, m, \ldots, m, 0, \ldots, 0)$ with $r$ components $m$).
The first application: Counting standard Young tableaux

Aitken’s Formula

The number of all standard Young tableaux of shape $\lambda/\mu$ equals

$$\left(\sum_{i=1}^{n}(\lambda_i - \mu_i)\right)! \cdot \det_{1\leq i,j\leq n} \left(\frac{1}{(\lambda_i - i - \mu_j + j)!}\right).$$
The first application: Counting standard Young tableaux

We substitute in Aitken’s formula:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)! \det_{1\leq i,j \leq n} \begin{pmatrix}
\frac{1}{(N+1-2i-m+j)!} & j \leq r \\
\frac{1}{(N+1-2i+j)!} & j > r
\end{pmatrix}.
\]
The first application: Counting standard Young tableaux

We substitute in Aitken’s formula:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)! \det_{1 \leq i,j \leq n} \begin{pmatrix} 1 & j \leq r \\ \frac{1}{(N + 1 - 2i - m + j)!} & j > r \end{pmatrix}.
\]

We now do a Laplace expansion with respect to the first \( r \) columns:

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \ldots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^{r} k_i} \det_{1 \leq i,j \leq r} \left( \frac{1}{(N + 1 - 2k_i - m + j)!} \right)
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\}} \left( \frac{1}{(N + 1 - 2i + j)!} \right).
\]
The first application: Counting standard Young tableaux

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \cdots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^{r} k_i} \det_{1 \leq i, j \leq r} \left( \frac{1}{(N + 1 - 2k_i - m + j)!} \right)
\]

\[
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\}} \left( \frac{1}{(N + 1 - 2i + j)!} \right) .
\]
The first application: Counting standard Young tableaux

\[
\left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\times \sum_{1 \leq k_1 < \cdots < k_r \leq n} (-1)^{\binom{r+1}{2} + \sum_{i=1}^r k_i} \det_{1 \leq i, j \leq r} \left( \frac{1}{(N + 1 - 2k_i - m + j)!} \right)
\cdot \det_{1 \leq i \leq n, i \notin \{k_1, \ldots, k_r\}} \left( \frac{1}{(N + 1 - 2i + j)!} \right).
\]

Both determinants can be evaluated by means of

\[
\det_{1 \leq i, j \leq s} \left( \frac{1}{(X_i + j)!} \right) = \prod_{i=1}^s \frac{1}{(X_i + s)!} \prod_{1 \leq i < j \leq s} (X_i - X_j),
\]
After a lot of simplification, one arrives at

\[
(-1)^r \binom{r}{2} 2 \binom{r}{2} + \binom{n-r}{2} \left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\]

\[
\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)!(N-m+r-1)!}
\]

\[
\times \sum_{0 \leq k_1 < \cdots < k_r \leq n-1 \ 1 \leq i < j \leq r} \prod (k_j - k_i)^2
\]

\[
\cdot \prod_{i=1}^{r} \frac{(-\frac{N-m+r-1}{2})_{k_i}}{(-\frac{N+n-1}{2})_{k_i} (-\frac{N+n-2}{2})_{k_i} k_i!}
\]
The first application: Counting standard Young tableaux

After a lot of simplification, one arrives at

\[
(-1)\binom{r}{2}2\binom{r}{2} + \binom{n-r}{2} \left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\]

\[
\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)! (N-m+r-1)!} \]

\[
\times \sum_{0 \leq k_1 < \cdots < k_r \leq n-1} \prod_{1 \leq i < j \leq r} (k_j - k_i)^2
\]

\[
\prod_{i=1}^{r} \frac{(-N-m+r-1)_k}{(-N+n-1)_k} \frac{(-N-m+r-2)_k}{(-N+n-2)_k} (-n+1)_k k_i!
\]

\[\rightarrow\] multiple hypergeometric series associated to root systems!

