# Nonlinear Functional Analysis I, SS 1993

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This manuscript is the outcome of a lecture course given at the University of Vienna in the Summer Semester 1993. So I want to use this opportunity to thank all those who attended these lectures and inspired me with the feedback I got from them. In particular I want to thank Cornelia Vizman who posed well selected and highly relevant questions after reading parts of my manuscript. My special thanks go to Konni Rietsch, who not only strongly influenced the selection of the covered topics but also sacrificed a huge amount of time during her holidays and lots of energy in order to make sense out of a preliminary version of these lecture notes. This way she supplied me with an extensive list of misprints, Germanisms, and imprecise or even incorrect mathematical formulations. All the remaining (and newly inserted) faux pas are of course all my own responsibility. And, as always, I explicitly ask the readers not only to pardon them but also to inform me about anything which sounds weird including possibly missing definitions and explanations.

Thank you all in advance,

Andreas Kriegl, August 1993

In the second edition an extensive list of misprints and corrections provided by Eva Adam has been taken gratefully into account.

Andreas Kriegl, September 1994

After some minor corrections I ported to source to LaTeX. Note that chapter 1 and 2 are outdated, since they have been incorperated into the book [**15**].

Andreas Kriegl, Jänner 2006

Essentially two topics from non-linear functional analysis will be treated. Firstly calculus will be extended from finite dimensions (or from Banach spaces, depending on the readers background) to general locally convex spaces. Secondly tensor-products will be discussed and their relationship to multi-linear mappings and to function spaces will be investigated. Of course these two topics are closely related to one another. Just note that the derivatives of smooth maps are multi-linear, and the spaces of smooth functions can be analyzed using various tensor products.

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# 1. Motivation

## Equations on Function Spaces

### 1.1

It should be unnecessary to convince the reader, that differential calculus is an important tool in mathematics. But probably some motivation is necessary why one should extend it to infinite dimensional spaces. One of our main tasks as mathematicians is, like it or not, to solve equations like

$$f(u) = 0.$$

However quite often one has to consider functions $f$ which don't take (real) numbers as arguments $u$ but functions. Let us just mention *differential equation*s, where $f$ is of the following form

$$f(u)(t) := F(t, u(t), u'(t), \dots, u^{(n)}(t)).$$

Note that this is not the most general form of a differential equation, consider for example the function $f$ given by $f(u) = u' - u \circ u$, which is not treated by the standard theory. If the arguments $t$ of $u$ are (real) numbers, then this is the general form of an *ordinary differential equation*. In the generic case one can solve the implicit equation $f(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0$ with respect to $u^{(n)}(t)$ and hence obtains an equation of the form

$$u^{(n)}(t) = g(t, u(t), u'(t), \dots, u^{(n-1)}(t)).$$

By substituting $u_0(t) := u(t), u_1(t) := u^{(1)}(t), \dots, u_{n-1}(t) := u^{(n-1)}(t)$ one obtains a (vector valued) equation

$$\begin{aligned}
u_0'(t) &= u_1(t)\\
u_1'(t) &= u_2(t)\\
&\vdots\\
u_{n-2}'(t) &= u_{n-1}(t)\\
u_{n-1}'(t) &= g(t, u_0(t), \dots, u_{n-1}(t))
\end{aligned}$$

And if we write $\mathbf{u} := (u_0, \dots, u_{n-1})$ and

$$\mathbf{g}(t, \mathbf{u}) := (u_1(t), \dots, u_{n-1}(t), g(t, u_0(t), \dots, u_{n-1}(t))),$$

we arrive at a *ordinary differential equation of order 1*

$$\mathbf{u}'(t) = \mathbf{g}(t, \mathbf{u}(t)).$$

So we are searching for a solution $\mathbf{u}$ of the equation $u' = G(u)$, where $G(u)(t) := \mathbf{g}(t, \mathbf{u}(t))$. The usual general existence and uniqueness results of an equation all have the requirement, that the domain and the range space are the same or at least isomorphic. Recall, for example, that in order to solve an equation $0 = g(u)$ one often transforms it into a fixed point equation $u = u - g(u)$ and then tries to apply some fixed point theorem. So we need that $u \mapsto u' - G(u)$ is a selfmapping. In order to apply it to a function $u$, we need that $u$ is 1-times differentiable, but in order that the image $u' - G(u)$ is 1-times differentiable, we need that $u$ is twice differentiable. Inductively we come to the conclusion that $u$ should be smooth. So are there spaces of smooth functions, to which we can apply some fixed point theorem?

## Spaces of Differentiable Functions

### 1.2

In [**2**, 3.2.5] we have shown that the space $C(X, \mathbb{R})$ of continuous real-valued functions on $X$ is a Banach-space with respect to the supremum-norm, provided $X$ is compact. Recall the proof goes as follows: If $f_n$ is a Cauchy-sequence, then it converges pointwise (using that the point-evaluations $\mathrm{ev}_x = \delta_x : C(X, \mathbb{R}) \to \mathbb{R}$ are continuous linear functionals), by the triangle inequality the convergence is uniform and by elementary analysis (e.g. see [**13**, 104.2]) a uniform limit of continuous functions is continuous.

If $X$ is not compact, one can nevertheless consider the linear restriction maps $C(X, \mathbb{R}) \to C(K, \mathbb{R})$ for compact subsets $K \subseteq X$ and then use the initial structure on $X$, given by the seminorms $f \mapsto \|f|_K\|_\infty$, where $K$ runs through some basis of the compact sets. If $X$ has a countable basis of compact sets, then we obtain a countable seminormed space. If we try to show completeness, we get a function $f$, which is on compact sets the uniform limit of the Cauchy-sequence $f_n$, and hence is continuous on these sets. If $X$ is *Kelley* (= *compactly generated*, i.e. a set is open if its trace to all compact subsets is open, or equivalently if $X$ carries the final topology with respect to all the inclusions of compact subsets) then we can conclude that $f$ is continuous and hence $C(X, \mathbb{R})$ is complete. So under these assumptions (and in particular if $X$ is locally compact) the space $C(X, \mathbb{R})$ is a Fréchet-space.

Is it really necessary to use countably many seminorms for non-compact $X$? -
There is no norm which defines an equivalent structure on $C(X, \mathbb{R})$. Otherwise some seminorm $p_K := \|\_|_K\|_\infty$ must dominate it. However this is not possible, since $p_K$ is not a norm: Since $X$ is not compact there is some point $a \in X \setminus K$ and hence the function $f$ defined by $f|_K = 0$ and $f(a) = 1$ is continuous on $K \cup \{a\}$. By Tietze-Urysohn it can be extended to a continuous function on $X$, which is obviously in the kernel of $p_K$ but not zero.
Is there some other reasonable norm turning $C(X, \mathbb{R})$ into a Banach space $E$? - By reasonable we mean that at least the point-evaluations should be continuous (i.e. the topology should be finer than that of pointwise convergence). Then the identity mapping $E \to C(X, \mathbb{R})$ is continuous by the application in [**2**, 5.3.8] of the closed graph theorem. Hence by the open mapping theorem for Fréchet spaces the identity is an isomorphism, and thus $E$ is not Banach. Note that this shows that in a certain sense the structure of $C(X, \mathbb{R})$ is unique.

Now what can be said about spaces of differentiable functions? - Of course the space $D^1(X, \mathbb{R})$ of differentiable functions on some interval $X$ is contained in $C(X, \mathbb{R})$. However it is not closed in $C(X, \mathbb{R})$ and hence not complete in the supremum-norm, since a uniform limit of differentiable functions need not be differentiable anymore. We need some control on the derivative. So we consider the space $C^1(X, \mathbb{R})$ of continuously differentiable functions with the initial topology induced by the inclusion in $C(X, \mathbb{R})$ and by the map $C^1(X, \mathbb{R}) \to C(X, \mathbb{R})$ given by $f \mapsto f'$. If $X$ is compact we can consider equally well instead of the corresponding two seminorms $f \mapsto \|f\|_\infty$ and $f \mapsto \|f'\|_\infty$ their maximum and obtain a norm on $C^1(X, \mathbb{R})$. Again elementary analysis gives completeness, since for a Cauchy-sequence $f_n$ we have a uniform limit $f$ of $f_n$ and a uniform limit $f^1$ of $f_n'$, and hence (e.g. see [**13**, 104.3] or 2.40) $f$ is differentiable with derivative $f^1$. Inductively, we obtain that for compact intervals $X$ and natural numbers $n$ the spaces $C^n(X,\mathbb{R})$ can be made canonically into Banach-spaces, see [**2**, 4.2.5].

### 1.3

What about the space $C^\infty(X,\mathbb{R})$ of infinite differentiable maps? - Then we have countably many seminorms $f \mapsto \|f^{(n)}\|_\infty$, and as before we obtain completeness. So we have a Fréchet space.

Again the question arises: Is it really necessary to use countably many seminorms? This time we have continuous norms, if we use the sum of the supremum norm and the supremum norm of some derivative. So we cannot argue as for $C(X)$. So let us assume that there is some norm on $C^\infty(X,\mathbb{R})$ defining an equivalent structure. In particular it has to be continuous and hence has to be dominated by the maximum of the suprema of finitely many derivatives. Let us take an even higher derivative. Then the supremum of this derivative must be dominated by the norm. This is however not possible, since there exist smooth functions $f$, for which all derivatives of order less than $n$ are globally bounded by 1, but which have arbitrarily large $n$-th derivative at a given point, say 0. In fact, without loss of generality, we may assume assume that $n$ is even and let $b \geq 1$. Take $f(x) := a \cos bx$ with $a := 1/b^{n-1}$. Then $|f^{(k)}(x)| = b^{k+1-n} \leq 1$ for $k < n$, but $f^n(0) = \pm b \cos 0$.
Is there some reasonable (inequivalent) norm which turns $C^\infty(X,\mathbb{R})$ into a Banach-space? - Well the same arguments as before show that any reasonable Fréchet-structure on $C^\infty(X,\mathbb{R})$ is identical to the standard one and hence not normable.

### 1.4

So the naive formulation of a fixed point equation for a general differential equation, does not lead to Banach spaces. There is however a way around this difficulty. The idea can be seen from the simplest differential equation, namely when $F$ doesn't depend on $u$, i.e. $u'(t) = F(t)$. Then the (initial value) problem can be solved by integration: $u(t) = u(0) + \int_0^t F(s)\,ds$ and in fact similar methods work in the case of separated variables, i.e. $u'(t) = F_1(t)\,F_2(u)$, since then $G_2(u) := \int \frac{1}{F_2(u)}\,du = c + \int F_1(t)\,dt =: G_1(t)$ and hence $u(t) = G_2^{-1}(G_1(t))$.
In [**2**, 1.3.2] we have seen how to prove an existence and uniqueness result for a differential equation with initial value conditions $u(0) = a$. Namely by integration one transforms it into the *integral equation*

$$u(t) = a + \int_0^t g(s, u(s))\,ds.$$

Thus one has to find a fixed point $u$ of $u = G(u)$, where $G$ is the integral operator given by

$$G(u)(t) := a + \int_0^t g(s, u(s))\,ds.$$

As space of possible solutions $u$ one can now take the vector space $C(I)$ of continuous functions on some interval $I$ around 0. If one takes $I$ sufficiently small then it is easily seen that $G$ is a contraction provided $g$ is sufficiently smooth, e.g. locally Lipschitz. Hence the existence of a fixed point follows from Banach's fixed point theorem [**2**, 1.2.2].

In [**2**, 3.5.1] we have elaborated on these ideas in the case of linear differential equations. We have seen that the solution of a *linear differential equation with constant coefficients* $u' = Au$ and initial condition $u(0) = u_0$ is given by $u(t) := e^{tA}\, u_0$. Furthermore the solution of a general initial value problem of a *linear differential equation of order* $n$

$$u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)}(t) = s(t), \quad u(0) = u_0, \dots, u^{(n-1)}(0) = u_{n-1}.$$

is given by an integral operator $G : f \mapsto u$ defined by $(Gf)(t) := f(t) + \int_0^1 g(t,\tau)\, d\tau$, with a certain continuous integral kernel $g$. We have also seen that a *boundary value problem of second order*

$$u''(t) + a_1(t)\, u'(t) + a_0(t)\, u(t) = s(t), \quad R_a(u) = 0 = R_b(u),$$

where the boundary conditions are $R_a(u) := r_{a,0}\, u(a) + r_{a,1}\, u'(a)$ and $R_b(u) := r_{b,0}\, u(b) + r_{b,1}\, u'(b)$ is also solved in the generic case by an integral operator

$$u(t) = \int_a^b g(t,\tau)\, f(\tau)\, d\tau,$$

with continuous integral kernel obtained from the solutions of corresponding initial value problems.

## Partial Differential Equations

### 1.5

Now what happens, if the arguments $u$ are functions of several numerical variables. Then the derivatives $u^{(k)}$ are given by the corresponding Jacobi-matrices of partial derivatives, and our equation is a *partial differential equation*, see [**2**, 4.7.1].

Even if we have a *partial linear differential equation with constant coefficients*

$$D(u)(x) := \sum_{|\alpha|\le n} a_\alpha \cdot \partial^\alpha u(x) = s(x)$$

we cannot apply the trick above. The first problem is, that we no longer have a natural candidate, with respect to which we could pass to a explicit equation. In some special cases one can do. An example is the *equation of heat-conduction*

$$\frac{\partial}{\partial t} u = \Delta u,$$

where $u : \mathbb{R} \times X \to \mathbb{R}$ is the heat-distribution at the time $t$ in the point $x$ and $\Delta$ denotes the *Laplace-operator* given on $X = \mathbb{R}^n$ by $\Delta := \sum_{k=1}^n \left(\frac{\partial}{\partial x^k}\right)^2$. So this is just an ordinary differential equation in an infinite dimensional space of functions on $X$. If we want $\Delta$ to be a self-mapping, we need smooth functions. But if we want to solve the equation as $u(t) = e^{t\Delta} u_0$ we need the functional calculus and hence a Hilbert space of functions. But then $\Delta$ becomes an unbounded symmetric operator. This we treated in [**3**, 12.48]. Another example of such a situation is the *Schrödinger equation*

$$i\hbar \frac{d}{dt} u = S\, u.$$

where the *Schrödinger-operator* is given by $S = -\frac{\hbar^2}{2m}\Delta + U(x)$ for some potential $U$.

A third important equation is the *wave-equation* $(\frac{\partial}{\partial t})^2 u = \Delta u$. If one makes an Ansatz of separated variables $u(t,x) = u_1(t)\, u_2(x)$ one obtains an Eigen-value equation

$\Delta u(x) = \lambda u(x)$ for $\Delta$ and after having obtained the Eigen-functions $u_n : X \to \mathbb{R}$, one is lead to the problem of finding coefficients $a_k$ and $b_k$ such that

$$u(t,x) := \sum_k \Big( a_k \cos(\sqrt{\lambda_k}\, t) + b_k \sin(\sqrt{\lambda_k}\, t) \Big)\, u_k(x)$$

solves the initial conditions

$$u(0,x) = \sum_k a_k u_k(x) \qquad \text{and} \qquad \partial_1 u(0,x) = \sum_k \sqrt{\lambda_k}\, b_k\, u_k(x)$$

If we have an inner-product, for which the $u_k$ are orthonormal, we can easily calculate the coefficients $a_k$ and $b_k$. The space $C_{2\pi}$ of $2\pi$-periodic functions is however not a Hilbert space. Otherwise it would be isomorphic to its dual, by the Riesz Representation theorem [**2**, 6.2.9]. However for $t \neq s$ we have that $\|\operatorname{ev_t} - \operatorname{ev}_s\| = \sup\{|f(t) - f(s) : \|f\|_\infty \leq 1\} = 1$ if we chose $f(t) = 1$ and $f(s) = 0$. Thus $C(X,\mathbb{R})'$ is not separable, since otherwise for every $t$ there would be an $\ell_t$ in a fixed dense countable subset with $\|\operatorname{ev}_t - \ell_t\| < \frac{1}{2}$. Since the $t$ are uncountable there have to be $t \neq s$ for which $\ell_t = \ell_s$, a contradiction. Another method to see this is to use Krein-Milman [**2**, 7.5.1]: If $C(X)$ were a dual-space, then its unit-ball would have to be contained in the closed convex hull of its extremal points. A function $f$ in the unit-ball, which is not everywhere of absolute value 1, is is not extremal. In fact, take a $t_0$ with $|f(t_0)| < 1$ and a function $v$ with support in a neighborhood of $t_0$. Then $f + s\,v$ lies in the unit ball for all values of $s$ near 0. Hence we have by far too few extremal points, since those real-valued functions have to be constant on connectivity components.

In analogy to the inner product on $\mathbb{R}^n$ we can however consider the continuous positive definite hermitian bilinear map $(f,g) \mapsto \int_X f(x)\,\overline{g(x)}\,dx$. By what we said above, it cannot give a complete norm on $C(X,\mathbb{R})$. But we can take the completion of $C(X,\mathbb{R})$ with respect to this norm and arrive at $L^2(X)$, a space not consisting of functions, but equivalence classes thereof. Now for the one-dimensional wave-equation, i.e. the equation of an *vibrating string*, we can solve the Eigenvalue-problem directly (it is given by an ordinary differential equation). And *Fourier-series* solves the problem, see [**2**, 5.4] and [**2**, 6.3.8].

In general the Laplace operator will be symmetric with respect to that inner product. If it were bounded, then it would be selfadjoint and one could apply geometry in order to find Eigen-values and Eigen-vectors by minimizing the angle between $x$ and $Tx$, or equivalently by maximizing $|\langle Tx, x\rangle|$, see [**2**, 6.5.3]. It is quite obvious that for a selfadjoint bounded operator the supremum of $|\langle Tx, x\rangle|$ is its norm, and that a point were it is attained is an Eigen-vector with maximal absolute Eigen-value. So one needs compactness to show the existence of such a point. Since Eigen-vectors to different Eigen-values are orthogonal to each other, one can then proceed recursively, provided the operator is compact.

Again the idea is that although the linear differential-operator $D$ is not even bounded, its inverse should be an integral operator $G$ (the *Green-operator*) with continuous kernel $\varepsilon$ and hence compact. And obviously instead of solving $Du = \lambda u$ we can equally well solve $\frac{1}{\lambda} u = Gu$.

In order to find the Green operator, we have seen in [**2**, 4.7.7] that a possible solution operator $G : s \mapsto u$ would be given by convolution of $s$ with a *Green-function* $\varepsilon$, i.e. a solution of $D(\varepsilon) = \delta$, where $\delta$ is the neutral element with respect to convolution. In fact, since $u := \varepsilon \star s$ should be a solution of $D(u) = s$, we conclude that $s = D(u) = D(\varepsilon \star s) = D(\varepsilon) \star s$. However such an element doesn't exist in the algebra of smooth functions, and one has to extend the notion of function to include so called generalized functions or distributions. These are the continuous linear functionals on the space $\mathcal{D}$ of smooth functions with compact support.

As we have seen in [**2**, 4.8.2] the space $\mathcal{D}$ is no longer a Fréchet space, but a strict inductive limit of the Fréchet spaces $C^\infty_K(X) := \{f \in C^\infty : \operatorname{supp} f \subseteq K\}$. Assume that there is some Fréchet structure on $C^\infty_c$. Then by the same arguments as before the identity from $\mathcal{D}$ to $C^\infty_c$ would be continuous, hence closed, and hence the inverse to the webbed space $\mathcal{D}$ would be continuous too, i.e. a homeomorphism. Remains to show that the standard structure is not a Fréchet structure. If it were, then $\mathcal{D}$ would be Baire. However the closed linear subspaces $C^\infty_K$ have as union $\mathcal{D}$ and have empty interior, since non-empty open sets are absorbing. A contradiction to the Baire-property.

By passing to the transposed, we have seen in [**2**, 4.9.1] that every linear partial differential operator $D$ can be extended to a continuous linear map $\tilde{D} : \mathcal{D}' \to \mathcal{D}'$, and so one can consider distributional solutions of such differential equations. In [**2**, 8.3.1] we have proven the Malgrange Ehrenpreis theorem on the existence of distributional fundamental solutions using the generalization of Fourier-series, namely the Fourier-transform $\mathcal{F}$. The idea is that $1 = \mathcal{F}(\delta) = \mathcal{F}(D(\varepsilon)) = F(P(\partial)(\varepsilon)) = P \cdot \mathcal{F}(\varepsilon)$ and hence $\varepsilon = \mathcal{F}^{-1}(1/P)$. For this we have to consider the Schwartz-space $\mathcal{S}$ of rapidly decreasing smooth functions, which is a Fréchet space, and its dual $\mathcal{S}'$. In order that the poles of $1/P$ make no trouble we had to show that the Fourier-transform of smooth functions with compact support and even of distributions with compact support are entire functions.

If we want to solve *linear partial differential equations with non-constant coefficients* or even *non-linear partial differential equations*, we have to consider not only the linear theory of $\mathcal{D}$ but the non-linear one. See [**7**] for an approach to this.

## Exponential Law

### 1.6

But let us consider a much more elementary result. In fact even in the introductory courses in analysis one considers infinite dimensional results, but of course disguised. Recall the result about *differentiation under the integral sign*. There one considers a function $f$ of two real variables and takes the integral $\int_0^1 f(t,s)\,ds$ with respect to one variable, and then one asks the question under what assumptions is the resulting function differentiable with respect to $t$ and what is its derivative? Before we try to remember the correct answer let us reformulate this result without being afraid of infinite dimensions. We are given the function $f : \mathbb{R} \times I \to \mathbb{R}$, $(t,s) \mapsto f(t,s)$. What do we actually mean by writing down $\int_0^1 f(t,s)\,ds$? - Well we keep $t$ fixed and consider the function $f_t : I \to \mathbb{R}$ given by $s \mapsto f(t,s)$ and integrate it, i.e. $\int_0^1 f(t,s)\,ds := I(f_t)$, where $I$ denotes the integration operator $I : C[0,1] \to \mathbb{R}$, $g \mapsto \int_0^1 g(s)\,ds$. But now we want to vary $t$, so we have to consider the result as a function $t \mapsto I(f_t)$, so we have to consider $t \mapsto f_t$ and we denote this function by $\check{f}$. It is given by the formula $\check{f}(t)(s) = f_t(s) = f(t,s)$. Then $I(f_t) = (I \circ \check{f})(t)$. Thus what we actually are interested in is, whether the composition $I \circ \check{f}$ is differentiable and what its derivative is. This problem is usually solved by the *chain-rule*, but the situation here is much easier. In fact recall that integration is linear and continuous with respect to the supremum norm (or even the 1-norm) and $\check{f}$ is a curve (into some function space). Now if $\ell$ is continuous and linear and $c$ is a differentiable curve then $\ell \circ c$ is differentiable with derivative $\ell(c'(t))$ at $t$: In fact

$$\lim_{s\to 0} \frac{\ell(c(t+s)) - \ell(c(t))}{s} = \lim_{s\to 0} \ell\left(\frac{c(t+s)-c(t)}{s}\right) = \\ = \ell\left(\lim_{s\to 0}\frac{c(t+s)-c(t)}{s}\right) = \ell(c'(t)).$$

So it remains to show that $\check{f}: \mathbb{R} \to C(I,\mathbb{R})$ is differentiable and to find its derivative. Let us assume it is differentiable and try to determine the derivative. On $C(I,\mathbb{R})$ we have nice continuous linear functionals, namely the *point evaluations* $\delta_s = \mathrm{ev}_s : g \mapsto g(s)$. These are continuous and linear and separate points (they are far from being all continuous linear functionals, see Riesz's Representation theorem [**2**, 7.3.3] and [**2**, 7.3.4]). Applying what we said before to $\ell := \delta_s$ and $c := \check{f}$ we obtain $\delta_s(\check{f}'(t)) = (\delta_s \circ \check{f})'(t)$, and $(\delta_s \circ \check{f})(t) = \delta_s(\check{f}(t)) = \check{f}(t)(s) = f(t,s)$. Hence $\delta_s(\check{f}'(t))$ is nothing else but the first partial derivative $\frac{\partial}{\partial t} f(t,s)$. Conversely, assume that the first partial derivative of $f$ exists on $\mathbb{R}\times I$ and is continuous, then we want to show, that $\check{f}$ is differentiable, and $(\check{f})'(t)(s) = \frac{\partial}{\partial t}f(t,s) = \partial_1 f(t,s)$, or in other words $(\partial_1 f)^\vee = (\check{f})'$.

For this we first consider the corresponding topological problem: Are the continuous mappings $f : X \times Y \to Z$ exactly the continuous maps $\check{f} : X \to C(Y,Z)$? This has been solved in the calculus courses. In fact a mapping $\check{f}$ is well-defined and continuous provided

1. $f(x, \_)$ is continuous for all $x$;
2. $f(\_, y)$ is equi-continuous with respect to $y$.

This however is equivalent to the continuity of $f$, as can be seen for example in [**13**, 107.2], provided $Y$ is compact.

Let us try to generalize this result. We will write $Y^X$ for the function spaces $C(X,Y)$ for reasons of cardinality. How is $\hat{g} : X \times Y \to Z$ constructed from a continuous $g : X \to Z^Y$. Well, one can consider $g \times Y : X \times Y \to Z^Y \times Y$ and compose it with the *evaluation map* $\mathrm{ev} : Z^Y \times Y \to Z$. Since the product of continuous maps is continuous, it remains to show that the evaluation map is continuous in order to obtain that $\hat{g}$ is continuous. So let $f \in Z^Y$ and $y \in Y$ and let $U$ be a neighborhood of $f(y)$. If $Y$ is locally compact, we can find a compact neighborhood $W$ of $y$ and then $f \in N_{W,U} := \{g : g(W) \subset U\}$ and $\mathrm{ev}(N_{W,U} \times W) \subseteq U$.

Conversely let a continuous $f : X \times Y \to Z$ be given. Then we consider $f_* := f^Y : (X \times Y)^Y \to Z^Y$ and compose it from the right with the *insertion map* $\mathrm{ins} : X \to (X\times Y)^Y$ given by $x \mapsto (y \mapsto (x,y))$. Then we arrive at $\check{f}$. Obviously $f_*$ is continuous since $(f_*)^{-1} N_{K,U} = N_{K,f^{-1}U}$. The insertion map is continuous, since $\mathrm{ins}^{-1}(N_{K,U\times V}) = U$ if $K \subseteq V$ and is empty otherwise, so $\check{f}$ is continuous. Thus the only difficult part was the continuity of the evaluation map.

Moreover we have the

**1.7 Lemma.**
*Let $X$, $Y$ and $Z$ be topological spaces with $Y$ being locally compact. Then we have a homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$, given by $f \mapsto \check{f}$, where the function spaces carry the compact open topology.*

**Proof.** We have already proved that we have a bijection. That this gives a homeomorphism follows, since the corresponding subbases $N_{K_1\times K_2,U}$ and $N_{K_1,N_{K_2,U}}$ correspond to each other. □

In general the compact open topology on $Z^Y$ will not be locally compact even for locally compact spaces $Y$ and $Z$ (e.g. $C([0,1],\mathbb{R})$ is an infinite dimensional and hence not locally compact Banach space). So in order to get an intrinsic exponential law, one can modify the notion of continuity and call a mapping $f : X \to Y$ between Hausdorff topological spaces *compactly-continuous* iff its restriction to every compact subset $K \subseteq X$ is continuous. Thus $f : X \times Y \to Z$ is continuous iff $f|_{K\times L} : K \times L \to Z$ is continuous for all compact subsets $K \subseteq X$ and $L \subseteq Y$. By the exponential law for compact sets this is equivalent to $\check{f} : K \to Z^L$ being continuous. Since $Z^Y$ carries the initial structure with respect to $Z^L \to Z^Y$, this is furthermore equivalent to the continuity of $\check{f} : K \to Z^Y$, and thus to $\check{f} : X \to Z^Y$ being compactly-continuous, but for this we have to denote with $Z^Y$ the space of compactly continuous maps from $Y \to Z$.

Instead of the category of compactly continuous maps between Hausdorff topological spaces, one can use the *equivalent category* of continuous mappings between compactly generated spaces. Recall that a Hausdorff topological space is called *compactly generated* or a *Kelley space* iff it carries the final topology with respect to the inclusions of its compact subsets with their trace topology. The equivalence between these two categories is given by the identity functor one one side, and on the other side by the Kelley-fication, i.e. by replacing the topology by the final topology with respect to the compact subsets. Note that the identity is compactly continuous in both directions. However the natural topology on product in this category is the Kelley-fication of the product topology and also on the function space one has to consider the Kelly-fication of the compact open topology.

Now back to the differentiability question. We assume that $\partial_1 f$ exists and is continuous. Hence $(\partial_1 f)^\vee : \mathbb{R} \to C(I,\mathbb{R})$ is continuous. We want to show that $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$ is differentiable (say at 0) with this function (at 0) as derivative. So we have to show that the mapping $t \mapsto \frac{\check{f}(t)-\check{f}(0)}{t}$ is continuously extendable to $\mathbb{R}$ by defining its value at 0 as $(\partial_1 f)^\vee(0)$. Or equivalently, by what we have shown for continuous maps before, that the map

$$(t,s) \mapsto \begin{cases} \frac{f(t,s)-f(0,s)}{t} & \text{for } t \neq 0 \\ \partial_1 f(0,s) & \text{otherwise} \end{cases}$$

is continuous. This follows immediately from the continuity of $\partial_1$ and that of $\int_0^1 \_\, dr$, since it can be written as $\int_0^1 \partial_1 f(r\,t,s)\,dr$ by the fundamental theorem.

So we arrive under this assumption at the conclusion, that $\int_0^1 f(t,s)\,ds$ is differentiable with derivative

$$\frac{d}{dt}\int_0^1 f(t,s)\,ds = I((\check{f})'(t)) = \int_0^1 \frac{\partial}{\partial t} f(t,s)\,ds.$$

Thus we have proved the

**1.8 Proposition.**
*For a continuous map $f : \mathbb{R} \times I \to \mathbb{R}$ the partial derivative $\partial_1 f$ exists and is continuous iff $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$ is continuously differentiable. And in this situation $I((\check{f})'(t)) = \frac{d}{dt}\int_0^1 f(t,s)\,ds = \int_0^1 \frac{\partial}{\partial t} f(t,s)\,ds$.*

And we see, that it is much more natural to formulate and prove this result with the help of the infinite dimensional space $C([0,1],\mathbb{R})$. But this not only simplifies the proof, but is of importance for its own sake, as we will show now.

## Variational Calculus

**1.9** In physics one is not a priori given an equation $f(x) = 0$, but often some *optimization problem*. One is searching for those $x$, for which the values $f(x)$ of some real-valued function (like the *Lagrange function* in classical mechanics, which is given by the difference of kinematic energy and the potential) attain an extremum (i.e. are minimal or maximal). Again $x$ is often not a finite dimensional vector but functions and then $f$ is often given by some integral (like the action (german: Wirkungsintegral) in classical mechanics)

$$f(x) := \int_0^1 F(t, x(t), x'(t))\, dt.$$

For finite dimensional vectors $x$ one finds solutions of the problem $f(x) \to \min$ by applying differential calculus and searching for solutions of $f'(x) = 0$. In infinite dimensions one proceeds similarly in the calculus of variations, by finding those points $x$, where the directional derivatives $f'(x)(v)$ vanish for all directions $v$. Since the boundary values of $x$ are given, the variation $v$ has to vanish on the boundary $\{0, 1\}$. One can calculate the directional derivative by what we have shown before as follows:

$$\begin{aligned}
f'(x)(v) &:= \frac{d}{dt}\Big|_{t=0} f(x+tv) \\
&= \frac{d}{dt}\Big|_{t=0} \int_0^1 F\Big(s, (x+tv)(s), (x+tv)'(s)\Big)\, ds \\
&= \int_0^1 \frac{\partial}{\partial t}\Big|_{t=0} F\Big(s, (x+tv)(s), (x+tv)'(s)\Big)\, ds \\
&= \int_0^1 \Big(\partial_2 F(s, x(s), x'(s)) \cdot v(s) + \partial_3 F(s, x(s), x'(s)) \cdot v'(s)\Big)\, ds \\
&= \int_0^1 \left(\partial_2 F(s, x(s), x'(s)) - \frac{d}{ds}\partial_3 F(s, x(s), x'(s))\right) \cdot v(s)\, ds
\end{aligned}$$

We have used partial integration and that the variation $v$ has to vanish at the boundary points 0 and 1. Since $f'(x)(v)$ has to be 0 for all such $v$ we arrive at the *Euler-Lagrange partial differential equation*

$$\partial_2 F(s, x(s), x'(s)) = \frac{d}{ds}\partial_3 F(s, x(s), x'(s)),$$

or with slight abuse of notation:

$$\frac{\partial}{\partial x} F = \left(\frac{\partial}{\partial \dot{x}} F\right)^{\cdot},$$

where $(\_)^{\cdot}$ denotes the derivative with respect to time $s$.

Warning: abuse may lead to disaster! In physics for example one has the *gas-equation* $p \cdot V \cdot t = 1$, where $p$ is pressure, $V$ the volume and $t$ the temperature scaled appropriately. So we obtain the following partial derivatives:

$$\begin{aligned}
\frac{\partial p}{\partial V} &= \frac{\partial}{\partial V}\frac{1}{V\,t} = -\frac{1}{t\,V^2} \\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{t\,p} = -\frac{1}{p\,t^2} \\
\frac{\partial t}{\partial p} &= \frac{\partial}{\partial p}\frac{1}{p\,V} = -\frac{1}{V\,p^2}
\end{aligned}$$

And hence cancellation yields

$$1=\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial p}=(-1)^3\frac{1}{t\,V^2}\cdot\frac{1}{p\,t^2}\cdot\frac{1}{V\,p^2}=-\frac{1}{(pVt)^3}=-1.$$

Try to find the mistake!

### 1.10 Flows

Another situation, where it is natural to consider differentiable curves into function spaces, are flows. So we are considering the ordinary time-independent differential equation is, i.e. equations of the form $\dot u = f(u)$. For given initial value $u(0)=a$ we can consider the solution $u_a$ and obtain a mapping $u:\mathbb{R}\times X\to X$ given by $(t,a)\mapsto u_a(t)$. Obviously $u(0,x)=x$ and by uniqueness we have $u(t+s,x)=u(t,u(s,x))$, i.e. $u$ is a *flow* on $X$. Conversely we can reconstruct the differential equation by differentiating with respect to $t$ at $t=0$, i.e. $\frac{\partial}{\partial t}|_{t=0}u(t,x)=f(u(t,x))|_{t=0}=f(x)$. It would be more natural to consider the associate mapping $\check u$ with values in some space of mappings from $X\to X$. The flow property translates into the assumption that $t\mapsto\check u(t)$ is a group-homomorphism from $\mathbb{R}$ into the group of invertible maps on $X$. The *vector field* $f$ can thus be interpreted as the tangent vector at 0 of the curve $\check u$. Thus we should have that $\check u$ is differentiable into the group $\mathrm{Diff}(X)$ *of diffeomorphisms* on $X$, and this group should carry some smooth structure, analogously to classical Lie-groups. In particular the composition $\mathrm{Diff}(X)\times\mathrm{Diff}(X)\to\mathrm{Diff}(X)$ map should be differentiable. Since $(f,g)\mapsto f\circ g$ is linear in the first variable, the difficult part is the differentiability in the second variable, i.e. that of the map $f_*:g\mapsto f\circ g$. If we compose this map with the linear functionals given by point-evaluations $\delta_x$ we obtain $g\mapsto(\delta_x\circ f_*)(g)=f(g(x))$. And if we calculate the directional derivative $(\delta_x\circ f_*)'(g)(h)$ we obtain

$$(\delta_x\circ f_*)'(g)(h)=\frac{\partial}{\partial t}\Big|_{t=0}(\delta_x\circ f_*)(g+t\,h)=\frac{\partial}{\partial t}\Big|_{t=0}f(g(x)+t\,h(x))=f'(g(x))(h(x)),$$

by using the chain-rule. Thus in order that the composition map is differentiable, we need that the first variable is differentiable, hence Diff should mean at least 1-times differentiable. But then in order that the derivative of the composition map has 1-time differentiable values we need that $f'$ is 1-times differentiable, i.e. $f$ is twice differentiable. Inductively we arrive at the smoothness of $f$, i.e. infinite often differentiability. But as we have mentioned before, even in the simplest case $C^\infty([0,1],\mathbb{R})$, these function spaces are not Banach-spaces anymore, but Fréchet-spaces.

### 1.11 Exponential law

A similar thing happens when searching for an exponential law for differentiable functions. If we want a nice correspondence between differentiable functions on a product and differentiable functions into a function space, we have seen that the curve associated to a $C^1$-function on a product is differentiable into the space of continuous functions (using the first partial derivative). The second partial derivative says, that this curve has even values in the space of $C^1$-functions. And is continuous therein. So if we want to use a single function space, we should at least take $C^1$-functions, and the curve should be $C^1$ into this space. However one easily sees that this amounts to the existence and continuity of $f$, $\partial_2 f$, $\partial_1 f$ and $\partial_2\partial_1 f$. If we want this concept to be independent on the choice of a basis, we need at least a $C^2$-function $f$. Inductively we arrive that only for $C^\infty$-functions will it be possible to obtain a nice correspondence.

We have learned a few things from these introductory words.

1. Problems in finite dimensions often lead to infinite dimensional function-spaces, which are quite often not Banach spaces, but Fréchet spaces or even more general ones.
2. Functions of 2 variables $f : X \times Y \to Z$, should often be considered as maps $\check{f}$ from $X$ to a space of functions from $Y$ to $Z$ and properties such as continuity or differentiability should translate nicely.
3. At least for continuous linear operators $T$ one should have the chain rule $(T \circ c)'(t) = T(c'(t))$.

## Continuous and Higher Order Differentiability

### 1.12

Well, as has been discovered at the turn of the last century, the derivative should be a linear (more precisely, an affine) approximation to the function. Assume we have already defined the concept of *derivative* $f'(x) \in L(E, F)$ for functions $f : E \supseteq U \to F$ at a given point $x \in U$. By collecting for all $x$ in the open domain $U$ of $f$ these derivatives $f'(x)$, we obtain a mapping $x \mapsto f'(x)$, the derivative $f' : E \supseteq U \to L(E, F)$ with values in the space of continuous linear mappings. In order to speak about continuous differentiable (short: $C^1$) mappings, we need some topology on $L(E, F)$ and then this amounts to the assumption, that $f' : U \to L(E, F)$ is continuous. For $C^1$-maps we should have a CHAIN-RULE, which guarantees that the composite $f \circ g$ of $C^1$-maps is again $C^1$ and the derivative should be $(f \circ g)'(x) = f'(g(x)) \circ g'(x)$. This map is thus given by the following description: Given $x$ then first calculate $g(x)$ and then $f'(g(x)) \in L(F, G)$ and $g'(x) \in L(E, F)$, and finally apply the composition map $L(F, G) \times L(E, F) \to L(E, G)$ to obtain $f'(g(x)) \circ g'(x)$. Since $f$ and $g$ are assumed to be $C^1$ the components $f' \circ g$ and $g'$ are continuous. So it remains to show the continuity of the composition mapping. Let us consider the simplified case where $G = E = \mathbb{R}$. Then composition reduces to the evaluation map $\mathrm{ev} : F' \times F \to \mathbb{R}$ and we are looking for a topology on $F'$ such that this map is continuous. Assume we have found such a topology. Then there exists 0-neighborhoods $V$ in $F'$ and $U$ in $F$ such that $\mathrm{ev}(V \times U) \subseteq [-1, 1]$. Since scalar-multiplication on $F'$ should be continuous, we can find for every $\ell \in F'$ a number $K > 0$, such that $\ell \in K\,V$. Thus for $x \in U$ we have $\ell(x) = \mathrm{ev}(K \frac{1}{K}\ell, x) = K\,\mathrm{ev}(\frac{1}{K}\ell, x) \in K\,\mathrm{ev}(V \times U) \subseteq [-K, K]$. This shows that $U$ is scalarly bounded, and hence is bounded by the corollary in [**2**, 5.2.7]. However, a seminormed space, which has a bounded 0-neighborhood has to be normed, by Kolmogoroff's theorem [**2**, 2.6.2]. So it seems that there is no reasonable notion of $C^1$, which applies to more than just functions between Banach spaces. However, we have assumed that continuity is meant with respect to topologies, so there have been several (more or less successful) attempts in the past to remedy this situation by considering convergence structures on $L(E, F)$. If one defines that a net (or a filter) $f_\alpha$ should converge to $f$ in $L(E, F)$ iff for nets (or filters) $x_\beta$ converging to some $x$ in $E$ the net (or filter) $f_\alpha(x_\beta)$ should converge to $f(x)$, then the evaluation map, and more generally the composition map becomes continuous. A second way to come around this problem, is to assume for $C^1$ the continuity of $\widehat{f'} : U \times E \to F$ instead. Then the chain-rule becomes easy. However this notion is bad, since it will not give the *inverse function theorem* for $C^1$ even in Banach spaces.

### 1.13

If we want to define higher derivatives - as we need them in conditions for local extrema and the like - we would call a function $f$ by recursion $(n+1)$-times differentiable iff $f'$ exists and is $n$-times differentiable. In order to show that the composite $f\circ g$ of two $D^2$-maps is again $D^2$, we have to show that $(f\circ g)' : x \mapsto f'(g(x))\circ g'(x)$ is again $D^1$. Now this map is given by the following composition: Given $x$ then first calculate $g(x)$ and then $f'(g(x)) \in L(F,G)$ and $g'(x) \in L(E,F)$, and finally apply the composition map $L(F,G) \times L(E,F) \to L(E,G)$ to obtain $f'(g(x)) \circ g'(x)$. By the chain-rule for $D^1$-mappings, we would obtain that $f' \circ g \in D^1$ and by assumption $g' \in D^1$. So it remains to differentiate the bilinear composition map. Since it is linear in both entries separately, its partial derivatives should obviously exist and the derivative also. But recall that it is not even continuous.

### 1.14 Resumé

1. The composition map, or at least the evaluation map, should be differentiable, although it is not continuous.
2. There is no reasonable notion of $C^1$ for functions between spaces beyond Banach spaces.
3. It is not clear, how to obtain a chain-rule, although this is essential.

After having found lots of difficulties, let's look what can be done easily:

1. It is obvious what differentiability for a curve $c$ into any locally convex space means.
2. Hence we have also the notion of $C^1$, $n$-times differentiable and smoothness for such curves.
3. Continuous (multi-)linear maps preserve smoothness, and satisfy the chain-rule.
4. Directional derivatives can be easily defined for arbitrary functions.
5. Derivatives can be detected by reducing to 1-dimensional spaces via affine mappings.

# 2. Calculus

## Curves

As we have already seen in the introduction curves pose no big problems and in particular we can give the following definitions.

### 2.1 Differentiable curves

Let $E$ be a locally convex vector space. A curve $c:\mathbb{R}\to E$ is called *differentiable* if the *derivative* $c'(t) := \lim_{s\to 0}\frac{1}{s}(c(t+s)-c(t))$ at $t$ exists for all $t$. A curve $c:\mathbb{R}\to E$ is called *smooth* or $C^\infty$ if all iterated derivatives exist. It is called $C^n$ for some finite $n$ iff its iterated derivatives up to order $n$ exist and are continuous.

Likewise a mapping $f:\mathbb{R}^n\to E$ is called smooth if all iterated partial derivatives $\partial_{i_1,\dots,i_p}f := \frac{\partial}{\partial x^{i_1}}\dots\frac{\partial}{\partial x^{i_p}}f$ exist for all $i_1,\dots,i_p\in\{1,\dots,n\}$.

A curve $c:\mathbb{R}\to E$ is called *locally Lipschitzian* if every point $r\in\mathbb{R}$ has a neighborhood $U$ such that the *Lipschitz condition* is satisfied on $U$, i.e. the set $\left\{\frac{1}{t-s}\Big(c(t)-c(s)\Big) : t\neq s; t,s\in U\right\}$ is bounded. Note that this implies that the curve satisfies the Lipschitz condition on each bounded interval, since

$$\frac{c(t_n)-c(t_0)}{t_n-t_0} = \sum \frac{t_{i+1}-t_i}{t_n-t_0}\,\frac{c(t_{i+1})-c(t_i)}{t_{i+1}-t_i}$$

is in the absolutely convex hull of a finite union of bounded sets.

A curve $c:\mathbb{R}\to E$ is called *Lip*$^k$ or $C^{k+1-}$ if all derivatives up to order $k$ exist and are locally Lipschitzian.

We have the following implications:

$$C^{n+1} \implies Lip^n \implies C^n,$$
$$\text{differentiable} \implies C.$$

In fact continuity of the derivative implies locally its boundedness, and since this can be tested by continuous linear functionals (see [**2**, 5.2.7]) we conclude from the 1-dimensional mean value-theorem the boundedness of the difference quotient.

**2.2 Lemma. Continuous linear mappings are smooth.**
*A continuous linear mapping $\ell: E\to F$ between locally convex vector spaces maps $\mathcal{L}ip^k$-curves in $E$ to $\mathcal{L}ip^k$-curves in $F$, for all $0\le k\le\infty$ and for $k>0$ one has $(\ell\circ c)'(t)=\ell(c'(t))$.*

**Proof.** As a linear map $\ell$ commutes with difference quotients, hence the image of a Lipschitz curve is Lipschitz since $\ell$ is bounded. As a continuous map it commutes with the formation of the respective limits. Hence $(\ell\circ c)'(t)=\ell(c'(t))$.

Note that a differentiable curve is continuous, and that a continuously differentiable curve is locally Lipschitz: For $\ell \in E'$ we have

$$\ell\left(\frac{c(t)-c(s)}{t-s}\right) = \frac{(\ell\circ c)(t)-(\ell\circ c)(s)}{t-s} = \int_0^1 (\ell\circ c)'(s+(t-s)r)dr,$$

which is bounded, since $(\ell\circ c)'$ is locally bounded.

Now the rest follows by induction. □

## 2.3 The mean value theorem

In classical analysis the basic tool for using the derivative to get statements on the original curve is the mean value theorem. So we try to generalize it to infinite dimensions. For this let $c : \mathbb{R} \to E$ be a differentiable curve. If $E = \mathbb{R}$ the classical mean value theorem states, that the difference quotient $\frac{c(a)-c(b)}{a-b}$ equals some intermediate value of $c'$. Already if $E$ is two dimensional this is no longer true. Take for example a parameterization of the circle by arc-length. However, we will show that $\frac{c(a)-c(b)}{a-b}$ lies still in the closed convex hull of $\{c'(r) : r\}$. Having weakened the conclusion, we can try to weaken the assumption. And in fact $c$ may be not differentiable in at most countably many points. Recall however, that there exist strictly monotonous functions $f : \mathbb{R} \to \mathbb{R}$, which have vanishing derivative outside a Cantor set (which is uncountable, but has still measure 0).

Sometimes one uses in 1-dimensional analysis a generalized version of the mean value theorem, where one has a second differentiable function $h$ with non-vanishing derivative and the conclusion says that $\frac{c(a)-c(b)}{h(a)-h(b)}$ equals some intermediate value of $\frac{c'}{h'}$. A more-dimensional version would be that $\frac{c(a)-c(b)}{h(a)-h(b)}$ lies in the closed convex hull of $\{\frac{c'(r)}{h'(r)} : r\}$. Here one obviously has to assume that $h$ is scalar valued. If we do not assume that $h'$ exists everywhere, we should replace the assumption that $h'$ doesn't vanish by the assumption that $h'$ has constant sign, or, more generally, that $h$ is monotonic. And if we allow $h'$ to vanish somewhere, we can not form the quotients. Therefore we should assume that $c'(t) \in h'(t)\cdot A$, where $A$ is some closed convex set, and should be able to conclude that $c(b)-c(a) \in (h(b)-h(a))\cdot A$. This is the version of the *mean value theorem* that we will prove now. However we will make use of it only in the case where $h = \mathrm{id}$ and $c$ is everywhere differentiable in the interior.

**Proposition. Mean value theorem.**
*Let $c : [a,b] =: I \to E$ be a continuous curve, which is differentiable except at points in a countable subset $D \subseteq I$. And let $h$ be a continuous monotone function $h : I \to \mathbb{R}$, which is differentiable on $I \setminus D$. Let $A$ be a convex closed subset of $E$, such that $c'(t) \in h'(t)\cdot A$ for all $t \notin D$. Then $c(b)-c(a) \in (h(b)-h(a))\cdot A$.*

**Proof.** Assume that this is not the case. By Hahn Banach (see [**2**, 7.2.4]) there exists a continuous linear functional $\ell$ with $\ell(c(b)-c(a)) \notin \overline{\ell((h(b)-h(a))\cdot A)}$. But then $\ell\circ c$ and $\overline{\ell(A)}$ satisfy the same assumptions as $c$ and $A$ and hence we may assume that $c$ is real-valued and $A$ is just a closed interval $[\alpha,\beta]$. We may furthermore assume that $h$ is monotonely increasing. Then $h'(t) \geq 0$ and $h(b)-h(a) \geq 0$. Thus the assumption says that $\alpha h'(t) \leq c'(t) \leq \beta h'(t)$ and we want to conclude that $\alpha(h(b)-h(a)) \leq c(b)-c(a) \leq \beta(h(b)-h(a))$. If we replace $c$ by $c-\beta h$ or by $\alpha h - c$ it is enough to show that $c'(t) \leq 0$ implies that $c(b)-c(a) \leq 0$. For given $\varepsilon > 0$ we will show that $c(b)-c(a) \leq \varepsilon(b-a+1)$. For this let $A$ be the set $\{t \in [a,b] : c(s)-c(a) \leq \varepsilon\left((s-a)+\sum_{t_n<s} 2^{-n}\right) \text{ for all } 0 \leq s < t\}$, where

$D := \{t_n : n \in \mathbb{N}\}$. Obviously $A$ is a closed interval containing 0, say $[0, t]$. By continuity of $c$ we obtain that $c(t) - c(0) \le \varepsilon\left((t-a) + \sum_{t_n<t} 2^{-n}\right)$. Suppose $t < b$. If $t \notin D$, then there exists a subinterval $[t, t+\delta]$ of $[a, b]$ such that for $t \le s < t+\delta$ we have

$$c(s) - c(t) - c'(t)(s-t) \le \varepsilon(s-t)$$

Hence

$$c(s) - c(t) \le c'(t)(s-t) + \varepsilon(s-t) \le \varepsilon(s-t).$$

Thus

$$\begin{aligned} c(s) - c(a) &\le c(s) - c(t) + c(t) - c(a) \\ &\le \varepsilon(s-t) + \varepsilon\Big(t - a + \sum_{t_n<t} 2^{-n}\Big) \le \varepsilon\Big(s - a + \sum_{t_n<s} 2^{-n}\Big). \end{aligned}$$

On the other hand if $t \in D$, i.e. $t = t_m$ for some $m$, then by continuity of $c$ we can find an interval $[t, t+\delta]$ contained in $[a, b]$ such that for all $t \le s < t+\delta$ we have

$$c(s) - c(t) \le \varepsilon 2^{-m}.$$

Again we deduce that

$$c(s) - c(a) \le \varepsilon 2^{-m} + \varepsilon\Big(t - a + \sum_{t_n<t} 2^{-n}\Big) \le \varepsilon\Big(s - a + \sum_{t_n<s} 2^{-n}\Big).$$

So we reach in both cases a contradiction to the maximality of $t$. □

Warning: One cannot drop the monotonicity assumption. In fact take $h(t) := t^2$, $c(t) := t^3$ and $[a, b] = [-1, 1]$. Then $c'(t) \in h'(t)[-2, 2]$, but $c(1) - c(-1) = 2 \notin \{0\} = (h(1) - h(-1))[-2, 2]$.

## 2.4 Testing with functionals

Recall that in classical analysis vector valued curves $c : \mathbb{R} \to \mathbb{R}^n$ are often treated by considering their components $c_k := \mathrm{pr_k} \circ c$, where $\mathrm{pr}_k : \mathbb{R}^n \to \mathbb{R}$ denotes the canonical projection onto the $k$-th factor $\mathbb{R}$. Since general locally convex spaces, do not have appropriate bases, we have to make this independent on the base and hence use all continuous linear functionals instead of the projections $\mathrm{pr}_k$. We will say that a property of a curve $c : \mathbb{R} \to E$ is *scalarly true*, iff $\ell \circ c : \mathbb{R} \to E \to \mathbb{R}$ has this property for all continuous linear functionals $\ell$ on $E$.

We want to compare scalar differentiability with differentiability. For finite dimensional spaces we know the trivial fact that these to notions coincide. For infinite dimensions we first consider $\mathcal{L}ip$-curves $c : \mathbb{R} \to E$. Since by [**2**, 5.2.7] boundedness can be tested by the continuous linear functionals we see, that $c$ is $\mathcal{L}ip$ iff $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}ip$ for all $\ell \in E^*$. Recall that $E^*$ denotes the *space of all continuous linear functionals*, whereas $E'$ denotes *space of all bounded linear functionals* on $E$. Moreover if for a bounded interval $J \subset \mathbb{R}$ we take as $B$ the absolutely convex hull of the bounded set $c(J) \cup \{\frac{c(t)-c(s)}{t-s} : t \ne s; t, s \in J\}$, then we see that $c|_J : J \to E_B$ is a well-defined $\mathcal{L}ip$-curve into $E_B$. Where by $E_B$ we denote the linear span of $B$ in $E$, equipped with the *Minkowski functional* $p_B(v) := \inf\{\lambda > 0 : v \in \lambda.B\}$. This is a normed space. Thus we have the following equivalent characterizations of $\mathcal{L}ip$-curves:

1. locally $c$ factors over a $\mathcal{L}ip$-curve into some $E_B$;
2. $c$ is $\mathcal{L}ip$;
3. $\ell \circ c$ is $\mathcal{L}ip$ for all $\ell \in E^*$.

For continuous instead of Lipschitz curves we obviously have the analogous implications ($1 \Rightarrow 2 \Rightarrow 3$). However if we take a non-convergent sequence $(x_n)_n$, which converges weakly (e.g. take an orthonormal base in a separable Hilbert space), and consider an infinite polygon $c$ through these points $x_n$, say with $c(\frac{1}{n}) = x_n$ and $c(0) = 0$. Then this is obviously a non-continuous curve but $\ell \circ c$ is continuous for all $\ell \in E^*$.

Furthermore the "worst" continuous curve - i.e. $c : \mathbb{R} \to \prod_{C(\mathbb{R},\mathbb{R})} \mathbb{R} =: E$ given by $(c(t))_f := f(t)$ for all $t \in \mathbb{R}$ and $f \in C(\mathbb{R}, \mathbb{R})$ - cannot be factored locally as a continuous curve over some $E_B$. Otherwise $c(t_n)$ would converge into some $E_B$ to $c(0)$, where $t_n$ is a given sequence converging to 0, say $t_n := \frac{1}{n}$. So $c(t_n)$ would converge *Mackey* to $c(0)$, i.e. there have to be $\mu_n \to \infty$ with $\{\mu_n(c(t_n) - c(0)) : n \in \mathbb{N}\}$ bounded in $E$. Since a set is bounded in the product iff its coordinates are bounded, we conclude that for all $f \in C(\mathbb{R}, \mathbb{R})$ the sequence $\mu_n(f(t_n) - f(0))$ has to be bounded. But we can choose a continuous function $f$ with $f(0) = 0$ and $f(t_n) = \frac{1}{\sqrt{\mu_n}}$ and conclude that $\mu_n(f(t_n) - f(0)) = \sqrt{\mu_n}$ is unbounded.

Similarly one shows that the reverse implications do not hold for differentiable, for $C^1$-curves and for $C^n$-curves.

However if we put instead some Lipschitz condition on the derivatives, there should be some chance, since this is a bornological concept. But in order to obtain this result, we should study convergence of sequences in $E_B$.

**2.5 Lemma. Mackey-convergence.**
*Let $B$ be a bounded and absolutely convex subset of $E$ and $(x_\gamma)_{\gamma \in \Gamma}$ be a net in $E_B$ and $x \in E_B$. Then the following two conditions are equivalent:*

1. *$x_\gamma$ converges to $x$ in the normed space $E_B$;*
2. *There exists a net $\mu_\gamma \to 0$ in $\mathbb{R}$, such that $x_\gamma - x \in \mu_\gamma \cdot B$.*

*In (2) we may assume that $\mu \geq 0$ and is monotonely decreasing. In the particular case of a sequence (or where we have a cofinal countable subset of $\Gamma$) we can choose all $\mu_n > 0$ and hence may divide.*

*A net $(x_\gamma)$ for which a bounded absolutely convex $B \subseteq E$ exists, such that $x_\gamma$ converges to $x$ in $E_B$ is called* MACKEY CONVERGENT *or short $M$-convergent.*

**Proof.** ($\Downarrow$) Let $\delta > 1$ and set $\mu_\gamma := \delta\, p_B(x_\gamma - x)$. By assumption $\mu_\gamma \to 0$ and $x_\gamma - x = \mu_\gamma \frac{x_\gamma - x}{\mu_\gamma}$, where $\frac{x_\gamma - x}{\mu_\gamma} := 0$ if $\mu_\gamma = 0$. Since $p_B(\frac{x_\gamma - x}{\mu_\gamma}) = \frac{1}{\delta} < 1$ or is 0, we conclude that $\frac{x_\gamma - x}{\mu_\gamma} \in B$.

If we replace $\mu_\gamma$ by $\min\{1, \sup\{|\mu_{\gamma'}| : \gamma' \geq \gamma\}\}$ we see that we may choose $\mu \geq 0$ and monotonely decreasing with respect to $\gamma$.

If we have a sequence $(\gamma_n)_{n \in \mathbb{N}}$ which is cofinal in $\Gamma$, i.e. for every $\gamma \in \Gamma$ there exists an $n \in \mathbb{N}$ with $\gamma \leq \gamma_n$, then we may replace $\mu_\gamma$ by $\max(\{\mu_\gamma : \gamma_m \geq \gamma\} \cup \{\frac{1}{m}\})$ to conclude that $\mu_\gamma \neq 0$ for all $\gamma$.

($\Uparrow$) Let $x_\gamma - x = \mu_\gamma \cdot b_\gamma$ with $b_\gamma \in B$ and $\mu_\gamma \to 0$. Then $p_B(x_\gamma - x) = |\mu_\gamma|\, p_B(b_\gamma) \leq |\mu_\gamma| \to 0$, i.e. $x_\gamma \to x$ in $E_B$. □

If $\Gamma$ is the ordered set of all countable ordinals, then it is not possible to find a net $(\mu_\gamma)_{\gamma \in \Gamma}$, which is positive everywhere and converges to 0.

Now we show to describe the quality of convergence of the difference quotient.

**2.6 Corollary. The difference quotient converges Mackey.**
*Let $c : \mathbb{R} \to E$ be a $\mathcal{L}ip^1$-curve. Then the curve $t \mapsto \frac{1}{t}(\frac{1}{t}(c(t) - c(0)) - c'(0))$ is bounded on bounded subsets of $\mathbb{R} \setminus \{0\}$.*

**Proof.** We apply 2.3 to $c$ and obtain:

$$\begin{aligned}\frac{c(t) - c(0)}{t} - c'(0) &\in \Big\langle c'(r) : 0 < |r| < |t| \Big\rangle_{closed,convex} - c'(0) \\ &= \Big\langle c'(r) - c'(0) : 0 < |r| < |t| \Big\rangle_{closed,convex} \\ &= \Big\langle r\,\frac{c'(r) - c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{closed,convex}\end{aligned}$$

Let $a > 0$. Since $\{\frac{c'(r)-c'(0)}{r} : 0 < |r| < a\}$ is bounded and hence is contained in a closed absolutely convex and bounded set $B$, we can conclude that

$$\frac{1}{t}\left(\frac{c(t) - c(0)}{t} - c'(0)\right) \in \Big\langle \frac{r}{t}\,\frac{c'(r) - c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{closed,convex} \subseteq B. \quad \square$$

**2.7 Corollary. Smoothness of curves is a bornological concept.**
*For $0 \le k < \infty$ a curve $c$ in a locally convex vector space $E$ is $\mathcal{L}ip^k$ if and only if for each bounded open interval $J \subset \mathbb{R}$ there exists an absolutely convex bounded set $B \subset E$ such that $c|_J$ is a $\mathcal{L}ip^k$-curve in the normed space $E_B$.*

Attention: A smooth curve factors locally into some $E_B$ as a $\mathcal{L}ip^k$-curve for each finite $k$ only, in general. Take the "worst" smooth curve $c : \mathbb{R} \to \prod_{C^\infty(\mathbb{R},\mathbb{R})} \mathbb{R}$, analogously to 2.4, and, using Borel's theorem, deduce from $c^{(k)}(0) \in E_B$ for all $k \in \mathbb{N}$ a contradiction.

**Proof.** For $k = 0$ this was shown before. For $k \ge 1$ take a closed absolutely convex bounded set $B \subseteq E$ containing all derivatives $c^{(i)}$ on $J$ up to order $k$ as well as their difference quotients on $\{(t, s) : t \ne s, t, s \in J\}$. We show first that $c$ is differentiable, say at 0, with derivative $c'(0)$. By the previous corollary we have that the expression $\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right)$ lies in $B$. So $\frac{c(t)-c(0)}{t} - c'(0)$ converges to 0 in $E_B$. For the higher order derivatives we can now proceed by induction.

The converse follows from lemma 2.2. $\square$

A consequence of this is, that smoothness doesn't depend on the topology but only on the dual (so all topologies with the same dual have the same smooth curves), and in fact it depends only on the bounded sets. Since on $L(E, F)$ there is essentially only one bornology (by the uniform boundedness principle) there is only one notion of $\mathcal{L}ip^n$-curves into $L(E, F)$. Furthermore the class of $\mathcal{L}ip^n$-curves doesn't change if we pass from a given locally convex topology to its *bornologification*, i.e. finest locally convex topology having the same bounded sets.

**2.8 Lemma. Bornologification.**
*The bornologification $E_{born}$ of a locally convex space can be described in the following equivalent ways:*

1. *It is the finest locally convex structure having the same bounded sets;*
2. *It is the final locally convex structure with respect to the inclusions $E_B \to E$, where $B$ runs through all bounded (closed) absolutely convex subsets.*

*Moreover, $E_{born}$ is bornological. Its continuous seminorms are exactly the bounded seminorms of $E$. An absolutely convex set is a 0-neighborhood in $E_{born}$ iff it is bornivorous, i.e. absorbs bounded sets.*

**Proof.** Let $E_{\mathrm{born}}$ be the vector space $E$ supplied with the finest locally convex structure having the same bounded sets as $E$.

($\Uparrow$) Since all bounded sets $B$ in $E$ are bounded in $E_{\mathrm{born}}$, the inclusions $E_B \to E_{\mathrm{born}}$ are bounded and hence continuous. Thus the final structure on $E$ induced by the inclusions $E_B \to E$ is finer than the structure of $E_{\mathrm{born}}$.

($\Downarrow$) Since every bounded subset of $E$ is contained in some absolutely convex bounded set $B \subseteq E$ it has to be bounded in the final structure given by all inclusions $E_B \to E$. Hence this final structure has exactly the same bounded sets as $E$. And we have equality between the final structure and that of $E_{\mathrm{born}}$.

A seminorm $p$ on $E$ is bounded, iff $p(B)$ is bounded for all bounded $B$, and this is exactly the case if $p|_{E_B}$ is a continuous seminorm on $E_B$ for all $B$, or equivalently that $p$ is a continuous seminorm for the final structure $E_{\mathrm{born}}$ on $E$ induced by the inclusions $E_B \to E$, see [**2**, 4.3.2].

As a consequence all bounded seminorms on $E_{\mathrm{born}}$ are continuous, and hence $E_{\mathrm{born}}$ is bornological. Recall the following equivalent properties characterizing bornological spaces $E$:

1. Every bounded linear mapping $T : E \to F$ is continuous;
2. Every bounded seminorm on $E$ is continuous;
3. Every absolutely convex bornivorous subset is a 0-neighborhood.

($3 \Rightarrow 2$), since for $a > 0$ the inverse images under bounded seminorms of intervals $(-\infty, a)$ are absolutely convex and bornivorous. In fact let $B$ be bounded and $a > 0$. Then by assumption $p(B)$ is bounded and hence there exists a $C > 0$ with $p(B) \subseteq C \cdot (-\infty, a)$. Hence $B \subseteq C \cdot p^{-1}(-\infty, a)$.

($2 \Rightarrow 1$), since $p \circ T$ is a bounded seminorm, for every continuous seminorm on $F$.

($2 \Rightarrow 3$), since the Minkowski-functional $p$ generated by an absolutely convex bornivorous subset is a bounded seminorm.

($1 \Rightarrow 2$) Since the canonical projection $T : E \to E/\ker p$ is bounded, for any bounded seminorm $p$, it is by assumption continuous. Hence $p = \tilde{p} \circ T$ is continuous, where $\tilde{p}$ denotes the canonical norm on $E/\ker p$ induced from $p$.

An absolutely convex subset $U$ is a 0-neighborhood for the final structure induced by $E_B \to E$ iff $U \cap E_B$ is a 0-neighborhood, or equivalently if $U$ absorbs $B$, for all bounded absolutely convex $B$, i.e. $U$ is bornivorous. □

Let us now return to the scalar differentiability. Corollary 2.6 gives us $\mathcal{L}ip^n$-ness provided we have appropriate candidates for the derivatives.

**2.9 Corollary. Scalar testing of curves.**
*Let $c^k : \mathbb{R} \to E$ for $k < n+1$ be curves such that $\ell \circ c^0$ is $\mathcal{L}ip^n$ and $(\ell \circ c^0)^{(k)} = \ell \circ c^k$ for all $k < n+1$ and all $\ell \in E^*$. Then $c^0$ is $\mathcal{L}ip^n$ and $(c^0)^{(k)} = c^k$.*

**Proof.** For $n = 0$ this was shown in 2.4. For $n \geq 1$, we have by 2.6 applied to $\ell \circ c$ that $\ell\left(\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c^1(0)\right)\right)$ is locally bounded and hence by [**2**, 5.2.7] the set $\left\{\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c^1(0)\right) : t \in I\right\}$ is bounded. Thus $\frac{c(t)-c(0)}{t}$ converges even Mackey to $c^1(0)$. Now the general statement follows by induction. □

## Completeness

But do we really need the knowledge of a candidate for the derivative? In finite dimensional analysis one uses often the Cauchy-condition to prove convergence. Here we will replace the Cauchy-condition again by a stronger condition, which provides information about the quality of Cauchy-ness:

A net $(x_\gamma)_{\gamma\in\Gamma}$ in $E$ is called *Mackey-Cauchy* provided that there exists a bounded (absolutely convex) set $B$ and a net $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbb{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$. As in 2.3 one shows that for a net $x_\gamma$ in $E_B$ this is equivalent to the condition that $x_\gamma$ is Cauchy in the normed space $E_B$.

**2.10 Lemma. The difference quotient is Mackey-Cauchy.**
*Let $c : \mathbb{R} \to E$ be scalarly a $\mathcal{L}ip^1$-curve. Then $t \mapsto \frac{c(t)-c(0)}{t}$ is a Mackey-Cauchy net for $t \to 0$.*

**Proof.** For $\mathcal{L}ip^1$-curves this is a immediate consequence of 2.6 but we only assume it to be scalarly $\mathcal{L}ip^1$. It is enough to show that $\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right)$ is bounded on bounded subsets on $\mathbb{R}\setminus\{0\}$. We may test this with continuous linear functionals, and hence may assume that $E = \mathbb{R}$. Then by the fundamental theorem of calculus we have

$$\begin{aligned}\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right) &= \int_0^1 \frac{c'(tr)-c'(sr)}{t-s}\,dr\\ &= \int_0^1 \frac{c'(tr)-c'(sr)}{tr-sr}\,r\,dr.\end{aligned}$$

Since $\frac{c'(tr)-c'(sr)}{tr-sr}$ is locally bounded by assumption, the same is true for the integral, and we are done. □

**2.11 Lemma. Mackey-Completeness.**
*For a space $E$ the following conditions are equivalent:*

1. *Every Mackey-Cauchy net converges in $E$;*
2. *Every Mackey-Cauchy sequence converges in $E$;*
3. *For every absolutely convex closed bounded set $B$ the space $E_B$ is complete;*
4. *For every bounded set $B$ there exists an absolutely convex bounded set $B' \supseteq B$ such that $E_{B'}$ is complete.*

*A space satisfying the equivalent conditions is called Mackey-complete. Note that a sequentially complete space is Mackey-complete.*

**Proof.** $(1 \Rightarrow 2)$ and $(3 \Rightarrow 4)$ are trivial.

$(2 \Rightarrow 3)$ Since $E_B$ is normed, it is enough to show sequential-completeness. So let $x_n$ be a Cauchy-sequence in $E_B$. Then $x_n$ is Mackey-Cauchy in $E$ and hence converges in $E$ to some point $x$. Since $p_B(x_n - x_m) \to 0$ there exists for every $\varepsilon > 0$ an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $p_B(x_n - x_m) < \varepsilon$ and hence $x_n - x_m \in \varepsilon B$. Taking the limit for $m \to \infty$ and using closedness of $B$ we conclude that $x_n - x \in \varepsilon B$ for all $n > N$. In particular $x \in E_B$ and $x_n \to x$ in $E_B$.

$(4 \Rightarrow 1)$ Let $(x_\gamma)_{\gamma\in\Gamma}$ be a Mackey-Cauchy net in $E$. So there is some net $\mu_{\gamma,\gamma'} \to 0$, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$ for some bounded set $B$. Let $\gamma_0$ be arbitrary. By (4) we may assume that $B$ is absolutely convex, and contains $x_{\gamma_0}$ and $E_B$ is complete. For $\gamma \in \Gamma$ we have that $x_\gamma = x_{\gamma_0} + x_\gamma - x_{\gamma_0} \in x_{\gamma_0} + \mu_{\gamma,\gamma_0} B \in E_B$, and $p_B(x_\gamma - x_{\gamma'}) \leq \mu_{\gamma,\gamma'} \to 0$. So $x_\gamma$ is a Cauchy net in $E_B$ and hence converges in $E_B$ and thus also in $E$. □

**2.12 Corollary. Scalar testing of differentiable curves.**
*Let $E$ be Mackey-complete and $c : \mathbb{R} \to E$ be a curve for which $\ell \circ c$ is $\mathcal{L}ip^n$ for all $\ell \in E^*$. Then $c$ is $\mathcal{L}ip^n$.*

**Proof.** For $n = 0$ this was shown in 2.4 without using any completeness, so let $n \geq 1$. Since we have shown in 2.10 that the difference quotient is a Mackey-Cauchy net we conclude that the derivative $c'$ exists and hence $(\ell \circ c)' = \ell \circ c'$. So we may apply the induction hypothesis to conclude that $c'$ is $\mathcal{L}ip^{n-1}$ and consequently $c$ is $\mathcal{L}ip^n$. □

Next we turn to integration. For continuous curves $c : [0, 1] \to E$ one can show completely analogously to 1-dimensional analysis that the *Riemann sums* $R(c, \mathcal{Z}, \xi)$, defined by $\sum_k (t_k - t_{k-1}) c(\xi_k)$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition $\mathcal{Z}$ of $[0, 1]$ and $\xi_k \in [t_{k-1}, t_k]$, form a Cauchy net with respect to the partial ordering given by the size of the *mesh* $\max\{|t_k - t_{k-1}| : 0 < k < n\}$. So under the assumption of sequential completeness we have a *Riemann-integral* of curves. A second way to see this is the following reduction to the 1-dimensional case.

**2.13 Lemma.**
*Let $L(E^*_{equi}, \mathbb{R})$ be the space of all linear functionals on $E^*$ which are bounded on equi-continuous sets, equipped with the complete locally convex topology of uniform convergence on these sets. There is a natural topological embedding $\delta : E \to L(E^*_{equi}, \mathbb{R})$ given by $\delta(x)(\ell) := \ell(x)$.*

**Proof.** Let $\mathcal{U}$ be a basis of absolutely convex closed 0-neighborhoods in $E$. Then the family of *polars* $U^o := \{\ell \in E^* : |\ell(x)| \leq 1 \text{ for all } x \in U\}$, with $U \in \mathcal{U}$ form a basis for the equi-continuous sets. And hence the *bipolars* $U^{oo} := \{\ell^* \in L(E^*_{\text{equi}}, \mathbb{R}) : |\ell^*(\ell)| \leq 1 \text{ for all } \ell \in U^o\}$ form a basis of 0-neighborhoods in $L(E^*_{\text{equi}}, \mathbb{R})$. By the bipolar-theorem [**2**, 7.4.7] we have $U = \delta^{-1}(U^{oo})$ for all $U \in \mathcal{U}$. This shows that $\delta$ is a homeomorphism onto its image. □

**2.14 Lemma. Integral of continuous curves.**
*Let $c : \mathbb{R} \to E$ be a continuous curve in a locally convex vector space. Then there is a unique differentiable curve $\int c : \mathbb{R} \to \widehat{E}$ in the completion $\widehat{E}$ of $E$ such that $(\int c)(0) = 0$ and $(\int c)' = c$.*

**Proof.** We show uniqueness first. Let $c_1 : \mathbb{R} \to \widehat{E}$ be a curve with derivative $c$ and $c_1(0) = 0$. For every $\ell \in E^*$ the composite $\ell \circ c_1$ is an antiderivative of $\ell \circ c$ with initial value 0, so it is uniquely determined, and since $E^*$ separates points $c_1$ is also uniquely determined.

Now we show the existence. By the previous lemma we have that $\widehat{E}$ is (isomorphic to) the closure of $E$ in the obviously complete space $L(E^*_{\text{equi}}, \mathbb{R})$. We define $(\int c)(t) : E^* \to \mathbb{R}$ by $\ell \mapsto \int_0^t (\ell \circ c)(s) ds$. It is a bounded linear functional on $E^*_{\text{equi}}$ since for an equi-continuous subset $\mathcal{E} \subset E^*$ the set $\{(\ell \circ c)(s) : \ell \in \mathcal{E}, s \in [0, t]\}$ is bounded. So $\int c : \mathbb{R} \to L(E^*_{\text{equi}}, \mathbb{R})$.

Now we show that $\int c$ is differentiable with derivative $\delta \circ c$.

$$\begin{aligned}
&\left(\frac{(\int c)(t + r) - (\int c)(r)}{t} - (\delta \circ c)(r)\right)(\ell) = \\
&\qquad = \frac{1}{t}\left(\int_0^{t+r} (\ell \circ c)(s) ds - \int_0^r (\ell \circ c)(s) ds - t(\ell \circ c)(r)\right) = \\
&= \frac{1}{t}\int_r^{r+t} \Big((\ell \circ c)(s) - (\ell \circ c)(r)\Big) ds = \int_0^1 \Big(\delta(c(r + ts)) - \delta(c(r))\Big)(\ell) ds.
\end{aligned}$$

Let $\mathcal{E} \subset E^*$ be equi-continuous and let $\varepsilon > 0$. Then there exists a neighborhood $U$ of 0 such that $|\ell(U)| < \varepsilon$ for all $\ell \in \mathcal{E}$. For sufficiently small $t$, all $s \in [0,1]$ and fixed $r$ we have $c(r+ts) - c(r) \in U$. So $|\int_0^1 \ell(c(r+ts) - c(r))ds| < \varepsilon$. This shows that the difference quotient of $\int c$ at $r$ converges to $c(r)$ uniformly on equi-continuous subsets.

