A dual of MacMahon's theorem on plane partitions

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By definition, a *plane partition* is an array of non-negative integers $(\pi_{i,j})_{i,j}$ such that entries along rows and along columns are non-increasing, that is,

$$\pi_{i,j} \geq \pi_{i,j+1} \quad \text{and} \quad \pi_{i,j} \geq \pi_{i+1,j}.$$

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Plane Partitions - modern perspective

$\longrightarrow$ rhombus tilings of a hexagon!

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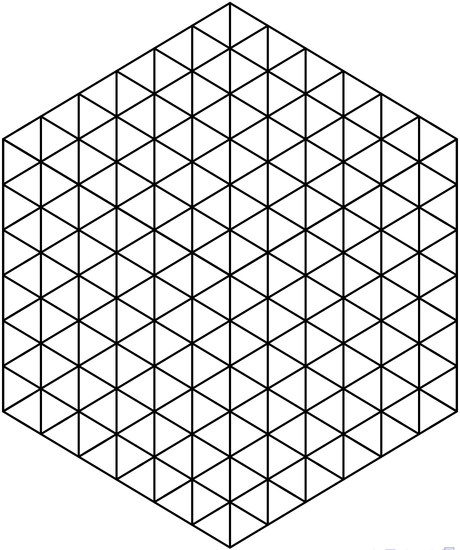
⟶ rhombus tilings of a hexagon!



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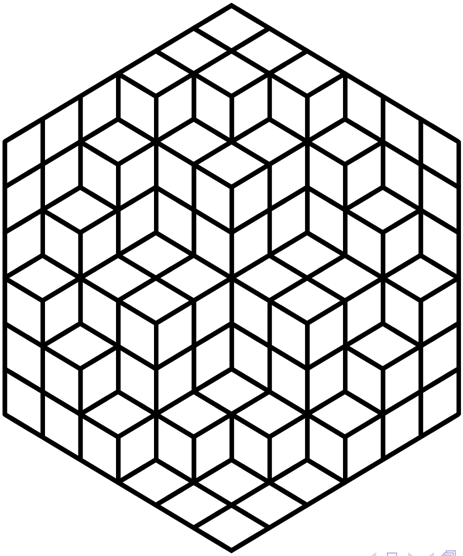

MacMahon's Plane Partition Theorem

The number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c equals

$$\frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(a+c)\,\mathrm{H}(b+c)},$$

where the hyperfactorials $\mathrm{H}(n)$ *are defined by*

$$\mathrm{H}(n) := 0!\,1!\cdots(n-1)!\,.$$

A dual?

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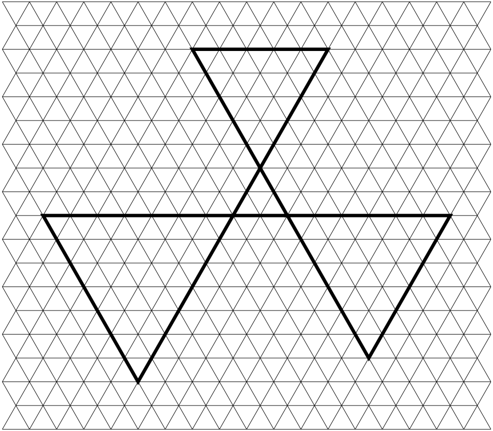

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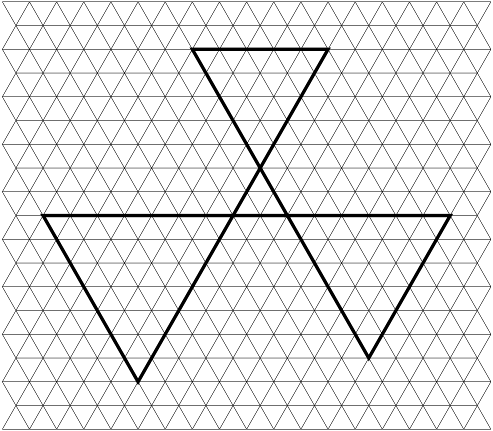