Christian Krattenthaler
Two applications of useful functions
The first application: Counting standard Young tableaux

An elliptic transformation formula (Rains, Coskun and Gustafson)

Let $a, b, c, d, e, f$ be indeterminates, let $m$ be a nonnegative integer, and $r \geq 1$. Then

$$
\sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(aq^{k_i+k_j}; p)^2 \\
\times \prod_{i=1}^r \frac{\theta(aq^{2k_i}; p)(a, b, c, d, e, f; q, p)_{k_i}}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f; q, p)_{k_i}} \\
\times \prod_{i=1}^r \frac{(\lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{(efq^{r-1-m}/\lambda, aq^{1+m}; q, p)_{k_i}}
$$
The first application: Counting standard Young tableaux

\[
= \prod_{i=1}^{r} \frac{(b, c, d, ef/a; q, p)_{i-1}}{(\lambda b/a, \lambda c/a, \lambda d/a, ef/\lambda; q, p)_{i-1}} \\
\times \prod_{i=1}^{r} \frac{(aq; q, p)_m (aq/ef; q, p)_{m+1-r} (\lambda q/e, \lambda q/f; q, p)_{m-i+1}}{(\lambda q; q, p)_m (\lambda q/ef; q, p)_{m+1-r} (aq/e, aq/f; q, p)_{m-i+1}} \\
\times \sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} q^{\sum_{i=1}^{r}(2i-1)k_i} \prod_{1 \leq i < j \leq r} \theta(q^{k_i-k_j}; p)^2 \theta(\lambda q^{k_i+k_j}; p)^2 \\
\times \prod_{i=1}^{r} \frac{\theta(\lambda q^{2k_i}; p)(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f; q, p)_{k_i}}{\theta(\lambda; p)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f; q, p)_{k_i}} \\
\times \prod_{i=1}^{r} \frac{(\lambda aq^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{(efq^{r-1-m}/\lambda, \lambda q^{1+m}; q, p)_{k_i}},
\]

where \( \lambda = a^2 q^{2-r} /bcd. \)
In the elliptic transformation formula, we let $p = 0$, $d \rightarrow aq/d$, $f \rightarrow aq/f$, and then $a \rightarrow 0$. Next we perform the substitutions $b \rightarrow q^b$, $c \rightarrow q^c$, etc., we divide both sides of the identity obtained so far by $(1 - q)^{(r)}$, and we let $q \rightarrow 1$. 
Corollary

For all non-negative integers $m$, $r$ and $s$, we have

\[
\sum_{0 \leq k_1 < k_2 < \ldots < k_r \leq m} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \frac{(d + k_i)_s (b)_{k_i} (-m)_{k_i}}{k_i! (f)_{k_i}}
\]

\[
= \frac{(-1)^{r \choose 2}}{(r + s - 1)!^{s-1}} \prod_{i=1}^{r} (b)_{i-1} \frac{(-f + b + s + 2r - i - m)_{m-r+1}}{(-f - m + i)_{m-i+1}}
\]

\[
\times \prod_{i=1}^{r+s-1} \frac{(i - 1)!}{(m - i)!} \prod_{i=r}^{r+s-1} \frac{(d - b + 1 - r)_{i}}{(r + s - i - 1)! (d)_{i-r} (f - b - s + 1 - r)_{i}}
\]

\[
\times \sum_{0 \leq \ell_1 < \ell_2 < \ldots < \ell_s \leq r + s - 1} \prod_{1 \leq i < j \leq s} (\ell_i - \ell_j)^2
\]

\[
\times \prod_{i=1}^{s} \frac{(d)_{\ell_i} (f - b - s + 1 - r)_{\ell_i} (-r - s + 1)_{\ell_i}}{\ell_i! (d - b + 1 - r)_{\ell_i} (-m)_{\ell_i}}.
\]
Corollary

