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Contents

Chapter III
Partitions of Unity

The main aim of this chapter is to discuss the abundance or scarcity of smooth functions on a convenient vector space: E.g. existence of bump functions and partitions of unity. This question is intimately related to differentiability of seminorms and norms, and in many examples these are, if at all, only finitely often differentiable. So we start this chapter with a short (but complete) account of finite order differentiability, based on Lipschitz conditions on higher derivatives, since with this notion we can get as close as possible to exponential laws. A more comprehensive exposition of finite order Lipschitz differentiability can be found in the monograph [Frölicher, Kriegl, 1988].

Then we treat differentiability of seminorms and convex functions, and we have tried to collect all relevant information from the literature. We give full proofs of all what will be needed later on or is of central interest. We also collect related results, mainly on 'generic differentiability', i.e. differentiability on a dense G_δ-set.

If enough smooth bump functions exist on a convenient vector space, we call it 'smoothly regular'. Although the smooth (i.e. bounded) linear functionals separate points on any convenient vector space, stronger separation properties depend very much on the geometry. In particular, we show that ℓ^1 and $C[0,1]$ are not even C^1-regular. We also treat more general 'smooth spaces' here since most results do not depend on a linear structure, and since we will later apply them to manifolds.

In many problems like E. Borel's theorem 15.4 that any power series appears as Taylor series of a smooth function, or the existence of smooth functions with given carrier 15.3, one uses in finite dimensions the existence of smooth functions with globally bounded derivatives. These do not exist in infinite dimensions in general; even for bump functions this need not be true globally. Extreme cases are Hilbert spaces where there are smooth bump functions with globally bounded derivatives, and c_0 which does not even admit C^2-bump functions with globally bounded derivatives.

In the final section of this chapter a space which admits smooth partitions of unity subordinated to any open cover is called smoothly paracompact. Fortunately, a wide class of convenient vector spaces has this property, among them all spaces of smooth sections of finite dimensional vector bundles which we shall need later as modeling spaces for manifolds of mappings. The theorem 16.15 of [Toruńczyk, 1973] characterizes smoothly paracompact metrizable spaces, and we will give a full proof. It is the only tool for investigating whether non-separable spaces are smoothly paracompact and we give its main applications.

12. Differentiability of Finite Order

12.1. Definition. A mapping $f : E \supseteq U \to F$, where E and F are convenient vector spaces, and $U \subseteq E$ is c^∞-open, is called $\mathcal{L}\mathrm{ip}^k$ if $f \circ c$ is a $\mathcal{L}\mathrm{ip}^k$-curve (see 1.2) for each $c \in C^\infty(\mathbb{R}, U)$.

This is equivalent to the property that $f \circ c$ is $\mathcal{L}\mathrm{ip}^k$ on $c^{-1}(U)$ for each $c \in C^\infty(\mathbb{R}, E)$. This can be seen by reparameterization.

12.2. General curve lemma. *Let E be a convenient vector space, and let $c_n \in C^\infty(\mathbb{R}, E)$ be a sequence of curves which converges fast to 0, i.e., for each $k \in \mathbb{N}$ the sequence $n^k c_n$ is bounded. Let $s_n \geq 0$ be reals with $\sum_n s_n < \infty$.*

Then there exists a smooth curve $c \in C^\infty(\mathbb{R}, E)$ and a converging sequence of reals t_n such that $c(t + t_n) = c_n(t)$ for $|t| \leq s_n$, for all n.

Proof. Let $r_n := \sum_{k<n}(\frac{2}{k^2} + 2s_k)$ and $t_n := \frac{r_n + r_{n+1}}{2}$. Let $h : \mathbb{R} \to [0, 1]$ be smooth with $h(t) = 1$ for $t \geq 0$ and $h(t) = 0$ for $t \leq -1$, and put $h_n(t) := h(n^2(s_n + t)).h(n^2(s_n - t))$. Then we have $h_n(t) = 0$ for $|t| \geq \frac{1}{n^2} + s_n$ and $h_n(t) = 1$ for $|t| \leq s_n$, and for the derivatives we have $|h_n^{(j)}(t)| \leq n^{2j}.H_j$, where $H_j := \max\{|h^{(j)}| : t \in \mathbb{R}\}$. Thus, in the sum

$$c(t) := \sum_n h_n(t - t_n).c_n(t - t_n)$$

at most one summand is non-zero for each $t \in \mathbb{R}$, and c is a smooth curve since for each $\ell \in E'$ we have

$$(\ell \circ c)(t) = \sum_n f_n(t), \qquad \text{where } f_n(t + t_n) := h_n(t).\ell(c_n(t)),$$

$$\begin{aligned} n^2. \sup_t |f_n^{(k)}(t)| &= n^2. \sup \left\{ |f_n^{(k)}(s + t_n)| : |s| \leq \tfrac{1}{n^2} + s_n \right\} \\ &\leq n^2 \sum_{j=0}^k \tbinom{k}{j} n^{2j} H_j. \sup \left\{ |(\ell \circ c_n)^{(k-j)}(s)| : |s| \leq \tfrac{1}{n^2} + s_n \right\} \\ &\leq \left(\sum_{j=0}^k \tbinom{k}{j} n^{2j+2} H_j \right). \sup \left\{ |(\ell \circ c_n)^{(i)}(s)| : |s| \leq \max_n(\tfrac{1}{n^2} + s_n) \text{ and } i \leq k \right\}, \end{aligned}$$

which is uniformly bounded with respect to n, since c_n converges to 0 fast. □

12.3. Corollary. *Let $c_n : \mathbb{R} \to E$ be polynomials of bounded degree with values in a convenient vector space E. If for each $\ell \in E'$ the sequence $n \mapsto \sup\{|(\ell \circ c_n)(t) : |t| \leq 1\}$ converges to 0 fast, then the sequence c_n converges to 0 fast in $C^\infty(\mathbb{R}, E)$, so the conclusion of 12.2 holds.*

Proof. The structure on $C^\infty(\mathbb{R}, E)$ is the initial one with respect to the cone $\ell_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E'$, by 3.9. So we only have to show the result for $E = \mathbb{R}$. On the finite dimensional space of all polynomials of degree at most d the expression in the assumption is a norm, and the inclusion into $C^\infty(\mathbb{R}, \mathbb{R})$ is bounded. □

12.4. Difference quotients. For a curve $c : \mathbb{R} \to E$ with values in a vector space E the *difference quotient* $\delta^k c$ of order k is given recursively by

$$\delta^0 c := c,$$
$$\delta^k c(t_0, \dots, t_k) := k\, \frac{\delta^{k-1}c(t_0, \dots, t_{k-1}) - \delta^{k-1}c(t_1, \dots, t_k)}{t_0 - t_k},$$

for pairwise different t_i. The constant factor k in the definition of δ^k is chosen in such a way that δ^k approximates the k-th derivative. By induction, one can easily see that

$$\delta^k c(t_0, \dots, t_k) = k! \sum_{i=0}^{k} c(t_i) \prod_{\substack{0 \le j \le k \\ j \ne i}} \tfrac{1}{t_i - t_j}.$$

We shall mainly need the *equidistant difference quotient* $\delta^k_{\text{eq}} c$ of order k, which is given by

$$\delta^k_{\text{eq}} c(t; v) := \delta^k c(t, t+v, \dots, t+kv) = \frac{k!}{v^k} \sum_{i=0}^{k} c(t+iv) \prod_{\substack{0 \le j \le k \\ j \ne i}} \tfrac{1}{i-j}.$$

Lemma. *For a convenient vector space E and a curve $c : \mathbb{R} \to E$ the following conditions are equivalent:*

(1) *c is $\mathcal{L}\text{ip}^{k-1}$.*
(2) *The difference quotient $\delta^k c$ of order k is bounded on bounded sets.*
(3) *$\ell \circ c$ is continuous for each $\ell \in E'$, and the equidistant difference quotient $\delta^k_{eq} c$ of order k is bounded on bounded sets in $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

Proof. All statements can be tested by composing with bounded linear functionals $\ell \in E'$, so we may assume that $E = \mathbb{R}$.

(3) $\Rightarrow$ (2) Let $I \subset \mathbb{R}$ be a bounded interval. Then there is some $K > 0$ such that $|\delta^k_{\text{eq}} c(x; v)| \le K$ for all $x \in I$ and $kv \in I$. Let $t_i \in I$ be pairwise different points. We claim that $|\delta^k c(t_0, \dots, t_k)| \le K$. Since $\delta^k c$ is symmetric we may assume that $t_0 < t_1 < \dots < t_k$, and since it is continuous (c is continuous) we may assume that all $\frac{t_i - t_0}{t_k - t_0}$ are of the form $\frac{n_i}{N}$ for $n_i, N \in \mathbb{N}$. Put $v := \frac{t_k - t_0}{N}$, then $\delta^k c(t_0, \dots, t_k) = \delta^k c(t_0, t_0 + n_1 v, \dots, t_0 + n_k v)$ is a convex combination of $\delta^k_{\text{eq}} c(t_0 + rv; v)$ for $0 \le r \le \max_i n_i - k$. This follows by recursively inserting intermediate points of the form $t_0 + mv$, and using

$$\begin{aligned}\delta^k(t_0 + m_0 v, \dots, \widehat{t_0 + m_i} v, \dots, t_0 + m_{k+1} v) = \\ = \frac{m_i - m_0}{m_{k+1} - m_0} \delta^k(t_0 + m_0 v, \dots, t_0 + m_k v) \\ + \frac{m_{k+1} - m_i}{m_{k+1} - m_0} \delta^k(t_1 + m_1 v, \dots, t_0 + m_{k+1} v)\end{aligned}$$

which itself may be proved by induction on k.

(2) $\Rightarrow$ (1) We have to show that c is k times differentiable and that $\delta^1 c^{(k)}$ is bounded on bounded sets. We use induction, $k = 0$ is clear.

Let $T \ne S$ be two subsets of $\mathbb{R}$ of cardinality $j + 1$. Then there exist enumerations $T = \{t_0, \dots, t_j\}$ and $S = \{s_0, \dots, s_j\}$ such that $t_i \ne s_j$ for $i \le j$; then we have

$$\delta^j c(t_0, \dots, t_j) - \delta^j c(s_0, \dots, s_j) = \tfrac{1}{j+1} \sum_{i=0}^{j} (t_i - s_i) \delta^{j+1} c(t_0, \dots, t_i, s_i, \dots, s_j).$$

For the enumerations we put the elements of $T \cap S$ at the end in T and at the beginning in S. Using the recursive definition of $\delta^{j+1}c$ and symmetry the right hand side becomes a telescoping sum.

Since $\delta^k c$ is bounded we see from the last equation that all $\delta^j c$ are also bounded, in particular this is true for $\delta^2 c$. Then

$$\frac{c(t+s)-c(t)}{s} - \frac{c(t+s')-c(t)}{s'} = \tfrac{s-s'}{2}\, \delta^2 c(t, t+s, t+s')$$

shows that the difference quotient of c forms a Mackey Cauchy net, and hence the limit $c'(t)$ exists.

Using the easily checked formula

$$c(t_j) = \sum_{i=0}^{j} \tfrac{1}{i!} \prod_{l=0}^{i-1} (t_j - t_l)\; \delta^j c(t_0, \dots, t_j),$$

induction on j and differentiability of c one shows that

$$\delta^j c'(t_0, \dots, t_j) = \tfrac{1}{j+1} \sum_{i=0}^{j} \delta^{j+1} c(t_0, \dots, t_j, t_i),$$

where $\delta^{j+1}c(t_0, \dots, t_j, t_i) := \lim_{t \to t_i} \delta^{j+1}c(t_0, \dots, t_j, t)$. The right hand side of 4 is bounded, so c' is $\mathcal{L}\mathrm{ip}^{k-2}$ by induction on k.

(1) $\Rightarrow$ (2) For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and $t_0 < \dots < t_j$ there exist s_i with $t_i < s_i < t_{i+1}$ such that

$$\delta^j f(t_0, \dots, t_j) = \delta^{j-1} f'(s_0, \dots, s_{j-1}).$$

Let p be the interpolation polynomial

$$p(t) := \sum_{i=0}^{j} \tfrac{1}{i!} \prod_{l=0}^{i-1} (t - t_l)\; \delta^j f(t_0, \dots, t_j).$$

Since f and p agree on all t_j, by Rolle's theorem the first derivatives of f and p agree on some intermediate points s_i. So p' is the interpolation polynomial for f' at these points s_i. Comparing the coefficient of highest order of p' and of the interpolation polynomial 6 for f' at the points s_i 5 follows.

Applying 5 recursively for $f = c^{(k-2)}, c^{(k-3)}, \dots, c$ shows that $\delta^k c$ is bounded on bounded sets, and (2) follows.

(2) $\Rightarrow$ (3) is obvious. □

12.5. Let $r_0, \dots, r_k$ be the unique rational solution of the linear equation

$$\sum_{i=0}^{k} i^j r_i = \begin{cases} 1 & \text{for } j = 1 \\ 0 & \text{for } j = 0, 2, 3, \dots, k. \end{cases}$$

Lemma. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}\mathrm{ip}^k$ for $k \ge 1$ and I is a compact interval then there exists M such that for all $t, v \in I$ we have*

$$\left| \tfrac{\partial}{\partial s}|_0 f(t,s).v - \sum_{i=0}^{k} r_i f(t, iv) \right| \le M |v|^{k+1}.$$

Proof. We consider first the case $0 \notin I$ so that v stays away from 0. For this it suffices to show that the derivative $\frac{\partial}{\partial s}|_0 f(t,s)$ is locally bounded. If it is unbounded near some point x_∞, there are x_n with $|x_n - x_\infty| \le \frac{1}{2^n}$ such that $\frac{\partial}{\partial s}|_0 f(x_n, s) \ge n.2^n$. We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := (x_n, \frac{t}{2^n})$ and to $s_n := \frac{1}{2^n}$ in order to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $|t| \le s_n$. Then $(f \circ c)'(t_n) = \frac{1}{2^n}\frac{\partial}{\partial s}|_0 f(x_n, s) \ge n$, which contradicts that f is $\mathcal{L}\text{ip}^1$.

Now we treat the case $0 \in I$. If the assertion does not hold there are $x_n, v_n \in I$, such that $\left|\frac{\partial}{\partial s}|_0 f(x_n, s).v_n - \sum_{i=0}^k r_i f(x_n, iv_n)\right| \ge n.2^{n(k+1)}|v_n|^{k+1}$. We may assume $x_n \to x_\infty$, and by the case $0 \notin I$ we may assume that $v_n \to 0$, even with $|x_n - x_\infty| \le \frac{1}{2^n}$ and $|v_n| \le \frac{1}{2^n}$. We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := (x_n, \frac{t}{2^n})$ and to $s_n := \frac{1}{2^n}$ to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $|t| \le s_n$. Then we have

$$\begin{aligned}\Big|(f \circ c)'(t_n)2^n v_n - \sum_{i=0}^k r_i (f \circ c)(t_n + i2^n v_n)\Big| &= \\ &= \Big|(f \circ c_n)'(0)2^n v_n - \sum_{i=0}^k r_i (f \circ c_n)(i2^n v_n)\Big| \\ &= \Big|\frac{1}{2^n}\frac{\partial}{\partial s}|_0 f(x_n, s)2^n v_n - \sum_{i=0}^k r_i f(x_n, iv_n)\Big| \ge n(2^n|v_n|)^{k+1}.\end{aligned}$$

This contradicts the next claim for $g = f \circ c$.

Claim. If $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}\text{ip}^k$ for $k \ge 1$ and I is a compact interval then there is $M > 0$ such that for $t, v \in I$ we have $\left|g'(t).v - \sum_{i=0}^k r_i g(t + iv)\right| \le M|v|^{k+1}$.

Consider $g_t(v) := g'(t).v - \sum_{i=0}^k r_i g(t + iv)$. Then the derivatives up to order k at $v = 0$ of g_t vanish by the choice of the r_i. Since $g^{(k)}$ is locally Lipschitzian there exists an M such that $|g_t^{(k)}(v)| \le M|v|$ for all $t, v \in I$, which we may integrate in turn to obtain $|g_t(v)| \le M\frac{|v|^{k+1}}{(k+1)!}$. □

12.6. Lemma. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{L}\text{ip}^{k+1}$. Then $t \mapsto \frac{\partial}{\partial s}|_0 f(t, s)$ is $\mathcal{L}\text{ip}^k$.*

Proof. Suppose that $g : t \mapsto \frac{\partial}{\partial s}|_0 f(t, s)$ is not $\mathcal{L}\text{ip}^k$. Then by lemma 12.4 the equidistant difference quotient $\delta_{\text{eq}}^{k+1} g$ is not locally bounded at some point which we may assume to be 0. Then there are x_n and v_n with $|x_n| \le 1/4^n$ and $0 < v_n < 1/4^n$ such that

$$|\delta_{\text{eq}}^{k+1} g(x_n; v_n)| > n.2^{n(k+2)}.$$

We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := e_n(\frac{t}{2^n} + x_n) := (\frac{t}{2^n} + x_n - v_n, \frac{t}{2^n})$ and to $s_n := \frac{k+2}{2^n}$ in order to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $0 \le t \le s_n$.

Put $f_0(t, s) := \sum_{i=0}^k r_i\, f(t, is)$ for r_i as in 12.5, put $f_1(t, s) := g(t)s$, finally put $f_2 := f_1 - f_0$. Then f_0 in $\mathcal{L}\text{ip}^{k+1}$, so $f_0 \circ c$ is $\mathcal{L}\text{ip}^{k+1}$, hence the equidistant difference quotient $\delta_{\text{eq}}^{k+2}(f_0 \circ c)(x_n; 2^n v_n)$ is bounded.

By lemma 12.5 there exists $M > 0$ such that $|f_2(t,s)| \le M|s|^{k+2}$ for all $t, s \in [-(k+1), k+1]$, so we get

$$\begin{aligned}|\delta_{\text{eq}}^{k+2}(f_2 \circ c)(x_n; 2^n v_n)| &= |\delta_{\text{eq}}^{k+2}(f_2 \circ c_n)(0; 2^n v_n)| \\ &= \tfrac{1}{2^{n(k+2)}} |\delta_{\text{eq}}^{k+2}(f_2 \circ e_n)(x_n; v_n)| \\ &\le \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=1}^{k+2} \frac{|f_2((i-1)v_n + x_n, iv_n)|}{|iv_n|^{(k+2)}} \frac{i^{(k+2)}}{\prod_{j\ne i} |i-j|} \\ &\le \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=1}^{k+2} M \frac{i^{(k+2)}}{\prod_{j\ne i} |i-j|}.\end{aligned}$$

This is bounded, and so for $f_1 = f_0 + f_2$ the expression $|\delta_{\text{eq}}^{k+2}(f_1 \circ c)(x_n; 2^n v_n)|$ is also bounded, with respect to n. However, on the other hand we get

$$\begin{aligned}\delta_{\text{eq}}^{k+2}(f_1 \circ c)(x_n; 2^n v_n) &= \delta_{\text{eq}}^{k+2}(f_1 \circ c_n)(0; 2^n v_n) \\ &= \tfrac{1}{2^{n(k+2)}} \delta_{\text{eq}}^{k+2}(f_1 \circ e_n)(x_n; v_n) \\ &= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=0}^{k+2} \frac{f_1((i-1)v_n + x_n, iv_n)}{v_n^{(k+2)}} \prod_{\substack{0\le j\le k+2\\ j\ne i}} \tfrac{1}{i-j} \\ &= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=0}^{k+2} \frac{g((i-1)v_n + x_n)iv_n}{v_n^{(k+2)}} \prod_{\substack{0\le j\le k+2\\ j\ne i}} \tfrac{1}{i-j} \\ &= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{l=0}^{k+1} \frac{g(lv_n + x_n)}{v_n^{(k+1)}} \prod_{\substack{0\le j\le k+1\\ j\ne l}} \tfrac{1}{l-j} \\ &= \tfrac{k+2}{2^{n(k+2)}} \delta_{\text{eq}}^{k+1} g(x_n; v_n),\end{aligned}$$

which in absolute value is larger than $(k+2)n$ by 1, a contradiction. □

12.7. Lemma. *Let $U \subseteq E$ be open in a normed space. Then, a mapping $f : U \to F$ into a convenient vector space is $\mathcal{L}\text{ip}^0$ if and only if f is Lipschitz on compact subsets K of U, i.e., $\{\frac{f(x)-f(y)}{\|x-y\|} : x \ne y \in K\}$ is bounded.*

A mapping $f : U \to F$ into a Banach space is $\mathcal{L}\text{ip}^0$ if and only if f is locally Lipschitz, i.e., for each $z \in U$ there exists a ball B_z around z such that $\{\frac{f(x)-f(y)}{\|x-y\|} : x \ne y \in B_z\}$ is bounded.

Proof. ($\Rightarrow$) If F is Banach and f is $\mathcal{L}\text{ip}^0$ but not locally Lipschitz near $z \in U$, there are points $x_n \ne y_n$ in U with $\|x_n - z\| \le 1/4^n$ and $\|y_n - z\| \le 1/4^n$, such that $\|f(y_n) - f(x_n)\| \ge n.2^n.\|y_n - x_n\|$. Now we apply the general curve lemma 12.2 with $s_n := 2^n.\|y_n - x_n\|$ and $c_n(t) := x_n - z + t\frac{y_n - x_n}{2^n\|y_n - x_n\|}$ to get a smooth curve c with $c(t + t_n) - z = c_n(t)$ for $0 \le t \le s_n$. Then $\frac{1}{s_n}\|(f \circ c)(t_n + s_n) - (f \circ c)(t_n)\| = \frac{1}{2^n.\|y_n - x_n\|}\|f(y_n) - f(x_n)\| \ge n$.

If F is convenient, f is $\mathcal{L}\text{ip}^0$ but not Lipschitz on a compact K, there exist $\ell \in F'$ such that $\ell \circ f$ is not Lipschitz on K. By the first part of the proof, $\ell \circ f$ is locally Lipschitz, a contradiction.

($\Leftarrow$) This is obvious, since the composition of Lipschitz mappings is again Lipschitz. □

12.8. Theorem. *Let $f : E \supseteq U \to F$ be a mapping, where E and F are convenient vector spaces, and $U \subseteq E$ is c^∞-open. Then the following assertions are equivalent for each $k \geq 0$:*

(1) *f is $\mathcal{L}\mathrm{ip}^{k+1}$.*
(2) *The directional derivative*
$$(d_v f)(x) := \tfrac{\partial}{\partial t}|_{t=0}(f(x+tv))$$
exists for $x \in U$ and $v \in E$ and defines a $\mathcal{L}\mathrm{ip}^k$-mapping $U \times E \to F$.

Note that this result gives a different (more algebraic) proof of Boman's theorem 3.4 and 3.14.

Proof. (1) $\Rightarrow$ (2) Clearly, $t \mapsto f(x+tv)$ is $\mathcal{L}\mathrm{ip}^{k+1}$, and so the directional derivative exists and is the Mackey-limit of the difference quotients, by lemma 1.7. In order to show that $df : (x,v) \mapsto d_v f(x)$ is $\mathcal{L}\mathrm{ip}^k$ we take a smooth curve $(x,v) : \mathbb{R} \to U \times E$ and $\ell \in F'$, and we consider $g(t,s) := x(t) + s.v(t)$, $g : \mathbb{R}^2 \to E$. Then $\ell \circ f \circ g : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}\mathrm{ip}^{k+1}$, so by lemma 12.6 the curve
$$t \mapsto \ell(df(x(t), v(t))) = \ell\left(\tfrac{\partial}{\partial s}|_0 f(g(t,s))\right) = \tfrac{\partial}{\partial s}|_0 \ell(f(g(t,s)))$$
is of class $\mathcal{L}\mathrm{ip}^k$.

