Nonlinear Functional Analysis I, SS 1993

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This manuscript is the outcome of a lecture course given at the University of Vienna in the Summer Semester 1993. So I want to use this opportunity to thank all those who attended these lectures and inspired me with the feedback I got from them. In particular I want to thank Cornelia Vizman who posed well selected and highly relevant questions after reading parts of my manuscript. My special thanks go to Konni Rietsch, who not only strongly influenced the selection of the covered topics but also sacrificed a huge amount of time during her holidays and lots of energy in order to make sense out of a preliminary version of these lecture notes. This way she supplied me with an extensive list of misprints, Germanisms, and imprecise or even incorrect mathematical formulations. All the remaining (and newly inserted) faux pas are of course all my own responsibility. And, as always, I explicitly ask the readers not only to pardon them but also to inform me about anything which sounds weird including possibly missing definitions and explanations.

Thank you all in advance,

Andreas Kriegl, August 1993

In the second edition an extensive list of misprints and corrections provided by Eva Adam has been taken gratefully into account.

Andreas Kriegl, September 1994

After some minor corrections I ported to source to LaTeX. Note that chapter 1 and 2 are outdated, since they have been incorperated into the book [**6**].

Andreas Kriegl, Jänner 2006

Essentially two topics from non-linear functional analysis will be treated. Firstly calculus will be extended from finite dimensions (or from Banach spaces, depending on the readers background) to general locally convex spaces. Secondly tensor-products will be discussed and their relationship to multi-linear mappings and to function spaces will be investigated. Of course these two topics are closely related to one another. Just note that the derivatives of smooth maps are multi-linear, and the spaces of smooth functions can be analyzed using various tensor products.

3. Tensor Products and Linearization

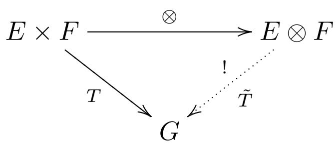
Algebraic Tensor Product

Remark.
The importance of the tensor product is twofold. First it allows linearizing of multi-linear mappings and secondly it allows to calculate function spaces.

We will consider the spaces of linear and multi-linear mappings between vector spaces. If we supply all vector spaces E, $E_1, \dots, E_n$, F with the finest locally convex topology (i.e. the final locally convex topology with respect to the inclusions of all finite dimensional subspaces - on which the topology is unique) then all linear mappings are continuous and all multi-linear mappings are bounded (but not necessarily continuous as the evaluation map $\mathrm{ev} : E^* \times E \to \mathbb{K}$ on an infinite dimensional vector space E shows) and hence it is consistent to denote the corresponding function spaces by $L(E, F) = \mathcal{L}(E, F)$ and $L(E_1, \dots E_n; F)$.

In more detail the first feature is:

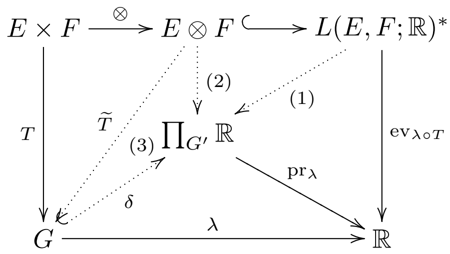
3.1 Proposition. *Linearization.*
Given two linear spaces E and F, then there exists a solution $\otimes : E \times F \to E \otimes F$ – called the algebraic tensor product of E and F – to the following universal problem:



Here $\otimes : E \times F \to E \otimes F$ and $T : E \times F \to G$ are bilinear and $\tilde{T}$ is linear.

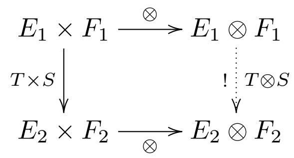
Proof. In order to find $E \otimes F$ one considers first the case, where $G = \mathbb{R}$. Then we have that $\otimes^* : (E \otimes F)^* \to L(E, F; \mathbb{R})$ should be an isomorphism. Hence $E \otimes F$ could be realized as subspace of $(E \otimes F)^{**} \cong L(E, F; \mathbb{R})^*$. Obviously to each bilinear functional $T : E \times F \to \mathbb{R}$ corresponds the linear map $\mathrm{ev}_T : L(E, F; \mathbb{R})^* \to \mathbb{R}$. The map $\otimes : E \times F \to E \otimes F \subseteq L(E, F; \mathbb{R})^*$ has to be such that $\mathrm{ev}_T \circ \otimes = T$ for all bilinear functionals $T : E \times F \to \mathbb{R}$, i.e. $\otimes(x, y)(T) = (\mathrm{ev}_T \circ \otimes)(x, y) = T(x, y)$. Thus we have proved the existence of $\tilde{T} := \mathrm{ev}_T$ for $G = \mathbb{R}$. But uniqueness can be true only on the linear subspace generated by the image of $\otimes$, and hence we denote this subspace $E \otimes F$.

For bilinear mappings $T : E \times F \to G$ into an arbitrary vector space G, we consider the following diagram, which has quite some similarities with that used in the construction of the c^∞-completion in 2.31:



The right dashed arrow (1) and δ exist uniquely by the universal property of the product in the center. The arrow (2) exists uniquely as restriction of (1) to the subspace $E \otimes F$. Finally (3) exists, since the generating subset $\otimes(E \times F)$ in $E \otimes F$ is mapped to $T(E \times F) \subseteq G$ and since δ is injective. □

Note that $\otimes$ extends to a functor, by defining $T \otimes S$ via the following diagram:



Furthermore one easily proves the existence of the following natural isomorphisms:

$$E \otimes \mathbb{R} \cong E$$
$$E \otimes F \cong F \otimes E$$
$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$$

In analogy to the exponential law for smooth mappings or continuous mappings, we show now the existence of a natural isomorphism

$$L(E, F; G) \cong L(E, L(F, G))$$

again denoted by $(\_)^\vee$ with inverse isomorphism $(\_)^\wedge$ given by the same formula as above.
In fact for a bilinear mapping $T : E \times F \to G$, the mapping $T^\vee$ has values in $L(F, G)$, since $T(x, \_)$ is linear, and it is linear, since $L(F, G)$ carries the initial vector space structure with respect to the evaluations $\mathrm{ev_y}$ and $\mathrm{ev_y} \circ T^\vee = T(\_, y)$ is also linear. The same way one shows that the converse implication is also true.

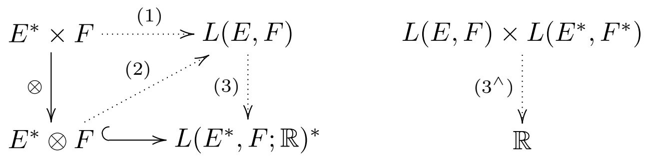
Note that if both spaces E and F are finite dimensional, then so is $L(E, F; \mathbb{R})$ and hence also the dual $L(E, F; \mathbb{R})^*$. But then $E \otimes F$ is finite dimensional too (in fact $\dim(E \otimes F) = \dim E \cdot \dim F$), as we will see in 3.30, and hence $E \otimes F = (E \otimes F)^{**} = L(E, F; \mathbb{R})^*$.

If one factor is infinite dimensional and the other one is not 0, then this is not true. In fact take $F = \mathbb{R}$, then $E \otimes \mathbb{R} \cong E$ whereas $L(E, \mathbb{R}; \mathbb{R})^* \cong L(E, L(\mathbb{R}, \mathbb{R}))^* \cong L(E, \mathbb{R})^* = E^{**}$.

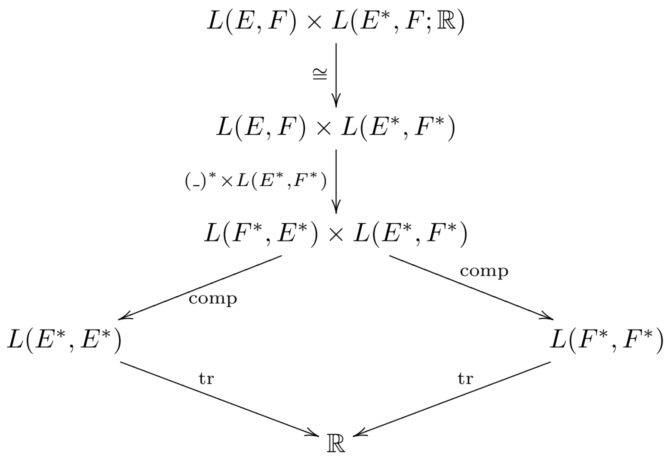
3.2 Vector-valued functions versus scalar valued ones

The second important usage of the tensor product lies in the possibility to express spaces of vector valued functions as tensor products of spaces of scalar valued functions times the space of values. In more detail this means, that given some type of function $f : X \to \mathbb{R}$ and a vector $y \in F$, then we can form the function $X \to F$ given by $x \mapsto f(x) \cdot y$. If we denote the space of functions $X \to F$ of some specific type by F^X then this means that we have a bilinear mapping $\mathbb{R}^X \times F \to F^X$. The question that arises is, whether it has the universal property of the tensor product, i.e. whether the natural mapping $\mathbb{R}^X \otimes F \to F^X$ is an isomorphism.

Let us consider the case where X itself is a linear space E and the considered functions are the linear ones. Then our claim is that $E^* \otimes F \cong L(E, F)$. For this we consider the following diagram:



The first dashed arrow is given by bilinear mapping discussed before, namely $(x^*, y) \mapsto (x \mapsto x^*(x)\, y)$. The second one exists by the universal property of the tensor product. And since the image of the first one generates $L(E, F)$ provided E or F is finite dimensional, we conclude that the second one is surjective. Remains to show that the third one exists and is a left-inverse. By the exponential law this mapping would correspond to a bilinear mapping $(3^\wedge) : L(E, F) \times L(E^*, F; \mathbb{R}) \to \mathbb{R}$, which we try to piece together as follows:



Clearly the transposition mapping $L(E, F) \to L(F^*, E^*)$ is linear, and if we apply the composition map from $L(F^*, E^*) \times L(E^*, F^*)$ to $L(E^*, E^*)$ or to $L(F^*, F^*)$ it remains to find for a vector space G a linear map $L(G, G) \to \mathbb{R}$. If G is finite dimensional such a map is given by the *trace*, i.e. the sum over the diagonal entries of a matrix-representation, or equivalently the derivative of the determinate at the identity, or equivalently the coefficient of $(-\lambda)^{\dim G - 1}$ in the *characteristic polynomial* $\det(T - \lambda)$. In order to show that the composite $L(E, F) \times L(E^*, F; \mathbb{R}) \to L(F^*, E^*) \times L(E^*, F^*) \to L(E^*, E^*) \to \mathbb{R}$ gives a left inverse, it is enough by the universal property of the tensor product to test on $x^* \otimes y$. This is mapped to $x \mapsto x^*(x) \cdot y =: S$ and furthermore to $T \mapsto \operatorname{trace}(S^* \circ T^\vee)$. So let us calculate $(S^* \circ T^\vee)(u^*)(u) = T(u^*, Su) = T(u^*, x^*(u) \cdot y) = T(u^*, y) \cdot x^*(u)$. Note that $x^* \in G := E^*$ and $T(\_, y) \in G^* := E^{**}$, and for $g \in G$ and $g^* \in G^*$ we have that the trace of $g^*(\_) \cdot g$ is $\operatorname{trace}(g^*(\_) \cdot g) = g^*(g)$. To show this, extend g to a basis and then the trace is the entry in the upper left corner, which is $g^*(g)$. So $\operatorname{trace}(S^* \circ T^\vee) = T(x^*, y)$, which was to be shown.

In particular we have shown, that $G^* \otimes G \cong L(G, G)$ for finite dimensional G. And the trace of $g^* \otimes g \in L(G, G)$ is just $g^*(g)$ and hence corresponds to the bilinear map $\operatorname{ev} : G^* \times G \to \mathbb{R}$ or the corresponding linear map $G^* \otimes G \to \mathbb{R}$.

If both factors are infinite dimensional this will no longer be true, even if we restrict to continuous mappings. However if we take some appropriate completion, there might be some chance.

Let us deduce some additional handy formulas for duals, in the case where at least one of E and F is finite dimensional:

$$(E \otimes F)^* \cong L(E, F; \mathbb{R}) \cong L(E, F^*) \cong E^* \otimes F^* \quad \text{and}$$
$$L(E, F)^* \cong (E^* \otimes F)^* \cong E^{**} \otimes F^* \cong L(F, E^{**}) \cong L(F, E^*; \mathbb{R}) \cong L(E^*, F^*).$$

Projective Tensor Product

We turn first to the property of making bilinear continuous mappings into linear ones. We call the corresponding solution the *projective tensor product* of E and F and denote it by $E \otimes_\pi F$. Obviously it can be obtained by taking the algebraic tensor product and supplying it with the finest locally convex topology such that $E \times F \to E \otimes F$ is continuous. This topology exists since the union of locally convex topologies is locally convex and $E \times F \to E \otimes F$ is continuous for the weak topology on $E \otimes F$ generated by those linear functionals which correspond to continuous bilinear functionals on $E \times F$. It has the universal property, since the inverse image of a locally convex topology under a linear mapping $\tilde{T}$ is again a locally convex topology, such that $\otimes$ is continuous, provided the associated bilinear mapping T is continuous. However, it is not obvious that this topology is separated, and we prove that now. We will denote the *space of continuous linear mappings* from E to F by $\mathcal{L}(E, F)$, and the *space of continuous multi-linear mappings* by $\mathcal{L}(E_1, \dots, E_n; F)$. If all $E_1, \dots, E_n$ are the same space E, we will also write $\mathcal{L}^n(E; F)$.

3.3 Lemma.
$E \otimes_\pi F$ is Hausdorff provided E and F are.

Proof. It is enough to show that the set $E^* \times F^*$ separates points in $E \otimes F$ or even in $L(E, F; \mathbb{R})^*$. So let $0 \neq z = \sum_k x_k \otimes y_k$ be given. By replacing linear dependent x_k by the corresponding linear combinations and using bilinearity of $\otimes$, we may assume that the x_k are linearly independent. Now choose $x^* \in E^*$ and $y^* \in F^*$ be such that $x^*(x_k) = \delta_{1,k}$ and $y^*(y_1) = 1$. Then $(x^* \otimes y^*)(z) = 1 \neq 0$. □

Since a bilinear mapping is continuous iff it is so at 0, a 0-neighborhood basis in $E \otimes_\pi F$ is given by all those absolutely convex sets, for which the inverse image under $\otimes$ is a 0-neighborhood in $E \times F$. A basis is thus given by the absolutely convex hulls denoted $U \otimes V$ of the images of $U \times V$ under $\otimes$, where U resp. V runs through a 0-neighborhood basis of E resp. F. We only have to show that these sets $U \otimes V$ are absorbing. So let $z = \sum_k x_k \otimes y_k \in E \otimes F$ be arbitrary. Then there are $a_k > 0$ and $b_k > 0$ such that $x_k \in a_k U$ and $y_k \in b_k V$ and hence $z = \sum_{k \leq K} a_k\, b_k \frac{x_k}{a_k} \otimes \frac{y_k}{b_k} \in (\sum_k a_k\, b_k) \cdot \langle U \otimes V \rangle_{\text{abs.conv.}}$. The Minkowski-functionals $p_{U \otimes V}$ form a base of the seminorms of $E \otimes_\pi F$ and we will denote them by $\pi_{U,V}$. In terms of the Minkowski-functionals p_U and p_V of U and V we obtain that $z \in (\sum_k p_U(x_k)\, p_V(y_k))\, U \otimes V$ for any $z = \sum_k x_k \otimes y_k$ since $x_k \in p_U(x_k) \cdot U$ for closed U, and thus $p_{U \otimes V}(z) \leq \inf\{\sum_k p_U(x_k)\, p_V(y_k) : z = \sum_k x_k \otimes y_k\}$. We now show the converse:

3.4 Proposition. Seminorms of the projective tensor product.

$$p_{U \otimes V}(z) = \inf \left\{ \sum_k p_U(x_k) \cdot p_V(y_k) : z = \sum_k x_k \otimes y_k \right\}.$$

Proof. Let $z \in \lambda \cdot U \otimes V$ with $\lambda > 0$. Then $z = \lambda \sum \lambda_k u_k \otimes v_k$ with $u_k \in U$, $v_k \in V$ and $\sum_k |\lambda_k| = 1$. Hence $z = \sum x_k \otimes v_k$, where $x_k = \lambda\lambda_k u_k$, and $\sum_k p_U(x_k) \cdot p_V(v_k) \leq \sum \lambda|\lambda_k| = \lambda$. Taking the infimum of all λ gives now that $p_{U\otimes V}(z)$ is greater or equal to the infimum on the right side. □

3.5 Corollary.
$E \otimes_\pi F$ is normable (metrizable) provided E and F are.

3.6 Lemma. The semi-norms of decomposable tensors.

$$p_{U,V}(x \otimes y) = p_U(x) \cdot p_V(y).$$

Proof. According to [**1**, 7.1.8] there are $x^* \in E^*$ and $y^* \in F^*$ such that $|x^*| \leq p_U$ and $|y^*| \leq p_V$ and $x^*(x) = p_U(x)$ and $y^*(y) = p_V(y)$. If $x \otimes y = \sum_k x_k \otimes y_k$, then

$$\begin{aligned} p_{U\otimes V}(x \otimes y) &\leq p_U(x) \cdot p_V(y) = x^*(x) \cdot y^*(y) = (x^* \otimes y^*)(x \otimes y) = \\ &= \sum_k x^*(x_k) \cdot y^*(y_k) \leq \sum_k p_U(x_k) \cdot p_V(y_k), \end{aligned}$$

and taking the infimum gives the desired result. □

3.7 Remark. Functorality.
Given two continuous linear maps $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ we can consider bilinear continuous map given by composing $T_1 \times T_2 : E_1 \times E_2 \to F_1 \times F_2$ with $\otimes : F_1 \times F_2 \to F_1 \otimes F_2$. By the universal property of $E_1 \times E_2 \to E_1 \otimes E_2$ we obtain a continuous linear map denoted by $T_1 \otimes T_2 : E_1 \otimes E_2 \to F_1 \otimes F_2$.

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{\otimes} & E_1 \otimes E_2 \\ {\scriptstyle T_1 \times T_2}\downarrow & & \downarrow{\scriptstyle T_1 \otimes T_2} \\ F_1 \times F_2 & \xrightarrow[\otimes]{} & F_1 \otimes F_2 \end{array}$$

By the uniqueness of the linearization one obtains immediately that $\otimes$ is a functor. Because of the uniqueness of universal solutions one sees easily that one has natural isomorphisms $\mathbb{R} \otimes E \cong E$, $E \otimes F \cong F \otimes E$ and $(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$.

3.10 Adjointness of the tensor functor.
In analogy to the algebraic tensor product we would expect that also for locally convex spaces $(\_) \otimes_\pi E$ is left-adjoint to the Hom-functor $\mathcal{L}(E, \_)$ supplied with some topology. Since $\mathcal{L}(E \otimes_\pi F, G) \cong \mathcal{L}(E, F; G)$ we would need a bijection $\mathcal{L}(E, F; G) \cong \mathcal{L}(E, \mathcal{L}(F, G))$. Obviously we have the linear injection $(\_)^\vee : \mathcal{L}(E, F; G) \to \mathcal{L}(E, \mathcal{L}(F, G))$ induced from the corresponding bijection of vector-spaces, since a jointly continuous map is separately continuous, and hence $T^\vee(x) = T(x, \_)$ is continuous. And if we supply $\mathcal{L}(F, G)$ with the topology of uniform convergence on bounded sets, then $T^\vee$ is continuous, since $(T^\vee)^{-1}(N_{B,W}) = \{x : T(x, B) \subseteq W\}$ contains the 0-neighborhood $\frac{1}{\lambda} U$, where U (and V) are chosen, such that $T(U \times V) \subseteq W$ and $\lambda > 0$ such that $B \subseteq \lambda V$.

Proposition.
If we supply also $\mathcal{L}(E, \mathcal{L}(F, G))$ and $\mathcal{L}(E, F; G)$ with the topology of uniform convergence on bounded sets then the mapping

$$(\_)^\vee : \mathcal{L}(E, F; G) \hookrightarrow \mathcal{L}(E, \mathcal{L}(F, G))$$

is a topological linear embedding.

In fact, a typical neighborhood of $\mathcal{L}(E,F;G)$ is $N_{B_1\times B_2,W}$ and one of $\mathcal{L}(E,\mathcal{L}(F,G))$ is $N_{B_1,N_{B_2,W}}$ and $N_{B_1\times B_2,W} = ((\_)^\vee)^{-1} N_{B_1,N_{B_2,W}}$, so it remains to show that $(\_)^\wedge$ is well defined. Recall that $f^\wedge$ is given by $\operatorname{ev}\circ(f\times F)$, where $\operatorname{ev}:\mathcal{L}(F,G)\times F\to G$. However this mappings is continuous only if F is normed. So only for normed F we have that $(\_)\otimes_\pi F$ is left-adjoint. If F is not normed, then in particular $\operatorname{id}\in\mathcal{L}(F^*,\mathcal{L}(F,\mathbb{R}))$ but $\operatorname{ev}=\operatorname{id}^\wedge\notin\mathcal{L}(F^*,F;\mathbb{R})$.

Corollary.
Let E be a normable space. Then $(\_)\otimes_\pi E$ preserves co-limits. □

From the exponential law for continuous and that for smooth mappings, we are used that one automatically gets an isomorphism between the corresponding function spaces, cf. 2.48. So one would expect that the linear isomorphism $\mathcal{L}(E\otimes_\pi F,G)\cong\mathcal{L}(E,F;G)$ is in fact a topological one. If one supplies both spaces with the topology of uniform convergence on bounded sets, then $\otimes^*:\mathcal{L}(E\otimes_\pi F,G)\to\mathcal{L}(E,F;G)$ is obviously continuous since $\otimes:E\times F\to E\otimes_\pi F$ is bounded. In order to prove that it is an embedding, we have to find for every bounded set $B\subseteq E\otimes_\pi F$ and 0-neighborhood $W\subseteq G$ two bounded sets $B_1\subseteq E$ and $B_2\subseteq F$ and a 0-neighborhoods $U\subseteq G$, such that $\otimes^*(N_{B,W})\supseteq N_{B_1\times B_2,U}$. In particular if $G=\mathbb{R}$ and $W=[-1,1]$, then $N_{B,W}$ is the polar B^o of B and for all bilinear continuous functionals, which map $B_1\times B_2$ to $U=[-K,K]$, the corresponding linear functional $\tilde{T}$ on $E\otimes_\pi F$ must be in B^0. By enlarging B_1 we may assume that $K=1$. Using the bipolar theorem we deduce from $(B_1\otimes B_2)^o\subseteq B^o$ that $B\subseteq(B_1\otimes B_2)^{oo}=\langle B_1\otimes B_2\rangle_{closed,abs.conv.}$. Thus the closed absolutely convex hull of the image of $B_1\times B_2$ must contain B. Whether this is true is even for Fréchet spaces unknown. This is also called *Grothendieck's problème de topologies*. For the corresponding result on compact subsets see 3.21.

However bornologically we have an isomorphism:

3.11 Lemma.
With respect to the equi-continuous bornology we have a bornological isomorphism

$$\mathcal{L}(E\otimes_\pi F,G)\cong\mathcal{L}(E,F;G).$$

Proof. Let us first show that $\mathcal{B}\subseteq\mathcal{L}(E,F;\mathbb{R})$ is equi-continuous iff there exist 0-neighborhoods U in E and V in F such that $B\subseteq(U\times V)^o$.
($\Leftarrow$) Let $(x_0,y_0)\in E\times F$ be given. Choose $\lambda\ge1$ and $\mu\ge1$ such that $x_0\in\lambda U$ and $y_0\in\mu V$. Then we have for $y-y_0\in\frac{1}{\lambda}V\subseteq V$ and for $x-x_0\in\frac{1}{\mu}U\subseteq U$ that

$$|b(x,y)-b(x_0,y_0)|\le|b(\underbrace{x-x_0}_{\in U},\underbrace{y-y_0}_{\in V})|+|b(\underbrace{x-x_0}_{\in\frac{1}{\mu}U},\underbrace{y_0}_{\in\mu V})|+|b(\underbrace{x_0}_{\in\lambda U},\underbrace{y-y_0}_{\in\frac{1}{\lambda}V})|\le3.$$

($\Rightarrow$) is obvious by the equi-continuity at 0 and since $b(0,0)=0$.