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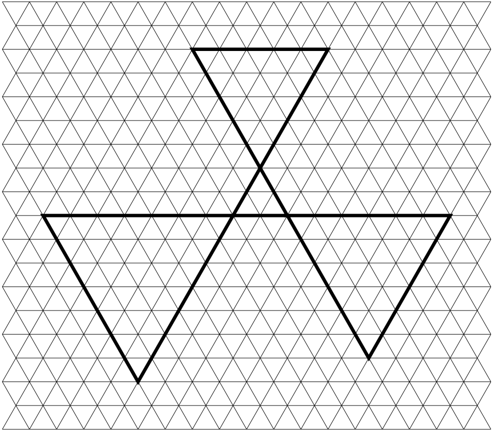

There are no rhombus tilings of such a thing!

"Dualise": "Count" the rhombus tilings of the outside!

A dual?

Let $S^*(a,b,c,m)$ denote the exterior of an (a,b,c,m)-shamrock. Furthermore, let $H_N(a,b,c,m)$ be the hexagonal region of side-lengths alternating between $N+a+b+c$ and $N+a+b+c+m$ (the top side being $N+a+b+c$), and having the shamrock $S(a,b,c,m)$ removed from its centre
Then we define

$$\frac{\mathrm{M}(S^*(a,b,c,m))}{\mathrm{M}(S^*(a+b+c,0,0,m))} := \lim_{N\to\infty} \frac{\mathrm{M}(H_N(a,b,c,m))}{\mathrm{M}(H_N(a+b+c,0,0,m))},$$

where $\mathrm{M}(R)$ denotes the number of rhombus tilings of the region R.

The dual MacMahon formula

Theorem

For any non-negative integers a, b, c and m we have

$$\frac{\mathrm{M}(S^*(a,b,c,m))}{\mathrm{M}(S^*(a+b+c,0,0,m))} = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c+m)\,\mathrm{H}(m)^2}{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c)} = P(a,b,m)\,P(a+b,c,m),$$

where $P(A,B,C)$ denotes the number of rhombus tilings of a hexagon with side lengths A, B, C, A, B, C.

Corollary

For any non-negative integers a, b, c, we have

$$\frac{\mathrm{M}(S^*(a, b, c, a + b + c))}{\mathrm{M}(S^*(a + b + c, 0, 0, a + b + c))} = P(a, b, c)\, P(a + b, b + c, c + a),$$

where $P(A, B, C)$ denotes the number of rhombus tilings of a hexagon with side lengths A, B, C, A, B, C.

$m = a + b + c$ — geometric interpretation

$$\frac{\mathrm{M}(S^*(a,b,c,a+b+c))}{\mathrm{M}(S^*(a+b+c,0,0,a+b+c))} = P(a,b,c)\,P(a+b,b+c,c+a)$$

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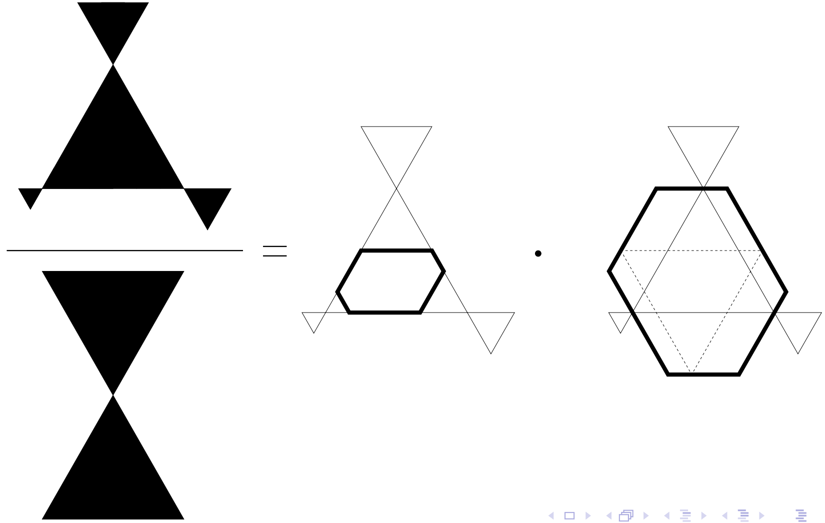
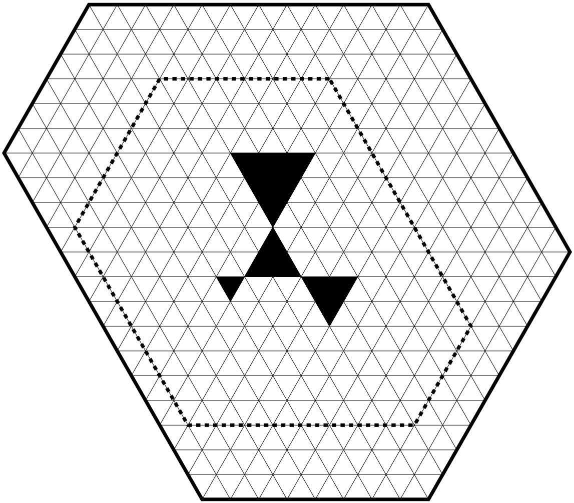