It remains to show that $(\int c)(t) \in \widehat{E}$. By the mean value theorem 2.3 the difference quotient $\frac{1}{t}((\int c)(t) - (\int c)(0))$ is contained in the closed convex hull in $L(E^*_{\text{equi}}, \mathbb{R})$ of the subset $\{c(s) : 0 < s < t\}$ of $E$. So it lies in $\widehat{E}$. $\square$

**Definition of the integral.**
For continuous curves $c : \mathbb{R} \to E$ the *definite integral* $\int_a^b c \in \widehat{E}$ is given by $\int_a^b c = (\int c)(b) - (\int c)(a)$.

**2.15 Corollary. Basics on the integral.**
*For a continuous curve $c : \mathbb{R} \to E$ we have:*

1. $\ell(\int_a^b c) = \int_a^b (\ell \circ c)$ *for all* $\ell \in E'$.
2. $\int_a^b c + \int_b^d c = \int_a^d c$.
3. $\int_a^b (c \circ \varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} c$ *for* $\varphi \in C^1(\mathbb{R}, \mathbb{R})$.
4. $\int_a^b c$ *lies in the closed convex hull in* $\widehat{E}$ *of the set* $\{(b-a)c(t) : a < t < b\}$ *in* $E$.
5. $\int_a^b : C(\mathbb{R}, E) \to \widehat{E}$ *is linear.*
6. *(Fundamental theorem of calculus.) For each $C^1$-curve $c : \mathbb{R} \to E$ we have* $c(s) - c(t) = \int_t^s c'$. $\square$

Since we are interested in smooth curves mainly and for the scalar testing we only needed Mackey-completeness, we should try to apply this here too. So let $c : [0,1] \to E$ be a $\mathcal{L}ip$-curve and take a partition $\mathcal{Z}$ with mesh $\mu(\mathcal{Z})$ at most $\delta$. If we have a second partition, then we can take the common refinement. Let $[a,b]$ be one interval of the original partition with intermediate point $t$, and let $a = t_0 < t_1 < \dots < t_n = b$ be the refinement. Note that $|b-a| \le \delta$ and hence $|t - t_k| \le \delta$ Then we can estimate as follows:

$$(b-a)\,c(t) - \sum_k (t_k - t_{k-1})c(t_k) = \sum_k (t_k - t_{k-1})\,(c(t) - c(t_k)) = \sum_k \mu_k b_k,$$

where $b_k := \frac{c(t)-c(t_k)}{\delta}$ is contained in the *absolutely convex Lipschitz bound* $B := \left\langle \left\{ \frac{c(t)-c(s)}{t-s} : t, s \in [0,1] \right\} \right\rangle_{abs.conv}$ of $c$ and $\mu_k := (t_k - t_{k-1})\delta \ge 0$ and satisfies $\sum_k \mu_k = (b-a)\delta$. Hence we have for the Riemann-sums with respect to the original partition $\mathcal{Z}_1$ and the refinement $\mathcal{Z}'$ that $R(c, \mathcal{Z}_1) - R(c, \mathcal{Z}')$ lies in $\delta \cdot B$. So $R(c, \mathcal{Z}_1) - R(c, \mathcal{Z}_2) \in 2\delta B$ for any two partitions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of mesh at most $\delta$, i.e. the Riemann-sums form a Mackey-Cauchy net with coefficients $\mu_{Z_1,\mathcal{Z}_2} := \frac{1}{\max\{\mu(\mathcal{Z}_1), \mu(\mathcal{Z}_2)\}}$ and we have proved:

**2.16 Proposition. Integral of Lipschitz curves.**
*Let $c : [0,1] \to E$ be a Lipschitz curve into a Mackey-complete space. Then the Riemann-integral exists in $E$ as (Mackey)-limit of the Riemann-sums.* $\square$

Next we have to discuss the relationship between differentiable curves and Mackey convergent sequences. Recall that for $t_n \to t$ and any continuous curve $c$ we have that $c(t_n) \to c(t)$ and, conversely, given $x_n \to x$ then there exists a continuous curve $c$ (i.e. a reparametrization of the infinite polygon) and $t_n \searrow 0$ with $c(t_n) = x_n$. The corresponding result for smooth curves is the

**2.17 Special curve lemma.**
*Let $x_n$ be a sequence which converges fast to $x$ in $E$, i.e. for each $k \in \mathbb{N}$ the sequence $n^k(x_n - x)$ is bounded.*

*Then the infinite polygon through the $x_n$ can be parameterized as a smooth curve $c : \mathbb{R} \to E$ such that $c(\frac{1}{n}) = x_n$ and $c(0) = x$.*

**Proof.** Let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth map, which is 0 on $\{t : t \le 0\}$ and 1 on $\{t : t \ge 1\}$. The parameterization $c$ is defined as follows:

$$c(t) := \begin{cases} x & \text{for } t \le 0, \\ x_{n+1} + \varphi\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(x_n - x_{n+1}) & \text{for } \frac{1}{n+1} \le t \le \frac{1}{n}, . \\ x_1 & \text{for } t \ge 1 \end{cases}$$

Obviously $c$ is smooth on $\mathbb{R} \setminus \{0\}$ and the $p$-th derivative of $c$ for $t \in [\frac{1}{n+1}, \frac{1}{n}]$ is given by

$$c^{(p)}(t) = \varphi^{(p)}\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(n(n+1))^p(x_n - x_{n+1}).$$

Since $x_n$ converges fast to $x$, we have that $c^{(p)}(t) \to 0$ for $t \to 0$, since the first factor is bounded and the second goes to zero. Hence $c$ is smooth on $\mathbb{R}$, by the following lemma. □

**2.18 Lemma. Differentiable extension to an isolated point.**
*Let $c : \mathbb{R} \to E$ be continuous and on $\mathbb{R} \setminus \{0\}$ differentiable, and assume that the derivative $c' : \mathbb{R} \setminus \{0\} \to E$ has a continuous extension to $\mathbb{R}$. Then $c$ is differentiable at 0 and $c'(0) = \lim_{t \to 0} c'(t)$.*

**Proof.** Let $a := \lim_{t \to 0} c'(t)$. By the mean value theorem 2.3 we have that $\frac{c(t)-c(0)}{t} \in \langle c'(s) : 0 \ne |s| \le |t| \rangle_{closed,abs.conv.}$. Since $c'$ is assumed to be continuously extendable to 0 we have that for any closed absolutely convex 0-neighborhood $U$ there exists a $\delta > 0$ such that $c'(t) \in a + U$ for all $|t| < \delta$. Hence $\frac{c(t)-c(0)}{t} - a \in U$, i.e. $c'(0) = a$. □

The next result shows that we can pass though certain sequences $x_n \to x$ even with given velocities $v_n \to 0$.

**2.19 Corollary.**
*If $x_n \to x$ fast and $v_n \to 0$ fast in $E$, then there is smoothly parameterized polygon $c : \mathbb{R} \to E$ and $t_n \to 0$ in $\mathbb{R}$ such that $c(t_n + t) = x_n + tv_n$ for $t$ in a neighborhood of 0 depending on $n$.*

**Proof.** Consider the sequence $y_n$ defined by $y_{2n} := x_n + \frac{1}{4n(2n+1)} v_n$ and $y_{2n+1} := x_n - \frac{1}{4n(2n+1)} v_n$. It is easy to show that $y_n$ converges fast to $x$ and the parameterization $c$ of the polygon through the $y_n$ (using a function $\varphi$ which satisfies $\varphi(t) = t$ for $t$ near $\frac{1}{2}$) has the claimed properties, where $t_n := \frac{4n+1}{4n(2n+1)} = \frac{1}{2}\left(\frac{1}{2n} + \frac{1}{2n+1}\right)$. □

As first application we can give the following sharpening of 2.2.

**2.20 Corollary. Bounded linear maps.**
*A linear mapping $\ell : E \to F$ between locally convex vector spaces is bounded (or bornological), i.e. maps bounded sets to bounded ones, if and only if it maps smooth curves in $E$ to smooth curves in $F$.*

**Proof.** As in the proof of 2.2 one shows using 2.6 that a bounded linear map preserves $\mathcal{L}ip^k$-curves. Conversely assume that a linear map $\ell : E \to F$ carries smooth curves to locally bounded curves. Take a bounded set $B$ and assume that $f(B)$ is unbounded. Then there is a sequence $b_n$ in $B$ and some $\lambda \in F'$ such that $|(\lambda \circ \ell)(b_n)| \geq n^{n+1}$. The sequence $n^{-n} b_n$ converges fast to 0, hence lies on some compact part of a smooth curve by 2.17. Consequently $(\lambda \circ \ell)(n^{-n} b_n) = n^{-n}(\lambda \circ \ell)(b_n)$ is bounded, a contradiction. □

With respect to non-linear mappings we have the following two results:

**2.2o. Lemma.**
*Let $U$ be a $c^\infty$-open subset of a locally convex space, let $\mu_n \to \infty$ be a real sequence, and let $f : U \to F$ be a mapping which is bounded on each $\mu$-converging sequence in $U$. Then $f$ is bounded on every* BORNOLOGICALLY COMPACT SUBSET *(i.e. compact in some $E_B$) of $U$.*

**Proof.** By composing with a linear functional we may assume that $F = \mathbb{R}$. Let $K \subset E_B \cap U$ be compact in $E_B$ for some bounded absolutely convex set $B$. Assume that $f(K)$ is not bounded. So there is a sequence $(x_n)$ in $K$ with $|f(x_n)| \to \infty$. Since $K$ is compact in the normed space $E_B$ we may assume that $(x_n)$ converges to $x \in K$. By passing to a subsequence we may even assume that $x_n$ is $\mu$-converging. Contradiction. □

**2.21 Corollary. Bounded seminorms.**
*For a seminorm $p$ and a sequence $\mu_n \to \infty$ the following statements are equivalent:*

1. *$p$ is bounded;*
2. *$p$ is bounded on compact sets;*
3. *$p$ is bounded on $M$-converging sequences;*
4. *$p$ is bounded on $\mu$-converging sequences;*
5. *$p$ is bounded on images of bounded intervals under $\mathcal{L}ip^k$-curves.*

*The corresponding statement for subsets of $E$ is the following. For a radial subset $U \subset E$ the following properties are equivalent:*

1. *$U$ is bornivorous.*
2. *For all absolutely convex bounded sets $B$, the trace $U \cap E_B$ is a 0-neighborhood in $E_B$.*
3. *$U$ absorbs all compact subsets in $E$.*
4. *$U$ absorbs all Mackey convergent sequences.*

(4'). *$U$ absorbs all sequences convering Mackey to 0.*

5. *$U$ absorbs all $\mu$-convergent sequences (for a fixed $\mu$).*

(5'). *$U$ absorbs all sequences being $\mu$-convergent to 0.*

6. *$U$ absorbs the images of compact sets under $\mathcal{L}ip^k$-curves (for a fixed $k$).*

A set $U$ is called *radial* if $[0, 1] \cdot U \subset U$.
A sequence $x_n$ is called *$\mu$-convergent* to $x$ iff $\{\mu_n(x_n - x) : n \in \mathbb{N}\}$ is bounded.

**Proof.** We prove the statement on radial subsets, for seminorms $p$ it then follows by using the radial set $U := \{x \in E : p(x) \leq 1\}$ and the equality $K \cdot U = \{x \in E : p(x) \leq K\}$.

(1) ⇔ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (5)', (4) ⇒ (4)', (3) ⇒ (6) are trivial.

(6) ⇒ (5', for $\mu$ fast falling). Such a sequence lies by the special curve lemma 2.17 on some compact part of a smooth curve, hence gets absorbed by (6).

(5', for any $\mu$) ⇒ (1). Suppose $U$ does not absorb some bounded $B$. Hence there are $b_n \in B$ with $b_n \notin \mu_n^2 U$. However, $\frac{b_n}{\mu_n}$ is $\mu$-convergent to 0, so it is contained in

$KU$ for some $K > 0$. Equivalently, $b_n \in \mu_n KU \subseteq \mu_n^2 U$ for all $\mu_n \geq K$, which gives a contradiction. $\square$

**2.22 Corollary. Bornologification as locally convexification.**

*The bornologification of $E$ is the finest locally convex topology with one (hence all) of the following properties:*

1. *It has the same bounded sets as $E$.*
2. *It has the same Mackey converging sequences as $E$.*
3. *It has the same $\mu$-converging sequences as $E$ (for some fixed $\mu$).*
4. *It has the same $\mathcal{L}ip^k$-curves as $E$ (for some fixed $k \leq \infty$).*
5. *It has the same bounded linear mappings from $E$ into arbitrary locally convex spaces.*
6. *It has the same continuous linear mappings from normed spaces into $E$.*

**Proof.** Since the bornologification has the same bounded sets as the original topology, the other objects are also the same: they depend only on the bornology – this would not be true for compact sets. Conversely we consider a topology $\tau$ which has for one of the above mentioned types the same objects as the original one. Then $\tau$ has by 2.21 the same bornivorous absolutely convex subsets as the original one. Hence any 0-neighborhood of $\tau$ has to be bornivorous for the original topology, and hence is a 0-neighborhood of the bornologification of the original topology. $\square$

**2.23 Definition.**
The *$c^\infty$-topology* on a locally convex space $E$ is the final topology with respect to all smooth curves $\mathbb{R} \to E$. Its open sets will be called *$c^\infty$-open*. We will see later that in general it is not a topological vector space topology. However, by 2.22 and 2.26 we get that the finest locally convex topology coarser than the $c^\infty$-topology is the bornologification of the locally convex topology.

**2.24 Theorem. $c^\infty$-open subsets.**
*Let $\mu \to \infty$ be a real-valued sequence. Then a subset $U \subset E$ is open for the $c^\infty$-topology if it satisfies any of the following equivalent conditions:*

1. *All inverse images under $\mathcal{L}ip^k$-curves are open in $\mathbb{R}$ (for fixed $k \in \mathbb{N}_\infty$).*
2. *All inverse images under $\mu$-converging sequences are open in $\mathbb{N}_\infty$.*
3. *The traces to $E_B$ are open in $E_B$ for all absolutely convex bounded subsets $B \subset E$.*

*Note for closed subsets an equivalent statement reads as follows: A set $A$ is $c^\infty$-closed iff for every sequence $x_n \in A$, which is $\mu$-converging (respectively $M$-converging, resp. fast falling) towards $x$, the point $x$ belongs to $A$.*

The topology described in (2) is also called *Mackey-closure topology.* It is not the Mackey topology discussed in duality theory.

**Proof.** (1) $\Rightarrow$ (2). Suppose $x_n$ is $\mu$-converging to $x \in U$, but $x_n \notin U$ for infinitely many $n$. Then we may choose a subsequence again denoted by $x_n$, which is fast falling to $x$, hence lies on some compact part of a smooth curve $c$ as described in 2.17. Then $c(\frac{1}{n}) = x_n \notin U$ but $c(0) = x \in U$. This is a contradiction.

(2) $\Rightarrow$ (3). A sequence $x_n$, which converges in $E_B$ to $x$ with respect to $p_B$, is Mackey convergent, hence has a $\mu$-converging subsequence. Note that $E_B$ is normed and hence it is enough to consider sequences.

(3) ⇒ (2). Suppose $x_n$ is $\mu$-converging to $x$. Then the absolutely convex hull $B$ of $\{\mu_n(x_n - x) : n \in \mathbb{N}\} \cup \{x\}$ is bounded, and $x_n \to x$ in $(E_B, p_B)$, since $\mu_n(x_n - x)$ is bounded.

(2) ⇒ (1). Use that for a converging sequence of parameters $t_n$ the images $x_n := c(t_n)$ under a $\mathcal{L}ip$-curve $c$ are Mackey converging. □

By 2.24 every $M$-convergent sequence gives a continuous mapping $\mathbb{N}_\infty \to c^\infty E$ and hence converges in $c^\infty E$. Conversely a sequence converging in $c^\infty E$ is not necessarily Mackey convergent, see [**9**].

However one has the following result:

**2.25 Lemma. c$^\infty$-convergent sequences.**
*A sequence $x_n$ is convergent to $x$ in the $c^\infty$-topology if and only if every subsequence has a subsequence which is Mackey convergent to $x$.*

**Proof.** (⇐) is true for any topological convergence. In fact if $x_n$ would not converge to $x$, then there would be a neighborhood $U$ of $x$ and a subsequence of $x_n$ which lies outside of $U$ and hence cannot have a subsequence converging to $x$.

(⇒) It is enough to show that $x_n$ has a subsequence converging to $x$, since every subsequence of a $c^\infty$-convergent sequence is clearly $c^\infty$-convergent to the same limit. Without loss of generality we may assume that $x \notin A := \{x_n : n \in \mathbb{N}\}$. Hence $A$ cannot be $c^\infty$-closed, and thus there is a sequence $n_k \in \mathbb{N}$ such that $x_{n_k}$ converges Mackey to some point $x'$. The set $\{n_k : k \in \mathbb{N}\}$ cannot be bounded and hence we may assume that the $n_k$ are strictly increasing by passing to a subsequence. But then $x_{n_k}$ is a subsequence of $x$ which converges in $c^\infty E$ to $x$ and Mackey to $x'$ hence also in $c^\infty E$. Thus $x' = x$ and we are done. □

## Remark

A consequence of this lemma is, that there is no topology having as convergent sequences exactly the $M$-convergent ones, since this topology obviously would have to be coarser than the $c^\infty$-topology.

One can use this lemma also to show that the $c^\infty$-topology on a locally convex vector space gives a so called *arc-generated vector space*. See [**10**, 2.3.9 and 2.3.13] for a discussion of this.

**2.26 Lemma.**
*Let $E$ be a bornological locally convex vector space, $U \subset E$ a convex subset. Then $U$ is open for the locally convex topology of $E$ iff $U$ is open for the $c^\infty$-topology. Furthermore, an absolutely convex subset $U$ of $E$ is a 0-neighborhood for the locally convex topology if and only if it is so for the $c^\infty$-topology.*

**Proof.** (⇒) The $c^\infty$-topology is finer than the locally convex topology, cf. 2.8.

(⇐) Let $U$ be a convex 0-neighborhood with respect to the $c^\infty$-topology. By passing to $U \cap -U \subset U$ we may assume that $U$ is absolutely convex. By 2.21 it is enough to show that $U$ is bornivorous, i.e. absorbs bounded subsets. Assume that some bounded $B$ does not get absorbed by $U$. Then for every $n \in \mathbb{N}$ there exists a $b_n \in B$ with $b_n \notin nU$. Since $\frac{1}{n}b_n$ is Mackey convergent to 0, we conclude that $\frac{1}{n}b_n \in U$ for sufficiently large $n$. This yields a contradiction.

Let now $U$ be an absolutely convex 0-neighborhood for the $c^\infty$-topology. Hence $U$ absorbs Mackey-0-sequences. By 2.22 we have to show that $U$ is bornivorous, in order to obtain that $U$ is a 0-neighborhood for the locally convex topology, but this follows immediately from 2.21. □

**2.26a Corollary.**
*The bornologification of a locally convex space $E$ is the finest locally convex topology coarser than the $c^\infty$-topology on $E$.* □

Let us show next that the $c^\infty$-topology and $c^\infty$-completeness are intimately related.

**2.27 Theorem. $\mathbf{c^\infty}$-completeness.**
*Let $E$ be a locally convex vector space. $E$ is said to be $c^\infty$-complete or convenient if one of the following equivalent (completeness) conditions is satisfied:*

1. *Any Lipschitz curve in $E$ is locally Riemann integrable.*
2. *For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_2' = c_1$ (existence of an antiderivative).*
3. *$E$ is $c^\infty$-closed in any locally convex space.*
4. *If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then $c$ is smooth.*
5. *Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges; i.e. $E$ Mackey-complete, see 2.11.*
6. *If $B$ is bounded closed absolutely convex, then $E_B$ is a Banach space. This property is called locally complete in* [**14**].
7. *Any continuous linear mapping from a normed space into $E$ has a continuous extension to the completion of the normed space.*

Condition 4 says that in a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

In [**10**] a convenient vector space is always considered with its bornological topology — an equivalent but not isomorphic category.

**Proof.** For Mackey-complete spaces, i.e. when (5) is satisfied, we have shown in 2.12 that (4), in 2.16 that (1) is true, and in 2.11 that (6) is true.

(1⇒2) A smooth curve is Lipschitz, thus locally Riemann integrable. The indefinite Riemann integral equals the “weakly defined” integral of lemma 2.14, hence is an antiderivative.

(2⇒3) Let $E$ be a topological subspace of $F$. To show that $E$ is closed we use 2.24. Let $x_n \to x_\infty$ be fast falling, $x_n \in E$ but $x_\infty \in F$. By 2.17 the polygon $c$ through $(x_n)$ can be smoothly parameterized. Hence $c'$ is smooth and has values in the vector space generated by $\{x_n : n \neq \infty\}$, which is contained in $E$. Its antiderivative $c_2$ is up to a constant equal to $c$ and by (2) $x_1 - x_\infty = c(1) - c(0) = c_2(1) - c_2(0)$ lies in $E$. So $x_\infty \in E$.

(3⇒5) Let $F$ be the completion $\hat{E}$ of $E$. Any Mackey Cauchy sequence in $E$ has a limit in $F$ and since $E$ is by assumption $c^\infty$-closed in $F$ the limit lies in $E$, and hence the sequence converges in $E$.

(5⇒6) This we have already proved in Lemma 2.11.

(6⇒7) Let $f : F \to E$ be a continuous mapping on a normed space $F$. Since the image of the unit ball is bounded, it is a bounded mapping into $E_B$ for some closed absolutely convex $B$. But into $E_B$ it can be extended to the completion, since $E_B$ is complete.

(7⇒1) Let $c : \mathbb{R} \to E$ be a Lipschitz curve. Then $c$ is locally a continuous curve into $E_B$ for some absolutely convex $B$. The inclusion of $E_B$ into $E$ has a continuous extension to the completion of $E_B$ and $c$ is Riemann integrable in this Banach space, so also in $E$.

(3⇒4) Let $c : \mathbb{R} \to E$ be scalarwise smooth. So $t \mapsto \frac{c(t)-c(0)}{t}$ is Lipschitz on each bounded subset of $\mathbb{R} \setminus \{0\}$. Then $\frac{c(t_n)-c(0)}{t_n}$ is a Mackey Cauchy sequence for each sequence $t_n \to 0$, so it converges by (3), so $c$ is differentiable at 0 and its derivative is again scalarwise smooth by 2.20. So by induction $c$ is smooth.

(4⇒3) Let $E$ be embedded in some space $F$. We use again 2.24 in order to show that $E$ is $c^\infty$-closed in $F$. So let $x_n \to x_0$ fast falling, $x_n \in E$ for $n \neq 0$, but $x_0 \in F$. By 2.17 the polygon $c$ through $(x_n)$ can be smoothly parameterized in $F$, and $c(t) \in E$ for $t \neq 0$. We consider $\tilde{c}(t) := tc(t)$. This is a curve in $E$ which is smooth in $F$, so it is scalarwise smooth in $E$, thus smooth in $E$ by (4). Then $x_0 = \tilde{c}'(0) \in E$. □

**2.28 Theorem. Inheritance of $\mathbf{c}^\infty$-completeness.**
*The following constructions preserve $c^\infty$-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings, as well as formation of $\ell^\infty(X, \_)$, where $X$ is a set together with a family $\mathcal{B}$ of subsets of $X$ containing the finite ones, which are called bounded and $\ell^\infty(X, F)$ denotes the space of all functions $f : X \to F$, which are bounded on all $B \in \mathcal{B}$, supplied with the topology of uniform convergence on the sets in $\mathcal{B}$.*

See 3.23 for an introduction to the categorical concept of limits.

**Proof.** The projective limit of a *cone* $\mathcal{F}$ is the $c^\infty$-closed linear subspace

$$\Big\{(x_\alpha) \in \prod \mathcal{F}(\alpha) : \mathcal{F}(f)x_\alpha = x_\beta \text{ for all } f : \alpha \to \beta\Big\},$$

hence is $c^\infty$-complete, since the product of $c^\infty$-complete factors is obviously $c^\infty$-complete.

Since the coproduct of spaces $X_\alpha$ is the topological direct sum, and has as bounded sets those which are contained and bounded in some finite subproduct, it is $c^\infty$-complete if all factors are.

For colimits this is in general not true. For strict inductive limits of sequences of closed embeddings it is true, since bounded sets are contained and bounded in some step, see [**2**, 4.8.1].

For the result on $\ell^\infty(X, F)$ we consider first the case, where $X$ itself is bounded. Then $c^\infty$-completeness can be proved as in [**2**, 3.2.3] or reduced to this result. In fact let $\mathcal{B}$ be bounded in $\ell^\infty(X, F)$. Then $\mathcal{B}(X)$ is bounded in $F$ and hence contained in some absolutely convex bounded set $B$, for which $F_B$ is a Banach space. So we may assume that $\mathcal{B} := \{f \in \ell^\infty(X, F) : f(X) \subseteq B\}$. The space $\ell^\infty(X, F)_\mathcal{B}$ is just the space $\ell^\infty(X, F_B)$ with the supremum norm, which is a Banach space by [**2**, 3.2.3].

Let now $X$ and $\mathcal{B}$ be arbitrary. Then the restriction maps $\ell^\infty(X, F) \to \ell^\infty(B, F)$ give an embedding $\iota$ of $\ell^\infty(X, F)$ into the product $\prod_{B \in \mathcal{B}} \ell^\infty(B, F)$. Since this product is complete, by what we have shown above, it is enough to show that this embedding has a closed image. So let $f_\alpha|_B$ converge to some $f_B$ in $\ell^\infty(B, F)$. Define $f(x) := f_{\{x\}}(x)$. For any $B \in \mathcal{B}$ containing $x$ we have that $f_B(x) = (\lim_\alpha f_\alpha|_B)(x) = \lim_\alpha(f_\alpha(x)) = \lim_\alpha f_\alpha|_{\{x\}} = f_{\{x\}}(x) = f(x)$. And $f(B)$ is bounded for all $B \in \mathcal{B}$, since $f|_B = f_B \in \ell^\infty(B, F)$.

Note that the definition of the topology of uniform convergence as initial topology shows, that adding all subsets of finite unions of elements in $\mathcal{B}$ to $\mathcal{B}$ does not change this topology. Hence we may always assume that $\mathcal{B}$ has this stability property. □

**Example.**
In general a quotient and an inductive limit of $c^\infty$-complete spaces need not be $c^\infty$-complete. In fact let $E_D := \{x \in \mathbb{R}^{\mathbb{N}} : \operatorname{supp} x \subseteq D\}$ for any subset $D \subseteq \mathbb{N}$ of *density* $\operatorname{dens} D := \limsup\{\frac{|D\cap[1,n]|}{n}\} = 0$. It can be shown that $E := \bigcup_{n\in\mathbb{N}} E_D$ is the inductive limit of the Fréchet subspaces $E_D \cong \mathbb{R}^D$. It cannot be $c^\infty$-complete, since finite sequences are contained in $E$ and are dense in $\mathbb{R}^{\mathbb{N}} \supset E$.

In general the trace of the $c^\infty$-topology on a linear subspace is not its $c^\infty$-topology. However for closed subspaces this is true:

**2.29 Lemma. Closed embedding lemma.**
*Let $E$ be a linear $c^\infty$-closed subspace of $F$. Then the trace of the $c^\infty$-topology of $F$ on $E$ is the $c^\infty$-topology on $E$*

**Proof.** Since the inclusion is continuous and hence bounded it is $c^\infty$-continuous. Hence it is enough to show that it is closed for the $c^\infty$-topologies. So let $A \subset E$ be $c^\infty E$-closed. And let $x_n \in A$ converge Mackey towards $x$ in $F$. Then $x \in E$, since $E$ is assumed to be $c^\infty$-closed, and hence $x_n$ converges Mackey to $x$ in $E$. Since $A$ is $c^\infty$-closed in $E$, we have that $x \in A$. □

Let us give an example which shows that $c^\infty$-closedness of the subspace is essential for this result. Another example will be given in 2.34.

**2.30 Example.**
*The trace of the $c^\infty$-topology is not the $c^\infty$-topology.*

**Proof.** Consider $E = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{(\mathbb{N})}$, $A := \{a_{n,k} := (\frac{1}{n}\chi_{\{1,..,k\}}, \frac{1}{k}\chi_{\{n\}}) : n, k \in \mathbb{N}\} \subset E$. Let $F$ be the linear subspace of $E$ generated by A. We show that the closure of $A$ with respect to the $c^\infty$-topology of $F$ is strictly smaller than that with respect to the trace topology of the $c^\infty$-topology of $E$.

$A$ is closed in the $c^\infty$-topology of $F$: Assume that a sequence $(a_{n_j,k_j})$ is M-converging to $(x, y)$. Then the second component of $a_{n_j,k_j}$ has to be bounded. Thus $j \mapsto n_j$ has to be bounded and may be assumed to have constant value $n_\infty$. If $j \mapsto k_j$ were unbounded, then $(x, y) = (\frac{1}{n_\infty}\chi_{\mathbb{N}}, 0)$, which is not an element of $F$. Thus $j \mapsto k_j$ has to be bounded too and may be assumed to have constant value $k_\infty$. Thus $(x, y) = a_{n_\infty,k_\infty} \in A$.

$A$ is not closed in the trace topology since (0,0) is contained in the closure of $A$ with respect to the $c^\infty$-topology of $E$: For $k \to \infty$ and fixed $n$ the sequence $a_{n,k}$ is M-converging to $(\frac{1}{n}\chi_{\mathbb{N}}, 0)$ and $\frac{1}{n}\chi_{\mathbb{N}}$ is M-converging to 0 for $n \to \infty$. □

**2.31 Theorem. The $\mathbf{c^\infty}$-completion.**
*For any locally convex space $E$ there exists a unique (up to a bounded isomorphism) convenient vector space $\tilde{E}$ and a bounded linear injection $i : E \to \tilde{E}$ with the following universal property:*

1. *Each bounded linear mapping $\ell : E \to F$ into a convenient vector space $F$ has a unique bounded extension $\tilde{\ell} : \tilde{E} \to F$ such that $\tilde{\ell} \circ i = \ell$.*

*Furthermore $i(E)$ is dense for the $c^\infty$-topology in $\tilde{E}$.*

**Proof.** Let $\tilde{E}$ be the $c^\infty$-closure of $E$ in the locally convex completion $\widehat{E_{\text{born}}}$ of the bornologification $E_{\text{born}}$ of $E$. The inclusion $i : E \to \tilde{E}$ is bounded (not continuous in general). By 2.29 the $c^\infty$-topology on $\tilde{E}$ is the trace of the $c^\infty$-topology on $\widehat{E_{\text{born}}}$. Hence $i(E)$ is dense also for the $c^\infty$-topology in $\tilde{E}$.

Using the universal property of the locally convex completion the mapping $\ell$ has a unique extension $\hat{\ell} : \widehat{E_{\text{born}}} \to \widehat{F}$ into the locally convex completion of $F$, whose restriction to $\tilde{E}$ has values in $F$, since $F$ is $c^\infty$-closed in $\widehat{F}$, so it is the desired $\tilde{\ell}$. Uniqueness follows, since $i(E)$ is dense for the $c^\infty$-topology in $\tilde{E}$. $\square$

**Example.**
We consider the space $\ell^\infty(X) := \ell^\infty(X, \mathbb{R})$ as defined in 2.28 for a set $X$ together with a family $\mathcal{B}$ of subsets called bounded. We have the subspace $C_c(X) := \{f \in \ell^\infty(X) : \operatorname{supp} f \text{ is finite}\}$. And we want to calculate its $c^\infty$-closure in $\ell^\infty(X)$.

**Claim:** The $c^\infty$-closure of $C_c(X)$ equals $C_0(X) := \{f \in \ell^\infty(X) : f|_B \in c_0(B) \text{ for all } B \in \mathcal{B}\}$ provided $X$ is countable.

**Proof.** The right hand side is just the intersection $c_0(X) := \bigcap_{B\in\mathcal{B}} \iota_B^{-1}(c_0(B))$, where $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ denotes the restriction map. We use the notation $c_0(X)$, since in the case where $X$ is bounded this is exactly the space $\{f \in \ell^\infty(X) : \forall\varepsilon > 0 \{x : |f(x)| \ge \varepsilon\} \text{ is finite}\}$. In particular this applies to the bounded space $\mathbb{N}$, where $c_0(\mathbb{N}) = c_0$. Since $\ell^\infty(X)$ carries the initial structure with respect to these maps $c_0(X)$ is closed. It remains to show that $C_c(X)$ is $c^\infty$-dense in $c_0(X)$. So let $f \in c_0(X)$. Let $\{x_1, x_2, \dots\} := \{x : f(x) \ne 0\}$.

We consider first the case, where there exists some $\delta > 0$ such that $|f(x_n)| \ge \delta$ for all $n$. Then we consider the functions $f_n := f \cdot \chi_{x_1,\dots,x_n} \in C_c(X)$. We claim that $n(f - f_n)$ is bounded in $\ell^\infty(X, \mathbb{R})$. In fact let $B \in \mathcal{B}$. Then $\{n : x_n \in B\} = \{n : x_n \in B \text{ and } |f(x_n)| \ge \delta\}$ is finite. Hence $\{n(f - f_n)(x) : x \in B\}$ is finite and thus bounded, i.e. $f_n$ converges Mackey to $f$.

Now the general case. We set $X_n := \{x \in X : |f(x)| \ge \frac{1}{n}\}$ and define $f_n := f \cdot \chi_{X_n}$. Then each $f_n$ satisfies the assumption of the particular case with $\delta = \frac{1}{n}$ and hence is a Mackey limit of a sequence in $C_c(X)$. Furthermore $n(f - f_n)$ is uniformly bounded by 1, since for $x \in X_n$ it is 0 and otherwise $|n(f - f_n)(x)| = n|f(x)| < 1$. So after forming the Mackey adherence (i.e. adding the limits of all Mackey-convergent sequences contained in the set, see 2.36 for a formal definition) twice, we obtain $c_0(X)$. $\square$

Now we want to show that $c_0(X)$ is in fact the $c^\infty$-completion of $C_c(X)$. For this we need the following proposition:

**2.32 Proposition. $c^\infty$-completion via $c^\infty$-dense embeddings.**
*Let $E$ be $c^\infty$-dense and bornologically embedded into a $c^\infty$-complete seminormed space $F$. If $E \to F$ has the extension property for bounded linear functionals, then $F$ is bornologically isomorphic to the $c^\infty$-completion of $E$.*

**Proof.** We have to show that $E \to F$ has the universal property for extending bounded linear maps $T$ into $c^\infty$-complete seminormed spaces $G$. Since we are only interested in bounded mappings, we may take the bornologification of $G$ and hence may assume that $G$ is bornological. Consider the following diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[dd, "T"'] & F \arrow[d, dotted] \arrow[ddl, dotted, "\widetilde{T}"'] \arrow[ddr, dotted, "\widetilde{\lambda\circ T}"] & \\
& \prod_{G'} \mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, "\delta"] \arrow[rr, hook, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

The arrow $\delta$, given by $\delta(x)_\lambda := \lambda(x)$, is a *bornological embedding*, i.e. the image of a set is bounded iff the set is bounded, since $B \subseteq G$ is bounded iff $\lambda(B) \subseteq \mathbb{R}$ is bounded for all $\ell \in G'$, i.e. $\delta(B) \subseteq \prod_{G'} \mathbb{R}$ is bounded.

By assumption the dashed arrow on the right hand side exists, hence by the universal property of the product the dashed vertical arrow exists. Remains to show that it has values in the image of $\delta$. But it is bounded, hence we have

$$\tilde{T}(F) = \tilde{T}(\overline{E}^{c^\infty}) \subseteq \overline{\tilde{T}(E)}^{c^\infty} \subseteq \overline{\delta(G)}^{c^\infty} = \delta(G),$$

since $G$ is $c^\infty$-complete and hence also $\delta(G)$, which is thus $c^\infty$-closed.

The uniqueness follows, since as a bounded linear map $\tilde{T}$ has to be continuous for the $c^\infty$-topology (since it preserves the smooth curves by 2.20 which in turn generate the $c^\infty$-topology) and $E$ lies dense in $F$ with respect to this topology. $\square$

**2.33 Example. $\mathbf{c_0(X)}$.**
We claim that $c_0(X)$ is the $c^\infty$-completion of the subspace $C_c(X)$ in $\ell^\infty(X)$ formed by the finite sequences.
We may assume that the bounded sets of $X$ are formed by those subsets $B$, for which $f(B)$ is bounded for all $f \in \ell^\infty(X)$. Obviously any bounded set has this property and the space $\ell^\infty(X)$ is not changed by adding these sets. Furthermore the restriction map $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ is also bounded for such a $B$, since using the closed graph theorem we only have to show that $\mathrm{ev}_b \circ \iota_B = \iota_{\{b\}}$ is bounded for every $b \in B$, see [**2**, 5.3.8], which is obviously the case.

By the previous proposition it is enough to show the universal property for bounded linear functionals. In analogy to Banach-theory, we only have to show that the dual $C_c(X)'$ is just

$$\ell^1(X) := \{g : X \to \mathbb{R} : \operatorname{supp} g \text{ is bounded and } g \text{ is absolutely summable}\}.$$

In fact any such $g$ acts even as bounded linear functional on $\ell^\infty(X, \mathbb{R})$ by $\langle g, f\rangle := \sum_x g(x)\, f(x)$, since a subset is bounded in $\ell^\infty(X)$ iff it is uniformly bounded on all bounded sets $B \subseteq X$. Conversely let $\ell : C_c(X) \to \mathbb{R}$ be bounded and linear and define $g : X \to \mathbb{R}$, by $g(x) := \ell(e_x)$, where $e_x$ denotes the function given by $e_x(y) := 1$ for $x = y$ and 0 otherwise. Obviously $\ell(f) = \langle g, f\rangle$ for all $f \in C_c(X)$. Suppose indirectly $\operatorname{supp} g = \{x : \ell(e_x) \neq 0\}$ is not bounded. Then there exists a sequence $x_n \in \operatorname{supp} g$ and a function $f \in \ell^\infty(X)$ such that $|f(x_n)| \geq n$. In particular the only bounded subsets of $\{x_n : n \in \mathbb{N}\}$ are the finite ones. Hence $\{\frac{n}{|g(x_n)|} e_{x_n}\}$ is bounded in $C_c(X)$ but the image under $\ell$ is not. Furthermore $g$ has to be absolutely summable, since the set of finite subsums of $\sum_x \operatorname{sign} g(x)\, e_x$ is bounded in $C_c(X)$ and its image under $\ell$ are the subsums of $\sum_x |g(x)|$.

**2.34 Corollary. Counter-examples on $\mathbf{c^\infty}$-topology.**
*The following statements are* **false***:*

1. *The $c^\infty$-closure of a subset (or of a linear subspace) is given by the Mackey adherence.*
2. *A subset $U$ of $E$ that contains a point $x$ and has the property, that every sequence which $M$-converges to $x$ belongs to it finally, is a $c^\infty$-neighborhood of $x$.*
3. *A $c^\infty$-dense subspace of a $c^\infty$-complete space has this space as $c^\infty$-completion.*
4. *If a subspace $E$ is $c^\infty$-dense in the total space, then it is also $c^\infty$-dense in each linear subspace lying inbetween.*

5. *The $c^\infty$-topology of a linear subspace is the trace of the $c^\infty$-topology of the whole space.*
6. *Every bounded linear functional on a linear subspace can be extended to such a functional on the whole space.*
7. *A linear subspace of a bornological locally convex space is bornological.*
8. *The $c^\infty$-completion preserves embeddings.*

**Proof.** (1) For this we give an example, where the $M$-adherence of $C_c(X)$ is not all of $c_0(X)$. By $M$-adherence we mean the set formed by all limits of sequences in this subset which are $M$-convergent in the total space.
Let $X = \mathbb{N} \times \mathbb{N}$ and take as bounded sets all sets of the form $B_\mu := \{(n,k) : n \leq \mu(k)\}$, where $\mu$ runs through all functions $\mathbb{N} \to \mathbb{N}$. Let $f : X \to \mathbb{R}$ be defined by $f(n,k) := \frac{1}{k}$. Obviously $f \in c_0(X)$, since for given $j \in \mathbb{N}$ and function $\mu$ the set of points $(n,k) \in B_\mu$ for which $f(n,k) = \frac{1}{k} \geq \frac{1}{j}$ is the finite set $\{(n,k) : k \leq j, n \leq \mu(k)\}$.
Assume there were some sequence $f_n \in C_c(X)$ Mackey-convergent to $f$. By passing to a subsequence we may assume that $n^2(f - f_n)$ is bounded. Now choose $\mu(k)$ to be larger than all of the finitely many $n$, with $f_k(n,k) \neq 0$. If $k^2(f - f_k)$ is bounded on $B_\mu$, then in particular $\{k^2(f - f_k)(\mu(k), k) : k \in \mathbb{N}\}$ has to be bounded, but $k^2(f - f_k)(\mu(k), k) = k^2\frac{1}{k} - 0 = k$.

(2) Let $A$ be a set for which (1) fails, and choose $x$ in the $c^\infty$-closure of $A$ but not in the $M$-adherence of $A$. Then $U := E \setminus A$ satisfies the assumptions of (2). In fact let $x_n$ be a sequence, which converges Mackey to $x$ and assume that it is not finally in $U$. So we may assume without loss of generality that $x_n \notin U$ for all $n$, but then $A \ni x_n \to x$ would imply that $x$ is in the Mackey adherence of $A$. However $U$ cannot be a $c^\infty$-neighborhood of $x$. In fact such a neighborhood must meet $A$, since $x$ is assumed to be in the $c^\infty$-closure of $A$.

(3) Let $F$ be a locally convex vector space whose *M-adherence* in its $c^\infty$-completion $E$ is not all of $E$, e.g. $C_c(X) \subseteq c_0(X)$ as in the previous counter-example. Choose a $y \in E$ that is not contained in the M-adherence of $F$ and let $F_1$ be the subspace of $E$ generated by $F \cup \{y\}$. We claim that $F_1 \subset E$ can not be the $c^\infty$-completion although $F_1$ is obviously Mackey dense in the convenient vector space $E$. In order to see this we consider the linear map $\ell : F_1 \to \mathbb{R}$ characterized by $\ell(F) = 0$ and $\ell(y) = 1$. Clearly $\ell$ is well defined.

$\ell : F_1 \to \mathbb{R}$ is bornological: For any bounded $B \subset F_1$ there exists an $N$ such that $B \subseteq F + [-N, N]y$. Otherwise $b_n = x_n + t_n y \in B$ would exist with $t_n \to \infty$ and $x_n \in F$. This would imply that $b_n = t_n(\frac{x_n}{t_n} + y)$ and thus $-\frac{x_n}{t_n}$ would converge Mackey to $y$; contradiction.

Now assume that a bornological extension $\bar{\ell}$ to $E$ exists. Then $F \subseteq \ker(\bar{\ell})$ and $\ker(\bar{\ell})$ is $c^\infty$-closed, which is a contradiction to the $c^\infty$-denseness of $F$ in $E$. So $F_1 \subset E$ does not have the universal property of a $c^\infty$-completion.

This shows also that (6) fails.

(4) Furthermore it follows that $F$ is $c^\infty F_1$-closed in $F_1$ although $F$ and hence $F_1$ are $c^\infty$-dense in $E$.

(5) The trace of the $c^\infty$-topology of $E$ to $F_1$ cannot be the $c^\infty$-topology of $F_1$, since for the first one $F$ is obviously dense.

(7) Obviously the trace topology of the bornological topology on $E$ cannot be bornological on $F_1$, since otherwise the bounded linear functionals on $F_1$ would be continuous and hence extendable to $E$.

(8) Furthermore, the extension of the inclusion $\iota : F \oplus \mathbb{R} \cong F_1 \to E$ to the completion is given by $(x,t) \in E \oplus \mathbb{R} \cong \tilde{F} \oplus \mathbb{R} = \tilde{F}_1 \mapsto x + ty \in E$ and has as kernel the linear subspace generated by $(y,-1)$. Hence the extension of an embedding to the $c^\infty$-completions need not be an embedding anymore, in particular the inclusion functor does not preserve injectivity of morphisms. $\square$

**2.35 Proposition. Inductive representation of bornological lcs.**
*For a locally convex space $E$ the bornologification $E_{born}$ is the colimit of all the normed spaces $E_B$ for the absolutely convex bounded sets $B$. The colimit of the respective completions $\tilde{E}_B$ is the linear subspace of the $c^\infty$-completion $\tilde{E}$ consisting of all limits in $\tilde{E}$ of Mackey Cauchy sequences in $E$.*

**Proof.** Let $E_1$ be the Mackey adherence of $E$ in the $c^\infty$-completion $\tilde{E}$, which is a subspace of the locally convex completion $\widehat{E_{\text{born}}}$. For every absolutely convex bounded set $B$ we have the continuous inclusion $E_B \to E_{\text{born}}$, and by passing to the $c^\infty$-completion we get mappings $\widehat{E_B} = \widetilde{E_B} \to \tilde{E}$. These mappings commute with the inclusions $\widehat{E_B} \to \widehat{E_{B'}}$ for $B \subseteq B'$ and have values in the Mackey adherence of $E$, since every point in $\widehat{E_B}$ is the limit of a sequence in $E_B$ and hence its image is the limit of this Mackey-Cauchy-sequence in $E$.

$$\begin{tikzcd}
\widehat{E_B} \arrow[rr] \arrow[dr] & & \widehat{E_{B'}} \arrow[dl] \\
& E_1 \arrow[d, hook] & \\
& \tilde{E} &
\end{tikzcd}$$

We claim that the Mackey adherence $E_1$ together with these mappings has the universal property of the colimit $\operatorname{colim}_B \widehat{E_B}$. In fact let $T : E_1 \to F$ be a linear mapping, such that $\widehat{E_B} \to E_1 \to F$ is continuous for all $B$. In particular $T|_E : E \to F$ has to be bounded, and hence $T|_{E_{\text{born}}} : E_{\text{born}} \to F$ is continuous. Thus it has a unique continuous extension $\hat{T} : E_1 \to \hat{F}$ and it remains to show that this extension is $T$. So take a point $x \in E_1$. Then there exists a sequence $(x_n)$ in $E$, which converges Mackey to $x$. Thus the $x_n$ form a Cauchy-sequence in some $E_B$ and hence converge to some $y$ in $\widehat{E_B}$. Then $\iota_B(y) = x$, since the mapping $\iota_B : \widehat{E_B} \to E_1$ is continuous. Since the trace of $T$ to $\widehat{E_B}$ is continuous $T(x_n)$ converges to $T(\iota_B(y)) = T(x)$ and $T(x_n) = \hat{T}(x_n)$ converges to $\hat{T}(x)$, i.e. $T(x) = \hat{T}(x)$. $\square$

In spite of (1) in 2.34 we can use the Mackey adherence to describe the $c^\infty$-closure in the following inductive way:

**2.36 Proposition. Mackey adherences.**
*For ordinals $\alpha$ let the Mackey-adherence $A^{(\alpha)}$ of order $\alpha$ be defined recursively by:*

$$A^{(\alpha)} := \begin{cases} \text{M-Adh}(A^{(\beta)}) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} A^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

*Then $\overline{A}^{c^\infty} = A^{\Omega}$, where $\Omega$ denotes the first uncountable ordinal, i.e. the set of all countable ordinals.*

**Proof.** Let us first show that $A^{(\Omega)}$ is $c^\infty$-closed. So take a sequence $x_n \in A^{(\Omega)} = \bigcup_{\alpha<\omega} A^{(\alpha)}$, which converges Mackey to some $x$. Then there are $\alpha_n < \Omega$ with $x_n \in A^{(\alpha_n)}$. Let $\alpha := \sup_n \alpha_n$. Then $\alpha$ is a again countable and hence less than $\Omega$.

Thus $x_n \in A^{(\alpha_n)} \subseteq A^{(\alpha)}$ and therefore $x \in \text{M-Adh}(A^{(\alpha)}) = A^{(\alpha+1)} \subseteq A^{(\Omega)}$ since $\alpha + 1 \leq \Omega$.

It remains to show that $A^{(\alpha)}$ is contained in $\overline{A}^{c^\infty}$ for all $\alpha$. We prove this by transfinite induction. So assume that for all $\beta < \alpha$ we have $A^{(\beta)} \subseteq \overline{A}^{c^\infty}$. If $\alpha$ is a limit ordinal then $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)} \subseteq \overline{A}^{c^\infty}$. If $\alpha = \beta + 1$ then every point in $A^{(\alpha)} = \text{M-Adh}(A^{(\beta)})$ is the Mackey-limit of some sequence in $A^{(\beta)} \subseteq \overline{A}^{c^\infty}$, and since $\overline{A}^{c^\infty}$ is $c^\infty$-closed, this limit has to belong to it. So $A^{(\alpha)} \subseteq \overline{A}^{c^\infty}$ in all cases. $\square$

## Smooth Maps and the Exponential Law

Now let us start proving the exponential law $C^\infty(U \times V, F) \cong C^\infty(U, C^\infty(V, F))$ first for $U = V = F = \mathbb{R}$.

**2.37 Theorem. Simplest case of exponential law.**
*Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping. Then all iterated partial derivatives exist and are locally bounded if and only if the associated mapping $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ exists as a smooth curve, where $c^\infty(\mathbb{R}, \mathbb{R})$ is considered as the Fréchet space with the topology of uniform convergence of each derivative on compact sets. Furthermore we have $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d \circ f^\vee = d_*(f^\vee)$.*

**Proof.** We have several possibilities to prove this result. Either we show Mackey-convergence of the difference quotients, via the boundedness of $\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right)$. And then use the trivial exponential law $\ell^\infty(X \times Y, \mathbb{R}) \cong \ell^\infty(X, \ell^\infty(Y, \mathbb{R}))$, or we use the induction step proved in 1.8, namely that $f^\vee : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ is differentiable iff $\partial_1 f$ exists and is continuous $\mathbb{R}^2 \to \mathbb{R}$, together with the exponential law $C(\mathbb{R}^2, \mathbb{R}) \cong C(\mathbb{R}, C(\mathbb{R}, \mathbb{R}))$. We choose the latter method.

For this we have to note first that if for a function $g$ the partial derivatives $\partial_1 g$ and $\partial_2 g$ exist and are locally bounded then $g$ is continuous:

$$\begin{aligned} g(x,y) - g(0,0) &= g(x,y) - g(x,0) + g(x,0) - g(0,0) \\ &= y\partial_2 g(x, r_2 y) + x\partial_1 g(r_1 x, 0) \end{aligned}$$

for suitable $r_1$, $r_2 \in [0,1]$, which goes to 0 with $(x,y)$.

($\Rightarrow$) By what we just said, all iterated partial derivatives of $f$ are continuous.

First observe that $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ makes sense and $d^q(f^\vee(t)) = (\partial_2^q f)^\vee(t)$ for all $t \in \mathbb{R}$.

Next we claim that $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is differentiable, with derivative $d(f^\vee) = (\partial_1 f)^\vee$, or equivalently that for all $a$ the curve

$$c : t \mapsto \begin{cases} \frac{f^\vee(t+a) - f^\vee(a)}{t} & \text{for } t \neq 0 \\ (\partial_1 f)^\vee(a) & \text{otherwise} \end{cases}$$

is continuous as curve $\mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$. Without loss of generality we may assume that $a = 0$. Since $C^\infty(\mathbb{R}, \mathbb{R})$ carries the initial structure with respect to the linear mappings $d^p : C^\infty(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ we have to show that $d^p \circ c : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ is continuous, or equivalently by the exponential law for continuous maps, that

$(d^p \circ c)^\wedge : \mathbb{R}^2 \to \mathbb{R}$ is continuous. For $t \neq 0$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}(d^p \circ c)^\wedge(t,s) &= d^p(c(t))(s)\\ &= d^p\left(\frac{f^\vee(t) - f^\vee(0)}{t}\right)(s)\\ &= \frac{\partial_2^p f(t,s) - \partial_2^p f(0,s)}{t} && \text{by what we mentioned above}\\ &= \int_0^1 \partial_1\partial_2^p f(t\,\tau, s)\,d\tau && \text{by the fundamental theorem.}\end{aligned}$$

For $t = 0$ we have

$$\begin{aligned}(d^p \circ c)^\wedge(0,s) &= d^p(c(0))(s)\\ &= d^p((\partial_1 f)^\vee(0))(s)\\ &= (\partial_2^p(\partial_1 f))^\vee(0)(s) && \text{by what we mentioned above}\\ &= \partial_2^p(\partial_1 f(0,s))\\ &= \partial_1(\partial_2^p f(0,s)) && \text{by the theorem of Schwarz}\end{aligned}$$

So we see that $(d^p \circ c)^\wedge(t,s) = \int_0^1 \partial_1\partial_2^p f(t\,\tau, s)\,d\tau$ for all $(t,s)$. This function is continuous in $(t,s)$, since $\partial_1\partial_2^p f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, hence $(t,s,\tau) \mapsto \partial_1\partial_2^p f(t\,\tau, s)$ is continuous, and therefore also $(t,s) \mapsto (\tau \mapsto \partial_1\partial_2^p f(t\,\tau, s))$ from $\mathbb{R}^2 \to C([0,1],\mathbb{R})$. Composition with the continuous linear mapping $\int_0^1 : C([0,1],\mathbb{R}) \to \mathbb{R}$ gives the continuity of $(d^p \circ c)^\wedge$.

Now we proceed by induction. By the induction hypothesis applied to $\partial_1 f$, we obtain that $d(f^\vee) = (\partial_1 f)^\vee$ and $(\partial_1 f)^\vee : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ is n-times differentiable, so $f^\vee$ is $(n+1)$-times differentiable.

($\Leftarrow$) First remark that for a smooth map $f : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ the associated map $f^\wedge : \mathbb{R}^2 \to \mathbb{R}$ is locally bounded: Since $f$ is smooth $f(I_1)$ is compact hence bounded in $C^\infty(\mathbb{R},\mathbb{R})$ for all compact intervals $I_1$. In particular $f(I_1)(I_2) = f^\wedge(I_1 \times I_2)$ has to be bounded in $\mathbb{R}$ for all compact intervals $I_1$ and $I_2$.

Since $f$ is smooth both curves $df$ and $d \circ f = d_* f$ are smooth (use 2.2 and that $d$ is continuous and linear). An easy calculation shows that the partial derivatives of $f^\wedge$ exist and are given by $\partial_1 f^\wedge = (df)^\wedge$ and $\partial_2 f^\wedge = (d \circ f)^\wedge$. So one obtains inductively that all iterated derivatives of $f^\wedge$ exist and are locally bounded, since they are associated to smooth curves $\mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$. $\square$

In order to proceed to more general cases of the exponential law we need a definition of $C^\infty$-maps defined on infinite dimensional spaces. This definition should at least guarantee the chain rule. And so one could take the weakest notion that satisfies the chain rule. However consider the following

**2.38 Example.**
We consider the following 3-fold "singular covering" $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in polar coordinates by $(r,\varphi) \mapsto (r, 3\varphi)$. In cartesian coordinates we obtain the following formula for the values of $f$:

$$\begin{aligned}(r\,\cos(3\varphi), r\sin(3\varphi)) &= r\Big((\cos\varphi)^3 - 3\cos\varphi(\sin\varphi)^2, 3\sin\varphi(\cos\varphi)^2 - (\sin\varphi)^3\Big)\\ &= \left(\frac{x^3 - 3xy^2}{x^2+y^2}, \frac{3x^2y - y^3}{x^2+y^2}\right).\end{aligned}$$

Note that the composite from the left with any orthonormal projection is just the composite of the first component of $f$ with a rotation from the right (Use that $f$ intertwines the rotation with angle $\delta$ and the rotation with angle $3\delta$).

Obviously the map $f$ is smooth on $\mathbb{R}^2 \setminus \{0,0\}$. It is homogeneous of degree 1, and hence the directional derivative is $f'(0)(v) = \frac{\partial}{\partial t}|_{t=0} f(tv) = f(v)$. However both components are not linear with respect to $v$ and thus are not differentiable at $(0,0)$.

Obviously $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

We claim that $f$ is differentiable along differentiable curves, i.e. $(f \circ c)'(0)$ exists, provided $c'(0)$ exists.
Only the case $c(0) = 0$ is not trivial. Since $c$ is differentiable at 0 the curve $c_1$ defined by $c_1(t) := \frac{c(t)}{t}$ for $t \neq 0$ and $c'(0)$ for $t = 0$ is continuous at 0. Hence $\frac{f(c(t))-f(c(0))}{t} = \frac{f(t\, c_1(t))-0}{t} = f(c_1(t))$. This converges to $f(c_1(0))$, since $f$ is continuous.

Furthermore, if $f'(x)(v)$ denotes the directional derivative, which exists everywhere, then $(f \circ c)'(t) = f'(c(t))(c'(t))$. Indeed for $c(t) \neq 0$ this is clear and for $c(t) = 0$ it follows from $f'(0)(v) = f(v)$.

The directional derivative of the 1-homogeneous mapping $f$ is 0-homogeneous: In fact for $s \neq 0$ we have

$$
\begin{aligned}
f'(sx)(v) = \frac{\partial}{\partial t}\Big|_{t=0} f(s\,x + tv) = \\
= s\ \frac{\partial}{\partial t}\Big|_{t=0} f(x + \frac{t}{s} v) = s\, f'(x)(\frac{1}{s} v) = f'(x)(v)
\end{aligned}
$$

For any $s \in \mathbb{R}$ we have $f'(s\,v)(v) = \frac{\partial}{\partial t}|_{t=0} f(s\,v + tv) = \frac{\partial}{\partial t}|_{t=s} t\, f(v) = f(v)$.

Using this homogeneity we show next, that it is also continuously differentiable along continuously differentiable curves. So we have to show that $(f \circ c)'(t) \to (f \circ c)'(0)$ for $t \to 0$. Again only the case $c(0) = 0$ is interesting. As before we factor $c$ as $c(t) = t\, c_1(t)$. In the case, where $c'(0) = c_1(0) \neq 0$ we have for $t \neq 0$ that

$$
\begin{aligned}
(f \circ c)'(t) - (f \circ c)'(0) &= f'(t\, c_1(t))(c'(t)) - f'(0)(c_1(0)) \\
&= f'(c_1(t))(c'(t)) - f'(c_1(0))(c_1(0)) \\
&= f'(c_1(t))(c'(t)) - f'(c_1(0))(c'(0)),
\end{aligned}
$$

which converges to 0 for $t \to 0$, since $(f')^\wedge$ is continuous (and even smooth) on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$.
In the other case, where $c'(0) = c_1(0) = 0$ we consider first the values of $t$, for which $c(t) = 0$. Then

$$
\begin{aligned}
(f \circ c)'(t) - (f \circ c)'(0) &= f'(0)(c'(t)) - f'(0)(c'(0)) \\
&= f(c'(t)) - f(c'(0)) \to 0,
\end{aligned}
$$

since $f$ is continuous. For the remaining values of $t$, where $c(t) \neq 0$, we factor $c(t) = |c(t)|\, c_1(t)$, with $c_1(t) \in \{x : \|x\| = 1\}$. Then

$$
(f \circ c)'(t) - (f \circ c)'(0) = f'(c_1(t))(c'(t)) - 0 \to 0,
$$

since $f'(x)(c'(t)) \to 0$ for $t \to 0$ uniformly for $\|x\| = 1$, since $c'(t) \to 0$.

Furthermore $f \circ c$ is smooth for all $c$ which are smooth and nowhere infinitely flat. In fact a smooth curve $c$ with $c^{(k)}(0) = 0$ for $k < n$ can be factored as $c(t) = t^n c_n(t)$ with smooth $c_n$, by Taylor's formula with integral remainder. Since $c^{(n)}(0) = n!\, c_n(0)$, we may assume that $n$ is chosen maximal and hence $c_n(0) \neq 0$. But then $(f \circ c)(t) = t^n \cdot (f \circ c_n)(t)$, and $f \circ c_n$ is smooth.

A completely analogous argument shows also that $f \circ c$ is real analytic for all real analytic curves $c : \mathbb{R} \to \mathbb{R}^2$.

However, let us show that $f \circ c$ is not Lipschitz-differentiable even for smooth curves $c$. For $x \neq 0$ we have

$$(\partial_2)^2 f(x,0) = \left(\tfrac{\partial}{\partial s}\right)^2 |_{s=0} f(x,s) = x \left(\tfrac{\partial}{\partial s}\right)^2 |_{s=0} f(1, \tfrac{1}{x}s) = \\ = \tfrac{1}{x} \left(\tfrac{\partial}{\partial s}\right)^2 |_{s=0} f(1,s) =: \tfrac{a}{x} \neq 0.$$

Now we choose a smooth curve which passes for each $n$ in finite time $t_n$ through $(\frac{1}{n^{2n+1}}, 0)$ with locally constant velocity vector $(0, \frac{1}{n^n})$. Then

$$(f \circ c)'(t_n + t) = \partial_1 f(c(t_n+t)) \underbrace{\mathrm{pr}_1(c'(t_n+t))}_{=0} + \partial_2 f(c(t_n+t)) \, \mathrm{pr}_2(c'(t_n+t))$$

$$(f \circ c)''(t_n) = (\partial_2)^2 f(c(t_n)) \, (\mathrm{pr}_2(c''(t_n)))^2 = a \, \frac{n^{2n+1}}{n^{2n}} = n\, a,$$

which is unbounded.

So although preservation of (continuous) differentiability is not enough to ensure differentiability of a function $\mathbb{R}^2 \to \mathbb{R}$ we now prove that smoothness can be tested with smooth curves.

**2.39 Boman's theorem.**
[**5**] *For a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ the following assertions are equivalent:*

1. *All iterated partial derivatives exist and are continuous.*
2. *All iterated partial derivatives exist and are locally bounded.*
3. *For $v \in \mathbb{R}^2$ the directional derivatives*
$$d_v^n f(x) := (\tfrac{\partial}{\partial t})^n|_{t=0}(f(x+tv))$$
*exist and are locally bounded with respect to $x$.*
4. *For all smooth curves $c : \mathbb{R} \to \mathbb{R}^2$ the composite $f \circ c$ is smooth.*

**Proof.** (1⇒2) is obvious.

(2⇒1) follows immediately, since the local boundedness of $\partial_1 f$ and $\partial_2 f$ imply the continuity of $f$ (see also the proof of 2.37):

$$f(t,s) - f(0,0) = t \int_0^1 \partial_1 f(\tau t, s) d\tau + s \int_0^1 \partial_2 f(0, \sigma s) d\sigma.$$

(1⇒4) is a direct consequence of the chain rule, namely that $(f \circ c)'(t) = \partial_1 f(c(t)) \cdot x'(t) + \partial_2 f(c(t)) \cdot y'(t)$, where $c = (x,y)$.

(4⇒3) Obviously $d_v^p f(x) := (\frac{d}{dt})^p|_{t=0} f(x+tv)$ exists, since $t \mapsto x + tv$ is a smooth curve. Suppose $d_v^p f$ is not locally bounded. So we may assume w.l.o.g. that a fast converging sequence $x_n$ to $x$ exists such that $|d_v^p f(x_n)| \geq 2^{n^2}$. Let $c$ be a smooth curve with $c(t + t_n) = x_n + \frac{t}{2^n} v$ locally for some sequence $t_n \to 0$. Then $(f \circ c)^{(p)}(t_n) = d_v^p f(x_n) \frac{1}{2^{np}}$ is unbounded, which is a contradiction.

(3⇒1) First we claim that $d_v^p f$ is continuous. We prove this by induction on $p$: $d_v^p f(. + tv) - dv^p f(.) = t \int_0^1 d_v^{p+1} f(. + t\tau v) d\tau \to 0$ for $t \to 0$ uniformly on bounded sets. Suppose now that $|d_v^p f(x_n) - d_v^p f(x)| \geq \varepsilon$ for some sequence $x_n \to x$. Without loss of generality we may assume that $d_v^p f(x_n) - d_v^p f(x) \geq \varepsilon$. Then by the uniform convergence there exists a $\delta > 0$ such that $d_v^p f(x_n + tv) - d_v^p f(x + tv) \geq \frac{\varepsilon}{2}$ for $|t| \leq \delta$. Integration $\int_0^\delta dt$ yields

$$d_v^{p-1} f(x_n + \delta v) - d_v^{p-1} f(x_n) - \Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) \geq \tfrac{\varepsilon\delta}{2},$$

but by induction hypothesis the left hand side converges towards

$$\Big(d_v^{p-1}f(x+\delta v)-d_v^{p-1}f(x)\Big)-\Big(d_v^{p-1}f(x+\delta v)-d_v^{p-1}f(x)\Big)=0$$

To complete the proof we use convolution by an *approximation of unity*. So let $\varphi\in C^\infty(\mathbb{R}^2,\mathbb{R})$ have compact support, $\int\varphi=1$, and $\varphi(y)\ge 0$ for all $y$. Define $\varphi_\varepsilon(x):=\frac{1}{\varepsilon^2}\varphi(\frac{1}{\varepsilon}x)$ and let

$$f_\varepsilon(x):=(f\star\varphi_\varepsilon)(x)=\int_{\mathbb{R}^2}f(x-y)\,\varphi_\varepsilon(y)\,dy=\int_{\mathbb{R}^2}f(x-\varepsilon y)\varphi(y)dy.$$

Since the *convolution* $f_\varepsilon:=f\star\varphi_\varepsilon$ of a continuous function $f$ with a smooth function $\varphi_\varepsilon$ with compact support is differentiable with directional derivative $d_v(f\star\varphi_\varepsilon)(x)=(f\star d_v\varphi_\varepsilon)(x)$, we obtain that $f_\varepsilon$ is smooth. And since $f\star\varphi_\varepsilon\to f$ in $C(\mathbb{R}^2,\mathbb{R})$ for $\varepsilon\to 0$ and any continuous function $f$, we conclude $d_v^pf_\varepsilon=d_v^pf\star\varphi_\varepsilon\to d_v^pf$ uniformly on compact sets.

We remark now that for a smooth map $g:\mathbb{R}^2\to\mathbb{R}$ we have by the chain rule

$$d_vg(x+tv)=\frac{d}{dt}g(x+tv)=\partial_1g(x+tv)\cdot v_1+\partial_2g(x+tv)\cdot v_2$$

that

$$d_v^pg(x)=\sum_{j=0}^{p}\binom{p}{i}v_1^iv_2^{p-i}\partial_1^i\partial_2^{p-i}g(x).$$

Hence we can calculate the iterated derivatives $\partial_1^i\partial_2^{p-i}$ for $0\le i\le p$ from $p+1$ many derivatives $d_{v^j}^pg(x)$ provided the $v^j$ are chosen in such a way, that the *Vandermonde's determinant* $\det((v_1^j)^i(v_2^j)^{p-i})_{ij}\ne 0$. For this choose $v_2=1$ and all the $v_1$ pairwise distinct, then $\det((v_1^j)^i(v_2^j)^{p-i})_{ij}=\prod_{j>k}(v_{1,j}-v_{1,k})\ne 0$.

Hence the iterated derivatives of $f_\varepsilon$ are linear combinations of the derivatives $d_v^pf_\varepsilon$ for $p+1$ many vectors $v$, where the coefficients depend only on the $v$'s. So we conclude that the iterated derivatives of $f_\varepsilon$ form a Cauchy-sequence in $C(\mathbb{R}^2,\mathbb{R})$ and hence converge to continuous functions $f^\alpha$. Hence all iterated derivatives $\partial^\alpha f$ of $f$ exist and are equal to these continuous functions $f^\alpha$, by the following lemma. $\square$

**2.40 Lemma.**
*Let $f_\varepsilon\to f$ in $C(\mathbb{R}^2,\mathbb{R})$ and $d_vf_\varepsilon\to f_v$ in $C(\mathbb{R}^2,\mathbb{R})$. Then $d_vf$ exists and equals $f_v$.*

**Proof.** We have to show that for fixed $x,v\in\mathbb{R}^2$ the curve

$$c:t\mapsto\begin{cases}\frac{f(x+tv)-f(x)}{t} & \text{for } t\ne 0\\ f_v(x) & \text{otherwise}\end{cases}$$

is continuous from $\mathbb{R}\to\mathbb{R}$. The corresponding curve $c_\varepsilon$ for $f_\varepsilon$ can be rewritten as $c_\varepsilon(t)=\int_0^1 d_vf_\varepsilon(x+\tau\,t\,v)\,d\tau$, which converges by assumption uniformly for $t$ in compact sets to the continuous curve $t\mapsto\int_0^1 f_v(x+\tau\,t\,v)\,d\tau$. Pointwise it converges to $c(t)$, hence $c$ is continuous. $\square$

For the vector valued case of the exponential law we need a locally convex structure on $C^\infty(\mathbb{R},E)$.

### 2.41 Definition. Space of curves

Let $C^\infty(\mathbb{R}, E)$ be the locally convex vector space of all smooth curves in $E$, with the pointwise vector operations and with the topology of uniform convergence on compact sets of each derivative separately. This is the initial topology with respect to the linear mappings $C^\infty(\mathbb{R}, E) \xrightarrow{d^k} C^\infty(\mathbb{R}, E) \to \ell^\infty(K, E)$, where $k$ runs through $\mathbb{N}$, where $K$ runs through all compact subsets of $\mathbb{R}$, and where $\ell^\infty(K, E)$ carries the topology of uniform convergence.

Note that the derivatives $d^k : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$, the point evaluations $\mathrm{ev}_t : C^\infty(\mathbb{R}, E) \to E$ and the pull backs $g^* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$ for all $g \in C^\infty(\mathbb{R}, \mathbb{R})$ are continuous and linear.

**2.42 Lemma.**
*A space $E$ is $c^\infty$-complete iff $C^\infty(\mathbb{R}, E)$ is.*

**Proof.** ($\Rightarrow$) The mapping $c \mapsto (c^{(n)})_{n\in\mathbb{N}}$ gives by definition an embedding of $C^\infty(\mathbb{R}, E)$ into the $c^\infty$-complete product $\prod_{n\in\mathbb{N}} \ell^\infty(\mathbb{R}, E)$. Its image is a closed subspace, since the previous lemma can be easily generalized to curves $c : \mathbb{R} \to E$.

($\Leftarrow$) Consider the continuous linear mapping $\mathrm{const} : E \to C^\infty(\mathbb{R}, E)$ given by $x \mapsto (t \mapsto x)$. It has as continuous left-inverse the evaluation at any point (e.g. $\mathrm{ev}_0 : C^\infty(\mathbb{R}, E) \to E$, $c \mapsto c(0)$). Hence $E$ can be identified with the closed subspace of $C^\infty(\mathbb{R}, E)$, given by the constant curves, and is thereby itself $c^\infty$-complete. $\square$

**2.43 Lemma. Curves into limits.**
*A curve into a $c^\infty$-closed subspace of a space is smooth if and only if it is smooth into the total space. In particular a curve is smooth into a projective limit, if all its components are smooth.*

**Proof.** Since the derivative of a smooth curve is the Mackey-limit of the difference quotient, the $c^\infty$-closedness implies that this limit belongs to the subspace. Thus we deduce inductively that all derivatives belong to the subspace, and hence the curve is smooth into the subspace.

The result on projective limits now follows, since obviously a curve is smooth into a product, if all its components are smooth. $\square$

We show now that the bornology, but obviously not the topology, on function spaces can be tested with the linear functionals on the range space.

**2.44 Lemma. Bornology of $C^\infty(\mathbb{R}, E)$.**
*The family*
$$\{\ell_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, \mathbb{R}) : \ell \in E^*\}$$
*generates the bornology of $C^\infty(\mathbb{R}, E)$.*

*A set is bounded if and only if each derivative is uniformly bounded on compact subsets.*

**Proof.** A set $\mathcal{B} \subset C^\infty(\mathbb{R}, E)$ is bounded if and only if the sets $\{d^n c(x) : x \in I, c \in \mathcal{B}\}$ are bounded in $E$ for all $n \in \mathbb{N}$ and compact subsets $I \subset \mathbb{R}$.

This is further equivalent to the condition that the set $\{\ell(d^n c(x)) = d^n(\ell \circ c)(x) : x \in I, c \in \mathcal{B}\}$ is bounded in $\mathbb{R}$ for all $\ell \in E'$, $n \in \mathbb{N}$, and compact subsets $I \subset \mathbb{R}$ and in turn equivalent to: $\{\ell \circ c : c \in \mathcal{B}\}$ is bounded in $C^\infty(\mathbb{R}, \mathbb{R})$. $\square$

**2.45 Proposition. Vector valued simplest exponential law.**
*For a mapping $f : \mathbb{R}^2 \to E$ the following assertions are equivalent:*

1. *$f$ is smooth along smooth curves.*
2. *All iterated directional derivatives $d_v^p f$ exist and are locally bounded.*
3. *All $\partial_\alpha f$ exist and are locally bounded.*
4. *$f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ exists as a smooth curve.*

**Proof.** We prove this result first for $c^\infty$-complete spaces $E$. Then each of the statements (1-4) are valid iff the corresponding statement for $\ell \circ f$ is valid for all $\ell \in E^*$. Only (4) needs some arguments: In fact $f^\vee(t) \in C^\infty(\mathbb{R}, E)$ iff $\ell_*(f^\vee(t)) = (\ell \circ f)^\vee(t) \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E^*$ by 2.27. Since $C^\infty(\mathbb{R}, E)$ is $c^\infty$-complete its bornologically isomorphic image in $\prod_{\ell \in E^*} C^\infty(\mathbb{R}, \mathbb{R})$ is $c^\infty$-closed. So $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ is smooth, iff $\ell_* \circ f^\vee = (\ell \circ f)^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is smooth for all $\ell \in E^*$. So the proof is reduced to the scalar valid case, which was proved in 2.37 and 2.39.

Now the general case. For the existence of certain derivatives we know by 2.9 that it is enough that we have some candidate in the space, which is the corresponding derivative of the map considered as map into the $c^\infty$-completion (or even some larger space). Since the derivatives required in (1-4) depend linearly on each other, this is true. In more detail this means:

(1$\Rightarrow$ 2) is obvious.

(2$\Rightarrow$ 3) is the fact that $\partial^\alpha$ is a universal linear combination of $d_v^{|\alpha|} f$.

(3$\Rightarrow$ 1) follows from the chain rule, saying that $(f \circ c)^{(p)}(t)$ is a universal linear combination of $\partial^\alpha f(c(t)) c_{i_1}^{(p_1)}(t) \dots c_{i_q}^{(p_q)}(t)$ for $\alpha = (i_1, \dots, i_q)$ and $\sum p_j = p$.