(2) $\Rightarrow$ (1) If $c \in C^\infty(\mathbb{R}, U)$ then
$$\frac{f(c(t)) - f(c(0))}{t} - df(c(0), c'(0)) =$$
$$= \int_0^1 \Big(df(c(0) + s(c(t) - c(0)), \tfrac{c(t)-c(0)}{t}) - df(c(0), c'(0)) \Big)\, ds$$
converges to 0 for $t \to 0$ since $g : (t,s) \mapsto df(c(0) + s(c(t) - c(0)), \frac{c(t)-c(0)}{t}) - df(c(0), c'(0))$ is $\mathcal{L}\mathrm{ip}^k$, thus by lemma 12.7 g is locally Lipschitz, so the set of all $\frac{g(t_1,s)-g(t_2,s)}{t_1-t_2}$ is locally bounded, and finally $t \mapsto \int_0^1 g(t,s)ds$ is locally Lipschitz. Thus, $f \circ c$ is differentiable with derivative $(f \circ c)'(0) = df(c(0), c'(0))$.

Since df is $\mathcal{L}\mathrm{ip}^k$ and (c, c') is smooth we get that $(f \circ c)'$ is $\mathcal{L}\mathrm{ip}^k$, hence $f \circ c$ is $\mathcal{L}\mathrm{ip}^{k+1}$. □

12.9. Corollary. Chain rule. *The composition of $\mathcal{L}\mathrm{ip}^k$-mappings is again $\mathcal{L}\mathrm{ip}^k$, and the usual formula for the derivative of the composite holds.*

Proof. We have to compose $f \circ g$ with a smooth curve c, but then $g \circ c$ is a $\mathcal{L}\mathrm{ip}^k$-curve, thus it is sufficient to show that the composition of a $\mathcal{L}\mathrm{ip}^k$ curve $c : \mathbb{R} \to U \subseteq E$ with a $\mathcal{L}\mathrm{ip}^k$-mapping $f : U \to F$ is again $\mathcal{L}\mathrm{ip}^k$, and that $(f \circ c)'(t) = df(c(t), c'(t))$.

This follows by induction on k for $k \geq 1$ in the same way as we proved theorem 12.8.2 $\Rightarrow$ 12.8.1, using theorem 12.8 itself. □

12.10. Definition and Proposition. *Let F be a convenient vector space. The space $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ of all $\mathcal{L}\mathrm{ip}^k$-curves in F is again a convenient vector space with the following equivalent structures:*

(1) *The initial structure with respect to the $k+2$ linear mappings (for $0 \leq j \leq k+1$) $c \mapsto \delta^j c$ from $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ into the space of all F-valued maps in $j+1$ pairwise different real variables $(t_0, \ldots, t_j)$ which are bounded on bounded subsets, with the c^∞-complete locally convex topology of uniform*

convergence on bounded subsets. In fact, the mappings δ^0 and δ^{k+1} are sufficient.

(2) *The initial structure with respect to the $k+2$ linear mappings (for $0 \le j \le k+1$) $c \mapsto \delta^j_{eq} c$ from $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ into the space of all maps from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ into F which are bounded on bounded subsets, with the c^∞-complete locally convex topology of uniform convergence on bounded subsets. In fact, the mappings δ^0_{eq} and δ^{k+1}_{eq} are sufficient.*

(3) *The initial structure with respect to the derivatives of order $j \le k$ considered as linear mappings into the space of $\mathcal{L}\mathrm{ip}^0$-curves, with the locally convex topology of uniform convergence of the curve on bounded subsets of $\mathbb{R}$ and of the difference quotient on bounded subsets of $\{(t,s) \in \mathbb{R}^2 : t \ne s\}$.*

The convenient vector space $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ satisfies the uniform boundedness principle with respect to the point evaluations.

Proof. All three structures describe closed embeddings into finite products of spaces, which in (1) and (2) are obviously c^∞-complete. For (3) this follows, since by (1) the structure on $\mathcal{L}\mathrm{ip}^0(\mathbb{R}, E)$ is convenient.

All structures satisfy the uniform boundedness principle for the point evaluations by 5.25, and since spaces of all bounded mappings on some (bounded) set satisfy this principle. This can be seen by composing with ℓ_* for all $\ell \in E'$, since Banach spaces do this by 5.24.

By applying this uniform boundedness principle one sees that all these structures are indeed equivalent. □

12.11. Definition and Proposition. *Let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. The space $\mathcal{L}\mathrm{ip}^k(U, F)$ of all $\mathcal{L}\mathrm{ip}^k$-mappings from U to F is again a convenient vector space with the following equivalent structures:*

(1) *The initial structure with respect to the linear mappings $c^* : \mathcal{L}\mathrm{ip}^k(U, F) \to \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, F)$.*

(2) *The initial structure with respect to the linear mappings $c^* : \mathcal{L}\mathrm{ip}^k(U, F) \to \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ for all $c \in \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$.*

This space satisfies the uniform boundedness principle with respect to the evaluations $\mathrm{ev}_x : \mathcal{L}\mathrm{ip}^k(U, F) \to F$ for all $x \in U$.

Proof. The structure (1) is convenient since by 12.1 it is a closed subspace of the product space which is convenient by 12.10. The structure in (2) is convenient since it is closed by 12.9. The uniform boundedness principle for the point evaluations now follows from 5.25 and 12.10, and this in turn gives us the equivalence of the two structures. □

12.12. Remark. We want to call the attention of the reader to the fact that there is no general exponential law for $\mathcal{L}\mathrm{ip}^k$-mappings. In fact, if $f \in \mathcal{L}\mathrm{ip}^k(\mathbb{R}, \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F))$ then $(\frac{\partial}{\partial t})^p (\frac{\partial}{\partial s})^q f^\wedge(t,s)$ exists if $\max(p,q) \le k$. This describes a smaller space than $\mathcal{L}\mathrm{ip}^k(\mathbb{R}^2, F)$, which is not invariantly describable.

However, some partial results still hold, namely for convenient vector spaces E, F, and G, and for c^∞-open sets $U \subseteq E$, $V \subseteq F$ we have

$$\mathcal{L}\mathrm{ip}^k(U, L(F,G)) \cong L(F, \mathcal{L}\mathrm{ip}^k(U,G)),$$
$$\mathcal{L}\mathrm{ip}^k(U, \mathcal{L}\mathrm{ip}^l(V,G)) \cong \mathcal{L}\mathrm{ip}^l(V, \mathcal{L}\mathrm{ip}^k(U,G)),$$

see [Frölicher, Kriegl, 1988, 4.4.5, 4.5.1, 4.5.2]. For a mapping $f : U \times F \to G$ which is linear in F we have: $f \in \mathcal{L}\mathrm{ip}^k(U \times F, G)$ if and only if $f^\vee \in \mathcal{L}\mathrm{ip}^k(U, L(E,F))$, see [Frölicher, Kriegl, 1988, 4.3.5]. The last property fails if we weaken Lipschitz to continuous, see the following example.

12.13. Smolyanov's Example. Let $f : \ell^2 \to \mathbb{R}$ be defined by $f := \sum_{k\geq 1} \frac{1}{k^2} f_k$, where $f_k(x) := \varphi(k(kx_k - 1)) \cdot \prod_{j<k} \varphi(jx_j)$ and $\varphi : \mathbb{R} \to [0,1]$ is smooth with $\varphi(0) = 1$ and $\varphi(t) = 0$ for $|t| \geq \frac{1}{4}$. We shall show that

(1) $f : \ell^2 \to \mathbb{R}$ is Fréchet differentiable.
(2) $f' : \ell^2 \to (\ell^2)'$ is not continuous.
(3) $f' : \ell^2 \times \ell^2 \to \mathbb{R}$ is continuous.

Proof. Let $A := \{x \in \ell^2 : |kx_k| \leq \frac{1}{4} \text{ for all } k\}$. This is a closed subset of ℓ^2.

(1) Remark that for $x \in \ell^2$ at most one $f_k(x)$ can be unequal to 0. In fact $f_k(x) \neq 0$ implies that $|kx_k - 1| \leq \frac{1}{4k} \leq \frac{1}{4}$, and hence $kx_k \geq \frac{3}{4}$ and thus $f_j(x) = 0$ for $j > k$.

For $x \notin A$ there exists a $k > 0$ with $|kx_k| > \frac{1}{4}$ and the set of points satisfying this condition is open. It follows that $\varphi(kx_k) = 0$ and hence $f = \sum_{j<k} \frac{1}{j^2} f_j$ is smooth on this open set.

On the other hand let $x \in A$. Then $|kx_k - 1| \geq \frac{3}{4} > \frac{1}{4}$ and hence $\varphi(k(kx_k - 1)) = 0$ for all k and thus $f(x) = 0$. Let $v \in \ell^2$ be such that $f(x+v) \neq 0$. Then there exists a unique k such that $f_k(x+v) \neq 0$ and therefore $|j(x_j + v_j)| < \frac{1}{4}$ for $j < k$ and $|k(x_k + v_k) - 1| < \frac{1}{4k} \leq \frac{1}{4}$. Since $|kx_k| \leq \frac{1}{4}$ we conclude $|kv_k| \geq 1 - |k(x_k+v_k) - 1| - |kx_k| \geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. Hence $|f(x+v)| = \frac{1}{k^2}|f_k(x+v)| \leq \frac{1}{k^2} \leq (2|v_k|)^2 \leq 4\|v\|^2$. Thus $\frac{\|f(x+v)-0-0\|}{\|v\|} \leq 4\|v\| \to 0$ for $\|v\| \to 0$, i.e. f is Fréchet differentiable at x with derivative 0.

(2) If fact take $a \in \mathbb{R}$ with $\varphi'(a) \neq 0$. Then $f'(t\,e^k)(e^k) = \frac{d}{dt}\frac{1}{k^2} f_k(t\,e^k) = \frac{d}{dt}\frac{1}{k^2}\varphi(k^2\,t - k) = \varphi'(k\,(k\,t-1)) = \varphi'(a)$ if $t = t_k := \frac{1}{k}\left(\frac{a}{k} + 1\right)$, which goes to 0 for $k \to \infty$. However $f'(0)(e^k) = 0$ since $0 \in A$.

(3) We have to show that $f'(x^n)(v^n) \to f'(x)(v)$ for $(x^n, v^n) \to (x, v)$. For $x \notin A$ this is obviously satisfied, since then there exists a k with $|kx_k| > \frac{1}{4}$ and hence $f = \sum_{j\leq k} \frac{1}{j^2} f_j$ locally around x.

If $x \in A$ then $f'(x) = 0$ and thus it remains to consider the case, where $x^n \notin A$. Let $\varepsilon > 0$ be given. Locally around x^n at most one summand f_k does not vanish: If $x^n \notin A$ then there is some k with $|kx^k| > 1/4$ which we may choose minimal. Thus $|jx^j| \leq 1/4$ for all $j < k$, so $|j(jx^j - 1)| \geq 3j/4$ and hence $f_j = 0$ locally since the first factor vanishes. For $j > k$ we get $f_j = 0$ locally since the second factor vanishes. Thus we can evaluate the derivative:

$$|f'(x^n)(v^n)| = \Big|\frac{1}{k^2} f_k'(x^n)(v^n)\Big| \leq \frac{\|\varphi'\|_\infty}{k^2}\Big(k^2|v_k^n| + \sum_{j<k} j|v_j^n|\Big).$$

Since $v \in \ell^2$ we find a K_1 such that $(\sum_{j\geq K_1} |v_j|^2)^{1/2} \leq \frac{\varepsilon}{2\|\varphi'\|_\infty}$. Thus we conclude from $\|v^n - v\|_2 \to 0$ that $|v_j^n| \leq \frac{\varepsilon}{\|\varphi'\|_\infty}$ for $j \geq K_1$ and large n. For the finitely many small n we can increase K_1 such that for these n and $j \geq K_1$ also $|v_j^n| \leq \frac{\varepsilon}{\|\varphi'\|_\infty}$. Furthermore there is a constant $K_2 \geq 1$ such that $\|v^n\|_\infty \leq \|v^n\|_2 \leq K_2$ for all n. Now choose $N \geq K_1$ so large that $N^2 \geq \frac{1}{\varepsilon}\|\varphi'\|_\infty K_2\,K_1^2$. Obviously $\sum_{n<N} \frac{1}{n^2} f_n$ is

smooth. So it remains to consider those n for which the non-vanishing term has index $k \geq N$. For those terms we have

$$\begin{aligned}
|f'(x^n)(v^n)| &= \Big|\frac{1}{k^2} f_k'(x^n)(v^n)\Big| \leq \|\varphi'\|_\infty \Big(|v_k^n| + \frac{1}{k^2}\sum_{j<k} j|v_j^n|\Big) \\
&\leq |v_k^n| \|\varphi'\|_\infty + \|\varphi'\|_\infty \frac{1}{k^2} \sum_{j<K_1} j|v_j^n| + \frac{1}{k^2} \sum_{K_1 \leq j < k} j|v_j^n| \, \|\varphi'\|_\infty \\
&\leq \varepsilon + \|\varphi'\|_\infty \frac{K_1^2}{N^2} \|v^n\|_\infty + \frac{1}{k^2} \sum_{K_1 \leq j < k} j\,\varepsilon \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

This shows the continuity. □

13. Differentiability of Seminorms

A desired separation property is that the smooth functions generate the topology. Since a locally convex topology is generated by the continuous seminorms it is natural to look for smooth seminorms. Note that every seminorm $p : E \to \mathbb{R}$ on a vector space E factors over $E_p := E/\ker p$ and gives a norm on this space. Hence, it can be extended to a norm $\tilde{p} : \tilde{E}_p \to \mathbb{R}$ on the completion $\tilde{E}_p$ of the space E_p which is normed by this factorization. If E is a locally convex space and p is continuous, then the canonical quotient mapping $E \to E_p$ is continuous. Thus, smoothness of $\tilde{p}$ off 0 implies smoothness of p on its carrier, and so the case where E is a Banach space is of central importance.

Obviously, every seminorm is a convex function, and hence we can generalize our treatment slightly by considering convex functions instead. The question of their differentiability properties is exactly the topic of this section.

Note that since the smooth functions depend only on the bornology and not on the locally convex topology the same is true for the initial topology induced by all smooth functions. Hence, it is appropriate to make the following

Convention. In this chapter the locally convex topology on all convenient vector spaces is assumed to be the bornological one.

13.1. Remark. It can be easily seen that for a function $f : E \to \mathbb{R}$ on a vector space E the following statements are equivalent (see for example [Frölicher, Kriegl, 1988, p. 199]):

(1) The function f is convex, i.e. $f(\sum_{i=1}^n \lambda_i x_i) \le \sum_{i=1}^n \lambda_i f(x_i)$ for $\lambda_i \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$;
(2) The set $U_f := \{(x,t) \in E \times \mathbb{R} : f(x) < t\}$ is convex;
(3) The set $A_f := \{(x,t) \in E \times \mathbb{R} : f(x) \le t\}$ is convex.

Moreover, for any translation invariant topology on E (and hence in particular for the locally convex topology or the c^∞-topology on a convenient vector space) and any convex function $f : E \to \mathbb{R}$ the following statements are equivalent:

(1) The function f is continuous;
(2) The set U_f is open in $E \times \mathbb{R}$;
(3) The set $f_{<t} := \{x \in E : f(x) < t\}$ is open in E for all $t \in \mathbb{R}$, i.e. f is upper semi-continuous.

Moreover the following statements are equivalent:

(1) The function f is lower semicontinuous, i.e. the set $f_{>t} := \{x \in E : f(x) > t\}$ is open in E for all $t \in \mathbb{R}$;
(2) The set A_f is closed in $E \times \mathbb{R}$.

13.2. Result. Convex Lipschitz functions. *Let $f : E \to \mathbb{R}$ be a convex function on a convenient vector space E. Then the following statements are equivalent:*

(1) *It is $\mathcal{L}\mathrm{ip}^0$;*
(2) *It is continuous for the bornological locally convex topology;*
(3) *It is continuous for the c^∞-topology;*
(4) *It is bounded on Mackey converging sequences;*

If f is a seminorm, then these further are equivalent to

(5) *It is bounded on bounded sets.*

If E is normed this further is equivalent to

(6) *It is locally bounded.*

Proof is commented out

The proof is due to [Aronszajn, 1976] for Banach spaces and [Frölicher, Kriegl, 1988, p. 200], for convenient vector spaces.

13.3. Basic definitions. Let $f : E \supseteq U \to F$ be a mapping defined on a c^∞-open subset of a convenient vector space E with values in another one F. Let $x \in U$ and $v \in E$. Then the (one sided) *directional derivative* of f at x in direction v is defined as

$$f'(x)(v) = d_v f(x) := \lim_{t \searrow 0} \frac{f(x + t\,v) - f(x)}{t}.$$

Obviously, if $f'(x)(v)$ exists, then so does $f'(x)(s\,v)$ for $s > 0$ and equals $s\,f'(x)(v)$.

Even if $f'(x)(v)$ exists for all $v \in E$ the mapping $v \mapsto f'(x)(v)$ may not be linear in general, and if it is linear it will not be bounded in general. Hence, f is called *Gâteaux-differentiable* at x, if the directional derivatives $f'(x)(v)$ exist for all $v \in E$ and $v \mapsto f'(x)(v)$ is a bounded linear mapping from $E \to F$.

Even for Gâteaux-differentiable mappings the difference quotient $\frac{f(x+t\,v)-f(x)}{t}$ need not converge uniformly for v in bounded sets (or even in compact sets). Hence, one defines f to be *Fréchet-differentiable* at x if f is Gâteaux-differentiable at x and $\frac{f(x+t\,v)-f(x)}{t} - f'(x)(v) \to 0$ uniformly for v in any bounded set. For a Banach space E this is equivalent to the existence of a bounded linear mapping denoted $f'(x) : E \to F$ such that

$$\lim_{v \to 0} \frac{f(x + v) - f(x) - f'(x)(v)}{\|v\|} = 0.$$

If $f : E \supseteq U \to F$ is Gâteaux-differentiable and the derivative $f' : E \supseteq U \to L(E, F)$ is continuous, then f is Fréchet-differentiable, and we will call such a function C^1. In fact, the fundamental theorem applied to $t \mapsto f(x + t\,v)$ gives us

$$f(x + v) - f(x) = \int_0^1 f'(x + t\,v)(v)\,dt,$$

and hence

$$\frac{f(x + s\,v) - f(x)}{s} - f'(x)(v) = \int_0^1 \Big(f'(x + t\,s\,v) - f'(x)\Big)(v)\,dt \to 0,$$

which converges to 0 for $s \to 0$ uniformly for v in any bounded set, since $f'(x + t\,s\,v) \to f'(x)$ uniformly on bounded sets for $s \to 0$ and uniformly for $t \in [0, 1]$ and v in any bounded set, since f' is assumed to be continuous.

Recall furthermore that a mapping $f : E \supseteq U \to F$ on a Banach space E is called *Lipschitz* if

$$\left\{\frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} : x_1, x_2 \in U, x_1 \neq x_2\right\} \text{ is bounded in } F.$$

It is called *Hölder* of order $0 < p \le 1$ if

$$\left\{\frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|^p} : x_1, x_2 \in U, x_1 \neq x_2\right\} \text{ is bounded in } F.$$

13.4. Lemma. Gâteaux-differentiability of convex functions. *Every convex function $q : E \to \mathbb{R}$ has one sided directional derivatives. The derivative $q'(x)$ is sublinear and locally bounded (or continuous at 0) if q is locally bounded (or* *continuous). In particular, such a locally bounded function is Gâteaux-differentiable at x if and only if $q'(x)$ is an odd function, i.e. $q'(x)(-v) = -q'(x)(v)$.*

If E is not normed, then locally bounded-ness should mean bounded on bornologically compact sets.

Proof. For $0 < t < t'$ we have by convexity that

$$q(x + t\,v) = q\big((1 - \frac{t}{t'})x + \frac{t}{t'}\,(x + t'\,v)\big) \le (1 - \frac{t}{t'})\,q(x) + \frac{t}{t'}\,q(x + t'v).$$

Hence $\frac{q(x+t\,v)-q(x)}{t} \le \frac{q(x+t'\,v)-q(x)}{t'}$. Thus, the difference quotient is monotone falling for $t \to 0$. It is also bounded from below, since for $t' < 0 < t$ we have

$$\begin{aligned} q(x) &= q\big(\frac{t}{t - t'}\,(x + t'\,v) + (1 - \frac{t}{t - t'})\,(x + t\,v)\big) \\ &\le \frac{t}{t - t'}\,q(x + t'\,v) + (1 - \frac{t}{t - t'})\,q(x + t\,v), \end{aligned}$$

and hence $\frac{q(x+t'\,v)-q(x)}{t'} \le \frac{q(x+t\,v)-q(x)}{t}$. Thus, the one sided derivative

$$q'(x)(v) := \lim_{t\searrow 0} \frac{q(x + t\,v) - q(x)}{t}$$

exists.

As a derivative $q'(x)$ automatically satisfies $q'(x)(t\,v) = t\,q'(x)(v)$ for all $t \ge 0$. The derivative $q'(x)$ is convex as limit of the convex functions $v \mapsto \frac{q(x+tv)-q(x)}{t}$. Hence it is sublinear.

The convexity of q implies that

$$q(x) - q(x - v) \le q'(x)(v) \le q(x + v) - q(x).$$

Therefore, the local boundedness of q at x implies that of $q'(x)$ at 0. Let $\ell := f'(x)$, then subadditivity and odd-ness implies $\ell(a) \le \ell(a + b) + \ell(-b) = \ell(a + b) - \ell(b)$ and hence the converse triangle inequality. □

Remark. If q is a seminorm, then $\frac{q(x+tv)-q(x)}{t} \le \frac{q(x)+t\,q(v)-q(x)}{t} = q(v)$, hence $q'(x)(v) \le q(v)$, and furthermore $q'(x)(x) = \lim_{t\searrow 0} \frac{q(x+t\,x)-q(x)}{t} = \lim_{t\searrow 0} q(x) = q(x)$. Hence we have

$$\|q'(x)\| := \sup\{|q'(x)(v)| : q(v) \le 1\} = \sup\{q'(x)(v) : q(v) \le 1\} = 1.$$

Convention. Let $q \ne 0$ be a seminorm and let $q(x) = 0$. Then there exists a $v \in E$ with $q(v) \ne 0$, and we have $q(x + tv) = |t|\,q(v)$, hence $q'(x)(\pm v) = q(v)$. So q is not Gâteaux differentiable at x. Therefore, we call a seminorm *smooth* for some differentiability class, if and only if it is smooth on its carrier $\{x : q(x) > 0\}$.

13.5. Differentiability properties of convex functions f can be translated in geometric properties of A_f:

Lemma. Differentiability of convex functions. *Let $f : E \to \mathbb{R}$ be a continuous convex function on a Banach space E, and let $x_0 \in E$. Then the following statements are equivalent:*

1. *The function f is Gâteaux differentiable at x_0;*
2. *There exists a unique $\ell \in E'$ with*

$$\ell(v) \le f(x_0 + v) - f(x_0) \text{ for all } v \in E;$$

3. *There exists a unique affine hyperplane tangent to A_f through $(x_0, f(x_0))$.*

(4) *The Minkowski functional of (some translate of)* A_f *is Gâteaux differentiable at* $(x_0, f(x_0))$.

