Exercises for Algebraic Topology

SS 2006

Andreas Kriegl

1.1.A1.
Prove the following statements:

(a) Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. Then $\overline{A} \times \dot{B} \cup \dot{A} \times \overline{B}$ is the boundary of $A \times B$ in $X \times Y$.

(b) Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be convex. Then $A \times B \subseteq \mathbb{R}^{n+m}$ is convex.

1.1.A2.
The convex hull $\langle A \rangle_{\mathrm{cv}}$ of $A \subseteq \mathbb{R}^n$ is defined to be the smallest convex subset of $\mathbb{R}^n$ which contains A. This is the intersection of all convex subsets of $\mathbb{R}^n$ containing A. Show that

$$A = \Big\{ \sum_{i=0}^{q} \lambda_i \, x_i : q \in \mathbb{N}, \lambda_i \geq 0, x_i \in A, \sum_{i=0}^{q} \lambda_i = 1 \Big\}.$$

1.1.A3.
For $R > r > 0$ let X be the subset of $\mathbb{R}^3$ obtained by rotating a circle in the x-z-plane with center $(R, 0, 0)$ and radius r around the z-axes. Prove that

(a) X is given by the equation $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$.

(b) $(x, y) = (x_1, x_2; y_1, y_2) \mapsto ((R + r y_1)x, r y_2)$ is an embedding of $S^1 \times S^1$ onto X.

(c) The filled torus $V \subseteq \mathbb{R}^3$ is the union $\{(x, y, z) : (\sqrt{x^2 + y^2} - R)^2 + z^2 \leq r^2\}$ of X and its "interior". Show that the formula in (b) gives a homeomorphism $S^1 \times D^2 \cong V$.

1.1.A4.
Show that for any $x, y \in \mathring{D}^n$ there is a homeomorphism of pairs $(D^n, \{x\}) \cong (D^n, \{y\})$.

1.3.A1.
Show that the mapping $(i_1, \ldots, i_n) : X_1 \vee \cdots \vee X_n \to X_1 \times \ldots \times X_n$ defined in (1.41) is an embedding.

1.3.A2.
Show: $(S^1 \times S^1)/(S^1 \vee S^1) \cong S^2$.

1.3.A3.
Show that $\mathbb{R}^n / D^n \cong \mathbb{R}^n$ and that $\mathbb{R}^n / \mathring{D}^n$ is not Hausdorff.

1.3.A4.
Show that any continuous $f : X \to Y$ induces a continuous mapping $C(f) : C(X) \to C(Y)$ between the cones, via $f \times I : X \times I \to Y \times I$.

1.3.A5.
The suspension (dt. Einhängung) of a topological space X is $E(X) := C(X)/X$, where X is embedded

into $C(X)$ via $x \mapsto (x, 1)$. Show that $f : X \to Y$ induces a mapping $E(f) : E(X) \to E(Y)$. Show furthermore, that $E(D^n) \cong D^{n+1}$ and $E(S^n) \cong S^{n+1}$.

1.5.A4.
Show that the lens space $L(\frac{1}{2})$ is homeomorphic to $\mathbb{P}^3_{\mathbb{R}}$.

1.6.A2.
Describe a mapping $f : S^2 \to S^2 \vee S^1$ such that $(S^2 \vee S^1) \cup_f D^3 \cong S^2 \times S^1$. **Hint:** (1.12).

1.6.A3.
Consider the subspace $X := S^1 \cup D^1 \subseteq \mathbb{C}$ and a mapping $f : S^1 \to X$ which runs through the top half circle, the diameter D^1, the bottom half circle, and again the diameter. Show that $X \cup_f D^2$ is homeomorphic to the Möbius strip. **Hint:** Use (1.94).

1.7.A3.
Let $\mathbb{Z}$ act on $\mathbb{R}^2$ by $n : (x_1, x_2) \mapsto (x_1 + n, (-1)^n x_2)$. Show that $\mathbb{R}^2/\mathbb{Z}$ is homeomorphic to the open Möbius strip (i.e. the Möbius strip from (1.59) without ist boundary S^1).

1.7.A5.
Let G be the subgroup of homeomorphisms on $\mathbb{R}^2$ generated by $(x_1, x_2) \mapsto (x_1 + 1, x_2)$ and $(x_1, x_2) \mapsto (-x_1, x_2 + 1)$. Show that $\mathbb{R}^2/G$ is homeomorphic to Kleins bottle.