(3$\Leftrightarrow$4) since $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d \circ f^\vee = d_*(f^\vee)$. In order to apply 2.9 we have to use that $C^\infty(\mathbb{R}, E)$ is embedded into $C^\infty(\mathbb{R}, \tilde{E})$, which is obvious. □

For the general case of the exponential law we need a notion of smooth mappings and a locally convex topology on the corresponding function spaces. Of coarse it would be also handy to have a notion of smoothness for locally defined mappings. Since the idea is to test smoothness with smooth curves, such curves should have locally values in the domains of definition, and hence these domains should be $c^\infty$-open.

### 2.46 Definition. Smooth mappings and spaces of them

A mapping $f : E \supseteq U \to F$ defined on a $c^\infty$-open subset $U$ is called *smooth* (or $C^\infty$) if it maps smooth curves in $U$ to smooth curves in $F$.

Let $C^\infty(U, F)$ denote the locally convex *space of all smooth mappings* $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$.

For $U = E = \mathbb{R}$ this coincides with our old definition. Obviously any composition of smooth mappings is also smooth.

**Lemma.**
*The space $C^\infty(U, F)$ is the limit of spaces $C^\infty(\mathbb{R}, F)$, one for each $c \in C^\infty(\mathbb{R}, U)$ where the connecting mappings are pull backs $g^*$ along reparametrizations $g \in C^\infty(\mathbb{R}, \mathbb{R})$.*

Note that this limit is the closed linear subspace in the product

$$\prod_{c \in C^\infty(\mathbb{R}, U)} C^\infty(\mathbb{R}, F)$$

consisting of all $(f_c)$ with $f_{c \circ g} = f_c \circ g$ for all $c$ and all $g \in C^\infty(\mathbb{R}, \mathbb{R})$.

**Proof.** The mappings $c^* : C^\infty(U,F) \to C^\infty(\mathbb{R},F)$ define a continuous linear embedding $C^\infty(U,F) \to \lim_c\{C^\infty(\mathbb{R},F) \xrightarrow{g^*} C^\infty(\mathbb{R},F)\}$, since $c^*(f) \circ g = f \circ c \circ g = (c \circ g)^*(f)$. It is surjective since for any $(f_c) \in \lim_c C^\infty(\mathbb{R},F)$ one has $f_c = f \circ c$ where $f$ is defined as $x \mapsto f_{\mathrm{const}_x}(0)$. □

**2.47 Theorem.** *Cartesian closedness.*
*Let $U_i \subset E_i$ be $c^\infty$-open subsets. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to C^\infty(U_2,F)$ exists and is smooth.*

**Proof.** We have the following implications:

1. $f^\vee : U_1 \to C^\infty(U_2,F)$ is smooth.
2. "$\Leftrightarrow$" $f^\vee \circ c_1 : \mathbb{R} \to C^\infty(U_2,F)$ is smooth for all smooth curves $c_1$ in $U_1$, by 2.46.
3. "$\Leftrightarrow$" $c_2^* \circ f^\vee \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R},F)$ is smooth for all smooth curves $c_i$ in $U_i$, by 2.46 and 2.43.
4. "$\Leftrightarrow$" $f \circ (c_1 \times c_2) = (c_2^* \circ f^\vee \circ c_1)^\wedge : \mathbb{R}^2 \to F$ is smooth for all smooth curves $c_i$ in $U_i$, by 2.45.
5. "$\Rightarrow$" $f : U_1 \times U_2 \to F$ is smooth, since each curve into $U_1 \times U_2$ is of the form $(c_1,c_2) = (c_1 \times c_2) \circ \Delta$, where $\Delta$ is the diagonal mapping.
6. "$\Rightarrow$" $f \circ (c_1 \times c_2) : \mathbb{R}^2 \to F$ is smooth for all smooth curves $c_i$ in $U_i$, since the product and the composite of smooth mappings is smooth by 2.46 (and by 2.39). □

**2.48 Corollary. Consequences of cartesian closedness.**
*Let $E$, $F$, etc. be locally convex spaces and let $U$, $V$, etc. be $c^\infty$-open subsets of such. Then the following canonical mappings are smooth.*

1. $\mathrm{ev} : C^\infty(U,F) \times U \to F$, $(f,x) \mapsto f(x)$;
2. $\mathrm{ins} : E \to C^\infty(F, E \times F)$, $x \mapsto (y \mapsto (x,y))$;
3. $(\quad)^\wedge : C^\infty(U, C^\infty(V,G)) \to C^\infty(U \times V, G)$;
4. $(\quad)^\vee : C^\infty(U \times V, G) \to C^\infty(U, C^\infty(V,G))$;
5. $\mathrm{comp} : C^\infty(F,G) \times C^\infty(U,F) \to C^\infty(U,G)$, $(f,g) \mapsto f \circ g$;
6. $C^\infty(\quad,\quad) : C^\infty(E_2,E_1) \times C^\infty(F_1,F_2) \to$ $\to C^\infty(C^\infty(E_1,F_1), C^\infty(E_2,F_2))$, $(f,g) \mapsto (h \mapsto g \circ h \circ f)$;
7. $\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)$.

**Proof.** (1). The mapping associated to *ev* via cartesian closedness is the identity on $C^\infty(U,F)$, which is $C^\infty$, thus *ev* is also $C^\infty$.

(2). The mapping associated to *ins* via cartesian closedness is the identity on $E \times F$, hence *ins* is $C^\infty$.

(3). The mapping associated via cartesian closedness is $(f; x, y) \mapsto f(x)(y)$, which is the $C^\infty$-mapping $\mathrm{ev} \circ (\mathrm{ev} \times id)$.

(4). The mapping associated by applying cartesian closedness twice is $(f; x; y) \mapsto f(x,y)$, which is just a $C^\infty$ evaluation mapping.

(5). The mapping associated to *comp* via cartesian closedness is just $(f,g;x) \mapsto f(g(x))$, which is the $C^\infty$-mapping $\mathrm{ev} \circ (id \times \mathrm{ev})$.

(6). The mapping associated by applying cartesian closed twice is $(f,g;h,x) \mapsto g(h(f(x)))$, which is the $C^\infty$-mapping $\mathrm{ev} \circ (id \times \mathrm{ev}) \circ (id \times id \times \mathrm{ev})$.

(7). Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $C^\infty(E_i,F_i) \times E_i \to F_i$. □

Next we generalize 2.39 to each finite dimensions.

**2.49 Corollary.**
[**5**]. *The smooth mappings on open subsets of $\mathbb{R}^n$ in the sense of definition 2.46 are exactly the usual smooth mappings.*

**Proof.** Both conditions are of local nature, so we may assume that the open subset of $\mathbb{R}^n$ is an open box and in turn even $\mathbb{R}^n$ itself.

($\Rightarrow$) If $f : \mathbb{R}^n \to F$ is smooth then by cartesian closedness 2.47, for each coordinate the respective associated mapping $f^{\vee_i} : \mathbb{R}^{n-1} \to C^\infty(\mathbb{R}, F)$ is smooth, so again by 2.47 we have $\partial_i f = (d_* f^{\vee_i})^\wedge$, so all first partial derivatives exist and are smooth. Inductively all iterated partial derivatives exist and are smooth, thus continuous, so $f$ is smooth in the usual sense.

($\Leftarrow$) Obviously $f$ is smooth along smooth curves, by the usual chain rule. $\square$

## 2.50 Differentiation of an integral

We turn now again to the question of differentiating an integral. So let $f : E \times \mathbb{R} \to F$ be smooth. Then we may form the function $f_0 : E \to \tilde{F}$ defined by $x \mapsto \int_0^1 f(x,t)\,dt$. We claim that it is smooth and the directional derivative $d_v f_0(x) = \int_0^1 d_v(f(\_,t))(x)\,dt$. By cartesian closedness 2.47 the associated mapping $f^\vee : E \to C^\infty(\mathbb{R}, F)$ is smooth, so the mapping $\int_0^1 \circ f^\vee : E \to \tilde{F}$ is smooth since integration is a bounded linear operator and

$$\begin{aligned} d_v f_0(x) &= \tfrac{\partial}{\partial s}\big|_{s=0} f_0(x+sv) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \int_0^1 f(x+sv,t)dt \\ &= \int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} f(x+sv,t)dt \\ &= \int_0^1 d_v(f(\_,t))(x)\,dt. \end{aligned}$$

But we want to generalize this to functions $f$ defined only on some $c^\infty$-open subset $U \subseteq E \times \mathbb{R}$. We have to show that the natural domain $U_0 := \{x \in E : \{x\} \times [0,1] \subseteq U\}$ of $f_0$ is $c^\infty$-open in $E$. We will do this in the following lemma. But then the proof runs exactly the same way as for globally defined functions. So we obtain the

**Proposition.**
*Let $f : E \times \mathbb{R} \supseteq U \to F$ be smooth with $c^\infty$-open $U \subseteq E$. Then $x \mapsto \int_a^b f(x,t)\,dt$ is smooth $U_0 := \{x \in E : \{x\} \times [0,1] \subseteq U\} \to \tilde{F}$ and $d_v f_0(x) = \int_0^1 d_v(f(\_,t))(x)\,dt$ for all $x \in U_0$ and $v \in E$.* $\square$

**2.51 Lemma.**
*Let $U$ be $c^\infty$-open in $E \times \mathbb{R}$ and $K \subseteq \mathbb{R}$ be compact. Then $U_0 := \{x \in E : \{x\} \times K \subseteq U\}$ is $c^\infty$-open in $E$.*

**Proof.** Let $x : \mathbb{R} \to E$ be a smooth curve in $E$ with $x(0) \in U_0$, i.e. $(x(0),t) \in U$ for all $t \in K$. We have to show that this is true for all $t$ sufficiently close to 0. So consider the smooth map $x \times \mathbb{R} : \mathbb{R} \times \mathbb{R} \to E \times \mathbb{R}$. By assumption $(x \times \mathbb{R})^{-1}(U)$ is open in $c^\infty(\mathbb{R}^2) = \mathbb{R}^2$. It contains the compact set $\{0\} \times K$ and hence also a $W \times K$ for some neighborhood $W$ of 0 in $\mathbb{R}$. But this amounts in saying that $x(W) \subseteq U_0$. $\square$

Although the $c^\infty$-topology of a product is in general not the product of the $c^\infty$-topologies we have:

**Corollary. The $c^\infty$-topology of a product with $\mathbb{R}^n$.**
*The $c^\infty$-topology of $E \times \mathbb{R}^n$ is the product-topology of the $c^\infty$-topologies of the two factors, i.e. $c^\infty(E \times \mathbb{R}^n) = c^\infty(E) \times \mathbb{R}^n$.*

**Proof.** Since the projections to the factors are linear and continuous, hence smooth, we always have that the identity $c^\infty(E \times F) \to c^\infty(E) \times c^\infty(F)$ is continuous. It is not always a homeomorphism: Just take a bounded separately continuous bi-linear functional, which is not continuous (like the evaluation map) and such that the $c^\infty$-topology on both factors is the bornological topology.
The case of $F = \mathbb{R}^n$ follows recursively from the special case where $F = \mathbb{R}$, for which we can proceed as follows. Take a $c^\infty$-open neighborhood $U$ of some point $(x, t) \in E \times \mathbb{R}$. Since the inclusion map $s \mapsto (x, s)$ from $\mathbb{R}$ into $E \times \mathbb{R}$ is continuous and linear the inverse image of $U$ in $\mathbb{R}$ is a neighborhood of $t$. Let's take a smaller compact neighborhood $K$ of $t$. Then by the previous lemma $U_0 := \{y \in E : \{y\} \times K \subseteq U\}$ is a $c^\infty$-open neighborhood of $x$ and hence $U_0 \times K^o$ is a neighborhood of $(x, t)$ in $c^\infty(E) \times \mathbb{R}$, what was to be shown. □

Now we want to define the derivative of a general smooth map and prove the chain rule for them.

**2.52 Corollary. Smoothness of the difference quotient.**
*For a smooth curve $c : \mathbb{R} \to E$ the difference quotient*

$$(t, s) \mapsto \begin{cases} \dfrac{c(t) - c(s)}{t - s} & \text{for } t \neq s \\ c'(t) & \text{for } t = s \end{cases}$$

*is a smooth mapping $\mathbb{R}^2 \to E$.*

**Proof.** By 2.14 we have $f : (t, s) \mapsto \frac{c(t)-c(s)}{t-s} = \int_0^1 c'(s + r(t - s))dr$ and by 2.50 it is smooth $\mathbb{R}^2 \to \widehat{E}$. The left hand side has values in $E$ and for $t \neq s$ this is also true for all iterated directional derivatives. Remains to consider the derivatives for $t = 0$. The iterated partial derivatives are given by 2.50 as

$$\begin{aligned} d_v^p f(t, s) &= d_v^p \int_0^1 c'(s + r(t - s))\, dr \\ &= \int_0^1 d_v^p c'(s + r(t - s))\, dr, \end{aligned}$$

where $d_v$ acts on the $(t, s)$-variable. The later integrand is for $t = s$ just a linear combination of derivatives of $c$ which are independent on $r$, hence $d_v^p f(t, s) \in E$. Since the derivatives of $f \circ c_0$ can be expressed as linear combinations of these directional derivatives by 2.45, these derivatives also belong to $E$, and thus $f$ is smooth into $E$. □

## 2.53 Definition. Spaces of linear mappings

Let $L(E, F)$ denote the *space of all bounded* (equivalently smooth by 2.20) *linear mappings* from $E$ to $F$. It is a closed linear subspace of $C^\infty(E, F)$ since $f$ is linear if and only if for all $x$, $y \in E$ and $\lambda \in \mathbb{R}$ we have $(\mathrm{ev}_x + \lambda\, \mathrm{ev}_y - \mathrm{ev}_{x+\lambda y})f = 0$ . We equip it with this topology and linear structure.

So $f : U \to L(E,F)$ is smooth if and only if the composite $U \xrightarrow{f} L(E,F) \to C^\infty(E,F)$ is smooth.

**2.54 Theorem. Chain rule.**
*Let $E$ and $F$ be locally convex spaces and let $U \subset E$ be $c^\infty$-open. Then the differentiation operator*

$$d : C^\infty(U,F) \to C^\infty(U, L(E,F)),$$
$$df(x)v := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t},$$

*exists and is linear and bounded (smooth). Also the chain rule holds:*

$$d(f \circ g)(x)v = df(g(x))dg(x)v.$$

**Proof.** Since $t \mapsto x+tv$ is a smooth curve we know that $\hat{\hat{d}} : C^\infty(U,F) \times U \times E \to F$ exists. We want to show that it is smooth, so let $(f,x,v) : \mathbb{R} \to C^\infty(U,F) \times U \times E$ be a smooth curve. Then $\hat{\hat{d}}(f(t),x(t),v(t)) = \lim_{s\to 0} \frac{f(t)(x(t)+sv(t))-f(t)(x(t))}{s} = \partial_2 h(t,0)$, which is smooth in $t$, where the smooth mapping $h : \mathbb{R}^2 \to F$ is given by $(t,s) \mapsto f^\wedge(t, x(t)+sv(t))$. By cartesian closedness 2.47 the mapping $d^\wedge : C^\infty(U,F) \times U \to C^\infty(E,F)$ is smooth.

Now we show that this mapping has values in the subspace $L(E,F)$: $d^\wedge(f,x)$ is obviously homogeneous. It is additive, because we may consider the smooth mapping $(t,s) \mapsto f(x+tv+sw)$ and compute as follows, using 2.49.

$$\begin{aligned} df(x)(v+w) &= \tfrac{\partial}{\partial t}\big|_{t=0} f(x+t(v+w)) \\ &= \tfrac{\partial}{\partial t}\big|_{t=0} f(x+tv+0w) + \tfrac{\partial}{\partial t}\big|_{t=0} f(x+0v+tw) \\ &= df(x)(v) + df(x)w. \end{aligned}$$

So $d^\wedge : C^\infty(U,F) \times U \to L(E,F)$ is smooth and by 2.47 the mapping $d : C^\infty(U,F) \to C^\infty(U, L(E,F))$ is smooth and obviously linear.

We first prove the chain rule for a smooth curve $c$ instead of $g$. We have to show that the curve

$$t \mapsto \begin{cases} \frac{f(c(t))-f(c(0))}{t} & \text{for } t \neq 0 \\ f'(c(0))(c'(0)) & \text{for } t = 0 \end{cases}$$

is continuous at 0. It can be rewritten as $t \mapsto \int_0^1 f'(c(0)+s(c(t)-c(0)))(c_1(t))\,ds$, where $c_1$ is the smooth curve given by

$$t \mapsto \begin{cases} \frac{c(t)-c(0)}{t} & \text{for } t \neq 0 \\ c'(0) & \text{for } t = 0 \end{cases}.$$

Since $h : \mathbb{R}^2 \to U \times E$ given by

$$(t,s) \mapsto (c(0)+s(c(t)-c(0)), c_1(t))$$

is smooth, the map $t \mapsto (s \mapsto f'(c(0)+s(c(t)-c(0)))(c_1(t)))$ is smooth $U \to C^\infty(\mathbb{R}, E)$, and hence $t \mapsto \int_0^1 f'(c(0)+s(c(t)-c(0)))(c_1(t))\,ds$ is smooth, and hence continuous.

For general $g$ we have $d(f\circ g)(x)(v) = \frac{\partial}{\partial t}\big|_{t=0} (f\circ g)(x+tv) = (df)(g(x+0v))(\frac{\partial}{\partial t}\big|_{t=0}(g(x+tv))) = (df)(g(x))(dg(x)(v))$. □

**2.55 Lemma.**
*Two locally convex spaces are locally diffeomorphic if and only if they are linearly diffeomorphic.*
*Any smooth and 1-homogeneous mapping is linear.*

**Proof.** By the chain rule the derivatives at corresponding points give the linear diffeomorphisms.

For a 1-homogeneous mapping $f$ one has $df(0)v = \frac{\partial}{\partial t}\big|_{t=0} f(tv) = f(v)$ and this is linear in $v$. $\square$

## 2.56 Exercises

1. Show that a differentiable curve at 0 is continuous at 0.

2. Show that for a locally Lipschitz curve $c$ the convergence

$$\frac{c(\pm\frac{1}{n}) - c(0)}{\pm\frac{1}{n}} \to x$$

is enough to ensure that $c$ is differentiable at 0 with derivative $x$.
Hint: Consider

$$\frac{c(t) - c(0)}{t} - \left(\lambda \frac{c(\frac{1}{n}) - c(0)}{\frac{1}{n}} + \mu \frac{c(\frac{1}{n+1}) - c(0)}{\frac{1}{n+1}}\right)$$

with $\lambda\frac{1}{n} + \mu\frac{1}{n+1} = t > 0$

3. Show that a mapping $f : \mathbb{R}^2 \to E$ is locally Lipschitz along Lipschitz curves if and only if the partial difference quotients $\frac{f(t,s)-f(t',s)}{t-t'}$ and $\frac{f(t,s)-f(t,s')}{s-s'}$ are locally bounded.

4. Show that for a map $f : \mathbb{R}^2 \to E$ the partial difference quotient $\frac{f(t,s)-f(t',s)}{t-t'}$ is locally bounded if and only if $f^\vee : \mathbb{R} \to C(\mathbb{R}, E)$ is locally Lipschitz.

5. Let $f(s,t) = s\varphi(\frac{t}{s^2})$ where $\varphi$ is smooth and has compact support. Then $f$ is continuous, the first partial difference quotient is locally bounded, $f$ is smooth in the second variable, but $f$ is not locally Lipschitz.

6. Let $E$ be a Banach space. Show that a mapping $f : E \to F$ is locally Lipschitz along locally Lipschitz curves if and only if $f$ is locally Lipschitz, i.e. for every $a \in E$ there is a neighborhood $U$ of $E$ such that the set $\{\frac{f(x)-f(y)}{\|x-y\|} : x, y \in U, x \neq y\}$ is bounded.

# 3. Tensor Products and Linearization

## Algebraic Tensor Product

**Remark.**
The importance of the tensor product is twofold. First it allows linearizing of multilinear mappings and secondly it allows to calculate function spaces.

We will consider the spaces of linear and multi-linear mappings between vector spaces. If we supply all vector spaces $E$, $E_1, \dots, E_n$, $F$ with the finest locally convex topology (i.e. the final locally convex topology with respect to the inclusions of all finite dimensional subspaces - on which the topology is unique) then all linear mappings are continuous and all multi-linear mappings are bounded (but not necessarily continuous as the evaluation map $\mathrm{ev} : E^* \times E \to \mathbb{K}$ on an infinite dimensional vector space $E$ shows) and hence it is consistent to denote the corresponding function spaces by $L(E, F) = \mathcal{L}(E, F)$ and $L(E_1, \dots E_n; F)$.

In more detail the first feature is:

**3.1 Proposition.** *Linearization.*
*Given two linear spaces $E$ and $F$, then there exists a solution $\otimes : E \times F \to E \otimes F$ – called the algebraic tensor product of $E$ and $F$ – to the following universal problem:*

$$\begin{tikzcd} E \times F \arrow[rr, "\otimes"] \arrow[dr, "T"'] & & E \otimes F \arrow[dl, dotted, "!"', "\tilde{T}"] \\ & G & \end{tikzcd}$$

*Here $\otimes : E \times F \to E \otimes F$ and $T : E \times F \to G$ are bilinear and $\tilde{T}$ is linear.*

**Proof.** In order to find $E \otimes F$ one considers first the case, where $G = \mathbb{R}$. Then we have that $\otimes^* : (E \otimes F)^* \to L(E, F; \mathbb{R})$ should be an isomorphism. Hence $E \otimes F$ could be realized as subspace of $(E \otimes F)^{**} \cong L(E, F; \mathbb{R})^*$. Obviously to each bilinear functional $T : E \times F \to \mathbb{R}$ corresponds the linear map $\mathrm{ev}_T : L(E, F; \mathbb{R})^* \to \mathbb{R}$. The map $\otimes : E \times F \to E \otimes F \subseteq L(E, F; \mathbb{R})^*$ has to be such that $\mathrm{ev}_T \circ \otimes = T$ for all bilinear functionals $T : E \times F \to \mathbb{R}$, i.e. $\otimes(x, y)(T) = (\mathrm{ev}_T \circ \otimes)(x, y) = T(x, y)$. Thus we have proved the existence of $\tilde{T} := \mathrm{ev}_T$ for $G = \mathbb{R}$. But uniqueness can be true only on the linear subspace generated by the image of $\otimes$, and hence we denote this subspace $E \otimes F$.

For bilinear mappings $T : E \times F \to G$ into an arbitrary vector space $G$, we consider the following diagram, which has quite some similarities with that used in the construction of the $c^\infty$-completion in 2.31:

$$\begin{tikzcd}
E \times F \arrow[r, "\otimes"] \arrow[dd, "T"'] & E \otimes F \arrow[r, hook] \arrow[d, dotted, "(2)"] \arrow[ddl, dotted, "\widetilde{T}"', "(3)"] & L(E,F;\mathbb{R})^* \arrow[dl, dotted, "(1)"'] \arrow[dd, "\mathrm{ev}_{\lambda\circ T}"] \\
 & \prod_{G'} \mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, dotted, "\delta"'] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

The right dashed arrow (1) and $\delta$ exist uniquely by the universal property of the product in the center. The arrow (2) exists uniquely as restriction of (1) to the subspace $E \otimes F$. Finally (3) exists, since the generating subset $\otimes(E \times F)$ in $E \otimes F$ is mapped to $T(E \times F) \subseteq G$ and since $\delta$ is injective. □

Note that $\otimes$ extends to a functor, by defining $T \otimes S$ via the following diagram:

$$\begin{tikzcd}
E_1 \times F_1 \arrow[r, "\otimes"] \arrow[d, "T\times S"'] & E_1 \otimes F_1 \arrow[d, dotted, "!"', "T\otimes S"] \\
E_2 \times F_2 \arrow[r, "\otimes"'] & E_2 \otimes F_2
\end{tikzcd}$$

Furthermore one easily proves the existence of the following natural isomorphisms:

$$E \otimes \mathbb{R} \cong E$$
$$E \otimes F \cong F \otimes E$$
$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$$

In analogy to the exponential law for smooth mappings or continuous mappings, we show now the existence of a natural isomorphism

$$L(E, F; G) \cong L(E, L(F, G))$$

again denoted by $(\_)^\vee$ with inverse isomorphism $(\_)^\wedge$ given by the same formula as above.
In fact for a bilinear mapping $T : E \times F \to G$, the mapping $T^\vee$ has values in $L(F, G)$, since $T(x, \_)$ is linear, and it is linear, since $L(F, G)$ carries the initial vector space structure with respect to the evaluations $\mathrm{ev_y}$ and $\mathrm{ev_y} \circ T^\vee = T(\_, y)$ is also linear. The same way one shows that the converse implication is also true.

Note that if both spaces $E$ and $F$ are finite dimensional, then so is $L(E, F; \mathbb{R})$ and hence also the dual $L(E, F; \mathbb{R})^*$. But then $E \otimes F$ is finite dimensional too (in fact $\dim(E \otimes F) = \dim E \cdot \dim F$), as we will see in 3.30, and hence $E \otimes F = (E \otimes F)^{**} = L(E, F; \mathbb{R})^*$.

If one factor is infinite dimensional and the other one is not 0, then this is not true. In fact take $F = \mathbb{R}$, then $E \otimes \mathbb{R} \cong E$ whereas $L(E, \mathbb{R}; \mathbb{R})^* \cong L(E, L(\mathbb{R}, \mathbb{R}))^* \cong L(E, \mathbb{R})^* = E^{**}$.

## 3.2 Vector-valued functions versus scalar valued ones

The second important usage of the tensor product lies in the possibility to express spaces of vector valued functions as tensor products of spaces of scalar valued functions times the space of values. In more detail this means, that given some type of function $f : X \to \mathbb{R}$ and a vector $y \in F$, then we can form the function $X \to F$ given by $x \mapsto f(x) \cdot y$. If we denote the space of functions $X \to F$ of some specific type by $F^X$ then this means that we have a bilinear mapping $\mathbb{R}^X \times F \to F^X$. The question that arises is, whether it has the universal property of the tensor product, i.e. whether the natural mapping $\mathbb{R}^X \otimes F \to F^X$ is an isomorphism.

Let us consider the case where $X$ itself is a linear space $E$ and the considered functions are the linear ones. Then our claim is that $E^* \otimes F \cong L(E,F)$. For this we consider the following diagram:

$$\begin{tikzcd}
E^* \times F \arrow[r, dotted, "(1)"] \arrow[d, "\otimes"'] & L(E,F) \arrow[d, dotted, "(3)"'] \\
E^* \otimes F \arrow[ur, dotted, "(2)"] \arrow[r, hook] & L(E^*,F;\mathbb{R})^*
\end{tikzcd}
\qquad
\begin{tikzcd}
L(E,F) \times L(E^*,F^*) \arrow[d, dotted, "(3^\wedge)"'] \\
\mathbb{R}
\end{tikzcd}$$

The first dashed arrow is given by bilinear mapping discussed before, namely $(x^*, y) \mapsto (x \mapsto x^*(x)\, y)$. The second one exists by the universal property of the tensor product. And since the image of the first one generates $L(E,F)$ provided $E$ or $F$ is finite dimensional, we conclude that the second one is surjective. Remains to show that the third one exists and is a left-inverse. By the exponential law this mapping would correspond to a bilinear mapping $(3^\wedge) : L(E,F) \times L(E^*,F;\mathbb{R}) \to \mathbb{R}$, which we try to piece together as follows:

$$\begin{tikzcd}
& L(E,F) \times L(E^*,F;\mathbb{R}) \arrow[d, "\cong"'] & \\
& L(E,F) \times L(E^*,F^*) \arrow[d, "(_)^* \times L(E^*,F^*)"'] & \\
& L(F^*,E^*) \times L(E^*,F^*) \arrow[dl, "\mathrm{comp}"] \arrow[dr, "\mathrm{comp}"] & \\
L(E^*,E^*) \arrow[dr, "\mathrm{tr}"] & & L(F^*,F^*) \arrow[dl, "\mathrm{tr}"'] \\
& \mathbb{R} &
\end{tikzcd}$$

Clearly the transposition mapping $L(E,F) \to L(F^*,E^*)$ is linear, and if we apply the composition map from $L(F^*,E^*) \times L(E^*,F^*)$ to $L(E^*,E^*)$ or to $L(F^*,F^*)$ it remains to find for a vector space $G$ a linear map $L(G,G) \to \mathbb{R}$. If $G$ is finite dimensional such a map is given by the *trace*, i.e. the sum over the diagonal entries of a matrix-representation, or equivalently the derivative of the determinate at the identity, or equivalently the coefficient of $(-\lambda)^{\dim G - 1}$ in the *characteristic polynomial* $\det(T - \lambda)$. In order to show that the composite $L(E,F) \times L(E^*,F;\mathbb{R}) \to L(F^*,E^*) \times L(E^*,F^*) \to L(E^*,E^*) \to \mathbb{R}$ gives a left inverse, it is enough by the universal property of the tensor product to test on $x^* \otimes y$. This is mapped to $x \mapsto x^*(x) \cdot y =: S$ and furthermore to $T \mapsto \operatorname{trace}(S^* \circ T^\vee)$. So let us calculate $(S^* \circ T^\vee)(u^*)(u) = T(u^*, Su) = T(u^*, x^*(u) \cdot y) = T(u^*, y) \cdot x^*(u)$. Note that $x^* \in G := E^*$ and $T(\_, y) \in G^* := E^{**}$, and for $g \in G$ and $g^* \in G^*$ we have that the trace of $g^*(\_) \cdot g$ is $\operatorname{trace}(g^*(\_) \cdot g) = g^*(g)$. To show this, extend $g$ to a basis and then the trace is the entry in the upper left corner, which is $g^*(g)$. So $\operatorname{trace}(S^* \circ T^\vee) = T(x^*, y)$, which was to be shown.

In particular we have shown, that $G^* \otimes G \cong L(G,G)$ for finite dimensional $G$. And the trace of $g^* \otimes g \in L(G,G)$ is just $g^*(g)$ and hence corresponds to the bilinear map $\mathrm{ev} : G^* \times G \to \mathbb{R}$ or the corresponding linear map $G^* \otimes G \to \mathbb{R}$.

If both factors are infinite dimensional this will no longer be true, even if we restrict to continuous mappings. However if we take some appropriate completion, there might be some chance.

Let us deduce some additional handy formulas for duals, in the case where at least one of $E$ and $F$ is finite dimensional:

$$(E\otimes F)^* \cong L(E,F;\mathbb{R}) \cong L(E,F^*) \cong E^*\otimes F^* \quad \text{and}$$
$$L(E,F)^* \cong (E^*\otimes F)^* \cong E^{**}\otimes F^* \cong L(F,E^{**}) \cong L(F,E^*;\mathbb{R}) \cong L(E^*,F^*).$$

## Projective Tensor Product

We turn first to the property of making bilinear continuous mappings into linear ones. We call the corresponding solution the *projective tensor product* of $E$ and $F$ and denote it by $E\otimes_\pi F$. Obviously it can be obtained by taking the algebraic tensor product and supplying it with the finest locally convex topology such that $E\times F\to E\otimes F$ is continuous. This topology exists since the union of locally convex topologies is locally convex and $E\times F\to E\otimes F$ is continuous for the weak topology on $E\otimes F$ generated by those linear functionals which correspond to continuous bilinear functionals on $E\times F$. It has the universal property, since the inverse image of a locally convex topology under a linear mapping $\tilde T$ is again a locally convex topology, such that $\otimes$ is continuous, provided the associated bilinear mapping $T$ is continuous. However, it is not obvious that this topology is separated, and we prove that now. We will denote the *space of continuous linear mappings* from $E$ to $F$ by $\mathcal{L}(E,F)$, and the *space of continuous multi-linear mappings* by $\mathcal{L}(E_1,\dots,E_n;F)$. If all $E_1,\dots,E_n$ are the same space $E$, we will also write $\mathcal{L}^n(E;F)$.

**3.3 Lemma.**
*$E\otimes_\pi F$ is Hausdorff provided $E$ and $F$ are.*

**Proof.** It is enough to show that the set $E^*\times F^*$ separates points in $E\otimes F$ or even in $L(E,F;\mathbb{R})^*$. So let $0\neq z=\sum_k x_k\otimes y_k$ be given. By replacing linear dependent $x_k$ by the corresponding linear combinations and using bilinearity of $\otimes$, we may assume that the $x_k$ are linearly independent. Now choose $x^*\in E^*$ and $y^*\in F^*$ be such that $x^*(x_k)=\delta_{1,k}$ and $y^*(y_1)=1$. Then $(x^*\otimes y^*)(z)=1\neq 0$. □

Since a bilinear mapping is continuous iff it is so at 0, a 0-neighborhood basis in $E\otimes_\pi F$ is given by all those absolutely convex sets, for which the inverse image under $\otimes$ is a 0-neighborhood in $E\times F$. A basis is thus given by the absolutely convex hulls denoted $U\otimes V$ of the images of $U\times V$ under $\otimes$, where $U$ resp. $V$ runs through a 0-neighborhood basis of $E$ resp. $F$. We only have to show that these sets $U\otimes V$ are absorbing. So let $z=\sum_k x_k\otimes y_k\in E\otimes F$ be arbitrary. Then there are $a_k>0$ and $b_k>0$ such that $x_k\in a_kU$ and $y_k\in b_kV$ and hence $z=\sum_{k\le K} a_k\,b_k\frac{x_k}{a_k}\otimes\frac{y_k}{b_k}\in(\sum_k a_k\,b_k)\cdot\langle U\otimes V\rangle_{\text{abs.conv.}}$. The Minkowski-functionals $p_{U\otimes V}$ form a base of the seminorms of $E\otimes_\pi F$ and we will denote them by $\pi_{U,V}$. In terms of the Minkowski-functionals $p_U$ and $p_V$ of $U$ and $V$ we obtain that $z\in(\sum_k p_U(x_k)\,p_V(y_k))\,U\otimes V$ for any $z=\sum_k x_k\otimes y_k$ since $x_k\in p_U(x_k)\cdot U$ for closed $U$, and thus $p_{U\otimes V}(z)\le\inf\{\sum_k p_U(x_k)\,p_V(y_k) : z=\sum_k x_k\otimes y_k\}$. We now show the converse:

**3.4 Proposition. Seminorms of the projective tensor product.**

$$p_{U\otimes V}(z)=\inf\Big\{\sum_k p_U(x_k)\cdot p_V(y_k) : z=\sum_k x_k\otimes y_k\Big\}.$$

**Proof.** Let $z \in \lambda \cdot U \otimes V$ with $\lambda > 0$. Then $z = \lambda \sum \lambda_k u_k \otimes v_k$ with $u_k \in U$, $v_k \in V$ and $\sum_k |\lambda_k| = 1$. Hence $z = \sum x_k \otimes v_k$, where $x_k = \lambda\lambda_k u_k$, and $\sum_k p_U(x_k) \cdot p_V(v_k) \leq \sum \lambda|\lambda_k| = \lambda$. Taking the infimum of all $\lambda$ gives now that $p_{U\otimes V}(z)$ is greater or equal to the infimum on the right side. □

**3.5 Corollary.**
*$E \otimes_\pi F$ is normable (metrizable) provided $E$ and $F$ are.*

**3.6 Lemma. The semi-norms of decomposable tensors.**

$$p_{U,V}(x \otimes y) = p_U(x) \cdot p_V(y).$$

**Proof.** According to [**2**, 7.1.8] there are $x^* \in E^*$ and $y^* \in F^*$ such that $|x^*| \leq p_U$ and $|y^*| \leq p_V$ and $x^*(x) = p_U(x)$ and $y^*(y) = p_V(y)$. If $x \otimes y = \sum_k x_k \otimes y_k$, then

$$\begin{aligned} p_{U\otimes V}(x \otimes y) &\leq p_U(x) \cdot p_V(y) = x^*(x) \cdot y^*(y) = (x^* \otimes y^*)(x \otimes y) = \\ &= \sum_k x^*(x_k) \cdot y^*(y_k) \leq \sum_k p_U(x_k) \cdot p_V(y_k), \end{aligned}$$

and taking the infimum gives the desired result. □

**3.7 Remark. Functorality.**
Given two continuous linear maps $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ we can consider bilinear continuous map given by composing $T_1 \times T_2 : E_1 \times E_2 \to F_1 \times F_2$ with $\otimes : F_1 \times F_2 \to F_1 \otimes F_2$. By the universal property of $E_1 \times E_2 \to E_1 \otimes E_2$ we obtain a continuous linear map denoted by $T_1 \otimes T_2 : E_1 \otimes E_2 \to F_1 \otimes F_2$.

$$\begin{tikzcd} E_1 \times E_2 \arrow[r, "\otimes"] \arrow[d, "T_1 \times T_2"'] & E_1 \otimes E_2 \arrow[d, dotted, "T_1 \otimes T_2"] \\ F_1 \times F_2 \arrow[r, "\otimes"'] & F_1 \otimes F_2 \end{tikzcd}$$

By the uniqueness of the linearization one obtains immediately that $\otimes$ is a functor. Because of the uniqueness of universal solutions one sees easily that one has natural isomorphisms $\mathbb{R} \otimes E \cong E$, $E \otimes F \cong F \otimes E$ and $(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$.

## 3.8 Adjoint functors

It would be nice to identify the tensor-product as a left-adjoint functor, since then several inheritance properties would automatically hold. In order to formulate adjointness of functors we need the notion of *Hom-functor*: Given a category $\underline{F}$ we have the set $\underline{F}(F_1, F_2)$ of all morphisms from $F_1$ to $F_2$ for any two objects $F_1$, $F_2$ in $\underline{F}$. This extends to a functor, called Hom-functor, from $\underline{F}^{op} \times \underline{F}$ to the category $\underline{Set}$ of mappings between sets, by defining $\underline{F}(f, g) := f^* \circ g_* = g_* \circ f^* : h \mapsto g \circ h \circ f$ for any two morphisms $f$ and $g$ in $\underline{F}$. Now two functors $L : \underline{G} \leftarrow \underline{F}$ and $R : \underline{G} \to \underline{F}$ are called *left* and *right adjoint* to each other, provided there exists a natural isomorphism $\underline{F}(F, R(G)) \cong \underline{G}(L(F), G)$, i.e. for every object $F$ in $\underline{F}$ and $G$ in $\underline{G}$ one has a bijection $(\_)^\vee : \underline{G}(L(F), G) \to \underline{F}(F, R(G))$, which makes the following diagrams commutative for all $\underline{F}$-morphisms $f$ and $\underline{G}$-morphisms $g$:

$$\begin{tikzcd} \underline{G}(L(F_1), G_1) \arrow[r, "\cong"] \arrow[d, "\underline{G}(L(f), g)"'] & \underline{F}(F_1, R(G_1)) \arrow[d, "\underline{F}(f, R(g))"] \\ \underline{G}(L(F_2), G_2) \arrow[r, "\cong"] & \underline{F}(F_2, R(G_2)) \end{tikzcd}$$

We have so far encountered the following examples of adjoint functors:

1. For any locally compact space $X$ the product functor $(\_) \times X$ from the category of continuous maps between topological spaces to itself is left-adjoint to the function space functor $C(X, \_)$ with the topology of uniform convergence on compact sets, as we have shown in 1.7.
$$C(Y \times X, Z) \cong C(Y, C(X, Z))$$
2. For any locally convex space $E$ the product functor $(\_) \times E$ from the category of smooth maps between locally convex spaces to itself is left-adjoint to the function space functor $C^\infty(E, \_)$. See 2.47 and 2.48.
$$C^\infty(F \times E, G) \cong C^\infty(F, C^\infty(E, G))$$
3. The $c^\infty$-completion functor $E \mapsto \tilde{E}$ from the category of bounded linear maps between locally convex spaces and into that of $c^\infty$-complete locally convex spaces is left-adjoint to the forgetful functor in the opposite direction. See 2.31.
$$L(\tilde{E}, F) \cong L(E, F) \text{ for all } c^\infty\text{-complete } F$$
4. The bornologification functor $E \mapsto E_{\text{born}}$ from the category of continuous linear maps between locally convex spaces to that of bornological locally convex spaces is right adjoint to the forgetful functor in the opposite direction. See 2.8.
$$\mathcal{L}(E, F) \cong \mathcal{L}(E, F_{\text{born}}) \text{ for all bornological } E$$
5. For any vector space $E$ the tensor product functor $(\_) \otimes E$ from the category of linear mappings between vector spaces to itself is left-adjoint to the Hom-functor $L(E, \_)$. See 3.1.
$$L(F \otimes E, G) \cong L(F, L(E, G))$$
6. The functor, which assigns to each vector space the same space with its finest locally convex topology is left adjoint to the forgetful functor, which forgets the topology.
7. Let $E$ be any vector space. Then $L(\_, E)$ becomes a functor from the category of linear maps between vector spaces, into its opposite category, where all arrows are reversed. The equation $L(E, L(F, G)) \cong L(F, L(E, G)) = L^{op}(L(E, G), F)$, shows that the functor $L(\_, E) : \underline{VS} \to \underline{VS}^{op}$ is left-adjoint to the functor $L(\_, E) : \underline{VS}^{op} \to \underline{VS}$.

One important property of adjoint functors is the following

**3.9 Proposition. Continuity of adjoint functors.**
*Let $R$ be a right adjoint functor. Then $R$ is continuous, i.e. preserves limits. Dually, let $L$ be a left adjoint functor. Then $L$ is co-continuous, i.e. preserves co-limits.*

**Proof.** It is enough to show the statement for a right-adjoint functor $R$. So let

$$\begin{tikzcd} A_i \arrow[rr, "f"] & & A_j \\ & A \arrow[ul, "p_i"] \arrow[ur, "p_j"'] & \end{tikzcd}$$

be a limit diagram in $\mathcal{A}$. We have to show that

$$\begin{tikzcd} R(A_i) \arrow[rr, "R(f)"] & & R(A_j) \\ & R(A) \arrow[ul, "R(p_i)"] \arrow[ur, "R(p_j)"'] & \end{tikzcd}$$

is a limit diagram in $\mathcal{B}$. So consider the following diagrams

$$\begin{tikzcd} R(A_i) \arrow[rr, "R(f)"] & & R(A_j) \\ & R(A) \arrow[ul, "R(p_i)"] \arrow[ur, "R(p_j)"'] & \\ & B \arrow[uul, "g_i"] \arrow[uur, "g_j"'] \arrow[u, dashed, "h^\vee"', "!"] & \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[rr, "f"] & & A_j \\ & A \arrow[ul, "p_i"] \arrow[ur, "p_j"'] & \\ & L(B) \arrow[uul, "g_i^\wedge"] \arrow[uur, "g_j^\wedge"'] \arrow[u, dashed, "h"', "!"] & \end{tikzcd}$$

Where the dashed arrow on the right exists by the limit property of $A$ and the one one the left by the natural isomorphism. $\square$

**3.10 Adjointness of the tensor functor.**
In analogy to the algebraic tensor product we would expect that also for locally convex spaces $(\_)\otimes_\pi E$ is left-adjoint to the Hom-functor $\mathcal{L}(E, \_)$ supplied with some topology. Since $\mathcal{L}(E\otimes_\pi F, G) \cong \mathcal{L}(E, F; G)$ we would need a bijection $\mathcal{L}(E, F; G) \cong \mathcal{L}(E, \mathcal{L}(F, G))$. Obviously we have the linear injection $(\_)^\vee : \mathcal{L}(E, F; G) \to \mathcal{L}(E, \mathcal{L}(F, G))$ induced from the corresponding bijection of vector-spaces, since a jointly continuous map is separately continuous, and hence $T^\vee(x) = T(x, \_)$ is continuous. And if we supply $\mathcal{L}(F, G)$ with the topology of uniform convergence on bounded sets, then $T^\vee$ is continuous, since $(T^\vee)^{-1}(N_{B,W}) = \{x : T(x, B) \subseteq W\}$ contains the 0-neighborhood $\frac{1}{\lambda} U$, where $U$ (and $V$) are chosen, such that $T(U \times V) \subseteq W$ and $\lambda > 0$ such that $B \subseteq \lambda V$.

**Proposition.**
*If we supply also $\mathcal{L}(E, \mathcal{L}(F, G))$ and $\mathcal{L}(E, F; G)$ with the topology of uniform convergence on bounded sets then the mapping*

$$(\_)^\vee : \mathcal{L}(E, F; G) \hookrightarrow \mathcal{L}(E, \mathcal{L}(F, G))$$

*is a topological linear embedding.*

In fact, a typical neighborhood of $\mathcal{L}(E, F; G)$ is $N_{B_1\times B_2,W}$ and one of $\mathcal{L}(E, \mathcal{L}(F, G))$ is $N_{B_1,N_{B_2,W}}$ and $N_{B_1\times B_2,W} = ((\_)^\vee)^{-1} N_{B_1,N_{B_2,W}}$, so it remains to show that $(\_)^\wedge$ is well defined. Recall that $f^\wedge$ is given by $\mathrm{ev} \circ (f \times F)$, where $\mathrm{ev} : \mathcal{L}(F, G) \times F \to G$. However this mappings is continuous only if $F$ is normed. So only for normed $F$ we have that $(\_) \otimes_\pi F$ is left-adjoint. If $F$ is not normed, then in particular $\mathrm{id} \in \mathcal{L}(F^*, \mathcal{L}(F, \mathbb{R}))$ but $\mathrm{ev} = \mathrm{id}^\wedge \notin \mathcal{L}(F^*, F; \mathbb{R})$.

**Corollary.**
*Let $E$ be a normable space. Then $(\_) \otimes_\pi E$ preserves co-limits.* $\square$

From the exponential law for continuous and that for smooth mappings, we are used that one automatically gets an isomorphism between the corresponding function spaces, cf. 2.48. So one would expect that the linear isomorphism $\mathcal{L}(E \otimes_\pi F, G) \cong \mathcal{L}(E, F; G)$ is in fact a topological one. If one supplies both spaces with the topology of uniform convergence on bounded sets, then $\otimes^* : \mathcal{L}(E \otimes_\pi F, G) \to \mathcal{L}(E, F; G)$ is obviously continuous since $\otimes : E \times F \to E \otimes_\pi F$ is bounded. In order to prove that it is an embedding, we have to find for every bounded set $B \subseteq E \otimes_\pi F$ and 0-neighborhood $W \subseteq G$ two bounded sets $B_1 \subseteq E$ and $B_2 \subseteq F$ and a 0-neighborhoods $U \subseteq G$, such that $\otimes^*(N_{B,W}) \supseteq N_{B_1 \times B_2, U}$. In particular if $G = \mathbb{R}$ and $W = [-1,1]$, then $N_{B,W}$ is the polar $B^o$ of $B$ and for all bilinear continuous functionals, which map $B_1 \times B_2$ to $U = [-K, K]$, the corresponding linear functional $\tilde{T}$ on $E \otimes_\pi F$ must be in $B^0$. By enlarging $B_1$ we may assume that $K = 1$. Using the bipolar theorem we deduce from $(B_1 \otimes B_2)^o \subseteq B^o$ that $B \subseteq (B_1 \otimes B_2)^{oo} = \langle B_1 \otimes B_2 \rangle_{closed,abs.conv.}$. Thus the closed absolutely convex hull of the image of $B_1 \times B_2$ must contain $B$. Whether this is true is even for Fréchet spaces unknown. This is also called *Grothendieck's problème de topologies*. For the corresponding result on compact subsets see 3.21.

However bornologically we have an isomorphism:

**3.11 Lemma.**
*With respect to the equi-continuous bornology we have a bornological isomorphism*

$$\mathcal{L}(E \otimes_\pi F, G) \cong \mathcal{L}(E, F; G).$$

**Proof.** Let us first show that $\mathcal{B} \subseteq \mathcal{L}(E, F; \mathbb{R})$ is equi-continuous iff there exist 0-neighborhoods $U$ in $E$ and $V$ in $F$ such that $B \subseteq (U \times V)^o$.
($\Leftarrow$) Let $(x_0, y_0) \in E \times F$ be given. Choose $\lambda \geq 1$ and $\mu \geq 1$ such that $x_0 \in \lambda U$ and $y_0 \in \mu V$. Then we have for $y - y_0 \in \frac{1}{\lambda} V \subseteq V$ and for $x - x_0 \in \frac{1}{\mu} U \subseteq U$ that

$$|b(x,y) - b(x_0,y_0)| \leq |b(\underbrace{x - x_0}_{\in U}, \underbrace{y - y_0}_{\in V})| + |b(\underbrace{x - x_0}_{\in \frac{1}{\mu} U}, \underbrace{y_0}_{\in \mu V})| + |b(\underbrace{x_0}_{\in \lambda U}, \underbrace{y - y_0}_{\in \frac{1}{\lambda} V})| \leq 3.$$

($\Rightarrow$) is obvious by the equi-continuity at 0 and since $b(0,0) = 0$.

Now the isomorphism is clear since the basis of the equi-continuous bornologies are $(U \otimes V)^o$ and $(U \times V)^o$ respectively, where $U$ and $V$ run through 0-neighborhood basis of $E$ and $F$. □

Since every injective mapping $f$ between vector spaces has a linear left inverse and every surjective one has a right inverse, the same is true for $f \otimes E$ and hence we have:

**3.12 Lemma.**
*The projective tensor product preserves injective and surjective continuous linear mappings.* □

**3.13 Proposition.**
*The projective tensor product preserves quotients.*

**Proof.** Let $F$ be a locally convex space and $f$ be a quotient mapping and hence open. We have to show, that $f \otimes_\pi F : E_1 \otimes_\pi F \to E_2 \otimes_\pi F$ is open. So let $U \otimes V$ be a typical 0-neighborhood of $E_1 \otimes F$. Since the image under a linear map of an absolutely convex hull is the absolutely convex hull of the image, we have that $(f \otimes F)(U \otimes V) = f(U) \otimes V$ and hence is a 0-neighborhood in $E_2 \otimes_\pi F$. □

Let us consider the dual situation next.

**3.14 Example.**
$\otimes_\pi$ does not preserve embeddings.
In fact consider the isometric embedding $\ell^2 \to C(K)$, where $K$ is the closed unit-ball of $(\ell^2)^*$ supplied with its compact topology of pointwise convergence, see the corollary to the Alaoğlu-Bourbaki-theorem in [**2**, 7.4.12]. This subspace has however no topological complement, since $C(K)$ has the *Dunford-Pettis property* (see [**14**,

20.7.8], i.e. $x_n^*(x_n) \to 0$ for every two sequences $x_n \to 0$ in $\sigma(E, E^*)$ and $x_n^* \to 0$ in $\sigma(E^*, E^{**})$), but no infinite dimensional reflexive Banach space like $\ell^2$ has it (e.g. $x_n := e_n$, $x_n^* := e_n$) and hence cannot be a complemented subspace of $C(K)$, see [**14**, 20.7].

Suppose now that $\ell^2 \otimes_\pi (\ell^2)^* \to C(K) \otimes_\pi (\ell^2)^*$ were an embedding. The duality mapping $\mathrm{ev} : \ell^2 \times (\ell^2)^* \to \mathbb{R}$ yields a continuous linear mapping $s : \ell^2 \otimes_\pi (\ell^2)^* \to \mathbb{R}$ and would hence have a continuous linear extension $\tilde{s} : C(K) \otimes (\ell^2)^* \to \mathbb{R}$. The corresponding bilinear map would give a continuous mapping $\tilde{s}^\vee : C(K) \to (\ell^2)^{**} \cong \ell^2$, which is a left inverse to the embedding $\ell^2 \to C(K)$, a contradiction.

In connection with the second usage of tensor products we would expect that for the product $E^{\mathbb{N}} = (\mathbb{R} \otimes_\pi E)^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}} \otimes_\pi E$, i.e. we are looking for preservation of certain products. But even purely algebraically this fails to be true. In fact take the coproduct $E = \mathbb{R}^{(\mathbb{N})}$. Using that $\mathbb{R}^{\mathbb{N}} \otimes (\_)$ is left-adjoint and hence preserves colimits we get $\mathbb{R}^{\mathbb{N}} \otimes \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$, which is strictly smaller than $(\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$. However in both spaces the union $\bigcup_n E^n$ is dense, so after taking completions there should be some chance. In order to work with completions we have to show preservation of dense embeddings. To obtain such a result we need a dual characterization of such mappings. And this we treat next.

## Duality between Topology and Bornology

### 3.15 Topologifying and Bornologifying

Let <u>*LCS*</u> be the category of continuous linear maps between locally convex spaces.

On the other hand we can consider *bornological spaces*. These are sets $X$ with a *bornology*, i.e. a set $\mathcal{B}$ of subsets of $X$, which contains all finite subsets and is closed under formation of finite unions and subsets. The elements of $\mathcal{B}$ are called the *bounded sets* of $X$. And a mapping between such sets is called *bounded*, iff it maps the bounded sets to bounded sets. If $X$ is in addition a vector space and addition and scalar multiplication are bounded, then $X$ is called *bornological vector space*. If furthermore the convex hull of each bounded set is bounded, then $X$ is called *convex bornological space*. Let <u>*CBS*</u> denote the category of bounded linear maps between convex bornological spaces.

To every locally convex space $(E,\mathcal{U})$ we can associate a a convex bornological space $(E, \mathcal{B})$, where the bornology $\mathcal{B}$ is given by the *von Neumann bounded sets*, i.e. those sets $B \subseteq E$ which are absorbed by all 0-neighborhoods $U \in \mathcal{U}$. This correspondence extends to a functor $b : \underline{LCS} \to \underline{CBS}$ which leaves the morphisms and the underlying vector spaces unchanged.

Conversely, we can associate to every convex bornological space $(E, \mathcal{B})$ a locally convex topology on $E$ given by the 0-neighborhood basis $\mathcal{U}$ formed by all *bornivorous subsets* $U \subseteq E$, i.e. those sets which absorb all the bounded sets $B \in \mathcal{B}$. This correspondence extends to a functor $t : \underline{CBS} \to \underline{LCS}$ which leaves the morphisms and the underlying vector spaces unchanged.

Let us show next that these functors are adjoint to each other. Since all 0-neighborhoods are obviously bornivorous with respect to the von Neumann bounded sets we have that the identity from $t(b(E)) \to E$ is continuous. So let $F$ be a convex bornological space and $T : t(F) \to E$ a continuous linear map. We have to show that $T : F \to b(E)$ is bounded. So let $B \subseteq F$ be bounded and $U \subseteq E$ a 0-neighborhood. We have to show that $T(B)$ gets absorbed by $U$. But since $T : t(F) \to E$ is continuous, we have that $T^{-1}(U) \subseteq t(F)$ absorbs $B$, i.e. $B \subseteq K \cdot T^{-1}(U)$ for some

$K > 0$, and hence $T(B) \subseteq K \cdot T(T^{-1}U) \subseteq K \cdot U$. Thus we have a natural bijection $\underline{LCS}(t(F), E) \cong \underline{CBS}(F, b(E))$, and hence $t$ is left adjoint to $b$. In particular it follows that $b$ preserves limits and $t$ colimits.

Recall that a locally convex space $E$ is called *bornological* if $tbE = E$ or equivalently that $E$ lies in the image of $t$. Conversely a convex bornological space $E$ is called *topological* if $btE = E$ or equivalently if $E$ lies in the image of $b$, i.e. the bornology is the von Neumann bornology of some locally convex topology. The two functors $t$ and $b$ restrict to an isomorphism between the category of continuous linear maps between bornological locally convex spaces and bounded linear maps between topological convex bornological spaces.

### 3.16 External duality

There is however a second pair of adjoint functors, which we have used already several times. Namely we can associate to every locally convex space $(E, \mathcal{U})$ the *dual space* $E^*$ formed by all continuous linear functionals on $E$ supplied with the *bornology of equi-continuous sets*. A base of this bornology is given by the polars $U^o$ of the 0-neighborhoods $U \in \mathcal{U}$. For every continuous linear map $T : E \to F$ we obtain a bounded linear map $T^* : F^* \to E^*$, since $T^*(V^o) \subseteq T^{-1}(V)^o$. In fact let $x^* \in T^*(V^o)$, i.e. $x^* = T^*(y^*) = y^* \circ T$ for some $y^* \in V^o$. Then $x^*(x) = y^*(Tx) \in [-1, 1]$ for all $x \in T^{-1}(V)$. This gives us a functor $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$.

Conversely we can associate to every bornological space $(X, \mathcal{B})$ the locally convex space $\ell^\infty(X, \mathbb{R})$ formed by all bounded functions on $X$ and supplied with the topology of uniform convergence on bounded sets of $X$. Every bounded map $T : X \to Y$ induces a continuous linear map $\ell^\infty(T, \mathbb{R}) : \ell^\infty(Y, \mathbb{R}) \to \ell^\infty(X, \mathbb{R})$ given by $f \mapsto f \circ T$. In fact a typical 0-neighborhood of $\ell^\infty(X, \mathbb{R})$ is given by the polar $B^o$ of a bounded set $B \subseteq X$ and we have $\ell^\infty(T, \mathbb{R})^{-1}(B^o) = T(B)^o$. This can be seen directly as follows:

$$\begin{aligned}\ell^\infty(T, \mathbb{R})^{-1}(B^o) &= \{y^* : (y^* \circ T) \in B^o\} = \{y^* : (y^* \circ T)(B) \subseteq [-1, 1]\} \\ &= \{y^* : y^*(T(B)) \subseteq [-1, 1]\} = T(B)^o.\end{aligned}$$

If $X$ is in addition a convex bornological space $E$, then we can restrict to the linear subspaces $E' \subseteq \ell^\infty(E, \mathbb{R})$ formed by the linear bounded functionals, and hence obtain a functor $(\_)' : \underline{CBS}^{op} \to \underline{LCS}$.

Again we show that these two functors form an adjoint pair. So let $E$ be a locally convex space and $F$ a convex bornological space and consider a linear $T : E \to F'$. It is continuous iff for every bounded set $B$ in $F$ there exists a 0-neighborhood $U$ in $E$ such that $T^{-1}(B^o) \supseteq U$ or equivalently that $T^\wedge(U \times B) \subseteq [-1, 1]$, where $T^\wedge$ denotes the associated bilinear map from $E \times F \to \mathbb{R}$. If we flip the coordinates we get a linear map $\tilde{T} : F \to E^*$. In fact $\tilde{T}(y) = \mathrm{ev}_y \circ T$ is continuous, since all $\mathrm{ev}_y : F' \to \mathbb{R}$ are so. This mapping is bounded, iff for every bounded $B \subseteq F$ there exists some 0-neighborhood $U \subseteq E$ such that $\tilde{T}(B) \subseteq U^o$, or equivalently such that $T^\wedge(U \times B) \subseteq [-1, 1]$. Since for any bounded linear map $T : F \to E^*$ the map $\tilde{T}$ obtained by changing the coordinates is bounded (since $\mathrm{ev}_x : E^* \to \mathbb{R}$ are) we have obtained a natural bijection

$$\underline{LCS}(E, F') \cong \underline{CBS}(F, E^*) = \underline{CBS}^{op}(E^*, F).$$

I.e. $(\_)' : \underline{CBS}^{op} \to \underline{LCS}$ is right adjoint to $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$ and hence carries limits in $\underline{CBS}^{op}$ (i.e. colimits in $\underline{CBS}$) to limits in $\underline{LCS}$ and $(\_)^*$ carries colimits in $\underline{LCS}$ to limits in $\underline{CBS}$.

## 3.17 Preservation of certain morphisms

Let us show next that $(\_)^*$ carries topological linear embeddings into *bornological quotient* mappings, i.e. mappings where each bounded set in the codomain is the image of a bounded set in the domain. Up to an isomorphism any topological linear embedding is given by the inclusion $T$ of a subspace $E$ in $F$. By 3.16 we know that $T^*$ is bounded. In order to show that it is a bornological quotient map let $U \subseteq E$ be a 0-neighborhood, which is without loss of generality closed and absolutely convex. We have to find a 0-neighborhood $V$ of $F$ such that $U^o \subseteq T^*(V^o)$. So let $p$ be a continuous seminorm on $F$ which extends the Minkowski functional of $U$ and let $V$ be the closed unit-ball of $p$. Then every continuous linear functional $x^* \in U^o$ satisfies $|x^*| \le p$ on $E$ and hence extends by Hahn-Banach to a continuous linear functional $y^* \in F^*$ with $|y^*| \le p$. Thus $y^* \in V^o$ and $T^*(y^*) = y^* \circ T = x^*$.

Conversely let us show that $(\_)'$ carries bornological quotient mappings into topological embeddings. Since a bornological quotient mapping $T : E \to F$ obviously has to be onto, we conclude that $T^* : F' \to E'$ is injective. Note that we refrain from denoting this map $T' : F' \to E'$ in order to avoid confusion with the derivative. Since $T^*(T(B)^o) = T^*((T^*)^{-1}(B^o)) = B^o \cap T^*(F')$, by what we proved above, and since the sets $T(B)^o$ form a 0-neighborhood basis of $F'$ we are done.

Thus if $T^* : F^* \to E^*$ is a bornological quotient map then $(T^*)^* : (F^*)' \to (E^*)'$ is a topological embedding and using the embedding $E \to L((E^*, \mathcal{E}), \mathbb{R})$ of [**2**, 7.4.11] and the commutative diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "T"'] & L(E^*,\mathbb{R}) \arrow[r, equal] \arrow[d, "L(T^*{,}\mathbb{R})"] & (E^*)' \arrow[d, "(T^*)^*"] \\
F \arrow[r, hook] & L(F^*,\mathbb{R}) \arrow[r, equal] & (F^*)'
\end{tikzcd}$$

shows that $T$ is an embedding as well. Hence we have proved

**3.18 Corollary.**
*A linear mapping $T : E \to F$ is a topological embedding iff the associated mapping $T^* : F^* \to E^*$ is a bornological quotient mapping for the equi-continuous bornologies.*
*It is a dense embedding iff the associated mapping $T^*$ is a bornological isomorphism, i.e. is invertible in the category of bounded linear mappings.* □

**3.19 Proposition.**
*The projective tensor product preserves dense mappings and dense embeddings.*

**Proof.** Obviously the tensor product $T_1 \otimes T_2$ of two dense mappings is dense. Otherwise there would exist a non-trivial continuous linear functional which vanishes on the image. The corresponding bilinear continuous map would then vanish on the dense image of $T_1 \times T_2$, and hence be 0, a contradiction.

Let now $T : E_2 \to E_1$ be in addition an embedding. By the previous proposition we have to show that $(T \otimes F)^* : (E_1 \otimes_\pi F)^* \to (E_2 \otimes_\pi F)^*$ is a quotient mapping for the equi-continuous bornologies. So let $\mathcal{B} := (U \otimes V)^o$ be a typical equi-continuous subset of $(E_2 \otimes_\pi F)^* \cong \mathcal{L}(E_2, F; \mathbb{R})$ formed by 0-neighborhoods $U$ and $V$. We may extend every $b \in \mathcal{B} \subseteq \mathcal{L}(E_2, F; \mathbb{R})$ to a continuous bilinear mapping $\tilde{b} \in \mathcal{L}(E_1, F; \mathbb{R})$ defined by $\mathrm{ev} \circ (E_1 \times \check{b}) : E_1 \times F \to E_1 \times E_2^* = E_1 \times E_1^* \to \mathbb{R}$. For this recall that $E_1^* = E_2^*$ by 3.18. This composition is continuous (although the last component is not), since $\bar{U} \times V$ is mapped to $\mathrm{ev}(\bar{U} \times U^o) = \mathrm{ev}(\bar{U} \times (\bar{U})^o) \subseteq [-1, 1]$ and $\bar{U}$ is a

0-neighborhood in $E_1$, see [**2**, 4.10.3]. Hence $\tilde{\mathcal{B}} := \{\tilde{b} : b \in \mathcal{B}\} \subseteq (\bar{U} \otimes V)^o$ is the required equi-continuous subset satisfying $(T \otimes F)^*(\tilde{\mathcal{B}}) \supseteq \mathcal{B}$. □

**3.20 Corollary. Completed projective tensor product.**
*The projective tensor product $E_1 \otimes_\pi E_2$ is a dense topological subspace of $\hat{E}_1 \otimes_\pi \hat{E}_2$. The completion of $E_1 \otimes_\pi E_2$ equals that of $\hat{E}_1 \otimes_\pi \hat{E}_2$. It will be denoted by $E_1 \hat{\otimes}_\pi E_2$, and will be called the completed projective tensor product.* □

**3.21 Theorem. Compact subsets of the projective tensor product.**

*Compact subsets of $E \hat{\otimes}_\pi F$ for metrizable spaces $E$ and $F$ are contained in the closed absolutely convex hull of a tensor product of precompact sets in $E$ and $F$.*

**Proof.** Every compact set $K$ in the Fréchet space $E \hat{\otimes}_\pi F$ is contained in the closed absolutely convex hull of a 0-sequence $z_n \in E \hat{\otimes}_\pi F$ by [**2**, 6.4.3]. For this 0-sequence we can choose $k_n$ strictly increasing, such that $z_k \in U_n \otimes V_n$ for all $k \geq k_n$, where $(U_n)_n$ and $(V_n)_n$ are countable 0-neighborhood bases of the topology of $E$ and $F$. For $k_n \leq k < k_{n+1}$ we can choose finite (disjoint) sets $N_k \subseteq \mathbb{N}$ and $\sum_{j \in N_k} |\lambda_j| = 1$, $x_j \in U_n$ and $y_j \in V_n$ such that $z_k = \sum_{j \in N_k} \lambda_j x_j \otimes y_j$. Let $A := \{x_j : j \in \bigcup_k N_k\}$ and $B := \{y_j : j \in \bigcup_k N_k\}$. These are two sequences converging to 0, and hence are precompact. Furthermore $z \in K$ can be written as

$$z = \sum_k \mu_k z_k = \sum_k \sum_{j \in N_k} \mu_k \lambda_j x_j \otimes y_j$$

with $\sum_k |\mu_k| \leq 1$ and $\sum_{j \in N_k} |\lambda_j| = 1$ and hence $\sum_k |\mu_k| \sum_{j \in N_k} |\lambda_j| \leq 1$. From this it easily follows that the series on the right hand side converges Mackey and hence $z$ is contained in the closed absolutely convex hull of $A \otimes B$. □

**3.22 Corollary. Elements of the completed tensor product as limits.**
*Every $z \in E \hat{\otimes}_\pi F$ for metrizable $E$ and $F$ has a representation of the form $z = \sum_n \lambda_n x_n \otimes y_n$, where $\lambda \in \ell^1$ and $x$ and $y$ are bounded (or even 0-)sequences.*

Since for every $\lambda \in \ell^1$ there exists a $\rho \in c_0$ and $\mu \in \ell^1$ with $\lambda_n = \rho_n^2 \mu_n$ it is enough to find bounded sequences $x_n$ and $y_n$.

**Proof.** In the previous proof we have just shown that $z = \sum_j \mu_{k_j} \lambda_j x_j \otimes y_j$. □

Next we will show some preservation properties with respect to limits. For this we need.

## Some Remarks on Limits

**3.23** In category theory one defines a *diagram* $X$ as a functor from a *small category* $\underline{I}$ (i.e. a category consisting only of a set of morphisms) into some category $\underline{A}$. A *cone* over such a diagram is an object $A \in \underline{A}$ together with morphisms $f_i : A \to X_i$ for all objects $i$ in $\underline{I}$ and such that for every morphism $\alpha : i \to j$ in $\underline{I}$ the following triangle commutes:

$$\begin{tikzcd} X_i \arrow[rr, "X_\alpha"] & & X_j \\ & A \arrow[ul, "f_i"] \arrow[ur, "f_j"'] & \end{tikzcd}$$

A *limit* is then a maximal cone $(X_\infty, (p_i)_i)$ in the following sense. For every cone $(A, (f_i)_i)$ there has to be a unique morphism $f : A \to X_\infty$ such that all triangles

$$\begin{tikzcd}
X_i & \\
 & X_\infty \arrow[ul, "p_i"'] \\
A \arrow[uu, "f_i"] \arrow[ur, dotted, "!", "f"'] &
\end{tikzcd}$$

commute. We can write this into one diagram

$$\begin{tikzcd}
X_i \arrow[rr, "X_\alpha"] & & X_j \\
 & X_\infty \arrow[ul, "p_i"] \arrow[ur, "p_j"'] & \\
 & A \arrow[uul, "f_i"] \arrow[uur, "f_j"'] \arrow[u, dotted, "!", "f"'] &
\end{tikzcd}$$

Special cases of limits (The reader is advised to draw the corresponding diagrams):

1. A *terminal object* $X$ is the limit of the empty diagram, i.e. an object $X$ such that for every object $Y$ there is a unique morphism $Y \to X$.
2. The *product* $\prod_i X_i$ is the limit of the diagram $X$ given by a discrete category $\underline{I}$, i.e. there are no morphisms beside the identities.
3. A *projective limit* $\varprojlim_i X_i$ is a limit of a diagram $X$ indexed by a category whose objects are the elements of a partially ordered set $(T, \preceq)$ and for all $i \preceq j$ a unique morphism say $(i, j)$ from $i$ to $j$ is given. In addition one assumes that any finitely many $i$ have a lower bound. It is easy to check that for any $i \in I$ one has that $\varprojlim_{j \preceq i} X_j \cong \varprojlim_j X_j$.
4. An *equalizer* is the limit of a diagram of the form $1 \rightrightarrows 2$ , i.e. two arrows from one object to another. Note that among the two required arrows the arrow $p_2 : X_\infty \to X_2$ is superfluous, since it can be obtained as composite $X_\infty \to X_1 \rightrightarrows X_2$.
5. A *pullback* $X_1 \times_{X_3} X_2$ is the limit of a diagram of the form

$$\begin{tikzcd}
 & X_3 & \\
X_1 \arrow[ur, "X_{\alpha_1}"] & & X_2 \arrow[ul, "X_{\alpha_2}"']
\end{tikzcd}$$

Similar to what we said before among the 3 required arrows the arrow $p_3 : X_\infty \to X_3$ is superfluous. And the condition of being a cone only says that $X_{\alpha_1} \circ p_1 = X_{\alpha_2} \circ p_2$.
The dual concepts, i.e. if all arrows are reversed, are the following:

1. Initial objects are dual to terminal ones;
2. Coproducts are dual to products;
3. Inductive limits are dual to projective ones;
4. Coequalizer are dual to equalizer;
5. Pushouts are dual to pullbacks.

Let us determine these limits in $\underline{LCS}$ and in $\underline{CBS}$.

We start with the equalizer of two maps $f_1, f_2 : X_1 \to X_2$. It is given as the subspace $X_\infty := \{x \in X_1 : f_1(x) = f_2(x)\}$ of $X_1$ with the initial structure inherited from $X_1$. Note that this is just the kernel of the morphism $f_1 - f_2$. Dual to this construction is the formation of quotients (i.e. cokernels).

Next the product is just the set theoretic product with the initial structure induced by the projections $\mathrm{pr}_i : \prod_i X_i \to X_i$. In case of an empty index set, we obtain the terminal object given by the 0 vector space. This is also an initial object. Coproducts are also called direct sums.

The pullback is given by the subspace $X_\infty := \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$ of the product $X_1 \times X_2$ with its initial structure.

A general limit is the subspace

$$X_\infty := \Big\{(x_i)_i \in \prod_i X_i : X_f(x_i) = x_j \text{ for all morphisms } f : i \to j \text{ in } \underline{I}\Big\}$$

of the product.

This is also true purely categorical. I.e. any limit can be constructed as the equalizer of the two maps $\alpha, \beta : \prod_i X_i \to \prod_f X_{\operatorname{cod} f}$, where the second product runs over all morphisms $f$ in $\underline{I}$, and $\alpha$ and $\beta$ are given given by the following diagrams (For a morphism $f : X \to Y$ we call $X$ the *domain* of $f$ and $Y$ the *codomain* of $X$ and denote them $\operatorname{dom} f$ and $\operatorname{cod} f$):

$$\begin{tikzcd}
\prod_i X_i \arrow[r, dotted, "\alpha"] \arrow[dr, "\mathrm{pr}_{\operatorname{cod} f}"'] & \prod_f X_{\operatorname{cod} f} \arrow[d, "\mathrm{pr}_f"] \\
 & X_{\operatorname{cod} f}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\prod_i X_i \arrow[r, dotted, "\beta"] \arrow[d, "\mathrm{pr}_{\operatorname{dom} f}"'] & \prod_f X_{\operatorname{cod} f} \arrow[d, "\mathrm{pr}_f"] \\
X_{\operatorname{dom} f} \arrow[r, "X_f"'] & X_{\operatorname{cod} f}.
\end{tikzcd}$$

A second way to construct an arbitrary limit is by taking the limits $\lim X|_{\underline{J}}$ of the restrictions of the diagram to all finite subcategories $\underline{J}$ of $\underline{I}$. Since these subcategories are ordered by $\underline{J'} \preceq \underline{J}$ iff $\underline{J'} \supseteq \underline{J}$ we may take the projective limit $\varprojlim_{\underline{J}} \lim X|_{\underline{J}}$. This is then the limit $\lim X$ of the full diagram $X$. In particular the product is given by the projective limit of all finite subproducts.

**3.24 Projective representation of a locally convex space.**
Let us consider an important particular case. Let $E$ be a locally convex space. We have shown in [**2**, 4.3.4] that $E$ carries the initial structure with respect to the family of projections $\pi_p : E \to E_p := E/\ker p$ indexed by all continuous seminorms where $E/\ker p$ is considered as normed space with respect to the norm induced by $p$. The natural partial ordering between seminorms turns these $E_p$ into a projective system. And since the projections $E \to E_p$ separate points we obtain an embedding $E \to \varprojlim_p E_p$. Let us show that this embedding has dense image. For this we use that a basis of 0-neighborhoods in a projective limit $\varprojlim_i E_i$ is given by the sets $\mathrm{pr}_i^{-1}(U_i)$ where $U_i$ runs through the 0-neighborhoods of $E_i$. In fact for finitely many $j$ and 0-neighborhoods $U_j \subseteq E_j$ we may choose an $i$ with $i \preceq j$ for all those $j$ and take $U_i := \bigcap_j T_{i,j}^{-1}(U_j)$, where $T_{i,j} : E_i \to E_j$ denotes the connecting morphism. Then $\mathrm{pr}_i^{-1}(U_i) \subseteq \bigcap_j \mathrm{pr}_j^{-1}(U_j)$. Now let $z \in \varprojlim_p E_p$ and take a 0-neighborhood $\mathrm{pr}_p^{-1}(U_p)$

in $\varprojlim_p E_p$. Since $\pi_p : E \to E_p$ is onto we can find a $x \in E$ with $\pi_p(x) = \mathrm{pr}_p(z)$. Thus $\iota(x) - z \in \mathrm{pr}_p^{-1}(U_p)$, i.e. $\iota$ has dense image. In general this mapping is however not onto. Take for example the subspace $E \subseteq \mathbb{R}^{\mathbb{N}}$ formed by all finite sequences. Then a basis of seminorms is given by $p_n : x \mapsto \max\{|x_i| : i \leq n\}$. The kernel of $p_n$ is $\{x : x_i = 0 \text{ for } i \leq n\}$ and hence $E_n := E_{p_n} \cong \mathbb{R}^n$. As we already mentioned the projective limit of $\mathbb{R}^n$ is just $\mathbb{R}^{\mathbb{N}}$ and $E$ is only dense in this space.