Moreover, for a sublinear function f *and* $f(x_0) \neq 0$ *the following statements are equivalent:*

(5) *The function* f *is Gâteaux (Fréchet) differentiable at* x_0*;*
(6) *The point* x_0 *(strongly) exposes the polar of the set* $\{x : f(x) \leq 1\}$.

In particular, the following statements are equivalent for a convex function f*:*

(7) *The function* f *is Gâteaux (Fréchet) differentiable at* x_0*;*
(8) *The Minkowski functional of (some translate of)* A_f *is Gâteaux (Fréchet) differentiable at the point* $(x_0, f(x_0))$*;*
(9) *The point* $(x_0, f(x_0))$ *(strongly) exposes the polar of some translate of* A_f.

An element $x^* \in E^*$ is said to *expose a subset* $K \subseteq E$ if there exists a unique point $k_0 \in K$ with $x^*(k_0) = \sup\{x^*(k) : k \in K\}$, i.e. x^* takes it supremum on K on a unique point k_0. It is said to *strongly expose* K, if satisfies in addition that $x^*(x_n) \to x^*(k_0)$ implies $x_n \to k_0$.

By an affine hyperplane H *tangent* to a convex set K at a point $x \in K$ we mean that $x \in H$ and K lies on one side of H.

Proof. Let f be a convex function. By the proof of 13.4 we have $f'(x_0)(v) \leq f(x_0 + v) - f(x_0)$. For any $\ell \in E'$ with $\ell(v) \leq f(x_0 + v) - f(x_0)$ for all $v \in E$ we have $\ell(v) = \frac{1}{t}\ell(tv) \leq \frac{f(x_0+t\,v)-f(x_0)}{t}$ for all $t > 0$, and hence $\ell \leq f'(x_0)$.

(1) $\Rightarrow$ (2) Let f be continuous and Gâteaux-differentiable at x_0, so $f'(x_0)$ is linear (and continous) and thus minimal among all sub-linear mappings. By what we said before $f'(x_0)$ is the unique linear functional satisfying (2).

(2) $\Rightarrow$ (1) By what we said before the unique ℓ in (2) satisfied $\ell \leq f'(x_0)$. So $f'(x_0) - \ell \geq 0$. If this is not identical zero, then there exists a $\mu \in E^*$ with $0 \neq \mu \leq f'(x_0) - \ell$ by Hahn-Banach. Thus $\ell + \mu$ satisfies (2) also, a contradiction to the uniqueness of ℓ.

(2) $\Leftrightarrow$ (3) Any hyperplane tangent to A_f at $(x_0, f(x_0))$ is described by a functional $0 \neq (\ell, s) \in E' \times \mathbb{R}$ such that $\ell(x) + s\,t \geq \ell(x_0) + s\,f(x_0)$ for all $t \geq f(x)$. Note that the scalar s cannot be 0, since this would imply that $\ell(x) \geq \ell(x_0)$ for all x. It has to be positive, since otherwise the left side would go to $-\infty$ for $f(x) \leq t \to +\infty$. Without loss of generality we may thus assume that $s = 1$, so the hyperplane uniquely determines the linear functional ℓ with $\ell(x-x_0) \geq f(x_0)-f(x)$ for all x or, by replacing ℓ by $-\ell$ and x by $x_0 + v$, we have a unique ℓ with $\ell(v) \leq f(x_0 + v) - f(x_0)$ for all $v \in E$.

(3) $\Leftrightarrow$ (4) A sublinear functional $p \geq 0$ is Gâteaux-differentiable at x_0 with $p(x_0) \neq 0$ if and only if there is a unique affine hyperplane tangent to $\{x : p(x) \leq p(x_0)\}$ at x_0:
By (1) $\Leftrightarrow$ (2) p is differentiable at x_0 iff there exists a unique $\ell \in E'$ with $\ell(v) \leq p(x_0 + v) - p(x_0)$ for all v, or, equivalently, $\ell(x - x_0) \leq p(x) - p(x_0)$ for all x. Thus $\ell(x) \leq \ell(x_0)$ for all $p(x) \leq p(x_0)$. Conversely let $0 \neq \ell \in E'$ satisfy this condition and x be arbitary. Since $\{x : p(x) \leq p(x_0)\}$ is absorbing, $\ell(x_0) > 0$ and we may replace ℓ by $\frac{p(x_0)}{\ell(x_0)}\ell$. If $p(x) = 0$ then $p(r\,x) = 0 \leq p(x_0)$ for all r and hence $\ell(r\,x) \leq \ell(x_0)$ for all r, i.e. $\ell(x) = 0$ and hence $\ell(x - x_0) = -\ell(x_0) = -p(x_0) = p(x) - p(x_0)$. Otherwise we may consider $x' := \frac{p(x_0)}{p(x)}x$ which satisfies

$p(x') = p(x_0)$ and hence $\ell(x_0) \geq \ell(x') = \frac{p(x_0)}{p(x)}\ell(x)$ so $\ell(x - x_0) = \ell(x) - \ell(x_0) \leq (p(x) - p(x_0))\frac{\ell(x_0)}{p(x_0)} = p(x) - p(x_0)$.

We translate A_f such that it becomes absorbing (e.g. by $-(0, f(0) + 1)$). The sublinear Minkowski functional p of this translated set A_f is by what we just showed Gâteaux-differentiable at $(x_0, f(x_0))$ with $p(x_0, f(x_0)) = 1$ iff there exists a unique affine hyperplane tangent to $\{(x,t) : p(x,t) \leq p(x_0, f(x_0))\} = f(x_0)\, A_f$ in $(x_0, f(x_0))$, since A_f is closed. Since $f(x_0) \neq 0$ this is equivalent with (3).

(5) $\Leftrightarrow$ (6) We show this for Gâteaux-differentiability. We have to show that there is a unique tangent hyperplane to $x_0 \in K := \{x : f(x) \leq 1\}$ if and only if x_0 exposes $K^o := \{\ell \in E^* : \ell(x) \leq 1 \text{ for all } x \in K\}$. Let us assume $0 \in K$ and $0 \neq x_0 \in \partial K$. Then a tangent hyperplane to K at x_0 is uniquely determined by a linear functional $\ell \in E^*$ with $\ell(x_0) = 1$ and $\ell(x) \leq 1$ for all $x \in K$. This is equivalent to $\ell \in K^o$ and $\ell(x_0) = 1$, since by Hahn-Banach there exists an $\ell \in K^o$ with $\ell(x_0) = 1$. From this the result follows.

This shows also (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) for Gâteaux-differentiability, since $\{(x,t) : p_{A_f}(x,t) \leq 1\} = A_f$.

In order to show the statements for Fréchet-differentiability one has to show that $\ell = f'(x)$ is a Fréchet derivative if and only if x_0 is a strongly exposing point. This is left to the reader, see also 13.19 for a more general result. □

13.6. Lemma. Duality for convex functions. [Moreau, 1965].
Let $\langle\ ,\ \rangle : G \times F \to \mathbb{R}$ be a dual pairing.

(1) *For $f : F \to \mathbb{R} \cup \{+\infty\}$, $f \neq +\infty$ one defines the dual function*
$$f^* : G \to \mathbb{R} \cup \{+\infty\}, \quad f^*(z) := \sup\{\langle z, y\rangle - f(y) : y \in F\}.$$
(2) *The dual function f^* is convex and lower semi-continuous with respect to the weak topology. Since a function g is lower semi-continuous if and only if for all $a \in \mathbb{R}$ the set $\{x : g(x) > a\}$ is open, equivalently the convex set $\{x : g(x) \leq a\}$ is closed, this is for convex functions the same for every topology which is compatible with the duality.*
(3) $f_1 \leq f_2 \;\Rightarrow\; f_1^* \geq f_2^*$.
(4) $f^* \leq g \;\Leftrightarrow\; g^* \leq f$.
(5) *$f^{**} = f$ if and only if f is lower semi-continuous and convex.*
(6) *Suppose $z \in G$ satisfies $f(x + v) \geq f(x) + \langle z, v\rangle$ for all v (in particular, this is true if $z = f'(x)$). Then $f(x) + f^*(z) = \langle z, x\rangle$.*
(7) *If $f_1(y) = f(y - a)$ for all y, then $f_1^*(z) = \langle z, a\rangle + f^*(z)$ for all z.*
(8) *If $f_1(y) = f(y) + a$ for all y, then $f_1^*(z) = f^*(z) - a$ for all z.*
(9) *If $f_1(y) = f(y) + \langle b, y\rangle$ for all y, then $f_1^*(z) = f^*(z - b)$ for all z.*
(10) *If $E = F = \mathbb{R}$ and $f \geq 0$ with $f(0) = 0$, then $f^*(s) = \sup\{ts - f(t) : t \geq 0\}$ for $t \geq 0$.*
(11) *If $\gamma : \mathbb{R} \to \mathbb{R}^+$ is convex and $\frac{\gamma(t)}{t} \to 0$, then $\gamma^*(t) > 0$ for $t > 0$.*
(12) *Let (F, G) be a Banach space and its dual. If $\gamma \geq 0$ is convex and $\gamma(0) = 0$, and $f(y) := \gamma(\|y\|)$, then $f^*(z) = \gamma^*(\|z\|)$.*
(13) *A convex function f on a Banach space is Fréchet differentiable at a with derivative $b := f'(a)$ if and only if there exists a convex non-negative function γ, with $\gamma(0) = 0$ and $\lim_{t\to 0} \frac{\gamma(t)}{t} = 0$, such that*
$$f(a + h) \leq f(a) + \langle f'(a), h\rangle + \gamma(\|h\|).$$

Proof. ([1]) Since $f \neq +\infty$, there is some y for which $\langle z, y\rangle - f(y)$ is finite, hence $f^*(z) > -\infty$.

([2]) The function $z \mapsto \langle z, y\rangle - f(y)$ is continuous and linear, and hence the supremum $f^*(z)$ is lower semi-continuous and convex. One would like to show that f^* is not constant $+\infty$: This is not true. In fact, take $f(t) = -t^2$ then $f^*(s) = \sup\{s\,t - f(t) : t \in \mathbb{R}\} = \sup\{s\,t + t^2 : t \in \mathbb{R}\} = +\infty$. More generally, $f^* \neq +\infty \Leftrightarrow f$ lies above some affine hyperplane, see ([5]).

([3]) If $f_1 \leq f_2$ then $\langle z, y\rangle - f_1(y) \geq \langle z, y\rangle - f_2(y)$, and hence $f_1^*(z) \geq f_2^*(z)$.

([4]) One has

$$\begin{aligned}\forall z : f^*(z) \leq g(z) &\Leftrightarrow \forall z, y : \langle z, y\rangle - f(y) \leq g(z)\\ &\Leftrightarrow \forall z, y : \langle z, y\rangle - g(z) \leq f(y)\\ &\Leftrightarrow \forall y : g^*(y) \leq f(y).\end{aligned}$$

([5]) Since $(f^*)^*$ is convex and lower semi-continuous, this is true for f provided $f = (f^*)^*$. Conversely, let $g(b) = -a$ and $g(z) = +\infty$ otherwise. Then $g^*(y) = \sup\{\langle z, y\rangle - g(z) : z \in G\} = \langle b, y\rangle + a$. Hence, $a + \langle b, \quad\rangle \leq f \Leftrightarrow f^*(b) \leq -a$. If f is convex and lower semi-continuous, then A_f is closed and convex and hence f is the supremum of all continuous linear functionals $a + \langle b, \quad\rangle$ below it by Hahn-Banach, and this is exactly the case if $f^*(b) \leq -a$. Hence, $f^{**}(y) = \sup\{\langle z, y\rangle - f^*(z) : z \in G\} \geq \langle b, y\rangle + a$ and thus $f = f^{**}$.

([6]) Let $f(a + y) \geq f(a) + \langle b, y\rangle$. Then $f^*(b) = \sup\{\langle b, y\rangle - f(y) : y \in F\} = \sup\{\langle b, a + v\rangle - f(a + v) : v \in F\} \leq \sup\{\langle b, a\rangle + \langle b, v\rangle - f(a) - \langle b, v\rangle : v \in F\} = \langle b, a\rangle - f(a)$.

([7]) Let $f_1(y) = f(y - a)$. Then

$$\begin{aligned}f_1^*(z) &= \sup\{\langle z, y\rangle - f(y - a) : y \in F\}\\ &= \sup\{\langle z, y + a\rangle - f(y) : y \in F\} = \langle z, a\rangle + f^*(z).\end{aligned}$$

([8]) Let $f_1(y) = f(y) + a$. Then

$$f_1^*(z) = \sup\{\langle z, y\rangle - f(y) - a : y \in F\} = f^*(z) - a.$$

([9]) Let $f_1(y) = f(y) + \langle b, y\rangle$. Then

$$\begin{aligned}f_1^*(z) &= \sup\{\langle z, y\rangle - f(y) - \langle b, y\rangle : y \in F\}\\ &= \sup\{\langle z - b, y\rangle - f(y) : y \in F\} = f^*(z - b).\end{aligned}$$

([10]) Let $E = F = \mathbb{R}$ and $f \geq 0$ with $f(0) = 0$, and let $s \geq 0$. Using that $s\,t - f(t) \leq 0$ for $t \leq 0$ and that $s\,0 - f(0) = 0$ we obtain

$$f^*(s) = \sup\{s\,t - f(t) : t \in \mathbb{R}\} = \sup\{s\,t - f(t) : t \geq 0\}.$$

([11]) Let $\gamma \geq 0$ with $\lim_{t\searrow 0} \frac{\gamma(t)}{t} = 0$, and let $s > 0$. Then there are t with $s > \frac{\gamma(t)}{t}$, and hence

$$\gamma^*(s) = \sup\{st - \gamma(t) : t \geq 0\} = \sup\{t(s - \frac{\gamma(t)}{t}) : t \geq 0\} > 0.$$

([12]) Let $f(y) = \gamma(\|y\|)$. Then

$$\begin{aligned} f^*(z) &= \sup\{\langle z, y\rangle - \gamma(\|y\|) : y \in F\} \\ &= \sup\{t\langle z, y\rangle - \gamma(t) : \|y\| = 1, t \geq 0\} \\ &= \sup\{\sup\{t\langle z, y\rangle - \gamma(t) : \|y\| = 1\}, t \geq 0\} \\ &= \sup\{t\|z\| - \gamma(t) : t \geq 0\} \\ &= \gamma^*(\|z\|). \end{aligned}$$

([13]) If $f(a+h) \leq f(a) + \langle b, h\rangle + \gamma(\|h\|)$ for all h, then we have for $t > 0$

$$\frac{f(a+t\,h) - f(a)}{t} \leq \langle b, h\rangle + \frac{\gamma(t\,\|h\|)}{t},$$

hence $f'(a)(h) \leq \langle b, h\rangle$. Since $h \mapsto f'(a)(h)$ is sub-linear and the linear functionals are minimal among the sublinear ones, we have equality. By convexity we have

$$\frac{f(a+t\,h) - f(a)}{t} \geq \langle b, h\rangle = f'(a)(h).$$

So f is Fréchet-differentiable at a with derivative $f'(a)(h) = \langle b, h\rangle$, since the remainder is bounded by $\gamma(\|h\|)$ which satisfies $\frac{\gamma(\|h\|)}{\|h\|} \to 0$ for $\|h\| \to 0$.

Conversely, assume that f is Fréchet-differentiable at a with derivative b. Then

$$\frac{|f(a+h) - f(a) - \langle b, h\rangle|}{\|h\|} \to 0 \text{ for } h \to 0,$$

and by convexity

$$g(h) := f(a+h) - f(a) - \langle b, h\rangle \geq 0.$$

Let $\gamma(t) := \sup\{g(u) : \|u\| = |t|\}$. Since g is convex γ is convex, and obviously $\gamma(t) \in [0, +\infty]$, $\gamma(0) = 0$ and $\frac{\gamma(t)}{t} \to 0$ for $t \to 0$. This is the required function. □

13.7. Proposition. Continuity of the Fréchet derivative. [Asplund, 1968]. *The differential f' of any continuous convex function f on a Banach space is continuous on the set of all points where f is Fréchet differentiable. In general, it is however neither uniformly continuous nor bounded, see [15.8].*

Proof. Let $f'(x)(h)$ denote the one sided derivative. From convexity we conclude that $f(x+v) \geq f(x) + f'(x)(v)$. Suppose $x_n \to x$ are points where f is Fréchet differentiable. Then we obtain $f'(x_n)(v) \leq f(x_n+v) - f(x_n)$ which is bounded in n. Hence, the $f'(x_n)$ form a bounded sequence. We get

$$\begin{aligned} f(x) &\geq \langle f'(x_n), x\rangle - f^*(f'(x_n)) && \text{since } f(y) + f^*(z) \geq \langle z, y\rangle \\ &= \langle f'(x_n), x\rangle + f(x_n) - \langle f'(x_n), x_n\rangle && \text{since } f^*(f'(z)) + f(z) = f'(z)(z) \\ &\geq \langle f'(x_n), x - x_n\rangle + f(x) + \langle f'(x), x_n - x\rangle && \text{since } f(x+h) \geq f(x) + f'(x)(h) \\ &= \langle f'(x_n) - f'(x), x - x_n\rangle + f(x). \end{aligned}$$

Since $x_n \to x$ and $f'(x_n)$ is bounded, both sides converge to $f(x)$, hence

$$\lim_{n\to\infty} \langle f'(x_n), x\rangle - f^*(f'(x_n)) = f(x).$$

Since f is convex and Fréchet-differentiable at $a := x$ with derivative $b := f'(x)$, there exists by [13.6.13] a γ with

$$f(h) \leq f(a) + \langle b, h - a\rangle + \gamma(\|h - a\|).$$

By duality we obtain using [13.6.3]

$$f^*(z) \geq \langle z, a\rangle - f(a) + \gamma^*(\|z - b\|).$$

If we apply this to $z := f'(x_n)$ we obtain

$$f^*(f'(x_n)) \geq \langle f'(x_n), x\rangle - f(x) + \gamma^*(\|f'(x_n) - f'(x)\|).$$

Hence

$$\gamma^*(\|f'(x_n) - f'(x)\|) \leq f^*(f'(x_n)) - \langle f'(x_n), x\rangle + f(x),$$

and since the right side converges to 0, we have that $\gamma^*(\|f'(x_n) - f'(x)\|) \to 0$. Then $\|f'(x_n) - f'(x)\| \to 0$ where we use that γ is convex, $\gamma(0) = 0$, and $\gamma(t) > 0$ for $t > 0$, thus γ is strictly monotone increasing. □

13.8. Asplund spaces and generic Fréchet differentiability. From 13.4 it follows easily that a convex function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at all except countably many points. This has been generalized by [Rademacher, 1919] to: Every Lipschitz mapping from an open subset of $\mathbb{R}^n$ to $\mathbb{R}$ is differentiable almost everywhere. Recall that a locally bounded convex function is locally Lipschitz, see 13.2.

Proposition. *For a Banach space E the following statements are equivalent:*

(1) *Every continuous convex function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense G_δ-subset of E;*
(2) *Every continuous convex function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense subset of E;*
(3) *Every locally Lipschitz function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense subset of E;*
(4) *Every equivalent norm is Fréchet-differentiable at least at one point;*
(5) *E has no equivalent rough norm;*
(6) *Every (closed) separable subspace has a separable dual;*
(7) *The dual E^* has the Radon-Nikodym property;*
(8) *Every linear mapping $E \to L^1(X, \Omega, \mu)$ which is integral is nuclear;*
(9) *Every closed convex bounded subset of E^* is the closed convex hull of its extremal points;*
(10) *Every bounded subset of E^* is dentable.*

A Banach space satisfying these equivalent conditions is called Asplund space.
Every Banach space with a Fréchet differentiable bump function is Asplund, [Ekeland, Lebourg, 1976, p. 203]*. It is an open question whether the converse is true.*
Every WCG-Banach-space (i.e. a Banach space for which a weakly compact subset K exists, whose linear hull is the whole space) is Asplund, [John, Zizler, 1976].
The Asplund property is inherited by subspaces, quotients, and short exact sequences, [Stegall, 1981].

About the proof. (1) [Asplund, 1968]: If a convex function is Fréchet differentiable on a dense subset then it is so on a dense G_δ-subset, i.e. a dense countable intersection of open subsets.

(2) is in fact a local property, since in [Borwein, Fitzpatrick, Kenderov, 1991] it is mentioned that for a Lipschitz function $f : E \to \mathbb{R}$ with Lipschitz constant L defined on a convex open set U the function

$$\tilde{f}(x) := \inf\{f(y) + L\|x - y\| : y \in U\}$$

is a Lipschitz extension with constant L, and it is convex if f is.

(2) ⇒ (3) is due to [Preiss, 1990], Every locally Lipschitz function on an Asplund space is Fréchet differentiable at points of a dense subset.

(3) ⇒ (2) follows from the fact that continuous convex functions are locally Lipschitz, see 13.2.

(2) ⇔ (4) is mentioned in [Preiss, 1990] without any proof or reference.

(2) ⇔ (10) is due to [Stegall, 1975]. A subset D of a Banach space is called *dentable*, if and only if for every $x \in D$ there exists an $\varepsilon > 0$ such that x is not in the closed convex hull of $\{y \in D : \|y - x\| \geq \varepsilon\}$.

(2) ⇔ (5) is due to [John, Zizler, 1978]. A norm p is called *rough*, see also 13.23, if and only if there exists an $\varepsilon > 0$ such that arbitrary close to each $x \in X$ there are points x_i and u with $\|u\| = 1$ such that $|p'(x_2)(u) - p'(x_1)(u)| \geq \varepsilon$. The usual norms on $C[0,1]$ and on ℓ^1 are rough by 13.12 and 13.13. A norm is not rough if and only if the dual ball is w^*-dentable. The unit ball is dentable if and only if the dual norm is not rough.

(2) ⇔ (6) is due to [Stegall, 1975].

(2) ⇔ (7) is due to [Stegall, 1978]. A closed bounded convex subset K of a Banach space E is said to have the *Radon-Nikodym property* if for any finite measure space (Ω, Σ, μ) every μ-continuous countably additive function $m : \Sigma \to E$ of finite variation with average range $\{\frac{m(S)}{\mu(S)} : S \in \Sigma, \mu(S) > 0\}$ contained in K is representable by a Bochner integrable function, i.e. there exists a Borel-measurable essentially separably valued function $f : \Omega \to E$ with $m(S) = \int_S f\, d\mu$. This function f is then called the Radon-Nikodym derivative of m. A Banach space is said to have the Radon-Nikodym property if every closed bounded convex subset has it. See also [Diestel, 1975]. A subset K is a Radon-Nikodym set if and only if every closed convex subset of K is the closed convex hull of its strongly exposed points.

(7) ⇔ (8) can be found in [Stegall, 1975] and is due to [Grothendieck, 1955]. A linear mapping $E \to F$ is called *integral* if and only if it has a factorization

$$\begin{array}{ccccc} E & \longrightarrow & F & \longrightarrow & F^{**} \\ \downarrow & & & & \uparrow \\ C(K) & & \longrightarrow & & L^1(K,\mu) \end{array}$$

for some Radon-measure μ on a compact space K.
A linear mapping $E \to F$ is called *nuclear* if and only if there are $x_n^* \in E^*$ and $y_n \in F$ such that $\sum_n \|x_n^*\|\,\|y_n\| < \infty$ and $T = \sum_n x_n^* \otimes y_n$.