1.7.A6.
Let T be the torus into $\mathbb{R}^3$ as in (1.18). Consider the action of the group $S^0 = \{\pm 1\}$ on T given by

(1) $(x, y, z) \overset{-1}{\longmapsto} (-x, -y, z)$ and show that $T/S^0 \cong S^1 \times S^1$.

(2) $(x, y, z) \overset{-1}{\longmapsto} (x, -y, -z)$ and show that $T/S^0 \cong S^2$.

(3) $(x, y, z) \overset{-1}{\longmapsto} (-x, -y, -z)$ and show that T/S^0 is homeomorphic to Kleins bottle.

2.1.A2.
Show that $X \times Y$ is contractible provided X and Y are contractible.

2.1.A5.
Two homeomorphisms $f_0, f_1 : X \to Y$ are called isotopic, iff there exists a homotopy $t \mapsto f_t$ consisting of homeomorphism $f_t : X \to Y$ only. Let $f : D^n \to D^n$ be a homeomorphism with $f|_{S^{n-1}} = \mathrm{id}$ and $f(0) = 0$. Show that id_{D_n} is isotopic f to via $f_t : x \mapsto t\, \tilde{f}(x/t)$, where $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ is an appropriate extension of f.

2.1.A7.
Show that X is contractible if and only if $\Delta : X \to X \times X$, $x \mapsto (x, x)$ is 0-homotopic.

2.2.A1.
Show that the pointwise multiplication defines an Abelian group structure on $[X, S^1]$ and, furthermore, that $\deg : [S^1, S^1] \to (\mathbb{Z}, +)$ is a group-homomorphism with respect to this group structure for $X := S^1$.

2.2.A2.
Let $f : D^2 \to \mathbb{R}^2$ be a continuous function with $f|_{S^1}$ odd. Show that there exists an $z \in D^2$ with $f(z) = 0$. Deduce the existence of a solution $(x, y) \in \mathbb{R}^2$ for

$$x \cos(y) = x^2 + y^2 - 1 \text{ and } y \cos(x) = \sin(2\pi(x^2 + y^2))$$

2.2.A4.
Show that S^∞ is contractible.
Hint: Let $p : \mathbb{R}^\infty \setminus \{0\} \to S^\infty$ given by $x \mapsto \frac{x}{\|x\|_2}$, where $\|x\|_2 := \sqrt{\sum_k x_k^2}$. Show that $h_t : (x_0, x_1, x_2, \dots) \mapsto p((1-t)x_0, tx_0 + (1-t)x_1, tx_1 + (1-t)x_2, tx_2 + (1-t)x_3, \dots)$ defines a homotopy between id_{S^∞} and the right shift $S^\infty \to \{x \in S^\infty : x_0 = 0\}$. Now consider the homotopy $(0, x_1, x_2, \dots) \mapsto p(t, (1-t)x_1, (1-t)x_2, \dots)$.

2.4.A3.
Let $p, q \in S^1 \times S^1$ be different points. Show that $S^1 \times S^1 \setminus \{p, q\} \sim S^1 \vee S^1 \vee S^1$.

2.4.A4.
Show that $\mathbb{R}^3 \setminus S^1 \sim S^1 \vee S^2$, where S^1 is the unit-circle in $\mathbb{R}^2 \times \{0\}$.

2.4.A5.
Show that $S^3 \setminus S^1 \sim S^1$, where S^1 is the unit-circle in $\mathbb{R}^2 \times \{(0,0)\}$.

2.4.A9.
Show that the mapping cylinder of $z \mapsto z^2$, $S^1 \to S^1$ is homeomorphic to the Möbius strip.

2.4.A10.
Show that for $f : S^{n-1} \to Y$ one has $M_f / S^{n-1} \sim Y \cup_f D^n$.

2.4.A13.
Show that $O(n) \subseteq GL(n)$ is an SDR. **Hint:** Apply Gram-Schmidt orthonormalization to the columns of $A \in GL(n)$ to obtain $r(A) \in O(n)$. This procedure is given by multiplication with an upper triangular matrix with positive diagonal entries depending smoothly on A. Now deform the matrix to the identity matrix.