Now the isomorphism is clear since the basis of the equi-continuous bornologies are $(U\otimes V)^o$ and $(U\times V)^o$ respectively, where U and V run through 0-neighborhood basis of E and F. □

Since every injective mapping f between vector spaces has a linear left inverse and every surjective one has a right inverse, the same is true for $f\otimes E$ and hence we have:

3.12 Lemma.
The projective tensor product preserves injective and surjective continuous linear mappings. □

3.13 Proposition.
The projective tensor product preserves quotients.

Proof. Let F be a locally convex space and f be a quotient mapping and hence open. We have to show, that $f \otimes_\pi F : E_1 \otimes_\pi F \to E_2 \otimes_\pi F$ is open. So let $U \otimes V$ be a typical 0-neighborhood of $E_1 \otimes F$. Since the image under a linear map of an absolutely convex hull is the absolutely convex hull of the image, we have that $(f \otimes F)(U \otimes V) = f(U) \otimes V$ and hence is a 0-neighborhood in $E_2 \otimes_\pi F$. □

Let us consider the dual situation next.

3.14 Example.
$\otimes_\pi$ does not preserve embeddings.
In fact consider the isometric embedding $\ell^2 \to C(K)$, where K is the closed unit-ball of $(\ell^2)^*$ supplied with its compact topology of pointwise convergence, see the corollary to the Alaoğlu-Bourbaki-theorem in [**1**, 7.4.12]. This subspace has however no topological complement, since $C(K)$ has the *Dunford-Pettis property* (see [**5**, 20.7.8], i.e. $x_n^*(x_n) \to 0$ for every two sequences $x_n \to 0$ in $\sigma(E, E^*)$ and $x_n^* \to 0$ in $\sigma(E^*, E^{**})$), but no infinite dimensional reflexive Banach space like ℓ^2 has it (e.g. $x_n := e_n$, $x_n^* := e_n$) and hence cannot be a complemented subspace of $C(K)$, see [**5**, 20.7].

Suppose now that $\ell^2 \otimes_\pi (\ell^2)^* \to C(K) \otimes_\pi (\ell^2)^*$ were an embedding. The duality mapping $\mathrm{ev} : \ell^2 \times (\ell^2)^* \to \mathbb{R}$ yields a continuous linear mapping $s : \ell^2 \otimes_\pi (\ell^2)^* \to \mathbb{R}$ and would hence have a continuous linear extension $\tilde{s} : C(K) \otimes (\ell^2)^* \to \mathbb{R}$. The corresponding bilinear map would give a continuous mapping $\tilde{s}^\vee : C(K) \to (\ell^2)^{**} \cong \ell^2$, which is a left inverse to the embedding $\ell^2 \to C(K)$, a contradiction.

In connection with the second usage of tensor products we would expect that for the product $E^\mathbb{N} = (\mathbb{R} \otimes_\pi E)^\mathbb{N} = \mathbb{R}^\mathbb{N} \otimes_\pi E$, i.e. we are looking for preservation of certain products. But even purely algebraically this fails to be true. In fact take the coproduct $E = \mathbb{R}^{(\mathbb{N})}$. Using that $\mathbb{R}^\mathbb{N} \otimes (\_)$ is left-adjoint and hence preserves colimits we get $\mathbb{R}^\mathbb{N} \otimes \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^\mathbb{N} \otimes \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^\mathbb{N})^{(\mathbb{N})}$, which is strictly smaller than $(\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$. However in both spaces the union $\bigcup_n E^n$ is dense, so after taking completions there should be some chance. In order to work with completions we have to show preservation of dense embeddings. To obtain such a result we need a dual characterization of such mappings. And this we treat next.

3.16 External duality

There is however a second pair of adjoint functors, which we have used already several times. Namely we can associate to every locally convex space $(E, \mathcal{U})$ the *dual space* E^* formed by all continuous linear functionals on E supplied with the *bornology of equi-continuous sets*. A base of this bornology is given by the polars U^o of the 0-neighborhoods $U \in \mathcal{U}$. For every continuous linear map $T : E \to F$ we obtain a bounded linear map $T^* : F^* \to E^*$, since $T^*(V^o) \subseteq T^{-1}(V)^o$. In fact let $x^* \in T^*(V^o)$, i.e. $x^* = T^*(y^*) = y^* \circ T$ for some $y^* \in V^o$. Then $x^*(x) = y^*(Tx) \in [-1, 1]$ for all $x \in T^{-1}(V)$. This gives us a functor $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$.

Conversely we can associate to every bornological space $(X, \mathcal{B})$ the locally convex space $\ell^\infty(X, \mathbb{R})$ formed by all bounded functions on X and supplied with the topology of uniform convergence on bounded sets of X. Every bounded map $T : X \to Y$ induces a continuous linear map $\ell^\infty(T, \mathbb{R}) : \ell^\infty(Y, \mathbb{R}) \to \ell^\infty(X, \mathbb{R})$ given by $f \mapsto f \circ T$. In fact a typical 0-neighborhood of $\ell^\infty(X, \mathbb{R})$ is given by the polar B^o of a bounded set $B \subseteq X$ and we have $\ell^\infty(T,\mathbb{R})^{-1}(B^o) = T(B)^o$. This can be seen directly as follows:

$$\begin{aligned}\ell^\infty(T,\mathbb{R})^{-1}(B^o) &= \{y^* : (y^* \circ T) \in B^o\} = \{y^* : (y^* \circ T)(B) \subseteq [-1,1]\}\\ &= \{y^* : y^*(T(B)) \subseteq [-1,1]\} = T(B)^o.\end{aligned}$$

If X is in addition a convex bornological space E, then we can restrict to the linear subspaces $E' \subseteq \ell^\infty(E,\mathbb{R})$ formed by the linear bounded functionals, and hence obtain a functor $(\_)' : \underline{CBS}^{op} \to \underline{LCS}$.

Again we show that these two functors form an adjoint pair. So let E be a locally convex space and F a convex bornological space and consider a linear $T : E \to F'$. It is continuous iff for every bounded set B in F there exists a 0-neighborhood U in E such that $T^{-1}(B^o) \supseteq U$ or equivalently that $T^\wedge(U \times B) \subseteq [-1,1]$, where $T^\wedge$ denotes the associated bilinear map from $E \times F \to \mathbb{R}$. If we flip the coordinates we get a linear map $\tilde{T} : F \to E^*$. In fact $\tilde{T}(y) = \mathrm{ev}_y \circ T$ is continuous, since all $\mathrm{ev}_y : F' \to \mathbb{R}$ are so. This mapping is bounded, iff for every bounded $B \subseteq F$ there exists some 0-neighborhood $U \subseteq E$ such that $\tilde{T}(B) \subseteq U^o$, or equivalently such that $T^\wedge(U \times B) \subseteq [-1,1]$. Since for any bounded linear map $T : F \to E^*$ the map $\tilde{T}$ obtained by changing the coordinates is bounded (since $\mathrm{ev}_x : E^* \to \mathbb{R}$ are) we have obtained a natural bijection

$$\underline{LCS}(E,F') \cong \underline{CBS}(F,E^*) = \underline{CBS}^{op}(E^*,F).$$

I.e. $(\_)' : \underline{CBS}^{op} \to \underline{LCS}$ is right adjoint to $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$ and hence carries limits in $\underline{CBS}^{op}$ (i.e. colimits in $\underline{CBS}$) to limits in $\underline{LCS}$ and $(\_)^*$ carries colimits in $\underline{LCS}$ to limits in $\underline{CBS}$.

3.17 Preservation of certain morphisms

Let us show next that $(\_)^*$ carries topological linear embeddings into *bornological quotient* mappings, i.e. mappings where each bounded set in the codomain is the image of a bounded set in the domain. Up to an isomorphism any topological linear embedding is given by the inclusion T of a subspace E in F. By 3.16 we know that T^* is bounded. In order to show that it is a bornological quotient map let $U \subseteq E$ be a 0-neighborhood, which is without loss of generality closed and absolutely convex. We have to find a 0-neighborhood V of F such that $U^o \subseteq T^*(V^o)$. So let p be a continuous seminorm on F which extends the Minkowski functional of U and let V be the closed unit-ball of p. Then every continuous linear functional $x^* \in U^o$ satisfies $|x^*| \le p$ on E and hence extends by Hahn-Banach to a continuous linear functional $y^* \in F^*$ with $|y^*| \le p$. Thus $y^* \in V^o$ and $T^*(y^*) = y^* \circ T = x^*$.

Conversely let us show that $(\_)'$ carries bornological quotient mappings into topological embeddings. Since a bornological quotient mapping $T : E \to F$ obviously has to be onto, we conclude that $T^* : F' \to E'$ is injective. Note that we refrain from denoting this map $T' : F' \to E'$ in order to avoid confusion with the derivative. Since $T^*(T(B)^o) = T^*((T^*)^{-1}(B^o)) = B^o \cap T^*(F')$, by what we proved above, and since the sets $T(B)^o$ form a 0-neighborhood basis of F' we are done.

Thus if $T^* : F^* \to E^*$ is a bornological quotient map then $(T^*)^* : (F^*)' \to (E^*)'$ is a topological embedding and using the embedding $E \to L((E^*,\mathcal{E}),\mathbb{R})$ of [**1**, 7.4.11] and the commutative diagram

$$\begin{array}{ccccc} E & \hookrightarrow & L(E^*,\mathbb{R}) & = & (E^*)' \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle L(T^*,\mathbb{R})} & & \downarrow{\scriptstyle (T^*)^*} \\ F & \hookrightarrow & L(F^*,\mathbb{R}) & = & (F^*)' \end{array}$$

shows that T is an embedding as well. Hence we have proved

3.18 Corollary.
A linear mapping $T : E \to F$ is a topological embedding iff the associated mapping $T^* : F^* \to E^*$ is a bornological quotient mapping for the equi-continuous bornologies.
It is a dense embedding iff the associated mapping T^* is a bornological isomorphism, i.e. is invertible in the category of bounded linear mappings. □

3.19 Proposition.
The projective tensor product preserves dense mappings and dense embeddings.

Proof. Obviously the tensor product $T_1 \otimes T_2$ of two dense mappings is dense. Otherwise there would exist a non-trivial continuous linear functional which vanishes on the image. The corresponding bilinear continuous map would then vanish on the dense image of $T_1 \times T_2$, and hence be 0, a contradiction.

Let now $T : E_2 \to E_1$ be in addition an embedding. By the previous proposition we have to show that $(T \otimes F)^* : (E_1 \otimes_\pi F)^* \to (E_2 \otimes_\pi F)^*$ is a quotient mapping for the equi-continuous bornologies. So let $\mathcal{B} := (U \otimes V)^o$ be a typical equi-continuous subset of $(E_2 \otimes_\pi F)^* \cong \mathcal{L}(E_2, F; \mathbb{R})$ formed by 0-neighborhoods U and V. We may extend every $b \in \mathcal{B} \subseteq \mathcal{L}(E_2, F; \mathbb{R})$ to a continuous bilinear mapping $\tilde{b} \in \mathcal{L}(E_1, F; \mathbb{R})$ defined by $\operatorname{ev} \circ (E_1 \times \check{b}) : E_1 \times F \to E_1 \times E_2^* = E_1 \times E_1^* \to \mathbb{R}$. For this recall that $E_1^* = E_2^*$ by 3.18. This composition is continuous (although the last component is not), since $\bar{U} \times V$ is mapped to $\operatorname{ev}(\bar{U} \times U^o) = \operatorname{ev}(\bar{U} \times (\bar{U})^o) \subseteq [-1, 1]$ and $\bar{U}$ is a 0-neighborhood in E_1, see [**1**, 4.10.3]. Hence $\tilde{\mathcal{B}} := \{\tilde{b} : b \in \mathcal{B}\} \subseteq (\bar{U} \otimes V)^o$ is the required equi-continuous subset satisfying $(T \otimes F)^*(\tilde{\mathcal{B}}) \supseteq \mathcal{B}$. □

3.20 Corollary. Completed projective tensor product.
The projective tensor product $E_1 \otimes_\pi E_2$ is a dense topological subspace of $\hat{E}_1 \otimes_\pi \hat{E}_2$. The completion of $E_1 \otimes_\pi E_2$ equals that of $\hat{E}_1 \otimes_\pi \hat{E}_2$. It will be denoted by $E_1 \hat{\otimes}_\pi E_2$, and will be called the completed projective tensor product. □

3.21 Theorem. Compact subsets of the projective tensor product.

Compact subsets of $E \hat{\otimes}_\pi F$ for metrizable spaces E and F are contained in the closed absolutely convex hull of a tensor product of precompact sets in E and F.

Proof. Every compact set K in the Fréchet space $E \hat{\otimes}_\pi F$ is contained in the closed absolutely convex hull of a 0-sequence $z_n \in E \hat{\otimes}_\pi F$ by [**1**, 6.4.3]. For this 0-sequence we can choose k_n strictly increasing, such that $z_k \in U_n \otimes V_n$ for all $k \geq k_n$, where $(U_n)_n$ and $(V_n)_n$ are countable 0-neighborhood bases of the topology of E and F. For $k_n \leq k < k_{n+1}$ we can choose finite (disjoint) sets $N_k \subseteq \mathbb{N}$ and $\sum_{j \in N_k} |\lambda_j| = 1$, $x_j \in U_n$ and $y_j \in V_n$ such that $z_k = \sum_{j \in N_k} \lambda_j x_j \otimes y_j$. Let $A := \{x_j : j \in \bigcup_k N_k\}$ and $B := \{y_j : j \in \bigcup_k N_k\}$. These are two sequences converging to 0, and hence are precompact. Furthermore $z \in K$ can be written as

$$z = \sum_k \mu_k z_k = \sum_k \sum_{j \in N_k} \mu_k \lambda_j x_j \otimes y_j$$

with $\sum_k |\mu_k| \leq 1$ and $\sum_{j \in N_k} |\lambda_j| = 1$ and hence $\sum_k |\mu_k| \sum_{j \in N_k} |\lambda_j| \leq 1$. From this it easily follows that the series on the right hand side converges Mackey and hence z is contained in the closed absolutely convex hull of $A \otimes B$. □

3.22 Corollary. Elements of the completed tensor product as limits.
Every $z \in E\hat{\otimes}_\pi F$ for metrizable E and F has a representation of the form $z = \sum_n \lambda_n x_n \otimes y_n$, where $\lambda \in \ell^1$ and x and y are bounded (or even 0-)sequences.

Since for every $\lambda \in \ell^1$ there exists a $\rho \in c_0$ and $\mu \in \ell^1$ with $\lambda_n = \rho_n^2 \mu_n$ it is enough to find bounded sequences x_n and y_n.

Proof. In the previous proof we have just shown that $z = \sum_j \mu_{k_j} \lambda_j x_j \otimes y_j$. □

Next we will show some preservation properties with respect to limits. For this we need.

3.28 Theorem.
The completed projective tensor product $(\_)\hat{\otimes}_\pi E$ preserves products.

Proof. The functoriality of $(\_)\hat{\otimes}_\pi F$ gives us a natural mapping

$$\iota : \Big(\prod_i E_i\Big)\hat{\otimes}_\pi F \to \prod_i (E_i \hat{\otimes}_\pi F).$$

We claim that this mapping is an embedding. As in 3.19 it is equivalent to show that the associated mapping $\iota^* : \big(\prod_i (E_i\hat{\otimes}_\pi F)\big)^* \to \big((\prod_i E_i)\hat{\otimes}_\pi F\big)^*$ is a quotient map for the equi-continuous bornologies. But this mapping is up to the natural isomorphisms from 3.11

$$\coprod_i \mathcal{L}(E_i, F; \mathbb{R}) \cong \coprod_i (E_i\hat{\otimes}_\pi F)^* \cong \Big(\prod_i (E_i\hat{\otimes}_\pi F)\Big)^*$$

and

$$\mathcal{L}\Big(\prod_i E_i, F; \mathbb{R}\Big) \cong \bigg(\Big(\prod_i E_i\Big)\hat{\otimes}_\pi F\bigg)^*$$

given by

$$\coprod_i \mathcal{L}(E_i, F; \mathbb{R}) \to \mathcal{L}\Big(\prod_i E_i, F; \mathbb{R}\Big).$$
$$(b_i)_i \mapsto \Big(((x_i)_i, y) \mapsto \sum_i b_i(x_i, y)\Big)$$

So let $B := ((\prod_i U_i) \times V)^o$ be a typical equi-continuous subset of $\mathcal{L}(\prod_i E_i, F; \mathbb{R})$, where U_i are 0-neighborhoods in E_i with $U_i = E_i$ for all i except those in some finite subset I of the index set, and V is a 0-neighborhood in F. In particular we have for every $b \in B$ that $b((x_i)_i, y) = 0$ provided $x_i = 0$ for all $i \in I$, since for all $\varepsilon > 0$ and $y \in V$ we have $b((\varepsilon x_i), y) = \varepsilon\, b((\frac{1}{\varepsilon} x_i)_i, \varepsilon y) \in \varepsilon\, b(\prod_i U_i \times V) \subseteq \varepsilon\,[-1, 1]$. Since V is absorbing it has to be 0. Thus b can be considered as element of $\coprod_{i\in I} \mathcal{L}(E_i, F; \mathbb{R})$ and lies moreover in the equi-continuous subset $\coprod_{i \in I}(U_i \times V)^o$.

Since the algebraic tensor product is left-adjoint to $L(E, \_)$ it commutes with coproducts. Hence algebraically we have that $(\coprod_i E_i) \otimes F \cong \coprod_i (E_i \otimes F)$. We will see later on that topologically this is not true in general. By the density of the coproducts, we obtain that ι is dense and hence we have the required isomorphism. □

3.29 Proposition.
The completed projective tensor product preserves reduced projective limits.

Proof. So let $E = \varprojlim_i E_i$ be a reduced projective limit, i.e. $\mathrm{pr}_i : E \to E_i$ has dense image. Then $\mathrm{pr}_i \otimes F : E \otimes F \to E_i \otimes_\pi F$ has dense image and consequently also $\mathrm{pr}_i \hat{\otimes}_\pi F : E \otimes F \to E_i \hat{\otimes}_\pi F$. Since this mapping factors over $\varprojlim_i (E_i \hat{\otimes}_\pi F) \to E_i \hat{\otimes}_\pi F$

the latter mapping has dense image as well. Thus the limit $\varprojlim_i(E_i\hat{\otimes}_\pi F)$ is a reduced one. Let us show next that the natural mapping

$$\Big(\varprojlim_i E_i\Big)\hat{\otimes}_\pi F \to \varprojlim_i(E_i\hat{\otimes}_\pi F)$$

is a dense embedding, or equivalently that the dual mapping

$$\Big(\varprojlim_i(E_i\hat{\otimes}_\pi F)\Big)^* \to \Big((\varprojlim_i E_i)\hat{\otimes}_\pi F\Big)^*$$

is a bornological isomorphism. The left side equals

$$\Big(\varprojlim_i(E_i\hat{\otimes}_\pi F)\Big)^* \cong \varinjlim_i(E_i\hat{\otimes}_\pi F)^* \cong \varinjlim_i \mathcal{L}(E_i,F;\mathbb{R}),$$

since the dual of a reduced projective limit is an injective one. The right hand side equals $\Big((\varprojlim_i E_i)\hat{\otimes}_\pi F\Big)^* \cong \mathcal{L}\Big(\varprojlim_i E_i, F;\mathbb{R}\Big)$. So let $(\mathrm{pr}_j^{-1}(U_j)\times V)^o$ be a typical bounded set in $\mathcal{L}(\varprojlim_i E_i, F;\mathbb{R})$. This is the image under the natural mapping of the bounded set $(\mathrm{pr}_i\times F)^*((U_i\times V)^o)$ in $\varinjlim_i \mathcal{L}(E_i,F;\mathbb{R}) = \bigcup_i(\mathrm{pr}_i\times F)^*(\mathcal{L}(E_i,F;\mathbb{R}))$. Thus the natural mapping is a bornological quotient mapping. It remains to show that it is injective. So let $T = (\mathrm{pr}_i\times F)^*(T_i) \in \varinjlim_i \mathcal{L}(E_i,F;\mathbb{R})$ be given with $T_i\in\mathcal{L}(E_i,F;\mathbb{R})$ and such that the associated element $\iota(T)=0$ in $\mathcal{L}\Big(\varprojlim_i E_i, F;\mathbb{R}\Big)$. Obviously $\iota(T) = T_i\circ(\mathrm{pr}_i\times F)$ and since the $\mathrm{pr}_i\times F$ has dense image in $E_i\times F$ we conclude that $T_i=0$ and hence $T=0$.

Since both sides of the natural dense embedding $\Big(\varprojlim_i E_i\Big)\hat{\otimes}_\pi F \to \varprojlim_i(E_i\hat{\otimes}_\pi F)$ are complete (limits of complete spaces are complete) we have equality. □

3.30 Corollary.
For the function space $C(X,E) = E^X = \prod_{x\in X} E$, where X is a discrete topological space, we have a natural isomorphism

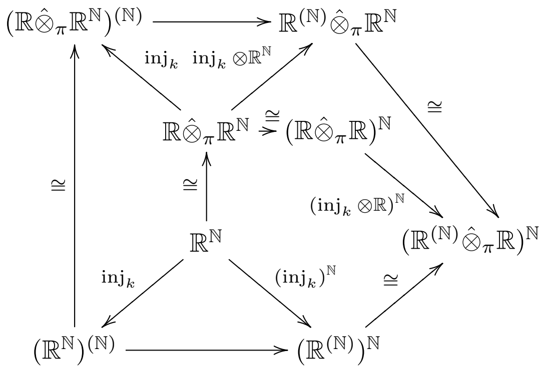
$$C(X)\hat{\otimes}_\pi E = \mathbb{R}^X\hat{\otimes}_\pi E \cong (\mathbb{R}\hat{\otimes}_\pi E)^X \cong E^X = C(X,E).$$

If X is finite and $E\cong\mathbb{R}^Y$ with finite Y we obtain in particular that $\mathbb{R}^X\hat{\otimes}_\pi\mathbb{R}^Y \cong (\mathbb{R}^Y)^X \cong \mathbb{R}^{X\times Y}$. Hence we have for finite dimensional spaces that $\dim(E\hat{\otimes}_\pi F) = \dim E\cdot\dim F$ thus also $\dim(E\otimes_\pi F) = \dim E\cdot\dim F$.

Note that for general projective limits the analogue to 3.29 is not true. In fact take the closed linear subspace $\ell^2\to C(K)$, which is the kernel (a special limit) of the quotient map $C(K)\to C(K)/\ell^2$. Since $\ell^2\otimes_\pi(\ell^2)^*\to C(K)\otimes_\pi(\ell^2)^*$ is not an embedding (see 3.14), also $\ell^2\hat{\otimes}_\pi(\ell^2)^*\to C(K)\hat{\otimes}_\pi(\ell^2)^*$ is not, and so both cannot be the kernel of some map.