For all non-negative integers $m$, $r$ and $s$, we have

$$\sum_{0 \leq k_1 < k_2 < \cdots < k_r \leq m} \prod_{1 \leq i < j \leq r} (k_i - k_j)^2 \prod_{i=1}^{r} \frac{(d + k_i)_s (b)_{k_i} (-m)_{k_i}}{k_i! (f)_{k_i}}$$

$$= \frac{(-1)^{r+1}}{(r + s - 1)!^{s-1}} \prod_{i=1}^{r} (b)_{i-1} \frac{(-f + b + s + 2r - i - m)_{m-r+1}}{(-f - m + i)_{m-i+1}}$$

$$\times \prod_{i=1}^{r+s-1} \frac{(i-1)! \ m!}{(m-i)!} \prod_{i=r}^{r+s-1} \frac{r+s-i-1)! \ (d)_{i-r}}{(r + s - i - 1)! \ (d)_{i-r}}$$

$$\times \sum_{0 \leq \ell_1 < \ell_2 < \cdots < \ell_s \leq r+s-1} \prod_{1 \leq i < j \leq s} (\ell_i - \ell_j)^2$$

$$\times \prod_{i=1}^{s} \frac{(d)_{\ell_i} \ (f - b - s + 1 - r)_{\ell_i} \ (-r - s + 1)_{\ell_i}}{\ell_i! \ (d - b + 1 - r)_{\ell_i} \ (-m)_{\ell_i}}.$$
The first application: Counting standard Young tableaux

**Theorem**

If $N - n$ is even, the number of standard Young tableaux of shape $(N, N - 1, \ldots, N - n + 1)/(m^r)$ equals

\[
(-1)^{\frac{(N-n)/2}{2}} + \frac{1}{2} r(N-n) \binom{n}{2} + (N-n-m)r \left( \binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!
\]

\[
\times \frac{1}{(r + \frac{N-n-2}{2})(N-n)/2 \frac{N+n-2}{2}(N-n)/2} \frac{\prod_{i=1}^{(N+n)/2} (i-1)!}{\prod_{i=1}^{n} (N - n + 2i - 1)!}
\]

\[
\times \prod_{i=1}^{r} \frac{(N-n)/2 + i - 1)! (n + m - r + 2i - 1)! (\frac{n+m-r+1}{2} + i)(N-n)/2}{(m+i-1)! (N - m - r + 2i - 1)!}
\]

\[
\times \sum_{0 \leq \ell_1 < \ell_2 < \cdots < \ell(N-n)/2 \leq r+\frac{N-n-2}{2}} (-1)^{\sum_{i=1}^{(N-n)/2} \ell_i} \left( \prod_{1 \leq i < j \leq \frac{N-n}{2}} (\ell_i - \ell_j)^2 \right)
\]

\[
\cdot \frac{N-n}{2} \prod_{i=1}^{N-n/2} \left( \frac{N-n}{2} - \ell_i \right) \frac{(N+n)/2 - \ell_i}{\binom{n+m-r+1}{2} - i} \frac{r+i-\ell_i-1}{(N+m-r+2)/2 - i} r+i-\ell_i-1,
\]

and there is a similar statement if $N - n$ is odd.
In the case of a full staircase (i.e., \( n = N \)), the formula reduces to DeWitt's original result.

**Corollary**

*The number of standard Young tableaux of shape\((n, n - 1, \ldots, 1)/(m^r)\) equals*

\[
2 \binom{n}{2}^{-rm} \left( \binom{n+1}{2} - mr \right)! \prod_{i=1}^{n} \frac{(i - 1)!}{(2i - 1)!} \times \prod_{i=1}^{r} \frac{(n + m - r + 2i - 1)! (i - 1)!}{(m + i - 1)! (n - m - r + 2i - 1)!'}
\]
The first application: Counting standard Young tableaux

The “next” case:

**Corollary**

The number of standard Young tableaux of shape 
\((n + 1, n, \ldots, 2)/(m^r)\) equals

\[
2^{n-m-1}r \left( \binom{n+2}{2} - mr - 1 \right)! \prod_{i=1}^{n} \frac{(i - 1)!}{(2i)!} 
\times \prod_{i=1}^{r} \frac{(n + m - r + 2i - 1)! (i - 1)!}{(m + i - 1)! (n - m - r + 2i)!} 
\times \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} \frac{(n - \ell + 1)\ell}{2} \frac{(n+m-r+1)}{2} \frac{(n+m-r+1)}{2} \frac{(n+m-r+1)}{2} \frac{(n+m-r+1)}{2} .
\]