(2) ⇔ (9) is due to [Stegall, 1981, p.516]. □

13.9. Results on generic Gâteaux differentiability of Lipschitz functions.

(1) [Mazur, 1933] & [Asplund, 1968] *A Banach space E with the property that every continuous convex function $f : E \to \mathbb{R}$ is Gâteaux-differentiable on a dense G_δ-subset is called weakly Asplund. Separable Banach spaces are weakly Asplund.*
(2) *In* [Živkov, 1983] *it is mentioned that there are Lipschitz functions on $\mathbb{R}$, which fail to be differentiable on a dense G_δ-subset.*
(3) *A Lipschitz function on a separable Banach space is "almost everywhere" Gâteaux-differentiable,* [Aronszajn, 1976].
(4) [Preiss, 1990] *If the norm on a Banach space is $\mathcal{B}$-differentiable then every Lipschitz function is $\mathcal{B}$-differentiable on a dense set. A function $f : E \supseteq U \to F$ is called $\mathcal{B}$-differentiable at $x \in U$ for some family $\mathcal{B}$ of bounded subsets, if there exists a continuous linear mapping (denoted $f'(x)$) in*

$L(E,F)$ such that for every $B \in \mathcal{B}$ one has $\frac{f(x+t\,v)-f(x)}{t} - f'(x)(v) \to 0$ for $t \to 0$ uniformly for $v \in B$.

(5) [Kenderov, 1974], *see* [Živkov, 1983]. *Every locally Lipschitzian function on a separable Banach space which has one sided directional derivatives for each direction in a dense subset is Gâteaux differentiable on a non-meager subset.*

(6) [Živkov, 1983]. *For every space with Fréchet differentiable norm any locally Lipschitzian function which has directional derivatives for a dense set of directions is generically Gâteaux differentiable.*

(7) *There exists a Lipschitz Gâteaux differentiable function $f : L^1[0,1] \to \mathbb{R}$ which is nowhere Fréchet differentiable,* [Sova, 1966a], *see also* [Gieraltowska-Kedzierska, Van Vleck, 1991]. *Hence, this is an example of a weakly Asplund but not Asplund space.*

Further references on generic differentiability are: [Phelps, 1989], [Preiss, 1984], and [Zhivkov, 1987].

13.10. Lemma. Smoothness of $2n$-norm. *For $n \in \mathbb{N}$ the $2n$-norm is smooth on $L^{2n} \setminus \{0\}$.*

Proof. Since $t \mapsto t^{1/2n}$ is smooth on $\mathbb{R}^+$ it is enough to show that $x \mapsto (\|x\|_{2n})^{2n}$ is smooth. Let $p := 2n$. Since $(x_1, \dots, x_n) \mapsto x_1 \cdot \dots \cdot x_n$ is a n-linear contraction from $L^p \times \dots \times L^p \to L^1$ by the Hölder-inequality ($\sum_{i=1}^p \frac{1}{p} = 1$) and $\int : L^1 \to \mathbb{R}$ is a linear contraction the mapping $x \mapsto (x, \dots, x) \mapsto \int x^{2n}$ is smooth. Note that since we have a real Banach space and $p = 2n$ is even we can drop the absolute value in the formula of the norm. □

13.11. Derivative of the 1-norm. Let $x \in \ell^1$ and $j \in \mathbb{N}$ be such that $x_j = 0$. Let e_j be the characteristic function of $\{j\}$. Then $\|x + t\,e_j\|_1 = \|x\|_1 + |t|$ since the supports of x and e_j are disjoint. Hence, the directional derivative of the norm $p : v \mapsto \|v\|_1$ is given by $p'(x)(e_i) = 1$ and $p'(x)(-e_i) = 1$, and p is not differentiable at x. More generally we have:

Lemma. [Mazur, 1933, p.79]. *Let Γ be some set, and let p be the* 1-norm *given by $\|x\|_1 = p(x) := \sum_{\gamma\in\Gamma} |x_\gamma|$ for $x \in \ell^1(\Gamma)$. Then $p'(x)(h) = \sum_{x_\gamma=0} |h_\gamma| + \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma$.*

The basic idea behind this result is, that the unit sphere of the 1-norm is a hyperoctahedra, and the points on the faces are those, for which no coordinate vanishes.

Proof. Without loss of generality we may assume that $p(x) = 1 = p(h)$, since for $r > 0$ and $s \geq 0$ we have $p'(r\,x)(s\,h) = \frac{d}{dt}|_{t=0} p(r\,x + t\,s\,h) = \frac{d}{dt}|_{t=0} r\,p(x + t\,(\frac{s}{r}h)) = r\,p'(x)(\frac{s}{r}h) = s\,p'(x)(h)$.

We have $|x_\gamma + h_\gamma| - |x_\gamma| = ||x_\gamma| + h_\gamma \operatorname{sign} x_\gamma| - |x_\gamma| \geq |x_\gamma| + h_\gamma \operatorname{sign} x_\gamma - |x_\gamma| = h_\gamma \operatorname{sign} x_\gamma$, and is equal to $|h_\gamma|$ if $x_\gamma = 0$. Summing up these (in)equalities we obtain

$$p(x+h) - p(x) - \sum_{x_\gamma=0} |h_\gamma| - \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma \geq 0.$$

For $\varepsilon > 0$ choose a finite set $F \subset \Gamma$, such that $\sum_{\gamma\notin F} |h_\gamma| < \frac{\varepsilon}{2}$. Now choose t so small that

$$|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma \geq 0 \text{ for all } \gamma \in F \text{ with } x_\gamma \neq 0.$$

We claim that

$$\frac{q(x+t\,h)-q(x)}{t} - \sum_{x_\gamma=0}|h_\gamma| - \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma \le \varepsilon.$$

Let first γ be such that $x_\gamma = 0$. Then $\frac{|x_\gamma+t\,h_\gamma|-|x_\gamma|}{t} = |h_\gamma|$, hence these terms cancel with $-\sum_{x_\gamma=0}|h_\gamma|$.

Let now $x_\gamma \neq 0$. For $|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma \ge 0$ (hence in particular for $\gamma \in F$ with $x_\gamma \neq 0$) we have

$$\frac{|x_\gamma+t\,h_\gamma|-|x_\gamma|}{t} = \frac{|x_\gamma|+t\,h_\gamma \operatorname{sign} x_\gamma - |x_\gamma|}{t} = h_\gamma \operatorname{sign} x_\gamma.$$

Thus, these terms sum up to the corresponding sum $\sum_\gamma h_\gamma \operatorname{sign} x_\gamma$.

It remains to consider γ with $x_\gamma \neq 0$ and $|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma < 0$. Then $\gamma \notin F$ and

$$\frac{|x_\gamma+t\,h_\gamma|-|x_\gamma|}{t} - h_\gamma \operatorname{sign} x_\gamma = \frac{-|x_\gamma| - t\,h_\gamma \operatorname{sign} x_\gamma - |x_\gamma| - t\,h_\gamma \operatorname{sign} x_\gamma}{t}$$
$$\le -2h_\gamma \operatorname{sign} x_\gamma,$$

and since $\sum_{\gamma\notin F}|h_\gamma| < \frac{\varepsilon}{2}$ these remaining terms sum up to something smaller than ε. □

Remark. The 1-norm is rough. This result shows that the 1-norm is Gâteaux-differentiable exactly at those points, where all coordinates are non-zero. Thus, if Γ is uncountable, the 1-norm is nowhere Gâteaux-differentiable.

In contrast to what is claimed in [Mazur, 1933, p.79], the 1-norm is nowhere Fréchet differentiable. In fact, take $0 \neq x \in \ell^1(\Gamma)$. For γ with $x_\gamma \neq 0$ and $t > 0$ we have that

$$p(x+t\,(-\operatorname{sign} x_\gamma\, e_\gamma)) - p(x) - t\,p'(x)(-\operatorname{sign} x_\gamma\, e_\gamma) =$$
$$= |x_\gamma - t \operatorname{sign} x_\gamma| - |x_\gamma| + t = \big||x_\gamma| - t\big| - |x_\gamma| + t \ge t\cdot 1,$$

provided $t \ge 2\,|x_\gamma|$, since then $\big||x_\gamma| - t\big| = t - |x_\gamma| \ge |x_\gamma|$. Obviously, for every $t > 0$ there are γ satisfying this required condition; either $x_\gamma = 0$ then we have a corner, or $x_\gamma \neq 0$ then it gets arbitrarily small. Thus, the directional difference quotient does not converge uniformly on the unit-sphere.

The set of points x in ℓ^1 where at least for one n the coordinate x_n vanishes is dense, and one has

$$p(x+t\,e^n) = p(x) + |t|, \text{ hence } p'(x+t\,e^n)(e^n) = \begin{cases} +1 & \text{for } t \ge 0 \\ -1 & \text{for } t < 0 \end{cases}.$$

Hence the derivative of p is uniformly discontinuous, i.e., in every non-empty open set there are points x_1, x_2 for which there exists an $h \in \ell^1$ with $\|h\| = 1$ and $|p'(x_1)(h) - p'(x_2)(h)| \ge 2$.

13.12. Derivative of the ∞-norm. On c_0 the norm is not differentiable at points x, where the norm is attained in at least two points. In fact let $|x(a)| = \|x\| = |x(b)|$ and let $h := \operatorname{sign} x(a)\, e_a$. Then $p(x+th) = |(x+th)(a)| = \|x\| + t$ for $t \ge 0$ and $p(x+th) = |(x+th)(b)| = \|x\|$ for $t \le 0$. Thus, $t \mapsto p(x+th)$ is not differentiable at 0 and thus p not at x.

If the norm of x is attained at a single coordinate a, then p is differentiable at x. In fact $p(x+th) = |(x+th)(a)| = |\operatorname{sign}(x(a))\|x\| + th(a)\operatorname{sign}^2(x(a))| = |\|x\| + th(a)\operatorname{sign}(x(a))| = \|x\| + th(a)\operatorname{sign}(x(a))$ for $|t|\,\|h\| \le \|x\| - \sup\{|x(t)| : t \neq a\}$. Hence the directional difference-quotient converges uniformly for h in the unit-ball.

Let $x \in C[0,1]$ be such that $\|x\|_\infty = |x(a)| = |x(b)|$ for $a \neq b$. Choose a y with $y(s)$ between 0 and $x(s)$ for all s and $y(a) = x(a)$ but $y(b) = 0$. For $t \geq 0$ we have $|(x + t\,y)(s)| \leq |x(a) + t\,y(a)| = (1+t)\,\|x\|_\infty$ and hence $\|x + t\,y\|_\infty = (1+t)\,\|x\|_\infty$. For $-1 \leq t \leq 0$ we have $|(x+t\,y)(s)| \leq |x(a)|$ and $\|(x+t\,y)(b)\| = \|x(a)\|$ and hence $\|x + t\,y\|_\infty = \|x\|_\infty$. Thus the directional derivative is given by $p'(x)(y) = \|x\|_\infty$ and $p'(x)(-y) = 0$. More precisely we have the following results.

Lemma. [Banach, 1932, p. 168]. *Let T be a compact metric space. Let $x \in C(T,\mathbb{R}) \setminus \{0\}$ and $h \in C(T,\mathbb{R})$. By p we denote the ∞-NORM $\|x\|_\infty = p(x) := \sup\{|x(t)| : t \in T\}$. Then $p'(x)(h) = \sup\{h(t)\ \mathrm{sign}\, x(t) : |x(t)| = p(x)$.*

The idea here is, that the unit-ball is a hyper-cube, and the points on the faces are exactly those for which the supremum is attained only in one point.

Proof. Without loss of generality we may assume that $p(x) = 1 = p(h)$, since for $r > 0$ and $s \geq 0$ we have $p'(r\,x)(s\,h) = \frac{d}{dt}|_{t=0} p(r\,x + t\,s\,h) = \frac{d}{dt}|_{t=0} r\,p(x + t\,(\frac{s}{r}h)) = r\,p'(x)(\frac{s}{r}h) = s\,p'(x)(h)$.

Let $A := \{t \in T : |x(t)| = p(x)\}$. For given $\varepsilon > 0$ we find by the uniform continuity of x and h a δ_1 such that $|x(t) - x(t')| < \frac{1}{2}$ and $|h(t) - h(t')| < \varepsilon$ for $\mathrm{dist}(t,t') < \delta_1$. Then $\{t : \mathrm{dist}(t,A) \geq \delta_1\}$ is closed, hence compact. Therefore $\delta := \|x\|_\infty - \sup\{|x(t)| : \mathrm{dist}(t,A) \geq \delta_1\} > 0$.

Now we claim that for $0 < t < \min\{\delta, 1\}$ we have

$$0 \leq \frac{p(x + t\,h) - p(x)}{t} - \sup\{h(r)\ \mathrm{sign}\, x(r) : r \in A\} \leq \varepsilon.$$

For all $s \in A$ we have

$$\begin{aligned} p(x + t\,h) &\geq |(x + t\,h)(s)| = \big||x(s)|\ \mathrm{sign}\, x(s) + t\,h(s)\ \mathrm{sign}\, x(s)^2\big| \\ &= \big||x(s)| + t\,h(s)\ \mathrm{sign}\, x(s)\big| = p(x) + t\,h(s)\ \mathrm{sign}\, x(s) \end{aligned}$$

for $0 \leq t \leq 1$, since $|h(s)| \leq p(h) = p(x)$. Hence

$$\frac{p(x + t\,h) - p(x)}{t} \geq \sup\{h(t)\ \mathrm{sign}\, x(t) : t \in A\}.$$

This shows the left inequality.

Let s be a point where the supremum $p(x+t\,h)$ is attained. From the left inequality it follows that

$$\begin{aligned} p(x + t\,h) &\geq p(x) + t\ \sup\{h(r)\ \mathrm{sign}\, x(r) : r \in A\}, \qquad \text{and hence} \\ |x(s)| &\geq |(x + th)(s)| - t\,|h(s)| \geq p(x + t\,h) - t\,p(h) \\ &\geq p(x) - t\ \underbrace{\Big(p(h) - \sup\{h(r)\ \mathrm{sign}\, x(r) : r \in A\}\Big)}_{\leq 1} \\ &> p(x) - \delta = \sup\{|x(r)| : \mathrm{dist}(r,A) \geq \delta_1\}. \end{aligned}$$

Therefore $\mathrm{dist}(s,A) < \delta_1$, and thus there exists an $a \in A$ with $\mathrm{dist}(s,a) < \delta_1$ and consequently $|x(s) - x(a)| < \frac{1}{2}$ and $|h(s) - h(a)| < \varepsilon$. In particular, $\mathrm{sign}\, x(s) = \mathrm{sign}\, x(a) \neq 0$. So we get

$$\begin{aligned} \frac{p(x + t\,h) - p(x)}{t} &= \frac{|(x + t\,h)(s)| - p(x)}{t} = \frac{\big||x(s)| + t\,h(s)\ \mathrm{sign}\, x(s)\big| - p(x)}{t} \\ &= \frac{|x(s)| + t\,h(s)\ \mathrm{sign}\, x(s) - p(x)}{t} \leq h(s)\ \mathrm{sign}\, x(a) \\ &\leq |h(s) - h(a)| + h(a)\ \mathrm{sign}\, x(a) \\ &< \varepsilon + \sup\{h(r)\ \mathrm{sign}\, x(r) : r \in A\}. \end{aligned}$$

This proves the claim which finally implies

$$p'(x)(v) = \lim_{t\searrow 0} \frac{p(x+t\,h)-p(x)}{t} = \sup\{h(r)\,\operatorname{sign} x(r) : r \in A\}. \quad \square$$

Remark. The ∞-norm is rough. This result shows that the points where the ∞-norm is Gâteaux-differentiable are exactly those x where the supremum $p(x)$ is attained in a single point a. The Gâteaux-derivative is then given by $p'(x)(h) = h(a)\,\operatorname{sign} x(a)$. In general, this is however not the Fréchet derivative:
Let $x \neq 0$. Without loss we may assume (that $p(x) = 1$ and) that there is a unique point a, where $|x(a)| = p(x)$. Moreover, we may assume $x(a) > 0$. Let $a_n \to a$ be such that $0 < x(a_n) < x(a)$ and let $0 < \delta_n := x(a) - x(a_n) < x(a)$. Now choose $s_n := 2\delta_n \to 0$ and $h_n \in C[0,1]$ with $p(h_n) \leq 1$, $h_n(a) = 0$ and $h_n(a_n) := 1$ and $p(x + s_n h_n) = (x + s_n h_n)(a_n) = x(a_n) + 2(x(a) - x(a_n)) = 2x(a) - x(a_n)$. For this choose $(x + s_n h_n)(t) \leq (x + s_n h_n)(a_n)$ locally, i.e.. $h_n(t) \leq 1 + (x(a_n) - x(t))/s_n$ and 0 far away from x. Then $p'(x)(h_n) = 0$ by 13.12 and

$$\frac{p(x+s_n\,h_n)-p(x)}{s_n} - p'(x)(h_n) = \frac{2x(a)-x(a_n)-x(a)}{s_n} = \frac{\delta_n}{2\delta_n} = \frac{1}{2} \not\to 0$$

Thus the limit is not uniform and p is not Fréchet differentiable at x.

The set of vectors $x \in C[0,1]$ which attain their norm at least at two points a and b is dense, and one has for appropriately chosen h with $h(a) = -x(a)$, $h(b) = x(b)$ that

$$p(x+t\,h) = (1+\max\{t,-t\})\,p(x), \text{ hence } p'(x+t\,h)(h) = \begin{cases} +1 & \text{for } t \geq 0 \\ -1 & \text{for } t < 0 \end{cases}.$$

Therefore, the derivative of the norm is uniformly discontinuous, i.e., in every non-empty open set there are points x_1, x_2 for which there exists an $h \in C[0,1]$ with $\|h\| = 1$ and $|p'(x_1)(h) - p'(x_2)(h)| \geq 2$.

13.13. Results on the differentiability of p-norms. [Bonic, Frampton, 1966, p.887].
For $1 < p < \infty$ not an even integer the function $t \mapsto |t|^p$ is differentiable of order n if $n < p$, and the highest derivative ($t \mapsto p\,(p-1)\dots(p-n+1)\,|t|^{p-n}$) satisfies a Hölder-condition with modulus $p - n$, one can show that the p-norm has exactly these differentiability properties, i.e.

1. *It is $(p-1)$-times differentiable with Lipschitzian highest derivative if p is an integer.*
2. *It is $[p]$-times differentiable with highest derivative being Hölderian of order $p - [p]$, otherwise.*
3. *The norm has no higher Hölder-differentiability properties.*

That the norm on L^p is C^1 for $1 < p < \infty$ was already shown by [Mazur, 1933].

13.14. Proposition. Smooth norms on a Banach space. *A norm on a Banach space is of class C^n on $E \setminus \{0\}$ if and only if its unit sphere is a C^n-submanifold of E.*

Proof. Let $p : E \to \mathbb{R}$ be a smooth norm. Since $p'(x)(x) = \frac{d}{dt}|_{t=0} p(x+tx) = \frac{d}{dt}|_{t=0}(1+t)p(x) = p(x)$, we see that $p(x) = 1$ is a regular equation and hence the unit sphere $S := p^{-1}(1)$ is a smooth submanifold (of codimension 1), see 27.11.

Explicitly, this can be shown as follows: For $a \in S$ let $\Phi : \ker(p'(a)) \times \mathbb{R}^+ \to E^+ := \{x \in E : p'(a)(x) > 0\}$ be given by $(v,t) \mapsto t\,\frac{a+v}{p(a+v)}$. This is well-defined, since $p(a+v) \geq p(a) + p'(a)(v) = p(a) = 0$ for $v \in \ker(p'(a))$. Note that $\Phi(v,t) = y$ implies that $t = p(y)$ and $v \in \ker(p'(a))$ is such that $a + v = \mu\, y$ for some $\mu \neq 0$, i.e. $\mu\, p'(a)(y) = p'(a)(a+v) = p'(a)(a) = p(a) = 1$ and hence $v = \frac{1}{p'(a)(y)}\, y - a$. Thus Φ is a diffeomorphism that maps $\ker(p'(a)) \times \{1\}$ onto $S \cap E^+$.

Conversely, let $x_0 \in E \setminus \{0\}$ and $a := \frac{x_0}{p(x_0)}$. Then a is in the unit sphere, hence there exists locally around a a diffeomorphism $\Phi : E \supseteq U \to \Phi(U) \subseteq E$ which maps $S \cap U \to F \cap \Phi(U)$ for some closed linear subspace $F \subseteq U$. Let $\lambda : E \to \mathbb{R}$ be a continuous linear functional with $\lambda(a) = 1$ and $\lambda \leq p$. Note that $b := \Phi'(a)(a) \neq F$, since otherwise $t \mapsto \Phi^{-1}(tb)$ is in S, but then $\lambda(\Phi^{-1}(tb)) \leq 0$ and hence $0 = \frac{d}{dt}|_{t=0}\lambda(\Phi^{-1}(tb)) = \lambda(\Phi'(a)^{-1}b) = \lambda(a) = 1$ gives a contradiction. Choose $\mu \in E'$ with $\mu|_F = 0$ and $\mu(b) = 1$. We have to show that $x \mapsto p(x)$ is C^n locally around x_0, or equivalently that this is true for $g : x \mapsto \frac{1}{p(x)}$. Then $g(x)$ is solution of the implicit equation $\varphi(x, g(x)) = 0$, where $\varphi : E \times \mathbb{R} \to F$ is given by $(x,g) \mapsto f(g \cdot x)$ with $f := \mu \circ \Phi$. This solution is C^n by the implicit function theorem, since $\partial_2\varphi(x_0, g(x_0)) = f'(g(x_0)x_0)(x_0) = p(x_0)\, f'(a)(a) = p(x_0)\,\mu(b) = p(x) \neq 0$, because f is a regular equation at a. □

Although this proof uses the implicit function theorem on Banach spaces we can do without as the following theorem shows:

13.15. Theorem. Characterization of smooth seminorms. *Let E be a convenient vector space.*

(1) Let $p : E \to \mathbb{R}$ be a convex function which is smooth on a neighborhood of $p^{-1}(1)$, and assume that $U := \{x \in E : p(x) < 1\}$ is not empty. Then U is open, and its boundary ∂U equals $\{x : p(x) = 1\}$, a smooth splitting submanifold of E.

(2) If U is a convex absorbing open subset of E whose boundary is a smooth submanifold of E then the Minkowski functional p_U is a smooth sublinear mapping, and $U = \{x \in E : p_U(x) < 1\}$.

Proof. (1) The set U is obviously convex and open by 4.5 and 13.1. Let $M := \{x : p(x) = 1\}$. We claim that $M = \partial U$. Let $x_0 \in U$ and $x_1 \in M$. Since $t \mapsto p(x_1 + t(x_0 - x_1))$ is convex it is strictly decreasing in a neighborhood of 0. Hence, there are points x close to x_1 with $p(x) < p(x_1)$ and such with $p(x) \geq 1$, i.e. x belongs to ∂U. Conversely, let $x \in \partial U$. Since U is open we have $p(x_1) \geq 1$. Suppose $p(x_1) > 1$, then $p(x) > 1$ locally around x_1, a contradiction to $x_1 \in \partial U$.