3.32 Example.
In general the projective tensor product does not commute with direct sums. Furthermore it does not preserve strict inductive limits (since $\mathbb{R}^{(\mathbb{N})} = \varinjlim_n \mathbb{R}^n$) and also not the function space $C_c(X) = \mathbb{R}^{(X)}$ for discrete X:
The natural injection of flipping the coordinates from $(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\to(\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ is obviously not onto. Moreover, it can be shown that $(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$ and $(\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ can not even be isomorphic by some non-canonical isomorphism, since both spaces are B-complete but their cartesian product is not, see [**5**, 15.5.1]. A locally convex spaces is called *B-complete*, iff every continuous *nearly open map* (i.e. the closure of the image of any 0-neighborhood is a 0-neighborhood) into some locally convex spaces has complete image, or equivalently if every such mapping is open onto its image. So isomorphy would imply that $(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\cong(\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\cong(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\times(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\cong(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}\times(\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$.

Then the natural mapping from $(\mathbb{R}\hat{\otimes}_\pi\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})} \to \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi\mathbb{R}^{\mathbb{N}}$ is not onto as can be seen also from the following commutative diagram



since the bottom arrow is obviously not onto.

3.33 Corollary.
Neither the projective tensor product nor the completed projective tensor product can be left adjoint functors. □

The Bornological Tensor Product

We have seen that the classical projective tensor product is not well behaved beyond normed spaces. And the main reason for that is that it is not longer a left-adjoint functor.

But we have already seen that bounded mappings are in many respects much nicer than continuous ones.

And if $L(E_1, \dots, E_n; F)$ denotes the *space of all bounded n-linear mappings* from $E_1 \times \dots \times E_n \to F$ with the topology of uniform convergence on bounded sets in $E_1 \times \dots \times E_n$ then we easily show the following.

3.34 Proposition. Exponential law for L.
There are natural topological linear isomorphisms

$$L(E_1, \dots, E_{n+k}; F) \cong L(E_1, \dots, E_n; L(E_{n+1}, \dots, E_{n+k}; F)).$$

Proof. We proof this for bilinear maps, the general case is completely analogous. We already know that bilinearity translates into linearity into the space of linear functions. Remains to prove boundedness. So let a set $\mathcal{B}$ of bilinear mappings $E_1 \times E_2 \to F$ be given. Then $\mathcal{B}$ is bounded in $L(E_1, E_2; F)$ iff $\mathcal{B}(B_1 \times B_2) \subseteq F$ is bounded for all bounded $B_i \subseteq E_i$. This however is equivalent to $\mathcal{B}^\vee(B_1)$ is contained and bounded in $L(E_2, F)$ for all bounded $B \subseteq E_1$, i.e. $\mathcal{B}^\vee$ is contained and bounded in $L(E_1, L(E_2, F))$.

That this even a topological isomorporphism follows by the arguments in 3.10. □

Hence it is natural to consider the universal problem of making bounded bilinear mappings into bounded linear ones. The solution is given by the *bornological tensor product* $E \otimes_\beta F$, i.e. the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded. A 0-neighborhood basis of this topology is given by those absolutely convex sets, which absorb $B_1 \otimes B_2$ for all bounded $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$. Note that this topology is bornological since it is the finest locally convex topology with given bounded linear mappings on it.

3.38 Theorem. Bornological tensor product.
The bornological tensor product is left adjoint to the Hom-functor $L(E,\_)$ on the category of bounded linear mappings between locally convex spaces and one has the following bornological isomorphisms:

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G))$$
$$E \otimes_\beta \mathbb{R} \cong E$$
$$E \otimes_\beta F \cong F \otimes_\beta E$$
$$(E \otimes_\beta F) \otimes_\beta G \cong E \otimes_\beta (F \otimes_\beta G)$$

Furthermore the bornological tensor product preserves co-limits. It neither preserves embeddings nor countable products.

Proof. We show first that this topology has the universal property for bounded bilinear mappings $f : E_1 \times E_2 \to F$. Let U be an absolutely convex zero neighborhood in F and let B_1, B_2 be bounded sets. Then $f(B_1 \times B_2)$ is bounded hence is absorbed by U. Then $\tilde{f}^{-1}(U)$ absorbs $\otimes(B_1 \times B_2)$, where $\tilde{f} : E_1 \otimes E_2 \to F$ is the canonically associated linear mapping. So $\tilde{f}^{-1}(U)$ is in the zero neighborhood basis of $E_1 \otimes_\beta E_2$ described above. Therefore $\tilde{f}$ is continuous.

A similar argument for sets of mappings shows that the first isomorphism $L(E \otimes_\beta F, G) \cong L(E, F; G)$ is bibounded.

The topology on $E_1 \otimes_\beta E_2$ is finer than the projective tensor product topology and so it is Hausdorff. The rest of the positive results is clear.

The counter example for embeddings given for the projective tensor product works also, since all spaces involved are Banach.

Since the bornological tensor-product preserves coproducts it cannot preserve products. In fact $(\mathbb{R} \otimes_\beta \mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ whereas $\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$. □

3.39 Proposition. Projective versus bornological tensor product.
If every bounded bi-linear mapping on $E \times F$ is continuous then $E \otimes_\pi F = E \otimes_\beta F$. In particular we have $E \otimes_\pi F = E \otimes_\beta F$ for any two metrizable spaces and for a normable space F we have $E_{born} \otimes_\pi F = E \otimes_\beta F$.

Proof. Recall that $E \otimes_\pi F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is continuous, whereas $E \otimes_\beta F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is bounded. So we have that $\otimes : E \times F \to E \otimes_\beta F$ is bounded and hence by assumption continuous and thus the topology of $E \otimes_\pi F$ is finer than that of $E \otimes_\beta F$. Since the converse is true n general, we have equality.

In [**1**, 3.1.6] we have shown that in metrizable locally convex spaces the convergent sequences coincide with the Mackey-convergent ones. Now let $T : E \times F \to G$ be bounded and bilinear. We have to show that T is continuous. So let (x_n, y_n) be a convergent sequence in $E \times F$. Without loss of generality we may assume that its limit is $(0, 0)$. So there are $\mu_n \to \infty$ such that $\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}$ is bounded and hence also $T\Big(\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}\Big) = \Big\{\mu_n^2 T(x_n, y_n) : n \in \mathbb{N}\Big\}$, i.e. $T(x_n, y_n)$ converges even Mackey to 0.

If F is normable, and $T : E_{born} \times F \to G$ is bi-linear and bounded, then $\check{T} : E_{born} \to L(F, G)$ is bounded, and since E_{born} is bornological it is even continuous. We have shown in 3.10 that for normed spaces F the evaluation map $\mathrm{ev} : L(F, G) \times F \to G$

is continuous, and hence $T = \operatorname{ev} \circ (\check{T} \times F) : E_{born} \times F \to G$ is continuous. Thus $E_{born} \otimes_\pi F = E \otimes_\beta F$. □

Note that the bornological tensor product is invariant under bornologification, i.e. $E_{born} \otimes_\beta F_{born} \cong E \otimes_\beta F$. So it is no loss of generality to assume that both spaces are bornological. Keep however in mind that the corresponding identity for the projective tensor product does not hold. Another possibility to obtain the identity $E \otimes_\pi F = E \otimes_\beta F$ is to assume that E and F are bornological and every separately continuous bi-linear mapping on $E \times F$ is continuous. In fact every bounded bi-linear mapping is obviously separately bounded and since E and F are assumed to be bornological it has to be separately continuous. We want to find another class beside the Fréchet spaces (see [**1**, Folgerung in 5.5]) which satisfies these assumptions.

3.47 Theorem. Continuity versus separately continuity.
Let E and F be two barreled spaces with a countable base of bornology. Then every separately continuous bilinear map $E \times F \to G$ is continuous.

Proof. Let A_n and B_n be a basis of the bornologies of E and F. Let $T : E \times F \to G$ be separately continuous. Then $T^\vee : E \to \mathcal{L}(F, G)$ is continuous for the topology of pointwise convergence on $\mathcal{L}(F, G)$. Thus $T^\vee(A_k)$ is bounded for this topology, and since F is barreled it is equi-continuous. Thus for every 0-neighborhood W in G there exists a 0-neighborhood V_k in F with $T(A_k \times V_k) \subseteq W$. By symmetry there exists a 0-neighborhood U_k in E with $T(U_k \times B_k) \subseteq W$. We have to show that this implies for gDF-spaces E and F the continuity of T, see [**5**, 15.6.1]. Since E is quasi-normable, we can find for every 0-neighborhood U_n a 0-neighborhood U_n' such that for every $\rho > 0$ there is some $k(n, \rho) \in \mathbb{N}$ with $U_n' \subseteq \rho U_n + A_{k(n,\rho)}$. Since A_k is a basis of bounded sets there exist $\rho_n > 0$ such that $U := \bigcap_n \rho_n U_n'$ is a 0-neighborhood in the topology generated by $\{A_n\}$, see [**5**, 12.3.2]. And this topology coincides with the given topology since E is gDF, by [**5**, 12.3.6]. Let $W_n := V_{k(n,1/\rho_n)}$. Then $V := \langle \bigcup_n \frac{1}{\rho_n} W_n \cap B_n$ is a 0-neighborhood again by [**5**, 12.3.6] and by the description of a 0-neighborhood basis of the topology induced by $\{B_n\}_n$ given in [**5**, 12.3.1]. We claim that $T(U \times V) \subseteq W$. In fact take $x \in U$ and $y \in V$. Then y is an absolutely convex combination of $y_n \in \frac{1}{\rho_n} W_n \cap B_n$. Since $x \in \rho_n U_n' \subseteq U_n + \rho_n A_{k(n,1/\rho_n)}$ there are $u_n \in U_n$ and $a_n \in A_{k(n,1/\rho_n)}$ with $x = u_n + \rho_n a_n$. So

$$T(x, y_n) = T(u_n, y_n) + T(\rho_n a_n, y_n) \in T(U_n \times B_n) + \rho_n T(A_{k(n,1/\rho_n)} \times \frac{1}{\rho_n} W_n) \subseteq 2\,W$$

Hence the same is true for the absolutely convex combination $T(x, y)$, i.e. $T(U \times V) \subseteq 2\,W$. □

3.48 Corollary. Projective versus bornological tensor product for LB-spaces.
Let E and F be countable inductive limits of Banach spaces (e.g. the duals of metrizable spaces with their bornological topology, i.e. the bornologification of the strong topology). Then $E \otimes_\pi F \cong E \otimes_\beta F$.

Proof. Let $T : E \times F \to G$ be bounded. Since both spaces are bornological T is separately continuous and since both spaces are barreled and DF it is continuous. This is enough to guarantee the equality of the two tensor products by 3.39. □

Spaces of Multi-Linear Mappings

3.49 Corollary.
The following mappings are bounded multi-linear.

1. $\lim : \mathrm{Nat}(\mathcal{F},\mathcal{G}) \to L(\lim\mathcal{F},\lim\mathcal{G})$, *where* $\mathcal{F}$ *and* $\mathcal{G}$ *are two functors on the same index category, and where* $\mathrm{Nat}(\mathcal{F},\mathcal{G})$ *denotes the space of all natural transformations with the structure induced by the embedding into* $\prod_i L(\mathcal{F}(i),\mathcal{G}(i))$.
2. $\mathrm{colim} : \mathrm{Nat}(\mathcal{F},\mathcal{G}) \to L(\mathrm{colim}\,\mathcal{F},\mathrm{colim}\,\mathcal{G})$.
3. $$\begin{aligned} L : L(E_1,F_1)\times\ldots\times L(E_n,F_n)\times L(F,E) &\to \\ &\to L(L(F_1,\ldots,F_n;F),L(E_1,\ldots,E_n;E)) \\ (T_1,\ldots,T_n,T) &\mapsto (S\mapsto T\circ S\circ(T_1\times\ldots\times T_n)); \end{aligned}$$
4. $\bigotimes^n_\beta : L(E_1,F_1)\times\ldots\times L(E_n,F_n)\to L(E_1\otimes_\beta\cdots\otimes_\beta E_n, F_1\otimes_\beta\cdots\otimes_\beta F_n)$.
5. $\bigwedge^n : L(E,F)\to L(\bigwedge^n E,\bigwedge^n F)$, *where* $\bigwedge^n E$ *is the linear subspace of all alternating tensors in* $\bigotimes^n_\beta E$. *It is the universal solution of*
$$L\Big(\bigwedge^n E,F\Big)\cong L^n_{alt}(E;F).$$
6. $\bigvee^n : L(E,F)\to L(\bigvee^n E,\bigvee^n F)$, *where* $\bigvee^n E$ *is the linear subspace of all symmetric tensors in* $\bigotimes^n_\beta E$. *It is the universal solution of*
$$L\Big(\bigvee^n E,F\Big)\cong L^n_{sym}(E;F).$$
7. $\bigotimes_\beta : L(E,F)\to L(\bigotimes_\beta E,\bigotimes_\beta F)$, *where* $\bigotimes_\beta E := \bigoplus_{n=0}^\infty \overset{n}{\otimes}_\beta E$ *is the tensor algebra of* E. *Note that is has the universal property of prolonging bounded linear mappings with values in locally convex spaces, which are algebras with bounded operations, to continuous algebra homomorphisms:*
$$L(E,F)\cong \mathrm{Alg}(\otimes E,F).$$
8. $\bigwedge : L(E,F)\to L(\bigwedge E,\bigwedge F)$, *where* $\bigwedge E := \bigoplus_{n=0}^\infty \bigwedge^n E$ *is the exterior algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into graded-commutative algebras, i.e. algebras in the sense above, which are as vector spaces a coproduct* $\coprod_{n\in\mathbb{N}} E_n$ *and the multiplication maps* $E_k\times E_l\to E_{k+l}$ *and for* $x\in E_k$ *and* $y\in E_l$ *one has* $x\cdot y=(-1)^{kl}y\cdot x$.
9. $\bigvee : L(E,F)\to L(\bigvee E,\bigvee F)$, *where* $\bigvee E := \bigoplus_{n=0}^\infty \bigvee^n E$ *is the symmetric algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into commutative algebras.*

Recall that for permutations π of $n := \{0,\ldots,n-1\}$ we have an associated linear mapping $\pi^* : E^n\to E^n$ and hence a linear mapping $\tilde{\pi^*} : \bigotimes^n E\to\bigotimes^n E$. The exterior product $\bigwedge^n E$ is the space invariant under $\mathrm{sign}(\pi)\,\tilde{\pi^*}$ for all permutations π and the symmetric product $\bigvee^n E$ is the space invariant under $\tilde{\pi^*}$ for all permutations π. The symmetric product is given as the image of the *symmetrizer* $\mathrm{sym} : E\otimes_\beta\cdots\otimes_\beta E\to E\otimes_\beta\cdots\otimes_\beta E$ given by
$$(x_1,\ldots,x_n)\to\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

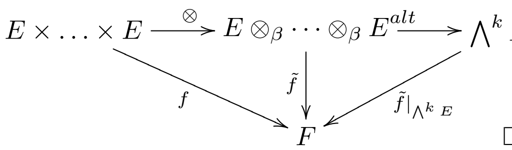
Similarly the wedge product is given as the image of the *alternator*

$$\operatorname{alt} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$$

$$\text{given by } (x_1, \ldots, x_n) \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\pi)\, (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Proof. All results follow easily by flipping coordinates until a composition of products of evaluation maps remains.

That the spaces in (5), and similar in (6), are universal solutions can be seen from the following diagram:



□

For (convenient) differential calculus the so-called convenient vector space, i.e. (bornological) locally convex spaces, for which every Mackey-Cauchy sequence converges, play an important role.

2.27 Theorem. c$^\infty$-completeness.
Let E be a locally convex vector space. E is said to be c^∞-complete or convenient if one of the following equivalent (completeness) conditions is satisfied:

1. *Any Lipschitz curve in E is locally Riemann integrable.*
2. *For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_2' = c_1$ (existence of an antiderivative).*
3. *E is c^∞-closed in any locally convex space.*
4. *If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.*
5. *Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges; i.e. E Mackey-complete, see 2.11.*
6. *If B is bounded closed absolutely convex, then E_B is a Banach space. This property is called locally complete in* [**5**].
7. *Any continuous linear mapping from a normed space into E has a continuous extension to the completion of the normed space.*

Condition 4 says that in a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

In [**3**] a convenient vector space is always considered with its bornological topology — an equivalent but not isomorphic category.

3.56 Theorem. Uniform boundedness principle.
If all E_i are convenient vector spaces and if F is a locally convex space, then the bornology on the space $L(E_1, \ldots, E_n; F)$ consists of all pointwise bounded sets.

Proof. Let us first consider the case $n = 1$. So let $\mathcal{B} \subset L(E, F)$ be a pointwise bounded subset. By lemma 3.35 we have to show that it is uniformly bounded on each bounded subset B of E. We may assume that B is closed absolutely convex and thus E_B is a Banach space, since E is convenient. By the classical Uniform Boundedness Principle, see [**1**, 5.2.2], the set $\{f|_{E_B} : f \in \mathcal{B}\}$ is bounded in $L(E_B, F)$ and thus $\mathcal{B}$ is bounded on B.

The multi-linear case follows from the exponential law 3.34 using induction on n. □

3.57 Theorem. Multi-linear mappings on convenient vector spaces. *A multi-linear mapping from convenient vector spaces to a locally convex space is bounded if and only if it is separately bounded.*

Proof. Let $f : E_1 \times \ldots \times E_n \to F$ be n-linear and separately bounded, i.e. $x_i \mapsto f(x_1, \ldots, x_n)$ is bounded for each i and fixed x_j for all $j \neq i$. Then $\check{f} : E_1 \times \ldots \times E_{n-1} \to L(E_n, F)$ is $(n-1)$-linear. By 3.56 the bornology on $L(E_n, F)$ consists of the pointwise bounded sets, so $\check{f}$ is separately bounded. By induction on n it is bounded. The bornology on $L(E_n, F)$ consists also of the subsets which are uniformly bounded on bounded sets by lemma 3.35, so f is bounded. □

4. Tensor Products and Function Spaces

Desired Isomorphisms

Let $\underline{LCS}$ be the category of continuous linear maps between locally convex spaces.

On the other hand we can consider *bornological space*s. These are sets X with a *bornology*, i.e. a set $\mathcal{B}$ of subsets of X, which contains all finite subsets and is closed under formation of finite unions and subsets. The elements of $\mathcal{B}$ are called the *bounded sets* of X. And a mapping between such sets is called *bounded*, iff it maps the bounded sets to bounded sets. If X is in addition a vector space and addition and scalar multiplication are bounded, then X is called *bornological vector space*. If furthermore the convex hull of each bounded set is bounded, then X is called *convex bornological space*. Let $\underline{CBS}$ denote the category of bounded linear maps between convex bornological spaces.

To every locally convex space $(E,\mathcal{U})$ we can associate a a convex bornological space $(E,\mathcal{B})$, where the bornology $\mathcal{B}$ is given by the *von Neumann bounded set*s, i.e. those sets $B \subseteq E$ which are absorbed by all 0-neighborhoods $U \in \mathcal{U}$. This correspondence extends to a functor $b : \underline{LCS} \to \underline{CBS}$ which leaves the morphisms and the underlying vector spaces unchanged.

Conversely, we can associate to every convex bornological space $(E,\mathcal{B})$ a locally convex topology on E given by the 0-neighborhood basis $\mathcal{U}$ formed by all *bornivorous subset*s $U \subseteq E$, i.e. those sets which absorb all the bounded sets $B \in \mathcal{B}$. This correspondence extends to a functor $t : \underline{CBS} \to \underline{LCS}$ which leaves the morphisms and the underlying vector spaces unchanged.

4.1. Suppose we are given some category $\underline{X}$ and a *forgetful functor* $V : \underline{LCS}_b \to \underline{X}$, where the index b indicates that we consider the bounded linear morphisms. Then for an object X in $\underline{X}$ and a locally convex space G we can consider the space of morphisms $\underline{X}(X, V(G))$ and we assume that this lifts to a functor $\mathcal{F} : \underline{X}^{op} \times \underline{LCS}_b \to \underline{LCS}_b$. Examples of that situation are

1. $\underline{X} := \underline{LCS}_b$, V the identity and $\mathcal{F}$ the *internal hom-functor* L.
2. $\underline{X}$ the category of mappings between sets and $\mathcal{F}(X,G) := \prod_X G = G^X$ the space of all mappings with the topology of pointwise convergence.
3. $\underline{X}$ the category of continuous maps between topological spaces and $\mathcal{F}(X,G)$ the space $C(X,G)$ of continuous mappings with the topology of uniform convergence on compact subsets. Here we have to restrict to continuous linear mappings to get a forgetful functor. Note that (2) is a particular case, where the topology on X is discrete.
4. $\underline{X}$ the bounded (better bornological) mappings between bornological spaces and $F(X,G) := \ell^\infty(X,G)$ the space of bornological mappings with the topology of uniform convergence on bounded sets. Note that (2) is a particular case, where the bounded sets are exactly the finite ones.

5. $\underline{X}$ the category of smooth mappings defined on c^∞-open subsets of locally convex spaces G, and $\mathcal{F}(X,G) := C^\infty(X,G)$ supplied with the locally convex topology described before.

Not completely fitting into this scheme but nevertheless interesting might be the following function spaces:

6. For sets X the space $\mathcal{F}(X,G) := G^{(X)}$ of all functions with finite support with the final topology induced by the subspaces G^A, where A runs through the finite subsets.
7. For topological spaces X the space $\mathcal{F}(X,G) := C_c(X,G)$ of all continuous functions with compact support with the final topology induced by the subspaces $C_K(X,G)$ formed by the continuous functions having support in K, where K runs through all compact subsets of X and where $C_K(X,G)$ carries the initial topology induced by the inclusion into $C(X,G)$. Note that (6) is a particular case, where the topology on X is discrete.
8. For a finite dimensional manifold X the space $\mathcal{F}(X,G) := C_c^\infty(X,G)$ of all smooth functions with compact support with the final topology induced by the subspaces $C_K^\infty(X,G)$ formed by the continuous functions having support in K, where K runs through all compact subsets of X and where $C_K^\infty(X,G)$ carries the initial topology induced by the inclusion into $C^\infty(X,G)$. Note that (6) is again a particular case, where the manifold is discrete.

Some desirable isomorphisms would then be the following, where we write $\mathcal{F}(X)$ as shortcut for $\mathcal{F}(X,\mathbb{R})$ and $\otimes_?$ denotes some appropriate tensor product.

(E) $\mathcal{F}(X,\mathcal{F}(Y,G)) \cong \mathcal{F}(X\times Y,G)$ *exponential law*

(V) $\mathcal{F}(X)\otimes_? G \cong \mathcal{F}(X,G)$ *vector valued versus scalar valued*

(P) $\mathcal{F}(X)\otimes_? \mathcal{F}(Y) \cong \mathcal{F}(X\times Y)$ *compatibility with products*

Note that (E) and (V) imply (P):

(P) $$\mathcal{F}(X)\otimes_? \mathcal{F}(Y) \overset{\text{(V)}}{\cong} \mathcal{F}(X,\mathcal{F}(Y)) \overset{\text{(E)}}{\cong} \mathcal{F}(X\times Y)$$

In the particular case (1), where the forgetful functor V forgets nothing, i.e. $\underline{X} = \underline{LCS_b}$, we would expect:

(E) $$L(E,L(F,G)) \cong L(E,F;G) \cong L(E\otimes_\beta F,G)$$

(V) $$E'\otimes_? G \cong L(E,G)$$

(P) $$E'\otimes_? F' \cong L(E,F') \cong (E\otimes_\beta F)'$$

Applying (P) for L to (V) and (P) for $\mathcal{F}$ we would obtain the dualized versions:

(V') $$\mathcal{F}(X)'\otimes_? G' \overset{(P_L)}{\cong} (\mathcal{F}(X)\otimes_\beta G)' \overset{(V)}{\cong} \mathcal{F}(X,G)'$$

(P') $$\mathcal{F}(X)'\otimes_? \mathcal{F}(Y)' \overset{(P_L)}{\cong} (\mathcal{F}(X)\otimes_\beta \mathcal{F}(Y))' \overset{(P)}{\cong} \mathcal{F}(X\times Y)'$$

Note again that (E) and (V') imply (P'):

(P') $$\mathcal{F}(X)'\otimes_? \mathcal{F}(Y)' \overset{\text{(V')}}{\cong} \mathcal{F}(X,\mathcal{F}(Y))' \overset{\text{(E)}}{\cong} \mathcal{F}(X\times Y)'$$

4.2. Exponential law

Lets us first determine in which situations we have the exponential law (E).