Christian Krattenthaler

Two applications of useful functions
In general:
The number of standard Young tableaux of shape
$(N, N - 1, \ldots, N - n)/(m^r)$ equals an $\lceil(N - n)/2\rceil$-fold
hypergeometric sum.
The first application: Counting standard Young tableaux

John Stembridge:

I think her approach is much simpler; but I don't think it would extend to the "next case" you mention.
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I think her approach is much simpler; but I don’t think it would extend to the ‘next case’ you mention.
The second application: Differential operators

(joint work with ANDEEAS JUHL)

A $GJMS$ $P$ is a specific rule which associates to any pseudo-Riemannian manifold $(M, g)$ a differential operator of the form $P^N(g) = \Delta^N g + \text{lower-order terms}$, where $\Delta g = -\delta g d$ is the Laplace–Beltrami operator of $g$. Our pseudo-Riemannian manifold is the Möbius spheres $S^{q,p}$, with the signature $(q, p)$-metric $g_{S^q} - g_{S^p}$ given by the round metrics on the factors.

The second application: Differential operators

(joint work with Andreas Juhl)

A GJMS\(^1\)-operator \(P_{2N}, N \geq 1\), is a specific rule which associates to any pseudo-Riemannian manifold \((M, g)\) a differential operator of the form

\[
P_{2N}(g) = \Delta^N_g + \text{lower-order terms},
\]

where \(\Delta_g = -\delta_g d\) is the Laplace–Beltrami operator of \(g\).

The second application: Differential operators

(joint work with Andreas Juhl)

A GJMS\textsuperscript{1}-operator $P_{2N}$, $N \geq 1$, is a specific rule which associates to any pseudo-Riemannian manifold $(M, g)$ a differential operator of the form

$$P_{2N}(g) = \Delta^N_g + \text{lower-order terms},$$

where $\Delta_g = -\delta_g d$ is the Laplace–Beltrami operator of $g$.

Our pseudo-Riemannian manifold is the Möbius spheres

$$\mathbb{S}^{q,p} = \mathbb{S}^q \times \mathbb{S}^p$$

with the signature $(q, p)$-metric $g_{\mathbb{S}^q} - g_{\mathbb{S}^p}$ given by the round metrics on the factors.

\textsuperscript{1}C. R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling (1992)
Andreas Juhl looked for relations between the GJMS-operators. By extensive computer experiments, he found a whole set of such relations. In order to state these, we need some notation first. For an $r$-tuple $I = (I_1, \ldots, I_r)$ of positive integers, we write $|I| := I_1 + \cdots + I_r$ and $P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}$. For $N \geq 1$, Juhl looked at sums of the form $\sum_{|I| = N} m_I P_{2I}$ with the multiplicities $m_I$, for $I = (I_1, \ldots, I_r)$, defined by $m_I = -(-1)^r |I|! (|I| - 1)!^r \prod_{j=1}^{r-1} 1/I_j! (I_j - 1)! \prod_{j=1}^r 1/I_j + I_j + 1$. 
Andreas Juhl looked for relations between the GJMS-operators. By extensive computer experiments, he found a whole set of such relations. In order to state these, we need some notation first.

For an $r$-tuple $I = (I_1, \ldots, I_r)$ of positive integers, we write $|I| := I_1 + \cdots + I_r$ and $P_{I_r}^{2I} = P_{2I_1}^I \circ \cdots \circ P_{2I_r}^I$.

For $N \geq 1$, Juhl looked at sums of the form $\sum_{|I| = N} m_I P_{2I}$ with the multiplicities $m_I$, for $I = (I_1, \ldots, I_r)$, defined by

$$m_I = -\left(\frac{-1}{r}\right)^r |I|! \left(|I| - 1\right)! \prod_{j=1}^r \frac{1}{I_j! \left(I_j - 1\right)!} \frac{r}{I_j + 1} \prod_{j=1}^r \frac{1}{I_j}.$$
Andreas Juhl looked for relations between the GJMS-operators.