3.1.A8.
Let K be a simplicial complex in $\mathbb{R}^n$ and $p \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$. The cone $C(K, p)$ is the set consisting of $\{p\}$, all simplices of K, and all simplicies $\langle p, x_0, \dots, x_i \rangle$ for $\langle x_0, \dots, x_i \rangle \in K$. The suspension is $E(K) := C(K, p) \cup C(K, -p)$. Show that $C(K, p)$ and $E(K)$ are simplicial complexes with $|C(K, p)| \cong C(|K|)$ and $|E(K)| = E(|K|)$.

3.1.A9.
The cartesian product of two polyeder is a polyeder. **Hint:** Show that the product of two closed simplices $\bar\sigma$ and $\bar\tau$ can be triangulated using $C((\sigma \times \tau)^\cdot) = \bar\sigma \times \bar\tau$.

3.1.A13.
Let K be a simplicial complex and α_i the number of i-simplices of K. The number $\chi(K) := \sum_{i \geq 0} (-1)^i \alpha_i$ is called Euler-characteristic of K. Show that

- For any triangulation K of S^1 we have $\chi(K) = 0$.
- $\chi(C(K, p)) = 1$ for the cone $C(K, p)$ given in exercise (3.1.A8).
- $\chi(E(K)) = 2 - \chi(K)$ for the suspension $E(K)$ given in exercise (3.1.A8).
- $\chi(\dot\sigma) = 1 + (-1)^n$ where $\dot\sigma := \{\tau : \tau < \sigma\}$ for any $n+1$-simplex σ.

3.2.A2.
Let $x_0, \dots, x_q$ be vertices of K. Show that $\mathrm{st}_K(x_0) \cap \dots \cap \mathrm{st}_K(x_q) \neq \emptyset \Leftrightarrow \langle x_0, \dots, x_q \rangle \in K$.

3.3.A1.
Show that $S^1 \not\sim S^n$ for $n > 1$ and deduce $\mathbb{R}^2 \not\cong \mathbb{R}^{n+1}$. **Hint:** (3.33).

4.1.A1.
Describe CW-decompositions with as few cells as possible for D^n, $S^1 \times I$, the closed Möbiusstrip, and the disk D^2_g with g holes as in (1.4.13).

4.1.A5.
Show that the lens space $L(\frac{q}{p})$ is a 3-dimensional CW-complex with exactly one cell in each dimension. **Hint:** Consider the CW-decomposition of D^3 given by the p-th unit-roots on $S^1 \subseteq S^2 \subseteq D^3$, the segments on S^1 between them, the two hemispheres of S^2 and the interior of D^3. Now take the images under the quotient mapping $D^3 \to L(\frac{q}{p})$.

4.2.A1.
Show that $S^n \times S^m / S^n \vee S^m$ is a CW-space which is homeomorphic to S^{n+m}.

4.2.A2.
Show that the cone and the suspension of a CW-space is also a CW-space.

4.2.A3.
Show that the mapping cylinder of a cellular mapping between CW-spaces is a CW-space.

4.3.A1.
Show that a CW-space X is path-connected if and only if X^1 is path-connected. **Hint:** (4.3.4)

4.3.A2.
Let X be a CW-space and $x_0 \in X^0$. Let Y be a connected CW-space without 1-cells and hence with only one 0-cell y_0. Then any two homotopic mappings $f_j : X \to Y$ which preserve the basepoints are homotopic relative $\{x_0\}$. **Hint:** (4.3.4).

4.3.A4.
Let X be a CW-space with $\dim(X) < n$. Show that $[X, S^n] = \{0\}$. **Hint:** (4.3.4) and cellular mappings $X \to S^n = e^0 \cup e^n$ are constant.

5.4.A4.
Determine the fundamental group of $S^1 \times \mathbb{P}^2$, $\mathbb{P}^2 \vee \mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^2$, $S^1 \times S^m$ for $m \geq 2$, and of $\mathbb{R}^3 \setminus S^1$.

The following exercises (5.3.7A)–(5.7.A4e) show, that the isomorphy problem is algorithmically unsolvable for m-manifolds with $m \geq 4$. For this it is enough to show that evevry finitely presented group appears as fundamental group of such a manifold.