Now we show that M is a smooth splitting submanifold of E, i.e. every point has a neighborhood, in which M is up to a diffeomorphism a complemented subspace. Let $x_1 \in M = \partial U$. We consider the convex mapping $t \mapsto p(x_0 + t(x_1 - x_0))$. It is locally around 1 differentiable, and its value at 0 is strictly less than that at 1. Thus, $p'(x_1)(x_1 - x_0) \geq p(x_1) - p(x_0) > 0$, and hence we may replace x_0 by some point on the segment from x_0 to x_1 closer to x_1, such that $p'(x_0)(x_1 - x_0) > 0$. Without loss of generality we may assume that $x_0 = 0$. Let $U := \{x \in E : p'(0)x > 0 \text{ and } p'(x_1)x > 0\}$ and $V := (U - x_1) \cap \ker p'(x_1) \subseteq \ker p'(x_1)$. A smooth mapping from the open set $U \subseteq E$ to the open set $V \times \mathbb{R} \subseteq \ker p'(x_1) \times (p(0), +\infty)$ is given by $x \mapsto (tx - x_1, p(x))$, where $t := \frac{p'(x_1)(x_1)}{p'(x_1)(x)}$. This mapping is a diffeomorphism, since for $(y,r) \in \ker p'(x_1) \times \mathbb{R}$ the inverse image is given as $t(y + x_1)$ where t can be calculated from $r = p(t\,(y + x_1))$. Since $t \mapsto p(t\,(y + x_1))$ is a diffeomorphism between the intervals $(0, +\infty) \to (p(0), +\infty)$ this t is uniquely determined. Furthermore, t depends smoothly on (y,r): Let $s \mapsto (y(s), r(s))$ be a smooth curve, then $t(s)$ is given by the implicit equation $p(t\,(y(s)+x_1)) = r(s)$, and by the 2-dimensional implicit function theorem the solution $s \mapsto t(s)$ is smooth.

(2) By general principles p_U is a sublinear mapping, and $U = \{x : p_U(x) < 1\}$ since U is open. Thus it remains to show that p_U is smooth on its open carrier. So let c be a smooth curve in the carrier. By assumption, there is a diffeomorphism v, locally defined on E near an intersection point a of the ray through $c(0)$ with the boundary $\partial U = \{x : p(x) = 1\}$, such that ∂U corresponds to a closed linear subspace $F \subseteq E$. Since U is convex there is a bounded linear functional $\lambda \in E'$ with $\lambda(a) = 1$ and $U \subseteq \{x \in E : \lambda(x) \leq 1\}$ by the theorem of Hahn-Banach. Then $\lambda(T_a(\partial U)) = 0$ since any smooth curve in ∂U through a stays inside $\{x : \lambda(x) \leq 1\}$. Furthermore, $b : \frac{\partial}{\partial t}|_1 v(ta) \notin F$, since otherwise $t \mapsto v^{-1}(tb) \in \partial U$ but $\frac{\partial}{\partial t}|_1 \lambda(v^{-1}(tb)) = \lambda(a) = 1$.

Put $f := 1/p_U \circ c : \mathbb{R} \to \mathbb{R}$. Then f is a solution of the implicit equation $(\lambda \circ dv^{-1}(0) \circ v)(f(t)c(t)) = 0$ which has a unique smooth solution by the implicit function theorem in dimension 2 since

$$\tfrac{\partial}{\partial s}|_{s=f(t)}(\lambda \circ dv^{-1}(0) \circ v)(sc(t)) = \lambda dv^{-1}(0)dv(f(t)c(t))c(t) \neq 0$$

for t near 0, since for $t = 0$ we get $\lambda(c(0)) = \frac{1}{f(0)}$. So p_U is smooth on its carrier. □

13.16. The space $c_0(\Gamma)$. For an arbitrary set Γ the space $c_0(\Gamma)$ is the closure of all functions on Γ with finite support in the Banach space $\ell^\infty(\Gamma)$ of globally bounded functions on Γ with the supremum norm. The supremum norm on $c_0(\Gamma)$ is not differentiable on its carrier, see 13.12. Nevertheless, it was shown in [Bonic, Frampton, 1965] that c_0 is C^∞-regular.

Proposition. Smooth norm on c_0. Due to Kuiper according to [Bonic, Frampton, 1966]. *There exists an equivalent norm on $c_0(\Gamma)$ which is smooth off 0.*

Proof. To prove this let $h : \mathbb{R} \to \mathbb{R}$ be an unbounded symmetric smooth convex function vanishing near 0. Let $f : c_0(\Gamma) \to \mathbb{R}$ be given by $f(x) := \sum_{\gamma\in\Gamma} h(x_\gamma)$. Locally on $c_0(\Gamma)$ the function f is just a finite sum, hence f is smooth. In fact let $h(t) = 0$ for $|t| \leq \delta$. For $x \in c_0(\Gamma)$ the set $F := \{\gamma : |x_\gamma| \geq \delta/2\}$ is finite, and for $\|y - x\| < \delta$ we have that $f(y) = \sum_{\gamma\in F} h(y_\gamma)$.

The set $U := \{x : f(x) < 1\}$ is open, and bounded: Let $h(t) \geq 1$ for $|t| \geq \Delta$ and $f(x) < 1$, then $h(x_\gamma) < 1$ and thus $|x_\gamma| \leq \Delta$ for all γ. The set U is also absolutely convex: Since h is convex, so is f and hence U. Since h is symmetric, so is f and hence U.

The boundary $\partial U = f^{-1}(1)$ is a splitting submanifold of $c_0(\Gamma)$ by the implicit function theorem on Banach spaces, since $df(x)x \neq 0$ for $x \in \partial U$. In fact $df(x)(x) = \sum_\gamma h'(x_\gamma)x_\gamma \geq 0$ and at least for one γ we have $h(x_\gamma) > 0$ and thus $h'(x_\gamma) \neq 0$. So by 13.14 the Minkowski functional p_U is smooth off 0. Obviously, it is an equivalent norm. □

13.17. Proposition. Inheritance properties for differentiable norms.

(1) *The product of two spaces with C^n-norm has again a C^n-norm given by $\|(x_1, x_2)\| := \sqrt{\|x_1\|^2 + \|x_2\|^2}$. More generally, the ℓ^2-sum of C^n-normable Banach spaces is C^n-normable.*
(2) *A subspace of a space with a C^n-norm has a C^n-norm.*
(3) [Godefroy, Pelant, et. al., 1988]. *If $c_0(\Gamma) \to E \to F$ is a short exact sequence of Banach spaces, and F has a C^k-norm, then E has a C^k-norm. See also 14.12.1 and 16.19.*

(4) *For a compact space K let K' be the set of all accumulation points of K. The operation $K \mapsto K'$ has the following properties:*

$$
\begin{aligned}
&(a) && A \subseteq B \Rightarrow A' \subseteq B' \\
&(b) && (A \cup B)' = A' \cup B' \\
&(c) && (A \times B)' = (A' \times B) \cup (A \times B') \\
&(d) && \big(\{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{N}\}\big)' = \{0\} \\
&(e) && K' = \emptyset \Leftrightarrow K \text{ discrete.}
\end{aligned}
$$

(5) *If K is compact and $K^{(\omega)} = \emptyset$ then $C(K)$ has an equivalent C^∞-norm, see also* 16.20.

Proof. (1) and (2) are obvious.

(4) (a) is obvious, since if $\{x\}$ is open in B and $x \in A$, then it is also open in A in the trace topology, hence $A \cap (B \setminus B') \subseteq A \setminus A'$ and hence $A' = A \setminus (A \setminus A') \subseteq (A \setminus A \cap (B \setminus B')) = A \cap B' \subseteq B'$.

(b) By monotonicity we have '$\supseteq$'. Conversely let $x \in A' \cup B'$, w.l.o.g. $x \in A'$, suppose $x \notin (A \cup B)'$, then $\{x\}$ is open in $A \cup B$ and hence $\{x\} = \{x\} \cap A$ would be open in A, i.e. $x \notin A'$, a contradiction.

(c) is obvious, since $\{(x, y)\}$ is open in $A \times B \Leftrightarrow \{x\}$ is open in A and $\{y\}$ is open in B.

(d) and (e) are trivial.

For (3) a construction is used similar to that of Kuiper's smooth norm for c_0. Let $\pi : E \to F$ be the quotient mapping and $\|\ \ \|$ the quotient norm on F. The dual sequence $\ell^1(A) \leftarrow E^* \leftarrow F^*$ splits (just define $T : \ell^1(A) \to E^*$ by selection of $x_a^* := T(e_a) \in E^*$ with $\|x_a^*\| = 1$ and $x_a^*|_{c_0(A)} = \mathrm{ev}_a$ using Hahn Banach). Note that for every $x \in E$ and $\varepsilon > 0$ the set $\{\alpha : |x_\alpha^*(x)| \geq \|\pi(x)\| + \varepsilon\}$ is finite. In fact, by definition of the quotient norm $\|\pi(x)\| := \sup\{\|x + y\| : y \in c_0(\Gamma)\}$ there is a $y \in c_0(\Gamma)$ such that $\|x + y\| \leq \|\pi(x)\| + \varepsilon/2$. The set $\Gamma_0 := \{\alpha : |y_\alpha| \geq \varepsilon/2\}$ is finite. For all other α we have

$$
\begin{aligned}
|x_\alpha^*(x)| \leq |x_\alpha^*(x + y)| + |x_\alpha^*(y)| \leq \|x_\alpha^*\|\,\|x + y\| + |y_\alpha| <& \\
< 1\,(\|\pi(x)\| + \varepsilon/2) + \varepsilon/2 = \|\pi(x)\| + \varepsilon.&
\end{aligned}
$$

Furthermore, we have

$$\|x\| \leq 2\|\pi(x)\| + \sup\{|x_\alpha^*(x)| : \alpha\}.$$

In fact,

$$
\begin{aligned}
\|x\| &= \sup\{|\langle x^*, x\rangle| : \|x^*\| \leq 1\} \\
&\leq \sup\{|\langle T(\lambda) + y^* \circ \pi, x\rangle| : \|\lambda\|_1 \leq 1, \|y^*\| \leq 2\} \\
&= \sup\{|x_\alpha^*(x)| : \alpha\} + 2\|\pi(x)\|,
\end{aligned}
$$

since $x^* = T(\lambda) + x^* - T(\lambda)$, where $\lambda := x^*|_{c_0(\Gamma)}$ and hence $\|\lambda\|_1 \leq \|x^*\| \leq 1$, and $|T(\lambda)(x)| \leq \|\lambda\|_1 \sup\{|x_\alpha^*(x)| : \alpha\} \leq \|x\|$ hence $\|T(\lambda)\| \leq \|\lambda\|_1$, and $y^* \circ \pi = x^* - T(\lambda)$. Let $\|\ \ \|$ denote a norm on F which is smooth and is larger than the quotient norm. Analogously to 13.16 we define

$$f(x) := h(4\|\pi(x)\|) \prod_{a \in A} h(x_a^*(x)),$$

where $h : \mathbb{R} \to [0, 1]$ is smooth, even, 1 for $|t| \leq 1$, 0 for $|t| \geq 2$ and concave on $\{t : h(t) \geq 1/2\}$. Then f is smooth, since if $\pi(x) > 1/2$ then the first factor vanishes locally, and if $\|\pi(x)\| < 1$ we have that $\Gamma_0 := \{\alpha : |x^*_\alpha(x)| \geq 1-\varepsilon\}$ is finite, where $\varepsilon := (1-\|\pi(x)\|)/2$, for $\|y-x\| < \varepsilon$ also $|x^*_\alpha(y) - x^*_\alpha(x)| < \varepsilon$ and hence $|x^*_\alpha(y)| < 1-\varepsilon+\varepsilon = 1$ for all $\alpha \notin \Gamma_0$. So the product is locally finite. The set $\{x : f(x) > \frac{1}{2}\}$ is open, bounded and absolutely convex and has a smooth boundary $\{x : f(x) = \frac{1}{2}\}$. It is symmetric since f is symmetric. It is bounded, since $f(x) > 1/2$ implies $h(4\|\pi(x)\|) \geq 1/2$ and $h(x^*_a(x)) \geq 1/2$ for all a. Thus $4\|\pi(x)\| \leq 2$ and $|x^*_a(x)| \leq 2$ and thus $\|x\| \leq 2 \cdot 1/2 + 2 = 3$. For the convexity note that $x_i \geq 0$, $y_i \geq 0$, $0 \leq t \leq 1$, $\prod_i x_i \geq 1/2$, $\prod_i y_i \geq 1/2$ imply $\prod_i (tx_i + (1-t)y_i) \geq 1/2$, since log is concave. Since all factors of f have to be $\geq 1/2$ and h is concave on this set, convexity follows. Since one factor of $f(x) = \prod_\alpha f_\alpha(x)$ has to be unequal to 1, the derivative $f'(x)(x) < 0$, since $f'_\alpha(x)(x) \leq 0$ for all α by concavity and $f'_\alpha(x)(x) < 0$ for all x with $f_\alpha(x) < 1$. So its Minkowski-functional is an equivalent smooth norm on E.

Statement (5) follows from (3). First recall that K' is the set of accumulation points of K, i.e. those points x for which every neighborhood meets $K \setminus \{x\}$, i.e. $\{x\}$ is not open. Thus $K \setminus K'$ is discrete. For successor ordinals $\alpha = \beta + 1$ one defines $K^{(\alpha)} := (K^{(\beta)})'$ and for limit ordinals α as $\bigcap_{\beta<\alpha} K^{(\beta)}$. For a compact space K the equality $K^{(\omega)} = \emptyset$ implies $K^{(n)} = \emptyset$ for some $n \in \omega$, since $K^{(n)}$ is closed. Now one shows this by induction. Let $E := \{f \in C(K) : f|_{K'} = 0\}$. By the Tietze-Urysohn theorem one has a short exact sequence $c_0(K \setminus K') \cong E \to C(K) \to C(K')$. The equality $E = c_0(K \setminus K_0)$ can be seen as follows:

Let $f \in C(K)$ with $f|_{K'} = 0$. Suppose there is some $\varepsilon > 0$ such that $\{x : |f(x)| \geq \varepsilon\}$ is not finite. Then there is some accumulation point x_∞ of this set and hence $|f(x_\infty)| \geq \varepsilon$ but $x_\infty \in K'$ and so $f(x_\infty) = 0$. Conversely let $f \in c_0(K \setminus K')$ and define $\tilde f$ by $\tilde f|_{K'} := 0$ and $\tilde f|_{K \setminus K'} = f$. Then $\tilde f$ is continuous on $K \setminus K'$, since $K \setminus K'$ is discrete. For $x \in K'$ we have that $\tilde f(x) = 0$ and for each $\varepsilon > 0$ the set $\{y : |\tilde f(y)| \geq \varepsilon\}$ is finite, hence its complement is a neighborhood of x, and $\tilde f$ is continuous at x. So the result follows by induction. □

13.18. Results.

(1) *We do not know whether the quotient of a C^n-normable space is again C^n-normable. Compare however with* [Fitzpatrick, 1980].

(2) *The statement 13.17.5 is quite sharp, since by* [Haydon, 1990] *there is a compact space K with $K^{(\omega)} = \{\infty\}$ but without a Gâteaux-differentiable norm.*

(3) [Talagrand, 1986] *proved that for every ordinal number γ, the compact and scattered space $[0, \gamma]$ with the order topology is C^1-normable.*

(4) *It was shown by* [Toruńczyk, 1981] *that two Banach spaces are homeomorphic if and only if their density number is the same. Hence, one can view Banach spaces as exotic (differentiable or linear) structures on Hilbert spaces. If two Banach spaces are even C^1-diffeomorphic then the differential (at 0) gives a continuous linear homeomorphism. It was for some time unknown if also uniformly homeomorphic (or at least Lipschitz homeomorphic) Banach spaces are already linearly homeomorphic. By* [Enflo, 1970] *a Banach space which is uniformly homeomorphic to a Hilbert space is linearly homeomorphic to it. A counter-example to the general statement was given by* [Aharoni, Lindenstrauss, 1978]*, and another one is due to* [Ciesielski, Pol, 1984]*: There exists a short exact sequence $c_0(\Gamma_1) \to C(K) \to c_0(\Gamma_2)$ where $C(K)$ cannot be continuously*

injected into some $c_0(\Gamma)$ but is Lipschitz equivalent to $c_0(\Gamma)$. For these and similar questions see [Tzafriri, 1980].

(5) *A space all of whose closed subspaces are complemented is a Hilbert space,* [Lindenstrauss, Tzafriri, 1971].

(6) [Enflo, Lindenstrauss, Pisier, 1975] *There exists a Banach space E not isomorphic to a Hilbert space and a short exact sequence $\ell^2 \to E \to \ell^2$.*

(7) [Bonic, Reis, 1966]. *If the norm of a Banach space and its dual norm are C^2 then the space is a Hilbert space.*

(8) [Deville, Godefroy, Zizler, 1990]. *This yields also an example that existence of smooth norms is not a three-space property, cf. 14.12.*

Notes. (2) Note that $K \setminus K'$ is discrete, open and dense in K. So we get for every $n \in \mathbb{N}$ by induction a space K_n with $K_n^{(n)} \neq \emptyset$ and $K_n^{(n+1)} = \emptyset$. In fact $(A \times B)^{(n)} = \bigcup_{i+j=n} A^{(i)} \times B^{(j)}$. Next consider the 1-point compactification K_∞ of the locally compact space $\bigsqcup_{n\in\mathbb{N}} K_n$. Then $K'_\infty = \{\infty\} \cup \bigsqcup_{n\in\mathbb{N}} K'_n$. In fact every neighborhood of $\{\infty\}$ contains all but finitely many of the K_n, thus we have $\supseteq$. The obvious relation is clear. Hence $K_\infty^{(n)} = \{\infty\} \cup \bigsqcup_{i\geq n} K_n^{(i)}$. And $K_\infty^{(\omega)} = \bigcap_{n<\omega} K_\infty^{(n)} = \{\infty\} \neq \emptyset$. The space of [Haydon, 1990] is the one-point compactification of a locally compact space L given as follows: $L := \bigsqcup_{\alpha<\omega_1} \omega_1^\alpha$, i.e. the space of functions $\omega_1 \to \omega_1$, which are defined on some countable ordinal. It is ordered by restriction, i.e. $s \preceq t :\Leftrightarrow \operatorname{dom} s \subseteq \operatorname{dom} t$ and $t|_{\operatorname{dom} s} = s$.

(3) The order topology on $X := [0, \gamma]$ has the sets $\{x : x < a\}$ and $\{x : x > a\}$ as basis. In particular open intervals $(a, b) := \{x : a < x < b\}$ are open. It is compact, since every subset has a greatest lower bound. In fact let $\mathcal{U}$ on X be a covering. Consider $S := \{x \in X : [\inf X, x)$ is covered by finitely many $U \in \mathcal{U}\}$. Let $s_\infty := \sup S$. Note that $x \in S$ implies that $[\inf X, x]$ is covered by finitely many sets in $\mathcal{U}$. We have that $s_\infty \in S$, since there is an $U \in \mathcal{U}$ with $s_\infty \in U$. Then there is an x with $s_\infty \in (x, s_\infty] \subseteq U$, hence $[\inf X, x]$ is covered by finitely many sets in $\mathcal{U}$ since there is an $s \in S$ with $x < s$, so $[\inf X, s_\infty] = [\inf X, x] \cup (x, s_\infty]$ is covered by finitely many sets, i.e. $s_\infty \in S$.

The space X is *scattered*, i.e. $X^{(\alpha)} = \emptyset$ for some ordinal α. For this we have to show that every closed non-empty subset $K \subseteq X$ has open points. For every subset K of X there is a minimum $\min K \in K$, hence $[\inf X, \min K + 1) \cap K = \{\min K\}$ is open in K.

For γ equal to the first infinite ordinal ω we have $[0, \gamma] = \mathbb{N}_\infty$, the one-point compactification of the discrete space $\mathbb{N}$. Thus $C([0, \gamma]) \cong c_0 \times \mathbb{R}$ and the result follows in this case from 13.16.

(5) For splitting short exact sequences the result analogous to 13.17.3 is by 13.17.1 obviously true. By (5) there are non-splitting exact sequences $0 \to F \to E \to E/F \to 0$ for every Banach space which is not Hilbertizable.

(8) By (6) there is a sort exact sequence with hilbertizable ends, but with middle term E not hilbertizable. So neither the sequence nor the dualized sequence splits. If E and E' would have a C^2-norm then E would be hilbertizable by (7).

13.19. Proposition. *Let E be a Banach space, $\|x\| = 1$. Then the following statements are equivalent:*

(1) *The norm is Fréchet differentiable at x;*

(2) *The following two equivalent conditions hold:*

$$\lim_{h\to 0} \frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|} = 0,$$

$$\lim_{t\to 0} \frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = 0 \text{ uniformly in } \|h\| \le 1;$$

(3) $\|y_n^*\| = 1$, $\|z_n^*\| = 1$, $y_n^*(x) \to 1$, $z_n^*(x) \to 1 \Rightarrow y_n^* - z_n^* \to 0$.

Proof. (1)⇒(2) This is obvious, since for the derivative ℓ of the norm at x we have $\lim_{h\to 0} \frac{\|x\pm h\|-\|x\|-l(\pm h)}{\|h\|} = 0$ and adding these equations gives (2).

(2) ⇒ (1) Since $\ell(h) := \lim_{t\searrow 0} \frac{\|x+th\|-\|x\|}{t}$ always exists, and since

$$\frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = \frac{\|x+th\|-\|x\|}{t} + \frac{\|x+t(-h)\|-\|x\|}{t}$$
$$\ge l(h) + l(-h) \ge 0$$

we have $\ell(-h) = \ell(h)$, thus ℓ is linear. Moreover $\frac{\|x\pm th\|-\|x\|}{t} - \ell(\pm h) \ge 0$, so the limit is uniform for $\|h\| \le 1$.

(2) ⇒ (3) By (2) we have that for $\varepsilon > 0$ there exists a δ such that $\|x+h\| + \|x-h\| \le 2 + \varepsilon\|h\|$ for all $\|h\| < \delta$. For $\|y_n^*\| = 1$ and $\|z_n^*\| = 1$ we have

$$y_n^*(x+h) + z_n^*(x-h) \le \|x+h\| + \|x-h\|.$$

Since $y_n^*(x) \to 1$ and $z_n^*(x) \to 1$ we get for large n that

$$(y_n^* - z_n^*)(h) \le 2 - y_n^*(x) - z_n^*(x) + \varepsilon\|h\| \le 2\varepsilon\delta,$$

hence $\|y_n^* - z_n^*\| \le 2\varepsilon$, i.e. $z_n^* - y_n^* \to 0$.