By extensive computer experiments, he found a whole set of such relations. In order to state these, we need some notation first.

For an \( r \)-tuple \( \mathbf{l} = (l_1, \ldots, l_r) \) of positive integers, we write

\[
|\mathbf{l}| := l_1 + \cdots + l_r
\]

and

\[
P_{2\mathbf{l}} = P_{2l_1} \circ \cdots \circ P_{2l_r}.
\]

For \( N \geq 1 \), Juhl looked at sums of the form

\[
\sum_{|\mathbf{l}| = N} m_{\mathbf{l}} P_{2\mathbf{l}}
\]

with the multiplicities \( m_{\mathbf{l}} \), for \( \mathbf{l} = (l_1, \ldots, l_r) \), defined by

\[
m_{\mathbf{l}} = -(-1)^r |\mathbf{l}|! (|\mathbf{l}| - 1)! \prod_{j=1}^{r} \frac{1}{l_j! (l_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{l_j + l_{j+1}}.
\]
The second application: Differential operators

Here is the first set of relations found by Juhl:

**Conjecture**

On $\mathbb{S}^q\,p$, we have

$$\sum_{|I|=2N} m_I P_{2I} = (2N)! (2N-1)! \left( \frac{1}{2} - B^2 - C^2 \right), \quad N \geq 1$$

and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)! (2N)! (-B^2 + C^2), \quad N \geq 0.$$

Here,

$$B^2 = -\Delta_{\mathbb{S}^q} + \left( \frac{q - 1}{2} \right)^2 \quad \text{and} \quad C^2 = -\Delta_{\mathbb{S}^p} + \left( \frac{p - 1}{2} \right)^2.$$

Recall:

$$m_I = -(-1)^r |I|! (|I| - 1)! \prod_{j=1}^{r} \frac{1}{l_j! (l_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{l_j + l_j+1}.$$
The second application: Differential operators

So far, so mysterious . . .
So far, so mysterious . . .

It is well-known that

\[ P_{4N} = \prod_{j=1}^{N} \left( (B^2 - C^2)^2 - 2(2j - 1)^2(B^2 + C^2) + (2j - 1)^4 \right) \]

and

\[ P_{4N+2} = (-B^2 + C^2) \prod_{j=1}^{N} \left( (B^2 - C^2)^2 - 2(2j)^2(B^2 + C^2) + (2j)^4 \right). \]
So far, so mysterious . . .

It is well-known that

\[ P_{4N} = \prod_{j=1}^{N} (B + C + (2j-1))(B - C - (2j-1))(B + C - (2j-1))(B - C + (2j-1)) \]

and

\[ P_{4N+2} = (-B^2 + C^2) \prod_{j=1}^{N} (B + C + 2j)(B - C - 2j)(B + C - 2j)(B - C + 2j). \]
So far, so mysterious . . .

It is well-known that

\[ P_{2N} = 2^{2N} \left( \frac{(C+B+1-N)}{2} \right)_N \left( \frac{(C-B+1-N)}{2} \right)_N, \]
So far, so mysterious . . .

It is well-known that

\[ P_{2N} = 2^{2^N} \binom{(C+B+1-N)/2}{N} \binom{(C-B+1-N)/2}{N}, \]

Writing \( X = C + B \) and \( Y = C - B \), this becomes

\[ P_{2N} = 2^{2^N} \binom{(X + 1 - N)/2}{N} \binom{(Y + 1 - N)/2}{N}. \]
The conjecture again:

**Conjecture**

*On $S^g,p$, we have*

$$
\sum_{|I|=2N} m_I P_{2I} = \frac{1}{2} (2N)! (2N-1)! \left(1 - X^2 - Y^2\right), \quad N \geq 1,
$$

*and*

$$
\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)! (2N)! XY, \quad N \geq 0,
$$

*where*

$$
P_{2N} = 2^{2N} \left(\frac{X + 1 - N}{2}\right)_N \left(\frac{Y + 1 - N}{2}\right)_N.
$$