1. For L we have proved in 3.34 that (E) is true.
2. For sets X and Y we obviously have $(G^Y)^X \cong G^{X\times Y}$.
3. For C we have shown in 2.3 that (E) is a bijection if Y is locally compact. That it is also true for the structure follows immediately since the 0-neighborhood $N_{A,N_{B,V}}$ corresponds to $N_{A\times B,V}$, where $A \subseteq X$ and $B \subseteq Y$ are compact and $V \subseteq G$ a 0-neighborhood.
4. That (E) is a bijection for ℓ^∞ is obvious, cf. 3.34. That it is also true for the structure follows the same way as in (3), where A and B are bounded instead.
5. That (E) is true for C^∞ has been shown in 2.47 and 2.48.
6. It is obvious that $(G^{(Y)})^{(X)} \cong G^{(X\times Y)}$ is true.
7. For C_c we use that $C_c(X,G)$ is the strict inductive limit of the spaces $C_K(X,G)$, where $K \subset X$ is compact. Obviously the closed subspace $C_{A\times B}(X\times Y,G) = \{f : f(x,y) = 0 \text{ if } x \notin A \text{ or } y \notin B\}$ of $C(X\times Y,G)$ corresponds to the closed subspace $C_A(X,C_B(Y,G)) = \{f^\vee \in C(X,C_B(Y,G)) : f^\vee(x) = 0 \text{ if } x \notin A\}$ of $C(X,C(Y,G))$ and hence we have a natural injection
$$\begin{array}{ccccc} C_c(X\times Y,G) & = & \bigcup_{A,B} C_{A\times B}(X\times Y,G) & \longrightarrow & \bigcup_A C_A(X,C_c(Y,G)) \\ & & \cong\downarrow & & \| \\ & & \bigcup_{A,B} C_A(X,C_B(Y,G)) & \longrightarrow & C_c(X,C_c(Y,G)) \end{array}$$
Conversely let $\mathcal{B} \subseteq C_A(X,C_c(Y,G))$ be bounded. Then $\mathcal{B}(A)$ is bounded in $C_c(Y,G)$. Now suppose that Y is in addition σ-compact. Then $C_c(Y,G)$ is the strict inductive limit of a sequence of spaces $C_B(Y,G)$ and hence $\mathcal{B}(A)$ has to be bounded in some step $C_B(Y,G)$ by [**1**, 4.8.1]. So $\mathcal{B}$ corresponds to a bounded subset of $C_A(X,C_B(Y,G))$. So these correspondences induce the required bornological isomorphism and hence (E) holds for C_c and σ-compact Y.
8. For C_c^∞ we can proceed completely analogously to (7) to obtain (E) for C^∞ and finite dimensional smooth manifolds.

Now let us come to the other desired isomorphisms. One could ask, whether we could deduce the case of a general $\mathcal{F}$ from that of L. For this we need:

Universal Linearization

4.3. Suppose we can solve the universal problem of linearizing maps in $\mathcal{F}(X,G)$, i.e. find a c^∞-complete locally convex space $\lambda(X)$, also called a *free convenient vector space*, and a map $\iota : X \to \lambda(X)$ which induces an isomorphism

(F) $\qquad \mathcal{F}(X,G) \cong L(\lambda(X),G)$ the forgetful functor is right adjoint

for all c^∞-complete locally convex spaces G, and hence in particular an isomorphism $\mathcal{F}(X) \cong \lambda(X)'$. A consequence of (E) for L and $\mathcal{F}$ is that

(P) $$\lambda(X\times Y) \cong \lambda X \tilde{\otimes}_\beta \lambda Y$$

This follows, since

$$L(\lambda(X)\tilde{\otimes}_\beta\lambda(Y),G) \overset{(E_L)}{\cong} L(\lambda(X),L(\lambda(Y),G)) \overset{\text{(F)}}{\cong} \mathcal{F}(X,L(\lambda(Y),G))$$
$$\overset{\text{(F)}}{\cong} \mathcal{F}(X,\mathcal{F}(Y,G)) \overset{(E_\mathcal{F})}{\cong} \mathcal{F}(X\times Y,G)$$

shows that $\lambda(X)\tilde{\otimes}_\beta\lambda(Y)$ has the universal property of $\lambda(X\times Y)$. Using all this we can translate the general case to that for L:

(E) $$\mathcal{F}(X,\mathcal{F}(Y,G)) \overset{\text{(F)}}{\cong} L(\lambda X,L(\lambda Y,G)) \overset{(E_L)}{\cong}$$
$$\cong L(\lambda X\otimes_\beta \lambda Y,G) \overset{(P_\lambda)}{\cong} L(\lambda(X\times Y),G) \overset{\text{(F)}}{\cong} \mathcal{F}(X\times Y,G)$$

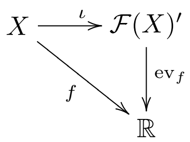
(V) $$\mathcal{F}(X)\otimes G \overset{\text{(F)}}{\cong} \lambda(X)'\otimes G \overset{(V_L)}{\cong} L(\lambda(X),G) \overset{\text{(F)}}{\cong} \mathcal{F}(X,G)$$

(P) $$\mathcal{F}(X)\otimes\mathcal{F}(Y) \overset{\text{(F)}}{\cong} \lambda(X)'\otimes\lambda(Y)' \overset{(P_L)}{\cong}$$
$$\cong (\lambda(X)\otimes_\beta\lambda(Y))' \overset{(P_\lambda)}{\cong} (\lambda(X\times Y))' \overset{\text{(F)}}{\cong} \mathcal{F}(X\times Y).$$

Let us try to construct $\lambda(X)$. Since $\mathcal{F}(X)=\mathcal{F}(X,\mathbb{R})\cong\lambda(X)'$ we have a candidate for the dual of $\lambda(X)$, and hence $\lambda(X)$ should be a subspace of $\lambda(X)''\cong\mathcal{F}(X)'$. Obviously we have a mapping $\iota: X\to\mathcal{F}(X)'$ given by $x\mapsto \mathrm{ev}_x$. So our first problem is to show that ι belongs to $\mathcal{F}$. Recall that for $\mathcal{F}=C^\infty$ and c^∞-complete locally convex spaces we have the following *uniform boundedness principle* 3.56:

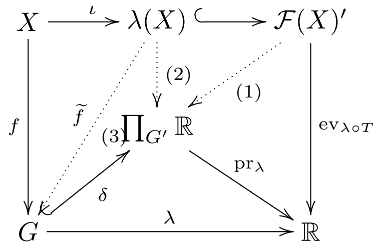
(U1) $$f:X\to L(E,F) \text{ is } \mathcal{F} \iff \mathrm{ev}_x\circ f: X\to F \text{ is } \mathcal{F} \text{ for all } x\in E$$

So let us assume that (U1) is satisfied for the $\mathcal{F}$ under consideration. From the commuting triangle



we conclude using (U1) for $L(\mathcal{F}(X),\mathbb{R})=\mathcal{F}(X)'$ that ι belongs to $\mathcal{F}$. In order to obtain the universal property (F) for scalar valued functions we only have to restrict ev_f to the subspace $\lambda(X)$ which is given by the c^∞-closure of the vector space generated by the image $\{\mathrm{ev}_x : x\in X\}$ of ι.

Now to the general case of G-valued functions, where G is at least c^∞-complete. Since ι belongs to $\mathcal{F}$ we have that $\iota^*: L(\lambda(X),G)\to\mathcal{F}(X,G)$ is well defined and injective since the linear subspace generated by the image of ι is c^∞-dense in $\lambda(X)$ by construction. To show surjectivity consider the following diagram:



Note that (2) has values in $\delta(G)$, since this is true on the ev_x, which generate by definition a c^∞-dense subspace of $\lambda(X)$. Note that this construction of $\tilde{f}$ works for every $f:X\to G$ which is scalarly in $\mathcal{F}$.

Remains to show that this bijection is a bornological isomorphism. In order to show that the linear mapping $\mathcal{F}(X,G) \to L(\lambda(X),G)$ is bounded we can reformulate this equivalently using (E) for L, the universal property of $\lambda(X)$ and (U1) as follows:

$$\begin{aligned} &\mathcal{F}(X,G) \to L(\lambda(X),G) \text{ is } L \\ \overset{(E_L)}{\Longleftrightarrow}\ &\lambda(X) \to L(\mathcal{F}(X,G),G) \text{ is } L \\ \overset{(F)}{\Longleftrightarrow}\ &X \to L(\mathcal{F}(X,G),G) \text{ is } \mathcal{F} \\ \overset{(U1)}{\Longleftrightarrow}\ &X \to L(\mathcal{F}(X,G),G) \overset{\mathrm{ev}_f}{\to} G \text{ is} \end{aligned}$$

and since the later map is f we are done. Another way to see this would be to show that $L(E,F) \subseteq \mathcal{F}(E,F)$ is initial even for $\mathcal{F}$-morphisms and then apply (E) for $\mathcal{F}$ to translate the map $X \to L(\mathcal{F}(X,G),G) \subseteq \mathcal{F}(\mathcal{F}(X,G),G)$ into the identity on $\mathcal{F}(X,G)$, which is a $\mathcal{F}$-map.

Conversely we have to show that $L(\lambda(X),G) \to \mathcal{F}(X,G)$ belongs to L. Composed with $\mathrm{ev}_x : F(X,G) \to G$ this yields the bounded linear map $\mathrm{ev}_{\delta(x)} : L(\lambda(X),G) \to G$. Thus we need the following kind of *uniform boundedness principle* for the function space $\mathcal{F}(X,G)$:

(U2) $\quad T : E \to \mathcal{F}(X,G)$ is $L \iff \mathrm{ev}_x \circ T : E \to G$ is L for all $x \in X$

A Uniform Boundedness Principle

4.4. Lemma. Uniform S-boundedness principle.
Let E be a locally convex space and let $\mathcal{S}$ be a point separating set of bounded linear mappings with common domain E. Then the following conditions are equivalent.

1. *If F is a Banach space (or even a c^∞-complete lcs) and $f : F \to E$ is linear and $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$, then f is bounded.*
2. *If $B \subseteq E$ is absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space E_B generated by B is complete, then B is bounded in E.*
3. *Let (b_n) be an unbounded sequence in E with $\lambda(b_n)$ bounded for all $\lambda \in \mathcal{S}$, then there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the initial locally convex topology induced by $\mathcal{S}$.*

Definition. We say that E satisfies the *uniform $\mathcal{S}$-boundedness principle* if these equivalent conditions are satisfied.

Proof. (1) ⇒ (3) : Suppose that (3) is not satisfied. So let (b_n) be an unbounded sequence in E such that $\lambda(b_n)$ is bounded for all $\lambda \in \mathcal{S}$, and such that for all $(t_n) \in \ell^1$ the series $\sum t_n b_n$ converges in E for the initial locally convex topology induced by $\mathcal{S}$. We define a linear mapping $f : \ell^1 \to E$ by $f((t_n)_n) = \sum t_n b_n$, i.e. $f(e_n) = b_n$. It is easily checked that $\lambda \circ f$ is bounded, hence by (1) the image of the closed unit ball, which contains all b_n, is bounded. Contradiction.

(3) ⇒ (2): Let $B \subseteq E$ be absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and that the normed space E_B generated by B is complete, and suppose that B is unbounded. Then B contains an unbounded sequence (b_n), so by (3) there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the weak topology induced by $\mathcal{S}$. But $\sum t_n b_n$ is a Cauchy sequence in E_B, since $\sum_{k=n}^m t_n b_n \in (\sum_{k=n}^m |t_n|) \cdot B$, and thus converges even bornologically, a contradiction.

(2) ⇒ (1): Let F be convenient, and let $f : F \to E$ be linear such that $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$. It suffices to show that $f(B)$, the image of an absolutely

convex bounded set B in F with F_B complete, is bounded. By assumption $\lambda(f(B))$ is bounded for all $\lambda \in \mathcal{S}$, the normed space $E_{f(B)}$ is a quotient of the Banach space F_B, hence complete. By (2) the set $f(B)$ is bounded. □

4.5. Theorem. Webbed spaces have the uniform boundedness property. *A locally convex space which is webbed satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear functionals.*

Proof. Since the bornologification of a webbed space is webbed, cf. [**5**, 13.3.3 and 13.3.1], we may assume that E is bornological, and hence that every bounded linear functional is continuous, cf. [**5**, 13.3.1]. Now the closed graph principle, cf. [**1**, 5.3.1] applies to any mapping satisfying the assumptions of 1 in 4.4. □

4.6. Lemma. Stability of the uniform boundedness principle. *Let $\mathcal{F}$ be a set of bounded linear mappings $f : E \to E_f$ between locally convex spaces, let $\mathcal{S}_f$ be a point separating set of bounded linear mappings on E_f for every $f \in \mathcal{F}$, and let $\mathcal{S} := \bigcup_{f\in\mathcal{F}} f^*(\mathcal{S}_f) = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{S}_f\}$. If $\mathcal{F}$ generates the bornology and E_f satisfies the uniform $\mathcal{S}_f$-boundedness principle for all $f \in \mathcal{F}$, then E satisfies the uniform $\mathcal{S}$-boundedness principle.*

Proof. We check the condition (1) of 4.4. So assume $h : F \to E$ is a linear mapping for which $g \circ f \circ h$ is bounded for all $f \in \mathcal{F}$ and $g \in \mathcal{S}_f$. Then $f \circ h$ is bounded by the uniform $\mathcal{S}_f$- boundedness principle for E_f. Consequently h is bounded since $\mathcal{F}$ generates the bornology of E. □

Note that the uniform boundedness principles (U1) and (U2) have the following bornological isomorphism as consequence:

(U3) $$L(E, \mathcal{F}(X, G)) \cong \mathcal{F}(X, L(E, G)) \quad \textit{flip of variables.}$$

In fact the mapping and its inverse are given by exchanging the coordinates, $f \mapsto \tilde{f} : (x \mapsto (y \mapsto f(y)(x)))$. For $f \in L(E, \mathcal{F}(X, G))$ we have $\tilde{f}(x) = \mathrm{ev}_x \circ f \in L(E, G)$, since $\mathrm{ev}_x : \mathcal{F}(X, G) \to G$ is bounded. Furthermore $\tilde{f} \in \mathcal{F}(X, L(E, G))$ since $\mathrm{ev}_e \circ \tilde{f} = f(e) \in \mathcal{F}(X, G)$ for all $e \in E$, using the uniform boundedness principle (U1). Conversely for $f \in \mathcal{F}(X, L(E, G))$ we have $\tilde{f}(e) = \mathrm{ev}_e \circ f \in \mathcal{F}(X, G)$, since $\mathrm{ev}_e : L(E, G) \to G$ is bounded and hence in $\mathcal{F}$. Furthermore $\tilde{f} \in L(E, \mathcal{F}(X, G))$ since $\mathrm{ev}_x \circ \tilde{f} = f(x) \in L(E, G)$, using the uniform boundedness principle (U2).

The bijection is bounded in both directions, since this can be tested by applying the uniform boundedness principles (U1) and (U2) and the equation $\mathrm{ev}_x \circ (\_)\tilde{} = (\mathrm{ev}_x)_*$.

On the other hand this isomorphism translates the two uniform boundedness principles into each other: For example $f \in L(E, \mathcal{F}(X, G))$ iff $\tilde{f} \in \mathcal{F}(X, L(E, G))$ and hence by (U1) iff $\tilde{f}(x) \in L(E, G)$ and $f(x) \in F(X, G)$, which are both satisfied by assumption.

Let us now discuss the situations where we have free convenient vector spaces λX, or the two related uniform boundedness principles.

4.7 Examples of free convenient vector spaces and the uniform boundedness principles.

1. *For L the uniform boundedness principles* (U1) *and* (U2) *are just a direct corollary of usual uniform boundedness principle, and of course $\lambda(X) = X$. The flipping isomorphism* (U3) *is $L(E, L(F, G)) \cong L(F, L(E, G))$.*

2. *The dual of* $\mathbb{R}^X$ *is* $\mathbb{R}^{(X)}$ *provided the cardinality of* X *is non-measurable. The evaluations* ev_x *correspond to the unit vectors* $e_x \in \mathbb{R}^{(X)}$*, hence* $\lambda(X) = \mathbb{R}^{(X)} = (\mathbb{R}^X)'$*. The uniform boundedness principle* (U2) *is just the universal property of the product and* (U1) *is trivial. The flipping isomorphism* (U3) : $L(E,G)^X \cong L(E,G^X)$ *is a particular case of the continuity of* $L(E,\_)$*.*
3. *For* C *there exists no* $\lambda(X)$*, a candidate for* $\lambda(X)$ *with locally compact* X *would be the space of Borel-measures on* X *being the dual of* $C(X)$*, however the uniform boundedness principle* (U1) *fails: Take* $X = \mathbb{N}_\infty$*,* $E = \ell^2$*,* $G = \mathbb{R}$ *and* $f : \mathbb{N}_\infty \to E$ *defined by* $f : n \mapsto e_n$ *and* $f(\infty) = 0$*. Then* f *is weakly continuous, but not continuous. Note however that the forgetful functor preserves limits hence is a candidate for a right adjoint. By* [**1**, 7.5.2] *neither* c_0 *nor* L^1 *is a dual of a normed space, hence there exists no Banach space* $\lambda(\mathbb{N}_\infty)$ *with* $C(\mathbb{N}_\infty,\mathbb{R}) = \lambda(\mathbb{N}_\infty)'$*. But since* $\lambda(\mathbb{N}_\infty)$ *is a subspace of* $C(\mathbb{N}_\infty,\mathbb{R})'$ *it has to be normable. However* (U2) *is valid, since it follows from the fact that* $C(K,\mathbb{R})$ *is a Banach space via 4.6.*
4. *For* ℓ^∞ *we have that* $\lambda(X) := \ell^1(X) \subseteq \ell^\infty(X)'$*. Recall that* $\ell^1(X)$ *is by 2.33 equal to the inductive limit of* $\ell^1(B)$ *over all bounded* $B \subseteq X$ *and it is not difficult to show that the* c^∞*-closure of the evaluations in* $\ell^\infty(B)'$ *is just* $\ell^1(B)$*. The boundedness principle* (U1) *is true, since the* ev_x *detect bounded sets. And* (U2) *is true, since* $\ell^\infty(B,\mathbb{R})$ *is a Banach space. The flipping isomorphism* (U3) *is* $\ell^\infty(X, L(E,G)) \cong L(E, \ell^\infty(X,G))$*.*
5. *For* C^∞ *we have* $\lambda(X)$*. And it can be shown that* $\lambda(X)$ *equals the distributions with compact support if* X *is a finite dimensional smooth manifold. No counterexample for* $\lambda(X) = C^\infty(X,\mathbb{R})'$ *is known for infinite dimensional spaces* X*. We already proved that* (U1) *and* (U2) *is true, since* $C^\infty(\mathbb{R},\mathbb{R})$ *is a Fréchet space. The flipping isomorphism* (U3) *is* $C^\infty(X, L(E,F)) \cong L(E, C^\infty(X,F))$*.*
6. *For* $G^{(X)}$ *we cannot apply the discussion above directly, since we have no forgetful functor in this situation. Here a candidate for* $\lambda(X)$ *would be* $\mathbb{R}^X$*, but the continuous linear functions* $\mathbb{R}^X \to G$ *have finite support only for spaces* G *admitting a continuous norm. We have no flipping isomorphism* (U3)*, since for* $X = \mathbb{N}$*,* $E = \mathbb{R}^{(\mathbb{N})}$ *and* $G = \mathbb{R}$ *we have* $L(E,G)^{(X)} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$ *but* $L(E,G^{(X)}) \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$*. However the uniform boundedness principle* (U2) *is true: In fact take a absolutely convex subset* $B \subseteq \coprod_i E_i$*, which is bounded in* $\prod_i E_i$ *and such that* $(\coprod_i E_i)_B$ *is complete. We claim that* B *is contained in some finite subproduct. Otherwise there would be a countable subset* $\mathbb{N}$ *of the index set and* $b^n \in B$ *with* $b^n(n) \neq 0$ *for all* $n \in \mathbb{N}$*. Choose furthermore* $\lambda_n \in E_n'$ *with* $\lambda_n(b^n(n)) = 1$*. Let* $p : \coprod_i E_i \to \mathbb{R}^{(\mathbb{N})}$ *be given by* $p((x_i)_{i\in I}) := (\lambda_n(x_n)_{n\in\mathbb{N}})$*. Then* $p(B) \subseteq \mathbb{R}^{(\mathbb{N})}$ *satisfies the same assumptions as* B*. But* $\coprod_{i\in\mathbb{N}} \mathbb{R}$ *is the strict inductive limit of the finite subproducts, hence is webbed and we may apply the closed graph theorem.*
7. *Since* C_c *is a generalization of the previous item, we have no* λ *here either. However if* Y *is* σ*-compact, then the space* $C_c(Y,\mathbb{R})$ *is webbed, and hence* $C_c(Y,G)$ *satisfies the uniform boundedness principle* (U2)*.*
8. *The same as in the previous item applies here.*

Example.
We consider the space $\ell^\infty(X) := \ell^\infty(X,\mathbb{R})$ as defined in 2.28 for a set X together with a family $\mathcal{B}$ of subsets called bounded. We have the subspace $C_c(X) := \{f \in \ell^\infty(X) : \operatorname{supp} f \text{ is finite}\}$. And we want to calculate its c^∞-closure in $\ell^\infty(X)$.

Claim: The c^∞-closure of $C_c(X)$ equals $C_0(X) := \{f \in \ell^\infty(X) : f|_B \in c_0(B) \text{ for all } B \in B\}$ provided X is countable.

Proof. The right hand side is just the intersection $c_0(X) := \bigcap_{B\in\mathcal{B}} \iota_B^{-1}(c_0(B))$, where $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ denotes the restriction map. We use the notation $c_0(X)$, since in the case where X is bounded this is exactly the space $\{f \in \ell^\infty(X) : \forall\varepsilon > 0\{x : |f(x)| \geq \varepsilon\}$ is finite$\}$. In particular this applies to the bounded space $\mathbb{N}$, where $c_0(\mathbb{N}) = c_0$. Since $\ell^\infty(X)$ carries the initial structure with respect to these maps $c_0(X)$ is closed. It remains to show that $C_c(X)$ is c^∞-dense in $c_0(X)$. So let $f \in c_0(X)$. Let $\{x_1, x_2, \dots\} := \{x : f(x) \neq 0\}$.

We consider first the case, where there exists some $\delta > 0$ such that $|f(x_n)| \geq \delta$ for all n. Then we consider the functions $f_n := f \cdot \chi_{x_1,\dots,x_n} \in C_c(X)$. We claim that $n(f - f_n)$ is bounded in $\ell^\infty(X, \mathbb{R})$. In fact let $B \in \mathcal{B}$. Then $\{n : x_n \in B\} = \{n : x_n \in B$ and $|f(x_n)| \geq \delta\}$ is finite. Hence $\{n(f - f_n)(x) : x \in B\}$ is finite and thus bounded, i.e. f_n converges Mackey to f.