(3) ⇒ (2) Otherwise, there exists an $\varepsilon > 0$ and $0 \ne h_n \to 0$, such that

$$\|x+h_n\| + \|x-h_n\| \ge 2 + \varepsilon\|h_n\|.$$

Now choose $\|y_n^*\| = 1$ and $\|z_n^*\| = 1$ with

$$y_n^*(x+h_n) \ge \|x+h_n\| - \frac{1}{n}\|h_n\| \text{ and } z_n^*(x-h_n) \ge \|x-h_n\| - \frac{1}{n}\|h_n\|.$$

Then $y_n^*(x) = y_n^*(x+h_n) - y_n^*(h_n) \to 1$ and similarly $z_n^*(x) \to 1$. Furthermore

$$y_n^*(x+h_n) + z_n^*(x-h_n) \ge 2 + (\varepsilon - \frac{2}{n})\,\|h_n\|,$$

hence

$$(y_n^* - z_n^*)(h_n) \ge 2 + (\varepsilon - \frac{2}{n})\,\|h_n\| - (y_n^* + z_n^*)(x) \ge (\varepsilon - \frac{2}{n})\,\|h_n\|,$$

thus $\|y_n^* - z_n^*\| \ge \varepsilon - \frac{2}{n}$, a contradiction. □

13.20. Proposition. Fréchet differentiable norms via locally uniformly rotund duals. [Lovaglia, 1955] *If the dual norm of a Banach space E is locally uniformly rotund on E' then the norm is Fréchet differentiable on E.*

A norm is called *locally uniformly rotund* if $\|x_n\| \to \|x\|$ and $\|x+x_n\| \to 2\|x\|$ implies $x_n \to x$. This is equivalent to $2(\|x\|^2 + \|x_n\|^2) - \|x+x_n\|^2 \to 0$ implies $x_n \to x$, since

$$2(\|x\|^2 + 2\|x_n\|^2) - \|x+x_n\|^2 \ge 2\|x\|^2 + 2\|x_n\|^2 - (\|x\| + \|x_n\|)^2 = (\|x\| - \|x_n\|)^2.$$

Proof. We use 13.19, so let $\|x\| = 1$, $\|y_n^*\| = 1$, $\|z_n^*\| = 1$, $y_n^*(x) \to 1$, $z_n^*(x) \to 1$. Let $\|x^*\| = 1$ with $x^*(x) = 1$. Then $2 \ge \|x^* + y_n^*\| \ge (x^* + y_n^*)(x) \to 2$. Since

$\| \quad \|_{E'}$ is locally uniformly rotund we get $y_n^* \to x$ and similarly $z_n^* \to z$, hence $y_n^* - z_n^* \to 0$. □

13.21. Remarks on locally uniformly rotund spaces. By [Kadec, 1959] and [Kadec, 1961] every separable Banach space is isomorphic to a locally uniformly rotund Banach space. By [Day, 1955] the space $\ell^\infty(\Gamma)$ is not isomorphic to a locally uniformly rotund Banach space. Every Banach space admitting a continuous linear injection into some $c_0(\Gamma)$ is locally uniformly rotund renormable, see [Troyanski, 1971]. By 53.21 every WCG-Banach space has such an injection, which is due to [Amir, Lindenstrauss, 1968]. By [Troyanski, 1968] every Banach space with unconditional basis (see [Jarchow, 1981, 14.7]) is isomorphic to a locally uniformly rotund Banach space.

In particular, it follows from these results that every reflexive Banach space has an equivalent Fréchet differentiable norm. In particular L^p has a Fréchet differentiable norm for $1 < p < \infty$ and in fact the p-norm is itself Fréchet differentiable, see 13.13.

13.22. Proposition. *If E' is separable then E admits an equivalent norm, whose dual norm is locally uniform rotund.*

Proof. Let E' be separable. Then there exists a bounded linear operator $T : E \to \ell^2$ such that $T^*((\ell^2)')$ is dense in E' (and obviously T^* is weak*-continuous):
Take a dense subset $\{x_i^* : i \in \mathbb{N}\} \subseteq E'$ of $\{x^* \in E' : \|x^*\| \le 1\}$ with $\|x_i^*\| \le 1$. Define $T : E \to \ell^2$ by

$$T(x)_i := \frac{x_i^*(x)}{2^i}.$$

Then for the basic unit vector $e_i \in (\ell^2)'$ we have

$$T^*(e_i)(x) = e_i(T(x)) = T(x)_i = \frac{x_i^*(x)}{2^i},$$

i.e. $T^*(e_i) = 2^{-i}\, x_i^*$.

Note that the canonical norm on ℓ^2 is locally uniformly rotund. We now claim that E' has a dual locally uniform rotund norm. For $x^* \in E'$ and $n \in \mathbb{N}$ we define

$$\|x^*\|_n^2 := \inf\{\|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 : y^* \in (\ell^2)'\} \text{ and}$$
$$\|x^*\|_\infty := \sum_{n=1}^\infty \frac{1}{2^n}\|x^*\|_n.$$

We claim that $\| \quad \|_\infty$ is the required norm.

So we show first, that it is an equivalent norm. For $\|x^*\| = 1$ we have $\|x^*\|_n \ge \min\{1/(2\sqrt{n}\|T^*\|), 1/2\}$. In fact if $\|y^*\| \ge 1/(2\|T^*\|)$ then $\|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 \ge 1/(2n^2\|T^*\|^2)$ and if $\|y^*\| \le 1/(2\|T^*\|)$ then $\|x^* - T^*y^*\| \ge \|x\| - \|T^*y^*\| \ge 1 - \frac{1}{2} = \frac{1}{2}$. Furthermore if we take $y := 0$ then we see that $\|x^*\|_n \le \|x\|$. Thus $\| \quad \|_n$ and $\| \quad \|$ are equivalent norms, and hence also $\| \quad \|_\infty$.

Note first, that a dual norm is the supremum of the weak* (lower semi-)continuous functions $x^* \mapsto |x^*(x)|$ for $\|x\| \le 1$. Conversely the unit ball B has to be weak* closed in E' since the norm is assumed to be weak* lower semi-continuous and B is convex. Let B_o be its polar in E. By the bipolar-theorem $(B_o)^o = B$, and thus the dual of the Minkowski functional of B_o is the given norm.

Next we show that the infimum defining $\| \quad \|_n$ is in fact a minimum, i.e. for each n and x^* there exists a y^* with $\|x^*\|_2^n = \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2$. Since $f_x : y^* \mapsto \|x^* -$

$T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2$ is weak* lower semi-continuous and satisfies $\lim_{y^*\to\infty} f_x(y^*) = +\infty$, hence it attains its minimum on some large (weak*-compact) ball.

We have that $\|x\|_n \to 0$ for $n \to \infty$.
In fact since the image of T^* is dense in E', there is for every $\varepsilon > 0$ a y^* with $\|x^* - T^*y^*\| < \varepsilon$, and so for large n we have $\|x^*\|_n^2 \le \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y\|^2 < \varepsilon^2$.

Let us next show that $\|\quad\|_\infty$ is a dual norm. For this it is enough to show that $\|\quad\|_n$ is a dual norm, i.e. is weak* lower semi-continuous. So let x_i^* be a net converging weak* to x^*. Then we may choose y_i^* with $\|x_i^*\|_n^2 = \|x_i^* - T^*y_i^*\|^2 + \frac{1}{n}\|y_i^*\|^2$. Then $\{x_i^* : i\}$ is bounded, and hence also $\|y_i^*\|^2$. Let thus y^* be a weak* cluster point of the (y_i^*). Without loss of generality we may assume that $y_i^* \to y^*$. Since the original norms are weak* lower semicontinuous we have

$$\|x^*\|_n^2 \le \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 \le \liminf_i (\|x_i^* - T^*y_i^*\|^2 + \frac{1}{n}\|y_i^*\|^2) = \liminf_i \|x_i^*\|_2^n.$$

So $\|\quad\|_n$ is weak* lower semicontinuous.
Here we use that a function $f : E \to \mathbb{R}$ is lower semicontinuous if and only if $x_\infty = \lim_i x_i \Rightarrow f(x_\infty) \le \liminf_i f(x_i)$.
($\Rightarrow$) otherwise for some subnet (which we again denote by x_i) we have $f(x_\infty) > \lim_i f(x_i)$ and this contradicts the fact that $f^{-1}((a,\infty))$ has to be a neighborhood of x_∞ for $2a := f(x_\infty) + \lim_i f(x_i)$.
($\Rightarrow$) otherwise there exists some x_∞ and an $a < f(x_\infty)$ such that in every neighborhood U of x_∞ there is some x_U with $f(x_U) \le a$. Hence $\lim_U x_U = x_\infty$ and $\liminf_U f(x_U) \le \limsup_U f(x_U) \le a < f(x_\infty)$.

Let us finally show that $\|\quad\|_\infty$ is locally uniform rotund.
So let $x^*, x_j^* \in E'$ with

$$2(\|x^*\|_\infty^2 + \|x_j^*\|_\infty^2) - \|x^* + x_j^*\|_\infty^2 \to 0,$$

or equivalently

$$\|x_j^*\|_\infty \to \|x^*\|_\infty \text{ and } \|x^* + x_j^*\|_\infty \to 2\|x^*\|_\infty.$$

Thus also

$$\|x_j^*\|_n \to \|x^*\|_n \text{ and } \|x^* + x_j^*\|_n \to 2\|x^*\|_n$$

and equivalently

$$2(\|x^*\|_n^2 + \|x_j^*\|_n^2) - \|x^* + x_j^*\|_n^2 \to 0.$$

Now we may choose y^* and y_j^* such that

$$\|x^*\|_2^n = \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 \text{ and } \|x_j^*\|_2^n = \|x_j^* - T^*y_j^*\|^2 + \frac{1}{n}\|y_j^*\|^2.$$

We calculate as follows:

$$\begin{aligned} 2(\|x^*\|_n^2 + \|x_j^*\|_n^2) - \|x^* + x_j^*\|^2 \ge \\ &\ge 2(\|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 + \|x_j^* - T^*y_j^*\|^2 + \frac{1}{n}\|y_j^*\|^2 \\ &\quad - \|x^* + x_j^* - T^*(y^* + y_j^*)\|^2 - \frac{1}{n}\|y^* + y_j^*\|^2 \\ &\ge 2(\|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 + \|x_j^* - T^*y_j^*\|^2 + \frac{1}{n}\|y_j^*\|^2 \\ &\quad - (\|x^* - T^*(y^*)\| + \|x_j^* - T^*(y_j^*)\|)^2 - \frac{1}{n}\|y^* + y_j^*\|^2 \\ &\ge (\|x^* - T^*y^*\| - \|x_j^* - T^*y_j^*\|)^2 + \\ &\quad + \frac{1}{n}(2\|y^*\|^2 + 2\|y_j^*\|^2 - \|y^* + y_j^*\|^2) \ge 0, \end{aligned}$$

hence

$$\|x_j^* - T^* y_j^*\| \to \|x^* - T^* y^*\| \text{ and } 2(\|y^*\|^2 + \|y_j^*\|^2) - \|y^* + y_j^*\|^2 \to 0.$$

Since $\|\ \ \|$ is locally uniformly rotund on $(\ell^2)^*$ we get that $y_j^* \to y^*$. Hence

$$\begin{aligned}\limsup_j \|x^* - x_j^*\| &\le \limsup_j (\|x^* - T^* y^*\| + \|T^*(y^* - y_j^*)\| + \|x_j^* - T^* y_j^*\|) \\ &= 2\,\|x^* - T^* y^*\| \le 2\|x^*\|_n.\end{aligned}$$

Since $\|x^*\|_n \to 0$ for $n \to \infty$ we get $x_j^* \to x^*$. □

13.23. Proposition. [Leach, Whitfield, 1972]. *For the norm $\|\ \ \| = p$ on a Banach space E the following statements are equivalent:*

1. *The norm is rough, i.e. p' is uniformly discontinuous, see 13.8.5.*
2. *There exists an $\varepsilon > 0$ such that for all $x \in E$ with $\|x\| = 1$ and all y_n^*, $z_n^* \in E'$ with $\|y_n^*\| = 1 = \|z_n^*\|$ and $\lim_n y_n^*(x) = 1 = \lim_n z_n^*(x)$ we have:*
$$\limsup_n \|y_n^* - z_n^*\| \ge \varepsilon;$$
3. *There exists an $\varepsilon > 0$ such that for all $x \in E$ with $\|x\| = 1$ we have that*
$$\limsup_{h \to 0} \frac{\|x+h\| + \|x-h\| - 2}{\|h\|} \ge \varepsilon;$$
4. *There exists an $\varepsilon > 0$ such that for every $x \in E$ with $\|x\| = 1$ and $\delta > 0$ there is an $h \in E$ with $\|h\| \le 1$ and $\|x + th\| \ge \|x\| + \varepsilon|t| - \delta$ for all $|t| \le 1$.*

Note that we always have

$$0 \le \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|x\|} \le 2,$$

hence ε in (3) satisfies $\varepsilon \le 2$. For ℓ^1 and $C[0,1]$ the best choice is $\varepsilon = 2$, see 13.11 and 13.12.

Proof. (3)⇒(2) is due to [Cudia, 1964]. Let $\varepsilon > 0$ such that for all $\|x\| = 1$ there are $0 \ne h_n \to 0$ with $\|x + h_n\| + \|x - h_n\| - 2 \ge \varepsilon \|h_n\|$. Now choose y_n^*, $z_n^* \in E'$ with $\|y_n^*\| = 1 = \|z_n\|^*$, $y_n^*(x + h_n) = \|x + h_n\|$ and $z_n^*(x - h_n) = \|x - h_n\|$. Then $\lim_n y_n^*(x) = \|x\| = 1$ and also $\lim_n z_n^*(x) = 1$. Moreover,

$$y_n^*(x + h_n) + z_n^*(x - h_n) \ge 2 + \varepsilon \|h_n\|$$

and hence

$$(y_n^* - z_n^*)(h_n) \ge 2 - y_n^*(x) - z_n^*(x) + \varepsilon \|h_n\| \ge \varepsilon \|h_n\|,$$

thus (2) is satisfied.

(2)⇒(1) By (2) we have an $\varepsilon > 0$ such that for all $\|x\| = 1$ there are y_n^* and z_n^* with $\|y_n^*\| = 1 = \|z_n^*\|$, $\lim_n y_n^*(x) = 1 = \lim_n z_n^*(x)$ and h_n with $\|h_n\| = 1$ and $(y_n^* - z_n^*)(h_n) \ge \varepsilon$. Let $0 < \delta < \varepsilon/2$ and $t > 0$. Then

$$y_n^*(x) > 1 - \frac{\delta^2}{4} \quad \text{and} \quad z_n^*(x) > 1 - \frac{\delta^2}{4} \text{ for large } n.$$

Thus

$$\|x + t h_n\| \ge y_n^*(x + t h_n) \ge 1 - \frac{\delta^2}{4} + t y_n^*(h_n)$$

and hence

$$
\begin{aligned}
t\,p'(x+th_n)(h_n) \ge \|x+th_n\| - \|x\| \ge ty_n^*(h_n) - \frac{\delta^2}{4} \quad &\Rightarrow \\
p'(x+th_n)(h_n) \ge y_n^*(h_n) - \frac{\delta^2}{4t}& \\
\text{and similarly } \; -p'(x-th_n)(h_n) \ge -z_n^*(h_n) - \frac{\delta^2}{4t}&
\end{aligned}
$$

If we choose $0 < t < \delta$ such that $\delta^2/(2t) < \delta$ we get

$$
p'(x+th_n)(h_n) - p'(x-th_n)(h_n) \ge (y_n^* - z_n^*)(h_n) - \frac{\delta^2}{2t} > \varepsilon - \delta > \frac{\varepsilon}{2}.
$$

(1)⇒(4) Using the uniform discontinuity assumption of p' we get $x_j \in E$ with $p(x_j - x) \le \eta/4$ and $u \in E$ with $p(u) = 1$ such that $(p'(x_2) - p'(x_1))(u) \ge \varepsilon$. Let $\mu := (p'(x_1) + p'(x_2))(u)/(2p(x))$ and $v := u - \mu\, x$.

Since $p'(x_1)(u) \le p'(x_2)(u) - \varepsilon$ we get $(p'(x_1) + p'(x_2))(u))/2 \le p'(x_2)(u) - \varepsilon/2 \le p(u) - \varepsilon/2 < 1$ and $(p'(x_1) + p'(x_2))(u)/2 \ge p'(x_1)(u) + \varepsilon \ge -p(u) + \varepsilon/2 > 1$, i.e. $|(p'(x_1) + p'(x_2))(u)/2| < 1$, so $0 < p(v) < 2$. For $0 \le t \le p(x)$ and $s := 1 - t\,\mu$ we get

$$
x + tv = sx + tu = s(x + \frac{t}{s}u) = s\Big((x_2 + \frac{t}{s}u) + (x - x_2)\Big).
$$

Thus $0 < s < 2$ and

$$
\begin{aligned}
p(x+tv) &\ge s(p(x_2 + \frac{t}{s}u) - p(x - x_2)) \\
&> s\Big(p(x_2) + \frac{t}{s}p'(x_2)u - \eta/4\Big) \quad \text{since } p(y+w) \ge p(y) + p'(y)(w) \\
&> s\,p(x) + t\,p'(x_2)(u) - s\,\eta/2 \quad \text{since } p(x) \le p(x_2) + p(x - x_2) \\
&= p(x) + (t/2)\,(p'(x_2) - p'(x_1))(u) - s\,\eta/2 \\
&> p(x) + t\varepsilon/2 - \eta.
\end{aligned}
$$

If $-p(x) \le t < 0$ we proceed with the role of x_1 and x_2 exchanged and obtain

$$
\begin{aligned}
p(x+tv) &> s\,p(x) + t\,p'(x_1)(u) - s\,\eta/2 \\
&= p(x) + (-t/2)\,(p'(x_2) - p'(x_1))(u) - s\,\eta/2 \\
&> p(x) + |t|\,\varepsilon/2 - \eta.
\end{aligned}
$$

Thus

$$
p(x+tv) \ge p(x) + |t|\,\varepsilon/2 - \eta.
$$

(4)⇒(3) By (4) there exists an $\varepsilon > 0$ such that for every $x \in E$ with $\|x\| = 1$ and $\delta > 0$ there is an $h \in E$ with $\|h\| \le 1$ and $\|x + th\| \ge \|x\| + \varepsilon|t| - \delta$ for all $|t| \le 1$. If we put $t := 1/n$ we have

$$
n(\|x + h_n/n\| + \|x - h_n/n\| - 2) \ge \varepsilon - 1/n > \varepsilon/2 \text{ for large } n. \qquad \square
$$

13.24. Results on the non-existence of C^1-norms on certain spaces.

(1) [Restrepo, 1964] *and* [Restrepo, 1965]. *A separable Banach space has an equivalent C^1-norm if and only if E^* is separable. This will be proved in* 16.11.

(2) [Kadec, 1965]. *More generally, if for a Banach space* $\operatorname{dens} E < \operatorname{dens} E^*$ *then no* C^1*-norm exists. This will be proved by showing the existence of a rough norm in* 14.10 *and then using* 14.9. *The density number* $\operatorname{dens} X$ *of a topological space* X *is the minimum of the cardinalities of all dense subsets of* X.

(3) [Haydon, 1990]. *There exists a compact space* K*, such that* $K^{(\omega_1)} = \{*\}$*, in particular* $K^{(\omega_1+1)} = \emptyset$*, but* $C(K)$ *has no equivalent Gâteaux differentiable norm, see also* 13.18.2.

One can interpret these results by saying that in these spaces every convex body necessarily has corners.

14. Smooth Bump Functions

In this section we return to the original question whether the smooth functions generate the topology. Since we will use the results given here also for manifolds, and since the existence of charts is of no help here, we consider fairly general non-linear spaces. This allows us at the same time to treat all considered differentiability classes in a unified way.

14.1. Convention. We consider a Hausdorff topological space X with a subalgebra $\mathcal{S} \subseteq C(X, \mathbb{R})$, whose elements will be called the *smooth* or $\mathcal{S}$*-functions* on X. We assume that for functions $h \in C^\infty(\mathbb{R}, \mathbb{R})$ (at least for those being constant off some compact set, in some cases) one has $h_*(\mathcal{S}) \subseteq \mathcal{S}$, and that $f \in \mathcal{S}$ provided it is locally in $\mathcal{S}$, i.e., there exists an open covering $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists an $f_U \in \mathcal{S}$ with $f = f_U$ on U. In particular, we will use for $\mathcal{S}$ the classes of C^∞- and of $\mathcal{L}\mathrm{ip}^k$-mappings on c^∞-open subsets X of convenient vector spaces with the c^∞-topology and the class of C^n-mappings on open subsets of Banach spaces, as well as subclasses formed by boundedness conditions on the derivatives or their difference quotients.

Under these assumptions on $\mathcal{S}$ one has that $\frac{1}{f} \in \mathcal{S}$ provided $f \in \mathcal{S}$ with $f(x) > 0$ for all $x \in X$: Just choose everywhere positive $h_n \in C^\infty(\mathbb{R}, \mathbb{R})$ with $h_n(t) = \frac{1}{t}$ for $t \geq \frac{1}{n}$. Then $h_n \circ f \in \mathcal{S}$ and $\frac{1}{f} = h_n \circ f$ on the open set $\{x : f(x) > \frac{1}{n}\}$. Hence, $\frac{1}{f} \in \mathcal{S}$.

For a (convenient) vector space F the *carrier* $\operatorname{carr}(f)$ of a mapping $f : X \to F$ is the set $\{x \in X : f(x) \neq 0\}$. The *zero set* of f is the set where f vanishes, $\{x \in X : f(x) = 0\}$. The *support* of f $\operatorname{support}(f)$ is the closure of $\operatorname{carr}(f)$ in X.

We say that X is *smoothly regular* (with respect to $\mathcal{S}$) or $\mathcal{S}$*-regular* if for any neighborhood U of a point x there exists a smooth function $f \in \mathcal{S}$ such that $f(x) = 1$ and $\operatorname{carr}(f) \subseteq U$. Such a function f is called a *bump function*.

14.2. Proposition. Bump functions and regularity. [Bonic, Frampton, 1966]. *A Hausdorff space is* $\mathcal{S}$*-regular if and only if its topology is initial with respect to* $\mathcal{S}$.

Proof. The initial topology with respect to $\mathcal{S}$ has as a subbasis the sets $f^{-1}(I)$, where $f \in \mathcal{S}$ and I is an open interval in $\mathbb{R}$. Let $x \in U$, with U open for the initial topology. Then there exist finitely many open intervals $I_1, \dots, I_n$ and $f_1, \dots, f_n \in \mathcal{S}$ with $x \in \bigcap_{i=1}^n f_i^{-1}(I_i)$. Without loss of generality we may assume that $I_i = \{t : |f_i(x) - t| < \varepsilon_i\}$ for certain $\varepsilon_i > 0$. Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be chosen such that $h(0) = 1$ and $h(t) = 0$ for $|t| \geq 1$. Set $f(x) := \prod_{i=1}^n h(\frac{f_i(x)}{\varepsilon_i})$. Then f is the required bump function. □

14.3. Corollary. Smooth regularity is inherited by products and subspaces. *Let X_i be topological spaces and $\mathcal{S}_i \subseteq C(X_i, \mathbb{R})$. On a space X we consider the initial topology with respect to mappings $f_i : X \to X_i$, and we assume that $\mathcal{S} \subseteq C(X, \mathbb{R})$ is given such that $f_i^*(\mathcal{S}_i) \subseteq \mathcal{S}$ for all i. If each X_i is $\mathcal{S}_i$-regular, then X is $\mathcal{S}$-regular.* □

Note however that the c^∞-topology on a locally convex subspace is not the trace of the c^∞-topology in general, see 4.33 and 4.36.5. However, for c^∞-closed subspaces this is true, see 4.28.