Recall:

$$
m_I = -(-1)^r |I|! |I|-1)! \prod_{j=1}^r \frac{1}{l_j! (l_j-1)!} \prod_{j=1}^{r-1} \frac{1}{l_j + l_j+1}.
$$
**Lemma**

For all non-negative integers $A$ and $B$, we have

\[
\frac{((X + 1 - A)/2)_A}{((X + 1 - B)/2)_B} = \sum_{j=0}^{[(A+B)/2]} (-1)^j \frac{(-A/2)_j (-B/2)_j (- (A + B)/2)_j}{j!} 
\times \frac{((X + 1 - A - B + 2j)/2)_{A+B-2j}}{j!}.
\]

Proof.

In hypergeometric notation, the sum on the right-hand side reads

\[
\]

The \(3\)\(F\)\(2\)-series is balanced and can hence be summed by means of the Pfaff–Saalschütz summation formula.
Lemma

For all non-negative integers $A$ and $B$, we have

\[
\frac{(X + 1 - A)/2}{A} \frac{(X + 1 - B)/2}{B}
= \sum_{j=0}^{[(A+B)/2]} (-1)^j \frac{(-A/2)_j (-B/2)_j (-A + B)/2)_j}{j!}
\times \frac{(X + 1 - A - B + 2j)/2}{A+B-2j}.
\]
Lemma

For all non-negative integers $A$ and $B$, we have

\[
\frac{\left((X + 1 - A)/2\right)_A \left((X + 1 - B)/2\right)_B}{\left[(A+B)/2\right]}
= \sum_{j=0}^{\left[(A+B)/2\right]} (-1)^j \frac{(-A/2)_j (-B/2)_j (-\left((A+B)/2\right)_j}{j!}
\times \left((X + 1 - A - B + 2j)/2\right)_A+B−2j.
\]

Proof.

In hypergeometric notation, the sum on the right-hand side reads

\[
\left((X+1-A-B)/2\right)_A+B \, _3F_2 \left[\begin{array}{c}
-(A+B)/2, -A/2, -B/2 \\
(1+X-A-B)/2, (1-X-A-B)/2
\end{array}; 1\right].
\]

The $\, _3F_2$-series is balanced and can hence be summed by means of the Pfaff–Saalschütz summation formula.
The conjecture again:

**Conjecture**

*On* $\mathbb{S}^q,p$, *we have*

\[
\sum_{|I|=2N} m_I P_{2I} = \frac{1}{2} (2N)! (2N - 1)! (1 - X^2 - Y^2), \; N \geq 1,
\]

and

\[
\sum_{|I|=2N+1} m_I P_{2I} = (2N + 1)! (2N)! XY, \; N \geq 0,
\]

where

\[
P_{2N} = 2^{2N} \left( (X + 1 - N)/2 \right)_N \left( (Y + 1 - N)/2 \right)_N.
\]

Recall:

\[
m_I = -(-1)^r |I|! (|I| - 1)! \prod_{j=1}^r \frac{1}{l_j! (l_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{l_j + l_j + 1}.
\]
The proof strategy: induction! First evaluate the partial sum, where in

\[ P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r} \]

the last component is fixed.
Lemma

For all positive integers \( a < N \), the partial sum

\[
S(N, a) = \sum_{J: |J| + a = N} m(J, a) P_{2J}
\]

satisfies

\[
S(N, a) = \binom{N-1}{a-1} \sum_{k=0}^{\lfloor (N-a)/2 \rfloor} \sum_{l=0}^{\lfloor (N-a)/2 \rfloor} (-1)^{N+k+l+a} 2^{N-2k-2l-2a} \cdot ((X + 1 - N + a + 2k)/2)_{N-a-2k} \cdot ((Y + 1 - N + a + 2l)/2)_{N-a-2l} \cdot (-N + a)_{2k} (-N + a)_{2l} (-N/2)_k (-N/2)_l \frac{k!}{l!} \times 4F_3 \left[ \begin{array}{c} -\frac{1}{2}, -k, -l, \frac{1}{2} - \frac{N}{2} \\ -N/2, a/2 - N/2, 1/2 + a/2 - N/2 \end{array} \right].
\]
The proof of the lemma is somewhat tedious . . .
The second application: Differential operators