Now the general case. We set $X_n := \{x \in X : |f(x)| \geq \frac{1}{n}\}$ and define $f_n := f \cdot \chi_{X_n}$. Then each f_n satisfies the assumption of the particular case with $\delta = \frac{1}{n}$ and hence is a Mackey limit of a sequence in $C_c(X)$. Furthermore $n(f-f_n)$ is uniformly bounded by 1, since for $x \in X_n$ it is 0 and otherwise $|n(f - f_n)(x)| = n|f(x)| < 1$. So after forming the Mackey adherence (i.e. adding the limits of all Mackey-convergent sequences contained in the set, see 2.36 for a formal definition) twice, we obtain $c_0(X)$. □

2.33 Example. $\mathbf{c_0(X)}$.
We claim that $c_0(X)$ is the c^∞-completion of the subspace $C_c(X)$ in $\ell^\infty(X)$ formed by the finite sequences.
We may assume that the bounded sets of X are formed by those subsets B, for which $f(B)$ is bounded for all $f \in \ell^\infty(X)$. Obviously any bounded set has this property and the space $\ell^\infty(X)$ is not changed by adding these sets. Furthermore the restriction map $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ is also bounded for such a B, since using the closed graph theorem we only have to show that $\mathrm{ev}_b \circ \iota_B = \iota_{\{b\}}$ is bounded for every $b \in B$, see [**1**, 5.3.8], which is obviously the case.

By the previous proposition it is enough to show the universal property for bounded linear functionals. In analogy to Banach-theory, we only have to show that the dual $C_c(X)'$ is just

$$\ell^1(X) := \{g : X \to \mathbb{R} : \operatorname{supp} g \text{ is bounded and } g \text{ is absolutely summable}\}.$$

In fact any such g acts even as bounded linear functional on $\ell^\infty(X, \mathbb{R})$ by $\langle g, f\rangle := \sum_x g(x)\, f(x)$, since a subset is bounded in $\ell^\infty(X)$ iff it is uniformly bounded on all bounded sets $B \subseteq X$. Conversely let $\ell : C_c(X) \to \mathbb{R}$ be bounded and linear and define $g : X \to \mathbb{R}$, by $g(x) := \ell(e_x)$, where e_x denotes the function given by $e_x(y) := 1$ for $x = y$ and 0 otherwise. Obviously $\ell(f) = \langle g, f\rangle$ for all $f \in C_c(X)$. Suppose indirectly $\operatorname{supp} g = \{x : \ell(e_x) \neq 0\}$ is not bounded. Then there exists a sequence $x_n \in \operatorname{supp} g$ and a function $f \in \ell^\infty(X)$ such that $|f(x_n)| \geq n$. In particular the only bounded subsets of $\{x_n : n \in \mathbb{N}\}$ are the finite ones. Hence $\{\frac{n}{|g(x_n)|} e_{x_n}\}$ is bounded in $C_c(X)$ but the image under ℓ is not. Furthermore g has to be absolutely summable, since the set of finite subsums of $\sum_x \operatorname{sign} g(x)\, e_x$ is bounded in $C_c(X)$ and its image under ℓ are the subsums of $\sum_x |g(x)|$.

Thus we should investigate the desired isomorphism (V) (and in particular (P)) for L. Obviously we have a bilinear mapping $E' \times G \to L(E, G)$ and this induces a linear map $\iota : E' \otimes G \to L(E, G)$. So we have to prove firstly that this map is an embedding for some topology on $E' \otimes G$ (which we can always achieve by taking the corresponding initial topology) and that secondly it has dense image. So let us calculate the image first:

4.8 Lemma. Algebraic tensor product as operators.
The image of the algebraic tensor product $E' \otimes G$ *in* $L(E, G)$ *consists exactly of the finite dimensional operators (i.e. those with finite dimensional image).*

Proof. Let $T : E \to G$ have finite dimensional image. Then choose a basis $(g_n)_n$ of $T(E)$ and continuous linear functionals $(\lambda_n)_n$ in G' dual to the g_n. Then $T = \sum_n (\lambda_n \circ T) \cdot g_n$. Conversely the image of $\sum_{n \le N} \lambda_n \otimes g_n$ is obviously contained in $\langle g_n : n \le N \rangle$. □

We have shown in [**1**, 6.4.8] that any limit of finite dimensional operators between Banach spaces is compact. Obviously the identity on a Banach space G is compact only if G is finite dimensional, so $E' \otimes E$ is not dense in $L(E, E)$ for any infinite dimensional Banach space E. Thus for no infinite dimensional Banach space $E = G$ there is a topology τ on the algebraic tensor-product such that

$$\text{(V)} \qquad E' \hat{\otimes}_\tau G \cong L(E, G)$$

is true.

Recall that with respect to the completed projective tensor product (V) is true for $\mathcal{F}(X, \_) := (\_)^X$ with discrete X by 3.28. But it fails with respect to the completed bornological tensor product for $G := \mathbb{R}^{(\mathbb{N})}$ and $X := \mathbb{N}$, since

$$G \hat{\otimes}_\beta \mathcal{F}(\mathbb{N}) = \mathbb{R}^{(\mathbb{N})} \hat{\otimes}_\beta \mathbb{R}^{\mathbb{N}} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})} \ncong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} = \mathcal{F}(\mathbb{N}, G)$$

By 3.38 we have that with respect to the completed bornological tensor product (V) is true for $\mathcal{F}(X, \_) := (\_)^{(X)}$. But it fails with respect to the completed projective tensor product, since

$$\mathcal{F}(\mathbb{N}) \hat{\otimes}_\pi G = \mathbb{R}^{(\mathbb{N})} \hat{\otimes}_\pi \mathbb{R}^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \ncong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})} = \mathcal{F}(\mathbb{N}, G).$$

So we see that the choice of the appropriate tensor topology depends on the function space functor $\mathcal{F}$ under consideration.

Let us now consider (V) (and in particular (P) for $G = \ell^\infty(Y)$) for the function spaces ℓ^∞. Using the free c^∞-complete vector space ℓ^1 this would translate into

$$\ell^1(X)' \otimes_? G \cong \ell^\infty(X) \otimes_? G \overset{\text{(V)}}{\cong} \ell^\infty(X, G) \overset{\text{(F)}}{\cong} L(\ell^1(X), G).$$

But since in particular case, where $X = \mathbb{N}$ and $G = \ell^\infty$, the natural inclusion of $\ell^1 \to \ell^\infty$ is not compact - (the image of) the standard basis is not precompact in ℓ^∞ - in cannot lie in the image under the composite of the completion of any topology on the algebraic tensor product by 4.8. Thus this composite is never onto and hence for $F = \ell^\infty$ neither (V) nor the particular case (P) can be true.

To C, C_c, C^∞ and C^∞_c we will come later.

Injective Tensor Product

Beside the few situations above the projective tensor product is not well suited for function spaces. So we need another topology ε on the algebraic tensor product, such that $F' \otimes_\varepsilon G \to L(F, G)$ is an embedding. We could take this as a definition, but not every locally convex space E is a dual space F'. However, since $L(F, \_)$ preserves embeddings (see below in 4.21), the same should be true for $E \otimes_\varepsilon (\_)$. And since the tensor product should be commutative, we only have to find an embedding of $E \to F'$ for some F and then $E \otimes_\varepsilon G \hookrightarrow F' \otimes_\varepsilon G \hookrightarrow L(F, G)$ should be an embedding. In fact we can take $F = E^*$ with the bornology of equi-continuous sets, see 2.15.

4.21 Lemma. $\ell^\infty(\mathbf{X}, .)$ preserves embeddings.
Let $T : F_1 \to F_2$ be an embedding and X be a bornological space. Then $T_* : \ell^\infty(X, F_1) \to \ell^\infty(X, F_2)$ is an embedding, and if E is a convex bornological space, then $T_* : L(E, F_1) \to L(E, F_2)$ is an embedding.

Proof. Since $L(E, F_i)$ is embedded into $\ell^\infty(E, F_i)$, only the first statement has to be shown. Clearly T_* is injective, provided T is injective: $T \circ f_1 = T_*(f_1) = T_*(f_2) = T \circ f_2$ implies that $f_1 = f_2$. Remains to show that T_* is a homeomorphism onto its image. So let $U \subseteq F_1$ be a 0-neighborhood and $B \subseteq X$ be bounded. Then $N_{B,U}$ is a typical 0-neighborhood in $\ell^\infty(X, F_1)$. By assumption there is some 0-neighborhood $V \subseteq F_2$, such that $T^{-1}(V) \subseteq U$. But then

$$(T_*)^{-1}(N_{B,V}) = \{f : T_*(f) \in N_{B,V}\} = \{f : T(f(B)) \subseteq V\}$$
$$\subseteq \{f : f(B) \subseteq U\} = N_{B,U}. \quad \square$$

Definition.
Thus we consider the bilinear mapping $E \times F \to L(E^*, F)$, given by $(x, y) \mapsto (x^* \mapsto x^*(x)y)$. It is well-defined, since $\mathrm{ev}_x : E^* \to \mathbb{R}$ is bounded. In fact $\mathrm{ev}_x : E^* \to \mathbb{R}$ is even continuous for the topology of uniform convergence on bounded sets, since the set $\{x^* : |x^*(x)| \leq 1\}$ is the polar of the bounded set $\{x\}$ and hence a 0-neighborhood for this topology. This induces a linear map $E \otimes F \to L(E^*, F)$, given by $x \otimes y \mapsto (x^* \mapsto x^*(x)y)$.

We claim that this mapping is injective. In fact take $\sum_i x_i \otimes y_i \in E \otimes F$ with x_i linearly independent. By Hahn-Banach we can find continuous linear functionals x_i^* with $x_i^*(x_j) = \delta_{i,j}$. Assume that the image of $\sum_i x_i \otimes y_i$ is 0 in $L(E^*, F)$. Since it has value y_i on x_i^*, we have that $y_i = 0$ for all i and hence $\sum_i x_i \otimes y_i = 0$.

We define the *injective tensor product* (also called *ε-tensor product* in [**11**]) $E \otimes_\varepsilon F$ to be the algebraic tensor product with the locally convex topology induced by the injective inclusion into $L(E^*, F)$. Since $L(E^*, F)$ is Hausdorff, the same is true for $E \otimes_\varepsilon F$.

Note that, since F embeds into $(F^*)'$ by 2.15, the structure of $E \otimes_\varepsilon F$ is also initial with respect to $E \otimes F \to L(E^*, F) \to L(E^*, (F^*)') \cong L(E^*, F^*; \mathbb{R})$, which gives a more symmetric form. Since the seminorms of $L(E^*, F^*; \mathbb{R})$ are given by suprema on $U^o \times V^o$, where U and V are 0-neighborhoods, we have for the corresponding seminorm $\varepsilon_{U,V}$ on $E \otimes_\varepsilon F$, that

$$\varepsilon_{U,V}\Big(\sum_k x_k \otimes y_k\Big) := \sup\Big\{\Big|\sum_k x^*(x_k)\, y^*(y_k)\Big| : x^* \in U^o, y^* \in V^o\Big\}$$

4.22 Corollary. Seminorms of the injective tensor product.

A defining family of seminorms on $E \otimes_\varepsilon F$ is given by $\varepsilon_{U,V} : \sum_i x_i \otimes y_i \mapsto \sup\{|\sum_i x^*(x_i)\, y^*(y_i)| : x^* \in U^o, y^* \in V^o\}$, where U and V run through the 0-neighborhoods of E and F. The injective tensor product $E \otimes_\varepsilon F$ is metrizable (resp. normable) if E and F are. $\square$

Let us show next, that the canonical bilinear mapping $E \times F \to L(E^*, F)$ is continuous, which implies that the identity $E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.
In fact, take an equi-continuous set $\mathcal{E} \subseteq E^*$, i.e. $\mathcal{E}$ is contained in the polar U^o of a 0-neighborhood U. And take furthermore an absolutely convex 0-neighborhood $V \subseteq F$. Then $U \times V$ is mapped into $\{T : T(\mathcal{E}) \subseteq V\}$, since $(x \otimes y)(x^*) = x^*(x)\, y \in [-1, 1] \cdot V \subseteq V$ for $x^* \in \mathcal{E} \subseteq U^o$.

4.23 Corollary.
$E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.

Proof. In the diagram

$$\begin{array}{ccc} E \otimes_\pi F & \longrightarrow & E \otimes_\varepsilon F \\ {\scriptstyle\otimes}\uparrow & & \downarrow \\ E \times F & \longrightarrow & L(E^*, F) \end{array}$$

continuity of the bilinear map at the bottom implies continuity of the top arrow. □

4.24 Definition.
A space E is called *nuclear* iff $E \otimes_\pi F = E \otimes_\varepsilon F$ for all F. We will come to this later on in more detail. Note that every product of $\mathbb{R}$ is nuclear, since $\mathbb{R}^X \otimes_\pi E$ embeds into $\mathbb{R}^X \hat{\otimes}_\pi \hat{E} \cong \hat{E}^X \cong L(\mathbb{R}^{(X)}, \hat{E}) \cong L((\mathbb{R}^X)^*, \hat{E})$ (the second isomorphism follows from the continuity of $L(\_, \hat{E})$) in which also $L((\mathbb{R}^X)^*, E)$ and thus $\mathbb{R}^X \otimes_\varepsilon E$ embeds.

Note however, that $E \times F \to L(E', F)$ is not continuous, even for $F = \mathbb{R}$, see [**1**, 7.4.20], where E' carries the topology of uniform convergence on bounded sets, and hence has as bounded sets those which are uniformly bounded on bounded sets.

4.25 Proposition.
The injective tensor product is commutative and associative.

Proof. Since the description of 0-neighborhoods in $E \otimes F$ is symmetric, we conclude that $\otimes_\varepsilon$ is commutative. This follows even more directly from the embedding $E \otimes_\varepsilon F \to L(E^*, F^*; \mathbb{R})$. For associativity, we consider the embeddings

$$(E \otimes_\varepsilon F) \otimes_\varepsilon G \cong G \otimes_\varepsilon (E \otimes_\varepsilon F) \hookrightarrow L(G^*, E \otimes_\varepsilon F) \hookrightarrow L(G^*, L(E^*, F)) \cong L(G^*, E^*; F)$$

and

$$\begin{aligned} E \otimes_\varepsilon (F \otimes_\varepsilon G) &\hookrightarrow L(E^*, F \otimes_\varepsilon G) \cong L(E^*, G \otimes_\varepsilon F) \hookrightarrow \\ &\hookrightarrow L(E^*, L(G^*, F)) \cong L(E^*, G^*; F) \cong L(G^*, E^*; F) \quad \square \end{aligned}$$

4.26 Corollary.
The space $E' \otimes_\varepsilon F$ embeds into $L(E, F)$.

Proof. In fact, since $E' \otimes_\varepsilon F \cong F \otimes_\varepsilon E'$ it embeds into $L(F^*, E') \cong L(E, (F^*)')$. This inclusion factors over the embedding $L(E, F) \to L(E, (F^*)')$, by $x^* \otimes y \mapsto (x \mapsto x^*(x)y)$. Hence this map $E' \otimes_\varepsilon F \to L(E, F)$ is an embedding.

$$\begin{array}{ccc} E' \otimes_\varepsilon F & \hookrightarrow & L(E, (F^*)') \\ & \searrow & \uparrow \\ & & L(E, F) \end{array} \quad \square$$

4.27 Proposition.
The injective tensor product is a functor, which preserves injective maps and embeddings.

Proof. That $T_1 \otimes_\varepsilon T_2$ is continuous and thus $\otimes_\varepsilon$ is a functor follows, since $T_1^* : E_2^* \to E_1^*$ is bounded and hence $L(T_1^*, T_2) = (T_2)_* \circ (T_1^*)^* : L(E_1^*, F_1) \to L(E_2^*, F_2)$ is continuous.

$$\begin{array}{ccc} E_1 \otimes_\varepsilon F_1 & \hookrightarrow & L(E_1^*, F_1) \\ {\scriptstyle T_1 \otimes_\varepsilon T_2}\downarrow & & \downarrow{\scriptstyle L(T_1^*, T_2)} \\ E_2 \otimes_\varepsilon F_2 & \hookrightarrow & L(E_2^*, F_2) \end{array}$$

Since $L(E^*, \_)$ preserves injectivity and embeddings, and since $\otimes_\varepsilon$ is commutative the claimed preservation properties follow. □

4.28 Corollary.
Let F_1 and F_2 be topological subspaces of E_1 and E_2. And assume that F_1 or F_2 is nuclear. Then $F_1 \otimes_\pi F_2$ is a topological subspace of $E_1 \otimes_\pi E_2$.

Proof. By 4.27 we have that $F_1 \otimes_\pi F_2 \cong F_1 \otimes_\varepsilon F_2$ is a subspace of $E_1 \otimes_\varepsilon E_2$. Since $F_1 \otimes_\pi F_2 \to E_1 \otimes_\pi E_2 \to E_1 \otimes_\varepsilon E_2$ is continuous, the result follows. □

4.29 Example.
The injective tensor product of quotient maps is not always a quotient map and it also doesn't preserve direct sums.

Proof. The first one follows by taking the tensor product of a quotient mapping $\ell^1 \to \ell^2$ with the identity on ℓ^2. Note that by [**5**, 6.9.4] every Banach space is a quotient space of some $\ell^1(X)$ with bounded X, and every separable Banach space is a quotient of ℓ^1.

The second follows from the example $\mathbb{R}^{(\mathbb{N})} \otimes_\varepsilon \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}^{(\mathbb{N})} \otimes_\pi \mathbb{R}^{\mathbb{N}}$, since $\mathbb{R}^{\mathbb{N}}$ is nuclear. Thus also the strict inductive limit $\varinjlim_{n\in\mathbb{N}} \mathbb{R}^n$ of the sequence $\mathbb{R}^n$ is not preserved. □

4.30 Proposition.
The injective tensor product preserves dense subspaces.

Proof. Let $E_1 \subseteq E_2$ be a dense topological subspace. Then $E_1 \otimes F$ is dense in $E_2 \otimes_\pi F$ and hence a fortiori in $E_2 \otimes_\varepsilon F$. By 4.27 we have that $E_1 \otimes_\varepsilon F$ is a subspace of $E_2 \otimes_\varepsilon F$. □

Remark.
Since $E \otimes_\varepsilon F$ embeds into $L(E^*, F)$ and in turn into the complete space $L(E^*, \hat{F})$, we have that the *completed injective tensor product* $E \hat{\otimes}_\varepsilon F$ is the closure of $E \otimes F$ in $L(E^*, \hat{F})$. Note that by 4.30 we have that

$$E \hat{\otimes}_\varepsilon F \cong \hat{E} \hat{\otimes}_\varepsilon \hat{F}.$$

4.31 Theorem.
The completed injective tensor product preserves products and reduced projective limits.

Proof. Since $\widehat{\prod_j F_j} = \prod_j \hat{F}_j$ (denseness and completeness), we may assume without loss of generality that E and all F_j are complete. The natural mapping $E \hat{\otimes}_\varepsilon \prod_j F_j \to \prod_j E \hat{\otimes}_\varepsilon F$ is induced from the isomorphism

$$L\Big(E^*, \prod_j F_j\Big) \to \prod_j L(E^*, F_j)$$

and hence is an embedding. Since for the algebraic tensor product we have $E \otimes \coprod_j F_j \cong \coprod_j E \otimes F_j$ and both sides are dense in the corresponding complete spaces above, we have an isomorphism.

By the corresponding result for the projective tensor product we have that $E \otimes \varprojlim_j F_j$ is dense in $\varprojlim_j E\hat{\otimes}_\pi F_j$ and hence a fortiori in $\varprojlim_j E\hat{\otimes}_\varepsilon F_j$, which is a subspace of $\prod_j E\hat{\otimes}_\varepsilon F_j \cong E\hat{\otimes}_\varepsilon \prod_j F_j$. Since $\varprojlim_j F_j$ is a subspace in $\prod_j F_j$, we have by 4.27 that $E\hat{\otimes}_\varepsilon \varprojlim_j F_j \to \varprojlim_j E\hat{\otimes}_\varepsilon F_j$ is a embedding and hence an isomorphism.

□

Some Function Spaces

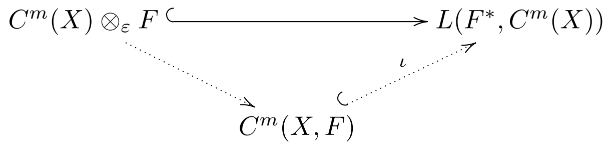
Let us show now that (V) is satisfied for C:

4.35 Theorem.
If F is complete, then we have

$$\text{(V)} \qquad C^m(X)\widehat{\otimes}_\varepsilon F \cong C^m(X,F).$$

provided X is an open subset in some $\mathbb{R}^n$ or $m = 0$ and X is a compactly generated completely regular space.

Proof. We try to factorize the natural embedding $C^m(X) \otimes_\varepsilon F \to L(F^*, C^m(X))$ in the following way:



The right hand side arrow is associated to the bilinear composition map $C^m(X,F) \times F^* \to C^m(X)$, and hence is given by $\iota : f \mapsto (y^* \mapsto y^* \circ f)$. Note that the other embedding $C^m(X) \otimes_\varepsilon F \to L(C^m(X)^*, F)$ cannot be factorized easily. The image $\iota(f)$ belongs to $L(F^*, C^m(X))$, since it maps the equi-continuous set V^o to $\{y^* \circ f : y^* \in V^o\}$, which is bounded in $C^m(X)$, since $\partial^\alpha(y^* \circ f)(x) = y^*(\partial^\alpha f(x))$. Furthermore ι is linear and injective, since F^* separates points of F. It is even a homeomorphic embedding, since a 0-neighborhood basis of $C^m(X,F)$ is given by $N_{p,K,V} := \{f : \partial^\alpha f(K) \subseteq V \text{ for } |\alpha| \le p\}$, where $p \le m$, the set $K \subseteq X$ is compact and $V \subseteq F$ a closed absolutely convex 0-neighborhood. And a 0-neighborhood basis of $L(F^*, C^m(X))$ is given by $N_{p,V^o,K^o} := \{T : |\partial^\alpha(T(y^*))(x)| \le 1 \text{ for } |\alpha| \le p,\ y^* \in V^o \text{ and } x \in K\}$ and $\iota^{-1}(N_{p,V^o,K^o}) = N_{p,V,K}$, since $\partial^\alpha f(K) \subseteq V$ iff for all $y^* \in V^o$ we have that $y^*(\partial^\alpha f(K)) \subseteq [-1, 1]$.

The arrow $C^m(X) \otimes_\varepsilon F \to C^m(X,F)$ on the left hand side is given by $f \otimes y \mapsto (x \mapsto f(x)\,y)$. Composed with the mapping ι from above we obtain the natural inclusion $C^m(X) \otimes_\varepsilon F \to L(F^*, C^m(X))$, given by $f \otimes y \mapsto (y^* \mapsto y^*(y)\,f)$. Hence $C^m(X) \otimes_\varepsilon F \to C^m(X,F)$ is an embedding as well.

We show now the required density properties, first for $m = 0$. So let $f \in C(X,F)$ be given as well as a 0-neighborhood $N_{K,V}$, with $K \subseteq X$ compact and $V = \{y : p(y) \le 1\} \subseteq F$ a 0-neighborhood. By continuity of f and compactness of K we can find a finite covering of K by open sets V_i and points $x_i \in V_i$, such that $p(f(x) - f(x_i)) \le 1$ for all $x \in V_i$. Let (h_i) be a partition of unity on K subordinated to this covering. By Tieze's extension theorem, we may assume that $h_i \in C(X)$. In fact we may extend h_i to a continuous function on the Stone-Čech compactification βX of X

and then restrict it to X. Now take $h := \sum h_i \otimes f(x_i) \in C(X) \otimes F$. Then for $x \in K$ we have $p\big(f(x) - \iota(h)(x)\big) \le \sum_i h_i(x)\, p\big(f(x) - f(x_i)\big) \le 1$, i.e. $\iota(h) \in f + N_{K,V}$.