So to get an isomorphism we could take completions on both sides and obtain

$$\hat{E} \cong \widehat{\varprojlim_p E_p}.$$

We want to show that the completion functor preserves this projective limit, i.e. that

$$\widehat{\varprojlim_p E_p} \cong \varprojlim_p \widehat{E_p}.$$

Obviously the limit $\lim_i f_i$ of embeddings $f_i$ is an embedding, since this is true for products, see the following diagram

$$\begin{tikzcd}
\lim_i X_i \arrow[r, hook] \arrow[d, "\lim_i f_i"'] & \prod_i X_i \arrow[d, "\prod_i f_i"] \\
\lim_i Y_i \arrow[r, hook] & \prod_i Y_i.
\end{tikzcd}$$

The user should try to give categorical definitions of $\prod_i f_i$ and $\lim_i f_i$ using universal properties.

Remains to show that certain projective limits of dense mappings are dense:

**3.25 Lemma. Reduced projective limits.**
*Let $\varprojlim_i X_i$ be a reduced projective limit, and $f_i : X_i \to Y_i$ be continuous linear mappings with dense image which intertwine with all connecting mappings. Then the canonical mapping $\varprojlim_i f_i : \varprojlim_i X_i \to \varprojlim_i Y_i$ has dense image.*

A projective limit is called *reduced* if all projections $\mathrm{pr}_j : \varprojlim_j E_j \to E_j$ have dense image. By replacing $E_j$ by the closure of the image of $\mathrm{pr}_j$ we see that every projective limit is the reduced projective limit of some modified diagram.

Note that the projective limit $\varprojlim_p E_p$ is obviously reduced, since $\pi_p : E \to E_p$ is onto and hence the same is true for $\mathrm{pr}_p : \varprojlim_p E_p \to E_p$. And consequently also the projective limit $\varprojlim_p \widehat{E_p}$ is reduced.

**Proof.** Let $z \in \varprojlim_i Y_i$ be given. Take an arbitrary 0-neighborhood $\mathrm{pr}_i^{-1}(2\,U_i)$. Since $f_i$ has dense image we may find an $x_i \in X_i$ with $f_i(x_i) - \mathrm{pr}_i(z) \in U_i$. Since the first limit is reduced we can find an $x \in E$ with $\mathrm{pr}_i(x) - x_i \in f_i^{-1}(U_i)$. But then

$$\mathrm{pr}_i\Big(\varprojlim_i f_i(x) - z\Big) = (f_i \circ \mathrm{pr}_i)(x) - f_i(x_i) + f_i(x_i) - \mathrm{pr}_i(z) \in 2U_i,$$

i.e. $\varprojlim_i f_i$ has dense image. □

**3.26 Lemma.**
*The functor $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$ preserves products.*

**Proof.** For any functor $\mathcal{F} : \underline{A} \to \underline{B}$ we have a natural mapping $\mathcal{F}(\lim X_i) \to \lim \mathcal{F}(X_i)$ by the universal property of the right side.

$$
\begin{tikzcd}
X_i \arrow[rr, "X_\alpha"] & & X_j \\
& \lim_k X_k \arrow[ul, "p_i"] \arrow[ur, "p_j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X_i) \arrow[rr, "\mathcal{F}(X_\alpha)"] & & \mathcal{F}(X_j) \\
& \lim_k \mathcal{F}(X_k) \arrow[ul, "p_i"] \arrow[ur, "p_j"'] & \\
& \mathcal{F}(\lim_k X_k) \arrow[uul, "\mathcal{F}(f_i)"] \arrow[uur, "\mathcal{F}(f_j)"'] \arrow[u, dotted, "! f"'] &
\end{tikzcd}
$$

Thus we have a mapping $\coprod_i X_i^* \to (\prod_i X_i)^*$, where $\coprod_i X_i^*$ denotes the coproduct in $\underline{CBS}$ and hence the product in $\underline{CBS}^{op}$. Since $\prod_i X_i$ obviously separates points in $\coprod_i X_i^*$ this mapping is injective. Let us show that it is a bornological quotient map. This implies that it is a isomorphism. So let $(\prod_i U_i)^o$ be a typical bounded subset of $(\prod_i X_i)^*$, i.e. the $U_i$ are 0-neighborhoods of $X_i$ and $U_i = X_i$ except $i$ in some finite subset $J$ of $I$. Let $T \in (\prod_i U_i)^o$. Then $T((x_i)_i) = 0$ for all $(x_i)_i$ with $x_j = 0$ for all $j \in J$ (use that for such $(x_i)_i$ every multiple belongs to $\prod_i U_i$). Let $T_i := T \circ \mathrm{inj}_i \in U_i^o \subseteq X_i^*$. Then $T = \sum_{j\in J} T_j \in \coprod_j U_j^o$ and $\coprod_j U_j^o$ is bounded in $\coprod_i X_i^*$. $\square$

Next we want to investigate stability properties of $(\_)^*$ with respect to projective limits.

**3.27 Lemma. The dual of a reduced projective limit.**
*The functor $(\_)^* : \underline{LCS} \to \underline{CBS}^{op}$ preserves reduced projective limits*

**Proof.** So let $E = \varprojlim_i E_i$ be a reduced projective limit. As in the proof of 3.26 we have a natural mapping $\varinjlim_i E_i^* \to (\varprojlim E_i)^*$. Since all projections $\mathrm{pr}_i : E \to E_i$ have dense image the dual cone $\mathrm{pr}_i^* : E_i^* \to E^*$ consists of injective mappings only. Let $x^* \in E^*$ be given. Then there has to exist an $i$ and a 0-neighborhood $U_i \subseteq E_i$ with $x^*(\mathrm{pr}_i^{-1}(U_i)) \subseteq [-1,1]$. In particular $x^*(\ker \mathrm{pr}_i|_E) = 0$ and hence there exists a linear $x_i^* : \mathrm{pr}_i(E) \to \mathbb{R}$ with $x_i^* \circ \mathrm{pr}_i = x^*$. Since $x_i^*(U_i \cap \mathrm{pr}_i(E)) \subseteq [-1,1]$ we may extend it to a continuous functional on the closure $E_i$ of $\mathrm{pr}_i(E)$ which lies in $U_i^o$. Thus the union of all images $\mathrm{pr}_i^*(E_i)^*$ is $E^*$. Moreover the same argument shows that every bounded set $(\mathrm{pr}_i^{-1}(U_i))^o$ is the image of the bounded set $U_i^o$ under $\mathrm{pr}_i^*$, i.e. the natural mapping is a bornological quotient mapping. From this it is clear that $(\varprojlim_i E_i)^*$ is the injective limit, since any family of bounded linear mappings $T_i : E_i^* \to F$ that commute with the connecting morphisms can be extended to a bounded linear mapping $T : E^* = \bigcup_i \mathrm{pr}_i^*(E_i^*) \to F$. $\square$

**3.28 Theorem.**
*The completed projective tensor product $(\_)\hat{\otimes}_\pi E$ preserves products.*

**Proof.** The functoriality of $(\_)\hat{\otimes}_\pi F$ gives us a natural mapping

$$\iota : \Big(\prod_i E_i\Big)\hat{\otimes}_\pi F \to \prod_i (E_i \hat{\otimes}_\pi F).$$

We claim that this mapping is an embedding. As in 3.19 it is equivalent to show that the associated mapping $\iota^* : \big(\prod_i (E_i \hat{\otimes}_\pi F)\big)^* \to \big((\prod_i E_i)\hat{\otimes}_\pi F\big)^*$ is a quotient map for the equi-continuous bornologies. But this mapping is up to the natural isomorphisms from 3.11

$$\coprod_i \mathcal{L}(E_i, F; \mathbb{R}) \cong \coprod_i (E_i \hat{\otimes}_\pi F)^* \cong \Big(\prod_i (E_i \hat{\otimes}_\pi F)\Big)^*$$

and

$$\mathcal{L}\Big(\prod_i E_i, F;\mathbb{R}\Big) \cong \bigg(\Big(\prod_i E_i\Big)\hat{\otimes}_\pi F\bigg)^*$$

given by

$$\begin{aligned}\prod_i \mathcal{L}(E_i,F;\mathbb{R}) &\to \mathcal{L}\Big(\prod_i E_i, F;\mathbb{R}\Big).\\ (b_i)_i &\mapsto \Big(((x_i)_i,y)\mapsto \sum_i b_i(x_i,y)\Big)\end{aligned}$$

So let $B := ((\prod_i U_i)\times V)^o$ be a typical equi-continuous subset of $\mathcal{L}(\prod_i E_i, F;\mathbb{R})$, where $U_i$ are 0-neighborhoods in $E_i$ with $U_i = E_i$ for all $i$ except those in some finite subset $I$ of the index set, and $V$ is a 0-neighborhood in $F$. In particular we have for every $b\in B$ that $b((x_i)_i,y)=0$ provided $x_i=0$ for all $i\in I$, since for all $\varepsilon>0$ and $y\in V$ we have $b((\varepsilon x_i),y)=\varepsilon\, b((\frac{1}{\varepsilon}x_i)_i,\varepsilon y)\in\varepsilon\, b(\prod_i U_i\times V)\subseteq\varepsilon\,[-1,1]$. Since $V$ is absorbing it has to be 0. Thus $b$ can be considered as element of $\coprod_{i\in I}\mathcal{L}(E_i,F;\mathbb{R})$ and lies moreover in the equi-continuous subset $\coprod_{i\in I}(U_i\times V)^o$.

Since the algebraic tensor product is left-adjoint to $L(E,\_)$ it commutes with coproducts. Hence algebraically we have that $(\coprod_i E_i)\otimes F\cong\coprod_i(E_i\otimes F)$. We will see later on that topologically this is not true in general. By the density of the coproducts, we obtain that $\iota$ is dense and hence we have the required isomorphism. □

**3.29 Proposition.**
*The completed projective tensor product preserves reduced projective limits.*

**Proof.** So let $E=\varprojlim_i E_i$ be a reduced projective limit, i.e. $\mathrm{pr}_i : E\to E_i$ has dense image. Then $\mathrm{pr}_i\otimes F : E\otimes F\to E_i\otimes_\pi F$ has dense image and consequently also $\mathrm{pr}_i\hat{\otimes}_\pi F : E\otimes F\to E_i\hat{\otimes}_\pi F$. Since this mapping factors over $\varprojlim_i(E_i\hat{\otimes}_\pi F)\to E_i\hat{\otimes}_\pi F$ the latter mapping has dense image as well. Thus the limit $\varprojlim_i(E_i\hat{\otimes}_\pi F)$ is a reduced one. Let us show next that the natural mapping

$$\Big(\varprojlim_i E_i\Big)\hat{\otimes}_\pi F\to\varprojlim_i(E_i\hat{\otimes}_\pi F)$$

is a dense embedding, or equivalently that the dual mapping

$$\Big(\varprojlim_i(E_i\hat{\otimes}_\pi F)\Big)^*\to\Big((\varprojlim_i E_i)\hat{\otimes}_\pi F\Big)^*$$

is a bornological isomorphism. The left side equals

$$\Big(\varprojlim_i(E_i\hat{\otimes}_\pi F)\Big)^*\cong\varinjlim_i(E_i\hat{\otimes}_\pi F)^*\cong\varinjlim_i\mathcal{L}(E_i,F;\mathbb{R}),$$

since the dual of a reduced projective limit is an injective one. The right hand side equals $\Big((\varprojlim_i E_i)\hat{\otimes}_\pi F\Big)^*\cong\mathcal{L}\Big(\varprojlim_i E_i,F;\mathbb{R}\Big)$. So let $(\mathrm{pr}_j^{-1}(U_j)\times V)^o$ be a typical bounded set in $\mathcal{L}(\varprojlim_i E_i,F;\mathbb{R})$. This is the image under the natural mapping of the bounded set $(\mathrm{pr}_i\times F)^*((U_i\times V)^o)$ in $\varinjlim_i\mathcal{L}(E_i,F;\mathbb{R})=\bigcup_i(\mathrm{pr}_i\times F)^*(\mathcal{L}(E_i,F;\mathbb{R}))$. Thus the natural mapping is a bornological quotient mapping. It remains to show that it is injective. So let $T=(\mathrm{pr}_i\times F)^*(T_i)\in\varinjlim_i\mathcal{L}(E_i,F;\mathbb{R})$ be given with $T_i\in\mathcal{L}(E_i,F;\mathbb{R})$ and such that the associated element $\iota(T)=0$ in $\mathcal{L}\Big(\varprojlim_i E_i,F;\mathbb{R}\Big)$. Obviously $\iota(T)=T_i\circ(\mathrm{pr}_i\times F)$ and since the $\mathrm{pr}_i\times F$ has dense image in $E_i\times F$ we conclude that $T_i=0$ and hence $T=0$.

Since both sides of the natural dense embedding $\Big(\varprojlim_i E_i\Big)\hat{\otimes}_\pi F\to\varprojlim_i(E_i\hat{\otimes}_\pi F)$ are complete (limits of complete spaces are complete) we have equality. □

**3.30 Corollary.**
*For the function space $C(X,E) = E^X = \prod_{x\in X} E$, where $X$ is a discrete topological space, we have a natural isomorphism*

$$C(X)\hat{\otimes}_\pi E = \mathbb{R}^X \hat{\otimes}_\pi E \cong (\mathbb{R}\hat{\otimes}_\pi E)^X \cong E^X = C(X,E).$$

*If $X$ is finite and $E \cong \mathbb{R}^Y$ with finite $Y$ we obtain in particular that $\mathbb{R}^X\hat{\otimes}_\pi\mathbb{R}^Y \cong (\mathbb{R}^Y)^X \cong \mathbb{R}^{X\times Y}$. Hence we have for finite dimensional spaces that $\dim(E\hat{\otimes}_\pi F) = \dim E \cdot \dim F$ thus also $\dim(E\otimes_\pi F) = \dim E\cdot\dim F$.*

Note that for general projective limits the analogue to 3.29 is not true. In fact take the closed linear subspace $\ell^2 \to C(K)$, which is the kernel (a special limit) of the quotient map $C(K) \to C(K)/\ell^2$. Since $\ell^2 \otimes_\pi (\ell^2)^* \to C(K)\otimes_\pi (\ell^2)^*$ is not an embedding (see 3.14), also $\ell^2\hat{\otimes}_\pi(\ell^2)^* \to C(K)\hat{\otimes}_\pi(\ell^2)^*$ is not, and so both cannot be the kernel of some map.

**3.32 Example.**
In general the projective tensor product does not commute with direct sums. Furthermore it does not preserve strict inductive limits (since $\mathbb{R}^{(\mathbb{N})} = \varinjlim_n \mathbb{R}^n$) and also not the function space $C_c(X) = \mathbb{R}^{(X)}$ for discrete $X$:
The natural injection of flipping the coordinates from $(\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \to (\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$ is obviously not onto. Moreover, it can be shown that $(\mathbb{R}^\mathbb{N})^{(\mathbb{N})}$ and $(\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$ can not even be isomorphic by some non-canonical isomorphism, since both spaces are $B$-complete but their cartesian product is not, see [**14**, 15.5.1]. A locally convex spaces is called *$B$-complete*, iff every continuous *nearly open map* (i.e. the closure of the image of any 0-neighborhood is a 0-neighborhood) into some locally convex spaces has complete image, or equivalently if every such mapping is open onto its image. So isomorphy would imply that $(\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \cong (\mathbb{R}^\mathbb{N}\times\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \cong (\mathbb{R}^\mathbb{N})^{(\mathbb{N})}\times(\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \cong (\mathbb{R}^\mathbb{N})^{(\mathbb{N})}\times(\mathbb{R}^{(\mathbb{N})})^\mathbb{N}$. Then the natural mapping from $(\mathbb{R}\hat{\otimes}_\pi\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \to \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi\mathbb{R}^\mathbb{N}$ is not onto as can be seen also from the following commutative diagram

$$\begin{tikzcd}
(\mathbb{R}\hat{\otimes}_\pi\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \arrow[rr] & & \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi\mathbb{R}^\mathbb{N} \arrow[ddr, "\cong"] & \\
& \mathbb{R}\hat{\otimes}_\pi\mathbb{R}^\mathbb{N} \arrow[ul, "\mathrm{inj}_k"] \arrow[ur, "\mathrm{inj}_k\otimes\mathbb{R}^\mathbb{N}"'] \arrow[r, "\cong"] & (\mathbb{R}\hat{\otimes}_\pi\mathbb{R})^\mathbb{N} \arrow[dr, "(\mathrm{inj}_k\otimes\mathbb{R})^\mathbb{N}"'] & \\
& & & (\mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi\mathbb{R})^\mathbb{N} \\
& \mathbb{R}^\mathbb{N} \arrow[uu, "\cong"] \arrow[dl, "\mathrm{inj}_k"'] \arrow[dr, "(\mathrm{inj}_k)^\mathbb{N}"] & & \\
(\mathbb{R}^\mathbb{N})^{(\mathbb{N})} \arrow[uuuu, "\cong"] \arrow[rr] & & (\mathbb{R}^{(\mathbb{N})})^\mathbb{N} \arrow[uur, "\cong"] &
\end{tikzcd}$$

since the bottom arrow is obviously not onto.

**3.33 Corollary.**
*Neither the projective tensor product nor the completed projective tensor product can be left adjoint functors.* □

## The Bornological Tensor Product

We have seen that the classical projective tensor product is not well behaved beyond normed spaces. And the main reason for that is that it is not longer a left-adjoint functor.

But we have already seen that bounded mappings are in many respects much nicer than continuous ones.

And if $L(E_1,\dots,E_n;F)$ denotes the *space of all bounded $n$-linear mappings* from $E_1\times\dots\times E_n\to F$ with the topology of uniform convergence on bounded sets in $E_1\times\dots\times E_n$ then we easily show the following.

**3.34 Proposition. Exponential law for $L$.**
*There are natural topological linear isomorphisms*

$$L(E_1,\dots,E_{n+k};F)\cong L(E_1,\dots,E_n;L(E_{n+1},\dots,E_{n+k};F)).$$

**Proof.** We proof this for bilinear maps, the general case is completely analogous. We already know that bilinearity translates into linearity into the space of linear functions. Remains to prove boundedness. So let a set $\mathcal{B}$ of bilinear mappings $E_1\times E_2\to F$ be given. Then $\mathcal{B}$ is bounded in $L(E_1,E_2;F)$ iff $\mathcal{B}(B_1\times B_2)\subseteq F$ is bounded for all bounded $B_i\subseteq E_i$. This however is equivalent to $\mathcal{B}^\vee(B_1)$ is contained and bounded in $L(E_2,F)$ for all bounded $B\subseteq E_1$, i.e. $\mathcal{B}^\vee$ is contained and bounded in $L(E_1,L(E_2,F))$.

That this even a topological isomorporphism follows by the arguments in 3.10. $\square$

Recall that we already put a structure on $L(E,F)$ in 2.53, namely the initial one with respect to the inclusion in $C^\infty(E,F)$. Let us now show that bornologically these definitions agree:

**3.35 Lemma. Structure on $L$.**
*A subset is bounded in $L(E,F)\subseteq C^\infty(E,F)$ if and only if it is uniformly bounded on bounded subsets of $E$, i.e. $L(E,F)\to C^\infty(E,F)$ is initial.*

**Proof.** Let $\mathcal{B}\subseteq L(E,F)$ be bounded in $C^\infty(E,F)$ and assume that it is not uniformly bounded on some bounded set $B\subset E$. So there are $f_n\in\mathcal{B}$, $b_n\in B$, and $\ell\in F^*$ with $|\ell(f_n(b_n))|\ge n^n$. Then the sequence $n^{1-n}b_n$ converges fast to 0 and hence lies on some compact part of a smooth curve $c$ by 2.18. So $\mathcal{B}$ cannot be bounded, since otherwise $C^\infty(\ell,c)=\ell_*\circ c^*:C^\infty(E,F)\to C^\infty(\mathbb{R},\mathbb{R})\to\ell^\infty(\mathbb{R},\mathbb{R})$ would have bounded image, i.e. $\{\ell\circ f_n\circ c:n\in\mathbb{N}\}$ would be uniformly bounded on any compact interval.

Conversely let $\mathcal{B}\subseteq L(E,F)$ be uniformly bounded on bounded sets and hence in particular on compact parts of smooth curves. We have to show that $d^n\circ c^*:L(E,F)\to C^\infty(\mathbb{R},F)\to\ell^\infty(\mathbb{R},F)$ has bounded image. But for linear smooth maps we have by the chain rule recursively applied that $d^n(f\circ c)(t)=f(c^{(n)}(t))$, and since $c^{(n)}$ is still a smooth curve we are done. $\square$

Let us now generalize this result to multi-linear mappings. For this we first characterize bounded multi-linear mappings in the following two ways:

**3.3o. Lemma.**
*A multilinear mapping is bounded if and only if it is bounded on each sequence which converges Mackey to 0.*

**Proof.** Suppose that $f:E_1\times\dots\times E_k\to F$ is not bounded on some bounded set $B\subset E_1\times\dots\times E_k$. By composing with a linear functional we may assume that $F=\mathbb{R}$. So there are $b_n\in B$ with $\lambda_n^{k+1}:=|f(b_n)|\to\infty$. Then $|f(\frac{1}{\lambda_n}b_n)|=\lambda_n\to\infty$, but $(\frac{1}{\lambda_n}b_n)$ is Mackey convergent to 0. $\square$

**3.36 Lemma. Bounded multi-linear mappings are smooth.**
*Let $f : E_1 \times \ldots \times E_n \to F$ be a multi-linear mapping. Then $f$ is bounded if and only if it is smooth. For the derivative we have the product rule:*

$$df(x_1, \ldots, x_n)(v_1, \ldots, v_n) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, v_i, x_{i+1}, \ldots, x_n).$$

*In particular we get for $f : E \supseteq U \to \mathbb{R}$, $g : E \supseteq U \to F$ and $x \in U$, $v \in E$ the Leibnitz formula*

$$(f \cdot g)'(x)(v) = f'(x)(v) \cdot g(x) + f(x) \cdot g'(x)(v).$$

**Proof.** We use induction on $n$. The case $n = 1$ is corollary 2.21. The induction goes as follows:

. $f$ is bounded
$\iff$ . $f(B_1 \times \ldots \times B_n) = f^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded sets $B_i$ in $E_i$;
$\iff$ . $f^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n, F) \subseteq C^\infty(E_n, F)$ is bounded, by 3.35;
$\iff$ . $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n, F)$ is bounded;
$\iff$ . $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n, F)$ is smooth by the inductive assumption;
$\iff$ . $f^\vee : E_1 \times \ldots \times E_n \to F$ is smooth by cartesian closedness 2.48. $\square$

The particular case follows by application to the scalar multiplication $\mathbb{R} \times E \to E$.

Now let us show that also the structures coincide:

**3.37 Proposition. Structure on space of multi-linear maps.**
*The injection of $L(E_1, \ldots, E_n; F) \to C^\infty(E_1 \times \ldots \times E_n, F)$ is a bornological embedding.*

**Proof.** We can show this by induction. In fact let $\mathcal{B} \subseteq L(E_1, \ldots, E_n; F)$. Then

. $\mathcal{B}$ is bounded
$\iff$ . $\mathcal{B}(B_1 \times \ldots \times B_n) = \mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded $B_i$ in $E_i$;
$\iff$ . $\mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n, F) \subseteq C^\infty(E_n, F)$ is bounded, by 3.35;
$\iff$ . $\mathcal{B}^\vee \subseteq C^\infty(E_1 \times \ldots \times E_{n-1}, C^\infty(E_n, F))$ is bounded by the inductive assumption;
$\iff$ . $\mathcal{B} \subseteq C^\infty(E_1 \times \ldots \times E_n, F)$ is bounded by cartesian closedness 2.48. $\square$

Hence it is natural to consider the universal problem of making bounded bilinear mappings into bounded linear ones. The solution is given by the *bornological tensor product* $E \otimes_\beta F$, i.e. the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded. A 0-neighborhood basis of this topology is given by those absolutely convex sets, which absorb $B_1 \otimes B_2$ for all bounded $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$. Note that this topology is bornological since it is the finest locally convex topology with given bounded linear mappings on it.

**3.38 Theorem. Bornological tensor product.**
*The bornological tensor product is left adjoint to the Hom-functor $L(E, \_)$ on the category of bounded linear mappings between locally convex spaces and one has the*

*following bornological isomorphisms:*

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G))$$
$$E \otimes_\beta \mathbb{R} \cong E$$
$$E \otimes_\beta F \cong F \otimes_\beta E$$
$$(E \otimes_\beta F) \otimes_\beta G \cong E \otimes_\beta (F \otimes_\beta G)$$

*Furthermore the bornological tensor product preserves co-limits. It neither preserves embeddings nor countable products.*

**Proof.** We show first that this topology has the universal property for bounded bilinear mappings $f : E_1 \times E_2 \to F$. Let $U$ be an absolutely convex zero neighborhood in $F$ and let $B_1$, $B_2$ be bounded sets. Then $f(B_1 \times B_2)$ is bounded hence is absorbed by $U$. Then $\tilde{f}^{-1}(U)$ absorbs $\otimes(B_1 \times B_2)$, where $\tilde{f} : E_1 \otimes E_2 \to F$ is the canonically associated linear mapping. So $\tilde{f}^{-1}(U)$ is in the zero neighborhood basis of $E_1 \otimes_\beta E_2$ described above. Therefore $\tilde{f}$ is continuous.

A similar argument for sets of mappings shows that the first isomorphism $L(E \otimes_\beta F, G) \cong L(E, F; G)$ is bibounded.

The topology on $E_1 \otimes_\beta E_2$ is finer than the projective tensor product topology and so it is Hausdorff. The rest of the positive results is clear.

The counter example for embeddings given for the projective tensor product works also, since all spaces involved are Banach.

Since the bornological tensor-product preserves coproducts it cannot preserve products. In fact $(\mathbb{R} \otimes_\beta \mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ whereas $\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$. □

**3.39 Proposition. Projective versus bornological tensor product.**
*If every bounded bi-linear mapping on $E \times F$ is continuous then $E \otimes_\pi F = E \otimes_\beta F$. In particular we have $E \otimes_\pi F = E \otimes_\beta F$ for any two metrizable spaces and for a normable space $F$ we have $E_{born} \otimes_\pi F = E \otimes_\beta F$.*

**Proof.** Recall that $E \otimes_\pi F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is continuous, whereas $E \otimes_\beta F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is bounded. So we have that $\otimes : E \times F \to E \otimes_\beta F$ is bounded and hence by assumption continuous and thus the topology of $E \otimes_\pi F$ is finer than that of $E \otimes_\beta F$. Since the converse is true n general, we have equality.

In [**2**, 3.1.6] we have shown that in metrizable locally convex spaces the convergent sequences coincide with the Mackey-convergent ones. Now let $T : E \times F \to G$ be bounded and bilinear. We have to show that $T$ is continuous. So let $(x_n, y_n)$ be a convergent sequence in $E \times F$. Without loss of generality we may assume that its limit is $(0, 0)$. So there are $\mu_n \to \infty$ such that $\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}$ is bounded and hence also $T\Big(\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}\Big) = \Big\{\mu_n^2 T(x_n, y_n) : n \in \mathbb{N}\Big\}$, i.e. $T(x_n, y_n)$ converges even Mackey to 0.

If $F$ is normable, and $T : E_{born} \times F \to G$ is bi-linear and bounded, then $\check{T} : E_{born} \to L(F, G)$ is bounded, and since $E_{born}$ is bornological it is even continuous. We have shown in 3.10 that for normed spaces $F$ the evaluation map $\mathrm{ev} : L(F, G) \times F \to G$ is continuous, and hence $T = \mathrm{ev} \circ (\check{T} \times F) : E_{born} \times F \to G$ is continuous. Thus $E_{born} \otimes_\pi F = E \otimes_\beta F$. □

Note that the bornological tensor product is invariant under bornologification, i.e. $E_{born} \otimes_\beta F_{born} \cong E \otimes_\beta F$. So it is no loss of generality to assume that both spaces are bornological. Keep however in mind that the corresponding identity for the projective tensor product does not hold. Another possibility to obtain the identity $E \otimes_\pi F = E \otimes_\beta F$ is to assume that $E$ and $F$ are bornological and every separately continuous bi-linear mapping on $E \times F$ is continuous. In fact every bounded bi-linear mapping is obviously separately bounded and since $E$ and $F$ are assumed to be bornological it has to be separately continuous. We want to find another class beside the Fréchet spaces (see [**2**, Folgerung in 5.5]) which satisfies these assumptions.

## Some Remarks on Duals of Fréchet Spaces

**3.40 Proposition. Dual of Fréchet spaces.**
*Let $E$ be metrizable and $F$ be the strong dual $E^*_\beta$ of $E$, i.e. the space of continuous linear functionals on $F$ with the topology of uniform convergence on bounded sets. The the following statements are equivalent:*

1. *$F$ is ultra-bornological, i.e. every absolutely convex set, which absorbs Banach-disks (that is absolutely convex bounded sets $B$ for which $F_B$ is complete) is a 0-neighborhood;*
2. *$F$ is bornological, i.e. every absolutely convex set which is bornivorous (that is, absorbs (absolutely convex) bounded sets) is a 0-neighborhood;*
3. *$F$ is barreled, i.e. every barrel (that is every closed absolutely convex absorbent subset) is a 0-neighborhood;*
4. *$F$ is infra-barreled, i.e. every bornivorous barrel is a 0-neighborhood.*

Recall that a space is barreled iff every pointwise bounded set of continuous linear functions into some lcs is equi-continuous, see [**14**, 11.1.1].
It is infra-barreled (or quasi-barreled) iff every bounded set of continuous linear functions is equi-continuous, see [**14**, 11.3.7].
It is bornological iff every bounded linear mapping into any lcs is continuous, see [**14**, 13.1.1]. These are exactly the inductive limits of normed spaces, see [**14**, 13.2.2].
It is ultra-bornological iff every linear map that is bounded on Banach-disks is continuous, see [**14**, 13.1.1]. These are exactly the inductive limits of Banach spaces, see [**14**, 13.2.2].

However the strong dual of a Fréchet space does not always satisfy these equivalent conditions. Thus one needs some weakenings:

**3.41 Definition.**
A space is called *$c_0$-barreled* iff every 0-sequence in $E'$ with respect to the weak topology $\sigma(E', E)$ is equi-continuous. It is called *quasi $c_0$-barreled* iff every 0-sequence in $E'$ with respect to the strong topology $\beta(E', E)$ is equi-continuous, see [**14**, 12.1].

For a $c_0$-barreled space $E$ every bounded subset of $(E', \sigma(E', E))$ is bounded in $(E', \beta(E', E))$. A space is (quasi-)$c_0$-barreled iff for every sequence $(U_n)$ of closed absolutely convex 0-neighborhoods in $(E, \sigma(E, E'))$ such that every finite (bounded) set is contained in $\bigcap_{n \geq m} U_n$ for some $m$ it holds that $\bigcap_n U_n$ is a 0-neighborhood, see [**14**, 12.2.6].

A space is called *(quasi) countably barreled)* iff every (bornivorous) barrel which is the intersection of countably many closed absolutely convex 0-neighborhoods is a 0-neighborhood, see [**14**, 12.2]. Equivalently if every pointwise bounded (uniformly on bounded sets bounded) countable set of continuous linear functions with values in some lcs are equi-continuous, see [**14**, 12.2.1].

For $c^\infty$-complete spaces one has the following reverse implications:

1. bornological $\Longrightarrow$ ultra-bornological, see also [**14**, 13.2.4].
2. quasi-barreled $\Longrightarrow$ barreled, [**14**, 11.2.5].
3. quasi-countably-barreled $\Longrightarrow$ countably-barreled, [**14**, 12.2.2].

Any $c_0$-barreled quasi countablybarreled resp. quasi-barreled space is countably-barreled resp. barreled, [**14**, 12.3.3].

Hence we have the following diagram, where the dotted implications are valid under the assumption of $c^\infty$-completeness:

$$\begin{tikzcd}
\text{ulta-bornological} \arrow[d] \arrow[dr, shift left] & \\
\text{barreled} \arrow[d] \arrow[dr, shift left] & \text{bornological} \arrow[d] \arrow[ul, dotted, shift left] \\
\text{countably barreled} \arrow[d] \arrow[dr, shift left] & \text{quasi-barreled} \arrow[d] \arrow[ul, dotted, shift left] \\
c_0\text{-barreled} \arrow[dr, shift left] & \text{quasi countably barreled} \arrow[d] \arrow[ul, dotted, shift left] \\
& \text{quasi-}c_0\text{-barreled} \arrow[ul, dotted, shift left]
\end{tikzcd}$$

**3.42 Proposition.**
*The strong dual of any metrizable locally convex space is countably barreled.*

**Proof.** Let $E$ be a metrizable space and $F$ its strong dual space. Since $F$ is complete it is enough to show quasi countablybarreledness. So suppose we are given closed absolutely convex 0-neighborhoods $V_n$ in $F$ such that $V := \bigcap_n V_n$ is a bornivorous barrel. We have to show that $V$ is a 0-neighborhood. For this we construct recursively bounded sets $B_n$ in $E$ and $\rho_n > 0$ such that

$$B_k^o \subseteq V_k \text{ and } \rho_k U_k^o \subseteq \tfrac{1}{2^{k+1}} V \cap B_j^o \text{ for } k, j \leq n.$$

For $n = 1$ this is easily done. For $n$ we first choose $\rho_n > 0$ such that $\rho_n U_n^o \subseteq \frac{1}{2^{n+1}} V \cap \bigcap_{k<n} B_k^o$. Then the set $K := \sum_{k \leq n} \rho_k U_k^o$ is absolutely convex, $\sigma(E', E)$-compact and is contained in $\sum_{j \leq n} \frac{1}{2^{j+1}} V \subseteq \frac{1}{2} V_n$. Now choose an absolutely convex and $\sigma(E', E)$-closed 0-neighborhood $V'$ of $F$ contained in $\frac{1}{2} V_n$. Then $B_n := (V' + K)^o$ is bounded in $E$ and by the bipolar-theorem $B_n^o = V' + K$. Since $K \subseteq \frac{1}{2} V_n$ we get $B_n^o \subseteq V_n$.

Finally the set $W := \bigcap_n B_n^o$ satisfies $W = W^{oo}$ and absorbs every $U_n^o$ hence it is a barrel in $(E', \sigma(E', E))$ and hence by [**14**, 8.3.2] is a 0-neighborhood in $F$. Since $W \subseteq V$ the same is true for $V$. $\square$

By an *absorbent* resp. *bornivorous sequence* one understands a sequence of absolutely convex subsets $A_n \subseteq E$ satisfying $2A_n \subseteq A_{n+1}$, $A_0 := \{0\}$ and every finite resp. bounded set is contained in $A_n$ for some $n$.

**3.43 Lemma. Final topology with respect to a absorbent sequence.**
*Let $(A_n)_n$ be an absorbent sequence in $E$ and let $\mathcal{U}$ be a 0-neighborhood basis in $E$. Let $\tau$ be the final locally convex topology induced by the inclusions of $A_n \to E$. Then the sets $\langle \bigcup_k A_k + U_k \rangle_{abs.conv}$ with $U_k \in \mathcal{U}$ form a 0-neighborhood basis of $\tau$. If each*

*$A_k$ is in addition bounded then for any sequence $U_n$ of 0-neighborhoods one can find $\rho_n > 0$ such that $\bigcap_k \rho_k U_k$ is a 0-neighborhood.* [**14**, 12.3.1] *and* [**14**, 12.3.2].

**3.44 Proposition. Barreledness and absorbent sequences.**

*If $E$ is (quasi) countablybarreled, then it carries the finest locally convex topology which coincides on every $A_n$ with the trace topology, where $A_n$ is some absorbent (bornivorous) sequence,* [**14**, 12.3.6].

A locally convex space is called *df-space* if it is quasi-$c_0$-barreled and has a countable basis of the bornology, [**14**, 12.4.1]. The strong dual of a space $E$ is Fréchet iff $(E, \mu(E, E'))$ is *df*, see [**14**, 12.4.1].

A space is called $DF$ (for dual-Fréchet) if it is quasi-countablybarreled and has a countable basis of the bornology, [**14**, 12.4.1].

A space is called $gDF$ (for generalized dual-Fréchet) if it has a countable base $\mathcal{B}$ of the bornology and its topology coincides with the final one induced by $\mathcal{B}$, [**14**, 12.4]. A space $E$ is $gDF$ if the space $\mathcal{L}(E, F)$ of continuous linear mappings is Fréchet for any Fréchet space $F$, [**14**, 12.4.2], and equivalently if it has a countable basis of its bornology and every 0-sequence in $L(E, F)$ is equi-continuous for every lcs $F$, [**14**, 12.4.3].

One has the implications $DF \overset{[\mathbf{14},\ 12.3.6]}{\Rightarrow} gDF \overset{[\mathbf{14},\ 12.4]}{\Rightarrow} df$.

**3.45 Corollary.**
*The strong dual of a metrizable lcs is a complete $DF$-space.*

**Proof.** This follows from 3.42 since a countable base of the bornology is given by the polars of a countable 0-neighborhood basis, see also [**14**, 12.4.5]. □

**Remark.**
A metrizable space that has a countable base of its bornology is already normed, [**14**, 12.4.4]. Every sequentially complete *df*-space is *strictly webbed*, [**14**, 12.4.6].

**3.46 Proposition.**
*Every $gDF$-space is quasi-normable, i.e. for every 0-neighborhood $U$ there exists a smaller 0-neighborhood $V$ such that for all $t > 0$ there exists a bounded set $B$ with $V \subseteq tU + B$, see* [**14**, 10.7.1] *and* [**14**, 12.4.7]. *Note that a normed space is quasi-normed. In fact we may take $V = B := U$.*

**Proof.** Let $(A_n)$ be a base of the bounded sets. Let $U$ be an absolutely convex closed 0-neighborhood. Let $D := \bigcup_n D_k$, where $D_k := A_k^o \cap kU^o$. We claim that $D$ is equi-continuous. So for $k \in \mathbb{N}$ we choose $n_k \le k$ such that $A_k \subseteq n_k U$. Then

$$\begin{aligned} A_k \cap \frac{1}{n_k} U &= A_k \cap \Big( \bigcap_{n \le n_k} \frac{1}{n} U \Big) \cap \Big( \bigcap_{n > n_k} A_n \Big) \\ &\subseteq A_k \cap \Big( \bigcap_{n \in \mathbb{N}} \frac{1}{n} U \cup A_n \Big) \\ &\subseteq A_k \cap D^o \subseteq D^o. \end{aligned}$$

Since $\langle \bigcup_k A_k \cap \frac{1}{n_k} U \rangle_{abs.conv} \subseteq D^o$ it follows from [**14**, 12.3.1] that $D^o$ is a 0-neighborhood in $E$ and hence $D$ is equi-continuous. One can now show that for $\rho > 0$ there exists some bounded set $B$ such that $D^o \subseteq \rho U + B$. □

A space is a bornological *df*- (or equivalently $DF$-) space iff it is the inductive limit of a sequence of normable spaces, [**14**, 13.2.6].

**3.47 Theorem. Continuity versus separately continuity.**
*Let $E$ and $F$ be two barreled spaces with a countable base of bornology. Then every separately continuous bilinear map $E \times F \to G$ is continuous.*

**Proof.** Let $A_n$ and $B_n$ be a basis of the bornologies of $E$ and $F$. Let $T : E \times F \to G$ be separately continuous. Then $T^\vee : E \to \mathcal{L}(F,G)$ is continuous for the topology of pointwise convergence on $\mathcal{L}(F,G)$. Thus $T^\vee(A_k)$ is bounded for this topology, and since $F$ is barreled it is equi-continuous. Thus for every 0-neighborhood $W$ in $G$ there exists a 0-neighborhood $V_k$ in $F$ with $T(A_k \times V_k) \subseteq W$. By symmetry there exists a 0-neighborhood $U_k$ in $E$ with $T(U_k \times B_k) \subseteq W$. We have to show that this implies for $gDF$-spaces $E$ and $F$ the continuity of $T$, see [**14**, 15.6.1]. Since $E$ is quasi-normable, we can find for every 0-neighborhood $U_n$ a 0-neighborhood $U'_n$ such that for every $\rho > 0$ there is some $k(n,\rho) \in \mathbb{N}$ with $U'_n \subseteq \rho U_n + A_{k(n,\rho)}$. Since $A_k$ is a basis of bounded sets there exist $\rho_n > 0$ such that $U := \bigcap_n \rho_n U'_n$ is a 0-neighborhood in the topology generated by $\{A_n\}$, see [**14**, 12.3.2]. And this topology coincides with the given topology since $E$ is $gDF$, by [**14**, 12.3.6]. Let $W_n := V_{k(n,1/\rho_n)}$. Then $V := \langle \bigcup_n \frac{1}{\rho_n} W_n \cap B_n$ is a 0-neighborhood again by [**14**, 12.3.6] and by the description of a 0-neighborhood basis of the topology induced by $\{B_n\}_n$ given in [**14**, 12.3.1]. We claim that $T(U \times V) \subseteq W$. In fact take $x \in U$ and $y \in V$. Then $y$ is an absolutely convex combination of $y_n \in \frac{1}{\rho_n} W_n \cap B_n$. Since $x \in \rho_n U'_n \subseteq U_n + \rho_n A_{k(n,1/\rho_n)}$ there are $u_n \in U_n$ and $a_n \in A_{k(n,1/\rho_n)}$ with $x = u_n + \rho_n a_n$. So

$$T(x,y_n) = T(u_n,y_n) + T(\rho_n a_n, y_n) \in T(U_n \times B_n) + \rho_n T(A_{k(n,1/\rho_n)} \times \frac{1}{\rho_n} W_n) \subseteq 2\,W$$

Hence the same is true for the absolutely convex combination $T(x,y)$, i.e. $T(U \times V) \subseteq 2\,W$. □

**3.48 Corollary. Projective versus bornological tensor product for $LB$-spaces.**
*Let $E$ and $F$ be countable inductive limits of Banach spaces (e.g. the duals of metrizable spaces with their bornological topology, i.e. the bornologification of the strong topology). Then $E \otimes_\pi F \cong E \otimes_\beta F$.*

**Proof.** Let $T : E \times F \to G$ be bounded. Since both spaces are bornological $T$ is separately continuous and since both spaces are barreled and $DF$ it is continuous. This is enough to guarantee the equality of the two tensor products by 3.39. □

One can show that the projective tensor product of two $gDF$ (resp. $DF$) spaces is again a $gDF$ (resp. $DF$) space, see [**14**, 15.6.2]. Long proof!

If $E$ and $F$ are two $DF$ spaces which are in addition barreled (quasi-barreled, bornological) then so is $E \otimes_\pi F$, [**14**, 15.6.8].

If $E$ and $F$ are metrizable and barreled, then $E \otimes_\pi F$ is barrelled, see [**14**, 15.6.6].

However the projective tensor product of barreled spaces is not barreled in general, e.g. $\mathbb{R}^{\mathbb{N}} \otimes_\pi \mathbb{R}^{(\mathbb{N})}$ see [**14**, 15.5.2].

The following spaces are preserved by passage to the completion (quasi) barreled spaces [**14**, 11.3.1], $DF$, $gDF$ and *df* spaces [**14**, 12.4.8].

## Spaces of Multi-Linear Mappings

**3.49 Corollary.**
*The following mappings are bounded multi-linear.*

1. $\lim : \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to L(\lim \mathcal{F}, \lim \mathcal{G})$, *where $\mathcal{F}$ and $\mathcal{G}$ are two functors on the same index category, and where $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$ denotes the space of all natural transformations with the structure induced by the embedding into $\prod_i L(\mathcal{F}(i), \mathcal{G}(i))$.*
2. $\mathrm{colim} : \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to L(\mathrm{colim}\, \mathcal{F}, \mathrm{colim}\, \mathcal{G})$.
3. $$\begin{aligned} L : L(E_1, F_1) \times \ldots \times & L(E_n, F_n) \times L(F, E) \to \\ & \to L(L(F_1, \ldots, F_n; F), L(E_1, \ldots, E_n; E)) \\ (T_1, \ldots, T_n, T) & \mapsto (S \mapsto T \circ S \circ (T_1 \times \ldots \times T_n)); \end{aligned}$$
4. $\bigotimes\limits^{n}_{\beta} : L(E_1, F_1) \times \ldots \times L(E_n, F_n) \to L(E_1 \otimes_\beta \cdots \otimes_\beta E_n, F_1 \otimes_\beta \cdots \otimes_\beta F_n)$.
5. $\bigwedge^n : L(E, F) \to L(\bigwedge^n E, \bigwedge^n F)$, *where $\bigwedge^n E$ is the linear subspace of all alternating tensors in $\bigotimes^n_\beta E$. It is the universal solution of*
$$L\Big(\bigwedge^n E, F\Big) \cong L^n_{alt}(E; F).$$
6. $\bigvee^n : L(E, F) \to L(\bigvee^n E, \bigvee^n F)$, *where $\bigvee^n E$ is the linear subspace of all symmetric tensors in $\bigotimes^n_\beta E$. It is the universal solution of*
$$L\Big(\bigvee^n E, F\Big) \cong L^n_{sym}(E; F).$$
7. $\bigotimes_\beta : L(E, F) \to L(\bigotimes_\beta E, \bigotimes_\beta F)$, *where $\bigotimes_\beta E := \bigoplus_{n=0}^\infty \overset{n}{\otimes}_\beta E$ is the tensor algebra of $E$. Note that is has the universal property of prolonging bounded linear mappings with values in locally convex spaces, which are algebras with bounded operations, to continuous algebra homomorphisms:*
$$L(E, F) \cong \mathrm{Alg}(\otimes E, F).$$
8. $\bigwedge : L(E, F) \to L(\bigwedge E, \bigwedge F)$, *where $\bigwedge E := \bigoplus_{n=0}^\infty \bigwedge^n E$ is the exterior algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into graded-commutative algebras, i.e. algebras in the sense above, which are as vector spaces a coproduct $\coprod_{n\in\mathbb{N}} E_n$ and the multiplication maps $E_k \times E_l \to E_{k+l}$ and for $x \in E_k$ and $y \in E_l$ one has $x \cdot y = (-1)^{kl} y \cdot x$.*
9. $\bigvee : L(E, F) \to L(\bigvee E, \bigvee F)$, *where $\bigvee E := \bigoplus_{n=0}^\infty \bigvee^n E$ is the symmetric algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into commutative algebras.*

Recall that for permutations $\pi$ of $n := \{0, \ldots, n-1\}$ we have an associated linear mapping $\pi^* : E^n \to E^n$ and hence a linear mapping $\tilde{\pi^*} : \bigotimes^n E \to \bigotimes^n E$. The exterior product $\bigwedge^n E$ is the space invariant under $\mathrm{sign}(\pi)\, \tilde{\pi^*}$ for all permutations $\pi$ and the symmetric product $\bigvee^n E$ is the space invariant under $\tilde{\pi^*}$ for all permutations $\pi$. The symmetric product is given as the image of the *symmetrizer* $\mathrm{sym} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$ given by
$$(x_1, \ldots, x_n) \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Similarly the wedge product is given as the image of the *alternator*

$$\begin{aligned} \mathrm{alt} : E \otimes_\beta \dots \otimes_\beta E \to E \otimes_\beta \dots \otimes_\beta E \\ \text{given by } (x_1, \dots, x_n) \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\pi)\, (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

**Proof.** All results follow easily by flipping coordinates until a composition of products of evaluation maps remains.

That the spaces in (5), and similar in (6), are universal solutions can be seen from the following diagram:

$$\begin{tikzcd} E \times \dots \times E \arrow[r, "\otimes"] \arrow[dr, "f"'] & E \otimes_\beta \dots \otimes_\beta E \arrow[r, "alt"] \arrow[d, "\tilde f"] & \bigwedge^k E \arrow[dl, "\tilde f|_{\bigwedge^k E}"] \\ & F & \end{tikzcd}$$

$\square$

**3.50 Corollary. Symmetry of higher derivatives.**
*Let $f : E \supset U \to F$ be smooth. The $n$-th derivative $f^{(n)}(x) = d^n f(x)$, considered as an element of $L^n(E; F)$, is symmetric, i.e. has values in the space $L_{sym}(E, \dots, E; F) \cong L(\bigvee^k E; F)$*

**Proof.** Recall that we can form iterated derivatives as follows:

$$\begin{aligned} f : E \supseteq U &\to F \\ df : E \supseteq U &\to L(E, F) \\ d(df) : E \supseteq U &\to L(E, L(E, F)) \cong L(E, E; F) \\ &\vdots \\ d(\dots (d(df)) \dots ) : E \supseteq U &\to L(E, \dots, L(E, F) \dots ) \cong L(E, \dots, E; F) \end{aligned}$$

Thus the iterated derivative $d^n f(x)(v_1, \dots, v_n)$ is given by

$$\tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x + t_1 v_1 + \dots + t_n v_n) = \partial_1 \dots \partial_n \tilde f(0, \dots, 0),$$

where $\tilde f(t_1, \dots, t_n) := f(x + t_1 v_1 + \dots + t_n v_n)$. The result now follows from the finite dimensional property. $\square$

**3.51 Corollary.**
*The following subspaces are direct summands:*

$$\begin{aligned} L(E_1, \dots, E_n; F) &\subseteq C^\infty(E_1 \times \dots \times E_n, F), \\ L^n_{sym}(E; F) &\subseteq L^n(E; F) := L(E, \dots, E; F), \\ L^n_{alt}(E; F) &\subseteq L^n(E; F), \\ L^n_{sym}(E; F) &\to C^\infty(E, F). \end{aligned}$$

Note that direct summand is meant in the bornological category, i.e. the embedding admits a left-inverse in the category of bounded linear mappings, or, equivalently, with respect to the bornological topology it is a topological direct summand.

**Proof.** The projection for $L(E, F) \subset C^\infty(E, F)$ is $f \mapsto df(0)$. The statement on $L^n$ follows by induction using cartesian closedness and 3.34. The projections for the next two subspaces are the symmetrizer and alternator, respectively.

The last embedding is given by $\triangle^*$, which is bounded and linear $C^\infty(E \times \dots \times E, F) \to C^\infty(E, F)$. Here $\Delta : E \to E \times \dots \times E$ denotes the diagonal mapping

$x \mapsto (x, \dots, x)$. A bounded linear left inverse $C^\infty(E,F) \to L^k_{\mathrm{sym}}(E;F)$ is given by $f \mapsto \frac{1}{k!} d^k f(0)$. See the following diagram:

$$\begin{tikzcd} L^k_{sym}(E;F) \arrow[r, hook] \arrow[d, dotted] & L(E,\dots,E;F) \arrow[d, hook] \\ C^\infty(E,F) & C^\infty(E\times\dots\times E;F) \arrow[l, "\Delta^*"] \end{tikzcd} \quad \square$$

**Remark.**
We are now going to discuss polynomials between locally convex spaces. Recall that for finite dimensional spaces $E = \mathbb{R}^n$ a polynomial is just a finite sum

$$\sum_{k\in\mathbb{N}^n} a_k x^k,$$

where $a_k \in F$ and $x^k := \prod_{i=1}^n x_i^{k_i}$. Thus it is just an element in the algebra generated by the coordinate projections $\mathrm{pr}_i$ tensorized with $F$. Since every (continuous) linear functional on $E = \mathbb{R}^n$ is a finite linear combination of coordinate projections, this algebra is also the algebra generated by $E^*$. For a general locally convex space $E$ we define the algebra of *finite type polynomials* to be the one generated by $E^*$.

However there is also another way to define polynomials, namely as those smooth functions for which some derivative is equal to 0. Take for example the square of the norm $\|\_\|^2 : E \to \mathbb{R}$ on some infinite dimensional Hilbert space $E$. Its derivative is given by $x \mapsto (v \mapsto 2\langle x, v\rangle)$ and hence is linear. The second derivative is $x \mapsto ((v,w) \mapsto 2\langle v, w\rangle)$ and hence constant. Thus the third derivative vanishes.
This function is not a finite type polynomial. Otherwise it would be continuous for the weak topology $\sigma(E,E^*)$. Hence the unit ball would be a 0-neighborhood for the weak topology, which is not true, since it is compact for it.
Note that the series $\sum_k x_k^2$ converges pointwise and even uniformly on compact sets. In fact, every compact set is contained in the absolutely convex hull of a 0-sequence $x^n$. In particular $\mu_k := \sup\{|x^n_k| : n \in \mathbb{N}\} \to 0$ for $k \to \infty$ (otherwise we can find an $\varepsilon > 0$ and $k_j \to \infty$ and $n_j \in \mathbb{N}$ with $\|x^{n_j}\|_2 \ge |x^{n_j}_{k_j}| \ge \varepsilon$. Since $x^n \in \ell^2 \subseteq c_0$, we conclude that $n_j \to \infty$, which yields a contradiction to $\|x^n\|_2 \to 0$.) Thus $K \subseteq \langle x^n : n \in \mathbb{N}\rangle_{abs.conv} \subseteq \langle \mu_n e^n\rangle_{abs.conv}$ and hence $\sum_{k\ge n} |x_k| \le \max\{\mu_k : k \ge n\}$ for all $x \in K$.
The series does not converge uniformly on bounded sets. To see this choose $x = e_k$.

## 3.52 Definition

A smooth mapping $f : E \to F$ is called a *polynomial* if some derivative $d^n f$ vanishes on $E$. The largest $p$ such that $d^p f \neq 0$ is called the degree of the polynomial. The mapping $f$ is called a *monomial of degree $p$* if it is of the form $f(x) = \tilde f(x,\dots,x)$ for some $\tilde f \in L^p_{\mathrm{sym}}(E;F)$.

**3.53 Lemma. Polynomials versus monomials.**

1. *The smooth $p$-homogeneous maps are exactly the monomials of degree $p$.*
2. *The symmetric multi-linear mapping representing a monomial is unique.*
3. *A smooth mapping is a polynomial of degree $\le p$ if and only if its restriction to each one dimensional subspace is a polynomial of degree $\le p$.*
4. *The polynomials are exactly the finite sums of monomials.*

**Proof.** (1) Every monomial of degree $p$ is clearly smooth and $p$-homogeneous. If $f$ is smooth and $p$-homogeneous, then

$$(d^p f)(0)(x,\dots,x) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} f(tx) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} t^p f(x) = p! f(x).$$

(2) The symmetric multi-linear mapping $g \in L^p_{\mathrm{sym}}(E;F)$ representing $f$ is uniquely determined, since we have $(d^p f)(0)(x_1,\dots,x_p) = p! g(x_1,\dots,x_p)$.

(3+4) Let the restriction of $f$ to each one dimensional subspace be a polynomial of degree $\le p$, i.e. for each $x \in E$ and $\ell \in F'$ we have $\ell(f(tx)) = \sum_{k=0}^p \frac{t^k}{k!} (\frac{\partial}{\partial t})^i\big|_{t=0} \ell(f(tx))$. So $f(x) = \sum_{k=0}^p \frac{1}{k!} d^k f(0.x)(x,\dots,x)$ and hence is a finite sum of monomials.
For the derivatives of a monomial $q$ of degree $k$ we have $q^{(j)}(tx)(v_1,\dots,v_j) = k(k-1)\cdot(k-j+1)t^{k-j}\tilde{q}(x,\dots,x,v_1,\dots,v_j)$. Hence any such finite sum is a polynomial in the sense of 3.52.
Finally any such polynomial has a polynomial as trace on each one dimensional subspace. □

**3.54 Lemma. Spaces of polynomials.**
*The space $Poly^p(E,F)$ of polynomials of degree $\le p$ is isomorphic to $\sum_{k\le p} L(\bigvee^k E;F)$ and is a direct summand in $C^\infty(E,F)$ with a complement given by the smooth functions which are p-flat at 0.*

**Proof.** We have already shown that $L(\bigvee^k E;F)$ embeds into $C^\infty(E,F)$ as a direct summand, where a retraction is given by the derivative of order $k$ at 0. Furthermore we have shown that the polynomials of degree $\le p$ are exactly the direct sums of homogeneous terms in $L(\bigvee^k E;F)$. A retraction to the inclusion $\bigoplus_{k\le p} L(\bigvee^k E;F) \to C^\infty(E,F)$ is hence given by $\bigoplus_{k\le p} \frac{1}{k!} d^k|_0$. □

**Remark.**
The corresponding statement is false for infinitely flat functions. I.e. the sequence $E \to C^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R}^\mathbb{N}$ does not split, where $E$ denotes the space of smooth functions which are *infinitely flat at 0*. Otherwise $\mathbb{R}^\mathbb{N}$ would be a subspace of $C^\infty(I,\mathbb{R})$ (compose the section with the restriction map from $C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(I,\mathbb{R})$) and hence would have a continuous norm. This is however easily seen to be not the case.

**3.55 Theorem. Taylor formula.**
*Let $f : U \to F$ be smooth, where $U$ is $c^\infty$-open in $E$. Then for each segment $[x, x+y] = \{x+ty : 0 \le t \le 1\} \subset U$ we have*

$$f(x+y) = \sum_{k=0}^n \frac{1}{k!} d^k f(x) y^k + \int_0^1 \frac{(1-t)^n}{n!} d^{n+1} f(x+ty) y^{n+1} dt,$$

*where $y^k = (y,\dots,y) \in E^k$.*

**Proof.** This is an assertion on the smooth curve $t \mapsto f(x+ty)$. Using functionals we can reduce it to the scalar valued case, or we proceed directly by induction on $n$: The first step is (6) in 2.15, and the induction step is partial integration of the remainder integral. □

**3.56 Theorem. Uniform boundedness principle.**
*If all $E_i$ are convenient vector spaces and if $F$ is a locally convex space, then the bornology on the space $L(E_1,\dots,E_n;F)$ consists of all pointwise bounded sets.*

*So a mapping into $L(E_1,\dots,E_n;F)$ is smooth if and only if all composites with evaluations at points in $E_1 \times \ldots \times E_n$ are smooth.*

**Proof.** Let us first consider the case $n = 1$. So let $\mathcal{B} \subset L(E, F)$ be a pointwise bounded subset. By lemma 3.35 we have to show that it is uniformly bounded on each bounded subset $B$ of $E$. We may assume that $B$ is closed absolutely convex and thus $E_B$ is a Banach space, since $E$ is convenient. By the classical Uniform Boundedness Principle, see [**2**, 5.2.2], the set $\{f|_{E_B} : f \in \mathcal{B}\}$ is bounded in $L(E_B, F)$ and thus $\mathcal{B}$ is bounded on $B$.

The smoothness detection principle: Clearly it suffices to recognize smooth curves. If $c : \mathbb{R} \to L(E, F)$ is such that $\mathrm{ev}_x \circ c : \mathbb{R} \to F$ is smooth for all $x \in E$, then clearly $\mathbb{R} \xrightarrow{c} L(E, F) \xrightarrow{j} \prod_E F$ is smooth. We will show that $(j \circ c)'$ has values in $L(E, F) \subset \prod_E F$. Clearly $(j \circ c)'(s)$ is linear $E \to F$. The family of mappings $\frac{c(s+t)-c(s)}{t} : E \to F$ is pointwise bounded for $s$ fixed and $t$ in a compact interval, so by the first part it is uniformly bounded on bounded subsets of $E$. It converges pointwise to $(j \circ c)'(s)$, so this is also a bounded linear mapping $E \to F$. By the first part $j : L(E, F) \to \prod_E F$ is a bornological embedding, so $c$ is differentiable into $L(E, F)$. Smoothness follows now by induction on the order of the derivative.

The multi-linear case follows from the exponential law 3.34 using induction on $n$. $\square$

**3.57 Theorem. Multi-linear mappings on convenient vector spaces.** *A multi-linear mapping from convenient vector spaces to a locally convex space is bounded if and only if it is separately bounded.*

**Proof.** Let $f : E_1 \times \ldots \times E_n \to F$ be $n$-linear and separately bounded, i.e. $x_i \mapsto f(x_1, \ldots, x_n)$ is bounded for each $i$ and fixed $x_j$ for all $j \neq i$. Then $\check{f} : E_1 \times \ldots \times E_{n-1} \to L(E_n, F)$ is $(n-1)$-linear. By 3.56 the bornology on $L(E_n, F)$ consists of the pointwise bounded sets, so $\check{f}$ is separately bounded. By induction on $n$ it is bounded. The bornology on $L(E_n, F)$ consists also of the subsets which are uniformly bounded on bounded sets by lemma 3.35, so $f$ is bounded. $\square$

We will now give an infinite dimensional version of 2.39, which gives as minimal requirements for a mapping to be smooth.

**3.58 Theorem.** *Let $E$ be a convenient vector space. An arbitrary mapping $f : E \supset U \to F$ is smooth if and only if all unidirectional iterated derivatives $d_v^p f(x) = (\frac{\partial}{\partial t})^p|_0 f(x+tv)$ exist, $x \mapsto d_v^p f(x)$ is bounded on sequences which are Mackey converging in $U$, and $v \mapsto d_v^p f(x)$ is bounded on sequences which are fast falling to 0.*

**Proof.** A smooth mapping obviously satisfies this requirement. Conversely from 2.39 we see that $f$ is smooth restricted to each finite dimensional subspace and the iterated directional derivatives $d_{v_1} \ldots d_{v_n} f(x)$ exist and are bounded multi-linear mappings in $v_1, \ldots, v_n$ by 3.3o, since they are universal linear combinations of the unidirectional iterated derivatives $d_v^p f(x)$, compare with the proof of 2.39. So $d^n f : U \to L^n(E; F)$ is bounded on Mackey converging sequences with respect to the pointwise bornology on $L^n(E; F)$. By the uniform boundedness principle 3.56 together with lemma 2.2o the mapping $d^n f : U \times E^n \to F$ is bounded on sets which are contained in a product of a BORNOLOGICALLY COMPACT SET in $U$ - i.e. a set in $U$ which is contained and compact in some $E_B$ - and a bounded set in $E^n$.

Now let $c : \mathbb{R} \to U$ be a smooth curve. We have to show that $\frac{f(c(t))-f(c(0))}{t}$ converges to $f'(c(0))(c'(0))$. It suffices to check that

$$\frac{1}{t}\left(\frac{f(c(t)) - f(c(0))}{t} - f'(c(0))(c'(0))\right)$$

is locally bounded with respect to $t$. Integrating along the segment from $c(0)$ to $c(t)$ we see that this expression equals

$$\begin{aligned}
&\frac{1}{t}\int_0^1 \Big(f'\Big(c(0)+s(c(t)-c(0))\Big)\left(\frac{c(t)-c(0)}{t}\right) - f'(c(0))(c'(0))\Big)\,ds \\
&= \int_0^1 f'\big(c(0)+s(c(t)-c(0))\big)\left(\frac{\frac{c(t)-c(0)}{t}-c'(0)}{t}\right)ds \\
&\quad + \int_0^1\int_0^1 f''\big(c(0)+rs(c(t)-c(0))\big)\left(s\frac{c(t)-c(0)}{t}, c'(0)\right)dr\,ds
\end{aligned}$$

The first integral is bounded since $df : U \times E \to F$ is bounded on the product of the bornologically compact set $\{c(0)+s(c(t)-c(0)) : 0 \le s \le 1, t \text{ near } 0\}$ in $U$ and the bounded set $\{\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right) : t \text{ near } 0\}$ in $E$ (use 2.5).

The second integral is bounded since $d^2 f : U \times E^2 \to F$ is bounded on the product of the bornologically compact set $\{c(0)+rs(c(t)-c(0)) : 0 \le r, s \le 1, t \text{ near } 0\}$ in $U$ and the bounded set $\{\left(s\frac{c(t)-c(0)}{t}, c'(0)\right) : 0s \le 1, t \text{ near } 0\}$ in $E^2$.

So $f \circ c$ is differentiable in $F$ with derivative $df \circ (c, c')$. Now $df : U \times E \to F$ satisfies again the assumptions of the theorem, so we may iterate. $\square$

# 4. Tensor Products and Function Spaces

## Desired Isomorphisms

**4.1.** Suppose we are given some category $\underline{X}$ and a *forgetful functor* $V : \underline{LCS}_b \to \underline{X}$, where the index $b$ indicates that we consider the bounded linear morphisms. Then for an object $X$ in $\underline{X}$ and a locally convex space $G$ we can consider the space of morphisms $\underline{X}(X, V(G))$ and we assume that this lifts to a functor $\mathcal{F} : \underline{X}^{op} \times \underline{LCS}_b \to \underline{LCS}_b$. Examples of that situation are

1. $\underline{X} := \underline{LCS}_b$, $V$ the identity and $\mathcal{F}$ the *internal hom-functor* $L$.
2. $\underline{X}$ the category of mappings between sets and $\mathcal{F}(X, G) := \prod_X G = G^X$ the space of all mappings with the topology of pointwise convergence.
3. $\underline{X}$ the category of continuous maps between topological spaces and $\mathcal{F}(X, G)$ the space $C(X, G)$ of continuous mappings with the topology of uniform convergence on compact subsets. Here we have to restrict to continuous linear mappings to get a forgetful functor. Note that (2) is a particular case, where the topology on $X$ is discrete.
4. $\underline{X}$ the bounded (better bornological) mappings between bornological spaces and $F(X, G) := \ell^\infty(X, G)$ the space of bornological mappings with the topology of uniform convergence on bounded sets. Note that (2) is a particular case, where the bounded sets are exactly the finite ones.
5. $\underline{X}$ the category of smooth mappings defined on $c^\infty$-open subsets of locally convex spaces $G$, and $\mathcal{F}(X, G) := C^\infty(X, G)$ supplied with the locally convex topology described before.

Not completely fitting into this scheme but nevertheless interesting might be the following function spaces:

6. For sets $X$ the space $\mathcal{F}(X, G) := G^{(X)}$ of all functions with finite support with the final topology induced by the subspaces $G^A$, where $A$ runs through the finite subsets.
7. For topological spaces $X$ the space $\mathcal{F}(X, G) := C_c(X, G)$ of all continuous functions with compact support with the final topology induced by the subspaces $C_K(X, G)$ formed by the continuous functions having support in $K$, where $K$ runs through all compact subsets of $X$ and where $C_K(X, G)$ carries the initial topology induced by the inclusion into $C(X, G)$. Note that (6) is a particular case, where the topology on $X$ is discrete.
8. For a finite dimensional manifold $X$ the space $\mathcal{F}(X, G) := C_c^\infty(X, G)$ of all smooth functions with compact support with the final topology induced by the subspaces $C_K^\infty(X, G)$ formed by the continuous functions having support in $K$, where $K$ runs through all compact subsets of $X$ and where $C_K^\infty(X, G)$ carries the initial topology induced by the inclusion into $C^\infty(X, G)$. Note that (6) is again a particular case, where the manifold is discrete.

Some desirable isomorphisms would then be the following, where we write $\mathcal{F}(X)$ as shortcut for $\mathcal{F}(X,\mathbb{R})$ and $\otimes_?$ denotes some appropriate tensor product.

$$\begin{aligned} \mathcal{F}(X,\mathcal{F}(Y,G)) &\cong \mathcal{F}(X\times Y,G) && \textit{exponential law}\\ \mathcal{F}(X)\otimes_? G &\cong \mathcal{F}(X,G) && \textit{vector valued versus scalar valued}\\ \mathcal{F}(X)\otimes_? \mathcal{F}(Y) &\cong \mathcal{F}(X\times Y) && \textit{compatibility with products} \end{aligned}$$

Note that (E) and (V) imply (P):

$$\mathcal{F}(X)\otimes_? \mathcal{F}(Y) \overset{\text{(V)}}{\cong} \mathcal{F}(X,\mathcal{F}(Y)) \overset{\text{(E)}}{\cong} \mathcal{F}(X\times Y)$$

In the particular case (1), where the forgetful functor $V$ forgets nothing, i.e. $\underline{X} = \underline{LCS}_b$, we would expect:

$$\begin{aligned} L(E,L(F,G)) &\cong L(E,F;G) \cong L(E\otimes_\beta F,G)\\ E'\otimes_? G &\cong L(E,G)\\ E'\otimes_? F' &\cong L(E,F') \cong (E\otimes_\beta F)' \end{aligned}$$

Applying (P) for $L$ to (V) and (P) for $\mathcal{F}$ we would obtain the dualized versions:

$$\begin{aligned} \mathcal{F}(X)'\otimes_? G' &\overset{(P_L)}{\cong} (\mathcal{F}(X)\otimes_\beta G)' \overset{(V)}{\cong} \mathcal{F}(X,G)'\\ \mathcal{F}(X)'\otimes_? \mathcal{F}(Y)' &\overset{(P_L)}{\cong} (\mathcal{F}(X)\otimes_\beta \mathcal{F}(Y))' \overset{(P)}{\cong} \mathcal{F}(X\times Y)' \end{aligned}$$

Note again that (E) and (V') imply (P'):

$$\mathcal{F}(X)'\otimes_? \mathcal{F}(Y)' \overset{\text{(V')}}{\cong} \mathcal{F}(X,\mathcal{F}(Y))' \overset{\text{(E)}}{\cong} \mathcal{F}(X\times Y)'$$

## 4.2. Exponential law

Lets us first determine in which situations we have the exponential law (E).

1. For $L$ we have proved in 3.34 that (E) is true.
2. For sets $X$ and $Y$ we obviously have $(G^Y)^X \cong G^{X\times Y}$.
3. For $C$ we have shown in 2.3 that (E) is a bijection if $Y$ is locally compact. That it is also true for the structure follows immediately since the 0-neighborhood $N_{A,N_{B,V}}$ corresponds to $N_{A\times B,V}$, where $A\subseteq X$ and $B\subseteq Y$ are compact and $V\subseteq G$ a 0-neighborhood.
4. That (E) is a bijection for $\ell^\infty$ is obvious, cf. 3.34. That it is also true for the structure follows the same way as in (3), where $A$ and $B$ are bounded instead.
5. That (E) is true for $C^\infty$ has been shown in 2.47 and 2.48.
6. It is obvious that $(G^{(Y)})^{(X)} \cong G^{(X\times Y)}$ is true.
7. For $C_c$ we use that $C_c(X,G)$ is the strict inductive limit of the spaces $C_K(X,G)$, where $K\subset X$ is compact. Obviously the closed subspace $C_{A\times B}(X\times Y,G) = \{f : f(x,y) = 0 \text{ if } x\notin A \text{ or } y\notin B\}$ of $C(X\times Y,G)$ corresponds to

the closed subspace $C_A(X, C_B(Y,G)) = \{f^\vee \in C(X, C_B(Y,G)) : f^\vee(x) = 0 \text{ if } x \notin A\}$ of $C(X, C(Y,G))$ and hence we have a natural injection

$$\begin{tikzcd} C_c(X\times Y,G) \arrow[r, equal] & \bigcup_{A,B} C_{A\times B}(X\times Y,G) \arrow[r] \arrow[d, "\cong"'] & \bigcup_A C_A(X, C_c(Y,G)) \arrow[d, equal] \\ & \bigcup_{A,B} C_A(X, C_B(Y,G)) \arrow[r] & C_c(X, C_c(Y,G)) \end{tikzcd}$$

Conversely let $\mathcal{B} \subseteq C_A(X, C_c(Y,G))$ be bounded. Then $\mathcal{B}(A)$ is bounded in $C_c(Y,G)$. Now suppose that $Y$ is in addition $\sigma$-compact. Then $C_c(Y,G)$ is the strict inductive limit of a sequence of spaces $C_B(Y,G)$ and hence $\mathcal{B}(A)$ has to be bounded in some step $C_B(Y,G)$ by [**2**, 4.8.1]. So $\mathcal{B}$ corresponds to a bounded subset of $C_A(X, C_B(Y,G))$. So these correspondences induce the required bornological isomorphism and hence (E) holds for $C_c$ and $\sigma$-compact $Y$.

8. For $C_c^\infty$ we can proceed completely analogously to (7) to obtain (E) for $C^\infty$ and finite dimensional smooth manifolds.

Now let us come to the other desired isomorphisms. One could ask, whether we could deduce the case of a general $\mathcal{F}$ from that of $L$. For this we need:

## Universal Linearization

**4.3.** Suppose we can solve the universal problem of linearizing maps in $\mathcal{F}(X,G)$, i.e. find a $c^\infty$-complete locally convex space $\lambda(X)$, also called a *free convenient vector space*, and a map $\iota : X \to \lambda(X)$ which induces an isomorphism

$$\mathcal{F}(X,G) \cong L(\lambda(X), G) \quad \text{the forgetful functor is right adjoint}$$

for all $c^\infty$-complete locally convex spaces $G$, and hence in particular an isomorphism $\mathcal{F}(X) \cong \lambda(X)'$. A consequence of (E) for $L$ and $\mathcal{F}$ is that

$$\lambda(X\times Y) \cong \lambda X \tilde{\otimes}_\beta \lambda Y$$

This follows, since

$$\begin{aligned} L(\lambda(X)\tilde{\otimes}_\beta\lambda(Y), G) &\overset{(E_L)}{\cong} L(\lambda(X), L(\lambda(Y),G)) \overset{\text{(F)}}{\cong} \mathcal{F}(X, L(\lambda(Y),G)) \\ &\overset{\text{(F)}}{\cong} \mathcal{F}(X, \mathcal{F}(Y,G)) \overset{(E_{\mathcal{F}})}{\cong} \mathcal{F}(X\times Y, G) \end{aligned}$$

shows that $\lambda(X)\tilde{\otimes}_\beta\lambda(Y)$ has the universal property of $\lambda(X\times Y)$. Using all this we can translate the general case to that for $L$:

$$\begin{aligned} \mathcal{F}(X, \mathcal{F}(Y,G)) &\overset{\text{(F)}}{\cong} L(\lambda X, L(\lambda Y, G)) \overset{(E_L)}{\cong} \\ &\cong L(\lambda X \otimes_\beta \lambda Y, G) \overset{(P_\lambda)}{\cong} L(\lambda(X\times Y), G) \overset{\text{(F)}}{\cong} \mathcal{F}(X\times Y, G) \\ \mathcal{F}(X)\otimes G &\overset{\text{(F)}}{\cong} \lambda(X)'\otimes G \overset{(V_L)}{\cong} L(\lambda(X), G) \overset{\text{(F)}}{\cong} \mathcal{F}(X,G) \\ \mathcal{F}(X)\otimes\mathcal{F}(Y) &\overset{\text{(F)}}{\cong} \lambda(X)'\otimes\lambda(Y)' \overset{(P_L)}{\cong} \\ &\cong (\lambda(X)\otimes_\beta\lambda(Y))' \overset{(P_\lambda)}{\cong} (\lambda(X\times Y))' \overset{\text{(F)}}{\cong} \mathcal{F}(X\times Y). \end{aligned}$$

Let us try to construct $\lambda(X)$. Since $\mathcal{F}(X) = \mathcal{F}(X,\mathbb{R}) \cong \lambda(X)'$ we have a candidate for the dual of $\lambda(X)$, and hence $\lambda(X)$ should be a subspace of $\lambda(X)'' \cong \mathcal{F}(X)'$.