A more general region - equal parity case

The region $SC_{x,y,z}(a, b, c, m)$:



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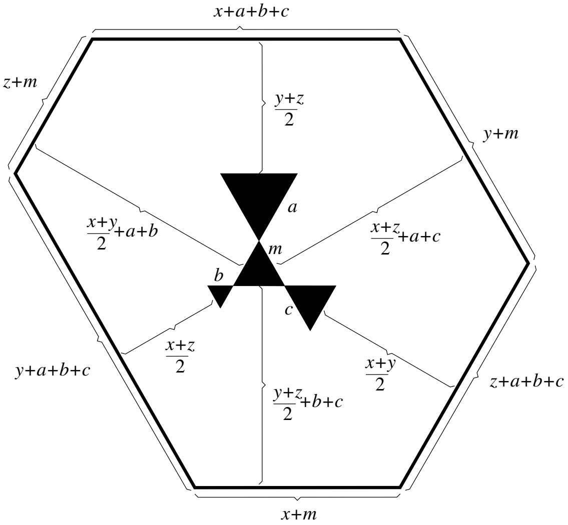


The main theorem: equal parity case

Theorem

Let x, y, z, a, b, c and m be nonnegative integers. If x, y and z have the same parity, we have

$$\mathrm{M}(SC_{x,y,z}(a,b,c,m)) = \frac{\mathrm{H}(m)^3\ \mathrm{H}(a)\ \mathrm{H}(b)\ \mathrm{H}(c)}{\mathrm{H}(m+a)\ \mathrm{H}(m+b)\ \mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(\frac{x+y}{2}+m+a+b)\ \mathrm{H}(\frac{x+z}{2}+m+a+c)\ \mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\frac{x+y}{2}+m+c)\ \mathrm{H}(\frac{x+z}{2}+m+b)\ \mathrm{H}(\frac{y+z}{2}+m+a)}$$
$$\times \frac{\mathrm{H}(\frac{x+y}{2}+c)\ \mathrm{H}(\frac{x+z}{2}+b)\ \mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\frac{x+y}{2}+a+b)\ \mathrm{H}(\frac{x+z}{2}+a+c)\ \mathrm{H}(\frac{y+z}{2}+b+c)}$$
$$\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}$$
$$\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)}$$

The main theorem: equal parity case

$$
\times \frac{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + m + a + b + c)}{\mathrm{H}(\frac{x+y}{2} + m + a + b + c)\,\mathrm{H}(\frac{x+z}{2} + m + a + b + c)}
$$

$$
\times \frac{\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + m + a + b + c)\,\mathrm{H}(\frac{y+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\frac{y+z}{2} + m + a + b + c)\,\mathrm{H}(\frac{x+y}{2})\,\mathrm{H}(\frac{x+z}{2})\,\mathrm{H}(\frac{y+z}{2})}
$$

$$
\times \frac{\mathrm{H}(\lceil \frac{x}{2} \rceil\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor)\,\mathrm{H}(\lceil \frac{y}{2} \rceil)}{\mathrm{H}(\lceil \frac{x}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{y}{2} \rceil + \frac{m+a+b+c}{2})}
$$

$$
\times \frac{\mathrm{H}(\lfloor \frac{y}{2} \rfloor)\,\mathrm{H}(\lceil \frac{z}{2} \rceil)\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor)}{\mathrm{H}(\lfloor \frac{y}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor + \frac{m+a+b+c}{2})}
$$