Now for arbitrary m and open $X \subseteq \mathbb{R}^n$. First note that $C^m_c(X,F)$ is dense in $C^m(X,F)$: In order to see this take a compact set $K \subset X$ and choose a bump-function $h \in C^\infty_c(X,F)$ with $h|_K = 1$. Then for $f \in C^m(X,F)$ we have $h \cdot f \in C^m_c(X,F)$ and $f - h \cdot f \in N_{p,K,V}$ for every p and V. So it is enough to show that $C^\infty_c(X) \otimes F$ is dense in $C^m_c(X,F)$, considered with its inductive limit topology.
For this let $f \in C^m_c(X,F)$ be given. Let $K \subseteq X$ be compact, such that the support of f is contained in the interior of K. The trace of an arbitrary neighborhood of f to $C^m_K(X,F)$ is a neighborhood in $C^m_K(X,F) \subseteq C^m(X,F)$. So it is enough to approximate f in $C^m_K(X,F)$ by elements in $C^\infty_K(X) \otimes F$. By what we have shown for C, we can find $f_j \in C(X) \otimes F$, which converge to f in $C(X,F)$. Let $h \in C^\infty_c(X,F)$ be such that $h|_{\operatorname{supp} f} = 1$ and $H := \operatorname{supp}(h)$ contained in the interior of K. Then $h \cdot f_j \in C_H(X) \otimes F$ converges to $h \cdot f = f$ in $C(X,F)$. In order to achieve convergence of the derivatives, we take *convolution with an approximation of unity* ρ_ε (see [**1**, 4.13.6]). Since $C^m_c(X,F) \subseteq C^m_c(\mathbb{R}^n,F)$, the convolutions $\rho_\varepsilon \star (h \cdot f_j) \in C^m(\mathbb{R}^n) \otimes F$ and $\rho_\varepsilon \star f \in C^m(\mathbb{R}^n,F)$ are well-defined, $\rho_\varepsilon \star f$ converges to f in $C^m(\mathbb{R}^n,F)$ for $\varepsilon \to 0$ and $\rho_\varepsilon \star (h \cdot f_j)$ converges to $\rho_\varepsilon \star f$ in $C^\infty(\mathbb{R}^n,F)$ for $j \to \infty$, since partial derivatives of a convolution can be moved to one factor (see [**1**, 4.7.6]). If we choose ε so small, that $\operatorname{supp}(\rho_\varepsilon) + H \subseteq K$, then $\rho_\varepsilon \star (h \cdot f_j) \in C^\infty_K(X) \otimes F$ and $\rho_\varepsilon \star f \in C^\infty_K(X,F)$. Hence the convergence takes place in $C^m_K(X,F)$.

The proof is now finished, since for complete F the space $C(X,F)$ is complete provided X is compactly generated and the space $C^m(X,F)$ is complete for any open subset X of $\mathbb{R}^n$. Hence the completion $C^m(X) \hat{\otimes}_\varepsilon F$ is isomorphic to the closure $C^m(X,F)$ of $C^m(X) \otimes F$ in $C^m(X,F)$. □

Note that on $C^\infty(X,F)$ the bornology discussed here is identical to that introduced in 2.46. In fact both structures are initial with respect to $\ell_* : C^\infty(X,F) \to C^\infty(X,\mathbb{R})$ for all $\ell \in F^*$ and on $C^\infty(X,\mathbb{R})$ both structures satisfy the uniform $\{\mathrm{ev}_x : x \in X\}$-boundedness principle.

4.36 Corollary.

(P) $\quad C(X \times Y) \cong C(X) \hat{\otimes}_\varepsilon C(Y)$ *for locally compact* X *and* Y.

(P) $\quad C^\infty(X \times Y) \cong C^\infty(X) \hat{\otimes}_\varepsilon C^\infty(Y)$ *for open* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$.

Proof. Under these assumptions we have the exponential law (E) and hence (P) follows from (V). □

Remark.
Very little about (V) and (P) is known for infinite dimensional spaces X and Y.

The corollary fails for C^m with $0 < m < \infty$. In fact we do not have an exponential law in this situation, since for every $f \in C^m(X, C^m(Y))$ the derivative $\partial_1^m \partial_2^m \hat{f}$ exists and is continuous, which is not the case for elements of $C^m(X \times Y)$. So the analogous proof will not work. Moreover from the validity of (P) we could deduce the scalar valued case of the exponential law (E) using (V):

$$\mathcal{F}(X \times Y) \overset{(P)}{\cong} \mathcal{F}(X) \hat{\otimes}_\varepsilon \mathcal{F}(Y) \overset{(V)}{\cong} \mathcal{F}(X, \mathcal{F}(Y)).$$

Note that for C we can not replace the ε-tensor product by the β- or π-tensor product, since we have shown in 4.13 that for $X = Y = \mathbb{N}_\infty$ we don't have equality.

We will show in 6.23 that $C^\infty(X,\mathbb{R})$ is nuclear, so we may replace the ε-tensor product by the π-tensor product. And since both factors are Fréchet also by the β-tensor product.

4.37 Proposition.
Let F be complete. Then

(E) $$\mathcal{S}(\mathbb{R}^n,\mathcal{S}(\mathbb{R}^m,F))\cong\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m,F)$$

(V) $$\mathcal{S}(\mathbb{R}^n)\hat{\otimes}_\varepsilon F\cong\mathcal{S}(\mathbb{R}^n,F)$$

(P) $$\mathcal{S}(\mathbb{R}^n)\hat{\otimes}_\varepsilon\mathcal{S}(\mathbb{R}^m)\cong S(\mathbb{R}^n\times\mathbb{R}^m)$$

Of course $\mathcal{S}(\mathbb{R}^n,F)$ is defined as all $f\in C^\infty(\mathbb{R}^n,F)$ for which $p\cdot\partial^\alpha f$ is globally bounded for all polynomials p on $\mathbb{R}^n$ and all multi-indices α. And we supply this space with the structure inherited from the linear mappings $f\mapsto p\cdot\partial^\alpha f$, from $\mathcal{S}(\mathbb{R}^m,F)$ into $\ell^\infty(\mathbb{R}^m,F)$, where $\mathbb{R}^m$ carries the trivial bornology. Since $\partial^\alpha(p\cdot f)=p\cdot\partial^\alpha f+\sum_{\beta>0}\binom{\alpha}{\beta}\partial^{\alpha-\beta}p\cdot\partial^\beta f$, we can show by induction that we could use equally well the expressions $\partial^\alpha(p\cdot f)$.

Proof. Note that $f\in\mathcal{S}(\mathbb{R}^n,\mathcal{S}(\mathbb{R}^m,F))$, iff for every polynomial p_1 on $\mathbb{R}^n$ and p_2 on $\mathbb{R}^m$ and all multi-indices α and β we have that $x\mapsto(y\mapsto\partial_y^\beta(p_2(y)\cdot\partial_x^\alpha(p_1(x)\cdot f(x))(y)))$ belongs to $\ell^\infty(\mathbb{R}^n,\ell^\infty(\mathbb{R}^m,F))\cong\ell^\infty(\mathbb{R}^n\times\mathbb{R}^m,F)$. This is equivalent to the assumption that $\partial^\gamma(p\cdot\hat{f})\in\ell^\infty(\mathbb{R}^n\times\mathbb{R}^m;F)$ for all γ and all polynomials p on $\mathbb{R}^n\times\mathbb{R}^m$.

The rest of the proof is completely analogous to that for C^∞. For the density use that $C_c^\infty(\mathbb{R}^n,F)$ is dense in $\mathcal{S}(\mathbb{R}^n,F)$ and $C_c^\infty(\mathbb{R}^n)\otimes F$ is dense in $C_c^\infty(\mathbb{R}^n,F)$ by what we have proved for C^∞. □

4.38 Theorem.
If F is complete, then $H(X,F)\cong H(X)\hat{\otimes}_\varepsilon F$ for every open domain $X\subseteq\mathbb{C}$ and complex locally convex space F.

For a proof of this result see [**5**, 16.7.5]. Here $H(X,F)$ denotes the space of all *holomorphic maps* $X\to F$ with the topology of uniform convergence on compact subsets of X.

Remark.
Let us consider C_c^m next for $m=0$ or $m=\infty$.

Note that for X compact we are in the situation of 4.35. So for $m=0$ we can neither use the bornological nor the projective tensor product. So we try again with the injective tensor product.

We try to find an embedding $C_c^m(X,F)\to L(F^*,C_c^m(X))$ as in the situations before. Since $C_c^m(X,F)$ is the inductive limit of $C_K^m(X,F)$, where K runs through a basis of the compact subsets of X and since $C_K^m(X,F)$ carries by definition the initial structure from the inclusion into $C^m(X,F)$, we obtain a continuous linear mapping as follows:

$$\begin{array}{ccccc}
C_c^m(X,F) & \hookleftarrow & C_K^m(X,F) & \hookrightarrow & C^m(X,F)\\
\downarrow{\scriptstyle(3)} & & \downarrow{\scriptstyle(2)} & & \downarrow{\scriptstyle(1)}\\
L(F^*,C_c^m(X)) & \hookleftarrow & L(F^*,C_K^m(X)) & \hookrightarrow & L(F^*,C^m(X)),
\end{array}$$

where (1) was given in the proof for C^m. The map (2) is just the restriction, which exists, since $\operatorname{supp}(f)\subseteq K$ implies that $f(x)=0$ for all $x\notin K$ and hence also

$y^*(f(x)) = 0$ for all $y^* \in F^*$, i.e. $y^* \circ f$ has support in K. The map (3) exists by the universal property of the inductive limit.

On the other hand we have a bounded bilinear mapping $C_c^m(X) \times F \to C_c^m(X, F)$ induced by

$$\begin{array}{ccccccc} C_c^m(X) & \hookleftarrow & C_K^m(X) & \hookrightarrow & C^m(X) & & C^m(X) \otimes F \\ \downarrow{\scriptstyle (3)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (0)} \\ L(F, C_c^m(X,F)) & \hookleftarrow & L(F, C_K^m(X,F)) & \hookrightarrow & L(F, C^m(X,F)) & & C^m(X,F), \end{array}$$

where the right most mapping (0) is the embedding given in 4.35. By the same arguments as before (2) and (3) exist. The associated mapping $C_c^m(X)\hat{\otimes}_\beta F \to C_c^m(X, F)$ clearly has dense image and the composite $C_c^m(X)\hat{\otimes}_\beta F \to C_c^m(X, F) \to L(F^*, C_c^m(X))$ is the natural mapping, which has values in $C_c^m(X)\hat{\otimes}_\varepsilon F$. Thus we conclude that also $C_c^m(X, F) \to L(F^*, C_c^m(X))$ has in values in $C_c^m(X)\hat{\otimes}_\varepsilon F$.

Could this be extended to give us the desired isomorphism (V) : $C_c^m(X)\hat{\otimes}_\beta F \cong C_c^m(X, F)$? This is not the case as the example $X = \mathbb{N}$ shows, since then we have $C_c^m(X, F) = F^{(\mathbb{N})}$ and we have already seen that for the nuclear space $F = \mathbb{R}^\mathbb{N}$ there is no isomorphism $C_c^m(X)\hat{\otimes}_\varepsilon F = C_c^m(X)\hat{\otimes}_\pi F = \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi \mathbb{R}^\mathbb{N} \cong (\mathbb{R}^{(\mathbb{N})})^\mathbb{N} \to (\mathbb{R}^\mathbb{N})^{(\mathbb{N})} = C_c^m(X, F)$.

But we should note that for $m > 0$ we assumed X to be open in some $\mathbb{R}^n$ in 4.35. So what about such a counter-example (in particular for $m = \infty$)? If $X = \mathbb{R}$, then we have a direct summand $F^{(\mathbb{N})} \subset C_c^\infty(\mathbb{R}, F)$ given by $(y_n)_n \mapsto \sum_n h(\_ - n)\, y_n$, where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ has support in $[-1, 1]$ and is equal to 1 at 0. A retraction is given by $f \mapsto (f(n))_{n\in\mathbb{N}}$. That both maps are continuous follows from the following diagram, since the restrictions to the bottom row are obviously continuous:

$$\begin{array}{ccccc} F^{(\mathbb{N})} & \dashrightarrow & C_c^\infty(\mathbb{R}, F) & \dashrightarrow & F^{(\mathbb{N})} \\ \uparrow & & \uparrow & & \uparrow \\ F^N & \longrightarrow & C^\infty_{[-1,N+1]}(\mathbb{R}, F) & \longrightarrow & F^N \end{array}$$

Now suppose we have some functorial topology τ on the tensor product, i.e. such that the tensor product becomes a functor with values in <u>*LCS*</u>. Then an isomorphism $C_c^\infty(X)\hat{\otimes}_\tau F \cong C_c^\infty(X, F)$ would induce an isomorphism $\mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\tau F \cong F^{(\mathbb{N})}$. Taking $F = \mathbb{R}^\mathbb{N}$ shows that this fails for $\tau = \pi = \varepsilon$. Note however, that for $\tau = \beta$ it is true.

What about the weaker statement (P) for $m = \infty$ (i.e. (V) for $F = C_c^\infty(Y)$). We have a quotient mapping $C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \to \mathbb{R}^\mathbb{N}$ given by $f \mapsto (f^{(n)}(0))_{n\in\mathbb{N}}$ (apply the open mapping theorem to the second map). Now suppose $C_c^\infty(\mathbb{R})\hat{\otimes}_\pi C_c^\infty(\mathbb{R}) \cong C_c^\infty(\mathbb{R}^2)$. Then we have the quotient mapping $C_c^\infty(\mathbb{R})\hat{\otimes}_\pi C_c^\infty(\mathbb{R}) \to \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi \mathbb{R}^\mathbb{N} \cong (\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$. This should correspond to a continuous mapping on $C_c^\infty(\mathbb{R}^2)$, whose (n, k)-th coordinate is given by $f \mapsto \partial_2^k f(n, 0)$. In fact $f \otimes g \in C_c^\infty(\mathbb{R}) \otimes_\pi C_c^\infty(\mathbb{R})$ are mapped to $(f(n))_n \otimes (g^{(k)}(0))_k$ and further to $((f(n)\, g^{(k)})_n)_k$. The corresponding map $h \in C_c^\infty(\mathbb{R}^2)$ is given by $h(x, y) = f(x)\, g(y)$ and hence $(f(n)\, g^{(k)}(0)) = \partial_2^k h(n, 0)$. Hence the linear mapping $C_c^\infty(\mathbb{R}^2) \to (\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$ has values in the strict subset $(\mathbb{R}^\mathbb{N})^{(\mathbb{N})}$, a contradiction. Since $C_c^\infty(X)$ is nuclear this shows at the same time that the result fails also for ε.

However, let us show now, that (P) is true for C_c^∞ with respect to the bornological tensor product:

4.39 Proposition.
Let X and Y be open in finite dimensional spaces. Then

(P) $$C_c^\infty(X)\hat{\otimes}_\beta C_c^\infty(Y) \cong C_c^\infty(X \times Y).$$

Proof. Since $C_c^\infty(Z)$ is the inductive limit of $C_K^\infty(Z) \subseteq C^\infty(Z)$, where K runs through a basis of the compact subsets of Z, and since the bornological tensor product preserves inductive limits it is enough to show that $C_A^\infty(X)\hat{\otimes}_\beta C_B^\infty(Y) \cong C_{A\times B}^\infty(X\times Y)$ for all compact subsets $A \subseteq X$ and $B \subseteq Y$. Since $C^\infty(Z)$ are nuclear Fréchet spaces, we have from what we have shown above $C^\infty(X)\hat{\otimes}_\beta C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\pi C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\varepsilon C^\infty(Y) \cong C^\infty(X \times Y)$. So the natural mapping $C_A^\infty(X) \otimes_\varepsilon C_B^\infty(Y) = C_A^\infty(X) \otimes_\beta C_B^\infty(Y) \to C_{A\times B}^\infty(X \times Y)$ is as restriction initial as well. Remains to show denseness of $C_A^\infty(X) \otimes C_B^\infty(Y)$ in $C_{A\times B}^\infty(X\times Y)$. For this we first show that $\bigcup_n C_{K_n}^\infty(Z)$ is dense in $C_K^\infty(Z)$ provided K_n are compact subsets of K such that their interiors cover K.
In fact, let $f \in C_K^\infty(Z)$. Then for all n and m we have that $f^{(n)}(z)$ is a $O(d(z, Z \setminus K)^m)$ for $z \to \partial K$. By [**10**, p.77] we may choose an $h \in C_K^\infty(Z)$ with $h = 1$ on $K_\varepsilon := \{z \in K : d(z, Z \setminus K) \geq \varepsilon\}$ and $h^{(n)}(z)$ is a $O(d(z, Z \setminus K)^{-n})$. Thus $(f \cdot h)^{(n)}(z) = O(d(z, Z))$, and hence is smooth on Z.

Note however, that it is even enough to embed $C_{A\times B}^\infty(X \times Y)$ into some space $C_{A'}^\infty(X)\hat{\otimes}_\varepsilon C_{B'}^\infty(Y)$, which is much easier to obtain. □

Kernel Theorems

4.40. We take up the discussion about the appropriate version of the *matrix-representation* of linear-operators
$$L(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{n\cdot m}.$$
We should replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by more general (function) spaces E and F. So we have to rewrite the right hand side in terms of $\mathbb{R}^n$ and $\mathbb{R}^m$, i.e.
$$\mathbb{R}^{n\cdot m} \cong \mathbb{R}^n \otimes \mathbb{R}^m.$$
Note that the left side is a functor on $\underline{VS}^{op} \times \underline{VS}$ and the right side on $\underline{VS} \times \underline{VS}$, so we have to dualize E on one side.

The simplest generalization seems to be from $\mathbb{R}^n = \mathbb{R}^{\{0,\dots,n-1\}}$ to $\mathbb{R}^X$ with arbitrary X. For the projective or injective tensor product we have $\mathbb{R}^X\hat{\otimes}_\pi\mathbb{R}^Y \cong (\mathbb{R}^Y)^X \cong L(\mathbb{R}^{(X)}, \mathbb{R}^Y)$. Recall that $\mathbb{R}^{(X)} = (\mathbb{R}^X)^*$. Hence we have $E\hat{\otimes}_\pi F \cong L(E^*, F)$, where $E = \mathbb{R}^X$ and $F = \mathbb{R}^Y$. But we can read this also as an isomorphism for $X \mapsto \mathbb{R}^{(X)}$, since $(\mathbb{R}^{(X)})' \cong \mathbb{R}^X$: $E'\hat{\otimes}_\pi F' \cong L(E, F')$, where $E = \mathbb{R}^{(X)}$ and $F = \mathbb{R}^{(Y)}$. Which seems more appropriate, since for function spaces E often $E \subseteq E'$ (e.g. $C_c^\infty \subseteq (C_c^\infty)'$, $\ell^1 \subseteq \ell^\infty$), and we are mainly interested in the operators $L(E, E)$.

Let us consider the corresponding result for $\mathcal{L}^p$ with $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (and hence $1 \leq q < \infty$):
$$\mathcal{L}^p(X \times Y) \cong \mathcal{L}^p(X, \mathcal{L}^p(Y)) \to L(\mathcal{L}^q(X), \mathcal{L}^p(Y)),$$
which is an isomorphism for $p = \infty$ and discrete X, but not otherwise, since the image consists of compact operators only, cf. [**1**, 6.4.8], where we proved this result for $p = q = 2$. Note that this mapping is given for discrete X and Y by $(k_{x,y})_{x,y} \mapsto \Big((f_x)_x \mapsto (\sum_{x\in X} k_{x,y} f_x\Big)$. Or, in general, k is mapped to the *integral operator* $K : f \mapsto (y \mapsto \int_X k(x,y)\, f(x)\, dx)$. So the question of surjectivity amounts to finding an *integral kernel* k for an operator K.

The existence of such a mapping for discrete X and Y can also be seen directly, since by Schwarz's Inequality we have

$$\begin{aligned}\|Kf\|_p^p &= \sum_y |\sum_x k(x,y)f(x)|^p = \sum_y |\langle k(\_,y), f\rangle|^p \\ &\le \sum_y \|k(\_,y)\|_p^p \cdot \|f\|_q^p = \sum_y \sum_x |k(x,y)|^p \cdot \|f\|_q^p.\end{aligned}$$

4.41 Remark.
We have shown in [**1**, 6.4.8] that $C(I \times I) \to L(C(I), C(I))$ is a well defined map with values in the compact operators for every interval I.

Furthermore we have the mapping

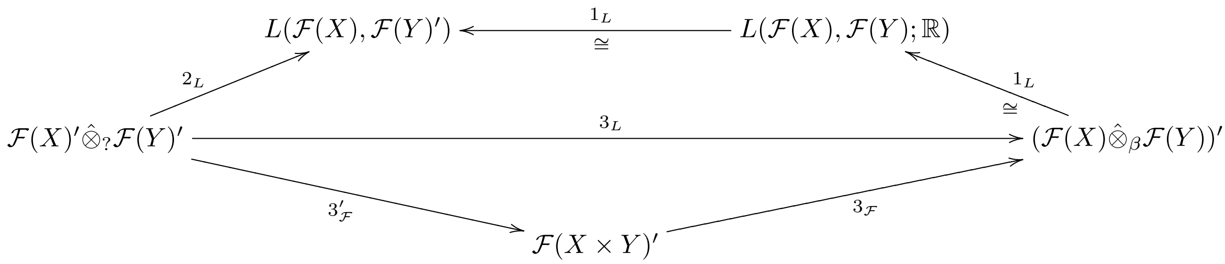
$$\mathcal{L}^p(X \times Y) \to L(\mathcal{L}^q(X), \mathcal{L}^p(Y)).$$

So one wants to extend this to some surjective mapping on some function space $\mathcal{F}(X \times Y)$. I.e. for every $K \in L(\mathcal{L}^q(X), \mathcal{L}^p(Y))$ there should be some "kernel" $k \in \mathcal{F}(X \times Y)$. This problem is unsolvable for functions. In fact take $p = 2$ and $X = Y = \mathbb{R}$ with the Lebesgue-measure. The kernel of the identity would be the Dirac delta function. Due to [**8**, 1966] it can be worked out for distributions. For this we rewrite the above mapping into

$$\mathcal{L}^q(X \times Y)' \to L(\mathcal{L}^q(X), \mathcal{L}^q(Y)').$$

This problem has shown to be of importance, in fact we constructed to every partial differential operator D with constant coefficients, and more generally to every continuous linear operator $D : C_c^\infty(\mathbb{R}^n) =: \mathcal{D} \to E := C^\infty(\mathbb{R}^n)$, which commutes with translations an integral-kernel $k \in \mathcal{D}'$, such that D is given by convolution with k, in [**1**, 4.13.5]. Moreover we found a solution operator of such equations as integral-operator with a distributional kernel ε in [**1**, 8.3.1]

Now, how could we show such an isomorphism:



Remark.
If one writes the action of a *distribution* T on a *test-function* f formally as an integral $Tf = \int_X T(x)\, f(x)\, dx$, then the mapping $\mathcal{D}(X \times Y)' \to L(\mathcal{D}(X), \mathcal{D}(Y)')$ is given by $k \mapsto (f \mapsto (g \mapsto \int_{X\times Y} k(x,y)\, f(x)\, g(y)\, d(x,y)))$. Conversely we now know that every continuous linear operator $K : D(X) \to \mathcal{D}(Y)'$ is of this form, i.e. has an distributional kernel $k \in \mathcal{D}(X \times Y)$. This is a strong generalization of the matrix representation of finite dimensional operators.