14.4. Proposition. [Bonic, Frampton, 1966]. *Every Banach space with $\mathcal{S}$-norm is $\mathcal{S}$-regular.*

More general, a convenient vector space is smoothly regular if its c^∞-topology is generated by seminorms which are smooth on their respective carriers. For example, nuclear Fréchet spaces have this property.

Proof. Namely, $g \circ p$ is a smooth bump function with carrier contained in $\{x : p(x) < 1\}$ if g is a suitably chosen real function, i.e., $g(t) = 1$ for $t \le 0$ and $g(t) = 0$ for $t \ge 1$.

Nuclear spaces have a basis of Hilbert-seminorms 52.34, and on Fréchet spaces the c^∞-topology coincides with the locally convex one 4.11.1, hence nuclear Fréchet spaces are c^∞-regular. □

14.5. Open problem. Has every non-separable $\mathcal{S}$-regular Banach space an equivalent $\mathcal{S}$-norm? Compare with 16.11.

A partial answer is given in:

14.6. Proposition. *Let E be a C^∞-regular Banach space. Then there exists a smooth function $h : E \to \mathbb{R}_+$, which is positively homogeneous and smooth on $E \setminus \{0\}$.*

Proof. Let $f : E \setminus \{0\} \to \{t \in \mathbb{R} : t \ge 0\}$ be a smooth function, such that $\operatorname{carr}(f)$ is bounded in E and $f(x) \ge 1$ for x near 0. Let $U := \{x : f(tx) \ne 0 \text{ for some } t \ge 1\}$. Then there exists a smooth function $Mf : E \setminus \{0\} \to \mathbb{R}$ with $(Mf)'(x)(x) < 0$ for $x \in U$, $\lim_{x\to 0} f(x) = +\infty$ and $\operatorname{carr} Mf \subseteq U$.
The idea is to construct out of the smooth function $f \ge 0$ another smooth function Mf with $(Mf)'(x)(x) = -f(x) \le 0$, i.e. $(Mf)'(tx)(tx) = -f(tx)$ and hence

$$\frac{d}{dt} Mf(tx) = (Mf)'(tx)(x) = -\frac{f(tx)}{t} \text{ for } t \ne 0.$$

Since we want bounded support for Mf, we get

$$Mf(x) = -\Big[Mf(tx)\Big]_{t=1}^{\infty} = -\int_1^\infty \frac{d}{dt} Mf(tx)\, dt = \int_1^\infty \frac{f(tx)}{t}\, dt,$$

and we take this as a definition of Mf. Since the support of f is bounded, we may replace the integral locally by $\int_1^N$ for some large N, hence Mf is smooth on $E \setminus \{0\}$ and $(Mf)'(x)(x) = -f(x)$.
Since $f(x) > \varepsilon$ for all $\|x\| < \delta$, we have that

$$Mf(x) \ge \int_1^N \frac{1}{t} f(tx)\, dt \ge \log(N)\varepsilon$$

for all $\|x\| < \frac{\delta}{N}$, i.e. $\lim_{x\to 0} Mf(x) = +\infty$.
Furthermore $\operatorname{carr}(Mf) \subseteq U$, since $f(tx) = 0$ for all $t \ge 1$ and $x \notin U$.

Now consider $M^2 f := M(Mf) : E \setminus \{0\} \to \mathbb{R}$. Since $(Mf)'(x)(x) \leq 0$, we have $(M^2 f)'(x)(x) = \int_1^\infty (Mf)'(tx)(x)\,dt \leq 0$ and it is < 0 if for some $t \geq 1$ we have $(Mf)'(tx)(x) < 0$, in particular this is the case if $M^2 f(x) > 0$.

Thus $U_\varepsilon := \{x : M^2 f(x) \geq \varepsilon\}$ is radial set with smooth boundary, and the Minkowski-functional is smooth on $E \setminus \{0\}$. Moreover $U_\varepsilon \cong E$ via $x \mapsto \frac{x}{M^2 f(x)}$. □

14.7. Lemma. Existence of smooth bump functions.
For a class $\mathcal{S}$ on a Banach space E in the sense of 14.1 the following statements are equivalent:

(1) *E is not $\mathcal{S}$-regular;*
(2) *For every $f \in \mathcal{S}$, every $0 < r_1 < r_2$ and $\varepsilon > 0$ there exists an x with $r_1 \leq \|x\| \leq r_2$ and $|f(x) - f(0)| < \varepsilon$;*
(3) *For every $f \in \mathcal{S}$ with $f(0) = 0$ there exists an x with $1 \leq \|x\| \leq 2$ and $|f(x)| \leq \|x\|$*

Proof. (1) ⇒ (2) Assume that there exists an f and $0 < r_1 < r_2$ and $\varepsilon > 0$ such that $|f(x) - f(0)| \geq \varepsilon$ for all $r_1 \leq \|x\| \leq r_2$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth bump function on $\mathbb{R}$. Let $g(x) := h(\frac{1}{\varepsilon} f(r_1 x) - f(0))$. Then g is of the corresponding class, $g(0) = h(0) = 1$, and for all x with $1 \leq \|x\| \leq \frac{r_2}{r_1}$ we have $|f(r_1 x) - f(0)| \geq \varepsilon$, and hence $g(x) = 0$. By redefining g on $\{x : \|x\| \geq \frac{r_2}{r_1}\}$ as 0, we obtain the required bump function.

(2) ⇒ (3) Take $r_1 = 1$ and $r_2 = 2$ and $\varepsilon = 1$.

(3) ⇒ (1) Assume a bump function g exists, i.e., $g(0) = 1$ and $g(x) = 0$ for all $\|x\| \geq 1$. Take $f := 2 - g$. Then $f(0) = 0$ and $f(x) = 2$ for $\|x\| \geq 1$, a contradiction to (3). □

14.8. Proposition. Boundary values for smooth mappings. [Bonic, Frampton, 1966] *Let E and F be convenient vector spaces, let F be $\mathcal{S}$-regular but E not $\mathcal{S}$-regular. Let $U \subseteq E$ be c^∞-open and $f \in C(\overline{U}, F)$ with $f^*(\mathcal{S}) \subseteq \mathcal{S}$. Then $\overline{f(\partial U)} \supseteq f(\overline{U})$. Hence, $f = 0$ on ∂U implies $f = 0$ on U.*

Proof. Since $f(\overline{U}) \subseteq \overline{f(U)}$ it is enough to show that $f(U) \subseteq \overline{f(\partial U)}$. Suppose $f(x) \notin \overline{f(\partial U)}$ for some $x \in U$. Choose a smooth h on F such that $h(f(x)) = 1$ and $h = 0$ on a neighborhood of $f(\partial U)$. Let $g = h \circ f$ on U and 0 outside. Then g is a smooth bump function on E, a contradiction. □

14.9. Theorem. C^1-regular spaces admit no rough norm. [Leach, Whitfield, 1972]. *Let E be a Banach space whose norm $p = \|\ \ \|$ has uniformly discontinuous directional derivative. If f is Fréchet differentiable with $f(0) = 0$ then there exists an $x \in E$ with $1 \leq \|x\| < 2$ and $f(x) \leq \|x\|$.*

By 14.7 this result implies that on a Banach space with rough norm there exists no Fréchet differentiable bump function. In particular, $C([0,1])$ and ℓ^1 are not C^1-regular by 13.11 and 13.12, which is due to [Kurzweil, 1954].

Proof. We try to reach the exterior of the unit ball by a recursively defined sequence x_n in $\{x : f(x) \leq p(x)\}$ starting at 0 with large step-length ≤ 1 in directions, where p' is large. Given x_n we consider the set

$$\mathcal{M}_n := \left\{ y \in E : \begin{array}{l} (1)\ f(y) \leq p(y), \\ (2)\ p(y - x_n) \leq 1 \text{ and} \\ (3)\ p(y) - p(x_n) \geq (\varepsilon/8)\, p(y - x_n) \end{array} \right\}.$$

Since $x_n \in \mathcal{M}_n$, this set is not empty and hence $M_n := \sup\{p(y-x_n) : y \in \mathcal{M}_n\} \le 1$ is well-defined and it is possible to choose $x_{n+1} \in \mathcal{M}_n$ with

$$(4)\ p(x_{n+1} - x_n) \ge M_n/2.$$

We claim that $p(x_n) \ge 1$ for some n, since then $x := x_n$ for the minimal n satisfies the conclusion of the theorem:
Otherwise $p(x_n)$ is bounded by 1 and increasing by (3), hence a Cauchy-sequence. By (3) we then get that (x_n) is a Cauchy-sequence. So let z be its limit. If $z = 0$ then $\mathcal{M}_n = \{0\}$ and hence $f(y) > p(y)$ for all $|y| \le 1$. Thus f is not differentiable. Then $p(z) \le 1$ and $f(z) \le p(z)$. Since f is Fréchet-differentiable at z there exists a $\delta > 0$ such that

$$f(z+u) - f(z) - f'(z)(u) \le \varepsilon p(u)/8 \text{ for all } p(u) < \delta.$$

Without loss of generality let $\delta \le 1$ and $\delta \le 2\,p(z)$. By 13.23.4 there exists a v such that $p(v) < 2$ and $p(z+tv) > p(z) + \varepsilon|t|/2 - \varepsilon\delta/8$ for all $|t| \le p(z)$. Now let $t := -\operatorname{sign}(f'(z)(v))\,\delta/2$. Then

$$(1)\quad p(z+tv) > p(z) + \varepsilon\delta/8 \ge f(z) + \varepsilon p(tv)/8 \ge f(z+tv),$$

$$(2)\quad p(z+tv-z) = |t|p(v) < \delta \le 1,$$

$$(3)\quad p(z+tv) - p(z) > \varepsilon\delta/8 > \varepsilon p(tv)/8.$$

Since f and p are continuous the $z + tv$ satisfy (1)-(3) for large n and hence $M_n \ge p(z + t\,v - x_n)$. From $p(z+tv-z) > \varepsilon\delta/8$ we get $M_n > \varepsilon\delta/8$ and so $p(x_{n+1} - x_n) > \varepsilon\delta/16$ by (4) contradicts the convergence of x_n. □

14.10. Proposition. *Let E be a Banach-space with* $\operatorname{dens} E < \operatorname{dens} E'$. *Then there is an equivalent rough norm on E.*

Proof. The idea is to describe the unit ball of a rough norm as intersection of hyper planes $\{x \in E : x^*(x) \le 1\}$ for certain functionals $x^* \in E'$. The fewer functionals we use the more 'corners' the unit ball will have, but we have to use sufficiently many in order that this ball is bounded and hence that its Minkowski-functional is an equivalent norm. We call a set X large, if and only if $|X| > \operatorname{dens}(E)$ and small otherwise. For $x \in E$ and $\varepsilon > 0$ let $B_\varepsilon(x) := \{y \in E : \|x-y\| \le \varepsilon\}$. Now we choose using Zorn's lemma a subset $D \subseteq E'$ maximal with respect to the following conditions:

(1) $0 \in D$;
(2) $x^* \in D \Rightarrow -x^* \in D$;
(3) $x^*, y^* \in D,\ x^* \ne y^* \Rightarrow \|x^* - y^*\| > 1$.

Note that D is then also maximal with respect to (3) alone, since otherwise, we could add a point x^* with $\|x^* - y^*\| > 1$ for all $y^* \in D$ and also add the point $-x^*$, and obtain a larger set satisfying all three conditions.

Claim. $D_\infty := \bigcup_{n\in\mathbb{N}} \frac{1}{n} D$ is dense in E', and hence $|D_\infty| \ge \operatorname{dens}(E')$:
Assume indirectly, that there is some $x^* \in E'$ and $n \in \mathbb{N}$ with $B_{1/n}(x^*) \cap D_\infty = \emptyset$. Then $B_1(n\,x^*) \cap D = \emptyset$ and hence we may add x^* to D, contradicting the maximality.

Without loss of generality we may assume that D is at least countable. Then $|D| = |\bigcup_{n\in\mathbb{N}} \frac{1}{n} D| \ge \operatorname{dens}(E') > \operatorname{dens}(E)$, i.e. D is large. Since $D = \bigcup_{n\in\mathbb{N}} D \cap B_n(0)$, we find some n such that $D \cap B_n(0)$ is large. Let $y^* \in E'$ be arbitrary and $w^* := \frac{1}{4n+2} y^*$. For every $x^* \in D$ there is a $z^* \in \frac{1}{2} D$ such that $\|x^* + w^* - z^*\| \le \frac{1}{2}$ (otherwise we could add $2(x^*+w^*)$ to D). Thus we may define a mapping $D\to\frac12 D$ by $x^*\mapsto z^*$. This mapping is injective, since $\|x_j^*+w^*-z^*\|\le\frac12$ for $j\in\{1,2\}$ implies $\|x_1^*-x_2^*\|\le 1$ and hence $x_1^*=x_2^*$. If we restrict it to the large set $D\cap B_n(0)$ it has image in $\frac12 D\cap B_{n+1/2}(w^*)$, since $\|z^*-w^*\|\le\|z^*+x^*-w^*\|+\|x^*\|\le\frac12+n$. Hence also $\frac{1}{2(4n+2)}D\cap B_{1/4}(y^*)=\frac{1}{4n+2}\frac12 D\cap B_{n+1/2}(w^*)$ is large.

In particular for $y^*:=0$ and $1/4$ replaced by 1 we get that $A:=\frac{1}{4(2n+1)}D\cap B_1(0)$ is large. Now let

$$U:=\Big\{x\in E:\exists A_0\subseteq A\text{ small},\forall x^*\in A\setminus A_0:x^*(x)\le 1\Big\}.$$

Since A is symmetric, the set U is absolutely convex (use that the union of two small exception sets is small). It is a 0-neighborhood, since $\{x:\|x\|\le 1\}\subseteq U$ ($x^*(x)\le\|x^*\|\cdot\|x\|=\|x\|\le 1$ for $x^*\in A$). It is bounded, since for $x\in E$ we may find by Hahn-Banach an $x^*\in E'$ with $x^*(x)=\|x\|$ and $\|x^*\|=1$. For all y^* in the large set $A\cap B_{1/4}(\frac34 x^*)$ we have $y^*(x)=(y^*-\frac34 x^*)(x)+\frac34 x^*(x)\ge\frac34\|x\|-\frac14\|x\|\ge\frac12\|x\|$. For $\|x\|>2$ we thus get $x\notin U$. Now let σ be the Minkowski-functional generated by U and σ^* the dual norm on E'. Let $\Delta\subseteq E$ be a small dense subset. Then $\{x^*\in A:\sigma^*(x^*)>1\}$ is small, since $\sigma^*(x^*)>1$ for $x^*\in A$ implies that there exists an $x\in\Delta$ with $x^*(x)>\sigma(x)$, but this is $\bigcup_{n\in\mathbb N}\{x^*\in A:x^*(x)>\sigma(x)+\frac1n\}$, and each of these sets is small by construction of $\sigma(x)$. Since Δ is small so is the union over all $x\in\Delta$. Thus $A_1:=\{x^*\in A:\sigma(x^*)\le 1\}$ is large.

Now let $\varepsilon:=\frac{1}{8(2n+1)}$, let $x\in E$, and let $0<\eta<\varepsilon$. We may choose two different $x_i^*\in A_1$ for $i\in\{1,2\}$ with $x_i^*(x)>\sigma(x)-\eta^2/2$. This is possible, since this is true for all but a small set of $x^*\in A$. Thus $\sigma^*(x_1^*-x_2^*)\ge\|x_1^*-x_2^*\|>2\varepsilon$, and hence there is an $h\in E$ with $\sigma(h)=1$ and $(x_1^*-x_2^*)(h)>2\varepsilon$. Let now $t>0$. Then

$$\sigma(x+th)\ge x_1^*(x+th)=x_1^*(x)+tx_1^*(h)>\sigma(x)-\frac{\eta^2}{2}+tx_1^*(h),$$
$$\sigma(x-th)\ge x_2^*(x-th)>\sigma(x)-\frac{\eta^2}{2}-tx_2^*(h).$$

Furthermore $\sigma(x)\ge\sigma(x+th)-t\sigma'(x+th)(h)$ implies

$$\sigma'(x+th)(h)\ge\frac{\sigma(x+th)-\sigma(x)}{t}>x_1^*(h)-\frac{\eta^2}{2t},$$
$$-\sigma'(x-th)(h)\ge -x_2^*(h)-\frac{\eta^2}{2t}.$$

Adding the last two inequalities gives

$$\sigma'(x+th)(h)-\sigma'(x-th)(h)\ge(x_2^*-x_1^*)(h)-\frac{\eta^2}{t}>\varepsilon,$$

since $(x_2^*-x_1^*)(h)>2\varepsilon$ and we choose $t<\eta$ such that $\frac{\eta^2}{t}<\varepsilon$. □

14.11. Results. Spaces which are not smoothly regular. *For Banach spaces one has the following results:*

(1) [Bonic, Frampton, 1965]. *By* 14.9 *no Fréchet-differentiable bump function exists on* $C[0,1]$ *and on* ℓ^1*. Hence, most infinite dimensional* C^**-algebras are not regular for 1-times Fréchet-differentiable functions, in particular those for which a normal operator exists whose spectrum contains an open interval.*

(2) [Leduc, 1970]. *If* $\operatorname{dens}E<\operatorname{dens}E^*$ *then no* C^1*-bump function exists. This follows from* 14.10, 14.9, and 14.7*. See also* 13.24.2.

(3) [John, Zizler, 1978]. *A norm is called strongly rough if and only if there exists an $\varepsilon > 0$ such that for every x with $\|x\| = 1$ there exists a unit vector y with $\limsup_{t\searrow 0} \frac{\|x+ty\|+\|x-ty\|-2}{t} \geq \varepsilon$. The usual norm on $\ell^1(\Gamma)$ is strongly rough, if Γ is uncountable. There is however an equivalent non-rough norm on $\ell^1(\Gamma)$ with no point of Gâteaux-differentiability. If a Banach space has Gâteaux differentiable bump functions then it does not admit a strongly rough norm.*

(4) [Day, 1955]. *On $\ell^1(\Gamma)$ with uncountable Γ there is no Gâteaux differentiable continuous bump function.*

(5) [Bonic, Frampton, 1965]. $E < \ell^p$, $\dim E = \infty$: *If $p = 2n + 1$ then E is not D^p-regular. If $p \notin \mathbb{N}$ then E is not $\mathcal{S}$-regular, where $\mathcal{S}$ denotes the $C^{[p]}$-functions whose highest derivative satisfies a Hölder like condition of order $p - [p]$ but with $o(\quad)$ instead of $O(\quad)$.*

14.12. Results.

(1) [Deville, Godefroy, Zizler, 1990]. *If $c_0(\Gamma) \to E \to F$ is a short exact sequence of Banach spaces and F has C^k-bump functions then also E has them. Compare with 16.19.*

(2) [Meshkov, 1978] *If a Banach space E and its dual E^* admit C^2-bump functions, then E is isomorphic to a Hilbert space. Compare with 13.18.7.*

(3) *Smooth bump functions are not inherited by short exact sequences.*

Notes. (1) As in 13.17.3 one chooses $x_a^* \in E^*$ with $x_a^*|_{c_0(\Gamma)} = \mathrm{ev}_a$. Let g be a smooth bump function on E/F and $h \in C^\infty(\mathbb{R}, [0,1])$ with compact support and equal to 1 near 0. Then $f(x) := g(x + F) \prod_{a\in\Gamma} h(x_a^*(x))$ is the required bump function.

(3) Use the example mentioned in 13.18.6, and apply (2).

Open problems. Is the product of C^∞-regular convenient vector spaces again C^∞-regular? Beware of the topology on the product!

Is every quotient of any $\mathcal{S}$-regular space again $\mathcal{S}$-regular?

15. Functions with Globally Bounded Derivatives

In many problems (like Borel's theorem 15.4, or the existence of smooth functions with given carrier 15.3) one uses in finite dimensions the existence of smooth functions with bounded derivatives. In infinite dimensions C^k-functions have locally bounded k-th derivatives, but even for bump functions this need not be true globally.

15.1. Definitions. For normed spaces we use the following notation: $C^k_B := \{f \in C^k : \|f^{(k)}(x)\| \le B \text{ for all } x \in E\}$ and $C^k_b := \bigcup_{B>0} C^k_B$. For general convenient vector spaces we may still define C^∞_b as those smooth functions $f : U \to F$ for which the image $d^k f(U)$ of each derivative is bounded in the space $L^k_{\text{sym}}(E, F)$ of bounded symmetric multilinear mappings.

Let $\mathcal{L}\text{ip}^k_K$ denote the space of C^k-functions with global Lipschitz-constant K for the k-th derivatives and $\mathcal{L}\text{ip}^k_{\text{global}} := \bigcup_{K>0} \mathcal{L}\text{ip}^k_K$. Note that $C^k_K = C^k \cap \mathcal{L}\text{ip}^{k-1}_K$.

15.2. Lemma. Completeness of C^n. *Let f_j be C^n-functions on some Banach space such that $f^{(k)}_j$ converges uniformly on bounded sets to some function f^k for each $k \le n$. Then $f := f^0$ is C^n, and $f^{(k)} = f^k$ for all $k \le n$.*

Proof. It is enough to show this for $n = 1$. Since $f'_n \to f^1$ uniformly, we have that f^1 is continuous, and hence $\int_0^1 f^1(x + t\,h)(h)\,dt$ makes sense and

$$f_n(x + h) - f_n(x) = \int_0^1 f'_n(x + t\,h)(h)\,dt \to \int_0^1 f^1(x + t\,h)(h)\,dt$$

for x and h fixed. Since $f_n \to f$ pointwise, this limit has to be $f(x + h) - f(x)$. Thus we have

$$\begin{aligned}\frac{\|f(x + h) - f(x) - f^1(x)(h)\|}{\|h\|} &= \frac{1}{\|h\|} \left\| \int_0^1 (f^1(x + t\,h) - f^1(x))(h)\,dt \right\| \\ &\le \int_0^1 \|f^1(x + t\,h) - f^1(x))\|\,dt\end{aligned}$$

which goes to 0 for $h \to 0$ and fixed x, since f^1 is continuous. Thus, f is differentiable and $f' = f^1$. □

15.3. Proposition. When are closed sets zero-sets of smooth functions. [Wells, 1973]. *Let E be a separable Banach space and $n \in \mathbb{N}$. Then E has a C^n_b-bump function if and only if every closed subset of E is the zero-set of a C^n-function.*

For $n = \infty$ and E a convenient vector space we still have ($\Rightarrow$), provided all $L^k(E; \mathbb{R})$ satisfy the SECOND COUNTABILITY CONDITION OF MACKEY, *i.e. for every countable family of bounded sets B_k there exist $t_k > 0$ such that $\bigcup_k t_k\,B_k$ is bounded.*

Proof. ($\Rightarrow$) Suppose first that E has a C^n_b-bump function. Let $A \subseteq E$ be closed and $U := E \setminus A$ be the open complement. For every $x \in U$ there exists an $f_x \in C^n_b(E)$ with $f_x(x) = 1$ and $\operatorname{carr}(f_x) \subseteq U$. The family of carriers of the f_x is an open covering of U. Since E is separable, those points in a countable dense subset that lie in U are dense in the metrizable space U. Thus, U is Lindelöf, and consequently we can find a sequence of points x_n such that for the corresponding functions $f_n := f_{x_n}$ the carriers still cover U. Now choose constants $t_n > 0$ such that $t_n \cdot \sup\{\|f^{(j)}_n(x)\| : x \in E\} \le \frac{1}{2^{n-j}}$ for all $j < n$. Then $f := \sum_n t_n\,f_n$ converges uniformly in all derivatives, hence represents by 15.2 a C^n-function on E that vanishes on A. Since the carriers of the f_n cover U, it is strictly positive on U, and hence the required function has as 0-set exactly A.