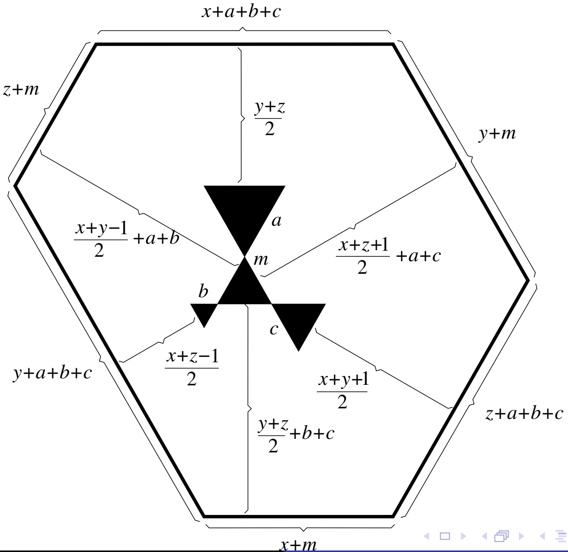
$$
\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\ \mathrm{H}(\frac{x+y}{2} + \frac{m+a+b+c}{2})^2\ \mathrm{H}(\frac{x+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + \frac{m+a+b+c}{2})},
$$

where

$$
\mathrm{H}(n) := \begin{cases} \prod_{k=0}^{n-1} \Gamma(k+1), & \textit{for } n \textit{ a positive integer,} \\ \prod_{k=0}^{n-\frac{1}{2}} \Gamma(k+\frac{1}{2}), & \textit{for } n \textit{ a positive half-integer,} \end{cases}
$$

A more general region - unequal parity case

The region $SC_{x,y,z}(a,b,c,m)$:



The main theorem: unequal parity case

Theorem

Let x, y, z, a, b, c and m be nonnegative integers. If x has parity different from the parity of y and z, we have

$$
\begin{aligned}
\mathrm{M}(SC_{x,y,z}(a,b,c,m)) &= \frac{\mathrm{H}(m)^3\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}\\
&\times \frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+m+a+b)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+m+a+c)\,\mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+m+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+m+b)\,\mathrm{H}(\frac{y+z}{2}+m+a)}\\
&\times \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil+c)\,\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor+b)\,\mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor+a+b)\,\mathrm{H}(\lceil\frac{x+z}{2}\rceil+a+c)\,\mathrm{H}(\frac{y+z}{2}+b+c)}\\
&\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}\\
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\end{aligned}
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The main theorem: unequal parity case

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\times \frac{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + m + a + b + c)}{\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + m + a + b + c)\,\mathrm{H}(\lceil \frac{x+z}{2} \rceil + m + a + b + c)}
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$$

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$$

Kuo's graphical condensation

Theorem (Kuo)

Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let vertices α, β, γ and δ appear cyclically on a face of G. If $\alpha, \gamma \in V_1$ and $\beta, \delta \in V_2$, then

$$\begin{aligned} &\mathrm{M}(G)\mathrm{M}(G - \{\alpha, \beta, \gamma, \delta\}) \\ =\ &\mathrm{M}(G - \{\alpha, \beta\})\mathrm{M}(G - \{\gamma, \delta\}) + \mathrm{M}(G - \{\alpha, \delta\})\mathrm{M}(G - \{\beta, \gamma\}), \end{aligned}$$

where $\mathrm{M}(H)$ denotes the number of perfect matchings of the bipartite graph H.

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\end{aligned}$$

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This is, as worked out by Fulmek, a combinatorial version of Dodgon's condensation formula for determinants ($=$ of a special case of a determinant formula of Jacobi ...).