We will show that $C^\infty(X)$, $\mathcal{S}(\mathbb{R}^p)$, $H(X)$ and $\mathcal{D}(X)$ are nuclear spaces and all except the last one are Fréchet. Hence

4.42 Corollary. (P) **for several function spaces.**

$$C^\infty(X)\hat{\otimes}_\varepsilon C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\pi C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\beta C^\infty(Y) \cong C^\infty(X\times Y)$$
$$\mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\varepsilon \mathcal{S}(\mathbb{R}^q) \cong \mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\pi \mathcal{S}(\mathbb{R}^q) \cong \mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\beta \mathcal{S}(\mathbb{R}^q) \cong \mathcal{S}(\mathbb{R}^p\times\mathbb{R}^q)$$
$$H(X)\hat{\otimes}_\varepsilon H(Y) \cong H(X)\hat{\otimes}_\pi H(Y) \cong H(X)\hat{\otimes}_\beta H(Y) \cong H(X\times Y)$$
$$\mathcal{D}(X)\hat{\otimes}_\varepsilon \mathcal{D}(Y) \cong \mathcal{D}(X)\hat{\otimes}_\pi \mathcal{D}(Y) \overset{!!!}{\ncong} \mathcal{D}(X)\hat{\otimes}_\beta \mathcal{D}(Y) \cong \mathcal{D}(X\times Y) \qquad \square$$

Combined with (E) for L this gives:

4.43 Theorem, Schwartz kernel theorem.
We have

$$C^\infty(X\times Y)' \cong L(C^\infty(X), C^\infty(Y)')$$
$$\mathcal{S}(\mathbb{R}^p\times\mathbb{R}^q)' \cong L(\mathcal{S}(\mathbb{R}^p), \mathcal{S}(\mathbb{R}^q)')$$
$$H(X\times Y)' \cong L(H(X), H(Y)')$$
$$\mathcal{D}(X\times Y)' \cong L(\mathcal{D}(X), \mathcal{D}(Y)')$$

Proof. For $\mathcal{D}$ we have to proceed differently:

$$\mathcal{D}(X\times Y)' \overset{4.39}{\cong} (\mathcal{D}(X)\hat{\otimes}_\beta \mathcal{D}(Y))' \cong L(\mathcal{D}(X),\mathcal{D}(Y);\mathbb{R}) \cong L(\mathcal{D}(X),\mathcal{D}(Y)'). \quad \square$$

Little is know about the validity of desired isomorphisms for C^∞ and λ in the infinite dimensional case. See [**7**] for partial results in the case of C^∞.

The Approximation Property

We turn now towards the question of density of the image of $E^*\otimes F$ in $\mathcal{L}(E,F)$.

4.44 Theorem. Density of finite dimensional operators.
Let E be a locally convex space and $\mathcal{B}$ be a bornology on E. And we consider on all function spaces $\mathcal{L}$ the uniform convergence on sets in $\mathcal{B}$, and hence denote them by $\mathcal{L}_\mathcal{B}$. Then the following statements are equivalent:

1. *$E^*\otimes F$ is dense in $\mathcal{L}_\mathcal{B}(E,F)$ for every locally convex space F;*
2. *$E^*\otimes F$ is dense in $\mathcal{L}_\mathcal{B}(E,F)$ for every Banach space F;*
3. *$E^*\otimes E$ is dense in $\mathcal{L}_\mathcal{B}(E,E)$;*
4. *id_E is a limit in $\mathcal{L}_\mathcal{B}(E,E)$ of a net in $E^*\otimes E$.*

Proof. (1⇒2) is trivial.
(2⇒1) A typical 0-neighborhood in $\mathcal{L}_\mathcal{B}(E,F)$ is given by $N_{B,V}$ with $B\in\mathcal{B}$ and V a 0-neighborhood in F. Let $p_V: F\to F_{(V)}$ be the canonical surjection. Since $F_{(V)}$ is a normed space $p_V\circ T: E\to F\to F_{(V)}\hookrightarrow \hat{F}_{(V)}$ can be uniformly approximated with respect to $p: F_{(V)}\to\mathbb{R}$ on B by finite operators $E\to\hat{F}_{(V)}$ by (2). Since $F_{(V)}$ is dense with respect to p in $\hat{F}_{(V)}$ we may assume that the finite operators belong to $\mathcal{L}(E,F_{(V)})$. Taking inverse images of the vector components, we may even assume that they belong to $\mathcal{L}(E,F)$.
(1⇒3) and (3⇒4) are trivial.
(4⇒1) Let T_i be a net of finite operators converging to id_E, then the net $T\circ T_i$ of finite operators converges to $T\circ\mathrm{id}=T$. $\square$

Let E be complete and assume that the equivalent statements are true for some bornology $\mathcal{B}$. And let $B \in \mathcal{B}$ w.l.o.g. be absolutely convex. Since the identity on E can be approximated uniformly on B by finite operators, we conclude that the inclusion $E_B \to E$ can be approximated by finite operators $E_B \to E$ uniformly on the unit ball of E_B. Hence it has to have relatively compact image on the unit ball, i.e. B has to be relatively compact:

In fact we have

4.45 Lemma.
The compact operators $\mathcal{K}(E,F)$ from a normed space E into a complete space F are closed in $\mathcal{L}(E,F)$.

Proof. To see this use that $F = \varprojlim_V \hat{F}_{(V)}$, hence a subset K of F is relatively compact iff $p_V(K)$ is relatively compact in $F_{(V)}$ for all V. Now let $T_i \in \mathcal{K}(E,F)$ converge to $T \in \mathcal{L}(E,F) = L(E,F)$. Then the $p_V \circ T_i \in \mathcal{K}(E, \hat{F}_{(V)})$ converge to $p_V \circ T$ in $L(E, \hat{F}_{(V)})$. Since $\hat{F}_{(V)}$ is a Banach spaces it can be shown as in [**1**, 6.4.8] that $p_V \circ T \in \mathcal{K}(E, \hat{F}_{(V)})$. Hence $p_V(T(oE))$ is relatively compact in $\hat{F}_{(V)}$ and thus $T(oE)$ is relatively compact in F. □

4.46 Definition.
We hence say that a complete space satisfies the *approximation property* if the equivalent statements from above are true for the bornology $B = \mathrm{cp}$ of all relatively compact subsets of E. A non-complete space E is said to have the *approximation property*, iff $\hat{E}$ has it. Note that the finite dimensional operators may be taken in $\mathcal{L}(E,E)$ in this situation.

A space E is said to have the *bornological approximation property*, iff $E^* \otimes F$ is dense in $L(E,F)$ with respect to the bornological topology, which is at least as fine as the topology of uniform convergence on bounded sets. So a necessary condition is that all bounded sets are relatively compact. A space with that property is called *semi-Montel* space. It is called *Montel*, iff it is in addition barreled. It is a classical result of P. Montel that every bounded sequence of holomorphic maps has a convergent subsequence, i.e. $H(X)$ is Montel for every domain $X \subseteq \mathbb{C}$. By Tychonoff's theorem $\mathbb{R}^X$ is semi-Montel for every X.

Reflexivity and Montel Spaces

Recall that a space is called *semi-reflexive*, iff the natural mapping $E \to (E^*)^*$ is onto, where E^* is considered with the strong topology. A space is called *reflexive*, iff the natural mapping $E \to (E^*)^*$ is an isomorphism for the strong topology on $(E^*)^*$. This is exactly the case when E is semi-reflexive and (infra-)barreled, since a space E is (quasi-)barreled iff each pointwise (uniformly) bounded set in $\mathcal{L}(E,F)$ is equi-continuous, see [**1**, 5.2.2], see also [**5**, 11.2.2].

One has the following

4.47 Lemma.
The strong dual of a semi-reflexive space is barreled.

Proof. See [**11**, 373]. Let E^*_β denote the strong dual of E and let B be a barrel in E^*_β. Since E is semi-reflexive the strong topology is compatible with the duality, and hence [**1**, 7.4.8] B is also closed for the weak-topology $\sigma(E^*,E)$. We show that B^o is a bounded subset of E (which implies that $B = B^{oo}$ is a 0-neighborhood in

E^*_β). For this it is enough to show that B^o is bounded in $\sigma(E, E^*)$. But since B is assumed to be absorbing, we find for every $x^* \in E^*$ a $c > 0$ with $cx^* \in B$. Thus $|cx^*(B^o)| \leq 1$. □

4.48 Proposition. Semi-reflexivity.
The following statements are equivalent:

1. *E is semi-reflexive;*
2. *Every closed bounded set is $\sigma(E, E^*)$-compact;*
3. *E is quasi-complete with respect to $\sigma(E, E^*)$.*

Proof. (1⇒2) Since E is as a vector space the dual of the barreled space E^*_β by the previous lemma it follows that every $\sigma(E, E^*)$-bounded set is equi-continuous and hence relatively compact for the topology $\sigma(E, E^*)$.

(2⇒3) Since every compact space is complete for any compatible uniformity this is obvious.

(3⇒1) We only have to show that the strong topology is compatible with the duality $\langle E^*, E\rangle$. By [**1**, 7.4.15] we must show that this topology coincides with the topology of uniform convergence on weakly compact sets. But since all bounded sets are weakly relatively-compact this is obvious. □

4.49 Proposition. (U1) **for** C**.**
Let X be compactly generated and E be semi-Montel and $f : X \to E$ be scalarly continuous. Then f is continuous.

Proof. Since X is compactly generated, it is enough to show that $f|_K : K \to E$ is continuous for every compact subset $K \subseteq X$. So let $x_i \to x$ be a convergent net in K. Then $f(x_i) \to f(x)$ with respect to the weak-topology, and since $f(K)$ is scalarly bounded, it is bounded, and hence is relatively compact. But on compact sets the weak and the given topology obviously coincide. So $f(x_i) \to f(x)$ in E. □

We have the implications:

4.50 Proposition.
semi-Montel ⇒ semi-reflexive ⇒ quasi-complete ⇒ sequentially complete ⇒ c^∞-complete.

Proof. (semi-Montel ⇒ semi-reflexive) Since every closed bounded set is relatively compact it is also relatively $\sigma(E, E^*)$-compact and hence E is semi-reflexive.

(semi-reflexive ⇒ quasi-complete) Since $\sigma(E, E^*)$ is quasi-complete by 4.48 it follows from [**5**, 3.2.4] that every bounded set is complete.

The other implications are clear. □

One has the following stability properties:

4.51 Proposition. Stability of reflexive and of Montel spaces.
Semi-reflexive and semi-Montel spaces are closed with respect to products, closed subspaces, direct sums, reduced regular inductive limits. Strong duals of reflexive and of Montel spaces are of the same type.

4.52. Definition

A locally convex vector space E is called *bornologically-reflexive* if the canonical embedding $\delta : E \to E''$ is surjective.

It is then even a bornological isomorphism, since δ is always a bornological embedding. Note that reflexivity as defined here is a bornological concept.

Note also that this notion is in general stronger than the usual locally convex notion of reflexivity, since the continuous functionals on the strong dual are bounded functionals on E' but not conversely.

4.53. Theorem. Bornological reflexivity.
For a bornological locally convex space E the following statements are equivalent.

1. *E is bornologically-reflexive.*
2. *E is reflexive and the strong dual of E is bornological.*
3. *E is η-reflexive (see* [**5**, p280]*).*
4. *E is completely reflexive (see* [**4**, 1977, p89]*).*
5. *The Schwartzification (or nuclearification) of E is a complete locally convex space.*

Proof. See [**3**, 5.4.6]. □

4.54. Corollary. Bornological reflexivity versus reflexivity.

1. *A Fréchet space is b-reflexive if and only if it is reflexive.*
2. *A convenient vector space with a countable base for its bornology is b-reflexive if and only if its bornological topology is reflexive.*

Proof. See [**3**, 5.4.7]. □

4.55. Proposition. Duals of bornologically reflexive spaces.
A locally convex vector space is b-reflexive if and only if its bornological topology is complete and its dual is b-reflexive.

Proof. See [**3**, 5.4.9]. □

4.56. Lemma. Subspaces of bornologically reflexive spaces.
A closed linear subspace of a b-reflexive bornological locally convex vector space is b-reflexive. Products and coproducts of b-reflexive convenient vector spaces are b-reflexive, if the index set is of non-measurable cardinality.

Proof. See [**3**, 5.4.8] and [**3**, 5.4.11] □

4.57. Theorem. Reflexivity of function spaces.
If E is a b-reflexive convenient vector space and M is a finite dimensional separable smooth manifold then $C^\infty(M, E)$ is b-reflexive.

Proof. See [**3**, 5.4.13]. □

4.59 Proposition. Stability of the approximation property.
The approximation property is preserved by products, complemented subspaces, reduced projective limits, direct sums, strict inductive limits of sequences of complete spaces and injective tensor products,

Proof. (Products) Every compact set $K \subseteq E = \prod_j E_j$ is contained in one of the form $\prod_j K_j$ with $K_j \subseteq E_j$ compact. A 0-neighborhood $U \subseteq E$ can be assumed to be of the form $\prod_j U_j$, with 0-neighborhoods $U_j \subseteq E_j$ and $U_j = E_j$ for all but finitely many j. For those finitely many j, we may find finite operators $T_j \in \mathcal{L}(E_j, E_j)$ with $(\mathrm{id}_{E_j} - T_j)(K_j) \subseteq U_j$. Then $T := \sum_j T_j \circ \mathrm{pr}_j \in \mathcal{L}(E, E)$ is finite dimensional with $(\mathrm{id}_E - T)(K) \subseteq U$.

Note that if $\mathcal{U}$ is a 0-neighborhood basis of closed absolutely convex sets, such that $E_{(U)}$ have the approximation property, then E has it, see [**5**, 18.2.2]. In fact we may assume that E is complete. Then E is the reduced projective limit of $\hat{E}_{(U)}$, and hence has the approximation property.

(Complemented subspaces) Let $E \subset F$ be a subspace admitting a continuous projection $p : F \to E$. Taking the completion, we may assume that E and F are complete. Let $K \subseteq E$ be compact and U a 0-neighborhood of E. Then there is a finite operator $T \in \mathcal{L}(F, F)$ with $(\mathrm{id}_F - T)(K) \subseteq p^{-1}(U)$. Then $(\mathrm{id}_E - (p \circ T)|_E)(K) \subseteq p(p^{-1}U) \subseteq U$.

(Projective limits) Let E be a reduced projective limit of E_j, we may assume that all spaces E_j and E are complete. Let $K \subseteq E$ be compact and U a 0-neighborhood in E. Since the limit is projective, we may assume that it is of the form $\mathrm{pr}_k^{-1}(U_k)$ for some k and 0-neighborhood U_k in E_k. Since the limit is reduced, $F_k := \mathrm{pr}_k(E)$ is dense in E_k and hence has the approximation property. In particular there exists a finite operator $T \in \mathcal{L}(F_k, F_k)$ such that $(\mathrm{id}_{F_k} - T)(\mathrm{pr}_k(K)) \subseteq U_k$. We may assume that T is of the form $T = \sum_j y_j^* \otimes \mathrm{pr}_k(x_j)$. Then $\tilde{T} := \sum_j (y_j^* \circ \mathrm{pr}_k) \otimes x_j$ is a finite operator in $\mathcal{L}(E, E)$, which satisfies $(\mathrm{id}_E - \tilde{T})(K) \subseteq U = \mathrm{pr}_k^{-1}(U_k)$.

(Inductive limits) By [**1**, 4.8.1] we know that such a limit is regular, and hence in particular every compact set K is contained and compact in some step E_k. Let U be a 0-neighborhood. Then we can find finite operators $T = \sum_j x_j^* \otimes x_j \in \mathcal{L}(E_k, E_k)$, such that $(\mathrm{id}_{E_k} - T)(K) \subseteq U$. Since E_k is a subspace of E we may assume that $x_j^* \in E^*$, hence $T \in \mathcal{L}(E, E)$.

(Direct sums) Let $E = \coprod_j E_j$. Then $\hat{E}$ is the direct sum of the $\hat{E}_j$, so we may assume that E_j is complete. Every compact subset of E is contained in some finite subsum. Since E is the strict inductive limit of the finite subsums and being products these have the approximation property, we may proceed as before to conclude that E has it.

(Injective tensor product) See [**5**, 18.2.8], this uses the associativity of the ε-product to be discussed later, see 4.68-4.71. □

4.60 Lemma. Topology on equicontinuous sets.
On equi-continuous subsets of $\mathcal{L}(E, F)$ the topology τ_{pc} of uniform convergence on precompact subsets of E and the topology of pointwise convergence coincide.

Proof. Let $H \subseteq \mathcal{L}(E, F)$ be equi-continuous and $T \in H$. Let $K \subseteq E$ be precompact and $V \subseteq F$ an absolutely convex 0-neighborhood of F. Since H is equi-continuous, there exists a 0-neighborhood $U \subseteq E$ with $S(U) \subseteq \frac{1}{2}V$ for all $S \in H$. Since K is precompact there is some finite subset $M \subseteq E$ such that $K \subseteq M + \frac{1}{2}U$. If $S \in H \cap (T + N_{2M,V})$, then $Su \in \frac{1}{2}V$ for all $u \in U$ and $(S-T)(x) \subseteq \frac{1}{2}V$ for all $x \in M$. Thus for all $k = x + \frac{1}{2}u \in K$ we have $(S - T)(k) = (S - T)(x) + \frac{1}{2}S(u) - \frac{1}{2}T(u) \in \frac{1}{2}V + \frac{1}{4}V + \frac{1}{4}V = V$, i.e. $S \in T + N_{K,V}$. □

4.61 Alaoğlu-Bourbaki Theorem.
Every equi-continuous set is relatively compact for the topology τ_{pc} of uniform convergence on precompact sets.

Proof. By 4.60 we only have to show that it is relatively compact for the topology $\sigma(E^*, E)$. Since $(E^*, \sigma(E^*, E)$ embeds into $\mathbb{R}^E$, and equi-continuous sets are pointwise bounded (see [**1**, 5.2.2]), it is bounded in $\mathbb{R}^E$ as well as its closure and hence is relatively compact there by Tychonoff's theorem. However the closure in $\mathbb{R}^E$ of an equi-continuous set is easily seen to be contained in E^*. □

4.62 Examples with the approximation property.
The following spaces have the approximation property:

1. *every complete space with an equi-continuous basis;*
2. c_0 *and* ℓ^p *for* $1 \le p < \infty$*;*
3. *every Hilbert space;*
4. $\mathcal{L}^p(X, \mathcal{A}, \mu)$ *for* $1 \le p \le \infty$*;*
5. $C(X)$ *for completely regular* X*;*
6. $C^k(X)$ *for open subsets* X *of finite dimensional spaces.*

Proof. (1) A space E is said to have an *equi-continuous basis*, if there are points $x_k \in E$ such that every x admits a unique representation $x = \sum_k \lambda_k x_k$ and the family of expansion operators $P_k : x \mapsto \sum_{j \le k} \lambda_k x_k$ is equi-continuous. Note that P_k is finite dimensional and $P_k \to \mathrm{id}_E$ pointwise. By 4.60 this equi-continuous family converges uniformly on precompact subsets, i.e. on relatively compact subsets since E is complete and hence the compact subsets are exactly the closed precompact ones.

(2) It is straight forward to show that the standard unit-vectors e_k form an equi-continuous base.

(3) Let x_i be an orthonormal basis in a Hilbert space. Then the projection operators $P_F(x) := \sum_{i \in F} \langle x, x_i \rangle x_i$ for finite F converge pointwise to the identity and are equi-continuous. Hence by 4.60 E has the approximation property.

(4) We skip the proof of this, see [**5**, p411].

(5) Since the completion $\widehat{C(X)}$ of $C(X)$ is the reduced projective limit of the spaces $C(K)$, with $K \subseteq X$ compact (use that $C(X) \to C(K)$ is onto for compact subsets $K \subseteq X$). It suffices to show that $C(X)$ has the approximation property for compact X. Let $\varepsilon > 0$ and let $K \subseteq C(X)$ be compact, thus by Arzela-Ascoli-theorem [**1**, 6.4.4] K is equi-continuous. Thus we can find a finite cover of X by open neighborhoods $U_j \subseteq X$ of some $x_j \in X$ such that $|f(x) - f(x_j)| \le \varepsilon$ for all $x \in U_j$ and $f \in K$. Let h_i be a subordinated partition of unity and set $T(f) := \sum_j f(x_j)\, h_j$. We claim that $(\mathrm{id}_E - T)(K) \subseteq U := \{f : \|f\|_\infty \le \varepsilon\}$. For $f \in K$ and $x \in X$ we have

$$|f(x) - T(f)(x)| \le \sum_j |f(x) - f(x_j)|\, h_j(x) = \sum_{x \in \mathrm{supp}(h_j) \subseteq U_j} |f(x) - f(x_j)|\, h_j(x)$$
$$\le \sup\{|f(x) - f(x_j)| : x \in U_j\} \le \varepsilon.$$

(6) This can be proved analogously to (5) using smooth partitions of unity. For $k = \infty$ we will give another proof in 6.23 together with 6.19. □

4.63 Remark.
For a long time it was unclear if there are spaces without the approximation property at all. It was known that, if such a Banach space exists, then there has to

be a subspace of c_0 failing this property. It was [**2**] who found a subspace of c_0 without this property. In [**9**] it was shown that $L(\ell^2,\ell^2)\cong L(\ell^2,(\ell^2)^*)\cong(\ell^2\hat{\otimes}_\pi\ell^2)^*$ doesn't have the approximation property. Note also, that $\ell^2\otimes_\pi\ell^2$ has the approximation property, since by [**5**, 18.2.9] every completed projective tensor product of Fréchet spaces with the approximation property has it. Note however that for Banach spaces one can show that if E^* has the approximation property then so does E, see [**5**, 18.3.5]. Due to [**4**] is the existence of a Fréchet-Montel space without the approximation property, see [**5**, p416].

We try to identify $E\hat{\otimes}_\varepsilon F$ as subspace of $L(E^*,F)$, and hence in particular, for $F=\mathbb{R}$, we try to find $E\hat{\otimes}_\varepsilon\mathbb{R}=\hat{E}$ in $L(E^*,\mathbb{R})$.

4.64 Grothendieck's completeness criterion.
The completion of E can be identified with $\mathcal{L}_{equi}(E^*_\gamma,\mathbb{R})$, where E^*_γ carries the finest locally convex topology which coincides with the weak topology on equi-continuous sets.

Proof. We note that the embedding $\delta:E\to(E^*)'$ factors over $L(E^*_\gamma,\mathbb{R})\subseteq(E^*)'$, since $\delta(x)$ is obviously continuous for $\sigma(E^*,E)$. Furthermore $\mathcal{L}(E^*_\gamma,\mathbb{R})$ is clearly closed in the complete space $(E^*)'$. So it remains to show that E is dense in $\mathcal{L}(E^*_\gamma,\mathbb{R})$. For this we use the following lemma. So let $\ell\in\mathcal{L}(E^*_\gamma,\mathbb{R})$ be given and a typical 0-neighborhood, which is of the form A^o with equi-continuous A. Since $\ell|_A$ is by assumption continuous with respect to $\sigma(E^*,E)$ we may apply 4.65 to obtain a $x\in(E^*,\sigma(E^*,E))^*=E$ with $|(x-\ell)(x^*)|\le 1$ for all $x^*\in A$. Hence $x-\ell\in A^o$. □

4.65 Lemma.
Let $A\subseteq E$ be absolutely convex and $\ell:E\to\mathbb{R}$ be linear. Then $\ell|_A$ is continuous iff for every $\varepsilon>0$ there exists an $x^*\in E^*$ with $|(x^*-\ell)(x)|\le\varepsilon$ for all $x\in A$.