Obviously we have a mapping $\iota : X \to \mathcal{F}(X)'$ given by $x \mapsto \mathrm{ev}_x$. So our first problem is to show that $\iota$ belongs to $\mathcal{F}$. Recall that for $\mathcal{F} = C^\infty$ and $c^\infty$-complete locally convex spaces we have the following *uniform boundedness principle* 3.56:

$$f : X \to L(E, F) \text{ is } \mathcal{F} \iff \mathrm{ev}_x \circ f : X \to F \text{ is } \mathcal{F} \text{ for all } x \in E$$

So let us assume that (U1) is satisfied for the $\mathcal{F}$ under consideration. From the commuting triangle

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{F}(X)' \arrow[d, "\mathrm{ev}_f"] \\
 & \mathbb{R}
\end{tikzcd}$$

we conclude using (U1) for $L(\mathcal{F}(X), \mathbb{R}) = \mathcal{F}(X)'$ that $\iota$ belongs to $\mathcal{F}$. In order to obtain the universal property (F) for scalar valued functions we only have to restrict $\mathrm{ev}_f$ to the subspace $\lambda(X)$ which is given by the $c^\infty$-closure of the vector space generated by the image $\{\mathrm{ev}_x : x \in X\}$ of $\iota$.

Now to the general case of $G$-valued functions, where $G$ is at least $c^\infty$-complete. Since $\iota$ belongs to $\mathcal{F}$ we have that $\iota^* : L(\lambda(X), G) \to \mathcal{F}(X, G)$ is well defined and injective since the linear subspace generated by the image of $\iota$ is $c^\infty$-dense in $\lambda(X)$ by construction. To show surjectivity consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dd, "f"'] & \lambda(X) \arrow[r, hook] \arrow[d, dotted, "(2)"] \arrow[ddl, dotted, "\tilde{f}"'] & \mathcal{F}(X)' \arrow[dd, "\mathrm{ev}_{\lambda\circ T}"] \arrow[dl, dotted, "(1)"] \\
 & (3)\prod_{G'} \mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, hook, "\delta"'] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

Note that (2) has values in $\delta(G)$, since this is true on the $\mathrm{ev}_x$, which generate by definition a $c^\infty$-dense subspace of $\lambda(X)$. Note that this construction of $\tilde{f}$ works for every $f : X \to G$ which is scalarly in $\mathcal{F}$.

Remains to show that this bijection is a bornological isomorphism. In order to show that the linear mapping $\mathcal{F}(X, G) \to L(\lambda(X), G)$ is bounded we can reformulate this equivalently using (E) for $L$, the universal property of $\lambda(X)$ and (U1) as follows:

$$\begin{aligned}
&\mathcal{F}(X, G) \to L(\lambda(X), G) \text{ is } L \\
\overset{(E_L)}{\iff}\ &\lambda(X) \to L(\mathcal{F}(X, G), G) \text{ is } L \\
\overset{\text{(F)}}{\iff}\ &X \to L(\mathcal{F}(X, G), G) \text{ is } \mathcal{F} \\
\overset{\text{(U1)}}{\iff}\ &X \to L(\mathcal{F}(X, G), G) \overset{\mathrm{ev}_f}{\to} G \text{ is}
\end{aligned}$$

and since the later map is $f$ we are done. Another way to see this would be to show that $L(E, F) \subseteq \mathcal{F}(E, F)$ is initial even for $\mathcal{F}$-morphisms and then apply (E) for $\mathcal{F}$ to translate the map $X \to L(\mathcal{F}(X, G), G) \subseteq \mathcal{F}(\mathcal{F}(X, G), G)$ into the identity on $\mathcal{F}(X, G)$, which is a $\mathcal{F}$-map.

Conversely we have to show that $L(\lambda(X), G) \to \mathcal{F}(X, G)$ belongs to $L$. Composed with $\mathrm{ev}_x : F(X, G) \to G$ this yields the bounded linear map $\mathrm{ev}_{\delta(x)} : L(\lambda(X), G) \to G$. Thus we need the following kind of *uniform boundedness principle* for the function space $\mathcal{F}(X, G)$:

$$T : E \to \mathcal{F}(X, G) \text{ is } L \iff \mathrm{ev}_x \circ T : E \to G \text{ is } L \text{ for all } x \in X$$

## A Uniform Boundedness Principle

**4.4. Lemma. Uniform $S$-boundedness principle.**
*Let $E$ be a locally convex space and let $\mathcal{S}$ be a point separating set of bounded linear mappings with common domain $E$. Then the following conditions are equivalent.*

1. *If $F$ is a Banach space (or even a $c^\infty$-complete lcs) and $f : F \to E$ is linear and $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$, then $f$ is bounded.*
2. *If $B \subseteq E$ is absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space $E_B$ generated by $B$ is complete, then $B$ is bounded in $E$.*
3. *Let $(b_n)$ be an unbounded sequence in $E$ with $\lambda(b_n)$ bounded for all $\lambda \in \mathcal{S}$, then there is some $(t_n) \in \ell^1$ such that $\sum t_n\, b_n$ does not converge in $E$ for the initial locally convex topology induced by $\mathcal{S}$.*

**Definition.** We say that $E$ satisfies the *uniform $\mathcal{S}$-boundedness principle* if these equivalent conditions are satisfied.

**Proof.** (1) $\Rightarrow$ (3) : Suppose that (3) is not satisfied. So let $(b_n)$ be an unbounded sequence in $E$ such that $\lambda(b_n)$ is bounded for all $\lambda \in \mathcal{S}$, and such that for all $(t_n) \in \ell^1$ the series $\sum t_n\, b_n$ converges in $E$ for the initial locally convex topology induced by $\mathcal{S}$. We define a linear mapping $f : \ell^1 \to E$ by $f((t_n)_n) = \sum t_n\, b_n$, i.e. $f(e_n) = b_n$. It is easily checked that $\lambda \circ f$ is bounded, hence by (1) the image of the closed unit ball, which contains all $b_n$, is bounded. Contradiction.

(3) $\Rightarrow$ (2): Let $B \subseteq E$ be absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and that the normed space $E_B$ generated by $B$ is complete, and suppose that $B$ is unbounded. Then $B$ contains an unbounded sequence $(b_n)$, so by (3) there is some $(t_n) \in \ell^1$ such that $\sum t_n\, b_n$ does not converge in $E$ for the weak topology induced by $\mathcal{S}$. But $\sum t_n\, b_n$ is a Cauchy sequence in $E_B$, since $\sum_{k=n}^m t_n b_n \in (\sum_{k=n}^m |t_n|) \cdot B$, and thus converges even bornologically, a contradiction.

(2) $\Rightarrow$ (1): Let $F$ be convenient, and let $f : F \to E$ be linear such that $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$. It suffices to show that $f(B)$, the image of an absolutely convex bounded set $B$ in $F$ with $F_B$ complete, is bounded. By assumption $\lambda(f(B))$ is bounded for all $\lambda \in \mathcal{S}$, the normed space $E_{f(B)}$ is a quotient of the Banach space $F_B$, hence complete. By (2) the set $f(B)$ is bounded. □

**4.5. Theorem. Webbed spaces have the uniform boundedness property.**
*A locally convex space which is webbed satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear functionals.*

**Proof.** Since the bornologification of a webbed space is webbed, cf. [**14**, 13.3.3 and 13.3.1], we may assume that $E$ is bornological, and hence that every bounded linear functional is continuous, cf. [**14**, 13.3.1]. Now the closed graph principle, cf. [**2**, 5.3.1] applies to any mapping satisfying the assumptions of 1 in 4.4. □

**4.6. Lemma. Stability of the uniform boundedness principle.**
*Let $\mathcal{F}$ be a set of bounded linear mappings $f : E \to E_f$ between locally convex spaces, let $\mathcal{S}_f$ be a point separating set of bounded linear mappings on $E_f$ for every $f \in \mathcal{F}$, and let $\mathcal{S} := \bigcup_{f \in \mathcal{F}} f^*(\mathcal{S}_f) = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{S}_f\}$. If $\mathcal{F}$ generates the bornology and $E_f$ satisfies the uniform $\mathcal{S}_f$-boundedness principle for all $f \in \mathcal{F}$, then $E$ satisfies the uniform $\mathcal{S}$-boundedness principle.*

**Proof.** We check the condition (1) of 4.4. So assume $h : F \to E$ is a linear mapping for which $g \circ f \circ h$ is bounded for all $f \in \mathcal{F}$ and $g \in \mathcal{S}_f$. Then $f \circ h$ is bounded by the uniform $\mathcal{S}_f$- boundedness principle for $E_f$. Consequently $h$ is bounded since $\mathcal{F}$ generates the bornology of $E$. $\square$

Note that the uniform boundedness principles (U1) and (U2) have the following bornological isomorphism as consequence:

$$L(E,\mathcal{F}(X,G)) \cong \mathcal{F}(X,L(E,G)) \quad \textit{flip of variables.}$$

In fact the mapping and its inverse are given by exchanging the coordinates, $f \mapsto \tilde{f} : (x \mapsto (y \mapsto f(y)(x)))$. For $f \in L(E,\mathcal{F}(X,G))$ we have $\tilde{f}(x) = \mathrm{ev}_x \circ f \in L(E,G)$, since $\mathrm{ev}_x : \mathcal{F}(X,G) \to G$ is bounded. Furthermore $\tilde{f} \in \mathcal{F}(X,L(E,G))$ since $\mathrm{ev}_e \circ \tilde{f} = f(e) \in \mathcal{F}(X,G)$ for all $e \in E$, using the uniform boundedness principle (U1). Conversely for $f \in \mathcal{F}(X,L(E,G))$ we have $\tilde{f}(e) = \mathrm{ev}_e \circ f \in \mathcal{F}(X,G)$, since $\mathrm{ev}_e : L(E,G) \to G$ is bounded and hence in $\mathcal{F}$. Furthermore $\tilde{f} \in L(E,\mathcal{F}(X,G))$ since $\mathrm{ev}_x \circ \tilde{f} = f(x) \in L(E,G)$, using the uniform boundedness principle (U2).

The bijection is bounded in both directions, since this can be tested by applying the uniform boundedness principles (U1) and (U2) and the equation $\mathrm{ev}_x \circ (\_)\tilde{} = (\mathrm{ev}_x)_*$.

On the other hand this isomorphism translates the two uniform boundedness principles into each other: For example $f \in L(E,\mathcal{F}(X,G))$ iff $\tilde{f} \in \mathcal{F}(X,L(E,G))$ and hence by (U1) iff $\tilde{f}(x) \in L(E,G)$ and $f(x) \in F(X,G)$, which are both satisfied by assumption.

Let us now discuss the situations where we have free convenient vector spaces $\lambda X$, or the two related uniform boundedness principles.

**4.7 Examples of free convenient vector spaces and the uniform boundedness principles.**

1. *For $L$ the uniform boundedness principles* (U1) *and* (U2) *are just a direct corollary of usual uniform boundedness principle, and of course $\lambda(X) = X$. The flipping isomorphism* (U3) *is $L(E,L(F,G)) \cong L(F,L(E,G))$.*
2. *The dual of $\mathbb{R}^X$ is $\mathbb{R}^{(X)}$ provided the cardinality of $X$ is non-measurable. The evaluations $\mathrm{ev}_x$ correspond to the unit vectors $e_x \in \mathbb{R}^{(X)}$, hence $\lambda(X) = \mathbb{R}^{(X)} = (\mathbb{R}^X)'$. The uniform boundedness principle* (U2) *is just the universal property of the product and* (U1) *is trivial. The flipping isomorphism* (U3) : *$L(E,G)^X \cong L(E,G^X)$ is a particular case of the continuity of $L(E,\_)$.*
3. *For $C$ there exists no $\lambda(X)$, a candidate for $\lambda(X)$ with locally compact $X$ would be the space of Borel-measures on $X$ being the dual of $C(X)$, however the uniform boundedness principle* (U1) *fails: Take $X = \mathbb{N}_\infty$, $E = \ell^2$, $G = \mathbb{R}$ and $f : \mathbb{N}_\infty \to E$ defined by $f : n \mapsto e_n$ and $f(\infty) = 0$. Then $f$ is weakly continuous, but not continuous. Note however that the forgetful functor preserves limits hence is a candidate for a right adjoint. By* [**2**, 7.5.2] *neither $c_0$ nor $L^1$ is a dual of a normed space, hence there exists no Banach space $\lambda(\mathbb{N}_\infty)$ with $C(\mathbb{N}_\infty,\mathbb{R}) = \lambda(\mathbb{N}_\infty)'$. But since $\lambda(\mathbb{N}_\infty)$ is a subspace of $C(\mathbb{N}_\infty,\mathbb{R})'$ it has to be normable. However* (U2) *is valid, since it follows from the fact that $C(K,\mathbb{R})$ is a Banach space via 4.6.*
4. *For $\ell^\infty$ we have that $\lambda(X) := \ell^1(X) \subseteq \ell^\infty(X)'$. Recall that $\ell^1(X)$ is by 2.33 equal to the inductive limit of $\ell^1(B)$ over all bounded $B \subseteq X$ and it is not difficult to show that the $c^\infty$-closure of the evaluations in $\ell^\infty(B)'$ is just $\ell^1(B)$. The boundedness principle* (U1) *is true, since the $\mathrm{ev}_x$ detect bounded sets. And* (U2) *is true, since $\ell^\infty(B,\mathbb{R})$ is a Banach space. The flipping isomorphism* (U3) *is $\ell^\infty(X,L(E,G)) \cong L(E,\ell^\infty(X,G))$.*
5. *For $C^\infty$ we have $\lambda(X)$. And it can be shown that $\lambda(X)$ equals the distributions with compact support if $X$ is a finite dimensional smooth manifold. No*

*counterexample for* $\lambda(X) = C^\infty(X,\mathbb{R})'$ *is known for infinite dimensional spaces* $X$*. We already proved that* (U1) *and* (U2) *is true, since* $C^\infty(\mathbb{R},\mathbb{R})$ *is a Fréchet space. The flipping isomorphism* (U3) *is* $C^\infty(X, L(E,F)) \cong L(E, C^\infty(X,F))$.

6. *For* $G^{(X)}$ *we cannot apply the discussion above directly, since we have no forgetful functor in this situation. Here a candidate for* $\lambda(X)$ *would be* $\mathbb{R}^X$*, but the continuous linear functions* $\mathbb{R}^X \to G$ *have finite support only for spaces* $G$ *admitting a continuous norm. We have no flipping isomorphism* (U3)*, since for* $X = \mathbb{N}$, $E = \mathbb{R}^{(\mathbb{N})}$ *and* $G = \mathbb{R}$ *we have* $L(E,G)^{(X)} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$ *but* $L(E, G^{(X)}) \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$*. However the uniform boundedness principle* (U2) *is true: In fact take a absolutely convex subset* $B \subseteq \coprod_i E_i$*, which is bounded in* $\prod_i E_i$ *and such that* $(\coprod_i E_i)_B$ *is complete. We claim that* $B$ *is contained in some finite subproduct. Otherwise there would be a countable subset* $\mathbb{N}$ *of the index set and* $b^n \in B$ *with* $b^n(n) \neq 0$ *for all* $n \in \mathbb{N}$*. Choose furthermore* $\lambda_n \in E_n'$ *with* $\lambda_n(b^n(n)) = 1$*. Let* $p : \coprod_i E_i \to \mathbb{R}^{(\mathbb{N})}$ *be given by* $p((x_i)_{i\in I}) := (\lambda_n(x_n)_{n\in\mathbb{N}})$*. Then* $p(B) \subseteq \mathbb{R}^{(\mathbb{N})}$ *satisfies the same assumptions as* $B$*. But* $\coprod_{i\in\mathbb{N}} \mathbb{R}$ *is the strict inductive limit of the finite subproducts, hence is webbed and we may apply the closed graph theorem.*
7. *Since* $C_c$ *is a generalization of the previous item, we have no* $\lambda$ *here either. However if* $Y$ *is* $\sigma$*-compact, then the space* $C_c(Y,\mathbb{R})$ *is webbed, and hence* $C_c(Y,G)$ *satisfies the uniform boundedness principle* (U2)*.*
8. *The same as in the previous item applies here.*

Thus we should investigate the desired isomorphism (V) (and in particular (P)) for $L$. Obviously we have a bilinear mapping $E' \times G \to L(E,G)$ and this induces a linear map $\iota : E' \otimes G \to L(E,G)$. So we have to prove firstly that this map is an embedding for some topology on $E' \otimes G$ (which we can always achieve by taking the corresponding initial topology) and that secondly it has dense image. So let us calculate the image first:

**4.8 Lemma. Algebraic tensor product as operators.**
*The image of the algebraic tensor product* $E' \otimes G$ *in* $L(E,G)$ *consists exactly of the finite dimensional operators (i.e. those with finite dimensional image).*

**Proof.** Let $T : E \to G$ have finite dimensional image. Then choose a basis $(g_n)_n$ of $T(E)$ and continuous linear functionals $(\lambda_n)_n$ in $G'$ dual to the $g_n$. Then $T = \sum_n (\lambda_n \circ T) \cdot g_n$. Conversely the image of $\sum_{n\le N} \lambda_n \otimes g_n$ is obviously contained in $\langle g_n : n \le N\rangle$. □

We have shown in [**2**, 6.4.8] that any limit of finite dimensional operators between Banach spaces is compact. Obviously the identity on a Banach space $G$ is compact only if $G$ is finite dimensional, so $E' \otimes E$ is not dense in $L(E,E)$ for any infinite dimensional Banach space $E$. Thus for no infinite dimensional Banach space $E = G$ there is a topology $\tau$ on the algebraic tensor-product such that

$$E' \hat{\otimes}_\tau G \cong L(E,G)$$

is true.

Recall that with respect to the completed projective tensor product (V) is true for $\mathcal{F}(X, \_) := (\_)^X$ with discrete $X$ by 3.28. But it fails with respect to the completed bornological tensor product for $G := \mathbb{R}^{(\mathbb{N})}$ and $X := \mathbb{N}$, since

$$G \hat{\otimes}_\beta \mathcal{F}(\mathbb{N}) = \mathbb{R}^{(\mathbb{N})} \hat{\otimes}_\beta \mathbb{R}^{\mathbb{N}} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})} \ncong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} = \mathcal{F}(\mathbb{N}, G)$$

By 3.38 we have that with respect to the completed bornological tensor product (V) is true for $\mathcal{F}(X, \_) := (\_)^{(X)}$. But it fails with respect to the completed projective tensor product, since

$$\mathcal{F}(\mathbb{N})\hat{\otimes}_\pi G = \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi \mathbb{R}^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \ncong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})} = \mathcal{F}(\mathbb{N}, G).$$

So we see that the choice of the appropriate tensor topology depends on the function space functor $\mathcal{F}$ under consideration.

Let us now consider (V) (and in particular (P) for $G = \ell^\infty(Y)$) for the function spaces $\ell^\infty$. Using the free $c^\infty$-complete vector space $\ell^1$ this would translate into

$$\ell^1(X)' \otimes_? G \cong \ell^\infty(X) \otimes_? G \overset{\text{(V)}}{\cong} \ell^\infty(X, G) \overset{\text{(F)}}{\cong} L(\ell^1(X), G).$$

But since in particular case, where $X = \mathbb{N}$ and $G = \ell^\infty$, the natural inclusion of $\ell^1 \to \ell^\infty$ is not compact - (the image of) the standard basis is not precompact in $\ell^\infty$ - in cannot lie in the image under the composite of the completion of any topology on the algebraic tensor product by 4.8. Thus this composite is never onto and hence for $F = \ell^\infty$ neither (V) nor the particular case (P) can be true.

To $C$, $C_c$, $C^\infty$ and $C^\infty_c$ we will come later.

## Integrable Functions

**4.9.** Since the free $c^\infty$-complete vector space $\ell^1(X)$ of a bornological set $X$ is a space of functions, we could try to generalize this space to the vector valued case. We will restrict ourselves to the case where $X$ is bounded. It would however be nice to treat also the case of a general bornological space $X$. It is quite natural to define $\ell^1(B, G)$ as those functions $f : B \to G$ for which $\tilde{p}(f) := \sum_{x\in B} p(f(x)) < \infty$ for every continuous seminorm $p$ on $G$. A basis of seminorms on $\ell^1(B, G)$ is the given by these $\tilde{p}$. (For general $X$ one could define $\ell^1(X, G)$ as the inductive limit of $\ell^1(B, G)$).

It is easy to show that (E) is true for $\ell^1$. In fact $\sum_x \sum_y p(g_{x,y})$ converges iff $\sum_{x,y} p(g_{x,y})$ does. Furthermore the seminorms correspond to each other.

Let us try to find a universal solution $\lambda(X)$ for linearization. For countable $X$ we have in 2.33 already shown that the closure of the finite sequences in $\ell^\infty(X) \cong \ell^1(X)'$ is given by

$$\begin{aligned} c_0(X) := \{f : X \to \mathbb{R} : &\operatorname{supp}(f) \text{ is countable and} \\ &\text{for every } \varepsilon > 0 \text{ and bounded } B \\ &\text{the set } \{x \in B : |f(x)| > \varepsilon\} \text{ is finite}\}. \end{aligned}$$

This can be similarly proved for general bornological spaces $X$. All scalar $\ell^1$-functions $f : X \to G$ induce functions in $L(c_0(X), G)$ via $\lambda \mapsto \sum_x \lambda(x) f(x)$, and in fact one can show the converse, see [**14**, 19.4.3]. Recall that according to 2.4 we call a function $f : X \to G$ to be scalarly $\ell^1$ iff $\ell \circ f \in \ell^1(X)$ for all $\ell \in E^*$. Thus we would need at least that all scalar $\ell^1$-functions are $\ell^1$, in order to obtain a universal solution. For $X = \mathbb{N}$ and sequentially complete $G$ one obviously has the following implications:

absolutely summable $\Longrightarrow$ unconditionally summ. $\Longrightarrow$ scalarly absolutely summ.

A series $\sum_k x_k$ is called *unconditionally summable*, iff for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_k x_{\sigma(k)}$ converges in $E$. Using the linear functionals shows that this limit does not depend on the permutation $\sigma$.

**4.10 Lemma.**
*The second implication can be reversed - i.e. the scalar $\ell^1$-functions on $\mathbb{N}$ are exactly the unconditionally summable series - provided $G'$ is countably-barreled for the Mackey-topology $\mu(G', G)$, see* [**14**, p.357].

**Lemma.**
*A sequence $x_n$ is unconditionally summable iff the net $F \mapsto \sum_{k\in F} x_k$ converges, where $F$ runs through all finite subsets directed by inclusion.*

**Proof.** ($\Rightarrow$) Let $x = \sum_k x_k$. Suppose indirectly that there is a 0-neighborhood $U$ such that for every finite set $F$ there exists a set $F' \supseteq F$ such that $\sum_{k\in F'} x_k \notin x+U$. By assumption there exists a $n_0$ such that $\sum_{k\le n} x_k \in x+U$ for all $n \ge n_0$. Let $F_0 := \{1, \dots, n_0\}$. We construct by induction $F_k := \{1, \dots, \max F'_{k-1}\} \supseteq F'_{k-1}$. Let $F'_k$ be the corresponding set $F'_k \supseteq F_k$ with $\sum_{k\in F'_k} x_k \notin x+U$. Enumerate the natural numbers in the order $F_0 \le F'_0 \le F_1 \le \dots \le F_k \le F'_k \le F_{k+1} \le \dots$. For this permutation $\sigma$ we have that $\sum_{k\in\sigma^{-1}(F'_k)} x_{\sigma(k)} \notin x+U$.

($\Leftarrow$) Let $x = \lim_F \sum_{x\in F} x_k$. Let $U$ be a 0-neighborhood and $\sigma$ be a permutation of $\mathbb{N}$. By assumption there is some finite set $F \subseteq \mathbb{N}$ such that $\sum_{k\in F'} x_k \in x+U$ for all finite $F' \supseteq F$. Let $N := \max \sigma^{-1}(F)$. Then we have $F' := \sigma(\{1, \dots, n\}) \supseteq \sigma(\sigma^{-1}F) = F$ for every $n \ge N$ and hence $\sum_{k\le n} x_{\sigma(k)} = \sum_{k\in F'} x_k \in x+U$. □

**4.11 Remark.**
There is however a *theorem by Dvoretzky-Rogers*, which we will prove later (for $p = 1$), and which says that a Banach space for which every scalar $\ell^p$-sequence is $\ell^p$ for some $1 \le p < \infty$ has to be finite dimensional, see [**14**, 19.6.9]. Take for example the sequence $x_n := \frac{1}{n} e_n \in c_0$. This sequence is not absolutely summable, but is unconditionally summable. So we have $L(c_0(\mathbb{N}), G) \supset \ell^1(\mathbb{N}, G)$ in particular for $G = \ell^1(\mathbb{N})$.

However one can show directly that (V) and hence (P) is true for complete spaces $G$ and $\mathcal{F} = \ell^1$:

**4.12 Lemma.**
*For bornological spaces $X$ and complete locally convex spaces $G$ we have*

$$\ell^1(X) \hat{\otimes}_\pi G \cong \ell^1(X, G)$$

**Proof.** We first show that the natural mapping $\ell^1_c(X) \otimes_\pi G \to \ell^1_c(X, G)$ is an isomorphism, where $\ell^1_c(X)$ is the dense subspace in $\ell^1(X)$ of finite sequences and $\ell^1_c(X, G)$ the analogous subspace in $\ell^1(X, G)$. Since $\mathbb{R}^k \otimes_\pi G \cong G^k$ we have a bijection. Let $z = \sum_k f_k \otimes y_k \in \ell^1_c(X) \otimes G$. Let $p$ be a seminorm on $G$ and $\|\_\|_1$ the usual norm on $\ell^1(X)$. We have for the corresponding norm $\tilde{p}$ on $\ell^1_c(X, G)$ (see 4.9) that

$$\tilde{p}(z) := \sum_{x\in X} p(z(x)) = \sum_{x\in X} p\left(\sum_k f_k(x)\, y_k\right) \le$$
$$\le \sum_k \sum_{x\in X} |f_k(x)|\, p(y_k) = \sum_k \|f_k\|_1 \cdot p(y_k),$$

Taking the infimum of the right side over all representations of $z$ shows that this norm $\tilde{p}$ is smaller than the corresponding projective tensor norm $p^\pi$. Conversely each $z \in \ell^1_c(X,G)$ can be written as finite sum $\sum_x e_x \otimes z(x)$, where $e_x$ denotes the standard unit vector in $\ell^1(X)$. Thus we have for the tensor norm $p^\pi$ that

$$p^\pi(z) \le \sum_x \|e_x\|_1 \cdot p(z(x)) = \sum_{x\in X} p(z(x)) = \tilde{p}(z)$$

which shows the converse relation.

Now since $\ell^1_c(X,G)$ is dense in $\ell^1(X,G)$ and the latter space is complete (as can be shown analogously to the case $\ell^1(\mathbb{N},\mathbb{R})$), we have the desired isomorphism:

$$\ell^1(X,G) = \widehat{\ell^1_c(X,G)} \cong \ell^1_c(X)\hat{\otimes}_\pi G \cong \ell^1(X)\hat{\otimes}_\pi G. \quad \square$$

**4.13 Corollary.**
*For the dual of $\ell^1(X,G)$ and a bornological and (complete) locally convex space $G$ we have:*

$$\begin{aligned}\ell^1(X,G)' &\overset{\text{(V)}}{\cong} (\ell^1(X)\otimes_\pi G)' = (\ell^1(X)\otimes_\beta G)' \overset{(E_L)}{\cong} L(\ell^1(X),G;\mathbb{R})\\ &\overset{(E_L)}{\cong} L(\ell^1(X),G') \overset{\text{(F)}}{\cong} \ell^\infty(X,G') \overset{\text{(U3)}}{\cong} L(G,\ell^\infty(X)) \quad \square\end{aligned}$$

Note however that the latter space is not isomorphic to $\ell^1(X)'\otimes_\beta G' = \ell^1(X)'\otimes_\pi G'$, as we have seen above for $X = \mathbb{N}$ and $G = \ell^1(X)$. Hence (V') and (P') fail to be true for $\ell^1$.

## 4.14. $\ell^p$-spaces

A consequence is that for $X = Y = \mathbb{N}$ we have

$$\begin{aligned}(c_0(X)\otimes_\beta c_0(Y))' &\cong L(c_0(X),c_0(Y)') \cong L(c_0(X),\ell^1(Y))\\ &\underset{4.11}{\supset} \ell^1(X,\ell^1(Y)) \cong \ell^1(X\times Y) \cong c_0(X\times Y)'\end{aligned}$$

Thus $c_0(X)\hat{\otimes}_\beta c_0(Y) \neq c_0(X\times Y)$. Which shows that (P) fails for $C$ with respect to the bornological and the projective tensor product. And so we should search for an appropriate different topology on the tensor product.

For $1 \le p < \infty$ one can show (see [**22**, 469]) that (E) holds, i.e. $\ell^p(X\times Y) \cong \ell^p(X,\ell^p(Y))$, and that $L(\ell^{p^*},G)$ is isomorphic to the space of scalar $\ell^p$-sequences in $G$ for $1 < p \le \infty$, see [**14**, 19.4.3]. However, as we mentioned before for $p < \infty$ these are strictly more than the $\ell^p$-sequences, and hence there is no free convenient vector space $\lambda(\mathbb{N})$ in this situation. One can furthermore show that (P) fails for all $\ell^p$ with $p > 1$. And in particular for the Hilbert-spaces $\ell^2$. This can be seen as follows: Suppose

$$\ell^2\hat{\otimes}\ell^2 \cong \ell^2(\mathbb{N}\times\mathbb{N})$$

then

$$L(\ell^2,\ell^2) \cong L(\ell^2,(\ell^2)') \cong (\ell^2\hat{\otimes}_\pi\ell^2)' \cong \ell^2(\mathbb{N}\times\mathbb{N})' \cong \ell^2(\mathbb{N}\times\mathbb{N})$$

But the identity corresponds to the characteristic function on the diagonal, which does not belong to $\ell^2(\mathbb{N}\times\mathbb{N})$.

### 4.15 Measure spaces

Let us consider another category, namely that of measure spaces $X$, i.e. sets together with a $\sigma$-algebra and a positive measure $\mu$ on $X$. Then we would like to define $\mathcal{L}^1(X,G)$ for all locally convex spaces $G$ and try to prove the desired isomorphisms (E), (V), (P), (V') and (P').

We want to generalize the spaces $\mathcal{L}^1(X)$ and $\mathcal{L}^\infty(X)$ of [**2**, 4.12.5] to the vector valued case. So let $(X,\mathcal{A},\mu)$ be a measure space and $G$ be an arbitrary locally convex space. As in [**2**, 4.12.1] we consider the space of simple functions, i.e. the vector space of functions $X\to G$ generated by the functions $\chi_A\otimes y : x\mapsto \chi_A(x)\,y$, where $y\in G$ and $A\in\mathcal{A}$ with $\mu(A)<\infty$ and $\chi_A$ denotes the *characteristic function* of $A$. For those functions $f=\sum_k \chi_{A_k}\otimes y_k$ we may define an integral $\int_X f\,d\mu := \sum_k \mu(A_k)\cdot y_k \in G$. And if $p$ is a continuous seminorm on $G$ we may define a seminorm $\tilde p$ given by $\tilde p(f) := \int_X p\circ f\,d\mu = \int_X p(f(x))\,d\mu(x)\in\mathbb{R}$. These seminorms have a common kernel, namely those simple functions, which are zero outside a set of $\mu$-measure 0. In fact if $f(x)=0$ for $x\notin A$, where $A$ has measure 0, then the same is true for the scalar valued elementary function $p\circ f$, and hence $\tilde p(f)=0$. Conversely suppose that the necessarily measurable set $A:=f^{-1}(G\setminus\{0\})$ is not a 0-set. Since $f$ takes only finitely many values on $A$, by shrinking $A$ we may assume w.l.o.g. that $f$ is constant on $A$ and $\mu(A)>0$. Now take a seminorm $p$ which does not vanish on $f(A)$, then $\tilde p(f)\ge \int_A p\circ f\,d\mu>0$. We denote the quotient of the space of simple functions supplied with all these seminorms modulo their common kernel by $\operatorname{Simp}(X;G)$. And it is now natural to define $\mathcal{L}^1(X;G)$ to be the completion of $\operatorname{Simp}(X;G)$. Our aim is to show how to obtain this completion from that of $\operatorname{Simp}(X;\mathbb{R}) =: \operatorname{Simp}(X)$.

**4.16 Lemma.**
$\operatorname{Simp}(X)\otimes_\pi G\cong \operatorname{Simp}(X;G)$.

**Proof.** Obviously we have a bilinear continuous map $\operatorname{Simp}(X)\times G\to\operatorname{Simp}(X;G)$ given by $(\sum_k \chi_{A_k}, y)\mapsto \sum_k \chi_{A_k}\otimes y$ and hence we obtain a map $\iota:\operatorname{Simp}(X)\otimes_\pi G\to \operatorname{Simp}(X;G)$. By construction of $\operatorname{Simp}(X;G)$ we have that $\iota$ is onto. Let us show that it is injective. So let $0\neq z=\sum_i f_i\otimes y_i\in\operatorname{Simp}(X)\otimes G$ be given, such that $\iota(z)=0$. W.l.o.g. we may assume that the $y_i$ are linearly independent and we can take $y_i^*\in G^*$ dual to those $y_i$. From $\iota(z)=0$ we get that $f_i=\sum_j f_j\cdot y_i^*(y_j)=y_i^*\circ\iota(z)=0$ $\mu$-a.e. and hence $z=0$.

For $z=\sum_i f_i\otimes y_i\in\operatorname{Simp}(X)\otimes G$ we obtain that

$$\tilde p(\iota(z))\le \int_X \sum_i |f_i(x)|\cdot p(y_i)\,d\mu(x)=\sum_i \|f_i\|_1\cdot p(y_i),$$

and hence $\tilde p\circ\iota$ is dominated by the projective tensor norm $\pi_{1,p}$ formed from the 1-norm on $\operatorname{Simp}(X)$ and $p$ on $F$.

Conversely we may assume that the $A_k$ are pairwise disjoint and then the tensor norm is less or equal to

$$\sum_j p(y_j)\cdot\mu(A_j)=\int_X p(\iota(z)(x))\,d\mu(x)=\tilde p(\iota(z)),$$

and hence we have equality. Thus the two topologies are the same. □

**4.17 Corollary.**
*For complete $G$ and measure space $X$ we have $\mathcal{L}^1(X)\hat\otimes_\pi G\cong\mathcal{L}^1(X;G)$*

**Proof.** This follows from 3.19 since

$$\mathcal{L}^1(X;G) := \widehat{\mathrm{Simp}(X;G)} \cong \widehat{\mathrm{Simp}(X) \otimes_\pi} G \cong \widehat{\mathrm{Simp}(X)}\hat{\otimes}_\pi G = \mathcal{L}^1(X)\hat{\otimes}_\pi G \quad \square$$

One can show that for Banach spaces $G$ the space $\mathcal{L}^1(X;G)$ can be realized as the space of all measurable functions $f$ for which $\int_X \|f\|\, d\mu < \infty$, the so called *Bochner integrable functions*, see [**14**, p340], modulo equality $\mu$-a.e..

**4.18 Corollary.**
*We have*

$$\mathcal{L}^1(X)\hat{\otimes}_\pi\mathcal{L}^1(Y) \cong \mathcal{L}^1(X;\mathcal{L}^1(Y)) \cong \mathcal{L}^1(X \times Y).$$

**Proof.** The first isomorphism follows directly from 4.17. The second one is valid for all $\mathcal{L}^p$-spaces with finite $p$. In fact $\mathrm{Simp}(X, \mathrm{Simp}(Y))$ embeds into $\mathrm{Simp}(X \times Y)$ via $\sum_i g_i \cdot \chi_{A_i} \mapsto \sum_i \sum_j c_{i,j}\chi_{A_i \times B_j}$, where $g_i = \sum_j c_{i,j} \cdot \chi_{B_j}$. And for the norms we have

$$\|\sum_i \sum_j c_{i,j} \cdot \chi_{A_i \times B_j}\| = \sum_{i,j} c_{i,j} \cdot \mu(A_i) \cdot \mu(B_j) = \sum_i \|g_i\| \cdot \mu(A_i).$$

The rest follows from density considerations. $\square$

**Remark.**
The corresponding result is not true for $\mathcal{L}^2$. In fact

$$(f,g) \mapsto \int_{\mathbb{R}^2} e^{-2\pi i x y}\, f(x)\, g(y)\, d(x,y)$$

defines a norm-1 bilinear mapping on $\mathcal{L}^2(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R})$ which cannot be extended to a continuous linear mapping on $\mathcal{L}^2(\mathbb{R}^2)$.

We have shown in [**2**, 6.4.8] that any $k \in \mathcal{L}^2(X \times X)$ defines a *compact linear operator* $K : \mathcal{L}^2(X) \to \mathcal{L}^2(X)$ by $Kf(x) := \int_X k(x,y)\, f(y)\, dy$. Furthermore $\mathcal{L}^2(X) \otimes \mathcal{L}^2(X)$ is dense in $\mathcal{L}^2(X \times X)$.

Consider the identity $\ell^2 \to \ell^2$. It is a non-compact continuous linear mapping. It gives us a continuous bilinear mapping $b : \ell^2 \times \ell^2 \cong \ell^2 \times (\ell^2)^* \to \mathbb{R}$, by $L(\ell^2, \ell^2) = L(\ell^2, (\ell^2)^{**}) \cong L(\ell^2, (\ell^2)^*; \mathbb{R}) \cong (\ell^2\hat{\otimes}_\pi(\ell^2)^*)^*$. Suppose this would correspond to a continuous linear functional on $\ell^2(\mathbb{N} \times \mathbb{N})$ and hence to a square-summable double-sequence $(b_{i,j})_{(i,j)}$. We can calculate the entries $b_{i,j}$ by applying this linear functional to $e_i \otimes e_j$, or equivalently by calculating $b(e_i, e^j) = \check{b}(e_i)(e^j) = \delta_i^j$. But this series is not square-summable. Thus $\ell^2\hat{\otimes}_\pi\ell^2 \not\cong \ell^2(\mathbb{N} \times \mathbb{N}) \cong \ell^2(\mathbb{N}, \ell^2(\mathbb{N}))$ (see 4.14) and also $\ell^2(X)\hat{\otimes}_\pi G \not\cong \ell^2(X, G)$.

Let us now turn towards $\mathcal{L}^\infty(X;G)$. This is defined to be the quotient space of all *bounded measurable functions* from $X \to G$ modulo equality $\mu$-a.e. supplied with the corresponding quotient semi-norms of the infinity norms $f \mapsto \sup\{p(f(x)) : x \in X\}$ on $\ell^\infty(X, G)$. If $G$ is a Fréchet space, then this space is Hausdorff, since the subspace formed by those functions, which are 0 $\mu$-a.e. is sequentially closed.

**4.19 Theorem, Dunford-Pettis theorem.**
*Let $G$ be a separable Banach space and $X$ a measure space. Then*

$$\mathcal{L}^1(X, G) \cong \mathcal{L}^1(X)\hat{\otimes}_\pi G$$

*and*

$$\begin{aligned}\mathcal{L}^1(X,G)^* &\cong (\mathcal{L}^1(X) \otimes_\pi G)^* \cong L(\mathcal{L}^1(X), G; \mathbb{R}) \cong L(\mathcal{L}^1(X), G^*)\\ &\cong L(G, \mathcal{L}^1(X)^*) = L(G, \mathcal{L}^\infty(X)) \cong \mathcal{L}^\infty(X, G^*),\end{aligned}$$

*the space of scalarly measurable bounded functions.*

For a proof see [**22**, 46.2].

It would be nice to show a corresponding statement for $\ell^1(X,G)$, where $X$ is a bornological space.

## Injective Tensor Product

Beside the few situations above the projective tensor product is not well suited for function spaces. So we need another topology $\varepsilon$ on the algebraic tensor product, such that $F' \otimes_\varepsilon G \to L(F,G)$ is an embedding. We could take this as a definition, but not every locally convex space $E$ is a dual space $F'$. However, since $L(F,\_)$ preserves embeddings (see below in 4.21), the same should be true for $E \otimes_\varepsilon (\_)$. And since the tensor product should be commutative, we only have to find an embedding of $E \to F'$ for some $F$ and then $E \otimes_\varepsilon G \hookrightarrow F' \otimes_\varepsilon G \hookrightarrow L(F,G)$ should be an embedding. In fact we can take $F = E^*$ with the bornology of equi-continuous sets, see 2.15.

**4.21 Lemma. $\ell^\infty(\mathbf{X},.)$ preserves embeddings.**
*Let $T : F_1 \to F_2$ be an embedding and $X$ be a bornological space. Then $T_* : \ell^\infty(X,F_1) \to \ell^\infty(X,F_2)$ is an embedding, and if $E$ is a convex bornological space, then $T_* : L(E,F_1) \to L(E,F_2)$ is an embedding.*

**Proof.** Since $L(E,F_i)$ is embedded into $\ell^\infty(E,F_i)$, only the first statement has to be shown. Clearly $T_*$ is injective, provided $T$ is injective: $T\circ f_1 = T_*(f_1) = T_*(f_2) = T\circ f_2$ implies that $f_1 = f_2$. Remains to show that $T_*$ is a homeomorphism onto its image. So let $U \subseteq F_1$ be a 0-neighborhood and $B \subseteq X$ be bounded. Then $N_{B,U}$ is a typical 0-neighborhood in $\ell^\infty(X,F_1)$. By assumption there is some 0-neighborhood $V \subseteq F_2$, such that $T^{-1}(V) \subseteq U$. But then

$$\begin{aligned}(T_*)^{-1}(N_{B,V}) &= \{f : T_*(f) \in N_{B,V}\} = \{f : T(f(B)) \subseteq V\}\\ &\subseteq \{f : f(B) \subseteq U\} = N_{B,U}. \quad \square\end{aligned}$$

**Definition.**
Thus we consider the bilinear mapping $E\times F \to L(E^*,F)$, given by $(x,y) \mapsto (x^* \mapsto x^*(x)y)$. It is well-defined, since $\mathrm{ev}_x : E^* \to \mathbb{R}$ is bounded. In fact $\mathrm{ev}_x : E^* \to \mathbb{R}$ is even continuous for the topology of uniform convergence on bounded sets, since the set $\{x^* : |x^*(x)| \le 1\}$ is the polar of the bounded set $\{x\}$ and hence a 0-neighborhood for this topology. This induces a linear map $E \otimes F \to L(E^*,F)$, given by $x \otimes y \mapsto (x^* \mapsto x^*(x)y)$.

We claim that this mapping is injective. In fact take $\sum_i x_i \otimes y_i \in E \otimes F$ with $x_i$ linearly independent. By Hahn-Banach we can find continuous linear functionals $x_i^*$ with $x_i^*(x_j) = \delta_{i,j}$. Assume that the image of $\sum_i x_i \otimes y_i$ is 0 in $L(E^*,F)$. Since it has value $y_i$ on $x_i^*$, we have that $y_i = 0$ for all $i$ and hence $\sum_i x_i \otimes y_i = 0$.

We define the *injective tensor product* (also called *$\varepsilon$-tensor product* in [**22**]) $E \otimes_\varepsilon F$ to be the algebraic tensor product with the locally convex topology induced by the injective inclusion into $L(E^*,F)$. Since $L(E^*,F)$ is Hausdorff, the same is true for $E \otimes_\varepsilon F$.

Note that, since $F$ embeds into $(F^*)'$ by 2.15, the structure of $E \otimes_\varepsilon F$ is also initial with respect to $E \otimes F \to L(E^*,F) \to L(E^*,(F^*)') \cong L(E^*,F^*;\mathbb{R})$, which gives a more symmetric form. Since the seminorms of $L(E^*,F^*;\mathbb{R})$ are given by suprema on $U^o \times V^o$, where $U$ and $V$ are 0-neighborhoods, we have for the corresponding seminorm $\varepsilon_{U,V}$ on $E \otimes_\varepsilon F$, that

$$\varepsilon_{U,V}\Big(\sum_k x_k \otimes y_k\Big) := \sup\Big\{\Big|\sum_k x^*(x_k)\, y^*(y_k)\Big| : x^* \in U^o, y^* \in V^o\Big\}$$

**4.22 Corollary. Seminorms of the injective tensor product.**

*A defining family of seminorms on $E \otimes_\varepsilon F$ is given by $\varepsilon_{U,V} : \sum_i x_i \otimes y_i \mapsto \sup\{|\sum_i x^*(x_i)\, y^*(y_i)| : x^* \in U^o, y^* \in V^o\}$, where $U$ and $V$ run through the 0-neighborhoods of $E$ and $F$. The injective tensor product $E \otimes_\varepsilon F$ is metrizable (resp. normable) if $E$ and $F$ are.* □

Let us show next, that the canonical bilinear mapping $E \times F \to L(E^*, F)$ is continuous, which implies that the identity $E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.
In fact, take an equi-continuous set $\mathcal{E} \subseteq E^*$, i.e. $\mathcal{E}$ is contained in the polar $U^o$ of a 0-neighborhood $U$. And take furthermore an absolutely convex 0-neighborhood $V \subseteq F$. Then $U \times V$ is mapped into $\{T : T(\mathcal{E}) \subseteq V\}$, since $(x \otimes y)(x^*) = x^*(x)\, y \in [-1, 1] \cdot V \subseteq V$ for $x^* \in \mathcal{E} \subseteq U^o$.

**4.23 Corollary.**
*$E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.*

**Proof.** In the diagram

$$\begin{tikzcd} E \otimes_\pi F \arrow[r] & E \otimes_\varepsilon F \arrow[d, hook] \\ E \times F \arrow[u, "\otimes"] \arrow[r] & L(E^*, F) \end{tikzcd}$$

continuity of the bilinear map at the bottom implies continuity of the top arrow. □

**4.24 Definition.**
A space $E$ is called *nuclear* iff $E \otimes_\pi F = E \otimes_\varepsilon F$ for all $F$. We will come to this later on in more detail. Note that every product of $\mathbb{R}$ is nuclear, since $\mathbb{R}^X \otimes_\pi E$ embeds into $\mathbb{R}^X \hat{\otimes}_\pi \hat{E} \cong \hat{E}^X \cong L(\mathbb{R}^{(X)}, \hat{E}) \cong L((\mathbb{R}^X)^*, \hat{E})$ (the second isomorphism follows from the continuity of $L(\_, \hat{E})$) in which also $L((\mathbb{R}^X)^*, E)$ and thus $\mathbb{R}^X \otimes_\varepsilon E$ embeds.

Note however, that $E \times F \to L(E', F)$ is not continuous, even for $F = \mathbb{R}$, see [**2**, 7.4.20], where $E'$ carries the topology of uniform convergence on bounded sets, and hence has as bounded sets those which are uniformly bounded on bounded sets.

**4.25 Proposition.**
*The injective tensor product is commutative and associative.*

**Proof.** Since the description of 0-neighborhoods in $E \otimes F$ is symmetric, we conclude that $\otimes_\varepsilon$ is commutative. This follows even more directly from the embedding $E \otimes_\varepsilon F \to L(E^*, F^*; \mathbb{R})$. For associativity, we consider the embeddings

$$(E \otimes_\varepsilon F) \otimes_\varepsilon G \cong G \otimes_\varepsilon (E \otimes_\varepsilon F) \hookrightarrow L(G^*, E \otimes_\varepsilon F) \hookrightarrow L(G^*, L(E^*, F)) \cong L(G^*, E^*; F)$$

and

$$\begin{aligned} E \otimes_\varepsilon (F \otimes_\varepsilon G) &\hookrightarrow L(E^*, F \otimes_\varepsilon G) \cong L(E^*, G \otimes_\varepsilon F) \hookrightarrow \\ &\hookrightarrow L(E^*, L(G^*, F)) \cong L(E^*, G^*; F) \cong L(G^*, E^*; F) \end{aligned}$$ □

**4.26 Corollary.**
*The space $E' \otimes_\varepsilon F$ embeds into $L(E, F)$.*

**Proof.** In fact, since $E' \otimes_\varepsilon F \cong F \otimes_\varepsilon E'$ it embeds into $L(F^*, E') \cong L(E, (F^*)')$. This inclusion factors over the embedding $L(E, F) \to L(E, (F^*)')$, by $x^* \otimes y \mapsto (x \mapsto x^*(x)y)$. Hence this map $E' \otimes_\varepsilon F \to L(E, F)$ is an embedding.

$$\begin{tikzcd} E' \otimes_\varepsilon F \arrow[r, hook] \arrow[dr, hook, dotted] & L(E, (F^*)') \\ & L(E, F) \arrow[u, hook] \end{tikzcd}$$ □

**4.27 Proposition.**
*The injective tensor product is a functor, which preserves injective maps and embeddings.*

**Proof.** That $T_1 \otimes_\varepsilon T_2$ is continuous and thus $\otimes_\varepsilon$ is a functor follows, since $T_1^* : E_2^* \to E_1^*$ is bounded and hence $L(T_1^*, T_2) = (T_2)_* \circ (T_1^*)^* : L(E_1^*, F_1) \to L(E_2^*, F_2)$ is continuous.

$$\begin{tikzcd} E_1 \otimes_\varepsilon F_1 \arrow[r, hook] \arrow[d, "T_1 \otimes_\varepsilon T_2"'] & L(E_1^*, F_1) \arrow[d, "L(T_1^*, T_2)"] \\ E_2 \otimes_\varepsilon F_2 \arrow[r, hook] & L(E_2^*, F_2) \end{tikzcd}$$

Since $L(E^*, \_)$ preserves injectivity and embeddings, and since $\otimes_\varepsilon$ is commutative the claimed preservation properties follow. □

**4.28 Corollary.**
*Let $F_1$ and $F_2$ be topological subspaces of $E_1$ and $E_2$. And assume that $F_1$ or $F_2$ is nuclear. Then $F_1 \otimes_\pi F_2$ is a topological subspace of $E_1 \otimes_\pi E_2$.*

**Proof.** By 4.27 we have that $F_1 \otimes_\pi F_2 \cong F_1 \otimes_\varepsilon F_2$ is a subspace of $E_1 \otimes_\varepsilon E_2$. Since $F_1 \otimes_\pi F_2 \to E_1 \otimes_\pi E_2 \to E_1 \otimes_\varepsilon E_2$ is continuous, the result follows. □

**4.29 Example.**
*The injective tensor product of quotient maps is not always a quotient map and it also doesn't preserve direct sums.*

**Proof.** The first one follows by taking the tensor product of a quotient mapping $\ell^1 \to \ell^2$ with the identity on $\ell^2$. Note that by [**14**, 6.9.4] every Banach space is a quotient space of some $\ell^1(X)$ with bounded $X$, and every separable Banach space is a quotient of $\ell^1$.

The second follows from the example $\mathbb{R}^{(\mathbb{N})} \otimes_\varepsilon \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}^{(\mathbb{N})} \otimes_\pi \mathbb{R}^{\mathbb{N}}$, since $\mathbb{R}^{\mathbb{N}}$ is nuclear. Thus also the strict inductive limit $\varinjlim_{n\in\mathbb{N}} \mathbb{R}^n$ of the sequence $\mathbb{R}^n$ is not preserved. □

**4.30 Proposition.**
*The injective tensor product preserves dense subspaces.*

**Proof.** Let $E_1 \subseteq E_2$ be a dense topological subspace. Then $E_1 \otimes F$ is dense in $E_2 \otimes_\pi F$ and hence a fortiori in $E_2 \otimes_\varepsilon F$. By 4.27 we have that $E_1 \otimes_\varepsilon F$ is a subspace of $E_2 \otimes_\varepsilon F$. □

**Remark.**
Since $E \otimes_\varepsilon F$ embeds into $L(E^*, F)$ and in turn into the complete space $L(E^*, \hat{F})$, we have that the *completed injective tensor product* $E\hat{\otimes}_\varepsilon F$ is the closure of $E \otimes F$ in $L(E^*, \hat{F})$. Note that by 4.30 we have that

$$E\hat{\otimes}_\varepsilon F \cong \hat{E}\hat{\otimes}_\varepsilon \hat{F}.$$

**4.31 Theorem.**
*The completed injective tensor product preserves products and reduced projective limits.*

**Proof.** Since $\widehat{\prod_j F_j} = \prod_j \hat{F}_j$ (denseness and completeness), we may assume without loss of generality that $E$ and all $F_j$ are complete. The natural mapping $E\hat{\otimes}_\varepsilon \prod_j F_j \to \prod_j E\hat{\otimes}_\varepsilon F$ is induced from the isomorphism

$$L\Big(E^*, \prod_j F_j\Big) \to \prod_j L(E^*, F_j)$$

and hence is an embedding. Since for the algebraic tensor product we have $E \otimes \coprod_j F_j \cong \coprod_j E \otimes F_j$ and both sides are dense in the corresponding complete spaces above, we have an isomorphism.

By the corresponding result for the projective tensor product we have that $E \otimes \varprojlim_j F_j$ is dense in $\varprojlim_j E\hat{\otimes}_\pi F_j$ and hence a fortiori in $\varprojlim_j E\hat{\otimes}_\varepsilon F_j$, which is a subspace of $\prod_j E\hat{\otimes}_\varepsilon F_j \cong E\hat{\otimes}_\varepsilon \prod_j F_j$. Since $\varprojlim_j F_j$ is a subspace in $\prod_j F_j$, we have by 4.27 that $E\hat{\otimes}_\varepsilon \varprojlim_j F_j \to \varprojlim_j E\hat{\otimes}_\varepsilon F_j$ is a embedding and hence an isomorphism. □

## Some Function Spaces

Let $F$ be complete. We denote with $\ell^1\{F\} := \ell^1(\mathbb{N}, F) \cong \ell^1\hat{\otimes}_\pi F$ (see 4.12) the space of all absolutely summable sequences in $F$, with $\ell^1\langle F\rangle$ the space of all unconditionally summable sequences in $F$, and with $\ell^1[F] \cong L(c_0, F)$ the space of all scalarly absolutely summable sequences. We have already shown the inclusions

$$\ell^1\{F\} \subseteq \ell^1\langle F\rangle \subseteq \ell^1[F].$$

Let us describe the structure on $\ell^1[F]$ induced by the bijection with $L(c_0, F)$.

**4.32 Lemma. The space of scalarly absolutely summable sequences.**
*The structure on $\ell^1[F]$ induced from $L(c_0, F)$ is given by the seminorms*

$$\tilde{p}(f) := \sup\Big\{\sum_{n=1}^\infty |y^*(f_n)| : |y^*| \le p\Big\},$$

*where $p$ runs through all continuous seminorms of $F$.*

**Proof.** Let $p$ be a continuous seminorm on $F$ and $V := \{y \in F : p(y) \le 1\}$. As in 4.22 we use that $p(y) = \sup\{|y^*(y)| : y^* \in V^o\}$. Thus we can calculate the seminorm $p_\infty$ on $L(c_0, F)$ associated to $p$ as follows, where $B$ denotes the closed unit-ball in

$c_0$ and $\iota : \ell^1[F] \to L(c_0, F)$, $\iota(f)(\lambda) = \sum_k f_k \lambda_k$ the canonical bijection:

$$\begin{aligned}
\tilde{p}(f) := p_\infty(\iota(f)) &:= \sup\{p(\iota(f)(\lambda)) : \lambda \in B\}\\
&= \sup\Big\{\Big|y^*\Big(\sum_{k=1}^{\infty} f_k\lambda_k\Big)\Big| : \lambda \in B, y^* \in V^o\Big\}\\
&= \sup\Big\{\Big|\sum_{k=1}^{\infty} \lambda_k y^*(f_k)\Big| : \lambda \in B, y^* \in V^o\Big\}\\
&\le \sup\Big\{\underbrace{\sup\{|\lambda_k|\}}_{\le 1} \sum_{k=1}^{\infty} |y^*(f_k)| : \lambda \in B, y^* \in V^o\Big\}\\
&\le \sup\Big\{\sum_{k=1}^{\infty} |y^*(f_k)| : y^* \in V^o\Big\}
\end{aligned}$$

Conversely we can find for $\varepsilon > 0$ an $n$ such that $\sum_{k>n}^{\infty} |y^*(f_k)| < \varepsilon$. Let $\lambda_k := \operatorname{sign}(y^*(f_k))$ for $k \le n$ and $0$ otherwise. Then $\lambda \in B$ and

$$\sum_{k=1}^{\infty} |y^*(f_k)| = \Big|\sum_{k\le n} \lambda_k\, y^*(f_k)\Big| + \sum_{k>n} |y^*(f_k)| \le \Big|\sum_{k=1}^{\infty} \lambda_k\, y^*(f_k)\Big| + \varepsilon \le p_\infty(\iota(f)) + \varepsilon.$$

Hence we have also the converse relation. □

**4.33 Lemma. The space of unconditionally summable sequences.**
*For complete $F$ the subspace $\ell^1\langle F\rangle$ of $\ell^1[F]$ is closed and both spaces are complete. Hence we will consider always the initial structure on $\ell^1\langle F\rangle$ induced from $\ell^1[F]$*

**Proof.** Since $\ell^1[F] \cong L(c_0, F)$, it is complete. So we only have to show that $\ell^1\langle F\rangle$ is closed in $\ell^1[F]$. Take an element $x$ in the closure of $\ell^1\langle F\rangle$. We have to show that the net $K \mapsto \sum_{k\in K} x_k$ converges, where $K$ runs through the finite subsets of $\mathbb{N}$. Since $F$ is complete, it is enough to show that this is a Cauchy-net. So let $p$ be a seminorm of $F$ and $\varepsilon > 0$. By the assumption we can find a $y \in \ell^1\langle F\rangle$ with $\tilde{p}(x-y) \le \varepsilon$. Thus the net $\sum_{k\in K} y_k$ converges in $F$, i.e. there is a finite $K_0 \subseteq \mathbb{N}$ such that $p\Big(\sum_{k\in\mathbb{N}} y_k - \sum_{k\in K} y_k\Big) \le \varepsilon$ for all $K \supseteq K_0$. Hence we have for $K_0 \subseteq K_1 \subset K_2$

$$\begin{aligned}
p\Big(\sum_{k\in K_2} x_k - \sum_{k\in K_1} x_k\Big) &= p\Big(\sum_{k\in K_2\setminus K_1} x_k\Big)\\
&\le p\Big(\sum_{k\in K_2\setminus K_1} (x_k - y_k)\Big) + p\Big(\sum_{k\in K_2} y_k - \sum_{k\in K_1} y_k\Big)\\
&\le \sup\Big\{\Big|y^*\Big(\sum_{k\in K_2\setminus K_1} (x_k - y_k)\Big)\Big| : |y^*| \le p\Big\}\\
&\qquad + p\Big(\sum_{k\in\mathbb{N}} y_k - \sum_{k\in K_1} y_k\Big) + p\Big(\sum_{k\in K_2} y_k - \sum_{k\in\mathbb{N}} y_k\Big)\\
&\overset{4.32}{\le} \tilde{p}(x-y) + p\Big(\sum_{k\in\mathbb{N}} y_k - \sum_{k\in K_1} y_k\Big) + p\Big(\sum_{k\in K_2} y_k - \sum_{k\in\mathbb{N}} y_k\Big)\\
&\le \varepsilon + \varepsilon + \varepsilon,
\end{aligned}$$

which shows that $K \mapsto \sum_{k\in K} x_k$ is a Cauchy-net. □

**4.34 Theorem.**
*Let $F$ be complete then $\ell^1 \widehat{\otimes}_\varepsilon F \cong \ell^1\langle F\rangle$, i.e.* (V) *is valid for $\ell^1\langle\_\rangle$, the space of unconditionally summable sequences.*

**Proof.** By 4.26 we have that $\ell^1(X)\otimes_\varepsilon F\cong c_0(X)'\otimes_\varepsilon F$ embeds into $L(c_0(X),F)$, the space of scalarly absolutely summable functions. Obviously $\lambda\otimes y\in\ell^1\otimes F$ is contained in $\ell^1\{F\}\subseteq\ell^1\langle F\rangle$. Since the latter space is complete, we only have to show that $\ell^1_c\otimes F=\mathbb{R}^{(\mathbb{N})}\otimes F\cong F^{(\mathbb{N})}$ is dense in $\ell^1\langle F\rangle$ with respect to the structure inherited from $\ell^1[F]$. So let $x\in\ell^1\langle F\rangle$ and consider $x^n:=x|_{[1,\dots,n]}\in F^n\subseteq F^{(\mathbb{N})}\subseteq\ell^1[F]$. We claim that $x^n\to x$ in $\ell^1[F]$. So let $p$ be a continuous seminorm on $F$. Since $K\mapsto\sum_{k\in K}x_k$ converges, we have that

$$\begin{aligned}
\tilde p(x-x^n)&=\sup\Big\{\sum_{k>n}|y^*(x_k)|:|y^*|\le p\Big\}\\
&=\sup\Big\{\Big|y^*\Big(\sum_{\substack{k>n\\ y^*(x_k)\ge0}}x_k\Big)\Big|+\Big|y^*\Big(\sum_{\substack{k>n\\ y^*(x_k)<0}}x_k\Big)\Big|:|y^*|\le p\Big\}\\
&\le\sup\Big\{p\Big(\sum_{\substack{k>n\\ y^*(x_k)\ge0}}x_k\Big)+p\Big(\sum_{\substack{k>n\\ y^*(x_k)<0}}x_k\Big)\Big\}\\
&\le 2\varepsilon
\end{aligned}$$

for $n$ sufficiently large. In the complex case we have to make a more involved estimation for $\sum_{k>n}|y^*(x_k)|$. Let $P:=\{z\in\mathbb{C}:\Re z>0\text{ and }-\Re z<\Im z\le\Re z\}$. For every $z\neq0$ there is a unique $j\in\{0,1,2,3\}$ with $i^j\,z\in P$. Then $|z|\le2\Re(i^j\,z)\le 2|z|$. Thus we can split the sum into 4 parts corresponding to $j\in\{0,1,2,3\}$, where $i^jy^*(x_k)\in P$. For each subsum we have

$$\begin{aligned}
\sum_{\substack{k>n\\ i^j\,y^*(x_k)\in P}}|y^*(x_k)|&=\sum_{\substack{k>n\\ i^j\,y^*(x_k)\in P}}2\Re(i^j\,y^*(x_k))=2\,\Re\Big(i^j\,y^*\Big(\sum_{\substack{k>n\\ i^j\,y^*(x_k)\in P}}x_k\Big)\Big)\\
&\le 2\,\Big|y^*\Big(\sum_{\substack{k>n\\ i^j\,y^*(x_k)\in P}}x_k\Big)\Big|\le 2\,p\Big(\sum_{\substack{k>n\\ i^j\,y^*(x_k)\in P}}x_k\Big)\le 2\,\varepsilon
\end{aligned}$$

Thus we have $\tilde p(x-x^n)\le 8\,\varepsilon$. □

Let us show now that (V) is satisfied for $C$:

**4.35 Theorem.**
*If $F$ is complete, then we have*

$$C^m(X)\widehat{\otimes}_\varepsilon F\cong C^m(X,F).$$

*provided $X$ is an open subset in some $\mathbb{R}^n$ or $m=0$ and $X$ is a compactly generated completely regular space.*

**Proof.** We try to factorize the natural embedding $C^m(X)\otimes_\varepsilon F\to L(F^*,C^m(X))$ in the following way:

$$\begin{tikzcd}
C^m(X)\otimes_\varepsilon F \arrow[rr, hook] \arrow[dr, dotted] & & L(F^*,C^m(X)) \\
& C^m(X,F) \arrow[ur, dotted, hook, "\iota"] &
\end{tikzcd}$$

The right hand side arrow is associated to the bilinear composition map $C^m(X,F)\times F^*\to C^m(X)$, and hence is given by $\iota:f\mapsto(y^*\mapsto y^*\circ f)$. Note that the other embedding $C^m(X)\otimes_\varepsilon F\to L(C^m(X)^*,F)$ cannot be factorized easily. The image $\iota(f)$ belongs to $L(F^*,C^m(X))$, since it maps the equi-continuous set $V^o$ to $\{y^*\circ f:y^*\in V^o\}$, which is bounded in $C^m(X)$, since $\partial^\alpha(y^*\circ f)(x)=y^*(\partial^\alpha f(x))$. Furthermore $\iota$ is linear and injective, since $F^*$ separates points of $F$. It is even a homeomorphic embedding, since a 0-neighborhood basis of $C^m(X,F)$ is given by $N_{p,K,V} := \{f : \partial^\alpha f(K) \subseteq V \text{ for } |\alpha| \le p\}$, where $p \le m$, the set $K \subseteq X$ is compact and $V \subseteq F$ a closed absolutely convex 0-neighborhood. And a 0-neighborhood basis of $L(F^*, C^m(X))$ is given by $N_{p,V^o,K^o} := \{T : |\partial^\alpha(T(y^*))(x)| \le 1 \text{ for } |\alpha| \le p,\ y^* \in V^o \text{ and } x \in K\}$ and $\iota^{-1}(N_{p,V^o,K^o}) = N_{p,V,K}$, since $\partial^\alpha f(K) \subseteq V$ iff for all $y^* \in V^o$ we have that $y^*(\partial^\alpha f(K)) \subseteq [-1,1]$.

The arrow $C^m(X) \otimes_\varepsilon F \to C^m(X,F)$ on the left hand side is given by $f \otimes y \mapsto (x \mapsto f(x)\, y)$. Composed with the mapping $\iota$ from above we obtain the natural inclusion $C^m(X) \otimes_\varepsilon F \to L(F^*, C^m(X))$, given by $f \otimes y \mapsto (y^* \mapsto y^*(y)\, f)$. Hence $C^m(X) \otimes_\varepsilon F \to C^m(X,F)$ is an embedding as well.

We show now the required density properties, first for $m = 0$. So let $f \in C(X,F)$ be given as well as a 0-neighborhood $N_{K,V}$, with $K \subseteq X$ compact and $V = \{y : p(y) \le 1\} \subseteq F$ a 0-neighborhood. By continuity of $f$ and compactness of $K$ we can find a finite covering of $K$ by open sets $V_i$ and points $x_i \in V_i$, such that $p(f(x) - f(x_i)) \le 1$ for all $x \in V_i$. Let $(h_i)$ be a partition of unity on $K$ subordinated to this covering. By Tieze's extension theorem, we may assume that $h_i \in C(X)$. In fact we may extend $h_i$ to a continuous function on the Stone-Čech compactification $\beta X$ of $X$ and then restrict it to $X$. Now take $h := \sum h_i \otimes f(x_i) \in C(X) \otimes F$. Then for $x \in K$ we have $p\big(f(x) - \iota(h)(x)\big) \le \sum_i h_i(x)\, p\big(f(x) - f(x_i)\big) \le 1$, i.e. $\iota(h) \in f + N_{K,V}$.

Now for arbitrary $m$ and open $X \subseteq \mathbb{R}^n$. First note that $C^m_c(X,F)$ is dense in $C^m(X,F)$: In order to see this take a compact set $K \subset X$ and choose a bump-function $h \in C^\infty_c(X,F)$ with $h|_K = 1$. Then for $f \in C^m(X,F)$ we have $h \cdot f \in C^m_c(X,F)$ and $f - h \cdot f \in N_{p,K,V}$ for every $p$ and $V$. So it is enough to show that $C^\infty_c(X) \otimes F$ is dense in $C^m_c(X,F)$, considered with its inductive limit topology.
For this let $f \in C^m_c(X,F)$ be given. Let $K \subseteq X$ be compact, such that the support of $f$ is contained in the interior of $K$. The trace of an arbitrary neighborhood of $f$ to $C^m_K(X,F)$ is a neighborhood in $C^m_K(X,F) \subseteq C^m(X,F)$. So it is enough to approximate $f$ in $C^m_K(X,F)$ by elements in $C^\infty_K(X) \otimes F$. By what we have shown for $C$, we can find $f_j \in C(X) \otimes F$, which converge to $f$ in $C(X,F)$. Let $h \in C^\infty_c(X,F)$ be such that $h|_{\operatorname{supp} f} = 1$ and $H := \operatorname{supp}(h)$ contained in the interior of $K$. Then $h \cdot f_j \in C_H(X) \otimes F$ converges to $h \cdot f = f$ in $C(X,F)$. In order to achieve convergence of the derivatives, we take *convolution with an approximation of unity* $\rho_\varepsilon$ (see [**2**, 4.13.6]). Since $C^m_c(X,F) \subseteq C^m_c(\mathbb{R}^n,F)$, the convolutions $\rho_\varepsilon \star (h \cdot f_j) \in C^m(\mathbb{R}^n) \otimes F$ and $\rho_\varepsilon \star f \in C^m(\mathbb{R}^n,F)$ are well-defined, $\rho_\varepsilon \star f$ converges to $f$ in $C^m(\mathbb{R}^n,F)$ for $\varepsilon \to 0$ and $\rho_\varepsilon \star (h \cdot f_j)$ converges to $\rho_\varepsilon \star f$ in $C^\infty(\mathbb{R}^n,F)$ for $j \to \infty$, since partial derivatives of a convolution can be moved to one factor (see [**2**, 4.7.6]). If we choose $\varepsilon$ so small, that $\operatorname{supp}(\rho_\varepsilon) + H \subseteq K$, then $\rho_\varepsilon \star (h \cdot f_j) \in C^\infty_K(X) \otimes F$ and $\rho_\varepsilon \star f \in C^\infty_K(X,F)$. Hence the convergence takes place in $C^m_K(X,F)$.

The proof is now finished, since for complete $F$ the space $C(X,F)$ is complete provided $X$ is compactly generated and the space $C^m(X,F)$ is complete for any open subset $X$ of $\mathbb{R}^n$. Hence the completion $C^m(X) \hat{\otimes}_\varepsilon F$ is isomorphic to the closure $C^m(X,F)$ of $C^m(X) \otimes F$ in $C^m(X,F)$. □

Note that on $C^\infty(X,F)$ the bornology discussed here is identical to that introduced in 2.46. In fact both structures are initial with respect to $\ell_* : C^\infty(X,F) \to C^\infty(X,\mathbb{R})$ for all $\ell \in F^*$ and on $C^\infty(X,\mathbb{R})$ both structures satisfy the uniform $\{\mathrm{ev}_x : x \in X\}$-boundedness principle.

**4.36 Corollary.**

$$C(X \times Y) \cong C(X)\hat{\otimes}_\varepsilon C(Y) \textit{ for locally compact } X \textit{ and } Y.$$
$$C^\infty(X \times Y) \cong C^\infty(X)\hat{\otimes}_\varepsilon C^\infty(Y) \textit{ for open } X \subseteq \mathbb{R}^n \textit{ and } Y \subseteq \mathbb{R}^m.$$

**Proof.** Under these assumptions we have the exponential law (E) and hence (P) follows from (V). □

**Remark.**
Very little about (V) and (P) is known for infinite dimensional spaces $X$ and $Y$.

The corollary fails for $C^m$ with $0 < m < \infty$. In fact we do not have an exponential law in this situation, since for every $f \in C^m(X, C^m(Y))$ the derivative $\partial_1^m \partial_2^m \hat{f}$ exists and is continuous, which is not the case for elements of $C^m(X \times Y)$. So the analogous proof will not work. Moreover from the validity of (P) we could deduce the scalar valued case of the exponential law (E) using (V):

$$\mathcal{F}(X \times Y) \overset{(\mathrm{P})}{\cong} \mathcal{F}(X)\hat{\otimes}_\varepsilon \mathcal{F}(Y) \overset{(\mathrm{V})}{\cong} \mathcal{F}(X, \mathcal{F}(Y)).$$

Note that for $C$ we can not replace the $\varepsilon$-tensor product by the $\beta$- or $\pi$-tensor product, since we have shown in 4.13 that for $X = Y = \mathbb{N}_\infty$ we don't have equality.

We will show in 6.23 that $C^\infty(X, \mathbb{R})$ is nuclear, so we may replace the $\varepsilon$-tensor product by the $\pi$-tensor product. And since both factors are Fréchet also by the $\beta$-tensor product.

**4.37 Proposition.**
*Let $F$ be complete. Then*

$$\mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^m, F)) \cong \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m, F)$$
$$\mathcal{S}(\mathbb{R}^n)\hat{\otimes}_\varepsilon F \cong \mathcal{S}(\mathbb{R}^n, F)$$
$$\mathcal{S}(\mathbb{R}^n)\hat{\otimes}_\varepsilon \mathcal{S}(\mathbb{R}^m) \cong S(\mathbb{R}^n \times \mathbb{R}^m)$$

Of course $\mathcal{S}(\mathbb{R}^n, F)$ is defined as all $f \in C^\infty(\mathbb{R}^n, F)$ for which $p \cdot \partial^\alpha f$ is globally bounded for all polynomials $p$ on $\mathbb{R}^n$ and all multi-indices $\alpha$. And we supply this space with the structure inherited from the linear mappings $f \mapsto p \cdot \partial^\alpha f$, from $\mathcal{S}(\mathbb{R}^m, F)$ into $\ell^\infty(\mathbb{R}^m, F)$, where $\mathbb{R}^m$ carries the trivial bornology. Since $\partial^\alpha(p \cdot f) = p \cdot \partial^\alpha f + \sum_{\beta > 0} \binom{\alpha}{\beta} \partial^{\alpha-\beta} p \cdot \partial^\beta f$, we can show by induction that we could use equally well the expressions $\partial^\alpha(p \cdot f)$.

**Proof.** Note that $f \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^m, F))$, iff for every polynomial $p_1$ on $\mathbb{R}^n$ and $p_2$ on $\mathbb{R}^m$ and all multi-indices $\alpha$ and $\beta$ we have that $x \mapsto (y \mapsto \partial_y^\beta(p_2(y) \cdot \partial_x^\alpha(p_1(x) \cdot f(x))(y)))$ belongs to $\ell^\infty(\mathbb{R}^n, \ell^\infty(\mathbb{R}^m, F)) \cong \ell^\infty(\mathbb{R}^n \times \mathbb{R}^m, F)$. This is equivalent to the assumption that $\partial^\gamma(p \cdot \hat{f}) \in \ell^\infty(\mathbb{R}^n \times \mathbb{R}^m; F)$ for all $\gamma$ and all polynomials $p$ on $\mathbb{R}^n \times \mathbb{R}^m$.

The rest of the proof is completely analogous to that for $C^\infty$. For the density use that $C_c^\infty(\mathbb{R}^n, F)$ is dense in $\mathcal{S}(\mathbb{R}^n, F)$ and $C_c^\infty(\mathbb{R}^n) \otimes F$ is dense in $C_c^\infty(\mathbb{R}^n, F)$ by what we have proved for $C^\infty$. □

**4.38 Theorem.**
*If $F$ is complete, then $H(X, F) \cong H(X)\hat{\otimes}_\varepsilon F$ for every open domain $X \subseteq \mathbb{C}$ and complex locally convex space $F$.*

For a proof of this result see [**14**, 16.7.5]. Here $H(X,F)$ denotes the space of all *holomorphic maps* $X \to F$ with the topology of uniform convergence on compact subsets of $X$.

**Remark.**
Let us consider $C^m_c$ next for $m = 0$ or $m = \infty$.

Note that for $X$ compact we are in the situation of 4.35. So for $m = 0$ we can neither use the bornological nor the projective tensor product. So we try again with the injective tensor product.