Both, the equal parity case and the unequal parity case, are simultaneously proved by induction on $x + y + z$.

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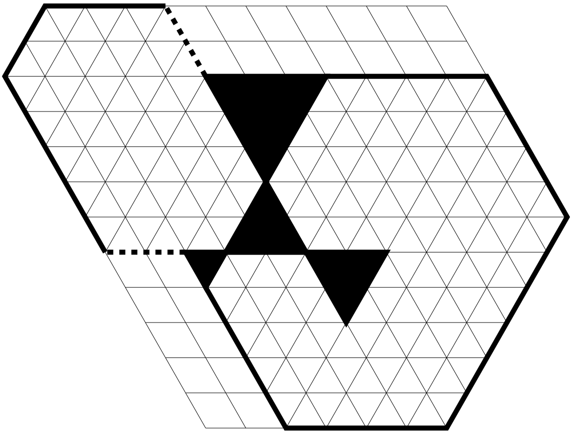
The induction base. These are the cases where $x = 0$, $y = 0$, or $z = 0$. Again, there are several cases. In the equal parity case, by symmetry of parameters it is sufficient to consider the case where $z = 0$.

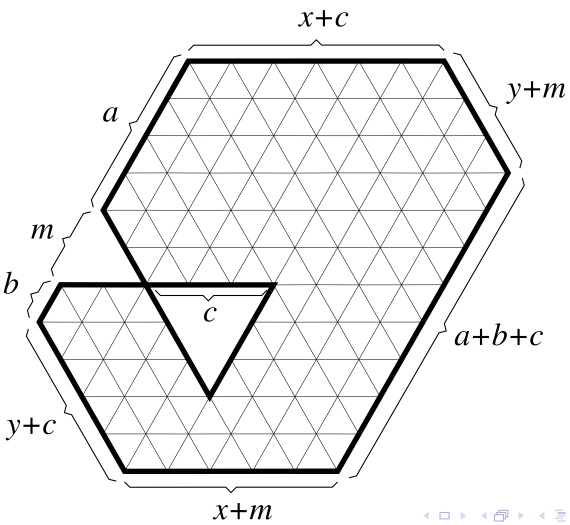
Proof of the main theorem: $z = 0$

The region $SC_{4,4,0}(3,1,2,2)$:



Proof of the main theorem: $z = 0$

¡ We need to know the number of rhombus tilings of regions of the form



Proof of the main theorem: $z = 0$

Theorem

For any nonnegative integers x, y, a, b, c and m, the number of rhombus tiling of this region equals

$$= \frac{\mathrm{H}(m)^2\,\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(m+a+b+c)}{\mathrm{H}(m+a)\,\mathrm{H}(m+b)\,\mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(x+m+a+c)\,\mathrm{H}(y+m+b+c)}{\mathrm{H}(x+y+m+c)} \frac{\mathrm{H}(x+y+c)}{\mathrm{H}(x+a+c)\,\mathrm{H}(y+b+c)}$$
$$\times \frac{\mathrm{H}(x+y+m+a+b+c)}{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)} \frac{\mathrm{H}(x)\,\mathrm{H}(y)}{\mathrm{H}(x+y)}.$$

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$$\times \frac{\mathrm{H}(x+y+m+a+b+c)}{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)} \frac{\mathrm{H}(x)\,\mathrm{H}(y)}{\mathrm{H}(x+y)}.$$

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$$\times \frac{\mathrm{H}(x+y+m+a+b+c)}{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}\frac{\mathrm{H}(x)\,\mathrm{H}(y)}{\mathrm{H}(x+y)}.$$

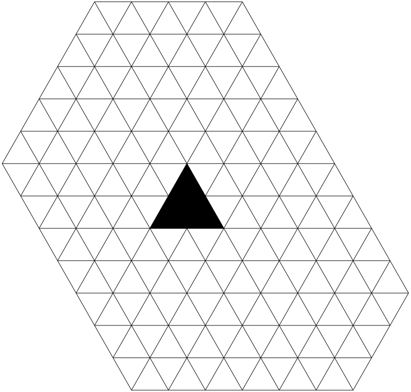
Proof.