5.3.A7.
Let M be a connected manifold of dimension $m \geq 3$. Show that $\pi_1(M \setminus D^0_1) \cong \pi_1(M)$ for $M \setminus D^0_1$ as in (1.5.5).

5.3.A8.
Let M and N be connected manifolds of dimension $m \geq 3$. Then for the connected sum we have $\pi_1(M \sharp N) \cong \pi_1(M) \coprod \pi_1(N)$.

5.7.A4a.
Show that for $m \geq 4$ the fundamental group of the connected sum M of k copies of $S^1 \times S^{m-1}$ is the free group $\langle \{s_1, \dots, s_k\} : \emptyset \rangle$ with k generators.

5.7.A4c.
Let $f : S^1 \times D^{m-1} \to M$ an embedding into an m-manifold M. Show that $\pi_1(M) \cong \pi_1(M \setminus f(S^1 \times D^{m-1}))$.

5.7.A4d.
Let f be as in (5.7.A4c) with M as in (5.7.A4a). Show that $\pi(M \cup_f (D^2 \times S^{m-2}) \cong \langle\{s_1,\dots,s_k\} : \{f|_{S^1\times\{0\}}\}\rangle$.

5.7.A4e.
Let $G = \langle\{s_1,\dots,s_k\} : \{r_1,\dots,r_l\}\rangle$ be a finitely represented group. Now construct a compact connected manifold without boundary recursively by starting with M from (5.7.A4a) and cutting for every $r_i \in \pi_1(M)$ a neighborhood homeomorphic to $S^1 \times D^{m-1}$ of a appropriately choosen representant of r_i and pasting a cylinder $D^2 \times S^{m-2}$ as in (5.7.A4d).

6.1.A1.
Show that for $a, b, c, d \in \mathbb{Z}$ with $m := ad-bc \neq 0$ the mapping $S^1 \times S^1 \to S^1 \times S^1$, $(z, w) \mapsto (z^a w^b, z^c w^d)$ is an m-fold covering.

6.1.A5.
Consider a torus $T \subseteq \mathbb{R}^3$ with the z-axes as rotation axis. Now glue $g \geq 2$ many handles to T such that the resulting surface F_{g+1} is symetric with respect to rotation R around the z-axes by the angle $2\pi/g$. Let G be the cyclic group generated by R. Show that $F_{g+1}/G \cong F_2$ and hence $F_{g+1} \to F_2$ is a covering.

6.3.A3.
Consider the covering $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi it}$. Let $Y := S^1 \vee S^1 \subseteq S^1 \times S^1$ and $X := (p \times p)^{-1}(Y) = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ oder } y \in \mathbb{Z}\}$. Show that:

1. $(p \times p)|_Y : X \to Y$ is an infinite covering.
2. $\pi_1(X)$ is a free group with infinite many generators (Hint: (5.5.14))
3. Show that the image of $\pi_1(X)$ in $\pi_1(Y)$ is the commutator subgroup of $\pi_1(Y) = \mathbb{Z} \coprod \mathbb{Z}$.
4. Note that this subgroup of the free group with 2 generators is a free group with infinite many generators.

7.2.10A.
Determine the homology of the Möbius strip M as in (7.2.10). Use this to calculate the relative homology $H(M, \partial M)$, see (7.4.7).

7.1.A6.
For a simplicial complex K let β_q be the Betti-number of $G := H_q(K)$, i.e. the rank of the free part $G/\operatorname{Tor}(G)$, where $\operatorname{Tor}(G) := \{g \in G : \exists n > 0 : n \cdot g = 0\}$ denotes the torsion subgroup. Show that the Euler-charakteristik from example (3.1.A13) is

$$\chi(K) = \sum_i (-1)^q \beta_q$$

Hint: Let α_q, ρ_q and γ_q denote the rank of the free abelian groups $C_q(K)$, $B_q(K)$ and $Z_q(K)$ then $\alpha_q = \gamma_q + \rho_{q-1}$ and $\gamma_q = \beta_q + \rho_q$. Use the formula $\operatorname{rank}(A/B) + \operatorname{rank}(B) = \operatorname{rank}(A)$ from the proof of (8.2.1a).

9.2.A4.
Let $A \subseteq X$ be path-connected. Show that $H_1(X) \to H_1(X, A) \to 0$ is exakt and give a geometric interpretation of this result.