We try to find an embedding $C^m_c(X,F) \to L(F^*, C^m_c(X))$ as in the situations before. Since $C^m_c(X,F)$ is the inductive limit of $C^m_K(X,F)$, where $K$ runs through a basis of the compact subsets of $X$ and since $C^m_K(X,F)$ carries by definition the initial structure from the inclusion into $C^m(X,F)$, we obtain a continuous linear mapping as follows:

$$\begin{tikzcd}
C^m_c(X,F) \arrow[d, dotted, "(3)"'] & C^m_K(X,F) \arrow[l, hook'] \arrow[r, hook] \arrow[d, dotted, "(2)"'] & C^m(X,F) \arrow[d, dotted, "(1)"'] \\
L(F^*, C^m_c(X)) & L(F^*, C^m_K(X)) \arrow[l, hook'] \arrow[r, hook] & L(F^*, C^m(X)),
\end{tikzcd}$$

where (1) was given in the proof for $C^m$. The map (2) is just the restriction, which exists, since $\operatorname{supp}(f) \subseteq K$ implies that $f(x) = 0$ for all $x \notin K$ and hence also $y^*(f(x)) = 0$ for all $y^* \in F^*$, i.e. $y^* \circ f$ has support in $K$. The map (3) exists by the universal property of the inductive limit.

On the other hand we have a bounded bilinear mapping $C^m_c(X) \times F \to C^m_c(X,F)$ induced by

$$\begin{tikzcd}
C^m_c(X) \arrow[d, dotted, "(3)"'] & C^m_K(X) \arrow[l, hook'] \arrow[r, hook] \arrow[d, dotted, "(2)"'] & C^m(X) \arrow[d, dotted, "(1)"'] & C^m(X) \otimes F \arrow[d, dotted, "(0)"'] \\
L(F, C^m_c(X,F)) & L(F, C^m_K(X,F)) \arrow[l, hook'] \arrow[r, hook] & L(F, C^m(X,F)) & C^m(X,F),
\end{tikzcd}$$

where the right most mapping (0) is the embedding given in 4.35. By the same arguments as before (2) and (3) exist. The associated mapping $C^m_c(X) \hat{\otimes}_\beta F \to C^m_c(X,F)$ clearly has dense image and the composite $C^m_c(X) \hat{\otimes}_\beta F \to C^m_c(X,F) \to L(F^*, C^m_c(X))$ is the natural mapping, which has values in $C^m_c(X) \hat{\otimes}_\varepsilon F$. Thus we conclude that also $C^m_c(X,F) \to L(F^*, C^m_c(X))$ has in values in $C^m_c(X) \hat{\otimes}_\varepsilon F$.

Could this be extended to give us the desired isomorphism (V) : $C^m_c(X) \hat{\otimes}_\beta F \cong C^m_c(X,F)$? This is not the case as the example $X = \mathbb{N}$ shows, since then we have $C^m_c(X,F) = F^{(\mathbb{N})}$ and we have already seen that for the nuclear space $F = \mathbb{R}^\mathbb{N}$ there is no isomorphism $C^m_c(X) \hat{\otimes}_\varepsilon F = C^m_c(X) \hat{\otimes}_\pi F = \mathbb{R}^{(\mathbb{N})} \hat{\otimes}_\pi \mathbb{R}^\mathbb{N} \cong (\mathbb{R}^{(\mathbb{N})})^\mathbb{N} \to (\mathbb{R}^\mathbb{N})^{(\mathbb{N})} = C^m_c(X,F)$.

But we should note that for $m > 0$ we assumed $X$ to be open in some $\mathbb{R}^n$ in 4.35. So what about such a counter-example (in particular for $m = \infty$)? If $X = \mathbb{R}$, then we have a direct summand $F^{(\mathbb{N})} \subset C^\infty_c(\mathbb{R}, F)$ given by $(y_n)_n \mapsto \sum_n h(\_ - n)\, y_n$, where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ has support in $[-1, 1]$ and is equal to 1 at 0. A retraction is given by $f \mapsto (f(n))_{n\in\mathbb{N}}$. That both maps are continuous follows from the following diagram, since the restrictions to the bottom row are obviously continuous:

$$\begin{tikzcd}
F^{(\mathbb{N})} \arrow[r, dotted] & C_c^\infty(\mathbb{R}, F) \arrow[r, dotted] & F^{(\mathbb{N})} \\
F^N \arrow[r] \arrow[u, hook] & C^\infty_{[-1,N+1]}(\mathbb{R}, F) \arrow[r] \arrow[u, hook] & F^N \arrow[u, hook]
\end{tikzcd}$$

Now suppose we have some functorial topology $\tau$ on the tensor product, i.e. such that the tensor product becomes a functor with values in <u>$LCS$</u>. Then an isomorphism $C_c^\infty(X)\hat{\otimes}_\tau F \cong C_c^\infty(X, F)$ would induce an isomorphism $\mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\tau F \cong F^{(\mathbb{N})}$. Taking $F = \mathbb{R}^{\mathbb{N}}$ shows that this fails for $\tau = \pi = \varepsilon$. Note however, that for $\tau = \beta$ it is true.

What about the weaker statement (P) for $m = \infty$ (i.e. (V) for $F = C_c^\infty(Y)$). We have a quotient mapping $C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ given by $f \mapsto (f^{(n)}(0))_{n\in\mathbb{N}}$ (apply the open mapping theorem to the second map). Now suppose $C_c^\infty(\mathbb{R})\hat{\otimes}_\pi C_c^\infty(\mathbb{R}) \cong C_c^\infty(\mathbb{R}^2)$. Then we have the quotient mapping $C_c^\infty(\mathbb{R})\hat{\otimes}_\pi C_c^\infty(\mathbb{R}) \to \mathbb{R}^{(\mathbb{N})}\hat{\otimes}_\pi \mathbb{R}^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$. This should correspond to a continuous mapping on $C_c^\infty(\mathbb{R}^2)$, whose $(n, k)$-th coordinate is given by $f \mapsto \partial_2^k f(n, 0)$. In fact $f \otimes g \in C_c^\infty(\mathbb{R}) \otimes_\pi C_c^\infty(\mathbb{R})$ are mapped to $(f(n))_n \otimes (g^{(k)}(0))_k$ and further to $((f(n)\, g^{(k)})_n)_k$. The corresponding map $h \in C_c^\infty(\mathbb{R}^2)$ is given by $h(x, y) = f(x)\, g(y)$ and hence $(f(n)\, g^{(k)}(0)) = \partial_2^k h(n, 0)$. Hence the linear mapping $C_c^\infty(\mathbb{R}^2) \to (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ has values in the strict subset $(\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$, a contradiction. Since $C_c^\infty(X)$ is nuclear this shows at the same time that the result fails also for $\varepsilon$.

However, let us show now, that (P) is true for $C_c^\infty$ with respect to the bornological tensor product:

**4.39 Proposition.**
*Let $X$ and $Y$ be open in finite dimensional spaces. Then*

$$C_c^\infty(X)\hat{\otimes}_\beta C_c^\infty(Y) \cong C_c^\infty(X \times Y).$$

**Proof.** Since $C_c^\infty(Z)$ is the inductive limit of $C_K^\infty(Z) \subseteq C^\infty(Z)$, where $K$ runs through a basis of the compact subsets of $Z$, and since the bornological tensor product preserves inductive limits it is enough to show that $C_A^\infty(X)\hat{\otimes}_\beta C_B^\infty(Y) \cong C_{A\times B}^\infty(X\times Y)$ for all compact subsets $A \subseteq X$ and $B \subseteq Y$. Since $C^\infty(Z)$ are nuclear Fréchet spaces, we have from what we have shown above $C^\infty(X)\hat{\otimes}_\beta C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\pi C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\varepsilon C^\infty(Y) \cong C^\infty(X \times Y)$. So the natural mapping $C_A^\infty(X) \otimes_\varepsilon C_B^\infty(Y) = C_A^\infty(X) \otimes_\beta C_B^\infty(Y) \to C_{A\times B}^\infty(X \times Y)$ is as restriction initial as well. Remains to show denseness of $C_A^\infty(X) \otimes C_B^\infty(Y)$ in $C_{A\times B}^\infty(X\times Y)$. For this we first show that $\bigcup_n C_{K_n}^\infty(Z)$ is dense in $C_K^\infty(Z)$ provided $K_n$ are compact subsets of $K$ such that their interiors cover $K$.
In fact, let $f \in C_K^\infty(Z)$. Then for all $n$ and $m$ we have that $f^{(n)}(z)$ is a $O(d(z, Z \setminus K)^m)$ for $z \to \partial K$. By [**21**, p.77] we may choose an $h \in C_K^\infty(Z)$ with $h = 1$ on $K_\varepsilon := \{z \in K : d(z, Z \setminus K) \geq \varepsilon\}$ and $h^{(n)}(z)$ is a $O(d(z, Z \setminus K)^{-n})$. Thus $(f \cdot h)^{(n)}(z) = O(d(z, Z))$, and hence is smooth on $Z$.

Note however, that it is even enough to embed $C_{A\times B}^\infty(X \times Y)$ into some space $C_{A'}^\infty(X)\hat{\otimes}_\varepsilon C_{B'}^\infty(Y)$, which is much easier to obtain. □

## Kernel Theorems

**4.40.** We take up the discussion about the appropriate version of the *matrix-representation* of linear-operators

$$L(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{n\cdot m}.$$

We should replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by more general (function) spaces $E$ and $F$. So we have to rewrite the right hand side in terms of $\mathbb{R}^n$ and $\mathbb{R}^m$, i.e.

$$\mathbb{R}^{n\cdot m} \cong \mathbb{R}^n \otimes \mathbb{R}^m.$$

Note that the left side is a functor on $\underline{VS}^{op} \times \underline{VS}$ and the right side on $\underline{VS} \times \underline{VS}$, so we have to dualize $E$ on one side.

The simplest generalization seems to be from $\mathbb{R}^n = \mathbb{R}^{\{0,\dots,n-1\}}$ to $\mathbb{R}^X$ with arbitrary $X$. For the projective or injective tensor product we have $\mathbb{R}^X \hat{\otimes}_\pi \mathbb{R}^Y \cong (\mathbb{R}^Y)^X \cong L(\mathbb{R}^{(X)}, \mathbb{R}^Y)$. Recall that $\mathbb{R}^{(X)} = (\mathbb{R}^X)^*$. Hence we have $E \hat{\otimes}_\pi F \cong L(E^*, F)$, where $E = \mathbb{R}^X$ and $F = \mathbb{R}^Y$. But we can read this also as an isomorphism for $X \mapsto \mathbb{R}^{(X)}$, since $(\mathbb{R}^{(X)})' \cong \mathbb{R}^X$: $E' \hat{\otimes}_\pi F' \cong L(E, F')$, where $E = \mathbb{R}^{(X)}$ and $F = \mathbb{R}^{(Y)}$. Which seems more appropriate, since for function spaces $E$ often $E \subseteq E'$ (e.g. $C_c^\infty \subseteq (C_c^\infty)'$, $\ell^1 \subseteq \ell^\infty$), and we are mainly interested in the operators $L(E, E)$.

Let us consider the corresponding result for $\mathcal{L}^p$ with $1 < p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (and hence $1 \le q < \infty$):

$$\mathcal{L}^p(X \times Y) \cong \mathcal{L}^p(X, \mathcal{L}^p(Y)) \to L(\mathcal{L}^q(X), \mathcal{L}^p(Y)),$$

which is an isomorphism for $p = \infty$ and discrete $X$, but not otherwise, since the image consists of compact operators only, cf. [**2**, 6.4.8], where we proved this result for $p = q = 2$. Note that this mapping is given for discrete $X$ and $Y$ by $(k_{x,y})_{x,y} \mapsto \Big((f_x)_x \mapsto (\sum_{x\in X} k_{x,y} f_x\Big)$. Or, in general, $k$ is mapped to the *integral operator* $K : f \mapsto (y \mapsto \int_X k(x,y)\, f(x)\, dx)$. So the question of surjectivity amounts to finding an *integral kernel* $k$ for an operator $K$.

The existence of such a mapping for discrete $X$ and $Y$ can also be seen directly, since by Schwarz's Inequality we have

$$\begin{aligned}\|Kf\|_p^p &= \sum_y |\sum_x k(x,y) f(x)|^p = \sum_y |\langle k(\_, y), f\rangle|^p \\ &\le \sum_y \|k(\_, y)\|_p^p \cdot \|f\|_q^p = \sum_y \sum_x |k(x,y)|^p \cdot \|f\|_q^p.\end{aligned}$$

**4.41 Remark.**
We have shown in [**2**, 6.4.8] that $C(I \times I) \to L(C(I), C(I))$ is a well defined map with values in the compact operators for every interval $I$.

Furthermore we have the mapping

$$\mathcal{L}^p(X \times Y) \to L(\mathcal{L}^q(X), \mathcal{L}^p(Y)).$$

So one wants to extend this to some surjective mapping on some function space $\mathcal{F}(X \times Y)$. I.e. for every $K \in L(\mathcal{L}^q(X), \mathcal{L}^p(Y))$ there should be some "kernel" $k \in \mathcal{F}(X \times Y)$. This problem is unsolvable for functions. In fact take $p = 2$ and $X = Y = \mathbb{R}$ with the Lebesgue-measure. The kernel of the identity would be the Dirac delta function. Due to [**19**, 1966] it can be worked out for distributions. For this we rewrite the above mapping into

$$\mathcal{L}^q(X \times Y)' \to L(\mathcal{L}^q(X), \mathcal{L}^q(Y)').$$

This problem has shown to be of importance, in fact we constructed to every partial differential operator $D$ with constant coefficients, and more generally to every continuous linear operator $D : C_c^\infty(\mathbb{R}^n) =: \mathcal{D} \to E := C^\infty(\mathbb{R}^n)$, which commutes with translations an integral-kernel $k \in \mathcal{D}'$, such that $D$ is given by convolution with $k$, in [**2**, 4.13.5]. Moreover we found a solution operator of such equations as integral-operator with a distributional kernel $\varepsilon$ in [**2**, 8.3.1]

Now, how could we show such an isomorphism:

$$\begin{tikzcd}
& L(\mathcal{F}(X),\mathcal{F}(Y)') & & L(\mathcal{F}(X),\mathcal{F}(Y);\mathbb{R}) \arrow[ll, "1_L", "\cong"'] & \\
\mathcal{F}(X)'\hat{\otimes}_? \mathcal{F}(Y)' \arrow[ur, "2_L"] \arrow[rrrr, "3_L"] \arrow[drr, "3'_{\mathcal{F}}"'] & & & & (\mathcal{F}(X)\hat{\otimes}_\beta \mathcal{F}(Y))' \arrow[ul, "1_L"', "\cong"] \\
& & \mathcal{F}(X\times Y)' \arrow[urr, "3_{\mathcal{F}}"'] & &
\end{tikzcd}$$

**Remark.**
If one writes the action of a *distribution* $T$ on a *test-function* $f$ formally as an integral $Tf = \int_X T(x)\, f(x)\, dx$, then the mapping $\mathcal{D}(X \times Y)' \to L(\mathcal{D}(X), \mathcal{D}(Y)')$ is given by $k \mapsto (f \mapsto (g \mapsto \int_{X\times Y} k(x,y)\, f(x)\, g(y)\, d(x,y)))$. Conversely we now know that every continuous linear operator $K : D(X) \to \mathcal{D}(Y)'$ is of this form, i.e. has an distributional kernel $k \in \mathcal{D}(X \times Y)$. This is a strong generalization of the matrix representation of finite dimensional operators.

We will show that $C^\infty(X)$, $\mathcal{S}(\mathbb{R}^p)$, $H(X)$ and $\mathcal{D}(X)$ are nuclear spaces and all except the last one are Fréchet. Hence

**4.42 Corollary.** (P) **for several function spaces.**

$$\begin{aligned}
C^\infty(X)\hat{\otimes}_\varepsilon C^\infty(Y) &\cong C^\infty(X)\hat{\otimes}_\pi C^\infty(Y) \cong C^\infty(X)\hat{\otimes}_\beta C^\infty(Y) \cong C^\infty(X\times Y)\\
\mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\varepsilon \mathcal{S}(\mathbb{R}^q) &\cong \mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\pi \mathcal{S}(\mathbb{R}^q) \cong \mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\beta \mathcal{S}(\mathbb{R}^q) \cong \mathcal{S}(\mathbb{R}^p\times\mathbb{R}^q)\\
H(X)\hat{\otimes}_\varepsilon H(Y) &\cong H(X)\hat{\otimes}_\pi H(Y) \cong H(X)\hat{\otimes}_\beta H(Y) \cong H(X\times Y)\\
\mathcal{D}(X)\hat{\otimes}_\varepsilon \mathcal{D}(Y) &\cong \mathcal{D}(X)\hat{\otimes}_\pi \mathcal{D}(Y) \overset{!!!}{\ncong} \mathcal{D}(X)\hat{\otimes}_\beta \mathcal{D}(Y) \cong \mathcal{D}(X\times Y) \qquad \square
\end{aligned}$$

Combined with (E) for $L$ this gives:

**4.43 Theorem, Schwartz kernel theorem.**
*We have*

$$\begin{aligned}
C^\infty(X\times Y)' &\cong L(C^\infty(X), C^\infty(Y)')\\
\mathcal{S}(\mathbb{R}^p\times\mathbb{R}^q)' &\cong L(\mathcal{S}(\mathbb{R}^p), \mathcal{S}(\mathbb{R}^q)')\\
H(X\times Y)' &\cong L(H(X), H(Y)')\\
\mathcal{D}(X\times Y)' &\cong L(\mathcal{D}(X), \mathcal{D}(Y)')
\end{aligned}$$

**Proof.** For $\mathcal{D}$ we have to proceed differently:

$$\mathcal{D}(X\times Y)' \overset{4.39}{\cong} (\mathcal{D}(X)\hat{\otimes}_\beta \mathcal{D}(Y))' \cong L(\mathcal{D}(X), \mathcal{D}(Y); \mathbb{R}) \cong L(\mathcal{D}(X), \mathcal{D}(Y)'). \quad \square$$

Little is know about the validity of desired isomorphisms for $C^\infty$ and $\lambda$ in the infinite dimensional case. See [**16**] for partial results in the case of $C^\infty$.

## The Approximation Property

We turn now towards the question of density of the image of $E^* \otimes F$ in $\mathcal{L}(E,F)$.

**4.44 Theorem. Density of finite dimensional operators.**
*Let $E$ be a locally convex space and $\mathcal{B}$ be a bornology on $E$. And we consider on all function spaces $\mathcal{L}$ the uniform convergence on sets in $\mathcal{B}$, and hence denote them by $\mathcal{L}_{\mathcal{B}}$. Then the following statements are equivalent:*

1. *$E^* \otimes F$ is dense in $\mathcal{L}_{\mathcal{B}}(E,F)$ for every locally convex space $F$;*
2. *$E^* \otimes F$ is dense in $\mathcal{L}_{\mathcal{B}}(E,F)$ for every Banach space $F$;*
3. *$E^* \otimes E$ is dense in $\mathcal{L}_{\mathcal{B}}(E,E)$;*
4. *$\mathrm{id}_E$ is a limit in $\mathcal{L}_{\mathcal{B}}(E,E)$ of a net in $E^* \otimes E$.*

**Proof.** (1⇒2) is trivial.
(2⇒1) A typical 0-neighborhood in $\mathcal{L}_{\mathcal{B}}(E,F)$ is given by $N_{B,V}$ with $B \in \mathcal{B}$ and $V$ a 0-neighborhood in $F$. Let $p_V : F \to F_{(V)}$ be the canonical surjection. Since $F_{(V)}$ is a normed space $p_V \circ T : E \to F \to F_{(V)} \hookrightarrow \hat{F}_{(V)}$ can be uniformly approximated with respect to $p : F_{(V)} \to \mathbb{R}$ on $B$ by finite operators $E \to \hat{F}_{(V)}$ by (2). Since $F_{(V)}$ is dense with respect to $p$ in $\hat{F}_{(V)}$ we may assume that the finite operators belong to $\mathcal{L}(E, F_{(V)})$. Taking inverse images of the vector components, we may even assume that they belong to $\mathcal{L}(E,F)$.
(1⇒3) and (3⇒4) are trivial.
(4⇒1) Let $T_i$ be a net of finite operators converging to $\mathrm{id}_E$, then the net $T \circ T_i$ of finite operators converges to $T \circ \mathrm{id} = T$. □

Let $E$ be complete and assume that the equivalent statements are true for some bornology $\mathcal{B}$. And let $B \in \mathcal{B}$ w.l.o.g. be absolutely convex. Since the identity on $E$ can be approximated uniformly on $B$ by finite operators, we conclude that the inclusion $E_B \to E$ can be approximated by finite operators $E_B \to E$ uniformly on the unit ball of $E_B$. Hence it has to have relatively compact image on the unit ball, i.e. $B$ has to be relatively compact:

In fact we have

**4.45 Lemma.**
*The compact operators $\mathcal{K}(E,F)$ from a normed space $E$ into a complete space $F$ are closed in $\mathcal{L}(E,F)$.*

**Proof.** To see this use that $F = \varprojlim_V \hat{F}_{(V)}$, hence a subset $K$ of $F$ is relatively compact iff $p_V(K)$ is relatively compact in $F_{(V)}$ for all $V$. Now let $T_i \in \mathcal{K}(E,F)$ converge to $T \in \mathcal{L}(E,F) = L(E,F)$. Then the $p_V \circ T_i \in \mathcal{K}(E, \hat{F}_{(V)})$ converge to $p_V \circ T$ in $L(E, \hat{F}_{(V)})$. Since $\hat{F}_{(V)}$ is a Banach spaces it can be shown as in [**2**, 6.4.8] that $p_V \circ T \in \mathcal{K}(E, \hat{F}_{(V)})$. Hence $p_V(T(oE))$ is relatively compact in $\hat{F}_{(V)}$ and thus $T(oE)$ is relatively compact in $F$. □

**4.46 Definition.**
We hence say that a complete space satisfies the *approximation property* if the equivalent statements from above are true for the bornology $B = \mathrm{cp}$ of all relatively compact subsets of $E$. A non-complete space $E$ is said to have the *approximation property*, iff $\hat{E}$ has it. Note that the finite dimensional operators may be taken in $\mathcal{L}(E,E)$ in this situation.

A space $E$ is said to have the *bornological approximation property*, iff $E^* \otimes F$ is dense in $L(E,F)$ with respect to the bornological topology, which is at least as fine as the topology of uniform convergence on bounded sets. So a necessary condition is that all bounded sets are relatively compact. A space with that property is called *semi-Montel* space. It is called *Montel*, iff it is in addition barreled. It is a classical result of P. Montel that every bounded sequence of holomorphic maps has a convergent subsequence, i.e. $H(X)$ is Montel for every domain $X \subseteq \mathbb{C}$. By Tychonoff's theorem $\mathbb{R}^X$ is semi-Montel for every $X$.

## Reflexivity and Montel Spaces

Recall that a space is called *semi-reflexive*, iff the natural mapping $E \to (E^*)^*$ is onto, where $E^*$ is considered with the strong topology. A space is called *reflexive*, iff the natural mapping $E \to (E^*)^*$ is an isomorphism for the strong topology on $(E^*)^*$. This is exactly the case when $E$ is semi-reflexive and (infra-)barreled, since a space $E$ is (quasi-)barreled iff each pointwise (uniformly) bounded set in $\mathcal{L}(E, F)$ is equi-continuous, see [**2**, 5.2.2], see also [**14**, 11.2.2].

One has the following

**4.47 Lemma.**
*The strong dual of a semi-reflexive space is barreled.*

**Proof.** See [**22**, 373]. Let $E^*_\beta$ denote the strong dual of $E$ and let $B$ be a barrel in $E^*_\beta$. Since $E$ is semi-reflexive the strong topology is compatible with the duality, and hence [**2**, 7.4.8] $B$ is also closed for the weak-topology $\sigma(E^*, E)$. We show that $B^o$ is a bounded subset of $E$ (which implies that $B = B^{oo}$ is a 0-neighborhood in $E^*_\beta$). For this it is enough to show that $B^o$ is bounded in $\sigma(E, E^*)$. But since $B$ is assumed to be absorbing, we find for every $x^* \in E^*$ a $c > 0$ with $cx^* \in B$. Thus $|cx^*(B^o)| \leq 1$. □

**4.48 Proposition. Semi-reflexivity.**
*The following statements are equivalent:*

1. *$E$ is semi-reflexive;*
2. *Every closed bounded set is $\sigma(E, E^*)$-compact;*
3. *$E$ is quasi-complete with respect to $\sigma(E, E^*)$.*

**Proof.** (1⇒2) Since $E$ is as a vector space the dual of the barreled space $E^*_\beta$ by the previous lemma it follows that every $\sigma(E, E^*)$-bounded set is equi-continuous and hence relatively compact for the topology $\sigma(E, E^*)$.

(2⇒3) Since every compact space is complete for any compatible uniformity this is obvious.

(3⇒1) We only have to show that the strong topology is compatible with the duality $\langle E^*, E\rangle$. By [**2**, 7.4.15] we must show that this topology coincides with the topology of uniform convergence on weakly compact sets. But since all bounded sets are weakly relatively-compact this is obvious. □

**4.49 Proposition.** (U1) **for $C$.**
*Let $X$ be compactly generated and $E$ be semi-Montel and $f : X \to E$ be scalarly continuous. Then $f$ is continuous.*

**Proof.** Since $X$ is compactly generated, it is enough to show that $f|_K : K \to E$ is continuous for every compact subset $K \subseteq X$. So let $x_i \to x$ be a convergent net in $K$. Then $f(x_i) \to f(x)$ with respect to the weak-topology, and since $f(K)$ is scalarly bounded, it is bounded, and hence is relatively compact. But on compact sets the weak and the given topology obviously coincide. So $f(x_i) \to f(x)$ in $E$. □

We have the implications:

**4.50 Proposition.**
*semi-Montel ⇒ semi-reflexive ⇒ quasi-complete ⇒ sequentially complete ⇒ $c^\infty$-complete.*

**Proof.** (semi-Montel ⇒ semi-reflexive) Since every closed bounded set is relatively compact it is also relatively $\sigma(E, E^*)$-compact and hence $E$ is semi-reflexive.

(semi-reflexive ⇒ quasi-complete) Since $\sigma(E, E^*)$ is quasi-complete by 4.48 it follows from [**14**, 3.2.4] that every bounded set is complete.

The other implications are clear. □

One has the following stability properties:

**4.51 Proposition. Stability of reflexive and of Montel spaces.**
*Semi-reflexive and semi-Montel spaces are closed with respect to products, closed subspaces, direct sums, reduced regular inductive limits. Strong duals of reflexive and of Montel spaces are of the same type.*

## 4.52. Definition

A locally convex vector space $E$ is called *bornologically-reflexive* if the canonical embedding $\delta : E \to E''$ is surjective.

It is then even a bornological isomorphism, since $\delta$ is always a bornological embedding. Note that reflexivity as defined here is a bornological concept.

Note also that this notion is in general stronger than the usual locally convex notion of reflexivity, since the continuous functionals on the strong dual are bounded functionals on $E'$ but not conversely.

**4.53. Theorem. Bornological reflexivity.**
*For a bornological locally convex space $E$ the following statements are equivalent.*

1. *$E$ is bornologically-reflexive.*
2. *$E$ is reflexive and the strong dual of $E$ is bornological.*
3. *$E$ is $\eta$-reflexive (see* [**14**, p280]*).*
4. *$E$ is completely reflexive (see* [**12**, 1977, p89]*).*
5. *The Schwartzification (or nuclearification) of $E$ is a complete locally convex space.*

**Proof.** See [**10**, 5.4.6]. □

**4.54. Corollary. Bornological reflexivity versus reflexivity.**

1. *A Fréchet space is b-reflexive if and only if it is reflexive.*
2. *A convenient vector space with a countable base for its bornology is b-reflexive if and only if its bornological topology is reflexive.*

**Proof.** See [**10**, 5.4.7]. □

**4.55. Proposition. Duals of bornologically reflexive spaces.**
*A locally convex vector space is b-reflexive if and only if its bornological topology is complete and its dual is b-reflexive.*

**Proof.** See [**10**, 5.4.9]. □

**4.56. Lemma. Subspaces of bornologically reflexive spaces.**
*A closed linear subspace of a b-reflexive bornological locally convex vector space is b-reflexive. Products and coproducts of b-reflexive convenient vector spaces are b-reflexive, if the index set is of non-measurable cardinality.*

**Proof.** See [**10**, 5.4.8] and [**10**, 5.4.11] □

**4.57. Theorem. Reflexivity of function spaces.**
*If $E$ is a b-reflexive convenient vector space and $M$ is a finite dimensional separable smooth manifold then $C^\infty(M,E)$ is b-reflexive.*

**Proof.** See [**10**, 5.4.13]. □

**4.59 Proposition. Stability of the approximation property.**
*The approximation property is preserved by products, complemented subspaces, reduced projective limits, direct sums, strict inductive limits of sequences of complete spaces and injective tensor products,*

**Proof.** (Products) Every compact set $K \subseteq E = \prod_j E_j$ is contained in one of the form $\prod_j K_j$ with $K_j \subseteq E_j$ compact. A 0-neighborhood $U \subseteq E$ can be assumed to be of the form $\prod_j U_j$, with 0-neighborhoods $U_j \subseteq E_j$ and $U_j = E_j$ for all but finitely many $j$. For those finitely many $j$, we may find finite operators $T_j \in \mathcal{L}(E_j, E_j)$ with $(\mathrm{id}_{E_j} - T_j)(K_j) \subseteq U_j$. Then $T := \sum_j T_j \circ \mathrm{pr}_j \in \mathcal{L}(E,E)$ is finite dimensional with $(\mathrm{id}_E - T)(K) \subseteq U$.

Note that if $\mathcal{U}$ is a 0-neighborhood basis of closed absolutely convex sets, such that $E_{(U)}$ have the approximation property, then $E$ has it, see [**14**, 18.2.2]. In fact we may assume that $E$ is complete. Then $E$ is the reduced projective limit of $\hat{E}_{(U)}$, and hence has the approximation property.

(Complemented subspaces) Let $E \subset F$ be a subspace admitting a continuous projection $p : F \to E$. Taking the completion, we may assume that $E$ and $F$ are complete. Let $K \subseteq E$ be compact and $U$ a 0-neighborhood of $E$. Then there is a finite operator $T \in \mathcal{L}(F,F)$ with $(\mathrm{id}_F - T)(K) \subseteq p^{-1}(U)$. Then $(\mathrm{id}_E - (p \circ T)|_E)(K) \subseteq p(p^{-1}U) \subseteq U$.

(Projective limits) Let $E$ be a reduced projective limit of $E_j$, we may assume that all spaces $E_j$ and $E$ are complete. Let $K \subseteq E$ be compact and $U$ a 0-neighborhood in $E$. Since the limit is projective, we may assume that it is of the form $\mathrm{pr}_k^{-1}(U_k)$ for some $k$ and 0-neighborhood $U_k$ in $E_k$. Since the limit is reduced, $F_k := \mathrm{pr}_k(E)$ is dense in $E_k$ and hence has the approximation property. In particular there exists a finite operator $T \in \mathcal{L}(F_k, F_k)$ such that $(\mathrm{id}_{F_k} - T)(\mathrm{pr}_k(K)) \subseteq U_k$. We may assume that $T$ is of the form $T = \sum_j y_j^* \otimes \mathrm{pr}_k(x_j)$. Then $\tilde{T} := \sum_j (y_j^* \circ \mathrm{pr}_k) \otimes x_j$ is a finite operator in $\mathcal{L}(E,E)$, which satisfies $(\mathrm{id}_E - \tilde{T})(K) \subseteq U = \mathrm{pr}_k^{-1}(U_k)$.

(Inductive limits) By [**2**, 4.8.1] we know that such a limit is regular, and hence in particular every compact set $K$ is contained and compact in some step $E_k$. Let $U$ be a 0-neighborhood. Then we can find finite operators $T = \sum_j x_j^* \otimes x_j \in \mathcal{L}(E_k, E_k)$, such that $(\mathrm{id}_{E_k} - T)(K) \subseteq U$. Since $E_k$ is a subspace of $E$ we may assume that $x_j^* \in E^*$, hence $T \in \mathcal{L}(E, E)$.

(Direct sums) Let $E = \coprod_j E_j$. Then $\hat{E}$ is the direct sum of the $\hat{E}_j$, so we may assume that $E_j$ is complete. Every compact subset of $E$ is contained in some finite subsum. Since $E$ is the strict inductive limit of the finite subsums and being products these have the approximation property, we may proceed as before to conclude that $E$ has it.

(Injective tensor product) See [**14**, 18.2.8], this uses the associativity of the $\varepsilon$-product to be discussed later, see 4.68-4.71. □

**4.60 Lemma. Topology on equicontinuous sets.**
*On equi-continuous subsets of $\mathcal{L}(E, F)$ the topology $\tau_{pc}$ of uniform convergence on precompact subsets of $E$ and the topology of pointwise convergence coincide.*

**Proof.** Let $H \subseteq \mathcal{L}(E, F)$ be equi-continuous and $T \in H$. Let $K \subseteq E$ be precompact and $V \subseteq F$ an absolutely convex 0-neighborhood of $F$. Since $H$ is equi-continuous, there exists a 0-neighborhood $U \subseteq E$ with $S(U) \subseteq \frac{1}{2}V$ for all $S \in H$. Since $K$ is precompact there is some finite subset $M \subseteq E$ such that $K \subseteq M + \frac{1}{2}U$. If $S \in H \cap (T + N_{2M,V})$, then $Su \in \frac{1}{2}V$ for all $u \in U$ and $(S-T)(x) \subseteq \frac{1}{2}V$ for all $x \in M$. Thus for all $k = x + \frac{1}{2}u \in K$ we have $(S-T)(k) = (S-T)(x) + \frac{1}{2}S(u) - \frac{1}{2}T(u) \in \frac{1}{2}V + \frac{1}{4}V + \frac{1}{4}V = V$, i.e. $S \in T + N_{K,V}$. □

**4.61 Alaoğlu-Bourbaki Theorem.**
*Every equi-continuous set is relatively compact for the topology $\tau_{pc}$ of uniform convergence on precompact sets.*

**Proof.** By 4.60 we only have to show that it is relatively compact for the topology $\sigma(E^*, E)$. Since $(E^*, \sigma(E^*, E)$ embeds into $\mathbb{R}^E$, and equi-continuous sets are pointwise bounded (see [**2**, 5.2.2]), it is bounded in $\mathbb{R}^E$ as well as its closure and hence is relatively compact there by Tychonoff's theorem. However the closure in $\mathbb{R}^E$ of an equi-continuous set is easily seen to be contained in $E^*$. □

**4.62 Examples with the approximation property.**
*The following spaces have the approximation property:*

1. *every complete space with an equi-continuous basis;*
2. *$c_0$ and $\ell^p$ for $1 \le p < \infty$;*
3. *every Hilbert space;*
4. *$\mathcal{L}^p(X, \mathcal{A}, \mu)$ for $1 \le p \le \infty$;*
5. *$C(X)$ for completely regular $X$;*
6. *$C^k(X)$ for open subsets $X$ of finite dimensional spaces.*

**Proof.** (1) A space $E$ is said to have an *equi-continuous basis*, if there are points $x_k \in E$ such that every $x$ admits a unique representation $x = \sum_k \lambda_k x_k$ and the family of expansion operators $P_k : x \mapsto \sum_{j \le k} \lambda_k x_k$ is equi-continuous. Note that $P_k$ is finite dimensional and $P_k \to \mathrm{id}_E$ pointwise. By 4.60 this equi-continuous family converges uniformly on precompact subsets, i.e. on relatively compact subsets since $E$ is complete and hence the compact subsets are exactly the closed precompact ones.

(2) It is straight forward to show that the standard unit-vectors $e_k$ form an equi-continuous base.

(3) Let $x_i$ be an orthonormal basis in a Hilbert space. Then the projection operators $P_F(x) := \sum_{i\in F}\langle x, x_i\rangle x_i$ for finite $F$ converge pointwise to the identity and are equi-continuous. Hence by 4.60 $E$ has the approximation property.

(4) We skip the proof of this, see [**14**, p411].

(5) Since the completion $\widehat{C(X)}$ of $C(X)$ is the reduced projective limit of the spaces $C(K)$, with $K \subseteq X$ compact (use that $C(X) \to C(K)$ is onto for compact subsets $K \subseteq X$). It suffices to show that $C(X)$ has the approximation property for compact $X$. Let $\varepsilon > 0$ and let $K \subseteq C(X)$ be compact, thus by Arzela-Ascoli-theorem [**2**, 6.4.4] $K$ is equi-continuous. Thus we can find a finite cover of $X$ by open neighborhoods $U_j \subseteq X$ of some $x_j \in X$ such that $|f(x) - f(x_j)| \le \varepsilon$ for all $x \in U_j$ and $f \in K$. Let $h_i$ be a subordinated partition of unity and set $T(f) := \sum_j f(x_j)\, h_j$. We claim that $(\mathrm{id}_E - T)(K) \subseteq U := \{f : \|f\|_\infty \le \varepsilon\}$. For $f \in K$ and $x \in X$ we have

$$\begin{aligned}|f(x) - T(f)(x)| &\le \sum_j |f(x) - f(x_j)|\, h_j(x) = \sum_{x\in\mathrm{supp}(h_j)\subseteq U_j} |f(x) - f(x_j)|\, h_j(x)\\ &\le \sup\{|f(x) - f(x_j)| : x \in U_j\} \le \varepsilon.\end{aligned}$$

(6) This can be proved analogously to (5) using smooth partitions of unity. For $k = \infty$ we will give another proof in 6.23 together with 6.19. □

**4.63 Remark.**
For a long time it was unclear if there are spaces without the approximation property at all. It was known that, if such a Banach space exists, then there has to be a subspace of $c_0$ failing this property. It was [**6**] who found a subspace of $c_0$ without this property. In [**20**] it was shown that $L(\ell^2, \ell^2) \cong L(\ell^2, (\ell^2)^*) \cong (\ell^2 \hat{\otimes}_\pi \ell^2)^*$ doesn't have the approximation property. Note also, that $\ell^2 \otimes_\pi \ell^2$ has the approximation property, since by [**14**, 18.2.9] every completed projective tensor product of Fréchet spaces with the approximation property has it. Note however that for Banach spaces one can show that if $E^*$ has the approximation property then so does $E$, see [**14**, 18.3.5]. Due to [**12**] is the existence of a Fréchet-Montel space without the approximation property, see [**14**, p416].

We try to identify $E\hat{\otimes}_\varepsilon F$ as subspace of $L(E^*, F)$, and hence in particular, for $F = \mathbb{R}$, we try to find $E\hat{\otimes}_\varepsilon\mathbb{R} = \hat{E}$ in $L(E^*, \mathbb{R})$.

**4.64 Grothendieck's completeness criterion.**
*The completion of $E$ can be identified with $\mathcal{L}_{equi}(E^*_\gamma, \mathbb{R})$, where $E^*_\gamma$ carries the finest locally convex topology which coincides with the weak topology on equi-continuous sets.*

**Proof.** We note that the embedding $\delta : E \to (E^*)'$ factors over $L(E^*_\gamma, \mathbb{R}) \subseteq (E^*)'$, since $\delta(x)$ is obviously continuous for $\sigma(E^*, E)$. Furthermore $\mathcal{L}(E^*_\gamma, \mathbb{R})$ is clearly closed in the complete space $(E^*)'$. So it remains to show that $E$ is dense in $\mathcal{L}(E^*_\gamma, \mathbb{R})$. For this we use the following lemma. So let $\ell \in \mathcal{L}(E^*_\gamma, \mathbb{R})$ be given and a typical 0-neighborhood, which is of the form $A^o$ with equi-continuous $A$. Since $\ell|_A$ is by assumption continuous with respect to $\sigma(E^*, E)$ we may apply 4.65 to obtain a $x \in (E^*, \sigma(E^*, E))^* = E$ with $|(x-\ell)(x^*)| \le 1$ for all $x^* \in A$. Hence $x-\ell \in A^o$. □

**4.65 Lemma.**
*Let $A \subseteq E$ be absolutely convex and $\ell : E \to \mathbb{R}$ be linear. Then $\ell|_A$ is continuous iff for every $\varepsilon > 0$ there exists an $x^* \in E^*$ with $|(x^* - \ell)(x)| \le \varepsilon$ for all $x \in A$.*

**Proof.** ($\Leftarrow$) is clear. ($\Rightarrow$) Since $\ell|_A$ is continuous there exists a 0-neighborhood $U$ such that $|\ell(x)| \le \varepsilon$ for all $x \in U \cap A$. Let $q_U$ and $q_A$ be the Minkowski-functionals of $U$ and $A$. Then $\ell \le \varepsilon(q_U + q_A)$ on $E_A$. Define

$$p(x) := \inf\{\varepsilon\, q_U(x-y) + \varepsilon\, q_A(y) + \ell(y) : y \in E_A\}.$$

Then $p$ is well-defined, since for all $(x,y) \in E \times E_A$

$$\begin{aligned} -\varepsilon\, q_U(x) &\le \varepsilon\, q_U(-y) + \varepsilon\, q_A(-y) - \ell(-y) - \varepsilon\, q_U(x) \\ &= \varepsilon\, q_U(y) + \varepsilon\, q_A(y) + \ell(y) - \varepsilon\, q_U(x) \\ &\le \varepsilon\, q_U(x-y) + \varepsilon\, q_A(y) + \ell(y). \end{aligned}$$

Since $p$ is sublinear there exists by [**2**, 7.1.1] a linear $x^* : E \to \mathbb{R}$ with $x^* \le p$. From $p(x) \le \varepsilon\, q_U(x)$ for all $x \in E$ it follows that $x^* \in E^*$. And from $p(y) \le \ell(y) + \varepsilon\, q_A(y)$ for all $y \in E_A$ we conclude that $(x^* - \ell)(y) \le \varepsilon$ for all $y \in A$. Thus $(\ell - x^*)(y) = (x^* - \ell)(-y) \le \varepsilon$ for all $y \in A$.
In the complex case use that $\mathcal{L}_{\mathbb{C}}(E,\mathbb{C}) \cong \mathcal{L}_{\mathbb{R}}(E,\mathbb{R})$, see [**2**, 6.1.5.2]. □

**4.66 Corollary.**
*We have* $E^*_\gamma =: \gamma(E^*, E) = \tau_c(E^*, \hat{E}) := \mathcal{L}_{cp}(\hat{E}, \mathbb{R})$.

**Proof.** First note that $\gamma(E^*, E) = \gamma(E^*, \hat{E})$. In fact, since the closures $\hat{U}$ in $\hat{E}$ of the 0-neighborhoods $U$ in $E$ form a 0-neighborhood basis of $\hat{E}$, the equi-continuous families on $E$ and on $\hat{E}$ coincide. Furthermore the topologies $\sigma(E^*, E)$ and $\sigma(E^*, \hat{E})$ coincide on equi-continuous subsets. Thus it is enough to prove the result for complete spaces $E$.

($\gamma \supseteq \tau_c$) Let us show first that $\gamma$ is finer than $\tau_c$. For this we only have to show that the inclusion from equi-continuous sets with the weak topology $\sigma(E^*, E)$ into $\tau_c(E^*, E) = \tau_{pc}(E^*, E)$ is continuous, which follows directly from 4.60.

($\tau_c \supseteq \gamma$) Conversely let $U$ be a closed 0-neighborhood for $\gamma$. Since by 4.64 $\gamma$ is compatible with the duality $(E^*, E)$ we have that $U^o$ is compact for the topology of uniform convergence on $\gamma$-precompact sets in $E^*$. Since every closed equi-continuous set for the original topology is by definition of $\gamma$ and because of 4.61 compact with respect to $\gamma$, we have that $U^o$ is also compact for this weaker topology of uniform convergence on equi-continuous subsets. But this is just the given topology on $E$, so $U^o$ is compact, and hence $U = U^{oo}$ is a 0-neighborhood for the topology $\tau_c$. □

**4.67 Corollary. "Kelley-fication" of the completion.**
*The space $(E^*_\gamma)^*_\gamma$ has the same compact subsets as $\hat{E}$ and carries the final locally convex topology with respect to these subsets. If $\hat{E}$ is compactly generated, and hence in particular if $E$ is metrizable, then we have equality.*

**Proof.** Since by 4.66 the 0-neighborhoods in $E^*_\gamma$ coincide with the 0-neighborhoods in $\tau_c(E^*, \hat{E})$, we have that the equi-continuous sets in $(E^*_\gamma)^*$ coincide with the subsets of polars of 0-neighborhoods in $\tau_c(E^*, \hat{E})$ and hence are just the subsets of compact sets in $\hat{E}$ (use that the bipolar of a compact set in $\hat{E}$ is compact). By the definition of $\sigma$ the topology $\hat{E}$ is finer than $\sigma(\hat{E}, E^*_\gamma)$ and hence they coincide on compact subsets of $\hat{E}$. □

**4.68 Proposition. Approximation property versus $\varepsilon$-product.**
*A complete space $E$ has the approximation property iff $F \otimes_\varepsilon E$ is dense in the* $\varepsilon$-PRODUCT $F \,\varepsilon\, E := \mathcal{L}_{equi}(F^*_\gamma, E)$ *for every locally convex space $F$.*

**Proof.** Note that $F \otimes E$ is mapped into $\mathcal{L}(F^*_\gamma, E)$, since for $y \in F$ we have $\delta(y) \in (F^*_\gamma)^*$ by 4.64.

($\Leftarrow$) Consider the following commuting diagram:

$$\begin{tikzcd}
E^*_\gamma \otimes E \arrow[rr] \arrow[dr] & & \mathcal{L}_{cp}(E,E) \arrow[dl, hook] \\
& \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, E) &
\end{tikzcd}$$

By assumption the arrow on the left hand side has dense image. The arrow on the right hand side is an embedding, since $(E^*_\gamma)^*_\gamma \to E$ is a continuous mapping, and the equi-continuous subsets in $(E^*_\gamma)^*_\gamma$ are exactly the relatively compact subsets of $\hat{E} = E$.

($\Rightarrow$) Let $T \in \mathcal{L}(F^*_\gamma, E)$ and let a 0-neighborhood $N_{V^o,U}$ in this space be given. Since $T$ is continuous on $V^o$, we have that $K := T(V^o)$ is compact in $E$. By assumption $E^* \otimes E$ is dense in $\mathcal{L}_{cp}(E,E)$. Hence there exists a finite operator $S \in \mathcal{L}(E,E)$ with $(\mathrm{id}_E - S)(K) \subseteq U$. Then $S \circ T : F^*_\gamma \to E \to E$ is finite dimensional and since $(F^*_\gamma)^* = \hat{F}$ by 4.64 it belongs to $\hat{F} \otimes E$ and $(T - ST)(V^o) = (1-S)(K) \subseteq U$. Thus $T - ST \in N_{V^o,U}$. Hence $\hat{F} \otimes_\varepsilon E$ is dense in $L(F^*_\gamma, E)$ and since $F \otimes E$ is dense in $\hat{F} \otimes_\varepsilon E$ it is also dense in $\mathcal{L}(F^*_\gamma, E)$. □

**4.69 Corollary.**
*Let $E$ be complete and satisfying the approximation property, then $F \hat{\otimes}_\varepsilon E = F \,\varepsilon\, E$.*

**Proof.** Recall that $F \,\varepsilon\, E = \mathcal{L}_{equi}(F^*_\gamma, E)$ is the subspace of $L(F^*, E)$ formed by all linear maps $T : F^* \to E$, which are continuous on equi-continuous subsets of $F^*$ with respect to the weak-topology $\sigma(F^*, F)$ on $F^*$. It is easily checked that for complete $E$ this space is complete, cf. [**14**, 16.1.5]. So $F \hat{\otimes}_\varepsilon E$ is the closure of $F \otimes E$ in $L(F^*, E)$, which is by 4.68 exactly $F \varepsilon E$. □

**4.70 Lemma.**
*For complete spaces $E$ and $F$ we have $F \,\varepsilon\, E \cong E \,\varepsilon\, F$.*

**Proof.** We only have to show bijectivity, since $F \,\varepsilon\, E = \mathcal{L}_{equi}(F^*_\gamma, E) \subseteq L(F^*, E)$ embeds into the space $L(F^*, E^{*\prime}) \cong L(F^*, E^*; \mathbb{R})$. To every continuous $T : F^*_\gamma \to E$ we associate the continuous $T^* : E^*_\gamma \to (F^*_\gamma)^*_\gamma$ (in fact every equi-continuous set $U^o$ of $E^*$ is mapped to $T^*(U^o) = \{x^* \circ T : x^* \in U^o\} \subseteq \{y^* : y^* \in (T^{-1}(U))^o\}$, the polar of a 0-neighborhood in $F^*_\gamma$). And by 4.64 we are done since by the lemma above the identity $(F^*_\gamma)^*_\gamma \to L_{equi}(F^*_\gamma, \mathbb{R}) = \hat{F}$ is continuous. □

**4.71 Remark.**
It can be shown (see [**14**, 16.2.6]) that for complete spaces also associativity of the $\varepsilon$-product is valid, i.e.

$$E \,\varepsilon\, (F \,\varepsilon\, G) \cong (E \,\varepsilon\, F) \,\varepsilon\, G.$$

**4.72 Property** (V) **for** $L$.
Remains to find situations where $E^*_\gamma$ coincides with $E^*_\beta$. By 4.66 this topology $\gamma$ coincides with the topology $\tau_c(E^*, \hat{E})$ of uniform convergence on compact subsets of $\hat{E}$, which is for metrizable spaces by [**14**, 9.4.3] identical to the topology of uniform convergence on precompact subsets of $E$. Thus if $E$ is complete and all bounded sets are precompact (like in Montel spaces) it coincides with the strong topology.

Remains to find situations, where the equi-continuous subsets coincide with the bounded ones in $E^*_\beta$. This is exactly the case, when $E$ is infra-barreled.

**4.73 Proposition.**
*If $E$ and $F$ are complete, $E$ is Montel and $F$ (or $E$) satisfies the approximation property, then*

$$E\hat{\otimes}_\varepsilon F \cong \mathcal{L}_{equi}(E^*_\gamma, F) \cong \mathcal{L}_\beta(E^*_\beta, F),$$

*For complete spaces $E$ and $F$ we have under the indicated assumptions the following version of* (V)*:*

$$\begin{aligned} E\hat{\otimes}_\varepsilon F \overset{app.P.}{\cong} E\,\varepsilon\, F = \mathcal{L}_{equi}(E^*_\gamma, F) \\ \overset{semi\text{-}Montel}{=} \mathcal{L}_{equi}(E^*_\beta, F) \overset{infra\text{-}barreled}{=} \mathcal{L}_b(E^*_\beta, F) \\ \overset{E^*_\beta\ bornological}{=} L(E^*_\beta, F) \end{aligned}$$

**Proof.** The first statement follows by what we said above, since Montel spaces are barreled.

Note that the strong dual of a semi-reflexive space is barreled [**14**, 11.4.1]. If $E$ is in addition metrizable, then by [**14**, 13.4.4] $E^*$ is bornological, and hence we have

$$\mathcal{L}_\beta(E^*_\beta, F) = L(E', F). \quad \square$$

**4.74 Proposition.**
*For complete spaces $E^*_\beta$ and $F$ we have the original version of* (V) *under the following conditions*

$$\begin{aligned} E^*_\beta\hat{\otimes}_\varepsilon F \overset{app.prop.}{\cong} E^*_\beta\,\varepsilon\, F = \\ \overset{E\ Montel}{=} \mathcal{L}_b((E^*_\beta)^*_\beta, F) \overset{E\ reflexive}{=} \mathcal{L}_b(E, F) = \\ \overset{E\ bornological}{=} L(E, F), \end{aligned}$$

**Proof.** This follows, since the strong dual $E^*_\beta$ of a Montel space $E$ is Montel. Note that a Montel-space $E$ is reflexive, i.e. $(E^*_\beta)^*_\beta = E$. Furthermore $E^*_\beta$ is complete, provided $E$ is bornological. $\square$

Now let us consider $E^*\hat{\otimes}_\varepsilon F$. If $F$ is complete and satisfies the approximation property, then $E^*_\gamma\hat{\otimes}_\varepsilon F \cong \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, F)$. By Grothendieck's completeness criterion we have $\hat{E} \cong \mathcal{L}_{equi}(E^*_\gamma, \mathbb{R})$.

The following result can be found in [**14**, 16.1.7]:

**4.75 Theorem.**
*One has the following natural isomorphisms for Fréchet spaces $E$ and $F$:*

$$\begin{aligned} (E\hat{\otimes}_\pi F)^*_\gamma &\cong E^*_\gamma \varepsilon F^*_\gamma & E\hat{\otimes}_\pi F &\cong (E^*_\gamma \varepsilon F^*_\gamma)^*_\gamma \\ (E\varepsilon F)^*_\gamma &\cong E^*_\gamma \hat{\otimes}_\pi F^*_\gamma & E\varepsilon F &\cong (E^*_\gamma \hat{\otimes}_\pi F^*_\gamma)^*_\gamma \end{aligned}$$

**Proof.** Note that the isomorphisms on the right follow from the ones on the left by applying $(\_)^*_\gamma$ and using that $(G^*_\gamma)^*_\gamma \cong \hat{G}$ for all metrizable spaces $G$.

(1) In fact $(E\hat{\otimes}_\pi F)^* = L(E,F;\mathbb{R})$ and

$$E^*_\gamma \varepsilon F^*_\gamma = \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, F^*_\gamma) = L_{cp}(E, \mathcal{L}_{cp}(F,\mathbb{R})).$$

Since $T : E \times F \to \mathbb{R}$ is continuous, iff it is continuous on compact sets, and hence iff $\check{T} : E \to \mathcal{L}_{cp}(F,\mathbb{R})$ is continuous, we obtain a bijection. That this is a homeomorphism follows since $\gamma$ is the topology of uniform convergence on compact sets, and the compact sets in $E\hat{\otimes}_\pi F$ are given by bipolars of tensor-products of two compact sets in $E$ and $F$.

For the second pair of isomorphisms see [**14**, 16.1.7] □

## The Approximation Property for Banach Spaces

For Banach spaces $E$, $F$ etc. we have $E^* = E'$ and we consider on $E^*$ the operator-norm topology induced by that of $E'$. Moreover $\mathcal{L}(E,F) = L(E,F)$.

**4.76 Proposition. Compact operators as tensor product.**
*For Banach spaces $E$ and $F$ one has*

$$E^*_\beta \varepsilon F \cong \mathcal{K}(E,F)$$

**Proof.** By completeness we have $E^*_\beta \varepsilon F = F \varepsilon E^*_\beta = \mathcal{L}_{equi}(F^*_\gamma, E^*_\beta)$. Remains to show that $T \mapsto T^*$ is a isomorphism $\mathcal{K}(E,F) \to \mathcal{L}_{equi}(F^*_\gamma, E^*_\beta)$. If $T$ is compact, then $T(oE)$ is relatively compact in $F$ and hence $(T^*)^{-1}(o(E^*)) = T(oE)^o$ is a 0-neighborhood in the topology $\tau_{cp}(F^*,F) = \gamma$, i.e. $T^* \in \mathcal{L}(F^*_\gamma, E^*_\beta)$. Conversely assume that $T^* : F^*_\gamma \to E^*_\beta$ is continuous. Then the set $T(oE)^o = (T^*)^{-1}(o(E^*))$ is a 0-neighborhood in $\gamma(F^*,F) = \tau_{cp}(F^*,F)$, and hence $T(oE)$ is contained in a compact subset of $F$. So $T \mapsto T^*$ is a bijection. That it is a homeomorphism follows immediately since $\{T^* : T \in N_{oE,U}\} = N_{U^o,oE^*}$. □

**4.77 Proposition. Approximation property and compact operators.**
*For a Banach space $E$ one has that:*

1. *$E$ has the approximation property iff $F^* \otimes E$ is dense in $\mathcal{K}(F,E)$ for every Banach space $F$, i.e. $F^*\hat{\otimes}_\varepsilon E = \mathcal{K}(F,E)$.*
2. *$E^*$ has the approximation property iff $E^* \otimes F$ is dense in $\mathcal{K}(E,F)$ for every Banach space $F$, i.e. $E^*\hat{\otimes}_\varepsilon F = \mathcal{K}(E,F)$.*

Recall that for Hilbert spaces $E$ we have shown in [**2**, 6.4.8] that $E^* \otimes E$ is dense in $\mathcal{K}(E,E)$.
Moreover one can show that in (1) and (2) it is enough to have denseness for all closed subspaces of $c_0$ or all reflexive separable Banach spaces.

**Proof.** (⇒) If $E$ or $F^*$ have the approximation property then $F^*\hat{\otimes}_\varepsilon E \cong F^*\varepsilon E$ by 4.68 and $F^*\varepsilon E \cong \mathcal{K}(F,E)$ by 4.76.

(⇐) Since in 4.68 it is enough to have denseness for all Banach spaces (see [**14**, 18.1.8]), this is true for the second statement. For the first one has to proceed more carefully, see [**14**].

For a proof of the second part see [**14**, 18.3.2]. □

**4.78 Lemma.**
*For Banach spaces $E$ and $F$ we have a natural surjective linear map $\iota : F^*\hat{\otimes}_\pi E \to L_{cp}(E,F)^*$, where $L_{cp}$ denotes $L$ with the topology of uniform convergence on compact sets in $E$.*

**Proof.** The map $\iota$ is associated to the bounded multi-linear composition map $F^* \times E \times L(E,F) \to \mathbb{R}$, hence is a well defined continuous map $F^*\hat{\otimes}_\pi E = F^*\hat{\otimes}_\beta E \to L(E,F)^*$ given by $y^* \otimes x \mapsto (T \mapsto y^*(Tx))$.

Its image is contained in $L_{cp}(E,F)^*$, since every $z \in F^*\hat{\otimes}_\pi E$ can be written as $z = \sum_k \lambda_k y_k^* \otimes x_k$ with $\lambda \in \ell^1$, $\|y_k^*\| \to 0$ and $\|x_k\| \to 0$, see [**22**, 15.6.4]. In other words $(\lambda_k y_k)_k \in \ell^1\{F^*\}$ and $(x_k)_k \in c_0\{E\}$. Without loss of generality we may assume $\sum_k \|\lambda_k y_k^*\| \le 1$ (move some factor to the $x_k$). By [**2**, 6.4.3] the closed absolutely convex hull $K$ of the $x_n$ is compact. Thus $N_{K,oF}$ is a 0-neighborhood in $L_{cp}(E,F)$. Let $T \in N_{K,oF}$. Then $|\iota(z)(T)| \le \sum_k |\lambda_k y_k^*(T(x_k))| \le \sum_k \|\lambda_k\, y_k^*\| \le 1$, i.e. $\iota(z) \in N_{K,oF}^o$.

Conversely let $\varphi \in L_{cp}(E,F)^*$. Then $\varphi \in (N_{K,oF})^o$ for some compact $K \subseteq E$. By [**2**, 6.4] we may assume that $K$ is contained in the closed absolutely convex hull of some sequence $x_n \to 0$ in $E$. Consider the Banach space $c_0(\mathbb{N},F) = c_0\{F\}$ with the supremum norm $\sup\{\|x_n\| : n \in \mathbb{N}\}$. Then $\psi : L(E,F) \to c_0(\mathbb{N},F)$ given by $T \mapsto (T(x_n))_n$ is continuous and linear. Hence its dual is $\psi^* : \ell^1\{F^*\} = \ell^1(\mathbb{N},F^*) \overset{!}{\cong} c_0(\mathbb{N},F)^* \to L(E,F)^*$ (see [**14**, p405] for the duality) is continuous for the weak$^*$-topologies. Thus the absolutely convex set $K_1 := \psi^*(o(\ell^1\{F^*\})) \subseteq L(E,F)^*$ is compact for this topology. We claim that $\varphi \in K_1$. Otherwise, by Hahn-Banach there is a $T \in L(E,F)$ with $\varphi(T) > 1$ and $|\varphi_1(T)| \le 1$ for all $\varphi_1 \in K_1$, i.e. $\sum_k |y_k^*(Tx_k)| \le 1$ for all $(y_k^*)_k \in o(\ell^1\{F^*\})$. In particular $|y^*(Tx_k)| \le 1$ for all $y^* \in o(F^*)$ and all $k$, i.e. $T \in N_{K,oF}$ and hence $|\varphi(T)| \le 1$, a contradiction.

Since $\varphi \in K_1 = \psi^*(o(\ell^1\{F^*\}))$ there is some $(y_k^*)_k \in o(\ell^1\{F^*\})$ with $\varphi = \psi^*((y_k^*)_k)$. Now $\sum_k y_k^* \otimes x_k \in F^*\hat{\otimes}_\pi E$ and

$$\psi^*((y_k^*)_k)(T) = \sum_k y_k^* T(x_k) = \iota(\sum_k y_k^* \otimes x_k)(T) \text{ for all } T \in L(E,F),$$

i.e. $\psi^*((y_k^*)) = \iota(\sum_k y_k^* \otimes x_k)$. □

**4.79 Proposition. Approximation property and tensor products.**

*For a Banach space $E$ the following properties are equivalent:*

1. *$E$ has the approximation property.*
2. *The map $F\hat{\otimes}_\beta E = F\hat{\otimes}_\pi E \to F\hat{\otimes}_\varepsilon E \subseteq L(F^*,E)$ is injective for every Banach space $F$.*
3. *The map $F^*\hat{\otimes}_\beta E = F^*\hat{\otimes}_\pi E \to F^*\hat{\otimes}_\varepsilon E \subseteq L(F,E)$ is injective for every Banach space $F$.*
4. *The map $E^*\hat{\otimes}_\beta E = E^*\hat{\otimes}_\pi E \to E^*\hat{\otimes}_\varepsilon E \subseteq L(E,E)$ is injective.*
5. *The evaluation map* $\mathrm{ev} : E^* \times E \to \mathbb{R}$ *extends to a linear functional* $\mathrm{Tr} : \mathcal{N}(E,E) \to \mathbb{R}$*, where $\mathcal{N}(E,E)$ denotes the image of $E^*\hat{\otimes}_\pi E$ in $L(E,E)$.*

**Proof.** (1⇒2) Consider the following commuting diagram:

$$\begin{tikzcd}
F\hat{\otimes}_\pi E \arrow[r] \arrow[dr, hook, "\delta"] \arrow[dd, hook, "\delta\otimes E"'] & F\hat{\otimes}_\varepsilon E \arrow[r, hook] & L(F^*,E) \arrow[r, hook, "\delta_*"] & L(F^*,E^{*\prime}) \arrow[d, "\cong"] \\
 & (F\hat{\otimes}_\pi E)^{*\prime} \arrow[r, "\cong"] & L(E,F;\mathbb{R})' \arrow[d, "\cong"] & L(F^*,E^*;\mathbb{R}) \arrow[d, "\cong"] \\
F^{**}\hat{\otimes}_\pi E \arrow[r, two heads] & L_{cp}(E,F^*)^* \arrow[r] & L(E,F^*)' \arrow[r] & (E^*\otimes_\beta F^*)'
\end{tikzcd}$$

The arrow $L(E,F^*)' \to (F^* \otimes_\beta E^*)'$ at the bottom is well-defined, since $E^* \times F^* \to L(E,F^*)$ is bounded.
Now start with $z_0$ in the top-left hand corner and assume it is mapped to 0 in $F \hat{\otimes}_\varepsilon E$. So it is mapped to 0 in the bottom-right hand corner. Since the composite of the last two arrows at the bottom is injective, because $E^* \otimes F^*$ is dense in $L_{cp}(E,F^*)$ by (1), it is mapped to 0 in $L_{cp}(E,F^*)^*$ and hence also in $L(E,F^*)'$. By the injectivity of the diagonal maps we conclude that $z_0 = 0$.

(2⇒3⇒4⇒5) are trivial.

(5⇒1) For this we consider the following commuting diagram:

$$\begin{tikzcd}
E^* \hat{\otimes}_\pi E \arrow[r, two heads] \arrow[dd, two heads] \arrow[dr, "\mathrm{ev}"'] & L_{cp}(E,E)^* \arrow[r] \arrow[d, "\mathrm{ev}_{\mathrm{id}}"] & (E^* \otimes_\beta E)^* \arrow[d, "\cong"'] \\
 & \mathbb{R} & (E \otimes_\beta E^*)' \arrow[d, "\cong"'] \\
\mathcal{N}(E,E) \arrow[r, hook] \arrow[ur, "\mathrm{tr}"] & L(E,E) \arrow[r, hook, "\delta_*"] & L(E,(E^*)')
\end{tikzcd}$$

Note that the second arrow on the top is well-defined, since the mapping $E^* \times E \to L_b(E,E) \to L_{cp}(E,E)$ is bounded, and the top-triangle commutes, since for $z = x^* \otimes x$ we have $\mathrm{ev}(z) = \mathrm{ev}(x^* \otimes x) = x^*(x) = x^*(\mathrm{id}(x)) = \iota(x^* \otimes x)(\mathrm{id}) = \varphi(\mathrm{id})$.
We have to show that $E^* \otimes E$ is dense in $L_{cp}(E,E)$. For this it is enough to show that all $\varphi \in L_{cp}(E,E)^*$ which vanish on $E^* \otimes E$ vanishes on $\mathrm{id}_E$. By [**14**, 18.3.3] every such $\varphi$ is in the image of some $z = \sum_{n=1}^\infty x_n^* \otimes x_n \in E^* \hat{\otimes}_\pi E$, i.e. $\varphi = \iota(z)$. For all $(x^*, x) \in E^* \times E$ we have $0 = \varphi(x^* \otimes x) = \iota(\sum_n x_n^* \otimes x_n)(x^* \otimes x) = \sum_n x_n^*((x^* \otimes x)(x_n)) = \sum_n x_n^*(x) \cdot x^*(x_n) = x^*\Big(\sum_n x_n^*(x) x_n\Big)$. Thus the image of $z$ in the top-right corner is 0, and hence also in the bottom-right corner. Since the bottom arrows are injective it is 0 in the bottom-left corner. Hence its image in the center is 0, which is exactly $\varphi(\mathrm{id})$. □

# 5. Operator Ideals

We will discuss spaces $E$ for which the *connecting maps* $\widehat{E_{(U)}} \to \widehat{E_{(V)}}$ for $U \subseteq V$ belong to certain subclasses $\mathcal{I}$ of all continuous linear maps.

So these classes $\mathcal{I}$ will be described by linear subspaces $\mathcal{I}(E,F)$ of $\mathcal{L}(E,F)$ for every Banach space $E$ and $F$. And that $\mathcal{I}$ is an *operator ideal* means that $RTS \in \mathcal{I}(E_1,F_1)$ for every $T \in \mathcal{I}(E,F)$ and all $R \in \mathcal{L}(F,F_1)$ and $S \in \mathcal{L}(E_1,E)$.

**5.1.** The smallest reasonable operator ideal $\mathcal{F}$ is given by the *finite operators*, i.e. the image of $E^* \otimes F$ in $\mathcal{L}(E,F)$. In fact this is an ideal, since for $T = \sum_k x_k^* \otimes y_k$ we have that $R \circ T \circ S = \sum_k (x_k^* \circ S) \otimes R(y_k) = \sum_k S^*(x_k^*) \otimes R(y_k)$.

## Compact Operators

Another ideal $\mathcal{K}$ is given by the *compact operators*, i.e. those operators $T : E \to F$ which map the unit-ball to a relatively compact set. This is an ideal, since $S(oF_1) \subseteq \|S\| \cdot oF$ and hence $(RTS)(oF_1)$ is contained in the compact image of $\|S\| \cdot \overline{T(oF)}$ under $R$.

Recall that we have shown in [**3**, 11.21]

**5.2 Schauder's theorem.**
*A continuous linear operator $T : E \to F$ between Banach spaces is compact iff its adjoint $T^*$ is compact.*

**5.3 Lemma. Orthogonal representation of compact operators.**
*An operator $T$ between Hilbert spaces is compact iff there are orthonormal sequences $e_n$ and $f_n$ and $\lambda_n \to 0$ such that $Tx = \sum_n \lambda_n \langle e_n, x\rangle f_n$.*

**Proof.** ($\Leftarrow$) If $T$ has such a representation, then the finite sums define finite dimensional operators which converge to $T$.

($\Rightarrow$) Since any compact $T : E \to F$ induces a compact injective operator $T : (\ker T)^\perp \to \overline{TE}$ with dense image, we may assume that $T$ is injective. Now we consider the positive compact operator $T^*T$. Its eigenvalues are all non-zero, since $T^*Tx = 0$ implies $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = 0$. By [**2**, 6.5.4] there is an orthonormal sequences of Eigen-vectors $e_n$ with Eigen-value $0 \neq \lambda_n^2 \to 0$ such that $T^*Tx = \sum_n \lambda_n^2 \langle e_n, x\rangle e_n$. Let $f_n := \frac{1}{\lambda_n} T e_n$. Then a simple direct calculation shows that the $f_n$ are orthonormal. Note that $x = \sum_n \langle e_n, x\rangle e_n$. Otherwise the compact positive operator $T^*T$ restricted to the orthogonal complement $\{e_k : k\}^\perp$ would have a unit Eigen-vector $e$ with positive Eigen-value $\lambda$. Which is impossible by definition of the $e_k$. So we obtain $Tx = \sum_n \langle e_n, x\rangle \lambda_n f_n$.

Anther way to prove this is to use the polar decomposition $T = U\,|T|$, see [**3**, 10.18], where $U$ is a partial isometry and $|T|$ a positive and also compact operator. The spectral theorem for $|T|$ gives an orthonormal family $e_n$ and $\lambda \in c_0$, such that $Tx = \sum_k \lambda_k \langle e_k, x\rangle e_k$. Applying $U$ to this equation, shows that we may take $f_k := U e_k$. $\square$

As a warning it should be mentioned that one can not read off from the spectrum whether an operator is compact.

**5.4 Corollary.**
*An operator $T$ between Hilbert spaces is compact iff $\langle Te_n, f_n\rangle \to 0$ holds for all orthonormal sequences $e_n$ and $f_n$.*

**Proof.** ($\Rightarrow$) Since $|\langle Te_n, f_n\rangle \le \|Te_n\| \cdot \|f_n\| = \|Te_n\|$ it is enough to show that $Te_n \to 0$. Since $e_n$ converges weakly to 0 (in fact $\langle x, e_k\rangle$ is even quadratic summable) we conclude that $Te_n$ converges to 0 weakly. Since $e_n$ is contained in the unit-ball and $T$ is compact, every subsequence of $Te_n$ has a subsequence, which is convergent. And the limit has to be 0, since this is true for the weak topology. But from this it easily follows that $Te_k \to 0$.

($\Leftarrow$) Given $\varepsilon > 0$ we choose maximal orthonormal sequences $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ such that $|\langle Te_i, f_i\rangle| \ge \varepsilon$. By assumption $I$ must be finite. We consider the orthonormal projections $P := \sum_i e_i \otimes e_i$ and $Q := \sum_i f_i \otimes f_i$. For the composition with the ortho-projections on the complement we obtain $(1-Q)T(1-P) = T-(TP+QT-QTP) =: T - S$. Hence $S$ is a finite dimensional operator and we claim that $\|T - S\| \le \varepsilon$. Suppose this were not true. Then there is an $x$ with $\|(T - S)x\| > \varepsilon\,\|x\|$ and hence an $y$ such that $|\langle T(1 - P)x, (1 - Q)y\rangle| = |\langle (T - S)x, y\rangle| > \varepsilon\,\|x\|\,\|y\|$. Let $e_0 := (1 - P)x$ and $f_0 := (1 - Q)y$. Obviously $e_0, f_0 \ne 0$ and hence we may assume without loss of generality that $\|e_0\| = 1 = \|f_0\|$ and hence $\|x\| \ge 1$ and $\|y\| \ge 1$. Since $e_0 \in (1-P)(E) \subseteq P(E)^\perp = \{e_i : i \in I\}^\perp$ and $f_0 \in (1-Q)(F) \subseteq \{f_i : i \in I\}^\perp$ we get a contradiction to the maximality of $I$. $\square$

In [**3**, 11.29] we have shown

**5.5 Proposition.**
*In a separable Hilbert space $H$ the compact operators form the unique non-trivial ideal, which is closed in $L(H, H)$.*

Note that this is not true for non-separable Hilbert spaces. In fact we may consider those operators, for which the image is contained in a separable subspace. These form obviously a non-trivial operator ideal, which is strictly larger than the compact operators, since it contains the ortho-projections to separable subspaces. Finally it is closed, since the image of the limit of a sequence of such operators is contained in the closure of the union of the images.

Recall furthermore that for Banach spaces $E$ and $F$ we have

$$\mathcal{K}(E, F) \cong E^* \varepsilon F,$$

and the latter space coincides with $E^* \hat{\otimes}_\varepsilon F$, if $F$ (or $E^*$) has the approximation property.

Note that one can generalize the notion of a compact operator to linear maps $T : E \to F$ between locally convex spaces, by assuming that there is a 0-neighborhood $U \subseteq E$ which is mapped to a relatively compact subset of $F$.

## Nuclear Operators

The corresponding ideal $\mathcal{N}$ for the projective tensor product is formed by the *nuclear operators*, i.e. those which are contained in the image of $E^* \hat{\otimes}_\pi F$ in $\mathcal{L}(E, F)$. Thus an operator $T$ between Banach spaces is nuclear iff there are $x_k^* \in E^*$, $y_k \in F$, $\lambda_k \in \mathbb{R}$ with $\|x_k^*\| \le 1$, $\|y_k\| \le 1$ and $\lambda \in \ell^1$ such that

$$Tx = \sum_{k=0}^{\infty} \lambda_k x_k^*(x) y_k \text{ for all } x \in E.$$

That this is an ideal follows as for the finite dimensional operators:

$$R\Big(\sum_{k=0}^{\infty} \lambda_k x_k^* \otimes y_k\Big)S = \sum_{k=0}^{\infty} \lambda_k S^*(x_k^*) \otimes R(y_k).$$

We have the following factorization result

**5.6 Proposition. Factorization property of $\mathcal{N}$.**
*A map $T : E \to F$ between Banach spaces is nuclear iff there are continuous linear operators $S : E \to \ell^\infty$ and $R : \ell^1 \to F$ such that $T$ factors as diagonal operator $\ell^\infty \to \ell^1$ with diagonal $\lambda \in \ell^1$, i.e.*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, dotted, "S"'] & F \\ \ell^\infty \arrow[r, dotted, "\lambda"'] & \ell^1 \arrow[u, dotted, "R"'] \end{tikzcd}$$

**Proof.** ($\Rightarrow$) Let $T$ be represented by $\sum_k \lambda_k\, x_k^* \otimes y_k$. Then $S(x) := (x_k^*(x))_k$ and $R((\mu_k)_k) := \sum_k \mu_k\, y_k$ define linear operators of norm $\le 1$ and $T = R\lambda S$, where $\lambda : \ell^\infty \to \ell^1$ denotes the diagonal operator, with diagonal $(\lambda_k)_k$.