Proof. ($\Leftarrow$) is clear. ($\Rightarrow$) Since $\ell|_A$ is continuous there exists a 0-neighborhood U such that $|\ell(x)|\le\varepsilon$ for all $x\in U\cap A$. Let q_U and q_A be the Minkowski-functionals of U and A. Then $\ell\le\varepsilon(q_U+q_A)$ on E_A. Define

$$p(x):=\inf\{\varepsilon\,q_U(x-y)+\varepsilon\,q_A(y)+\ell(y):y\in E_A\}.$$

Then p is well-defined, since for all $(x,y)\in E\times E_A$

$$\begin{aligned}-\varepsilon\,q_U(x)&\le\varepsilon\,q_U(-y)+\varepsilon\,q_A(-y)-\ell(-y)-\varepsilon\,q_U(x)\\&=\varepsilon\,q_U(y)+\varepsilon\,q_A(y)+\ell(y)-\varepsilon\,q_U(x)\\&\le\varepsilon\,q_U(x-y)+\varepsilon\,q_A(y)+\ell(y).\end{aligned}$$

Since p is sublinear there exists by [**1**, 7.1.1] a linear $x^*:E\to\mathbb{R}$ with $x^*\le p$. From $p(x)\le\varepsilon\,q_U(x)$ for all $x\in E$ it follows that $x^*\in E^*$. And from $p(y)\le\ell(y)+\varepsilon\,q_A(y)$ for all $y\in E_A$ we conclude that $(x^*-\ell)(y)\le\varepsilon$ for all $y\in A$. Thus $(\ell-x^*)(y)=(x^*-\ell)(-y)\le\varepsilon$ for all $y\in A$.
In the complex case use that $\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})\cong\mathcal{L}_{\mathbb{R}}(E,\mathbb{R})$, see [**1**, 6.1.5.2]. □

4.66 Corollary.
We have $E^*_\gamma=:\gamma(E^*,E)=\tau_c(E^*,\hat{E}):=\mathcal{L}_{cp}(\hat{E},\mathbb{R})$.

Proof. First note that $\gamma(E^*,E)=\gamma(E^*,\hat{E})$. In fact, since the closures $\hat{U}$ in $\hat{E}$ of the 0-neighborhoods U in E form a 0-neighborhood basis of $\hat{E}$, the equi-continuous families on E and on $\hat{E}$ coincide. Furthermore the topologies $\sigma(E^*,E)$ and $\sigma(E^*,\hat{E})$ coincide on equi-continuous subsets. Thus it is enough to prove the result for complete spaces E.

($\gamma \supseteq \tau_c$) Let us show first that γ is finer than τ_c. For this we only have to show that the inclusion from equi-continuous sets with the weak topology $\sigma(E^*, E)$ into $\tau_c(E^*, E) = \tau_{pc}(E^*, E)$ is continuous, which follows directly from 4.60.

($\tau_c \supseteq \gamma$) Conversely let U be a closed 0-neighborhood for γ. Since by 4.64 γ is compatible with the duality (E^*, E) we have that U^o is compact for the topology of uniform convergence on γ-precompact sets in E^*. Since every closed equi-continuous set for the original topology is by definition of γ and because of 4.61 compact with respect to γ, we have that U^o is also compact for this weaker topology of uniform convergence on equi-continuous subsets. But this is just the given topology on E, so U^o is compact, and hence $U = U^{oo}$ is a 0-neighborhood for the topology τ_c. □

4.67 Corollary. "Kelley-fication" of the completion.
The space $(E^*_\gamma)^*_\gamma$ has the same compact subsets as $\hat{E}$ and carries the final locally convex topology with respect to these subsets. If $\hat{E}$ is compactly generated, and hence in particular if E is metrizable, then we have equality.

Proof. Since by 4.66 the 0-neighborhoods in E^*_γ coincide with the 0-neighborhoods in $\tau_c(E^*, \hat{E})$, we have that the equi-continuous sets in $(E^*_\gamma)^*$ coincide with the subsets of polars of 0-neighborhoods in $\tau_c(E^*, \hat{E})$ and hence are just the subsets of compact sets in $\hat{E}$ (use that the bipolar of a compact set in $\hat{E}$ is compact). By the definition of σ the topology $\hat{E}$ is finer than $\sigma(\hat{E}, E^*_\gamma)$ and hence they coincide on compact subsets of $\hat{E}$. □

4.68 Proposition. Approximation property versus ε-product.
A complete space E has the approximation property iff $F \otimes_\varepsilon E$ is dense in the ε-PRODUCT *$F \,\varepsilon\, E := \mathcal{L}_{equi}(F^*_\gamma, E)$ for every locally convex space F.*

Proof. Note that $F \otimes E$ is mapped into $\mathcal{L}(F^*_\gamma, E)$, since for $y \in F$ we have $\delta(y) \in (F^*_\gamma)^*$ by 4.64.

($\Leftarrow$) Consider the following commuting diagram:

$$\begin{array}{ccc} E^*_\gamma \otimes E & \longrightarrow & \mathcal{L}_{cp}(E,E) \\ & \searrow \qquad \swarrow & \\ & \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, E) & \end{array}$$

By assumption the arrow on the left hand side has dense image. The arrow on the right hand side is an embedding, since $(E^*_\gamma)^*_\gamma \to E$ is a continuous mapping, and the equi-continuous subsets in $(E^*_\gamma)^*_\gamma$ are exactly the relatively compact subsets of $\hat{E} = E$.

($\Rightarrow$) Let $T \in \mathcal{L}(F^*_\gamma, E)$ and let a 0-neighborhood $N_{V^o,U}$ in this space be given. Since T is continuous on V^o, we have that $K := T(V^o)$ is compact in E. By assumption $E^* \otimes E$ is dense in $\mathcal{L}_{cp}(E, E)$. Hence there exists a finite operator $S \in \mathcal{L}(E, E)$ with $(\mathrm{id}_E - S)(K) \subseteq U$. Then $S \circ T : F^*_\gamma \to E \to E$ is finite dimensional and since $(F^*_\gamma)^* = \hat{F}$ by 4.64 it belongs to $\hat{F} \otimes E$ and $(T - ST)(V^o) = (1 - S)(K) \subseteq U$. Thus $T - ST \in N_{V^o,U}$. Hence $\hat{F} \otimes_\varepsilon E$ is dense in $L(F^*_\gamma, E)$ and since $F \otimes E$ is dense in $\hat{F} \otimes_\varepsilon E$ it is also dense in $\mathcal{L}(F^*_\gamma, E)$. □

4.69 Corollary.
Let E be complete and satisfying the approximation property, then $F \hat{\otimes}_\varepsilon E = F \,\varepsilon\, E$.

Proof. Recall that $F \varepsilon E = \mathcal{L}_{equi}(F^*_\gamma, E)$ is the subspace of $L(F^*, E)$ formed by all linear maps $T : F^* \to E$, which are continuous on equi-continuous subsets of F^* with respect to the weak-topology $\sigma(F^*, F)$ on F^*. It is easily checked that for complete E this space is complete, cf. [**5**, 16.1.5]. So $F \hat{\otimes}_\varepsilon E$ is the closure of $F \otimes E$ in $L(F^*, E)$, which is by 4.68 exactly $F\varepsilon E$. □

4.70 Lemma.
For complete spaces E and F we have $F \varepsilon E \cong E \varepsilon F$.

Proof. We only have to show bijectivity, since $F \varepsilon E = \mathcal{L}_{equi}(F^*_\gamma, E) \subseteq L(F^*, E)$ embeds into the space $L(F^*, E^{*\prime}) \cong L(F^*, E^*; \mathbb{R})$. To every continuous $T : F^*_\gamma \to E$ we associate the continuous $T^* : E^*_\gamma \to (F^*_\gamma)^*_\gamma$ (in fact every equi-continuous set U^o of E^* is mapped to $T^*(U^o) = \{x^* \circ T : x^* \in U^o\} \subseteq \{y^* : y^* \in (T^{-1}(U))^o\}$, the polar of a 0-neighborhood in F^*_γ). And by 4.64 we are done since by the lemma above the identity $(F^*_\gamma)^*_\gamma \to L_{equi}(F^*_\gamma, \mathbb{R}) = \hat{F}$ is continuous. □

4.71 Remark.
It can be shown (see [**5**, 16.2.6]) that for complete spaces also associativity of the ε-product is valid, i.e.

$$E \varepsilon (F \varepsilon G) \cong (E \varepsilon F) \varepsilon G.$$

4.72 Property (V) **for** L.
Remains to find situations where E^*_γ coincides with E^*_β. By 4.66 this topology γ coincides with the topology $\tau_c(E^*, \hat{E})$ of uniform convergence on compact subsets of $\hat{E}$, which is for metrizable spaces by [**5**, 9.4.3] identical to the topology of uniform convergence on precompact subsets of E. Thus if E is complete and all bounded sets are precompact (like in Montel spaces) it coincides with the strong topology. Remains to find situations, where the equi-continuous subsets coincide with the bounded ones in E^*_β. This is exactly the case, when E is infra-barreled.

4.73 Proposition.
If E and F are complete, E is Montel and F (or E) satisfies the approximation property, then

$$E \hat{\otimes}_\varepsilon F \cong \mathcal{L}_{equi}(E^*_\gamma, F) \cong \mathcal{L}_\beta(E^*_\beta, F),$$

For complete spaces E and F we have under the indicated assumptions the following version of (V)*:*

$$\begin{aligned} E \hat{\otimes}_\varepsilon F \overset{app.P.}{\cong} E \varepsilon F &= \mathcal{L}_{equi}(E^*_\gamma, F) \\ &\overset{semi\text{-}Montel}{=} \mathcal{L}_{equi}(E^*_\beta, F) \overset{infra\text{-}barreled}{=} \mathcal{L}_b(E^*_\beta, F) \\ &\overset{E^*_\beta \ bornological}{=} L(E^*_\beta, F) \end{aligned}$$

Proof. The first statement follows by what we said above, since Montel spaces are barreled.

Note that the strong dual of a semi-reflexive space is barreled [**5**, 11.4.1]. If E is in addition metrizable, then by [**5**, 13.4.4] E^* is bornological, and hence we have

$$\mathcal{L}_\beta(E^*_\beta, F) = L(E', F). \quad \square$$

4.74 Proposition.
For complete spaces E^*_β and F we have the original version of (V) *under the following conditions*

$$E^*_\beta\hat{\otimes}_\varepsilon F \overset{app.prop.}{\cong} E^*_\beta\,\varepsilon\,F = \overset{E\ Montel}{=} \mathcal{L}_b((E^*_\beta)^*_\beta, F) \overset{E\ reflexive}{=} \mathcal{L}_b(E,F) = \overset{E\ bornological}{=} L(E,F),$$

Proof. This follows, since the strong dual E^*_β of a Montel space E is Montel. Note that a Montel-space E is reflexive, i.e. $(E^*_\beta)^*_\beta = E$. Furthermore E^*_β is complete, provided E is bornological. □

Now let us consider $E^*\hat{\otimes}_\varepsilon F$. If F is complete and satisfies the approximation property, then $E^*_\gamma\hat{\otimes}_\varepsilon F \cong \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, F)$. By Grothendieck's completeness criterion we have $\hat{E} \cong \mathcal{L}_{equi}(E^*_\gamma, \mathbb{R})$.

The following result can be found in [**5**, 16.1.7]:

4.75 Theorem.
One has the following natural isomorphisms for Fréchet spaces E and F:

$$(E\hat{\otimes}_\pi F)^*_\gamma \cong E^*_\gamma\varepsilon F^*_\gamma \qquad E\hat{\otimes}_\pi F \cong (E^*_\gamma\varepsilon F^*_\gamma)^*_\gamma$$
$$(E\varepsilon F)^*_\gamma \cong E^*_\gamma\hat{\otimes}_\pi F^*_\gamma \qquad E\varepsilon F \cong (E^*_\gamma\hat{\otimes}_\pi F^*_\gamma)^*_\gamma$$

Proof. Note that the isomorphisms on the right follow from the ones on the left by applying $(\_)^*_\gamma$ and using that $(G^*_\gamma)^*_\gamma \cong \hat{G}$ for all metrizable spaces G.

(1) In fact $(E\hat{\otimes}_\pi F)^* = L(E,F;\mathbb{R})$ and

$$E^*_\gamma\varepsilon F^*_\gamma = \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, F^*_\gamma) = L_{cp}(E, \mathcal{L}_{cp}(F,\mathbb{R})).$$

Since $T : E\times F \to \mathbb{R}$ is continuous, iff it is continuous on compact sets, and hence iff $\check{T} : E \to \mathcal{L}_{cp}(F,\mathbb{R})$ is continuous, we obtain a bijection. That this is a homeomorphism follows since γ is the topology of uniform convergence on compact sets, and the compact sets in $E\hat{\otimes}_\pi F$ are given by bipolars of tensor-products of two compact sets in E and F.

For the second pair of isomorphisms see [**5**, 16.1.7] □

The Approximation Property for Banach Spaces

For Banach spaces E, F etc. we have $E^* = E'$ and we consider on E^* the operator-norm topology induced by that of E'. Moreover $\mathcal{L}(E,F) = L(E,F)$.

4.76 Proposition. Compact operators as tensor product.
For Banach spaces E and F one has

$$E^*_\beta\varepsilon F \cong \mathcal{K}(E,F)$$

Proof. By completeness we have $E^*_\beta\varepsilon F = F\varepsilon E^*_\beta = \mathcal{L}_{equi}(F^*_\gamma, E^*_\beta)$. Remains to show that $T \mapsto T^*$ is a isomorphism $\mathcal{K}(E,F) \to \mathcal{L}_{equi}(F^*_\gamma, E^*_\beta)$. If T is compact, then $T(oE)$ is relatively compact in F and hence $(T^*)^{-1}(o(E^*)) = T(oE)^o$ is a 0-neighborhood in the topology $\tau_{cp}(F^*, F) = \gamma$, i.e. $T^* \in \mathcal{L}(F^*_\gamma, E^*_\beta)$. Conversely assume that $T^* : F^*_\gamma \to E^*_\beta$ is continuous. Then the set $T(oE)^o = (T^*)^{-1}(o(E^*))$ is a 0-neighborhood in $\gamma(F^*, F) = \tau_{cp}(F^*, F)$, and hence $T(oE)$ is contained in a compact subset of F. So $T \mapsto T^*$ is a bijection. That it is a homeomorphism follows immediately since $\{T^* : T \in N_{oE,U}\} = N_{U^o,oE^*}$. □

4.77 Proposition. Approximation property and compact operators.
For a Banach space E one has that:

1. *E has the approximation property iff $F^* \otimes E$ is dense in $\mathcal{K}(F, E)$ for every Banach space F, i.e. $F^* \hat{\otimes}_\varepsilon E = \mathcal{K}(F, E)$.*
2. *E^* has the approximation property iff $E^* \otimes F$ is dense in $\mathcal{K}(E, F)$ for every Banach space F, i.e. $E^* \hat{\otimes}_\varepsilon F = \mathcal{K}(E, F)$.*

Recall that for Hilbert spaces E we have shown in [**1**, 6.4.8] that $E^* \otimes E$ is dense in $\mathcal{K}(E, E)$.
Moreover one can show that in (1) and (2) it is enough to have denseness for all closed subspaces of c_0 or all reflexive separable Banach spaces.

Proof. ($\Rightarrow$) If E or F^* have the approximation property then $F^* \hat{\otimes}_\varepsilon E \cong F^* \varepsilon E$ by 4.68 and $F^* \varepsilon E \cong \mathcal{K}(F, E)$ by 4.76.

($\Leftarrow$) Since in 4.68 it is enough to have denseness for all Banach spaces (see [**5**, 18.1.8]), this is true for the second statement. For the first one has to proceed more carefully, see [**5**].

For a proof of the second part see [**5**, 18.3.2]. □

4.78 Lemma.
For Banach spaces E and F we have a natural surjective linear map $\iota : F^* \hat{\otimes}_\pi E \to L_{cp}(E, F)^*$, where L_{cp} denotes L with the topology of uniform convergence on compact sets in E.

Proof. The map ι is associated to the bounded multi-linear composition map $F^* \times E \times L(E, F) \to \mathbb{R}$, hence is a well defined continuous map $F^* \hat{\otimes}_\pi E = F^* \hat{\otimes}_\beta E \to L(E, F)^*$ given by $y^* \otimes x \mapsto (T \mapsto y^*(Tx))$.

Its image is contained in $L_{cp}(E, F)^*$, since every $z \in F^* \hat{\otimes}_\pi E$ can be written as $z = \sum_k \lambda_k y^*_k \otimes x_k$ with $\lambda \in \ell^1$, $\|y^*_k\| \to 0$ and $\|x_k\| \to 0$, see [**11**, 15.6.4]. In other words $(\lambda_k y_k)_k \in \ell^1\{F^*\}$ and $(x_k)_k \in c_0\{E\}$. Without loss of generality we may assume $\sum_k \|\lambda_k y^*_k\| \le 1$ (move some factor to the x_k). By [**1**, 6.4.3] the closed absolutely convex hull K of the x_n is compact. Thus $N_{K,oF}$ is a 0-neighborhood in $L_{cp}(E, F)$. Let $T \in N_{K,oF}$. Then $|\iota(z)(T)| \le \sum_k |\lambda_k y^*_k(T(x_k))| \le \sum_k \|\lambda_k\, y^*_k\| \le 1$, i.e. $\iota(z) \in N^o_{K,oF}$.

Conversely let $\varphi \in L_{cp}(E, F)^*$. Then $\varphi \in (N_{K,oF})^o$ for some compact $K \subseteq E$. By [**1**, 6.4] we may assume that K is contained in the closed absolutely convex hull of some sequence $x_n \to 0$ in E. Consider the Banach space $c_0(\mathbb{N}, F) = c_0\{F\}$ with the supremum norm $\sup\{\|x_n\| : n \in \mathbb{N}\}$. Then $\psi : L(E, F) \to c_0(\mathbb{N}, F)$ given by $T \mapsto (T(x_n))_n$ is continuous and linear. Hence its dual is $\psi^* : \ell^1\{F^*\} = \ell^1(\mathbb{N}, F^*) \overset{!}{\cong} c_0(\mathbb{N}, F)^* \to L(E, F)^*$ (see [**5**, p405] for the duality) is continuous for the weak*-topologies. Thus the absolutely convex set $K_1 := \psi^*(o(\ell^1\{F^*\})) \subseteq L(E, F)^*$ is compact for this topology. We claim that $\varphi \in K_1$. Otherwise, by Hahn-Banach there is a $T \in L(E, F)$ with $\varphi(T) > 1$ and $|\varphi_1(T)| \le 1$ for all $\varphi_1 \in K_1$, i.e. $\sum_k |y^*_k(Tx_k)| \le 1$ for all $(y^*_k)_k \in o(\ell^1\{F^*\})$. In particular $|y^*(Tx_k)| \le 1$ for all $y^* \in o(F^*)$ and all k, i.e. $T \in N_{K,oF}$ and hence $|\varphi(T)| \le 1$, a contradiction.

Since $\varphi \in K_1 = \psi^*(o(\ell^1\{F^*\}))$ there is some $(y_k^*)_k \in o(\ell^1\{F^*\})$ with $\varphi = \psi^*((y_k^*)_k)$. Now $\sum_k y_k^* \otimes x_k \in F^* \hat{\otimes}_\pi E$ and

$$\psi^*((y_k^*)_k)(T) = \sum_k y_k^* T(x_k) = \iota(\sum_k y_k^* \otimes x_k)(T) \text{ for all } T \in L(E,F),$$

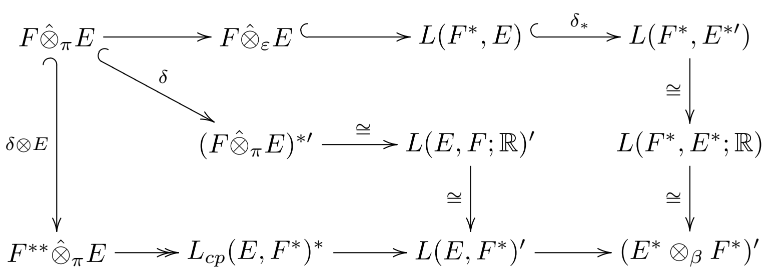
i.e. $\psi^*((y_k^*)) = \iota(\sum_k y_k^* \otimes x_k)$. □

4.79 Proposition. Approximation property and tensor products.

For a Banach space E the following properties are equivalent:

1. *E has the approximation property.*
2. *The map $F \hat{\otimes}_\beta E = F \hat{\otimes}_\pi E \to F \hat{\otimes}_\varepsilon E \subseteq L(F^*, E)$ is injective for every Banach space F.*
3. *The map $F^* \hat{\otimes}_\beta E = F^* \hat{\otimes}_\pi E \to F^* \hat{\otimes}_\varepsilon E \subseteq L(F, E)$ is injective for every Banach space F.*
4. *The map $E^* \hat{\otimes}_\beta E = E^* \hat{\otimes}_\pi E \to E^* \hat{\otimes}_\varepsilon E \subseteq L(E, E)$ is injective.*
5. *The evaluation map $\mathrm{ev} : E^* \times E \to \mathbb{R}$ extends to a linear functional $\mathrm{Tr} : \mathcal{N}(E,E) \to \mathbb{R}$, where $\mathcal{N}(E,E)$ denotes the image of $E^* \hat{\otimes}_\pi E$ in $L(E,E)$.*

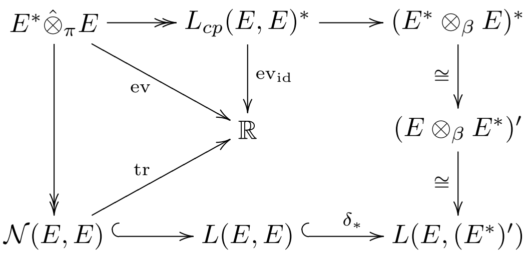
Proof. (1⇒2) Consider the following commuting diagram:



The arrow $L(E, F^*)' \to (F^* \otimes_\beta E^*)'$ at the bottom is well-defined, since $E^* \times F^* \to L(E, F^*)$ is bounded.
Now start with z_0 in the top-left hand corner and assume it is mapped to 0 in $F \hat{\otimes}_\varepsilon E$. So it is mapped to 0 in the bottom-right hand corner. Since the composite of the last two arrows at the bottom is injective, because $E^* \otimes F^*$ is dense in $L_{cp}(E, F^*)$ by (1), it is mapped to 0 in $L_{cp}(E, F^*)^*$ and hence also in $L(E, F^*)'$. By the injectivity of the diagonal maps we conclude that $z_0 = 0$.

(2⇒3⇒4⇒5) are trivial.

(5⇒1) For this we consider the following commuting diagram:



Note that the second arrow on the top is well-defined, since the mapping $E^* \times E \to L_b(E,E) \to L_{cp}(E,E)$ is bounded, and the top-triangle commutes, since for $z = x^* \otimes x$ we have $\mathrm{ev}(z) = \mathrm{ev}(x^* \otimes x) = x^*(x) = x^*(\mathrm{id}(x)) = \iota(x^* \otimes x)(\mathrm{id}) = \varphi(\mathrm{id})$. We have to show that $E^* \otimes E$ is dense in $L_{cp}(E,E)$. For this it is enough to show that all $\varphi \in L_{cp}(E,E)^*$ which vanish on $E^* \otimes E$ vanishes on id_E. By [**5**, 18.3.3] every such φ is in the image of some $z = \sum_{n=1}^\infty x_n^* \otimes x_n \in E^* \hat{\otimes}_\pi E$, i.e. $\varphi = \iota(z)$. For all $(x^*, x) \in E^* \times E$ we have $0 = \varphi(x^* \otimes x) = \iota(\sum_n x_n^* \otimes x_n)(x^* \otimes x) = \sum_n x_n^*((x^* \otimes x)(x_n)) = \sum_n x_n^*(x) \cdot x^*(x_n) = x^*\Big(\sum_n x_n^*(x) x_n\Big)$. Thus the image of z in the top-right corner is 0, and hence also in the bottom-right corner. Since the bottom arrows are injective it is 0 in the bottom-left corner. Hence its image in the center is 0, which is exactly $\varphi(\mathrm{id})$.

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