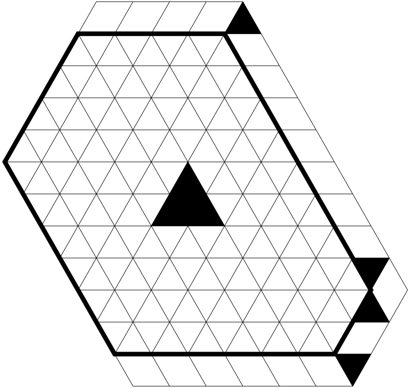
This can also be proved by means of Kuo's condensation. □

Proof of the main theorem

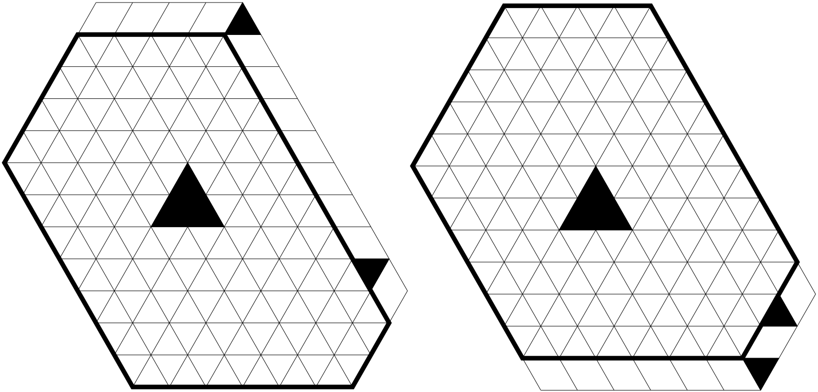
Induction step: We apply Kuo's condensation. For $\alpha, \beta, \gamma, \delta$ we choose triangles along in the corners of the outer boundary of the hexagon.

Proof of the main theorem





Proof of the main theorem



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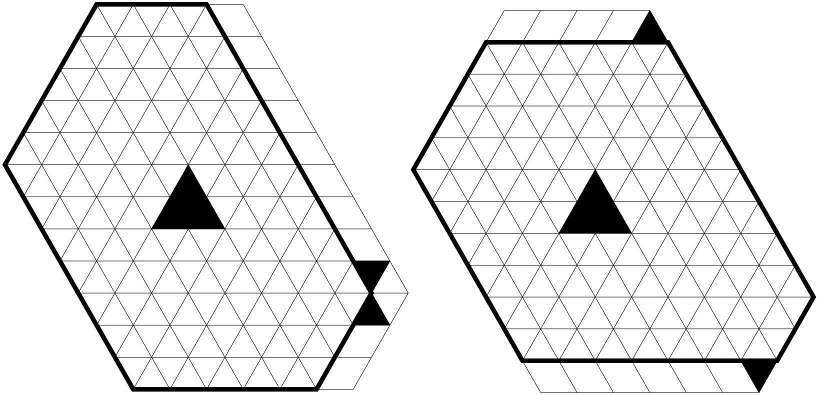

The implied recurrence:

$$
\begin{aligned}
&\mathrm{M}(SC_{x,y,z}(a,b,c,m))\mathrm{M}(SC_{x,y-1,z-1}(a,b,c,m)) \\
&= \mathrm{M}(SC_{y,x,z-1}(b,a,c,m))\mathrm{M}(SC_{z,y-1,x}(c,b,a,m)) \\
&\qquad + \mathrm{M}(SC_{x-1,y,z}(a,b,c,m))\mathrm{M}(SC_{x+1,y-1,z-1}(a,b,c,m)).
\end{aligned}
$$

Proof of the main theorem

The implied recurrence:

$$
\begin{aligned}
&\mathrm{M}(SC_{x,y,z}(a,b,c,m))\mathrm{M}(SC_{x,y-1,z-1}(a,b,c,m))\\
&= \mathrm{M}(SC_{y,x,z-1}(b,a,c,m))\mathrm{M}(SC_{z,y-1,x}(c,b,a,m))\\
&\qquad + \mathrm{M}(SC_{x-1,y,z}(a,b,c,m))\mathrm{M}(SC_{x+1,y-1,z-1}(a,b,c,m)).
\end{aligned}
$$

One needs to verify that the guessed formula satisfies the same recurrence.