($\Leftarrow$) Since the nuclear operators form an ideal, it is enough to show that such diagonal operators $T : (\mu_k)_k \mapsto (\lambda_k \mu_k)_k$ are nuclear, which is clear since they can be represented by $\sum_k \lambda_k x_k^* \otimes y_k$, where $x_k^* := e_k \in \ell^1 \subseteq (\ell^\infty)^*$ and $y_k := e_k \in \ell^1$. □

More generally one can call a linear map $T : E \to F$ between locally convex spaces $E$ and $F$ *nuclear* iff there is an absolutely convex 0-neighborhood $U \subseteq E$ and an absolutely convex bounded subset $B \subseteq F$, for which $F_B$ is complete, such that $T$ factors over a nuclear mapping $T_1 : \widehat{E_{(U)}} \to F_B$, i.e.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d] & F \\ \widehat{E_{(U)}} \arrow[r, dotted, "T_1"'] & F_B \arrow[u] \end{tikzcd}$$

Obviously that is exactly the case, iff there is an equi-continuous sequence $x_n^* \in E^*$ and a sequence $y_n$ which is contained in a bounded absolutely convex $B \subseteq F$ with complete $F_B$ and $\lambda \in \ell^1$, such that $T$ has a description of form $\sum_k \lambda_k x_k^* \otimes y_k$.

**5.7 Lemma. $\mathcal{N} \subseteq \mathcal{K}$.**
*Every nuclear mapping is compact.*

**Proof.** Let $T$ be a nuclear mapping. Since the compact mappings form an ideal, we may assume that $T$ is a diagonal-operator $\ell^\infty \to \ell^1$ with absolutely summable diagonal $(\lambda_k)_k$. Such an operator is compact, since the finite sub-sums $\sum_{k\le n} \lambda_k e_k \otimes e_k$ define finite dimensional operators, which converge to $T$ uniformly on the unit-ball of $\ell^\infty$. □

A first connection to nuclear spaces is given by the following

**5.8 Proposition.**
*Let $T : E \to F$ be nuclear. Then $T \otimes G : E \otimes_\varepsilon G \to F \otimes_\pi G$ is continuous for every locally convex space $G$.*

**Proof.** We use the factorization of $T$ as $T = R\lambda S$ given in 5.6. The diagram

$$\begin{tikzcd} E \otimes_\varepsilon G \arrow[r, "T\otimes G"] \arrow[d, "S\otimes_\varepsilon G"'] & F \otimes_\pi G \\ \ell^\infty \otimes_\varepsilon G \arrow[r, "\lambda\otimes G"] & \ell^1 \otimes_\pi G \arrow[u, "R\otimes_\pi G"'] \end{tikzcd}$$

shows that we may assume that $T$ is a diagonal operator $\lambda : \ell^\infty \to \ell^1$.
Moreover, since $\ell^1 \otimes_\pi G \to \ell^1 \hat{\otimes}_\pi \hat{G}$ is a dense embedding and $\hat{G}$ is a reduced projective limit $\varprojlim_W \hat{G}_{(W)}$ and thus $\ell^1 \hat{\otimes}_\pi G = \varprojlim_W \ell^1 \hat{\otimes}_\pi \widehat{G_{(W)}}$, it is enough to show that $\lambda \otimes \widehat{G_{(W)}} : \ell^\infty \otimes_\varepsilon \widehat{G_{(W)}} \to \ell^1 \otimes_\pi \widehat{G_{(W)}}$ is continuous. In fact consider the following diagram:

$$\begin{tikzcd} \ell^\infty \otimes_\varepsilon G \arrow[r, "\lambda\otimes G"] \arrow[d, "\ell^\infty\otimes_\varepsilon p_W"'] & \ell^1 \otimes_\pi G \arrow[d, "\ell^1\otimes_\pi p_W"] \\ \ell^\infty \hat{\otimes}_\varepsilon \widehat{G_{(W)}} \arrow[r, "\lambda\otimes\widehat{G_{(W)}}"] & \ell^1 \hat{\otimes}_\pi \widehat{G_{(W)}} \\ \ell^\infty \otimes \widehat{G_{(W)}} \arrow[r, "\lambda\otimes\widehat{G_{(W)}}"] \arrow[u, hook] & \ell^1 \otimes_\pi \widehat{G_{(W)}} \arrow[u, hook] \end{tikzcd}$$

Thus we may assume that $E = \ell^\infty$, $F = \ell^1$, $G$ is a Banach space, and $T$ a diagonal operator with absolutely summable diagonal $\lambda$.
Now let $z = \sum_{j\le n} x_j \otimes y_j \in \ell^\infty \otimes G$ be given. Then $x_j = \sum_{k=0}^\infty e_k^*(x_j)e_k$ and hence

$$T(x_j) = \sum_{k=0}^\infty e_k^*(x_j)T(e_k) = \sum_{k=0}^\infty \lambda_k e_k^*(x_j)e_k$$

and thus

$$\begin{aligned}(T\otimes G)(z) &= \sum_{j\le n} T(x_j)\otimes y_j = \sum_{j\le n}\sum_{k=0}^\infty \lambda_k e_k^*(x_j)e_k\otimes y_j \\ &= \sum_k \lambda_k e_k \otimes \sum_j e_k^*(x_j)y_j.\end{aligned}$$

So we get

$$\begin{aligned}\|(T\otimes G)(z)\|_\pi &\le \sum_{k=0}^\infty \|\lambda_k e_k \otimes \sum_j e_k^*(x_j)y_j\|_\pi \\ &\le \sum_{k=0}^\infty \underbrace{\|\lambda_k e_k\|_\infty}_{=|\lambda_k|} \cdot \underbrace{\|\sum_j e_k^*(x_j)y_j\|_G}_{\sup\{|y^*(\sum_j e_k^*(x_j)y_j)|:\|y^*\|\le 1\}} \\ &\le \sum_{k=0}^\infty |\lambda_k|\cdot \sup\{|\sum_j x^*(x_j)\,y^*(y_j)| : x^*\in o(\ell^\infty)^*, \|y^*\|\le 1\} \\ &\le \|\lambda\|_1\cdot\|z\|_\varepsilon. \quad\square\end{aligned}$$

It can be shown that the adjoint of a nuclear operator between Banach spaces is nuclear, see [**14**, 17.3.6]. The converse however is false.

Recall that $\mathcal{N}(E,F) \cong E^*\hat{\otimes}_\pi F$ by 4.79, provided $F$ has the approximation property. In this case we have that $\mathcal{N}(E,F)$ is a Banach space with dual

$$\mathcal{N}(E,F)^* \cong (E^*\hat{\otimes}_\pi F)^* \cong L(E^*,F;\mathbb{R}) \cong L(F,E^{*\prime}) \cong L(E^*,F').$$

Hence if in addition $E$ is reflexive, then $\mathcal{N}(E,F)^* \cong L(F,E)$.

In the general case, we will always consider $\mathcal{N}(E,F)$ as quotient of the Banach space $E^*\hat{\otimes}_\pi F$, with the corresponding quotient norm.

**5.9 Definition.**
More generally an operator $T : E \to F$ between Banach spaces is called *$p$-nuclear* (see [**14**, 19.7]) for $1 \le p < \infty$ iff there are bounded sequences $(x_n)_n \in \ell^p\{E^*\}$ and $(y_n) \in \ell^q\{F\}$ with $T = \sum_n x_n^* \otimes y_n$ and $\frac{1}{p} + \frac{1}{q} = 1$. One can extend this notion to $p = \infty$ using $c_0\{E^*\}$ instead of $\ell^p\{E^*\}$. The class of all $p$-nuclear operators will be denoted $N_p$. Note that this generalizes nuclear operators, i.e. $\mathcal{N}_1 = \mathcal{N}$.

One has the analogous factorization theorem:

$$\begin{tikzcd} E \arrow[r, "\mathcal{N}_p"] \arrow[d] & F \\ \ell^\infty \arrow[r, "\ell^p"'] & \ell^p \arrow[u] \end{tikzcd}$$

[**14**, 19.7.4]

## Integral Mappings

**5.10 Definition.**
Since the identity induces continuous bijections $E \otimes_\beta F \to E \otimes_\pi F \to E \otimes_\varepsilon F$, we get inclusions

$$(E \otimes_\varepsilon F)^* \to (E \otimes_\pi F)^* \to (E \otimes_\beta F)^*.$$

Recall that the last space coincides with the bounded bilinear forms, the second one with the continuous bilinear forms and for the first one we give the following definition:
A bilinear form $b : E \times F \to \mathbb{R}$ is called *integral*, iff it belongs to the dual of $E \otimes_\varepsilon F$.

**5.11 Proposition. Integral forms as integrals.**
*For a bilinear mapping $b : E \times F \to \mathbb{R}$ the following statements are equivalent:*

1. *$b$ is integral;*
2. *There are 0-neighborhoods $U \subseteq E$ and $V \subseteq F$ and a measure $\mu$ on $U^o \times V^o$ such that $b(x,y) = \int_{U^o \times V^o} x^*(x)\, y^*(y)\, d\mu(x^*,y^*)$.*

**Proof.** ($1\Rightarrow 2$) Let $b : E \otimes_\varepsilon F \to \mathbb{R}$ be continuous. So we may assume that $b : \hat{E}\hat{\otimes}_\varepsilon \hat{F} = E\hat{\otimes}_\varepsilon F \to \mathbb{R}$ is continuous. Recall that $\hat{E} \cong \varprojlim \widehat{E_{(U)}}$, is a reduced projective limit and similarly for $F$, so the linear continuous mapping $b$ on $\varprojlim_{U,V} \widehat{E_{(U)}}\hat{\otimes}_\varepsilon \widehat{F_{(V)}}$ factors over some $E \otimes_\varepsilon F \to E_{(U)} \otimes_\varepsilon F_{(V)} \to \widehat{E_{(U)}}\hat{\otimes}_\varepsilon \widehat{F_{(V)}}$. We now consider $U^o$ with its compact topology $\sigma(U^o,E)$ and define a linear mapping $\delta : E \to C(U^o)$ by $x \mapsto (x^* \mapsto x^*(x))$. We may assume that $U = \{x \in E : p(x) \le 1\}$ for some seminorm $p$ of $E$ and then $\|\delta(x)\|_\infty = \sup\{|x^*(x)| : x^* \in U^o\} = p(x)$. Thus this induces an isometric embedding $E_{(U)} \hookrightarrow C(U^o)$. Taking the tensor product we get an embedding

$$E_{(U)} \otimes_\varepsilon F_{(V)} \hookrightarrow C(U^o) \otimes_\varepsilon C(V^o) \hookrightarrow C(U^o)\hat{\otimes}_\varepsilon C(V^o) \overset{4.36}{\cong} C(U^o \times V^o).$$

By Hahn-Banach we may extend $b$ to a continuous linear functional $b \in C(U^o \times V^o)^*$, and hence there is a measure $\mu$ on $U^o \times V^o$, such that the required representation is valid.

(2⇒1) cf. [**22**, 502].
Let $b$ be given by a measure on $U^o \times V^o$. We have to show that $b : E \otimes_\varepsilon F \to \mathbb{R}$ is continuous. Consider the 0-neighborhood

$$W := \Big\{ z = \sum_{i=1}^n x_i \otimes y_i : \sup\Big\{ \Big| \sum_{i=1}^n x^*(x_i) y^*(y_i) \Big| : x^* \in U^o, y^* \in V^o \Big\} \leq 1 \Big\}$$

in $E \otimes_\varepsilon F$. Since

$$\begin{aligned} |b(z)| &= \Big| \sum_i \int_{U^o \times V^o} x^*(x_i) y^*(y_i)\, d\mu(x^*, y^*) \Big| \leq \\ &\leq \int_{U^o \times V^o} \Big| \sum_i x^*(x_i) y^*(y_i) \Big|\, d\mu(x^*, y^*) \leq \mu(U^o \times V^o) \end{aligned}$$

for all $z = \sum_i x_i \otimes y_i \in W$ we are done. □

**5.12 Factorization of integral forms.**
Note that we have shown that for an integral form $b : E \times F \to \mathbb{R}$ there is a compact space $K = U^o \times V^o$ and a measure $\mu$ on $K$, and two continuous linear mappings $R : E \to C(K)$ and $S : F \to C(K)$ given by $x \mapsto ((x^*, y^*) \mapsto x^*(x))$ and $y \mapsto ((x^*, y^*) \mapsto y^*(y))$ such that

$$b(x, y) = \int_K R(x) \cdot S(y) \cdot d\mu,$$

or in other words:

$$b = i \circ (R \times S) \text{ and hence } \check{b} = (i \circ (R \times S))^\vee = S^* \circ \check{i} \circ R,$$

where $i : C(K) \times C(K) \to \mathbb{R}$ denotes the continuous bilinear mapping $(f, g) \mapsto \int_K f(z)\, g(z)\, d\mu(z)$. Recall that for compact spaces we have natural continuous maps

$$C(K) \to \mathcal{L}^\infty(\mu) \to \mathcal{L}^1(\mu)$$

and by Hölder's inequality the map $i$ "extends" to a continuous bilinear map

$$i : \mathcal{L}^1(\mu) \times \mathcal{L}^\infty(\mu) \to \mathbb{R} \text{ and in particular } i : \mathcal{L}^1(\mu) \times C(K) \to \mathbb{R}.$$

So $\check{i} : \mathcal{L}^1(\mu) \to C(K)'$ is continuous and we get the following factorization for $\check{b}$

$$\begin{tikzcd} E \arrow[r, "R"] \arrow[dr, dotted] & C(K) \arrow[r, "\check{i}"] \arrow[d] & C(K)' \arrow[r, "S^*"] & F' \\ & \mathcal{L}^\infty(\mu) \arrow[r, "\check{i}"] & \mathcal{L}^1(\mu) \arrow[u, "\check{i}"] \arrow[ur, dotted] & \end{tikzcd}$$

**5.13 Definition.**
A linear operator $T : E \to F$ is called *integral* iff the associated bilinear form $E \times F^* \to \mathbb{R}$ given by $(\delta \circ T)\hat{}= \mathrm{ev} \circ (T \times F^*)$ is an integral bilinear form, where $\delta : F \to F^{**}$ denotes the natural embedding. In fact

$$\begin{aligned} (\mathrm{ev} \circ (T \times F^*))(x, y^*) &= \mathrm{ev}(T(x), y^*) = y^*(T(x)) \\ &= \delta(T(x))(y^*) = (\delta \circ T)(x)(y^*) = (\delta \circ T)\hat{}(x, y^*) \end{aligned}$$

**Lemma.**
*The integral operators form an ideal $\mathcal{I}$.*

**Proof.** Let $T : E \to F$ be integral and $R : F \to F_1$ and $S : E_1 \to E$ be continuous. Let $b : E \times F^* \to \mathbb{R}$ be the bilinear form associated to $T$. Then the bilinear form $E_1 \times F_1^* \to \mathbb{R}$ associated to $R \circ T \circ S$ is given by $b \circ (S \times R^*)$. In fact let $b := \operatorname{ev} \circ (T \times F^*)$ and $b_1 := \operatorname{ev} \circ (R \circ T \circ S \times F_1^*)$ be the associated bilinear forms. Then

$$b((S \times R^*)(x, y_1^*)) = b(Sx, R^* y_1^*) = R^*(y_1^*)(TSx) = y_1^*(RTSx) = b_1(x, y_1^*).$$

Since $S \otimes R^* : E_1 \otimes_\varepsilon F_1^* \to E \otimes_\varepsilon F^*$ is continuous, this form is integral. $\square$

**5.14 Lemma.**
*Let $T : E \to F$ be nuclear, then it is integral, i.e. $\mathcal{N} \subseteq \mathcal{I}$.*

**Proof.** Without loss of generality we may assume that $T : E \to F$ is a nuclear mapping between Banach spaces (In fact a diagonal mapping $\ell^\infty \to \ell^1$ would be enough). We have to show that the bilinear form $b = \operatorname{ev} \circ (T \times F^*) : E \times F^* \to F \times F^* \to \mathbb{R}$ is integral, which is clear, since it induces by 5.8 the continuous map

$$\operatorname{ev} \circ (T \otimes F^*) : E \otimes_\varepsilon F^* \to F \otimes_\pi F^* \to \mathbb{R}. \quad \square$$

**5.15 Lemma. Factorization of integral operators.**
*Every integral operator $T : E \to F$ between Banach spaces can be factored as*

$$\begin{tikzcd}
E \arrow[rr, "\mathcal{I}"] \arrow[d, dotted, "R"'] & & F \arrow[r, hook, "\delta"] & F'' \\
C(K) \arrow[r] & \mathcal{L}^\infty(\mu) \arrow[r, hook] & \mathcal{L}^1(\mu) \arrow[r, "\check{i}"] & C(K)' \arrow[u, dotted, "S^*"']
\end{tikzcd}$$

*where $K$ is some compact space and $\mu$ a measure on $K$.*

One can show that the converse is true as well, see [**14**, 17.4.2].

**Proof.** We consider the associated integral form $b : E \times F^* \to \mathbb{R}$ given by $b(x, y^*) = y^*(Tx)$. By what we said above there are continuous linear operators $R : E \to C(K) \to \mathcal{L}^\infty(\mu)$ and $S : F^* \to C(K)$, such that

$$(\delta T x)(y^*) = y^*(Tx) = b(x, y^*) = \check{b}(x)(y^*) = (S^* \check{i} R)(x)(y^*). \quad \square$$

**5.16 Proposition.**
*A mapping $T : E \to F$ between Banach spaces is integral iff the adjoint mapping $T^* : F_\beta^* \to E_\beta^*$ is.*

**Proof.** Let $T : E \to F$ be a linear operator and $b_T := \operatorname{ev} \circ (T \times F^*)$ and $b_{T^*} : \operatorname{ev} \circ (T^* \times E^{**})$ the associated bilinear forms. Then

$$b_{T^*}(y^*, \delta(x)) = \delta(x)(T^*(y^*)) = T^*(y^*)(x) = y^*(Tx) = b_T(x, y^*).$$

($\Leftarrow$) is now obvious, since integrality of $b_{T^*}$ implies that of $b_T = b_{T^*} \circ (F^* \times \delta) \circ \kappa$, where $\kappa$ denotes the flipping isomorphism exchanging the factors $F^*$ and $E$.

($\Rightarrow$) We use the factorization given in 5.15, i.e.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[dd, "S"'] & F \arrow[dr, "\delta_F"] & \\
& & F'' \\
\mathcal{L}^\infty(\mu) \arrow[r, "i"] & \mathcal{L}^1(\mu) \arrow[ur, "R"'] &
\end{tikzcd}$$

Dualizing it gives:

$$\begin{tikzcd}
E' & & F' \arrow[ll, "T^*"'] & & F' \arrow[ll, "F'", "\cong"'] \arrow[dl, "\delta_{F'}"] \\
& & & F'' \arrow[ul, "\delta_F^*"] \arrow[dl, "R^*"] & \\
\mathcal{L}^\infty(\mu)' \arrow[uu, "S^*"] \arrow[d, equal] & & \mathcal{L}^1(\mu)' \arrow[ll, "i^*"'] \arrow[d, equal] & & \\
(\mathcal{L}^1(\mu))'' & & \mathcal{L}^\infty(\mu) \arrow[dl, "i"] & & \\
& \mathcal{L}^1(\mu) \arrow[ul, hook, "\delta_{\mathcal{L}^1(\mu)}"] & & &
\end{tikzcd}$$

This factorization $T^* = (S^* \circ \delta_{\mathcal{L}^1}) \circ i \circ (R^* \circ \delta_{F'})$ shows that $T^*$ is integral, by the converse to 5.15. □

**5.17 Proposition.**
*Let $T : E \to F$ be linear between Banach spaces. Then $T$ is integral iff for every Banach space $G$ we have that $T \otimes G : E \otimes_\varepsilon G \to F \otimes_\pi G$ is continuous.*

For a proof see [**14**, 17.4.7].

It can be shown that all integral mappings are weakly compact, i.e. map the unit ball to a compact set in $(F, \sigma(F, F^*))$, see [**14**, 17.4.3].

Moreover one can prove the following result:

**5.18 Theorem.**
*If $E$ is a reflexive Banach space or a separable dual of some Banach space, then $\mathcal{N}(F, E) = \mathcal{I}(F, E)$ for all Banach spaces $F$.*

For a proof see [**14**, 17.6.5], [**14**, 17.6.6] and in particular case of Hilbert spaces also [**22**, 49.6,p506].

Recall that under the approximation property $\mathcal{K}(E, F) \cong E^* \,\varepsilon\, F = E^* \hat{\otimes}_\varepsilon F$. Hence $\mathcal{K}(E, F)^*$ coincides with the space of integral forms $E^* \times F \to \mathbb{R}$. Every integral operator $T : F \to E$ gives rise to an integral form $b_T : F \times E^* \to \mathbb{R}$ defined by $b_T(y, x^*) = x^*(Ty)$. And conversely every integral form $b : F \times E^* \to \mathbb{R}$ defines an operator $F \to E^{*\prime}$, and hence, if we assume reflexivity of $E$, an operator $T_b : F \to E$. Thus under these conditions we have:

$$\mathcal{K}(E, F)^* \cong \mathcal{I}(F, E)$$
$$\mathcal{K}(E, F)^{**} \cong \mathcal{I}(F, E)^* \cong \mathcal{N}(F, E)^* \cong \mathcal{L}(E, F)$$

Compare this to the dualities:

$$c_0' = \ell^1 \text{ and } c_0'' = (\ell^1)' = \ell^\infty$$

## Absolutely Summing Operators

**Definition.**
Let $T : E \to F$ be a linear operator between Banach spaces and $1 \le p < \infty$. Then $T$ is called *absolutely $p$-summing* (or $p$-summing for short) (see [**14**, 19.5])

iff $T$ maps weakly $p$-summing sequences to $p$-summing ones. With $\mathcal{S}_p$ we denote the class formed by these operators. Note that we could extend this definition to the case $p = \infty$, either by taking $c_0$-sequences and hence characterizing the fully complete operators, see [**14**, p420], or the $\ell^\infty$-sequences, and hence characterizing bounded (=continuous) linear operators.

Note furthermore that every $p$-summing operator $T : E \to F$ is continuous, since otherwise there would exists a bounded sequence $x_n \in E$ with $\|Tx_n\| > 2^n$. Hence $(2^{-n}x_n) \in \ell^p[E]$ but $(2^{-n}x_n) \notin \ell^p\{F\}$.

**5.19 Lemma.**
*Every p-summing operator induces a continuous linear map from $\ell^p[E] \to \ell^p\{F\}$. Thus we may consider $\mathcal{S}_p(E, F)$ as normed subspace of the space $L(\ell^p[E], \ell^p\{F\})$.*

Here we consider the space $\ell^p\{F\}$ supplied with the norm

$$\|(y_k)_k\|_\pi := \Big(\sum_k \|y_k\|_F^p\Big)^{1/p}.$$

As in 4.12 one can show that $\ell^p\{F\}$ is complete (see [**14**, 19.4.1]). For $p > 1$ it is however not isomorphic to $\ell^p \hat{\otimes}_\pi F$. Otherwise we would obtain for $E = \ell^p$, that $\ell^p \hat{\otimes}_\pi \ell^p = \ell^p\{\ell^p\} = \ell^p(\mathbb{N} \times \mathbb{N})$, which is not the case..

On $\ell^p[E]$ we consider the norm $\|(x_k)_k\|_\varepsilon := \sup\{(\sum_k |x^*(x_k)|^p)^{1/p} : x^* \in oE^*\}$. As in 4.33 one can show that $\ell^p[E]$ is complete and in fact isomorphic to $L(\ell^q, E)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1$, see [**14**, 19.4.3]. Furthermore $\ell^p \hat{\otimes}_\varepsilon E$ is an isometrically embedded subspace, see [**14**, 19.4.4].

It is obvious, that the inclusion $\ell^p\{E\} \to \ell^p[E]$ is a contraction (i.e. has norm $\leq 1$).

**Proof.** Let $T : E \to F$ be a $p$-summing operator. We will apply the closed graph theorem to $T_*$, so it is enough to consider a sequence $(x^{(k)}) \in \ell^p[E]$ which converges to $x$ in $\ell^p[E]$ and for which $T_*(x^{(k)})$ converges to $y$ in $\ell^p\{F\}$. Since $\|T_*(z)\|_\varepsilon \leq \|T\| \cdot \|z\|_\varepsilon$ we get $\|y - T_*x\|_\varepsilon \leq \|y - T_*x^{(k)}\|_\varepsilon + \|T(x^{(k)} - x)\|_\varepsilon \leq \|y - T_*x^{(k)}\|_\pi + \|T\|\,\|x^{(k)} - x\|_\varepsilon$, and hence $T_*x = y$. □

**5.20 Corollary.**
*An operator $T : E \to F$ is p-summing iff there exists a $R > 0$ such that*

$$\Big(\sum_k \|Tx_k\|^p\Big)^{1/p} \leq R \cdot \sup_{\|x^*\| \leq 1} \Big(\sum_k |x^*(x_k)|^p\Big)^{1/p}$$

*for all finite sequences $x_k$. The smallest such $R$ is the norm of $T_* : \ell^p[E] \to \ell^p\{F\}$, and is also denoted $\|T\|_{\mathcal{S}_p}$. In particular $\mathcal{S}_p(E, F)$ is a Banach space.*

**Proof.** ($\Rightarrow$) By 5.19 we have that $T_*$ is continuous, and hence we have the required property for $R := \|T_*\|$ and all (even the infinite) sequences in $\ell^p[E]$.

($\Leftarrow$) For $x = (x_k)_k \in \ell^p[E]$ we have $\|(Tx_k)_k\|_\pi = \sup_n (\sum_{k=1}^n \|Tx_k\|^p)^{1/p} \leq R \cdot \|(x_k)_{k \leq n}\|_\varepsilon \leq R \cdot \|(x_k)_k\|_\varepsilon < \infty$ and hence $(Tx_k)_k \in \ell^p\{F\}$. □

**5.21 Proposition.**
*For $p \leq q$ we have $\mathcal{S}_p \subseteq \mathcal{S}_q$.*

Under the same assumption it can be shown that also $\mathcal{N}_p \subseteq \mathcal{N}_q$, see [**14**, 19.7.5].

**Proof.** Let $r \geq p$ be given by $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$ and let $T \in \mathcal{S}_p$. Let $\lambda_k := \|Tx_k\|^{q/r}$. Then $\|Tx_k\| = \lambda_k^{r/q}$ and hence $\|\lambda_k T(x_k)\|^p = \|T(\lambda_k x_k)\|^p = \lambda_k^p \cdot \|Tx_k\|^p = \|Tx_k\|^{p(1+\frac{q}{r})} = \|Tx_k\|^q$ and so Hölder's inequality shows that

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/p} = \Big(\sum_k \|T(\lambda_k x_k)\|^p\Big)^{1/p} \leq \|T\|_{\mathcal{S}_p} \cdot \sup_{\|x^*\|\leq 1} \Big(\sum_k \lambda_k^p |x^*(x_k)|^p\Big)^{1/p}$$
$$\leq \|T\|_{\mathcal{S}_p} \cdot \Big(\sum_k \lambda_k^r\Big)^{1/r} \cdot \sup_{\|x^*\|\leq 1} \Big(\sum_k |x^*(x_k)|^q\Big)^{1/q}$$
$$\leq \|T\|_{\mathcal{S}_p} \cdot \Big(\sum_k \|Tx_k\|^q\Big)^{1/r} \cdot \sup_{\|x^*\|\leq 1} \Big(\sum_k |x^*(x_k)|^q\Big)^{1/q}$$

Dividing by $\Big(\sum_k \|Tx_k\|^q\Big)^{1/r} = \Big(\sum_k \|Tx_k\|^q\Big)^{1/p-1/q}$ gives

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/q} \leq \|T\|_{\mathcal{S}_p} \cdot \sup_{\|x^*\|\leq 1} \Big(|x^*(x_k)|^q\Big)^{1/q}.$$

Thus $T \in \mathcal{S}_q$ by 5.20. $\square$

**5.22 Lemma. Summing via measures.**
*An operator $T$ is $p$-summing iff there exists some probability measure $\mu$ on $oE^*$ and an $M > 0$ such that $\|Tx\| \leq M \cdot \Big(\int_{oE^*} |x^*(x)|^p \, d\mu(x^*)\Big)^{1/p}$.*

**Proof.** Note that the right hand side is nothing else but $M \cdot \|\delta(x)\|_p$, where $\delta(x) \in C(oE^*)$.

($\Leftarrow$) If $\mu$ is an probability measure with that property, we have

$$\sum_k \|Tx_k\|^p \leq M^p \int_{oE^*} \sum_k |x^*(x_k)|^p \, d\mu(x^*) \leq M^p \cdot \sup\{\sum_k |x^*(x_k)|^p : x^* \in oE^*\}.$$

So $T \in \mathcal{S}_p$ by [**14**, 19.5.2].

($\Rightarrow$) Let $T \in \mathcal{S}_p(E, F)$. For every finite sequence $x = (x_1, \dots, x_n)$ in $E$ let $f_x \in C(oE^*)$ be defined by

$$f_x(x^*) := \|T\|_{\mathcal{S}_p}^p \cdot \sum_i |x^*(x_i)|^p - \sum_i \|Tx_i\|^p = \sum_i \Big(\|T\|_{\mathcal{S}_p}^p \cdot |x^*(x_i)|^p - \|Tx_i\|^p\Big).$$

The set $B := \{f_M : M \subseteq E \text{ is finite}\}$ is convex in $C(oE^*)$. In fact let $x$ and $y$ be two finite sequences in $E$ and $\lambda + \mu = 1$ with $\lambda \geq 0$ and $\mu \geq 0$. Let $z$ be the sequence obtained by appending $\mu^{1/p} y$ to $\lambda^{1/p} x$. Then

$$(\lambda f_x + \mu f_y)(x^*) = \sum_i \lambda\Big(\|T\|_{\mathcal{S}_p}^p |x^*(x_i)|^p - \|Tx_i\|^p\Big)$$
$$+ \sum_j \mu\Big(\|T\|_{\mathcal{S}_p}^p |x^*(y_j)|^p - \|Ty_j\|^p\Big)$$
$$= \sum_i (\|T\|_{\mathcal{S}_p}^p) |x^*(\lambda^{1/p} x_i)|^p - \|T(\lambda^{1/p} x_i)\|^p$$
$$+ \sum_j (\|T\|_{\mathcal{S}_p}^p) |x^*(\mu^{1/p} y_j)|^p - \|T(\mu^{1/p} y_j)\|^p$$
$$= \sum_k (\|T\|_{\mathcal{S}_p}^p) |x^*(z_k)|^p - \|T(z_k)\|^p = f_z(x^*).$$

By 5.20 we have that $\sup_{x^* \in oE^*} f_x(x^*) \geq 0$. Thus the open set $A := \{f \in C(oE^*) : \sup_{x^* \in oE^*} f(x) < 0\}$ is disjoint from $B$. So by [**2**, 7.2.1] there exists a regular Borel measure $\mu$ on $oE^*$ and a constant $\alpha$ such that $\langle \mu, f\rangle < \alpha \le \langle \mu, g\rangle$ for all $f \in A$ and $g \in B$. Since $0 \in B$ we have $\alpha \le 0$. Since $A$ contains the constant negative functions we have $\alpha = 0$ and $\mu(oE^*) > 0$. Without loss of generality we may assume $\|\mu\| = 1$. Hence for every $x \in E$ we have

$$0 \le \langle \mu, f_x\rangle = \int_{oE^*} \Big( \|T\|_{\mathcal{S}_p}^p |x^*(x)|^p - \|Tx\|^p \Big)\, d\mu(x^*)$$

and thus $\|Tx\|^p \le \|T\|_{\mathcal{S}_p}^p \cdot \int_{oE^*} |x^*(x)|^p \, d\mu(x^*)$. □

**5.23 Lemma.**
*Every continuous linear operator $T : F \to \ell^\infty(X)$ defined on a subspace $F$ of a normed space $E$ extends to a continuous linear operator $\tilde{T} : E \to \ell^\infty(X)$ with $\|\tilde{T}\| = \|T\|$.*

**Proof.** Every $\mathrm{ev}_x \circ T$ admits a continuous extension $\ell_x : E \to \mathbb{R}$, with $\|\ell_x\| = \|ev_x \circ T\| \le \|T\|$. Thus $\tilde{T} := (\ell_x)_x$ is the required extension. □

**5.24 Theorem. Factorization of absolutely summing operators.**
*The operators $T$ in $\mathcal{S}_p$ are characterized by the existence of a compact space $K$ and a measure $\mu$ on $K$ such that we have the following factorization:*

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, dotted] & F \arrow[r, "\delta"] & \ell^\infty(oF^*) \\
C(K) \arrow[rr, hook, "i"] & & \mathcal{L}^p(\mu) \arrow[u, dotted]
\end{tikzcd}$$

*or, for $p = 2$, equivalently*

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, dotted] & F \\
C(K) \arrow[r, hook, "i"] & \mathcal{L}^2(\mu) \arrow[u, dotted]
\end{tikzcd}$$

**Proof.** ($\Leftarrow$) It is enough to show that the canonical mapping $i : C(K) \to \mathcal{L}^p(\mu)$ is absolutely $p$-summable. Since then $\delta \circ T$ is absolutely $p$-summable, since $\mathcal{S}_p$ is an ideal. And hence $T$ is in $\mathcal{S}_p$, since $\delta$ is an embedding. So let $\delta_x$ be the point measure at $x$. Then for $f_k \in C(K)$ we have

$$\begin{aligned}
\sum_k \|i(f_k)\|_p^p &= \int_K \sum_k |f_k(x)|^p \, d\mu(x) = \int_K \sum_k |\delta_x(f_k)|^p \, d\mu(x) \\
&\le \mu(K) \cdot \sup\Big\{ \sum_k |\nu(f_k)|^p : \nu \in oC(K)^* \Big\},
\end{aligned}$$

hence the natural mapping $i$ belongs to $\mathcal{S}_p$ by [**14**, 19.5.2].

($\Rightarrow$) By 5.22 there is some probability measure $\mu \in \mathcal{M}(oE^*)$ such that

$$\|Tx\| \le M \cdot \Big( \int_{oE^*} |x^*(x)|^p \, d\mu(x^*) \Big)^{1/p}.$$

The map $\delta : E \to C(oE^*)$, $x \mapsto \mathrm{ev}_x$ is isometric by what we have shown in the proof of 5.11. Now consider the diagram

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[dr, "S"] \arrow[dd, "\delta"'] & & F \arrow[dr, "\delta"] & \\
& H \arrow[ur, dotted, "R"] \arrow[dr, hook] & & \ell^\infty(oF^*) \\
C(oE^*) \arrow[rr, "i"] & & \mathcal{L}^p(\mu) \arrow[ur, dotted, "\tilde{R}"'] &
\end{tikzcd}$$

where $H$ denotes the image of $\delta \circ i$ in $\mathcal{L}^p(\mu)$. The operator $T$ factorizes via a continuous linear operator $R : H \to F$, since $\|Tx\| \leq M \cdot \|i(\delta(x))\|_p$ for some $M > 0$. By [**14**, 7.4.4] it extends to a continuous linear operator $\tilde{R} : \mathcal{L}^p(\mu) \to \ell^\infty(oF^*)$.

If $p = 2$ then the closure of the image of $H$ is a direct summand in $\mathcal{L}^2(\mu)$. Using the ortho-projection $P$ onto $H$ we get the factorization $R \circ P \circ (i \circ \delta) = R \circ i \circ \delta = R \circ S = T$. □

**5.25 Corollary.**
*Every integral operator is absolutely 1-summing, i.e. $\mathcal{I} \subseteq \mathcal{S}_1$.*

**Proof.** Let $T : E \to F$ be integral. By 5.15 we have a factorization

$$\begin{tikzcd}
E \arrow[rrr, "E"] \arrow[d, "R"'] & & & F \arrow[r, hook, "\delta"] & F'' \arrow[r, hook] & \ell^\infty(oF^*) \\
C(K) \arrow[r] & \mathcal{L}^\infty(\mu) \arrow[r, hook] & \mathcal{L}^1(\mu) \arrow[rr] & & C(K)' \arrow[u, "S^*"'] &
\end{tikzcd}$$

So $T \in \mathcal{S}_1$ by 5.24. □

## Approximable Operators

**Definition.**
For $T \in L(E, F)$ the *approximation numbers* $a_n(T)$ are defined by $a_n(T) := \inf\{\|T - S\| : \dim S(E) \leq n\}$. Obviously $\|T\| \geq a_n(T) \geq a_{n+1}(T)$ and $a_n(T) \to 0$ iff $T$ is in the closure of $E^* \otimes F$ in $L(E, F)$.

For $1 \leq p < \infty$ an operator $T$ is called *$p$-approximable* (see [**14**, 19.8]) iff the sequence of approximation-numbers $a_n(T)$ belongs to $\ell^p$. The class of all $p$-approximable operators will be denoted $\mathcal{A}_p$.

**5.26 Auerbach's Lemma.**
*Let $E$ be a finite dimensional Banach space. Then there are unit vectors $x_i \in E$ and $x_i^* \in E^*$ with $x_i^*(x_j) = \delta_{i,j}$ for $1 \leq i, j \leq \dim E$.*

**Proof.** Let $e_1, \dots, e_n$ be an algebraic basis of $E$. For the compact set $K := oE^*$ we consider the continuous map $f : K^n \to \mathbb{R}$, $(x_1^*, \dots, x_n^*) \mapsto |\det(x_j^*(e_i))|$. Let $(x_1^*, \dots, x_n^*)$ be a point where it attains its maximum. Since the $e_j$ are linearly independent this maximum is positive. Hence there is a unique solution with $x_j \in E$ of the equations

$$\sum_j x_j^*(e_i) x_j = e_i \text{ for } 1 \leq i \leq n.$$

Applying some $x_k^*$ to this equation, yields the equations

$$\sum_j x_j^*(e_i)\, x_k^*(x_j) = x_k^*(e_i) \text{ for } 1 \le i \le n.$$

whose unique solution is $x_j^*(x_i) = \delta_{i,j}$.

$$\begin{aligned} f(x_1^*,\dots,x_n^*)\cdot|\det(y_j^*(x_i))| &= |\det(x_j^*(e_i))\cdot\det(y_j^*(x_i))| \\ &= |\det(\sum_k x_k^*(e_i)\, y_j^*(x_k))| = |\det(y_j^*(e_i))| \\ &= f(y_1^*,\dots,y_n^*) \le f(x_1^*,\dots,x_n^*) \text{ for all } y_i^* \in K. \end{aligned}$$

Thus $|\det(y_j^*(x_i))| \le 1$. Choosing $y_j^* = x_j^*$ for all $j \ne k$ shows that $|y_k^*(x_k)| \le 1$ and hence $\|x_k\| \le 1$. From $1 = x_j^*(x_j) \le \|x_j^*\|\,\|x_j\|$ we conclude that $\|x_j\| = 1 = \|x_j^*\|$. □

**5.27 Lemma.**
*Let $T \in \mathcal{L}(E,F)$ be such that $\dim T(E) = k < \infty$. Then $T$ can be written as $T = \sum_{j=1}^k \lambda_j\, x_j^* \otimes y_j$ with $\|x_j\| \le 1$ and $\|y_j\| \le 1$ and $0 < \lambda_j \le \|T\|$.*

**Proof.** We may assume that $F = T(E)$. By 5.26 we have a biorthogonal sequence $y_j$ and $y_j^*$ for $F$. Let $\lambda_j := \|T^* y_j^*\|$. Then $0 < \lambda_j \le \|T^*\| = \|T\|$ and $x_j^* := \frac{1}{\lambda_j} T^* y_j^* \in oE^*$. So we have $Tx = \sum_j y_j^*(Tx)\, y_j = \sum_j \lambda_j\, x_j^*(x)\, y_j$ . □

**5.28 Corollary.**
*We have $\mathcal{A}_1 \subseteq \mathcal{N}$.*

**Proof.** See [**14**, 19.8.5]. Let $T \in \mathcal{A}_1(E,F)$. We have to show that it can be written as $T = \sum_n \lambda_n\, x_n^* \otimes y_n$ with $x_n^* \in oE^*$, $y_n \in oF$ and $\lambda \in \ell^1$.
Let $\varepsilon > 0$. Choose $T_n$ with $\dim T_n(E) \le 2^n$ and $\|T - T_n\| \le (1+\varepsilon)\, a_{2^n}(T)$. Let $D_n := T_{n+1} - T_n$. Then $d_n := \dim D_n(E) \le 3\cdot 2^n$ and since $a_n(T) \to 0$ we have $\|T - T_n\| \to 0$, hence $T = \sum_n D_n$. By 5.27 we have $T = \sum_{n=0}^\infty \sum_{i=1}^{d_n} \lambda_{n,i}\, x_{n,i}^* \otimes y_{n,i}$, with $x_{n,i}^* \in oE^*$, $y_{n,i} \in oF$ and $0 \le \lambda_{n,i} \le \|D_n\|$. We estimate as follows

$$\begin{aligned} \sum_n \sum_{i=1}^{d_n} \lambda_{n,i} &\le \sum_n d_n\, \|D_n\| \le 3 \sum_n 2^n (\|T_{n+1} - T\| + \|T_n - T\|) \\ &\le 3 \cdot \sum_n 2^n (1+\varepsilon)(a_{2^{n+1}}(T) + a_{2^n}(T)) \\ &\le 3 \cdot \sum_n 2^{n+1} (1+\varepsilon)\, a_{2^n}(T) \\ &\le 2^2\, 3\, (1+\varepsilon) \sum_n 2^{n-1}\, a_{2^n}(T) \\ &\le 2^2\, 3\, (1+\varepsilon) \sum_n a_n(T) \quad \text{since } a_n(T) \text{ is decreasing} \end{aligned}$$

to conclude that $(\lambda_{n,i})_{n,i} \in \ell^1$. □

**5.29 Proposition.**
*Let $0 < p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $\mathcal{A}_q \circ \mathcal{A}_p \subseteq \mathcal{A}_r$. In particular we will use $\mathcal{A}_2 \circ \mathcal{A}_2 \subseteq \mathcal{A}_1$.*

More generally one has for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$:

$$\begin{aligned}
\mathcal{A}_p \circ \mathcal{A}_q &\subseteq \mathcal{A}_r, \text{ see [\textbf{14}, 19.10.1]}\\
\mathcal{S}_p \circ \mathcal{S}_q &\subseteq \mathcal{S}_r, \text{ see [\textbf{14}, 19.10.3]}\\
\mathcal{N}_p \circ \mathcal{S}_q &\subseteq \mathcal{N}_r, \text{ see [\textbf{14}, 19.10.5]}\\
\mathcal{S}_p \circ \mathcal{N}_q &\subseteq \mathcal{N}_r, \text{ see [\textbf{14}, 19.10.5]}\\
\text{and in particular } \mathcal{N}_p \circ \mathcal{N}_q &\subseteq \mathcal{N}_r
\end{aligned}$$

**Proof.** We have $a_{n+m}(S \circ T) \le a_n(S)\, a_m(T)$:
In fact let $R := S_0 \circ T + (S - S_0) \circ T_0$ for finite dimensional $S_0$ and $T_0$. Then $a_{n+m}(S \circ T) \le \|S \circ T - R\| = \|(S - S_0) \circ (T - T_0)\| \le \|S - S_0\| \cdot \|T - T_0\|$ and hence $a_{n+m}(S \circ T) \le a_n(S) \cdot a_m(T)$.

Using the Hölder inequality for $\frac{r}{p} + \frac{r}{q} = 1$ we obtain:

$$\begin{aligned}
\Big(\sum_n a_n(S \circ T)^r\Big)^{1/r} &\le 2^{1/r}\Big(\sum_n a_{2n}(S \circ T)^r\Big)^{1/r}\\
&\le 2^{1/r}\Big(\sum_n a_n(S)^r \cdot a_n(T)^r\Big)^{1/r}\\
&\le 2^{1/r}\Big(\sum_n a_n(S)^p\Big)^{1/p} \cdot \Big(\sum_n a_n(T)^q\Big)^{1/q} \qquad \square
\end{aligned}$$

## Ideals for Hilbert Spaces

**5.30 Proposition.**
*An operator $T : E \to F$ between Hilbert spaces is $p$-approximable provided $(\langle Te_n, f_n\rangle)_n \in \ell^p$ for all orthonormal sequences $e_n$ and $f_n$.*

It can be shown that the converse is valid as well, see [**14**, 20.2.3]. Using the *polar decomposition* $T = U \cdot |T|$ one shows easily that

$$\begin{aligned}
T \in \mathcal{A}_p &\Leftrightarrow |T| \in \mathcal{A}_p\\
&\Leftrightarrow |T| \text{ is compact and the sequence of Eigen-values of } |T| \text{ belongs to } \ell^p.
\end{aligned}$$

Moreover one can show that the Eigen-values of $|T|$ are the approximation numbers $a_n(T)$. So those operators are up to isomorphisms just the multiplications with $\ell^p$-sequences. The operators in $\mathcal{A}_p$ are also called of *Schatten* or of *von Neumann class* $p$.

Note that a non-normal compact operator need not have any Eigen-values and the spectrum can be $\{0\}$. Take for example the operator $T : \ell^2 \to \ell^2$ given by $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$. Obviously $T$ is compact and from $Tx = \lambda x$ for $\lambda \ne 0$ one inductively obtains $x_i = 0$.

**Proof.** By 5.4 we conclude that $T$ is compact and hence admits by 5.3 a representation $Tx = \sum_n \lambda_n \langle e_n, x\rangle f_n$ with $\lambda_n \to 0$ and orthonormal sequences $e_n$ and $f_n$. Since $\lambda_n = \langle Te_n, f_n\rangle$ we have that $(\lambda_n)_n \in \ell^p$. By applying a permutation and putting signs to $f_n$ we may assume that $0 < \lambda_{n+1} \le \lambda_n$ Let $T_n(x) := \sum_{k<n} \lambda_k \langle e_k, x\rangle f_k$.

Then

$$a_n(T) \le \|T - T_n\| = \sup\Big\{\Big\|\sum_{k\ge n} \lambda_k \langle e_k, x\rangle f_k\Big\| : x \in oE\Big\}$$

$$= \sup\Big\{\Big(\sum_{k\ge n} \lambda_k^2 |\langle e_k, x\rangle|^2\Big)^{1/2} : x \in oE\Big\} \le \lambda_n,$$

hence $T \in \mathcal{A}_p$. □

**5.31 Proposition.**
*For Hilbert spaces we have $\mathcal{S}_2 \subseteq \mathcal{A}_2$.*

**Proof.** We estimate using 5.20 for orthonormal families $e_k$ and $f_k$ as follows

$$\Big(\sum_k \|Te_k\|^2\Big)^{1/2} \le \|T\|_{\mathcal{S}_2} \cdot \sup_{\|x\|\le 1} \Big(\sum_k |\langle x, e_k\rangle|^2\Big)^{1/2}$$

By the Cauchy-Schwarz inequality $|\langle Te_k, f_k\rangle| \le \|Te_k\| \cdot \|f_k\| = \|Te_k\|$ we have $\sum_k \langle Te_k, f_k\rangle^2 \le \sum_k \|Te_k\|^2 < \infty$ and hence $T \in \mathcal{A}_2$ by 5.20. □

In addition the following duality result holds:

**5.32 Proposition. Duals of the operator ideals.**
*See* [**14**, 20.2.5].

$$\mathcal{A}_p^* = \mathcal{A}_q \text{ for } 1 < p < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \quad \mathcal{K}' = \mathcal{A}_1, \quad \mathcal{A}_1' = L$$

One can show:

**5.33 Proposition, Hilbert-Schmidt-Operators.**
*See* [**14**, 20.5.1]. *For Hilbert spaces one has the following identities*

$$\mathcal{A}_2 = \mathcal{N}_q = \mathcal{S}_p \text{ for } 1 \le p < \infty \text{ and } 1 < q \le \infty.$$

*The operators in this class are called Hilbert-Schmidt operators.*

**5.34 Proposition.**
*See* [**14**, 20.5.2] *and* [**14**, 20.2.7]. *For a continuous linear operator $T : E \to F$ between Hilbert spaces the following statements are equivalent:*

1. *$T$ is Hilbert-Schmidt*
2. *$T$ has the following lifting property: For every quotient map $F_1 \to F$ between Banach spaces there exists a continuous linear lift $T_1 : E \to F_1$.*
3. *$T$ has the following extension property: For every embedding $E_1 \to E$ of a Banach space there exists a continuous linear extension $T_1 : E_1 \to F$.*
4. *$T$ can be factored over $\ell^1$;*
5. *$T$ can be factored over $c_0$;*
6. *$T^*$ is Hilbert Schmidt;*
7. *$(\|Te_i\|)_{i\in I}$ belongs to $\ell^2(I)$ for some (any) orthonormal basis $(e_i)_{i\in I}$.*

**5.35 Proposition, Trace-class operators.**
*See* [**14**, 20.2.4+20.2.5+20.2.8]. *One has the following identities*

$$\mathcal{A}_1 = \mathcal{N}_1 = \mathcal{A}_2 \circ \mathcal{A}_2.$$

*This class is also called trace-class.*

## Overview of Operator Ideals

**5.36.** We have the following inclusions:

$$
\begin{array}{ccccccc}
\mathcal{A}_1 & \overset{5.28}{\subseteq} & \mathcal{N}_1 & \overset{5.14}{\subseteq} & \mathcal{J}_1 & \overset{5.25}{\subseteq} & \mathcal{S}_1 \\
\text{(clear)} \cap\!| & & 5.21 \cap\!| & & & & 5.21 \cap\!| \\
\mathcal{A}_p & & \mathcal{N}_p & & \overset{[\mathbf{14},\ 19.7.8]}{\subseteq} & & \mathcal{S}_p
\end{array}
$$

For Hilbert spaces we have the following results:

$$
\begin{array}{cccccccc}
p = 1 & \mathcal{A}_1 & \overset{5.28}{=} & \mathcal{N}_1 & \overset{5.18}{=} & \mathcal{J}_1 & \subset & \mathcal{S}_1 \\
 & \cap & & \cap & & & & 5.33 \parallel \\
1 < p < \infty & \mathcal{A}_2 & \overset{5.33}{=} & \mathcal{N}_p & & \overset{5.33}{=} & & \mathcal{S}_p \\
 & \cap & & \parallel & & & & \cap \\
p = \infty & \mathcal{A}_\infty & & \mathcal{N}_\infty & & \subset & & \mathcal{S}_\infty
\end{array}
$$

# 6. Projective Representations

In this chapter we will consider locally convex spaces $E$ for which the projective representation $E = \varprojlim_p E_p$ has additional properties in the sense that the connecting morphisms $\Phi : E_p \to E_q$ can be chosen as elements of certain operator ideals.

## Precompact Sets

Recall that we have shown in [**2**, 6.4.2] that a subset of a complete space is compact iff it is closed and precompact. We also have:

**6.1 Lemma.**

1. *The continuous linear image of a precompact set is precompact.*
2. *The closure of a precompact set is precompact.*
3. *A subset of a precompact set is precompact.*
4. *If $E$ is embedded into $F$ and $K \subseteq E$ is precompact in $F$ then also in $E$.*

**Proof.** (1) Let $f : E \to F$ be continuous linear, $K \subseteq E$ precompact and $V$ a 0-neighborhood in $F$. Then there is some finite set $A \subseteq K$ such that $K \subseteq A + f^{-1}V$, and hence $f(K) \subseteq f(A + f^{-1}V) = f(\bigcup_{a\in A} a + f^{-1}V) = \bigcup_{a\in A} f(a) + f(f^{-1}V) \subseteq f(A) + V$.

(2) Let $K$ be precompact and $U$ be a closed 0-neighborhood in $E$. Then there is some finite set $A \subseteq K$ such that $K \subseteq A + U$. But then $\overline{K} \subseteq \overline{\bigcup_{a\in A} a + U} = \bigcup_{a\in A} a + \overline{U} = A + U$.

(3) Is obvious, since it is enough to find the finite set as subset of $E$.

(4) Let $U$ be a 0-neighborhood in $E$. Then there exists a 0-neighborhood $V$ in $F$ with $V \cap E = U$. By assumption there is some finite subset $A \subseteq K$ (!) such that $K \subseteq A + V$, hence $K = K \cap E = (A + V) \cap E = A + (V \cap E) = A + U$, since $A \subseteq K \subseteq E$. □

**6.2 Corollary.**
*A set $K \subseteq E$ is precompact in $E$ iff it is relatively compact in the completion $\hat{E}$.*

**Proof.** ($\Rightarrow$) Let $K \subseteq E$ be precompact. By the lemma the closure $\hat{K}$ of $K$ in $\hat{E}$ is precompact (and closed) hence compact.

($\Leftarrow$) Let $K \subseteq E$ be relatively compact in $\hat{E}$. Then $\bar{K}$ is compact and hence precompact in $\hat{E}$. By the lemma the subset $K$ is precompact in $\hat{E}$ and also in $E$. □

**6.3 Proposition.**
*A subset of a product is precompact iff it is contained in a product of pre-compact subsets.*

*A subset of a coproduct is precompact iff it is contained in a finite product of pre-compact sets.*

**Proof.** Since the completion of $\prod_i E_i$ is $\prod_i \hat{E}_i$ the first part follows from Tychonoff's theorem for compact sets.

Since the completion of $\coprod_i E_i$ is $\coprod_i \hat{E}_i$ the second part follows, since bounded sets are contained in a finite subproduct. $\square$

## Schwartz Spaces

**6.4 Definition.**
A locally convex space is called *Schwartz* iff for every 0-neighborhood $U$ there exist another one $V \subseteq U$, such that the connecting morphism $E_{(V)} \to E_{(U)}$ is pre-compact (i.e. the image of the unit-ball is pre-compact), or equivalently that $\widehat{E_{(V)}} \to \widehat{E_{(U)}}$ is compact.

Note that $E$ is Schwartz iff

$$\forall U \in \mathcal{U}\; \exists V \in \mathcal{U}, V \subset U,\; \forall \varepsilon > 0\; \exists M \overset{\text{finite}}{\subseteq} E : V \subseteq M + \varepsilon U.$$

This just expresses the fact, that the image of the unit-ball under the natural mapping from $E_{(V)} \to E_{(U)}$ is pre-compact. Comparison with the definition of quasi-normed spaces (see 3.46) shows that every Schwartz space is quasi-normed. And if every bounded set $B$ is precompact, then there is a finite set $M \subseteq E$ such that $B \subseteq M + \varepsilon U$, and we have the converse implication.
Let $B \subseteq E$ be bounded in a Schwartz space. Then for every 0-neighborhood $U$ there exists another one $V \subseteq U$ such that $E_{(V)} \to E_{(U)}$ has precompact image on every bounded set $B$. Hence the image is relatively compact in $\hat{E}_{(U)}$ and thus also in $\prod_U \hat{E}_{(U)}$. Since $E$ embeds into this product $B$ is precompact in $E$. Thus we have shown:

**6.5 Proposition. Schwartz versus quasi-normable spaces.**
*A locally convex space $E$ is Schwartz iff it is quasi-normable and every bounded set is precompact.* $\square$

**6.6 Corollary. Schwartz versus semi-Montel spaces.**
*A Schwartz space is semi-Montel iff it is quasi-complete, i.e. every bounded closed subset is complete.* $\square$

**6.7 Corollary.**
*A space $E$ is Schwartz iff $\mathcal{L}(E,F) = \mathcal{K}(E,F)$ for all Banach spaces $F$.*

**Proof.** ($\Rightarrow$) Let $T : E \to F$ be continuous linear with values in a Banach space $F$. Since $E$ is Schwartz, we can find a 0-neighborhood $V \subseteq U := T^{-1}(oF)$, such that the canonical mapping $E_{(V)} \to E_{(U)}$ is precompact. Since $U = T^{-1}(oF)$ we obtain a factorization:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d] & F \\ E_{(V)} \arrow[r, "pcp."'] & E_{(U)} \arrow[u, dotted] \end{tikzcd}$$

hence $T$ has precompact (=relatively compact) image on $V$.

($\Leftarrow$) By assumption the canonical linear map $E \to E_{(U)} \hookrightarrow \widehat{E_{(U)}}$ into the Banach space $\widehat{E_{(U)}}$ is compact for every 0-neighborhood $U$, i.e. there is some 0-neighborhood $V$ such that the image of $V$ in $\widehat{E_{(U)}}$ is relatively compact. Since we may assume that $V \subseteq U$, we obtain that the connecting map $\widehat{E_{(V)}} \to \widehat{E_{(U)}}$ is compact. $\square$

## Schwartz Function Spaces

**6.8 Proposition.**
*For every completely regular space $X$ the space $C(X)$ is quasi-normable.*

**Proof.** Let $U := \{f \in C(X) : |f(x)| \leq \varepsilon \text{ for } x \in K\}$ be a typical 0-neighborhood in $C(X)$ with $\varepsilon > 0$ and compact $K \subseteq X$. We choose $V = U$, and consider the bounded set $B := \{f : |f(x)| \leq 2\varepsilon \text{ for all } x\}$. We claim that $V \subseteq B + \lambda U$ for $0 < \lambda \leq 1$. Since $X$ is completely regular their exists an $h \in C(X, [0,1])$ with $h|_K = 1$ and $h|_A = 0$, where $A := \{x \in X : |f(x)| \geq 2\varepsilon\}$. Then $f = (1-\lambda)hf + (1-(1-\lambda)h)f \in B + \lambda U$, since $|(1-\lambda)(hf)(x)| \leq |(hf)(x)| \leq 2\varepsilon$ for all $x \in X$ and $|((1-(1-\lambda)h)f)(x)| = |((1-h)+\lambda h)f)(x)| = |\lambda|\,|f(x)| \leq \varepsilon\,\lambda$ for $x \in K$. $\square$

**6.9 Proposition.**
*For completely regular spaces $X$ the following statements are equivalent:*

1. *$C(X)$ is Schwartz;*
2. *Every bounded set in $C(X)$ is precompact;*
3. *Every compact set in $X$ is finite.*

Note that a compactly generated space with property (3) of 6.9 has to be discrete, since every subset has obviously open trace on the finite subsets.

**Proof.** ($1\Leftrightarrow 2$) is obvious from 6.5 and 6.8.

($2\Rightarrow 3$) Suppose there is some infinite compact $K \subseteq X$. By assumption the bounded set $B := \{f : \|f\| \leq 1\}$ has to be precompact. Thus for the 0-neighborhood $U := \{f : |f(x)| < 1 \text{ for all } x \in K\}$ there has to exist finitely many $f_1, \dots, f_n$ with $B \subseteq \{f_1, \dots, f_n\} + U$. Choose $n$ different points $x_j \in K$ and an $f \in B$ such that for $1 \leq j \leq n$ we have

$$f(x_j) = \begin{cases} -1 & \text{for } f_j(x_j) \geq 0 \\ +1 & \text{otherwise} \end{cases}.$$

Thus $\|f - f_j\|_\infty \geq |f(x_j) - f_j(x_j)| \geq 1$, i.e. $f \notin \{f_1, \dots, f_n\} + U$. A contradiction.

($3\Rightarrow 2$) Since compact sets $K$ are finite, we have that every bounded set in $C(K) = \mathbb{R}^K$ is precompact, and hence the same is true for the subspace $C(X)$ of $\prod_K C(K)$. $\square$

In a similar spirit is the following

**6.10 Proposition.**
*See* [**14**, 11.7.7].
*For completely regular spaces $X$ the following statements are equivalent:*

1. *$C(X)$ is (semi-)Montel;*
2. *$C(X)$ is (semi-)reflexive;*
3. *$X$ is discrete.*

In this context one has the following two results:

**6.11 Nachbin-Shirota-theorem.**
*See* [**14**, 13.6.1].
*For completely regular spaces $X$ the following statements are equivalent:*

1. *$C(X)$ is (ultra-)bornological;*
2. *$X$ is realcompact, i.e. every real-valued algebra homomorphism on $C(X)$ is a point evaluation.*

**6.12 Nachbin-Shirota-theorem.**
*See* [**14**, 11.7.5].
*For completely regular spaces $X$ the following statements are equivalent:*

1. *$C(X)$ is barreled;*
2. *every bounding closed subset is compact.*

Here a set $K \subseteq X$ is called *bounding*, iff $f(K)$ is bounded in $\mathbb{R}$ for every $f \in C(X)$.

**6.13 Proposition. FM-spaces.**
*See* [**14**, 11.6.1+11.6.2].
*For a Fréchet space $E$ the following statements are equivalent:*

1. *$E$ is Montel;*
2. *$E$ is separable and every $\sigma(E^*, E)$-convergent sequences in $E^*$ is $\beta(E^*, E)$-convergent.*
3. *$E^*_\beta$ is Schwartz;*

**Proof.** We show only that Fréchet Montel spaces are separable.
By assumption $E$ embeds into $\prod_n E_{U_n}$ for some countable 0-neighborhood basis $\{U_n\}_n$. Let $p_n$ be the corresponding seminorms. So it is enough to show that $E_{(U_n)}$ is separable, and without loss of generality we may assume $n = 1$. Suppose $E_{(U_1)}$ is not separable. Then there has to exist an $\varepsilon > 0$ and a countable subset $A_1 \subseteq E$, with $p_1(x - y) \geq \varepsilon$ for all $x \neq y$ in $A_1$ (Otherwise we could take for every $\varepsilon > 0$ a maximal and hence countable subset $A_\varepsilon$ and then $A := \bigcup_n A_{1/n}$ would be countable and dense in $E$). Since the sets $U_n$ have to be absorbing, we can choose inductively $\lambda_n > 0$ such that $A_n := A_{n-1} \cap \lambda_n U_n$ is still uncountable. Now choose recursively $x_n \in A_n \setminus \{x_1, \ldots, x_{n-1}\}$. For $n > m$ we have $x_n \in A_n \subseteq A_{m+1} \subseteq \lambda_m U_m$, and hence $p_m(x_n) \leq \lambda_m$. Thus $\{x_n : n \in \mathbb{N}\}$ is bounded and hence precompact. Since $E$ is Fréchet it is sequentially compact, i.e. we can find a subsequence $x_{n_j}$ converging to some $x \in E$. But then $p_1(x_{n_j} - x) < \frac{\varepsilon}{2}$ for sufficiently large $j$, a contradiction. □

Moreover one has the following characterization of duals:

**6.14 Proposition. Duals of FM-spaces.**
*See* [**10**, 4.4.38].
*For a locally convex space $E$ the following statements are equivalent:*

1. *$E$ is the strong dual of a Fréchet-Montel space (which can be chosen to be $E^*_\beta$);*
2. *$E$ is a (quasi-)complete DF-Schwartz space;*
3. *$E$ is bornological and the bornology of compact subsets has a countable base;*
4. *$E$ is bornological and Montel, and the von Neumann bornology has a countable base.*

We will show in (5) of 7.4 that on these spaces the topology is final with respect to all convergent sequences.

For Schwartz spaces one has:

**6.15 Proposition. FS-spaces.**
*See* [**14**, 11.6.3].
*For a Fréchet space $E$ the following statements are equivalent:*

1. *$E$ is Schwartz;*
2. *$E$ is separable and $\sigma(E^*, E)$-convergent sequences in $E^*$ converge Mackey with respect to the bornology of equi-continuous subsets.*

**6.16 Proposition. Duals of FS-spaces.**
*See* [**14**, 12.5.9] *and* [**10**, 4.4.39].
*For a locally convex space $E$ the following statements are equivalent:*

1. *$E$ is the strong dual of a Fréchet-Schwartz space (which can be chosen to be $E^*_\beta$);*
2. *$E$ is a (quasi-)complete $DF$-Schwartz space, and every 0-sequence converges Mackey;*
3. *$E$ is (ultra-)bornological and the bornology of bornologically compact subsets has a countable base;*
4. *$E$ is (ultra-)bornological, every bounded set is bornologically relatively compact, and the von Neumann bornology has a countable base.*
5. *$E$ is the inductive limit of a sequence of Banach spaces with compact connecting mappings.*

A space satisfying these equivalent conditions is called *Silva space*. From what we said about duals of Fréchet Montel spaces it follows that the $c^\infty$-topology on Silva spaces coincides with the locally convex topology.

Köthe gave an example of a Fréchet-Montel space which has $\ell^1$ as quotient, and hence cannot be Schwartz. See also [**14**, p233].

Hogbe-Nlend gave an example of a Fréchet Schwartz space without the approximation property, see [**14**, p416].

## Nuclear Spaces

**6.17 Theorem. Nuclear spaces.**
*The following statements are equivalent*

1. *$E$ is nuclear;*
2. *$E \otimes_\pi F = E \otimes_\varepsilon F$ for every Banach space $F$;*
3. *$\ell^1 \otimes_\pi E = \ell^1 \otimes_\varepsilon E$;*
4. *$\ell^1\{E\} = \ell^1\langle E\rangle$ topologically;*
5. *$\ell^1\{E\} = \ell^1[E]$ topologically;*
6. *The connecting maps of the projective representation can be chosen absolutely summable (or $\mathcal{S}_p$);*
7. *The connecting maps of the projective representation can be chosen nuclear (or $\mathcal{N}_p$);*
8. *The connecting maps of the projective representation can be chosen traceable (or $\mathcal{A}_p$);*
9. *The connecting maps of the projective representation can be chosen integral;*

10. *Every continuous linear map into a Banach space is nuclear.*

$$\begin{tikzcd}
2 \arrow[d] & 1 \arrow[l] & 10 \arrow[d, leftrightarrow] \\
3 \arrow[r] & 6 \arrow[r] \arrow[d] & 7 \arrow[ul] \arrow[d, leftrightarrow] \\
4 \arrow[u] & 5 \arrow[l] & 8,9
\end{tikzcd}$$

**Proof.** (1⇒2⇒3) and (5⇒4⇒3) are obvious.

(3⇒6) From (3) we obtain that $\ell^1\langle\hat{E}\rangle \cong \ell^1\{\hat{E}\}$. Thus for every $U \subseteq E$ there exists a $V \subseteq E$ and a $\delta > 0$ such that $\pi_U \leq \delta\varepsilon_V$, where

$$\pi_U((x_k)_k) := \sum_k p_U(x_k)$$

is the semi-norm associated to $U$ on $\ell^1\hat{\otimes}_\pi E \cong \ell^1\{E\}$, see 4.12, and where

$$\varepsilon_V((x_k)_k) := \sup\{\sum_k |y^*(x_k)| : y^* \in V^o\}$$

is the semi-norm associated to $U$ on $L(c_0, F) = \ell^1[E]$ and hence on the subspace $\ell^1\langle E\rangle \cong \ell^1\hat{\otimes}_\varepsilon E$, see 4.34. From this it follows by 5.20 that the connecting map is absolutely summable.

(6⇒5) For every $U$ we can find by assumption a $V$ such that the connecting map $\Phi : E_{(U)} \to E_{(V)}$ is absolutely summable. Hence if $(x_n) \in \ell^1[E]$, then the images are in $\ell^1[\hat{E}_{(V)}]$ and hence in $\ell^1\{\hat{E}_{(U)}\}$. Moreover

$$\begin{aligned}\|(x_k)_k\|_{\pi_U} &= \sum_k q_U(x_k) = \sum_k ||x_k||_U \\ &\leq \|\Phi\|_{\mathcal{S}_1} \cdot \sup\{\sum_k |x^*(x_k)| : x^* \in V^o\} \leq \|\Phi\|_{\mathcal{S}_1} \cdot \varepsilon_V((x_k)_k),\end{aligned}$$

Since $U$ was arbitrary we have (5).

(7⇒1) By assumption for every $U$ there exists a $U'$ such that the connecting map $\Phi_{U,U'}$ is nuclear (or integral). By [**14**, 17.3.8] we have that $\Phi_{U,U'} \otimes \hat{F}_{(V)} : \hat{E}_{(U')} \otimes_\varepsilon \hat{F}_{(V)} \to \hat{E}_{(U)} \otimes_\pi \hat{F}_{(V)}$ is continuous. Thus $\pi_{U,V} \leq c \cdot \varepsilon_{U,V}$ for some $c > 0$, i.e. $E \otimes_\varepsilon F = E \otimes_\pi F$. Recall that $\pi_{U,V}(x) := \inf\{\sum_k p_U(x_k)p_V(y_k) : z = \sum_k x_k \otimes y_k\}$ and $\varepsilon_{U,V}(\sum_k x_k \otimes y_k) := \sup\{|\sum_k x^*(x_k)y^*(y_k)| : x^* \in U^o, y^* \in V^o\}$ are the corresponding norms on $E_{(U)} \otimes_\pi E_{(V)}$ and on $E_{(U)} \otimes_\varepsilon E_{(V)}$.

Now let us show that for all mentioned ideals it is the same to assume that the connecting mappings belong to them.
In fact we have $\mathcal{A}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{I}_1 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2$. The composite of 3 maps in $\mathcal{S}_2$ belongs to $\mathcal{A}_2$, since using 5.24 the following diagram shows that it factors over a map between Hilbert spaces of class $\mathcal{S}_2 \subseteq \mathcal{A}_2$ by 5.31:

$$\begin{tikzcd}
E_3 \arrow[rr, "\mathcal{S}_2"] \arrow[dr] & & E_2 \arrow[r, "\mathcal{S}_2"] & E_1 \arrow[rr, "\mathcal{S}_2"] \arrow[dr] & & E_0 \\
& \mathcal{L}^2(\mu_3) \arrow[ur] \arrow[rrr, dotted, "\mathcal{S}_2\subseteq\mathcal{A}_2"] & & & \mathcal{L}^2(\mu_1) \arrow[ur] &
\end{tikzcd}$$

Since $(\mathcal{A}_2)^2 \subseteq \mathcal{A}_1$ we have $(\mathcal{S}_2)^6 \subseteq A_1$. Now choose for a given seminorm $p$ successively $p_6 \geq p_5 \geq \dots \geq p_1 \geq p$ such that the connecting maps all belong to $\mathcal{S}_2$. Then the connecting mapping $\widehat{E_{p_6}} \to \widehat{E_p}$ belongs to $A_1$.

This shows $(6 \Leftrightarrow 7 \Leftrightarrow 8 \Leftrightarrow 9)$.

$(7 \Leftrightarrow 10)$ Recall that a map $T : E \to F$ with values in a Banach space is called nuclear (see 5.6), iff it factors over a nuclear map $T_1 : E_1 \to F$ on some Banach space $E_1$. In fact for $E_1$ we may choose $\widehat{E_{(U)}}$ for some 0-neighborhood $U$. Now we can proceed as for the corresponding result 6.7 for compact mappings and Schwartz spaces. □

**6.18 Proposition. NF-spaces.**
*A Fréchet space is nuclear if $\ell^1\{E\} = \ell^1[E]$ or $\ell^1\{E\} = \ell^1\langle E\rangle$ holds algebraically.*

**Proof.** Since $\ell^1\{E\}$ and $\ell^1\langle E\rangle$ are Fréchet spaces it follows from the closed graph theorem that the identity is a homeomorphism. □

**6.19 Proposition. Consequences of nuclearity.**
*For a nuclear space the following is true:*

1. *It has a basis of Hilbert seminorms, i.e. seminorms which are induced from continuous bilinear symmetric forms.*
2. *It satisfies the approximation property.*
3. *It is Schwartz.*
4. *Bounded sets are precompact.*
5. *If it is Fréchet, then it is separable.*
6. *If it is normable, then it is finite dimensional.*

**Proof.** (1) By what we have shown in the proof of 6.17 every natural mapping $E \to E_p$ factors over some Hilbert-space $H$. Taking the norm $q$ of the Hilbert-space, we get a continuous seminorm $E \to H \to \mathbb{R}$, which dominates $p$.

(2) Since by (1) $E$ is a reduced projective limit of Hilbert-spaces, it satisfies the approximation property, by [**14**, 18.2.1].

(3) Since nuclear mappings are compact by 5.7, every nuclear space is Schwartz.

(4) is true for Schwartz spaces by 6.5.

(5) is true for Fréchet Montel spaces by 6.13.

(6) is true for normable Schwartz spaces by 6.5 together with [**2**, 4.4.5]. □

**6.20 Proposition.**
*A space $E$ is nuclear iff it has a basis of Hilbert-seminorms and $E \otimes_\varepsilon E = E \otimes_\pi E$.*

**Sketch of a proof.** By assumption given $U$ we can find $V$ such that the connecting map $\Phi_{U,V}$ is between Hilbert spaces and $\Phi_{U,V} \otimes \Phi_{U,V}$ is continuous from the $\varepsilon$-tensor product into the projective one. From this it is easily shown that $\Phi_{U,V}$ is Hilbert-Schmidt, and hence $E$ is nuclear. □

**6.21 Proposition.**
*Both the nuclear and the Schwartz spaces are stable with respect to initial structures, countable direct sums, quotients, completions, projective and injective tensor products and $\varepsilon$-products.*

**Proof.**

(Products) Let $E = \prod_i E_i$. Then a typical seminorm is of the form $p : x \mapsto \max_{i\in A} p_i(x_i)$, where $A$ is finite and $p_i$ are seminorms on $E_i$. Obviously $\widehat{E_p} = \prod_{i\in A} \widehat{(E_i)_{p_i}}$. For every $p_i$ we can find a seminorm $q_i \geq p_i$ such that the canonical mapping $\widehat{(E_i)_{q_i}} \to \widehat{(E_i)_{p_i}}$ is precompact/nuclear. Then the canonical mapping $\prod_{i\in A} \widehat{(E_i)_{q_i}} \to \prod_{i\in A} \widehat{(E_i)_{q_i}}$ is precompact/nuclear, in fact a finite product $\prod_{i\in A} T_i$ can be written as $\sum_{i\in A} \mathrm{inj}_i \circ T_i \circ \mathrm{pr}_i$ and hence belongs to the considered ideal. Hence we may use $q := \max_{i\in A} q_i$ as the required seminorm.

(Subspaces) First for Schwartz spaces. Let $E$ be a subspace of $F$. The seminorms on $E$ are the restrictions of seminorms $p$ on $F$. Let $q \geq p$ be a seminorm such that $F_q \to F_p$ is precompact. Since $E_{p|_E} \to F_p$ is an embedding we have the diagram:

$$\begin{tikzcd} E_{q|_E} \arrow[r] \arrow[d, hook] & E_{p|_E} \arrow[d, hook] \\ F_q \arrow[r] & F_p \end{tikzcd}$$

Since the bottom arrow is precompact, the same is true for the top arrow.

Now for nuclear spaces. The corresponding proof will not work for nuclear mappings, but for absolutely summable mappings, since the ideal $\mathcal{S}_1$ is obviously injective, i.e. if $T : E \to F_1 \hookrightarrow F$ belongs to $\mathcal{S}_1$ and $F_1$ is a closed subspace of $F$, then $T : E \to F_1$ belongs to $\mathcal{S}_1$.