\[
\frac{(-1)^{N-a-2s-1}}{(-N+2s)^{N-a-2s}} \left[ \frac{(-N+2s)/2)_{s_1-s}(N+2s)/2\right]_{s_2-s} \\
+ \frac{\chi(s_1 = s_2 = (N-a)/2 \cdot 2^{-N+a+2s}}{a!(N-2s)!} \\
= \frac{-a!(N/2)_{s_1}(-N/2)_{s_2}}{(N/2)^2_s} + \frac{\chi(s_1 = s_2 = (N-a)/2 \cdot 2^{-N+a+2s}}{a!(N-2s)!}
\]

where \(\chi(S) = 1\) if \(S\) is true and \(\chi(S) = 0\) otherwise. If we substitute this in (2.15), then we obtain

\[
\sum_{s_1=0}^{[(N-a)/2]} \sum_{s_2=0}^{[(N-a)/2]} ((X+1-N+a+2s_1)/2)_{N-a-2s_1} ((Y+1-N+a+2s_2)/2)_{N-a-2s_2}
\cdot (-1)^{N+s_1+s_2+a} 2^{N-2s_1-2s_2-2a} \frac{(N-1)!}{(a-1)!(N-a)!} \\
\cdot \frac{(-N+a)_{s_1}(-N+a)_{s_2}(-N/2)_{s_1}(-N/2)_{s_2}}{s_1!s_2!} \\
\cdot \sum_{s=0}^{N-a/2} \frac{(-1/2)_s(-s_1)_s(-s_2)_s(N-2s)!}{s! (N-2s)! (N-a-2s)!} \\
\chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} \\
\times \sum_{s=0}^{(N-a)/2} 2^{-N+a+2s} (-1/2)_s (-N-a/2)_s (N-a-2s)! \\
\cdot \frac{(N-a)/2}{s!}
\]

Here, the first sum is, upon rewriting, exactly equal to the right-hand side of (2.10) (except that \(s_1\) and \(s_2\) took over the role of \(k\) and \(l\)). On the other hand, if we write the second sum in hypergeometric notation, we obtain

\[
\chi(N-a \text{ is even}) \cdot \frac{N! (N-1)!}{a! (a-1)! ((N-a)/2)!^2} \\
\times \sum_{s=0}^{(N-a)/2} \frac{2^{-N+a+2s} (-1/2)_s (-N-a/2)_s (N-a-2s)!}{s!}
\]
The proof of the lemma is somewhat tedious . . . but it is not difficult. (The only summation formulas needed for simplification are the Chu–Vandermonde summation formula, the binomial theorem, and simple difference calculus.)
The proof of the lemma is somewhat tedious . . . but it is not difficult. (The only summation formulas needed for simplification are the Chu–Vandermonde summation formula, the binomial theorem, and simple difference calculus.)

Now the conjecture can be proved:

$$\sum_{|I|=N} m_I P_{2I} = P_{2N} + \sum_{a=1}^{N-1} S(N, a) P_{2a}.$$ 

One uses the lemma for the expansion of $S(N, a)$, applies the multiplication lemma, has to go through some more pages of the kind . . .
The second application: Differential operators

\[ ((Y + 1 - a)/2)_a \left( (Y + 1 - N + a + 2l)/2 \right)_{N-a-2l} \]
\[ = \sum_{j_2=0}^{[N/2][N/2]} (-1)^{j_2} \frac{(-a/2)_{j_2} \left( -(N - a - 2l)/2 \right)_{j_2} \left( -(N - 2l)/2 \right)_{j_2}}{j_2!} \]
\[ \cdot \left( (Y + 1 - N + 2l + 2j_2)/2 \right)_{N-2l-2j_2}. \]