One substitutes the guessed expression in the recurrence to be verified and forms the quotient of the two sides of the recurrence. After a lot of simplification, this quotient becomes

$$\frac{\Gamma(y+z+a+b+c+m)\,\Gamma(x+y+z+a+b+c+m-1)}{\Gamma(y+z+a+b+c+m-1)\,\Gamma(x+y+z+a+b+c+m)}$$
$$\times\frac{\Gamma\left(\left\lceil\frac{x+y+z-1}{2}\right\rceil+a+b+c+m\right)\,\Gamma\left(\left\lceil\frac{x+y+z}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}{\Gamma\left(\left\lceil\frac{x+y+z}{2}\right\rceil+a+b+c+m\right)\,\Gamma\left(\left\lceil\frac{x+y+z-1}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}$$
$$+\frac{\Gamma(x+a+b+c+m+1)\,\Gamma(x+y+z+a+b+c+m-1)}{\Gamma(x+a+b+c+m)\,\Gamma(x+y+z+a+b+c+m)}$$
$$\times\frac{\Gamma\left(\left\lceil\frac{x+1}{2}\right\rceil\right)\,\Gamma\left(\left\lceil\frac{x}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}{\Gamma\left(\left\lceil\frac{x}{2}\right\rceil\right)\,\Gamma\left(\left\lceil\frac{x+1}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}$$
$$\times\frac{\Gamma\left(\left\lceil\frac{x+y+z-1}{2}\right\rceil+a+b+c+m\right)\,\Gamma\left(\left\lceil\frac{x+y+z}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}{\Gamma\left(\left\lceil\frac{x+y+z}{2}\right\rceil+a+b+c+m\right)\,\Gamma\left(\left\lceil\frac{x+y+z-1}{2}\right\rceil+\frac{a+b+c+m}{2}\right)}.$$

Proof of the main theorem

If all of x, y, z are even, this condenses down to

$$\frac{(y+z+a+b+c+m-1)}{(x+y+z+a+b+c+m-1)} + \frac{(x+a+b+c+m)}{(x+y+z+a+b+c+m-1)} \cdot \frac{x/2}{(x+a+b+c+m)/2},$$

which indeed equals 1.

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which indeed equals 1.

The other cases are similar. $\square$

The dual MacMahon formula

For the dual of MacMahon's formula, we need to do asymptotics of hyperfactorials.

By the Glaisher–Kinkelin formula, which gives the asymptotics of the Barnes G-function, we have

$$\lim_{n\to\infty} \frac{0!\,1!\,\cdots\,(n-1)!}{n^{\frac{n^2}{2}-\frac{1}{12}}\,(2\pi)^{\frac{n}{2}}\,e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A},$$

where $A = 1.28242712...$ is the so-called the Glaisher–Kinkelin constant.

The dual MacMahon formula

$$\lim_{n\to\infty} \frac{\mathrm{H}(n)}{n^{\frac{n^2}{2}-\frac{1}{12}}\,(2\pi)^{\frac{n}{2}}\,e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A},$$

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$$\lim_{n\to\infty} \frac{\mathrm{H}(n)}{n^{\frac{n^2}{2}-\frac{1}{12}}\,(2\pi)^{\frac{n}{2}}\,e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A},$$

where $A = 1.28242712\ldots$ is the so-called the Glaisher–Kinkelin constant.

This leads (more or less) straightforwardly to the dual MacMahon formula:

Theorem

For any non-negative integers a, b, c and m we have

$$\begin{aligned}\frac{\mathrm{M}(S^*(a,b,c,m))}{\mathrm{M}(S^*(a+b+c,0,0,m))}\\
&= \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c+m)\,\mathrm{H}(m)^2}{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c)}\\
&= P(a,b,m)\,P(a+b,c,m).\end{aligned}$$

A dual of MacMahon's formula