(Countable Sums) first for Schwartz spaces. Recall that a basis of seminorms on a countable co-product $E = \coprod_k E_k$ is given by $\sup_k p_k$, where the $p_k$ run through the seminorms of $E_k$ and $\sup_k p_k : (x_k)_k \mapsto \sup_k p_k(x_k)$. By assumption we can find seminorms $q_k \geq p_k$ such that the connecting map $T_k : (E_k)_{q_k} \to (E_k)_{p_k}$ is precompact. Furthermore we may assume that its norm is less than $\frac{1}{2^k}$, by replacing $q_k$ with $2^k\|T_k\|q_k$. Now the following diagram shows that we get a natural bijection $\coprod_k (E_k)_{p_k} \cong (\coprod_k E_k)_{\sup_k p_k}$ which is an isometry iff we supply $\coprod_k (E_k)_{p_k}$ with the norm $(x_k)_k \mapsto \sup\{p_k(x_k) : k\}$ and analogously for the $q_k$.

$$\begin{tikzcd} \ker(\sup_k p_k) \arrow[rrr, equal] \arrow[d, hook] & & & \coprod_k \ker(p_k) \arrow[d, hook] \\ \coprod_k E_k \arrow[rrr, equal] \arrow[dr, "\sup_k p_k"] \arrow[dd] & & & \coprod_k E_k \arrow[dl, "\coprod_k p_k"'] \arrow[dd] \\ & \mathbb{R} & \coprod_k \mathbb{R} \arrow[l, "\sup"'] & \\ (\coprod_k E_k)_{\sup_k p_k} \arrow[ur] \arrow[rrr, leftrightarrow, "\cong"] & & & \coprod_k (E_k)_{p_k} \arrow[ul] \end{tikzcd}$$

Up to these isometries the connecting map is nothing else but

$$T := \coprod_k T_k : \coprod_k (E_k)_{q_k} \to \coprod_k (E_k)_{p_k}.$$

Since the finite subsums $\coprod_{k\leq n} T_k$ converge uniformly to $\coprod_k T_k$ on the unit-ball with respect to $p = \sup_k p_k$ and are precompact operators by the result on products, hence so is the infinite sum.

Now for nuclear spaces. We proceed as before using that the connecting mappings $T_k$ can be chosen of the form $T_k = \sum_j (\lambda_k)_j (x_k^*)_j \otimes (y_k)_j$ with $\lambda^k \in \ell^1$ and sequences $x_k^* \in o((E_k)_{q_k})^*$ and $y_k \in o(E_k)_{p_k}$. By replacing $q_k$ by $\|\lambda_k\|_1\, 2^k q_k$, we have that

$(\lambda_1, \lambda_2, \dots) \in \ell^1$ and hence the connecting mapping $T$ admits the representation $\sum_{k,j} (\lambda_k)_j (x_k^*)_j \otimes (y_k)_j$, where $(x_k^*)_j$ can be extended to the corresponding space, since $(E_k)_{q_k}$ embeds isometrically into it.

(Quotients) First for Schwartz spaces. Let $F := E/N$, where $N$ is a closed subspace and let $\pi : E \to F$ denote the quotient mapping. Let $\tilde{p}$ be a seminorm on $F$. By assumption there exists a seminorm $q$ on $E$ with $q \geq \tilde{p} \circ \pi$ and such that $E_q \to F_{\tilde{p}\circ\pi}$ is precompact. Let $\tilde{q}$ be the corresponding quotient semi-norm on $F$, see [**2**, 4.3.3]. Then $q \geq \tilde{q} \circ \pi \geq \tilde{p} \circ \pi$. Now the following diagram shows that we get a natural isometry $E_{\tilde{p}\circ\pi} \cong F_{\tilde{p}}$ and similarly for $\tilde{q}$.

$$\begin{tikzcd}
N \arrow[r, hook] \arrow[d, equal] & \ker\tilde{p}\circ\pi \arrow[r, equal] \arrow[d, hook] & \pi^{-1}(\ker\tilde{p}) \arrow[r, two heads] & \ker\tilde{p} \arrow[d, hook] \\
N \arrow[r, hook] & E \arrow[rr, two heads] \arrow[dr, "\tilde{p}\circ\pi"] \arrow[dd] & & F \arrow[dl, "\tilde{p}"'] \arrow[dd] \\
& & \mathbb{R} & \\
& E_{\tilde{p}\circ\pi} \arrow[ur, dotted, "\tilde{p}\circ\pi"] \arrow[rr, dotted, "\cong"] & & F_{\tilde{p}} \arrow[ul, dotted, "\tilde{p}"']
\end{tikzcd}$$

Another formulation of the same result would be an application of the isomorphy-theorem

$$F/\ker\tilde{p} \cong (E/N)/(\ker(\tilde{p}\circ\pi)/N) \cong E/\ker(\tilde{p}\circ\pi)$$

Hence we have the diagram:

$$\begin{tikzcd}
& F_{\tilde{q}} \arrow[r] & F_{\tilde{p}} \\
E_q \arrow[ur] \arrow[r, two heads] & E_{\tilde{q}\circ\pi} \arrow[u, "\cong"] \arrow[r, two heads] & E_{\tilde{p}\circ\pi}. \arrow[u, "\cong"]
\end{tikzcd}$$

Note that connecting morphisms are always quotient maps, since the projections $E \to E_q$ are. So the diagonal arrow is open, since it is up to the vertical isomorphism the connecting map $E_q \to E_{\tilde{q}\circ\pi}$. Hence the inverse image of the unit ball in $E_{\tilde{q}}$ is bounded in $E_q$. But then its image is precompact in $F_{\tilde{p}} \cong E_{\tilde{p}\circ\pi}$.

Now in order that the proof given for Schwartz spaces works for nuclear spaces, we can use the following: It is enough to consider the situation, where $E \to E_1 \to F$ is nuclear, $E \to E_1$ is a quotient map and $E$ a Hilbert space. But then the sequence $E_2 \to E \to E_1$ splits, where $E_2$ is the kernel of the quotient map $E \to E_1$, and hence $T : E_1 \to F$ can be written as $E_1 \to E \to E_1 \to F$ and thus is nuclear.

(Completions) Use that $\widehat{E_q} = \widehat{\hat{E}_{\hat{q}}}$, where $\hat{q}$ denotes the unique extension of $q$ to a seminorm on $\hat{E}$.

(Projective tensor product) for Schwartz spaces. Recall that the typical 0-neighborhoods of $E \otimes_\pi F$ are $U := \langle U_1 \otimes U_2 \rangle_{abs.conv.}$, where the $U_i$ are absolutely convex 0-neighborhoods in $E_i$. By assumption there are 0-neighborhoods $V_i \subseteq U_i$ in $E_i$ such that for every $0 < \varepsilon \leq 1$ there is a finite set $B_i$ such that $V_i \subseteq B_i + \varepsilon U_i$. Taking intersection with $U_i$ shows that $V_i \subseteq (B_i + \varepsilon U_i) \cap U_i \subseteq (B \cap 2U_i + \varepsilon U_i)$. In fact $b + \varepsilon u \in U$ implies that $b \in U - \varepsilon u \subseteq U - U \subseteq 2U$. Thus we may assume that $B_i \subseteq 2U_i$. Now we have that

$$\begin{aligned}
V_1 \otimes V_2 &\subseteq B_1 \otimes B_2 + \varepsilon B_1 \otimes U_2 + \varepsilon U_1 \otimes B_2 + \varepsilon^2 U_1 \otimes U_2 \\
&\subseteq B_1 \otimes B_2 + (2\varepsilon + 2\varepsilon + \varepsilon^2) U_1 \otimes U_2.
\end{aligned}$$

So let $V$ be the absolutely convex hull of $\frac{1}{5}V_1 \otimes V_2 \subseteq \frac{1}{4+\varepsilon}(V_1 \otimes V_2)$ and $B$ the finite dimensional bounded set $\frac{1}{4+\varepsilon}\langle B_1 \otimes B_2\rangle_{abs.conv.}$. Then $V \subseteq B + \varepsilon U$. Since $B$ is precompact, we can find a finite set $B_0 \subseteq B$ such that $B \subseteq B_0 + \varepsilon U$, and so $V \subseteq B_0 + 2\varepsilon U$.

For nuclear spaces $E$ and $F$ we take an arbitrary lcs $G$ and calculate as follows:

$$\begin{aligned}(E \otimes_\pi F) \otimes_\varepsilon G &\overset{E\text{ nucl.}}{\cong} (E \otimes_\varepsilon F) \otimes_\varepsilon G \cong E \otimes_\varepsilon (F \otimes_\varepsilon G) \\ &\overset{F\text{ nucl.}}{\cong} E \otimes_\varepsilon (F \otimes_\pi G) \overset{E\text{ nucl.}}{\cong} E \otimes_\pi (F \otimes_\pi G) \\ &\cong (E \otimes_\pi F) \otimes_\pi G.\end{aligned}$$

($\varepsilon$-product and $\varepsilon$-tensor product) For Schwartz spaces this follows from 6.33 since $E \otimes_\varepsilon F \subseteq E\,\varepsilon\, F \subseteq \mathcal{L}(E^*_\gamma, F)$ and $(E^*_\gamma)^*_\gamma = \hat{E}$ is Schwartz.

For nuclear spaces this follows, since by the approximation property $E\hat{\otimes}_\pi F = E\hat{\otimes}_\varepsilon F = \hat{E}\,\varepsilon\,\hat{F}$, and the first space is nuclear. □

## Examples of Nuclear Spaces

**6.22 Lemma.**
*The space $s$ of all fast falling sequences is nuclear.*

**Proof.** Recall that

$$s := \{x \in \mathbb{R}^{\mathbb{N}} : \|p \cdot x\|_\infty < \infty \text{ for all polynomials } p \text{ on } \mathbb{N}\}.$$

Since

$$\begin{aligned}\|p \cdot x\|_\infty &= \sup\Big\{|p(n)x_n| : n \in \mathbb{N}\Big\} \\ \le \|p \cdot x\|_1 &= \sum_n |p(n)x_n| \le \sum_n \Big|\frac{1}{n^2+1}(1+n^2)p(n)x_n\Big| \\ &\le \sup\Big\{\Big|(1+n^2)p(n)x_n\Big| : n \in \mathbb{N}\Big\} \cdot \sum_n \frac{1}{1+n^2}\end{aligned}$$

we may replace the supremum by the sum. Thus $x \mapsto p{\cdot}x$ is a well defined continuous linear mapping $s \to \ell^1$. If $p$ has no roots in $\mathbb{N}$ (which is no restriction to assume), then this mapping has dense image, since the finite sequences belong to $s$ and are dense in $\ell^1$. Moreover in this case the kernel of $\|\_\|_p$ is trivial, so the Banach space $\widehat{s_p}$ generated by $\|p \cdot \_\|_1$ is isomorphic to $\ell^1$. Given $p$ then $q : n \mapsto (1+n^2)p(n)$ induces the multiplication operator $\hat{s}_q \cong \ell^1 \to \ell^1 \cong \hat{s}_p$ by the summable sequence $\lambda := (\frac{1}{1+n^2})_n$, and hence is nuclear since it factors over the nuclear multiplication operator $\ell^\infty \to \ell^1$ with the same diagonal.

$$\begin{tikzcd} & s \arrow[dl, "q\cdot"'] \arrow[dr, "p\cdot"] & \\ \ell^1 \arrow[rr, "\lambda"] \arrow[dr] & & \ell^1 \\ & \ell^\infty \arrow[ur, "\lambda"'] & \end{tikzcd}$$ □

**6.23 Corollary. Nuclear function spaces.**
*All the spaces $C^\infty(X)$, $C^\infty_K(X)$, $C^\infty_c(X)$, $\mathcal{S}(\mathbb{R}^n)$, $H(X)$ as well as their strong duals*

*are nuclear, where $X$ denotes an open subset in a finite dimensional vector space and $K$ a compact subset of $X$.*

**Proof.** We have shown in [**2**, 5.4.5] that the Fourier-coefficients give an isomorphism of $s$ with $C^\infty_{2\pi}(\mathbb{R})$, the space of $2\pi$-periodic smooth functions. Thus $C^\infty_{2\pi}$ is nuclear.

Now let $K$ be a compact subset in $\mathbb{R}$. We choose an affine isomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ which maps $K$ to a subset $\alpha(K)$ of the open interval $(-\pi, \pi)$. Hence $\alpha^*$ gives an isomorphism $C^\infty_{\alpha(K)}(\mathbb{R}) \cong C^\infty_K(\mathbb{R})$. Since all functions in $C^\infty_{\alpha(K)}(\mathbb{R})$ are infinitely flat at $\pm\pi$ we can prolong them to $2\pi$-periodic functions on $\mathbb{R}$. Thus $C^\infty_{\alpha(K)}(\mathbb{R})$ can be considered as closed subspace of $C^\infty_{2\pi}(\mathbb{R})$ and hence is nuclear.

Recall that $C^\infty_{A\times B}(X\times Y) \cong C^\infty_A(X)\hat{\otimes}_\varepsilon C^\infty_B(Y)$ (or at least embeds into the injective tensor product of such spaces). Since the injective tensor product of nuclear spaces is nuclear as are closed subspaces we obtain that $C^\infty_{K_1\times\ldots\times K_n}(\mathbb{R}^n)$ is nuclear. Since every compact subset $K$ of an open subset $X \subseteq \mathbb{R}^n$ can be covered by finitely many compact boxes, we can embed $C^\infty_K(X)$ into a finite product of nuclear spaces $C^\infty_{I_1\times\ldots\times I_n}(\mathbb{R}^n)$, hence $C^\infty_K(X)$ is nuclear.

Thus also the strict inductive countable limit $C^\infty_c(X)$ of the spaces $C^\infty_K(X)$ with $K \subseteq X$ is nuclear.

Since $C^\infty(X)$ carries the initial structure with respect to multiplication mappings $C^\infty(X) \to C^\infty_c(X)$ by smooth-functions $h_n \in C^\infty_{K_{n+1}}(\mathbb{R}^m)$ with $h_n|_{K_n} = 1$ for some countable base $K_n$ of the compact subsets of $X$, we conclude that $C^\infty(X)$ is nuclear as a closed subspace of a product of nuclear spaces.

Being a subspace of $C^\infty(X)$ the space $H(X)$ is also nuclear.

Furthermore $\mathcal{S}(\mathbb{R})$ can be considered as subspace of $C^\infty_{2\pi}(\mathbb{R}) \cong C^\infty(S^1)$ (see [**22**, 429]).

For this we consider $\mathbb{R}$ as the stereographic image of $S^1 \setminus \{1\}$ with pole 1. This chart is easily seen to be given by $\mathbb{R} \ni r \mapsto (\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}) \in S^1$. The chart-change to the stereographic image of $S^1 \setminus \{-1\}$ with pole $-1$ is given by $t \mapsto \frac{1}{t}$, by a direct application of similar triangles, see [**1**, ???] or [**3**, I,9.20]. So in order to show that every $f \in \mathcal{S}$ induces a smooth map $\tilde{f} : S^1 \to \mathbb{R}$, which is infinitely flat at 1, it is enough to show that $f \circ i : \mathbb{R} \to \mathbb{R}$ is smooth and infinitely flat at 0, where $i : t \mapsto \frac{1}{t}$. Obviously $f \circ i$ is smooth on $\mathbb{R} \setminus \{0\}$. For the derivatives we have that $(f \circ i)^{(n)}(t)$ equals $(-\frac{1}{t^2})^n f^{(n)}(\frac{1}{t})$ plus a universal linear combination of $f^{(k)}(\frac{1}{t}) \cdot (\frac{1}{t})^{k+n}$ with $0 \le k < n$. Since $f \in \mathcal{S}$ we have that for $\frac{1}{t} \to \infty$ these terms go to 0. By the closed graph theorem the linear mapping $f \mapsto \tilde{f}$ is continuous from $\mathcal{S}$ to $C^\infty_0(S^1)$, the space of smooth mappings which are infinitely flat at 1.

Moreover this mapping is onto. Since given $\tilde{f} \in C^\infty_0(S^1)$, then the corresponding $f$ is obviously smooth on $\mathbb{R}$ and $f \circ i$ is smooth on $\mathbb{R}$ and infinite flat at 0. So by Taylor's theorem we have that $\frac{f^{(n)}(1/t)}{t^k} \to 0$ for all $n, k \ge 0$. By induction with respect to $n$ we have that

$$\pm f^{(n)}(\frac{1}{t})\frac{1}{t^{2n+k}} = \frac{(f\circ i)^{(n)}(t)}{t^k} + \text{ a linear comb. of } \Big\{ f^{(j)}(\frac{1}{t})\frac{1}{t^{j+n+k}} : j < n \Big\} \to 0$$

for $t \to 0$. But this means that $s^k f^{(n)}(s) \to 0$ for $s \to \infty$, i.e. $f \in \mathcal{S}$. By the open mapping theorem for Fréchet spaces we have that $\mathcal{S}$ as lcs is isomorphic with $C^\infty_0(S^1)$. Hence $\mathcal{S}(\mathbb{R})$ is nuclear, as is $\mathcal{S}(\mathbb{R}^n) \cong \mathcal{S}(\mathbb{R})\hat{\otimes}_\varepsilon \cdots \hat{\otimes}_\varepsilon \mathcal{S}(\mathbb{R})$.

Note that $C^\infty_K(\mathbb{R})$ is isomorphic to the subspace of $C^\infty_{2\pi}(\mathbb{R}, \mathbb{R}) \cong C^\infty(S^1)$ formed by those functions which are infinitely flat at 0, if $K$ is the closed interval $[0, 2\pi]$. In fact every $f \in C^\infty_K(\mathbb{R})$ has to be infinitely flat at 0 and at $2\pi$. Hence it can be prolonged to a $2\pi$-periodic function on $\mathbb{R}$ which is infinitely flat at 0. Conversely every such function can be turned into a function in $C^\infty_K(\mathbb{R})$ by replacing it by 0 outside $K$. The closed graph theorem shows that this bijection is a homeomorphism.

One can even show that $\mathcal{S}(\mathbb{R}) \cong s$. In fact [**2**, 6.3.9] we have discussed the *Hermite-polynomials* $H_n$, which are obtained by orthornormalizing the polynomials $t^n$ on $\mathcal{L}^2_\rho(\mathbb{R},\mathbb{R})$, where the inner product is given by $\langle f,g\rangle_\rho := \int_{\mathbb{R}} f(t)\,\overline{g(t)}\,\rho(t)\,dt$, where $\rho(t) := e^{-1/t^2}$. The Hermite-polynomials are solutions of the second order differential equation $H_n''(t) - t\,H_n'(t) + n\,H_n(t) = 0$ and are proportional to $t \mapsto e^{t^2/2}(\frac{d}{dt})^n e^{-t^2/2}$. Now one defines the *Hermite functions* $h_n$ to be proportional to $t \mapsto e^{-t^2/2} H_n(\sqrt{2}\cdot t)$. So they are solutions of the differential equation:

$$h_n''(t) - t^2\,h_n(t) + (2n+1)\,h_n = 0.$$

In fact

$$\begin{aligned}
h_n'(t) &= \frac{d}{dt}\Big(e^{-t^2/2} H_n(\sqrt{2}\,t)\Big) = e^{-t^2/2}\,(-t\,H_n(\sqrt{2}\,t) + \sqrt{2}H_n'(\sqrt{2}\,t)) \Rightarrow\\
h_n''(t) &= e^{-t^2/2}\,(-t(-t\,H_n(\sqrt{2}\,t) + \sqrt{2}H_n'(\sqrt{2}\,t))\\
&\quad - H_n(\sqrt{2}\,t) - \sqrt{2}\,t\,H_n'(\sqrt{2}\,t) + 2H_n''(\sqrt{2}\,t))\\
&= e^{-t^2/2}\,((t^2-1)\,H_n(\sqrt{2}\,t) - 2\sqrt{2}\,t\,H_n'(\sqrt{2}\,t) + 2H_n''(\sqrt{2}\,t)) \Rightarrow\\
h_n''(t) &+ (2n+1-t^2)\,h_n(t) =\\
&= e^{-t^2/2}\,((t^2-1)\,H_n(\sqrt{2}\,t) - 2\sqrt{2}\,t\,H_n'(\sqrt{2}\,t)\\
&\quad + 2H_n''(\sqrt{2}\,t) + (2n+1-t^2)\,H_n(\sqrt{2}\,t))\\
&= e^{-t^2/2}\,((t^2-1)\,H_n(\sqrt{2}\,t) - 2\sqrt{2}\,t\,H_n'(\sqrt{2}\,t)\\
&\quad + 2\sqrt{2}\,t\,H_n'(\sqrt{2}\,t) - 2n\,H_n(\sqrt{2}\,t) + (2n+1-t^2)\,H_n(\sqrt{2}\,t)) = 0
\end{aligned}$$

Thus the $h_n$ are Eigen-functions of the differential operator $D : h \mapsto \frac{(t^2+1)h_n - h_n''}{2}$ for the Eigen-value $(n+1)$. Note that the multiplication operator $s \to s$, $(x_n) \mapsto (n\,x_n)$ has $n$ as Eigen-values and $e_n$ as corresponding Eigen-vectors. One can show that the continuous seminorms $h \mapsto \|D^m h\|_2$ generate the topology, and that the Fourier-coefficients with respect to $h_n$ give an isomorphism of $\mathcal{S}(\mathbb{R})$ with $s$.

All but $C^\infty_c(X)$ are Fréchet spaces, hence co-nuclear by 6.30, i.e. their strong dual is nuclear. Since $C^\infty_c(X)$ is a strict inductive limit of nuclear Fréchet spaces it is also co-nuclear by 6.31. □

It can be shown that $C^\infty(\mathbb{R},\mathbb{R}) \cong s^{\mathbb{N}}$, which is a universal nuclear Fréchet space, i.e. the nuclear Fréchet spaces are exactly the closed subspaces of $C^\infty(\mathbb{R},\mathbb{R})$. Since $s^{\mathbb{N}}\hat{\otimes}_\varepsilon s^{\mathbb{N}} \cong s^{\mathbb{N}\times\mathbb{N}} \cong s^{\mathbb{N}}$, we have that $C^\infty(\mathbb{R}^n,\mathbb{R}) \cong s^{\mathbb{N}} \cong C^\infty(\mathbb{R},\mathbb{R})$. It is also not difficult to show that $s\hat{\otimes}_\varepsilon s \cong s$.

**Remark.**
Note that the weak-topology $\sigma(E,E^*)$ on any locally convex space is Schwartz. In fact the typical seminorms are given by $p := \max\{|x_j^*| : 1 \le j \le n\}$ for some $x_j^* \in E^*$. So $(x_j^*)_{j\le n : E\to\mathbb{R}^n}$ induces a injection $E_p \to \mathbb{R}^n$, and hence the connecting mappings are even finite dimensional. So we may consider the finest locally convex Schwartz topology $E_{\text{Schw.}}$ coarser than the given topology on $E$. Similarly there is a finest locally convex nuclear topology $E_{\text{nucl.}}$ coarser than the given topology of $E$. We will call these modifications the *Schwartzification* and the *nuclearification*.

**6.24 Proposition.**
*A locally convex space $E$ is Schwartz iff every equi-continuous set is contained in the closed convex hull of a sequence $x_n^*$, which converges Mackey to 0 in $E^*$*

*with respect to the equi-continuous bornology, i.e. there are $\lambda_n \to \infty$ and a 0-neighborhood $U \subseteq E$, such that $\lambda_n x_n^* \in U^o$ for all $n$.*
*A locally convex space $E$ is nuclear iff every equi-continuous set is contained in the closed convex hull of a sequence $x_n^*$, which is fast falling in $E^*$ with respect to the equi-continuous bornology, i.e. for every $k \in \mathbb{N}$ there exists a 0-neighborhood $U \subseteq E$ such that $n^k x_n^* \in U^o$ for all $n$.*

**Sketch of Proof.** ($\Rightarrow$) Let $U^o$ be equi-continuous. Then one may find $V_k \subseteq U$ such that the connecting map $E_{(V_k)} \to E_{(U)}$ belongs to $\mathcal{A}_{1/n}$ . We take representations via some orthonormal sequences. Together we obtain an orthonormal basis in $E^*_{(U^o)}$ and after appropriate renorming the required fast falling sequence in $E^*$.

($\Leftarrow$) From the assumption it follows that $E$ has a basis of polars of fast falling sequences in $E^*_{equi}$. Let $U := \{x_n^*\}^o$ and $V := \{n^2 x_n^*\}^o$. Then $R : \ell^1 \to E^*_{(U^o)}$ given by $\lambda \mapsto \sum_n \lambda_n x_n^*$ and $S : \ell^1 \to E^*_{(V^o)}$ given by $\lambda \mapsto \sum_n \lambda_n n^2 x_n^*$ are quotient mappings and the lift of the connecting morphism $\Phi$ along $R$ and $S$ is given by the nuclear diagonal operator $D$ with diagonal $(\frac{1}{n^2})_n$. Dualizing this gives the nuclear mapping $R^* \circ \Phi^* = (\Phi \circ R)^* = (S \circ D)^* = D^* \circ S^*$. Since $R^*$ is an isometric embedding, we have that the trace of $\Phi^*$ to $\widehat{E_{(V)}} \to E_{(U)}$ is absolutely summable. □

Using this one can show:

**6.25 Corollary.**
*The Schwartzification $E_{Schw.}$ of a locally convex space is the topology of uniform convergence on sequences which converge Mackey to 0 in $E^*$.*
*The nuclearification $E_{nucl.}$ of a locally convex space is the topology of uniform convergence on sequences which are fast falling in $E^*$.*

From this one can construct *universal Schwartz* and *nuclear space*s:

**6.26 Proposition.**
*The Schwartz spaces are exactly the subspaces of a product of the Schwartzification of $c_0$, or of its completion $\widehat{(c_0)_{Schw.}} = (\ell^\infty, \mu(\ell^\infty, \ell^1))$.*
*The nuclear spaces are exactly the subspaces of a product of $s$. A metrizable space $E$ is nuclear iff it embeds into $s^{\mathbb{N}} \cong C^\infty(\mathbb{R}, \mathbb{R})$*

The first statement is also called Schur's lemma, see [**14**, 218]. The second is Komura's Theorem, see [**14**, 21.7.1].

**Proof.** In fact by 6.24 one has a basis of 0-neighborhoods given by the polars of fast falling sequences $(x_n^*)_n$ in $E^*$. The map $x \mapsto (\langle x_n^*, x\rangle)_{n\in\mathbb{N}}$ thus defines a continuous linear map from $E \to s$, and together they give an embedding $E \to \prod_{(x_n^*)} s$. □

If $E$ is normable, then the Schwartzification is the topology of uniform convergence on compact subsets of $E^*_\beta$, see [**14**, 10.4.5].

This Schwartzification has several applications. Among them are:

**6.27 Proposition.**
*The dual of a Banach space $E$ has the approximation property iff $E_{Schw.}$ has it.*

We mentioned earlier that a bornological locally convex space is bornologically reflexive iff its Schwartzification is complete.

## Nuclear Function Spaces

Note that $\mathbb{R}^X$ is nuclear and hence Schwartz for all sets $X$. However its dual $\mathbb{R}^{(X)}$ is not quasi-normed (hence not Schwartz and not nuclear) if $X$ is uncountable. Recall that the typical seminorms on $\mathbb{R}^{(X)}$ are given by $f \mapsto \sum_x c_x \|f_x\|$ with $c_x \geq 0$. Thus for the seminorm with $c_x = 1$ for all $x$ there exist another seminorm given by some corresponding $c_x > 0$ such that for all $\varepsilon > 0$ there is some bounded set $B_\varepsilon$ with $\{f : \sum_x c_x |f_x| \leq 1\} \subseteq B_\varepsilon + \varepsilon \{f : \sum_x |f_x| \leq 1\}$. For some $\delta > 0$ the set $I := \{x : c_x \leq \frac{1}{\delta}\}$ has to be infinite. Now choose $\varepsilon = \frac{\delta}{2}$. Then $B_\varepsilon$ is contained in a finite subsum, so there is some $x \in I$ with $\mathrm{pr}_x(B_\varepsilon) = \{0\}$. Since $\delta \cdot e_x$ is an element of the left hand side hence also of the right hand side. Thus there has to exist a $b \in B_\varepsilon$ and $f$ with $\|f_x\| \leq 1$ with $\delta \cdot e_x = b + \varepsilon f$ and hence $\mathrm{pr}_x(b) \geq \delta - \frac{\delta}{2} > 0$, a contradiction.

**6.28 Definition.**
A lcs $E$ is called *co-nuclear* iff for every disk $B$ there exists a larger disk $C$ such that $\widehat{E_B} \to \widehat{E_C}$ is nuclear.

**6.29 Proposition.**
*A space is co-nuclear iff its strong dual $E^*_\beta$ is nuclear.*

**Proof.** We need and hence prove only ($\Rightarrow$). A typical 0-neighborhood in $E^*_\beta$ is $B^o$ for some bounded (absolutely convex and closed) $B \subseteq E$. By assumption there is some bounded $C \supseteq B$ such that $\widehat{E_B} \to \widehat{E_C}$ is nuclear. Then its dual mapping $(E_C)^* \to (E_B)^*$ is also nuclear. Now note that $(E^*)_{(B^o)}$ is isometrically embedded into $(E_B)^*$, since the inclusion $E_B \to E$ induces a morphism $E^* \to (E_B)^*$, which factors over $(E^*)_{(B^o)}$ via an embedding, since $\|x^*\|_{(E_B)^*} = \sup\{|x^*(x)| : p_B(x) \leq 1\} = \sup\{|x^*(x)| : x \in B = B^{oo}\} = p_{B^o}(x^*)$. So the connecting morphism from $(E^*)_{(C^o)} \to (E^*)_{(B^o)}$ is absolutely summable as restriction of the absolutely summable map $(E_C)^* \to (E_B)^*$, i.e. $E^*_\beta$ is nuclear. □

**6.30 Proposition.**
*For metrizable or df-spaces nuclearity and co-nuclearity are equivalent.*

**Proof.** We only need (nuclear $\overset{(F)}{\Rightarrow}$ conuclear) Let $p_n$ be an increasing sequence of seminorms defining the topology such that the connecting morphism $T_n : E_{p_{n+1}} \to E_{p_n}$ is nuclear, and hence admits a representation

$$T_n = \sum_k \lambda_{n,k} x^*_{n,k} \otimes y_{n,k}.$$

with $x_{n,k} \in o(E_{p_{n+1}})^*$, $y_{n,k} \in o(E_{p_n})$ and $\lambda_n := \sum_k |\lambda_{n,k}| < \infty$. Now let $B \subseteq E$ be a closed disk and choose $\sigma_n > 0$ with $p_{n+1}(B) \leq \sigma_n$. Let $\rho_n := \max\{\sigma_n, \lambda_n \cdot \sigma_n\}$ and set $C := \{x \in E : \sum_n \frac{1}{2^n \rho_n} p_n(x) \leq 1\}$. For $x \in B$ we have $p_n(x) \leq p_{n+1}(B) \leq \sigma_n$, hence $\sum_n \frac{1}{2^n \rho_n} p_n(x) \leq \sum_n \frac{1}{2^n \sigma_n} \sigma_n = 1$, i.e. $B \subseteq C$. Furthermore $C$ is bounded since $p_n(C) \leq 2^n$. The connecting morphism $E_B \to E_C$ is absolutely summable, since for arbitrary finitely many $x_i \in E_B \subseteq E$ we have

$$\begin{aligned}\sum_i p_n(x_i) = \sum_i p_n(T_n(x_i)) &\le \sum_i \sum_j p_n(\lambda_{n,j}\, x^*_{n,j}(x_i)\, y_{n,i}) \\ &\le \sum_i \sum_j |\lambda_{n,j}|\,|x^*_{n,j}(x_i)| \le \lambda_n \sup_{x^*\in U^o_{n+1}} \sum_i |x^*(x_i)| \\ &\le \lambda_n \sup_{x^*\in \sigma_n B^o_n} \sum_i |x^*(x_i)| \\ &\le \lambda_n \sigma_n \sup_{x^*\in B^o} \sum_j |x^*(x_i)| \\ &\le \rho_n \sup_{x^*\in B^o} \sum_j |x^*(x_i)|.\end{aligned}$$

Thus $\sum_i q_C(x_i) \le \sup\{\sum_i |x^*(x_i) : x^* \in B^o\}$ and hence the identity $E_B \to E_C$ is absolutely summable by 5.20. Since $\mathcal{S}_1^6 \subseteq \mathcal{A}_1 \subseteq \mathcal{N}$ we may assume that it is even nuclear, and hence $E$ is co-nuclear.

for the remaining implications see [**14**, 21.5.3]. □

**6.31 Proposition.**
*Every strict inductive limit of a sequence of nuclear Fréchet spaces is co-nuclear.*

**Proof.** In fact it is immediate that the strict inductive limit of a sequence of co-nuclear spaces is co-nuclear. □

**6.32 Theorem.**
*Let $E$ and $F$ be Fréchet spaces with $E$ nuclear. Then we have the following isomorphisms:*

1. $E\hat{\otimes}_\beta F \cong E\hat{\otimes}_\pi F \cong E\hat{\otimes}_\varepsilon F \cong L(E', F)$;
2. $E'\hat{\otimes}_\pi F \cong E'\hat{\otimes}_\varepsilon F \cong L(E, F)$;
3. $E'\hat{\otimes}_\beta F' \cong E'\hat{\otimes}_\pi F' \cong E'\hat{\otimes}_\varepsilon F' \cong L(E, F') \cong (E\hat{\otimes}_\pi F)'$;

**Proof.** (1) Recall that we have shown in 4.73 that for complete spaces we have $E\hat{\otimes}_\varepsilon F \cong L(E^*_\beta, F)$ provided $E$ satisfies the approximation property, is Montel and $E^*_\beta$ is bornological. These conditions are satisfied if $E$ is a nuclear Fréchet.

(2) Recall that we have shown in 4.74 that for complete spaces $E^*_\beta$ and $F$ we have $E^*_\beta\hat{\otimes}_\varepsilon F \cong L(E, F)$ provided $E^*_\beta$ satisfies the approximation property and $E$ is Montel and bornological. This is all satisfied if $E$ is a nuclear Fréchet space, since then $E^*_\beta$ is nuclear.

(3) the same argument as in (2) applies and hence $E'\hat{\otimes}_\varepsilon F' \cong L(E, F')$. In general we have $L(E, F') \cong L(E, F; \mathbb{R}) = (E\hat{\otimes}_\beta F)'$, and since $E$ and $F$ are Fréchet and $E$ is nuclear we can replace this by any of the other two tensor products. □

**6.33 Proposition.**
*Let $\mathcal{B}$ be a bornology on $E$. Then $\mathcal{L}_\mathcal{B}(E, F)$ is Schwartz iff $E^*_\mathcal{B}$ and $F$ are Schwartz. And $\mathcal{L}_\mathcal{B}(E, F)$ is nuclear iff $E^*_\mathcal{B}$ and $F$ are nuclear.*

**Proof.** ($\Rightarrow$) is obvious, since $F$ and $E^*_\mathcal{B}$ can be considered as subspaces.

($\Leftarrow$) First one shows that a 0-neighborhood basis in $\mathcal{L}_\mathcal{B}(E, F)$ is given by the sets $N := N_{\{x_n\},\{y^*_n\}^o}$, where $x_n$ is Mackey-convergent to 0 in $E$ with respect to $\mathcal{B}$ and $y^*_n$ is Mackey convergent to 0 in $F^*$ with respect to the bornology of equi-continuous sets, in fact the polars of these sequences form bases. Without loss of generality we may replace $x_n$ by $\lambda_n x_n$ and $y_n^*$ by $\mu_n y_n^*$ with $\lambda$ and $\mu$ in $c_0$. The functionals $\ell_{j,k} : \mathcal{L}_{\mathcal{B}}(E,F) \to \mathbb{R}$ given by $T \mapsto y_j^*(T(x_k))$ form an equi-continuous family, since $N$ is mapped into $[-1,1]$. Thus $\lambda_k \mu_j \ell_{j,k}$ are Mackey-convergent to 0 with respect to the bornology of equi-continuous subsets. Hence its polar is a neighborhood in the Schwartzification of $\mathcal{L}_{\mathcal{B}}(E,F)$ which is contained in $N$.

The proof for nuclearity is analogous using that by [**14**, 21.9.1] the nuclearification is given by the topology of uniform convergence on fast falling sequences $x_n^* \in E^*$. Note that that the sequences which are fast falling in a fixed space $E^*_{(U)}$ generate the strongly nuclear topology. □

In order to get results about spaces of smooth functions between infinite dimensional spaces, we need an explicit description of the bornological topology of $C^\infty(U,F)$. If we can find some finer Fréchet space structure on $C^\infty(U,F)$, then this has to be the bornological topology (see [**14**, 13.3.5]) by the closed graph theorem. In fact the compositions of the point-evaluations with the identity from the bornological topology to the Fréchet topology are continuous, and hence the identity is continuous by [**2**, 5.3.8]. Note that (for $U \neq \emptyset$) the space $L(E,F)$ is a complemented subspace of $C^\infty(U,F)$, where the retraction is given by $f \mapsto f'(x)$, for any $x \in U$. So in order for $C^\infty(U,F)$ to be Fréchet we need that $L(E,F)$ is Fréchet, hence that $F$ is Fréchet and $E$ has a countable base of bornology, i.e. $E'$ is Fréchet.

Note that (for $U \neq \emptyset$) the space $F$ is isomorphic to the complemented subspace of $C^\infty(U,F)$ formed by the constant functions. Hence we need that $F$ is Fréchet. Note that for every bornologically compact set $K \subseteq E_B \cap U \subseteq E$ and continuous seminorm $p$ on $F$ the seminorm

$$f \mapsto \sup\{p(f^{(n)}(x)(v_1,\dots,v_n)) : x, v_1,\dots,v_n \in K\}$$

is a bounded seminorm on $C^\infty(U,F)$. Hence if we can find a countable base of the b-compact sets in $U \subseteq E$ and a countable 0-neighborhood basis of $F$, then those seminorms define a finer Fréchet topology on $C^\infty(U,F)$. This shows the first part of

**6.34 Proposition. $\mathbf{C^\infty(X,F)}$ as Fréchet space.**
*See* [**8**] *and* [**10**, 5.4.16]. *If $E$ is the dual of a Fréchet-Schwartz space, $F$ is a Fréchet space, and $U \subseteq E$ is $c^\infty$-open, then the bornological topology of $C^\infty(U,F)$ is Fréchet. If $F$ is in addition Montel, then so is $C^\infty(U,F)$. If $F$ is in addition Schwartz, then so is $C^\infty(U,F)$.*

**Remark.**
In [**17**], see also [**8**], it is shown that for infinite dimensional domains $U \subseteq E$ the space $C^\infty(U,\mathbb{R})$ is most often not nuclear. It is shown that if $E$ contains an infinite dimensional compact absolutely convex subset, then $C^\infty(U,\mathbb{R})$ is not nuclear. This is proved for a different class of smooth functions and a different topology on the function space than the ones treated here. But if $E$ is the dual of a Fréchet Schwartz space $F$, then the $c^\infty$-topology on $E$ coincides with the locally convex topology by 7.4 and the natural topology on $C^\infty(E,\mathbb{R})$ is Fréchet by 6.34. Hence in this situation the concepts agree and by 6.16 every closed bounded set is (bornologically) compact. So it is enough to find a 0-neighborhood $U \subseteq F$ such that the equi-continuous and hence bounded set given by the polar of $U$ is infinite dimensional. This does not exist for $F = \mathbb{R}^{\mathbb{N}}$ but already $F = s$ or $F = C^\infty(\mathbb{R},\mathbb{R})$ yield examples. In the first case take $U := \{x \in s : \|x\|_\infty \leq 1\}$. Then the functionals $\mathrm{ev}_n$ belong to $U^o$ and are obviously linearly independent. In the second case take $U := \{f : |f(x)| \leq 1 \text{ for all } |x| \leq 1\}$. Then the distributions $\mathrm{ev}_{1/n} \in U^o$ for $n \in \mathbb{N}$ are linearly independent.

The proof of Meise's result goes as follows.

It is easily seen that for any non-empty open $U \subseteq E$ the natural mapping $C^\infty(E,\mathbb{R}) \to \prod_{x\in E} C^\infty(U,\mathbb{R})$ given by $f \mapsto (f(\_ - x)|_U)_{x\in E}$ is an embedding. Thus nuclearity of $C^\infty(U,\mathbb{R})$ would imply that also $C^\infty(E,\mathbb{R})$ is nuclear by 6.21. So it is enough to show this result for $U = E$.

Assume that $C^\infty(E,\mathbb{R})$ is nuclear and $K \subseteq E$ a (bornologically-)compact absolutely convex infinite dimensional subset and choose $\{x_n : n \in \mathbb{N}\} \subseteq K$ linearly independent. Then for the continuous seminorm $p : f \mapsto \|f|_K\|_\infty$ there has to exist a continuous seminorm $q \geq p$ such that the connecting mapping

$$\Phi : \widehat{C^\infty(E,\mathbb{R})_q} \to \widehat{C^\infty(E,\mathbb{R})_p}$$

is of class $\mathcal{A}_{1/2}$. Without loss of generality we may assume that $q$ is given by $f \mapsto \sup\{|f^{(i)}(x)(y^i)| : x \in Q, y \in Q, i \leq k\}$ for some bornologically compact $Q \subseteq E$ and $k \in \mathbb{N}$. Obviously the restriction map $C^\infty(E,\mathbb{R}) \to C(K,\mathbb{R})$ factors continuously over $C^\infty(E,\mathbb{R})_p$. Hence we obtain a natural extension

$$\widehat{C^\infty(E,\mathbb{R})_p} \to C(K).$$

Now let $E_n$ be the linear subspace of $E$ generated by $\{x_1,\dots,x_n\}$. Obviously $\iota : (t_1,\dots,t_n) \mapsto \sum_{k=1}^n t_k x_k$ defines an isomorphism $\mathbb{R}^n \cong E_n$. Let $P_n : E \to E_n$ be some continuous linear projection. Then the composite $C^\infty(\mathbb{R}^n,\mathbb{R}) \cong C^\infty(E_n,\mathbb{R}) \xrightarrow{P_n^*} C^\infty(E,\mathbb{R}) \to C^\infty(E,\mathbb{R})_q \subseteq \widehat{C^\infty(E,\mathbb{R})_q}$ is continuous with respect to the topology of $C^k(\mathbb{R}^n,\mathbb{R})$ induced on $C^\infty(\mathbb{R}^n,\mathbb{R})$. Since $C^\infty(\mathbb{R}^n,\mathbb{R})$ is dense for this topology, we may extend this composite to a continuous linear mapping

$$C^k(\mathbb{R}^n,\mathbb{R}) \to \widehat{C^\infty(E,\mathbb{R})_q}.$$

Now consider the separately $\frac{1}{n}$-periodic functions on $\mathbb{R}^n$. Obviously $C^k_{1/n\text{-per.}}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$. Since $K$ is absolutely convex the isomorphism $\iota : \mathbb{R}^n \to E_n$ restricts to a map $[0,\frac{1}{n}]^n \to K \cap E_n \subseteq K$. Thus we obtain a continuous linear mapping

$$C(K) \xrightarrow{\iota^*} C([0,\tfrac{1}{n}]^n) \to L^2([0,\tfrac{1}{n}]^n).$$

Using the ideal property of $\mathcal{A}_{1/2}$ we obtain a mapping of that type

$$C^k_{1/n\text{-per.}}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n) \to \widehat{C^\infty(E,\mathbb{R})_q} \to \widehat{C^\infty(E,\mathbb{R})_p} \to C(K) \to L^2([0,\tfrac{1}{n}]^n).$$

Note that this mapping is nothing else but the natural inclusion. Applying some homothety we may replace $\frac{1}{n}$ by $2\pi$.

Now this finite dimensional statement can be shown to fail using Fourier-series and an idea of [**11**]. Namely one considers the Hilbert space

$$H := \Big\{(a_j)_j \in \mathbb{R}^{\mathbb{N}^n} : \|a\|^2 := \sum_{j\in\mathbb{N}^n} \prod_{i=1}^n j_i^2 \sum_{\substack{i\in\mathbb{N}^n\\|i|\leq k}} j^{2i} a_j^2 < \infty\Big\}.$$

Then the mapping $T : H \to C^k_{2\pi\text{-per.}}(\mathbb{R}^n)$ given by

$$(a_j)_j \mapsto \Big((x_i)_{i=1}^n \mapsto \sum_{j\in\mathbb{N}^n} a_j \prod_{i=1}^n \sin(j_i x_i)\Big)$$

is continuous and linear. And the composite with the mapping from above gives a type $\mathcal{A}_{1/2}$-mapping between Hilbert spaces. Using that the functions

$$f_j : (x_i)_{i=1}^n \mapsto \prod_{i=1}^n \sin(j_i x_i)$$

are orthogonal in $L^2([0,2\pi]^n)$ one concludes that the approximation numbers of the composite are the monotone reordering of

$$\left(\left(2^n\prod_{i=1}^n j_i\sqrt{\sum_{|i|\le k} i^{2k}}\right)^{-1}\right)_{j\in\mathbb{N}^n}.$$

An easy evaluation shows that this does not belong to $\ell^{1/2}$ for $N\geq l+2$.

# 7. The $c^\infty$-topology

## Refinements of the Locally Convex Topology

In 2.23 we defined the $c^\infty$-topology on an arbitrary locally convex space $E$ as the final topology with respect to the smooth curves $c : \mathbb{R} \to E$. Now we will compare the $c^\infty$-topology with other refinements of a given locally convex topology. We first specify those refinements.

### 7.1. Definition

Let $E$ be a locally convex vector space.

(i) We denote with $kE$ the *Kelleyfication* of the locally convex topology of $E$, i.e. the vector space $E$ together with the final topology induced by the inclusions of the subsets being compact for the locally convex topology.

(ii) We denote with $sE$ the vector space $E$ with the final topology induced by the curves being continuous for the locally convex topology, or equivalently the sequences $\mathbb{N}_\infty \to E$ converging in the locally convex topology. The equivalence holds since the infinite polygon through a converging sequence can be continuously parametrized by a compact interval.

(iii) We recall that by $c^\infty E$ we denote the vector space $E$ with its M-closure topology, i.e. the final topology induced by the smooth curves.

Using that smooth curves are continuous and that converging sequences $\mathbb{N}_\infty \to E$ have compact images, the following identities are continuous: $c^\infty E \to sE \to kE \to E$.

If the locally convex topology of $E$ coincides with the topology of $c^\infty E$, resp. $sE$, resp. $kE$ then we call $E$ smoothly generated, resp. sequentially generated, resp. compactly generated.

### 7.2. Example

On $E = \mathbb{R}^J$ all these refinements are different, i.e. $c^\infty E \neq sE \neq kE \neq E$, provided the cardinality of the index set $J$ is at least that of the continuum. Proof. It is enough to show this for $J$ equipotent to the continuum, since $\mathbb{R}^{J_1}$ is a direct summand in $\mathbb{R}^{J_2}$ for $J_1 \subset J_2$.

($c^\infty E \neq sE$) We may take as index set $J$ the set $c_0$ of all real sequences converging to 0. Define a sequence $(x^n)$ in $E$ by $(x^n)_j := j_n$. Since every $j \in J$ is a 0-sequence we conclude that the $x^n$ converge to 0 in the locally convex topology of the product, hence also in $sE$. Assume now that the $x^n$ converge towards 0 in $c^\infty E$. Then by 2.7 some subsequence converges Mackey to 0. Thus there exists an unbounded sequence of reals $\lambda_n$ with $\{\lambda_n x^n : n \in \mathbb{N}\}$ bounded. Let $j$ be a 0-sequence with $\{j_n \lambda_n : n \in \mathbb{N}\}$

unbounded ( e.g. $(j_n)^{-2} := 1 + \max\{|\lambda_k| : k \le n\}$). Then the j-th coordinate $j_n\lambda_n$ of $\lambda_n x^n$ is not bounded with respect to $n$, contradiction.

($sE \neq kE$) Consider in $E$ the subset

$$A := \Big\{x \in \{0,1\}^J : x_j = 1 \text{ for at most countably many } j \in J\Big\}.$$

It is clearly closed with respect to the converging sequences, hence closed in $sE$. But it is not closed in $kE$ since it is dense in the compact set $\{0,1\}^J$.

($kE \neq E$) Consider in $E$ the subsets

$$A_n := \Big\{x \in E : |x_j| < n \text{ for at most } n \text{ many } j \in J\Big\}.$$

Each $A_n$ is closed in $E$ since its complement is the union of the open sets $\{x \in E : |x_j| < n \text{ for all } j \in J_o\}$ where $J_o$ runs through all subsets of $J$ with $n+1$ elements. We show that the union $A := \bigcup_{n\in\mathbb{N}} A_n$ is closed in $kE$. So let $K$ be a compact subset of $E$; then $K \subset \prod pr_j(K)$ and each $pr_j(K)$ is compact, hence bounded in $\mathbb{R}$. Since the family $\{j \in J : pr_j(K) \subset [-n,n]\}_{n\in\mathbb{N}}$ covers $J$ there has to exist an $N \in \mathbb{N}$ and infinitely many $j \in J$ with $pr_j(K) \subset [-N,N]$. Thus $K \cap A_n = \emptyset$ for all $n > N$. And hence $A \cap K = \bigcup_{n\in\mathbb{N}} A_n \cap K$ is closed. Nevertheless $A$ is not closed in $E$, since 0 is in $\bar{A}$ but not in $A$. □

Let us now describe several important situations where at least some of these topologies coincide. For the proof we will need the following

**7.3. Lemma.**
[**23**, 1968] *For any locally convex space $E$ the following statements are equivalent:*

1. *The sequential closure of any subset is formed by all limits of sequences in the subset.*
2. *For any given double sequence $(x_{n,k})$ in $E$ with $x_{n,k}$ convergent to some $x_k$ for $n \to \infty$ and $k$ fixed and $x_k$ convergent to some $x$, there are strictly increasing sequences $i \mapsto n(i)$ and $i \mapsto k(i)$ with $x_{n(i),k(i)} \to x$ for $i \to \infty$.*

**Proof.** (1⇒2) Take an $a_0 \in E$ different from $k \cdot (x_{n+k,k} - x)$ and from $k \cdot (x_k - x)$ for all $k$ and $n$. Define $A := \{a_{n,k} := x_{n+k,k} - \frac{1}{k} \cdot a_0 : n,k \in \mathbb{N}, n \ge k\}$. Then $x$ is in the sequential closure of A, since $x_{n,k} - \frac{1}{k} \cdot a_0$ converges to $x_k - \frac{1}{k} \cdot a_0$ as $n \to \infty$ and $x_k - \frac{1}{k} \cdot a_0$ converges to $x - 0 = x$ as $k \to \infty$. Hence by (1) there has to exist a sequence $i \mapsto (n(i), k(i))$ with $a_{n(i),k(i)}$ convergent to $x$. By passing to a subsequence we may suppose that $i \mapsto k(i)$ and $i \mapsto n(i)$ is monotonely increasing. Assume that $i \mapsto k(i)$ is bounded, hence finally constant. Then a subsequence of $x_{n(i),k} - \frac{1}{k(i)} \cdot a_0$ is converging to $x_k - \frac{1}{k} \cdot a_0 \neq x$ if $i \mapsto n(i)$ is unbounded and to $x_{n,k} - \frac{1}{k} \cdot a_0 \neq x$ if $i \mapsto n(i)$ is bounded, which both yields a contradiction. Thus $i \mapsto k(i)$ can be chosen strictly increasing. But then $x_{n(i)+k(i),k(i)} = a_{n(i),k(i)} + \frac{1}{k(i)} a_0 \to x$.

(1 ⇐ 2) is obvious. □

**7.4. Theorem.**
*For any bornological vector space $E$ the following implications hold:*

1. *$c^\infty E = E$ provided the closure of subsets in $E$ is formed by all limits of sequences in the subset; hence in particular if $E$ is metrizable.*
2. *$c^\infty E = E$ provided $E$ is the strong dual of a Fréchet Schwartz space;*
3. *$c^\infty E = kE$ provided $E$ is the strict inductive limit of a sequence of Fréchet spaces.*
4. *$c^\infty E = sE$ provided $E$ satisfies the $M$-convergence condition, i.e. every sequence converging in the locally convex topology is $M$-convergent.*

5. *$sE = E$ provided $E$ is the strong dual of a Fréchet Montel space;*

**Proof.** (1) Using the lemma above one obtains that the closure and the sequential closure coincide, hence $sE = E$. It remains to show that $sE \to c^\infty E$ is continuous. So suppose a sequence converging to $x$ is given and let $(x_n)$ be an arbitrary subsequence. Then $x_{n,k} := k(x_n - x) \to k \cdot 0 = 0$ for $n \to \infty$, and hence by lemma 7.3 there are subsequences $k(i)$, $n(i)$ with $k(i) \cdot (x_{n(i)} - x) \to 0$, i.e. $i \mapsto x_{n(i)}$ is M-convergent to $x$. Thus the original sequence converges in $c^\infty E$ by 2.5.

(3) Let $E$ be the strict inductive limit of the Fréchet spaces $E_n$. By [**2**, 4.8.1] every $E_n$ carries the trace topology of $E$, hence is closed in $E$ and every bounded subset of $E$ is contained in some $E_n$. Thus every compact subset of $E$ is contained as compact subset in some $E_n$. Since $E_n$ is a Fréchet space such a subset is even b-compact and hence compact in $c^\infty E$. Thus the identity $kE \to c^\infty E$ is continuous.

(4) is valid, since the M-closure topology is the final one induced by the M-converging sequences.

(5) Let $E$ be the dual of any Fréchet Montel space $F$. By 6.14 $E$ is bornological. Fréchet Montel spaces have a reflexive dual by 4.51. First we show that $kE = sE$. Let $K \subset E = F'$ be compact for the locally convex topology. Then $K$ is bounded, hence equicontinuous and since $F$ is separable by 6.13 $K$ is metrizable in the weak topology $\sigma(E, F)$ [**14**, p157]. Since $K$ is compact the weak topology and the locally convex topology of $E$ coincide on $E$, thus the topology on $K$ is the initial one induced by the converging sequences. Hence the identity $kE \to sE$ is continuous and therefore $sE = kE$.

It remains to show $kE = E$. Since $F$ is Montel the locally convex topology of the strong dual coincides with the topology of uniform convergence on precompact subsets of $F$. Since $F$ is metrizable this topology coincides with the so-called equicontinuous weak$^*$-topology, cf. [**14**, p182], which is the final topology induced by the inclusions of the equicontinuous subsets. These subsets are by the Alaoğlu-Bourbaki theorem 4.61 relatively compact in the topology of uniform convergence on precompact subsets. Thus the locally convex topology of $E$ is compactly generated.

(2). By (5) and since Fréchet Schwartz spaces are Montel by 6.6 we have $sE = E$ and it remains to show that $c^\infty E = sE$. So let $(x_n)$ be a sequence converging to 0 in $E$. Then $\{x_n : n \in \mathbb{N}\}$ is relatively compact and by [**10**, 4.4.39] this set is relatively compact in some Banach space $E_B$. Hence at least a subsequence has to be convergent in $E_B$. Clearly its Mackey limit has to be 0. This shows that $(x_n)$ is convergent to 0 in $c^\infty E$ and hence $c^\infty E = sE$. One can even show, see 6.15, that $E$ satisfies the Mackey convergence condition. □

We give now a non-metrizable example to which (1) applies.

### 7.5. Example

Let $E$ denote the subspace of $\mathbb{R}^J$ of all sequences with countable support. Then the closure of subsets of $E$ is given by all limits of sequences in the subset but for non-countable $J$ the space $E$ is not metrizable. This was proved in [**4**].

### 7.6. Remark

The conditions (1) and (2) in 7.4 are rather disjoint since every locally convex space that has a countable basis of its von Neumann bornology and for which the sequential adherence of subsets is sequentially closed is normable as the following proposition shows:

**Proposition.**
*Let $E$ be a non-normable bornological locally convex space that has a countable basis of its bornology. Then there exists a subset of $E$ whose sequential adherence is not sequentially closed.*

**Proof.** Let $\{B_k : k \in \mathbb{N}_0\}$ be an increasing basis of the von Neumann bornology with $B_0 = \{0\}$. Since $E$ is non-normable we may assume that $B_k$ does not absorb $B_{k+1}$ for all $k$. Now choose $b_{n,k} \in \frac{1}{n} B_{k+1}$ with $b_{n,k} \notin B_k$. We consider the double sequence $\{b_{k,0} - b_{n,k} : n,k \geq 1\}$. For fixed $k$ the sequence $b_{n,k}$ converges by construction (in $E_{B_{k+1}}$) to 0 for $n \to \infty$. Thus $b_{k,0} - 0$ is the limit of the sequence $b_{k,0} - b_{n,k}$ for $n \to \infty$ and $b_{k,0}$ converges to 0 for $k \to \infty$. Suppose $b_{k(i),0} - b_{n(i),k(i)}$ converges to 0. Thus it has to be bounded and so there must be an $N \in \mathbb{N}$ with $B_1 - \{b_{k(i),0} - b_{n(i),k(i)} : i \in \mathbb{N}\} \subset B_N$. Hence $b_{n(i),k(i)} = b_{k(i),0} - (b_{k(i),0} - b_{n(i),k(i)}) \in B_N$, i.e. $k(i) < N$. This contradicts 7.3(2). □

## Continuity of the Addition and Regularity

Now we describe classes of spaces where $c^\infty E \neq E$ or where $c^\infty E$ is not even a topological vector space. Finally we give an example where the Mackey-closure topology is not completely regular.

We begin with the relationship between the $c^\infty$-topology and the locally convex topology on locally convex vector spaces.

**7.7. Proposition.**
*Let $E$ and $F$ be bornological locally convex vector spaces. If there exists a bilinear bounded mapping $m : E \times F \to \mathbb{R}$ that is not continuous with respect to the locally convex topologies, then $c^\infty(E \times F)$ is not a topological vector space.*

**Proof.** Suppose that addition $c^\infty(E \times F) \times c^\infty(E \times F) \to c^\infty(E \times F)$ is continuous with respect to the product topology. Using the continuous inclusions $c^\infty E \to c^\infty(E \times F)$ and $c^\infty F \to c^\infty(E \times F)$ we can write $m$ as composite of continuous maps as follows: $c^\infty E \times c^\infty F \to c^\infty(E \times F) \times c^\infty(E \times F) \xrightarrow{+} c^\infty(E \times F) \xrightarrow{m} \mathbb{R}$. Thus for every $\varepsilon > 0$ there are 0-neighborhoods $U$ and $V$ with respect to the $c^\infty$-topology such that $m(U \times V) \subset (-\varepsilon, \varepsilon)$. Then also $m(\langle U \rangle \times \langle V \rangle) \subset (-\varepsilon, \varepsilon)$ where $\langle \quad \rangle$ denotes the absolutely convex hull. By 2.26 one concludes that $m$ is continuous with respect to the locally convex topology, a contradiction. □

**7.8. Corollary.**
*Let $E$ be a bornological non-normable locally convex space. Then $c^\infty(E \times E')$ is not a topological vector space.*

**Proof.** By 7.7 it is enough to show that $\mathrm{ev} : E \times E' \to \mathbb{R}$ is not continuous; if it were so then there would be a neighborhood $U$ of 0 in $E$ and a neighborhood $U'$ of 0 in $E'$ such that $\mathrm{ev}(U \times U') \subset [-1, 1]$. Since $U'$ is absorbing, $U$ is scalarwise bounded, hence a bounded neighborhood. Thus $E$ is normable. □

In order to get a large variety of spaces where the $c^\infty$-topology is not a topological vector space topology the next three technical lemmas will be useful.

**7.9. Proposition. General situations, where addition is not continuous.** *Let $E$ be a locally convex vector space. Suppose a double sequence $b_{n,k}$ in $E$ exists which satisfies the following two conditions:*

1. *"(b')For every sequence $k \mapsto n(k)$ the sequence $k \mapsto b_{n(k),k}$ has no accumulation point in $c^\infty E$.*

(b"). *For all $k$ the sequence $n \mapsto b_{n,k}$ converges to 0 in $c^\infty E$.*

*Suppose furthermore that a double sequence $c_{n,k}$ in $E$ exists that satisfies the following two conditions:*

1. *"(c')For every 0-neighborhood $U$ in $c^\infty E$ there exists a $k_0$ such that $c_{n,k} \in U$ for all $k \geq k_0$ and all $n$.*
2. *"(c")For all $k$ the sequence $n \mapsto c_{n,k}$ has no accumulation point in $c^\infty E$.*

*Then $c^\infty E$ is not a topological vector space.*

**Proof.** Here convergence is meant always with respect to $c^\infty E$. We may without loss of generality assume that $c_{n,k} \neq 0$ for all $n$, $k$, since by (c") we may delete all those $c_{n,k}$ which are equal to 0. Then we consider $A := \{b_{n,k} + \varepsilon_{n,k} c_{n,k} : n, k \in \mathbb{N}\}$ where the $\varepsilon_{n,k} \in \{-1, 1\}$ are chosen in such a way that $0 \notin A$.

We first show that $A$ is closed in the sequentially generated topology $c^\infty E$: Let $b_{n(i),k(i)} + \varepsilon_{n(i),k(i)} c_{n(i),k(i)} \to x$ and assume that $(k(i))$ is unbounded. By passing if necessary to a subsequence we may even assume that $i \mapsto k(i)$ is strictly increasing. Then $c_{n(i),k(i)} \to 0$ by (c'), hence by 2.7 $b_{n(i),k(i)} \to x$ which is a contradiction to (b'). Thus $(k(i))$ is bounded and we may assume constant. Now suppose that $(n(i))$ is unbounded. Then $b_{n(i),k} \to 0$ by (b") and hence $\varepsilon_{n(i),k} c_{n(i),k} \to x$ and for a subsequence where $\varepsilon$ is constant one has $c_{n(i),k} \to \pm x$, which is a contradiction to (c"). Thus $n(i)$ is bounded as well and we may assume constant. Hence $x = b_{n,k} + \varepsilon_{n,k} c_{n,k} \in A$.

Assume now that the addition $c^\infty E \times c^\infty E \to c^\infty E$ is continuous. Then there has to exist an open and symmetric 0-neighborhood $U$ in $c^\infty E$ with $U + U \subset E \setminus A$. For $K$ sufficiently large and $n$ arbitrary one has $c_{n,K} \in U$ by (c'). For such a fixed $K$ and $N$ sufficiently large $b_{N,K} \in U$ by (b'). Thus $b_{N,K} + \varepsilon_{N,K} c_{N,K} \notin A$, which is a contradiction. □

Let us now show that many spaces have a double sequence $c_{n,k}$ as in the above lemma.

**7.10. Lemma.**
*Let $E$ be an infinite dimensional metrizable locally convex space. Then a double sequence $c_{n,k}$ subject to the conditions (c') and (c") of 7.9 exists.*

**Proof.** If $E$ is normable we choose a sequence $c_n$ in the unit ball without accumulation point and define $c_{n,k} := \frac{1}{k} c_n$. If $E$ is not normable we take a countable increasing family of non-equivalent seminorms $p_k$ generating the locally convex topology, and we choose $c_{n,k}$ with $p_k(c_{n,k}) = \frac{1}{k}$ and $p_{k+1}(c_{n,k}) > n$. □

Next we show that many spaces have a double sequence $b_{n,k}$ as in lemma 2.20.

**7.11. Lemma.**
*Let $E$ be a non-normable bornological locally convex space having a countable basis of its bornology. Then a double sequence $b_{n,k}$ subject to the conditions (b') and (b") of 2.20 exists.*

**Proof.** Let $B_n$ ($n \in \mathbb{N}$) be absolutely convex sets forming an increasing basis of the bornology. Since $E$ is not normable the sets $B_n$ can be chosen such that $B_n$ does not absorb $B_{n+1}$. Now choose $b_{n,k} \in \frac{1}{n} B_{k+1}$ with $b_{n,k} \notin B_k$. □

Using these lemmas one obtains the

**7.12. Proposition. Examples, where addition is not continuous.**
*For the following bornological locally convex spaces the $c^\infty$-topology is not a vector space topology:*

1. *"(i)Ëvery bornological locally convex space that contains as M-closed subspaces an infinite dimensional Fréchet space and a space which is not normable and has a countable basis of its bornology.*
2. *"(ii)Ëvery strict inductive limit of a strictly increasing sequence of infinite dimensional Fréchet spaces.*
3. *"(iii)Ëvery product for which at least $2^{\aleph_0}$ many factors are non-zero.*
4. *"(iv)Ëvery coproduct for which at least $2^{\aleph_0}$ many summands are non-zero.*

**Proof.** (i) follows directly from the last 3 lemmas.

(ii) Let $E$ be the strict inductive limit of the spaces $E_n$ ($n \in \mathbb{N}$). Then $E$ contains the infinite dimensional Fréchet space $E_1$ as subspace. The subspace generated by points $x_n \in E_{n+1} \setminus E_n$ ($n \in \mathbb{N}$) is bornologically isomorphic to $\mathbb{R}^{(\mathbb{N})}$, hence its bornology has a countable basis. Thus by (i) we are done.

(iii) Such a product $E$ contains the Fréchet space $\mathbb{R}^{\mathbb{N}}$ as complemented subspace. We want to show that $\mathbb{R}^{(\mathbb{N})}$ is also a subspace of $E$. For this we may assume that the index set $J$ is $\mathbb{R}^{\mathbb{N}}$ and all factors are equal to $\mathbb{R}$. Now consider the linear subspace $E_1$ of the product that is generated by the sequence $x^n \in E = \mathbb{R}^{\mathbb{N}}$ where $(x^n)_j := j(n)$ for every $j \in J = \mathbb{R}^{\mathbb{N}}$. The linear map $\mathbb{R}^{(\mathbb{N})} \to E_1 \subset E$ that maps the $n$-th unit vector to $x^n$ is injective, since for a given finite linear combination $\sum t_n x^n = 0$ the $j$-th coordinate for $j(n) := \operatorname{sign}(t_n)$ equals $\sum |t_n|$. It is a morphism since $\mathbb{R}^{(\mathbb{N})}$ carries the finest structure. So it remains to show that it is a bornological embedding. We have to show that any bounded $B \subset E_1$ is contained in a subspace generated by finitely many $x^n$. Otherwise there would exist a strictly increasing sequence $(n_k)$ and $b^k = \sum_{n \le n_k} t^k_n x^n \in B$ with $t^k_{n_k} \neq 0$. Define an index $j$ recursively by $j(n) := n|t^k_n|^{-1} \cdot \operatorname{sign}\left(\sum_{m<n} t^k_m j(m)\right)$ if $n = n_k$ and $j(n) := 0$ if $n \neq n_k$ for all $k$. Then the absolute value of the $j$-th coordinate of $b^k$ evaluates as follows:

$$\begin{aligned} |(b^k)_j| &= \Big| \sum_{n \le n_k} t^k_n j(n) \Big| = \Big| \sum_{n<n_k} t^k_n j(n) + t^k_{n_k} j(n_k) \Big| \\ &= \Big| \sum_{n<n_k} t^k_n j(n) | + |t^k_{n_k} j(n_k) \Big| \ge |t^k_{n_k} j(n_k)| \ge n_k. \end{aligned}$$

Hence the $j$-th coordinates of $\{b^k : k \in \mathbb{N}\}$ are unbounded with respect to $k \in \mathbb{N}$ and thus $B$ is unbounded.

(iv) We can not apply lemma 2.20 since every double sequence has countable support and hence is contained in the dual $\mathbb{R}^{(A)}$ of a Fréchet Schwartz space $\mathbb{R}^A$ for some countable subset $A \subset J$. It is enough to show (iv) for $\mathbb{R}^{(J)}$ where $J = \mathbb{N} \cup c_0$. Let $A := \{j_n(e_n + e_j) : n \in \mathbb{N}, j \in c_0, j_n \neq 0 \text{ for all } n\}$, where $e_n$ and $e_j$ denote the unit vectors in the corresponding summand. The set $A$ is M-closed, since its intersection with finite subsums is finite. Suppose there exists a symmetric M-open 0-neighborhood $U$ with $U + U \subset E \setminus A$. Then for every $n$ there exists a $j_n \neq 0$ with $j_n e_n \in U$ and we may assume that $n \mapsto j_n$ converges to 0 and hence defines an element $j \in c_0$. Furthermore there has to be an $N \in \mathbb{N}$ with $j_N e_j \in U$, thus $j_N(e_N + e_j) \in (U+U) \cap A$, in contradiction to $U + U \subset E \setminus A$. $\square$

**Remark.**
A nice and simple example where one either uses (i) or (ii) is $\mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{(\mathbb{N})}$. The locally convex topology on both factors coincides with their Mackey-closure topology (the first being a Fréchet (Schwartz) space, cf. (i) of 7.4, the second as dual of the first, cf. (ii) of 7.4); but the $c^\infty$-topology on their product is not even a vector space topology.

Although the $c^\infty$-topology on a convenient vector space is always functionally separated, hence Hausdorff, it is not always completely regular as the following example shows.

**7.13. Theorem. $c^\infty$-topology is not completely regular.**
*The $c^\infty$-topology of $\mathbb{R}^J$ is not completely regular if the cardinality of $J$ is at least $2^{\aleph_0}$.*

**Proof.** It is enough to show this for an index set $J$ of cardinality $2^{\aleph_0}$, since the corresponding product is a complemented subspace in every product with larger index set. We prove the theorem by showing that every function $f : \mathbb{R}^J \to \mathbb{R}$ which is continuous for the Mackey-closure topology is also continuous with respect to the locally convex topology. Hence the completely regular topology associated to the Mackey-closure topology is the locally convex topology of $E$. That these two topologies are different was shown in 7.2. We use the following theorem of [**18**, 1952]: Let $E_0 := \{x \in \mathbb{R}^J : \operatorname{supp}(x) \text{ is countable}\}$ and let $f : E_0 \to \mathbb{R}$ be sequentially continuous. Then there is some countable subset $A \subset J$ such that $f(x) = f(x_A)$, where in this proof $x_A$ is defined as $x_A(j) := x(j)$ for $j \in A$ and $x_A(j) = 0$ for $j \notin A$. Every sequence which is converging in the locally convex topology of $E_0$ is contained in a metrizable complemented subspace $\mathbb{R}^A$ for some countable $A$ and therefore is even M-convergent. Thus this theorem of Mazur remains true if $f$ is assumed to be continuous for the M-closure topology. This generalization follows also from the fact that $c^\infty E_0 = E_0$, cf. 7.5. Now let $f : \mathbb{R}^J \to \mathbb{R}$ be continuous for the $c^\infty$-topology. Then $f|E_0 : E_0 \to \mathbb{R}$ is continuous for the $c^\infty$-topology and hence there exists a countable set $A_0 \subset J$ such that $f(x) = f(x_{A_0})$ for any $x \in E_0$. We want to show that the same is true for arbitrary $x \in \mathbb{R}^J$. In order to show this we consider for $x \in \mathbb{R}^J$ the map $\varphi_x : 2^J \to \mathbb{R}$ defined by $\varphi_x(A) := f(x_A) - f(x_{A \cap A_0})$ for any $A \subset J$, i.e. $A \in 2^J$. For countable A one has $x_A \in E_0$, hence $\varphi_x(A) = 0$. Furthermore $\varphi_x$ is sequentially continuous where one considers on $2^J$ the product topology of the discrete factors 2. In order to see this consider a converging sequence of subsets $A_n \to A$, i.e. for every $j \in J$ one has for the characteristic functions $\chi_{A_n}(j) = \chi_A(j)$ for $n$ sufficiently large. Then $\{n(x_{A_n} - x_A) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^J$ since for fixed $j \in J$ the $j$-th coordinate equals 0 for $n$ sufficiently large. Thus $x_{A_n}$ converges Mackey to $x_A$ and since $f$ is continuous for the $c^\infty$-topology $\varphi_x(A_n) \to \varphi_x(A)$. Now we can apply another theorem of [**18**, 1952]: Any function $f : 2^J \to \mathbb{R}$ that is sequentially continuous and is zero on all countable subsets of $J$ is identically 0 provided the cardinality of $J$ is smaller than the first inaccessible cardinal. Thus we conclude that $0 = \varphi_x(J) = f(x) - f(x_{A_n})$ for all $x \in \mathbb{R}^J$. Hence $f$ factors over the metrizable space $\mathbb{R}^{A_0}$ and is therefore continuous for the locally convex topology. $\square$

# 8. Additional Stuff

I will end this lecture notes by an overview about related and - as I think - interesting topics which could be touched upon in future seminars.

## 8.1 Partitions of unity

The existence of sufficiently many smooth functions (like bump-functions or partitions of unity) depends heavily on the geometry of the locally convex or even of the Banach space under consideration. Some of the available results have been discussed in my seminar in the Winter semester 1992/93.

## 8.2 Differentiability of convex functions

This is related to the previous topic. It circles around the question of differentiability of semi-norms. Recall that every Hilbert-seminorm is smooth outside its kernel, since its square is induced from a continuous bilinear from. In general it is not always the case that the smooth seminorms generate the topology.

## 8.3 Solvability of equations

Beyond Banach spaces there are no inverse function theorems for general smooth functions. For tame Fréchet spaces one has the famous Hamilton-Nash-Moser inverse function theorem.

## 8.4 Algebras of smooth functions: characters and derivations

This topic concerns the question of classifying those spaces $X$, where every algebra homomorphism $C^\infty(X,\mathbb{R}) \to \mathbb{R}$ is given by a point-evaluation at some point $x \in X$. Similarly one wants to know whether every derivation $C^\infty(X,\mathbb{R}) \to \mathbb{R}$ over some point $x$ is given by a (tangent-)vector to $X$ at $x$.

## 8.5 Bounded seminorms and bounding sets

Of course it is important to determine the bornological topology on a given function space $C^\infty(X,F)$. In general there seems to be no direct description. This question is closely related to the bounding sets, since suprema over such sets give bounded seminorms.

### 8.6 Extension and lifting properties

This concerns the question of a Hahn-Banach/Tietze-Urysohn like extension results for smooth mappings. For real valued functions on finite dimensional spaces this is provided by Whitney's extension theorem. In general however it is not true even for closed linear subspaces of nuclear Fréchet spaces. The dual property is of course a lifting problem.

### 8.7 Approximation results

As we have seen it is quite often important to have denseness of certain subspaces of function spaces. Again there are only partial extensions to the smooth case in infinite dimensions of results like the Stone-Weierstrass theorem.

### 8.8 Non-linear spaces, manifolds

Obviously this frame work for calculus in locally convex spaces is only the starting point for a theory of non-linear infinite dimensional objects generalizing smooth manifolds. It has been shown that several function spaces carry natural infinite dimensional manifold structures. There have been also attempts to introduce smooth structures which are not manifolds, since this notion seems to be to restrictive in infinite dimensions.

### 8.9 Holomorphic mappings

The calculus presented here has been extended to complex differentiability by calling a function holomorphic iff its composites with holomorphic curves (i.e. maps from the complex unit disk into the space) are holomorphic curves. In fact life is much easier in the holomorphic theory since 1-times differentiable implies smoothness. So the problems discussed in 1.12 are not so severe here.

### 8.10 Real-analytic mappings

The calculus also extends to real analytic mappings. But this is more difficult than the smooth case, since even in 2 dimensions a function which is real analytic along real analytic curves is not necessarily real analytic and real analyticity can not be tested with continuous linear functionals in general.

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