We use these in (2.18) and, in addition, perform the index transformation \( s_1 = k + j_1 \) and \( s_2 = l + j_2 \). Thus, the left-hand side in (2.16) can be written in the form

\[ \sum_{s_1=0}^{[N/2]} \sum_{s_2=0}^{[N/2]} \sum_{a=1}^{N} (-1)^{N+s_1+s_2+a} 2^{N-2k-2l} \left( \frac{N-1}{a-1} \right) \]
\[ \cdot \left( (X + 1 - N + 2s_1)/2 \right)_{N-2s_1} \left( (Y + 1 - N + 2s_2)/2 \right)_{N-2s_2} \]
\[ \cdot \sum_{k=0}^{[N-a/2]} \sum_{l=0}^{[N-a/2]} \frac{(-a/2)_{s_1-k} \left( -(N-a)/2 \right)_{s_2-l} \left( -(N-a)/2 \right)_{s_1} \left( -(N-a)/2 \right)_{s_2}}{(s_1-k)! (s_2-l)! \left( -(N-a)/2 \right)_k \left( -(N-a)/2 \right)_l} \]
\[ \cdot \frac{(-N+a)_{2k} \left( -(N-a)/2 \right)_{2l} \left( -(N/2) \right)_{s_1} \left( -(N/2) \right)_{s_2}}{k! l!} \]
\[ 4F_3 \left[ \frac{-1}{2}, -k, -l, \frac{1}{2} - \frac{N}{2}; 1 \right]. \quad (2.19) \]

In this expression, we now concentrate on the terms involving the summation index \( k \) only:

\[ \sum_{k=0}^{[N-a/2]} (-a/2)_{s_1-k} \left( -(N-a)/2 \right)_k (-k)_s \]
\[ = \sum_{k=0}^{s_1} \frac{(-a/2)_{s_1-k} \left( -(N-a-1)/2 \right)_k (-k)_s}{k! (s_1-k)!}, \quad (2.20) \]

where \( s \) stands for the summation index of the \( 4F_3 \)-series in (2.19). Because of the term \((-k)_s\) in the numerator of the summand, we may start the summation at \( k = s \) (instead of at \( k = 0 \)). Hence, if we write this sum in hypergeometric notation, we obtain

\[ \frac{(-a/2)_{s_1-s} \left( -(N-a-1)/2 \right)_s (-s)_s}{2F_1 \left[ -s_1 + s, \frac{1}{2} - \frac{N}{2}, \frac{a}{2} + \frac{s}{2} + 1 \right]} \].
The proof of the lemma is somewhat tedious . . . but it is not difficult. (The only summation formulas needed for simplification are the Chu–Vandermonde summation formula, the binomial theorem, and simple difference calculus.) Now the conjecture can be proved:

\[
\sum_{|I|=N} m_I P_{2I} = P_{2N} + \sum_{a=1}^{N-1} S(N, a) P_{2a}.
\]

One uses the lemma for the expansion of \( S(N, a) \), applies the multiplication lemma, has to go through some more pages of the kind . . . until one arrives at the desired conclusion.
The second application: Differential operators

**Theorem**

On $\mathbb{S}^q,p$, we have

$$\sum_{|I|=2N} m_I P_{2I} = \frac{1}{2} (2N)! (2N - 1)! \left(1 - X^2 - Y^2\right), \quad N \geq 1,$$

and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N + 1)! (2N)! XY, \quad N \geq 0,$$

where

$$P_{2N} = 2^{2N} \left(\frac{X + 1 - N}{2}\right)_N \left(\frac{Y + 1 - N}{2}\right)_N.$$

Recall:

$$m_I = -(-1)^r |I|! (|I| - 1)! \prod_{j=1}^{r} \frac{1}{l_j! (l_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{l_j + l_{j+1}}.$$
A second theorem providing relations for the GJMS-operators:

**Theorem**

*On $S^{q,p}$, we have*

\[
\sum_{|I|=N} m_I \frac{P_{2I}(1)}{\frac{n}{2} - I_{\text{last}}} = N! (N-1)! \sum_{M=0}^{N} (-1)^M \binom{q}{M} \binom{p}{N-M}
\]

*for all $N \geq 1$.\]
A belated

Happy Birthday!