($\Leftarrow$) Consider a vector $a \neq 0$, and let $A := E \setminus \bigcup_{n\in\mathbb{N}}\{x : \|x - \frac{1}{2^n}a\| < \frac{1}{2^{n+1}}\}$. Since A is closed there exists by assumption a C^n-function $f : E \to \mathbb{R}$ with $f^{-1}(0) = A$ (without loss of generality we may assume $f(E) \subseteq [0,1]$). By continuity of the derivatives we may assume that $f^{(n)}$ is bounded on some neighborhood U of 0. Choose n so large that $D := \{x : \|x - \frac{1}{2^n}a\| < \frac{1}{2^n}\} \subseteq U$, and let $g := f$ on $A \cup D$ and 0 on $E \setminus D$. Then $f \in C^n$ and $f^{(n)}$ is bounded. Up to affine transformations this is the required bump function. □

15.4. Borel's theorem. [Wells, 1973]. *Suppose a Banach space E has C_b^∞-bump functions. Then every formal power series with coefficients in $L^n_{sym}(E;F)$ for another Banach space F is the Taylor-series of a smooth mapping $E \to F$.*

Moreover, if G is a second Banach space, and if for some open set $U \subseteq G$ we are given $b_k \in C_b^\infty(U, L^k_{\mathrm{sym}}(E,F))$, then there is a smooth $f \in C^\infty(E \times U, F)$ with $d^k(f(\quad, y))(0) = b_k(y)$ for all $y \in U$ and $k \in \mathbb{N}$. In particular, smooth curves can be lifted along the mapping $C^\infty(E,F) \to \prod_k L^k_{sym}(E;F)$.

Proof. Let $\rho \in C_b^\infty(E,\mathbb{R})$ be a C_b^∞-bump function, which equals 1 locally at 0. We shall use the notation $b_k(x,y) := b_k(y)(x^k)$. Define

$$f_k(x,y) := \frac{1}{k!} b_k(x,y)\,\rho(x)$$

and

$$f(x,y) := \sum_{k\geq 0} \frac{1}{t_k^k} f_k(t_k \cdot x, y)$$

with appropriately chosen $t_k > 0$. Then $f_k \in C^\infty(E \times U, F)$ and f_k has carrier inside of $\mathrm{carr}(\rho) \times U$, i.e. inside $\{x : \|x\| < 1\} \times U$. For the derivatives of b_k we have

$$\partial_1^j \partial_2^i b_k(x,y)(\xi,\eta) = k\,(k-1)\,\dots\,(k-j)\,(d^i b_k(y)(\eta))(x^{k-j},\xi^j).$$

Hence, for $\|x\| \leq 1$ this derivative is bounded by

$$(k)_j \sup_{y\in U} \|d^i b_k(y)\|_{L(F,L^k_{\mathrm{sym}}(E;G))},$$

where $(k)_j := k(k-1)\dots(k-j)$. Using the product rule we see that for $j \geq k$ the derivative $\partial_1^j\partial_2^i f_k$ of f_k is globally bounded by

$$\sum_{l\leq k} \binom{j}{l} \sup\{\|\rho^{(j-l)}(x)\| : x \in E\}\,(k)_l \sup_{y\in U}\|d^i b_k(y)\| < \infty.$$

The partial derivatives of f would be

$$\partial_1^j\partial_2^i f_k(x,y) = \sum_k \frac{t_k^j}{t_k^k}\partial_1^j\partial_2^i f_k(t_k x, y).$$

We now choose the $t_k \geq 1$ such that these series converge uniformly. This is the case if,

$$\frac{1}{t_k^{k-j}} \sup\{\|\partial_1^j\partial_2^i f_k(x,y)\| : x \in E, y \in U\} \leq$$

$$\leq \frac{1}{t_k^{k-(j+i)}} \sup\{\|\partial_1^j\partial_2^i f_k(x,y)\| : x \in E, y \in U\} \leq \frac{1}{2^{k-(j+i)}},$$

and thus if

$$t_k \geq 2.\sup\{\|\partial_1^j\partial_2^i f_k(x,y)\|^{\frac{1}{k-(j+i)}} : x \in E, y \in U, j+i < k\}.$$

Since we have $\partial_1^j f_k(0,y)(\xi) = \frac{1}{k!}(k)_j b_k(y)(0^{k-j},\xi^j)\,\rho(0) = \delta_k^j\, b_k(y)$, we conclude the desired result $\partial_1^j f(0,y) = b_k(y)$. □

Remarks on Borel's theorem.

(1) [Colombeau, 1979]. Let E be a strict inductive limit of a non-trivial sequence of Fréchet spaces E_n. Then Borel's theorem is wrong for $f : \mathbb{R} \to E$. The idea is to choose $b_n = f^{(n)}(0) \in E_{n+1} \setminus E_n$ and to use that locally every smooth curve has to have values in some E_n.

(2) [Colombeau, 1979]. Let $E = \mathbb{R}^{\mathbb{N}}$. Then Borel's theorem is wrong for $f : E \to \mathbb{R}$. In fact, let $b_n : E \times \ldots \times E \to \mathbb{R}$ be given by $b_n := \mathrm{pr}_n \otimes \cdots \otimes \mathrm{pr}_n$. Assume $f \in C^\infty(E,\mathbb{R})$ exists with $f^{(n)}(0) = b_n$. Let f_n be the restriction of f to the n-th factor $\mathbb{R}$ in E. Then $f_n \in C^\infty(\mathbb{R},\mathbb{R})$ and $f_n^{(n)}(0) = 1$. Since $f' : \mathbb{R}^n \to (\mathbb{R}^n)' = \mathbb{R}^{(\mathbb{N})}$ is continuous, the image of $B := \{x : |x_n| \le 1 \text{ for all } n\}$ in $\mathbb{R}^{(\mathbb{N})}$ is bounded, hence contained in some $\mathbb{R}^{N-1}$. Since f_N is not constant on the interval $(-1,1)$ there exists some $|t_N| < 1$ with $f_N'(t_N) \ne 0$. For $x_N := (0,\ldots,0,t_N,0,\ldots)$ we obtain

$$f'(x_N)(y) = f_N'(t_N)(y_N) + \sum_{i\ne N} a_i\, y_i,$$

a contradiction to $f'(x_n) \in \mathbb{R}^{N-1}$.

(3) [Colombeau, 1979] showed that Borel's theorem is true for mappings $f : E \to F$, where E has a basis of Hilbert-seminorms and for any countable family of 0-neighborhoods U_n there exist $t_n > 0$ such that $\bigcap_{n=1}^\infty t_n U_n$ is a 0-neighborhood.

(4) If theorem 15.4 would be true for $G = \prod_k L^k_{\mathrm{sym}}(E;F)$ and $b_k = \mathrm{pr}_k$, then the quotient mapping $C^\infty(E,F) \to G = \prod_k L^k_{\mathrm{sym}}(E;F)$ would admit a smooth and hence a linear section. This is well know to be wrong even for $E = F = \mathbb{R}$, see 21.5.

15.5. Proposition. Hilbert spaces have C_b^∞-bump functions. [Wells, 1973] *If the norm is given by the n-th root of a homogeneous polynomial b of even degree n, then $x \mapsto \rho(b(x^n))$ is a C_b^∞-bump function, where $\rho : \mathbb{R} \to \mathbb{R}$ is smooth with $\rho(t) = 1$ for $t \le 0$ and $\rho(t) = 0$ for $t \ge 1$.*

Proof. As before in the proof of 15.4 we see that the j-th derivative of $x \mapsto b(x^n)$ is bounded by $(n)_j$ on the closed unit ball. Hence, by the chain-rule and the global boundedness of all derivatives of ρ separately, the composite has bounded derivatives on the unit ball, and since it is zero outside, even everywhere. Obviously, $\rho(b(0)) = \rho(0) = 1$. □

In [Bonic, Frampton, 1966] it is shown that L^p is $\mathcal{L}\mathrm{ip}^n_{\mathrm{global}}$-smooth for all n if p is an even integer and is $\mathcal{L}\mathrm{ip}^{[p-1]}_{\mathrm{global}}$-smooth otherwise. This follows from the fact (see loc. cit., p. 140) that $d^{(p+1)}\|x\|^p = 0$ for even integers p and

$$\left\| d^k\|x+h\|^p - d^k\|x\|^p \right\| \le \frac{p!}{k!}\|h\|^{p-k}$$

otherwise, cf. 13.13.

15.6. Estimates for the remainder in the Taylor-expansion. The Taylor formula of order k of a C^{k+1}-function is given by

$$f(x+h) = \sum_{j=0}^k \frac{1}{j!} f^{(j)}(x)(h^j) + \int_0^1 \frac{(1-t)^k}{k!}\, f^{(k+1)}(x+th)(h^{k+1})\,dt,$$

which can easily be seen by repeated partial integration of $\int_0^1 f'(x+th)(h)\,dt = f(x+h)-f(x)$.

For a C_B^2 function we have

$$|f(x+h)-f(x)-f'(x)(h)| \le \int_0^1 (1-t)\|f^{(2)}(x+th)\|\,\|h\|^2\,dt \le B\frac{1}{2!}\,\|h\|^2.$$

If we take the Taylor formula of f up to order 0 instead, we obtain

$$f(x+h) = f(x) + \int_0^1 f'(x+th)(h)\,dt$$

and usage of $f'(x)(h) = \int_0^1 f'(x)(h)\,dt$ gives

$$|f(x+h)-f(x)-f'(x)(h)| \le \int_0^1 \frac{\|f'(x+th)-f'(x)\|}{\|th\|}\,\|h\|^2\,dt \le B\frac{1}{2!}\,\|h\|^2,$$

so it is in fact enough to assume $f\in C^1$ with f' satisfying a Lipschitz-condition with constant B.

For a C_B^3 function we have

$$\begin{aligned}|f(x+h)-f(x)-f'(x)(h)-\frac{1}{2}f''(x)(h^2)| &\le\\ &\le \int_0^1 \frac{(1-t)^2}{2!}\|f^{(3)}(x+th)\|\,\|h\|^3\,dt \le B\frac{1}{3!}\,\|h\|^3.\end{aligned}$$

If we take the Taylor formula of f up to order 1 instead, we obtain

$$f(x+h) = f(x) + f'(x)(h) + \int_0^1 (1-t)\,f''(x+th)(h^2)\,dt,$$

and using $\frac{1}{2}f''(x)(h^2) = \int_0^1 (1-t)\,f''(x)(h^2)\,dt$ we get

$$\begin{aligned}|f(x+h)-f(x)-f'(x)(h)-\frac{1}{2}f''(x)(h^2)| &\le\\ &\le \int_0^1 (1-t)t\,\frac{\|f''(x+th)-f''(x)\|}{\|th\|}\,\|h\|^3\,dt \le B\frac{1}{3!}\,\|h\|^3.\end{aligned}$$

Hence, it is in fact enough to assume $f\in C^2$ with f'' satisfying a Lipschitz-condition with constant B.

Let $f\in C_B^k$ be flat of order k at 0. Applying $\|f(h)-f(0)\| = \|\int_0^1 f'(th)(h)\,dt\| \le \sup\{\|f'(th)\| : t\in[0,1]\}\,\|h\|$ to $f^{(j)}(\quad)(h_1,\dots,h_j)$ gives using $\|f^{(k)}(x)\|\le B$ inductively

$$\begin{aligned}\|f^{(k-1)}(x)\| &\le B\cdot\|x\|\\ \|f^{(k-2)}(x)\| &\le \int_0^1 \|f^{(k-1)}(tx)(x,\dots)\|\,dt \le B\int_0^1 t\,dt\,\|x\|^2 = \frac{B}{2}\,\|x\|^2\\ &\;\vdots\\ \|f^{(j)}(x)\| &\le \frac{B}{(k-j)!}\,\|x\|^{k-j}.\end{aligned}$$

15.7. Lemma. $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-functions on $\mathbb{R}^n$. [Wells, 1973]. *Let $n := 2^N$ and $E=\mathbb{R}^n$ with the ∞-norm. Suppose $f\in\mathcal{L}\mathrm{ip}^1_M(E,\mathbb{R})$ with $f(0)=0$ and $f(x)\ge 1$ for $\|x\|\ge 1$. Then $M\ge 2N$.*

The idea behind the proof is to construct recursively a sequence of points $x_k := \sum_{j<k} \sigma_j h_j$ of norm $\frac{k-1}{N}$ starting at $x_0 = 0$, such that the increment along the segment is as small as possible. In order to evaluate this increment one uses the Taylor-formula and chooses the direction h_k such that the derivative at x_k vanishes.

Proof. Let A be the set of all edges of a hyper-cube, i.e.

$$A := \{x : x_i = \pm 1 \text{ for all } i \text{ except one } i_0 \text{ and } |x_{i_0}| \leq 1\}.$$

Then A is symmetric. Let $x \in E$ be arbitrary. We want to find $h \in A$ with $f'(x)(h) = 0$. By permuting the coordinates we may assume that $i \mapsto |f'(x)(e^i)|$ is monotone decreasing. For $2 \leq i \leq n$ we choose recursively $h_i \in \{\pm 1\}$ such that $\sum_{j=2}^{i} h_j\, f'(x)(e_j)$ is an alternating sum. Then $|\sum_{j=2}^{i} f'(x)(e^j)h_j| \leq |f'(x)(e^1)|$. Finally, we choose $\|h_1\| \leq 1$ such that $f'(x)(h) = 0$.

Now we choose inductively $h_i \in \frac{1}{N} A$ and $\sigma_i \in \{\pm 1\}$ such that $f'(x_i)(h_i) = 0$ for $x := \sum_{j<i} \sigma_j h_j$ and x_i has at least 2^{N-i} coordinates equal to $\frac{i}{N}$. For the last statement we have that $x_{i+1} = x_i + \sigma_i\, h_i$ and at least 2^{N-i} coordinates of x_i are $\frac{i}{N}$. Among those coordinates all but at most 1 of the h_i are $\pm\frac{1}{N}$. Now let σ_i be the sign which occurs more often and hence at least $2^{N-i}/2$ times. Then those $2^{N-(i+1)}$ many coordinates of x_{i+1} are $\frac{i+1}{N}$.

Thus $\|x_i\| = \frac{i}{N}$ for $i \leq N$, since at least one coordinate has this value. Furthermore we have

$$\begin{aligned} 1 = |f(x_N) - f(x_0)| &\leq \sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k) - f'(x_k)(h_k)| \\ &\leq \sum_{k=1}^{N} \frac{M}{2}\, \|h_k\|^2 \leq N\, \frac{M}{2}\, \frac{1}{N^2}, \end{aligned}$$

hence $M \geq 2N$. □

15.8. Corollary. c_0 is not $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-regular. [Wells, 1973]. *The space c_0 is not $\mathcal{L}\mathrm{ip}^1_{global}$-smooth.*

Proof. Suppose there exists an $f \in \mathcal{L}\mathrm{ip}^1_{\mathrm{global}}$ with $f(0) = 1$ and $f(x) = 0$ for all $\|x\| \geq 1$. Then the previous lemma applied to $1 - f$ restricted to finite dimensional subspaces shows that the Lipschitz constant M of the derivative has to be greater or equal to N for all N, a contradiction. □

This shows even that there exist no differentiable bump functions on $c_0(A)$ which have uniformly continuous derivative. Since otherwise there would exist an $N \in \mathbb{N}$ such that

$$\|f(x+h) - f(x) - f'(x)h\| \leq \int_0^1 \|f'(x + t\,h) - f'(x)\|\, \|h\|\, dt \leq \frac{1}{2}\|h\|,$$

for $\|h\| \leq \frac{1}{N}$. Hence, the estimation in the proof of 15.7 would give $1 \leq N\frac{1}{2}\frac{1}{N} = \frac{1}{2}$, a contradiction.

15.9. Positive results on $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-functions. [Wells, 1973].

(1) *Every closed subset of a Hilbert space is the zero-set of a $\mathcal{L}\mathrm{ip}^1_{global}$-function.*
(2) *For every two closed subsets of a Hilbert space which have distance $d > 0$ there exists a $\mathcal{L}\mathrm{ip}^1_{4/d^2}$-function which has value 0 on one set and 1 on the other.*

(3) *Whitney's extension theorem is true for* $\mathcal{L}\mathrm{ip}^1_{global}$*-functions on closed subsets of Hilbert spaces.*

16. Smooth Partitions of Unity and Smooth Normality

16.1. Definitions. We say that a Hausdorff space X is *smoothly normal* with respect to a subalgebra $\mathcal{S} \subseteq C(X,\mathbb{R})$ or $\mathcal{S}$-*normal*, if for two disjoint closed subsets A_0 and A_1 of X there exists a function $f : X \to \mathbb{R}$ in $\mathcal{S}$ with $f|A_i = i$ for $i = 0, 1$. If an algebra $\mathcal{S}$ is specified, then by a smooth function we will mean an element of $\mathcal{S}$. Otherwise it is a C^∞-function.

A $\mathcal{S}$-*partition of unity* on a space X is a set $\mathcal{F}$ of smooth functions $f : X \to \mathbb{R}$ which satisfy the following conditions:

(1) For all $f \in \mathcal{F}$ and $x \in X$ one has $f(x) \geq 0$.
(2) The set $\{\mathrm{carr}(f) : f \in \mathcal{F}\}$ of all carriers is a locally finite covering of X.
(3) The sum $\sum_{f\in\mathcal{F}} f(x)$ equals 1 for all $x \in X$.

Since a family of open sets is locally finite if and only if the family of the closures is locally finite, the foregoing condition (2) is equivalent to:

(2') The set $\{\mathrm{supp}(f) : f \in \mathcal{F}\}$ of all supports is a locally finite covering of X.

The partition of unity is called *subordinated* to an open covering $\mathcal{U}$ of X, if for every $f \in \mathcal{F}$ there exists an $U \in \mathcal{U}$ with $\mathrm{carr}(f) \subseteq U$.

We say that X is *smoothly paracompact* with respect to $\mathcal{S}$ or $\mathcal{S}$-*paracompact* if every open cover $\mathcal{U}$ admits a $\mathcal{S}$-partition $\mathcal{F}$ of unity subordinated to it. This implies that X is $\mathcal{S}$-normal.

The partition of unity can then even be chosen in such a way that for every $f \in \mathcal{F}$ there exists a $U \in \mathcal{U}$ with $\mathrm{supp}(f) \subseteq U$. This is seen as follows. Since the family of carriers is a locally finite open refinement of $\mathcal{U}$, the topology of X is paracompact. So we may find a finer open cover $\{\tilde{U} : U \in \mathcal{U}\}$ such that the closure of $\tilde{U}$ is contained in U for all $U \in \mathcal{U}$, see [Bourbaki, 1966, IX.4.3]. The partition of unity subordinated to this finer cover has the support property for the original one.

Lemma. *Let $\mathcal{S}$ be an algebra which is closed under sums of locally finite families of functions. If $\mathcal{F}$ is an $\mathcal{S}$-partition of unity subordinated to an open covering $\mathcal{U}$, then we may find an $\mathcal{S}$-partition of unity $(f_U)_{U\in\mathcal{U}}$ with* $\mathrm{carr}(f_U) \subseteq U$.

Proof. For every $f \in \mathcal{F}$ we choose a $U_f \in \mathcal{U}$ with $\mathrm{carr}(f) \in U_f$. For $U \in \mathcal{U}$ put $\mathcal{F}_U := \{f : U_f = U\}$ and let $f_U := \sum_{f\in\mathcal{F}_U} f \in \mathcal{S}$. □

16.2. Proposition. Characterization of smooth normality. *Let X be a Hausdorff space with $\mathcal{S} \subseteq C(X,\mathbb{R})$ as in 14.1 Consider the following statements:*

(1) *X is $\mathcal{S}$-normal;*
(2) *For any two closed disjoint subsets $A_i \subseteq X$ there is a function $f \in \mathcal{S}$ with $f|A_0 = 0$ and $0 \notin f(A_1)$;*
(3) *Every locally finite open covering admits $\mathcal{S}$-partitions of unity subordinated to it.*
(4) *For any two disjoint zero-sets A_0 and A_1 of continuous functions there exists a function $g \in \mathcal{S}$ with $g|_{A_j} = j$ for $j = 0, 1$ and $g(X) \subseteq [0, 1]$;*
(5) *For any continuous function $f : X \to \mathbb{R}$ there exists a function $g \in \mathcal{S}$ with $f^{-1}(0) \subseteq g^{-1}(0) \subseteq f^{-1}(\mathbb{R} \setminus \{1\})$.*
(6) *The set $\mathcal{S}$ is dense in the algebra of continuous functions with respect to the topology of uniform convergence;*
(7) *The set of all bounded functions in $\mathcal{S}$ is dense in the algebra of continuous bounded functions on X with respect to the supremum norm;*

(8) *The bounded functions in $\mathcal{S}$ separate points in the Stone-Čech-compactification βX of X.*

The statements (1)-(3) are equivalent, and (4)-(8) are equivalent as well. If X is metrizable all statements are equivalent.
If every open set is the carrier set of a smooth function then X is $\mathcal{S}$-normal. If X is $\mathcal{S}$-normal, then it is $\mathcal{S}$-regular.
A space is $\mathcal{S}$-paracompact if and only if it is paracompact and $\mathcal{S}$-normal.

Proof. (2) $\Rightarrow$ (1). By assumption, there is a smooth function f_0 with $f_0|A_1 = 0$ and $0 \notin f_0(A_0)$, and again by assumption, there is a smooth function f_1 with $f_1|A_0 = 0$ and $0 \notin f_1(\{x : f_0(x) = 0\})$. The function $f = \frac{f_1}{f_0+f_1}$ has the required properties.

(1) $\Rightarrow$ (2) is obvious.

(3) $\Rightarrow$ (1) Let A_0 and A_1 be two disjoint closed subset. Then $\mathcal{U} := \{X \setminus A_1, X \setminus A_0\}$ admits a $\mathcal{S}$-partition of unity $\mathcal{F}$ subordinated to it, and

$$\sum \{f \in \mathcal{F} : \operatorname{carr} f \subseteq X \setminus A_0\}$$

is the required bump function.

(1) $\Rightarrow$ (3) Let $\mathcal{U}$ be a locally finite covering of X. The space X is $\mathcal{S}$-normal, so its topology is also normal, and therefore for every $U \in \mathcal{U}$ there exists an open set V_U such that $\overline{V_U} \subseteq U$ and $\{V_U : U \in \mathcal{U}\}$ is still an open cover. By assumption, there exist smooth functions $g_U \in \mathcal{S}$ such that $V_U \subseteq \operatorname{carr}(g_U) \subseteq U$, cf. 16.1. The function $g := \sum_U g_U$ is well defined, positive, and smooth since $\mathcal{U}$ is locally finite, and $\{f_U := g_U/g : U \in \mathcal{U}\}$ is the required partition of unity.

(5) $\Rightarrow$ (4) Let $A_j := f_j^{-1}(a_j)$ for $j = 0, 1$. By replacing f_j by $(f_j - a_j)^2$ we may assume that $f_j \geq 0$ and $A_j = f_j^{-1}(0)$. Then $(f_1 + f_2)(x) > 0$ for all $x \in X$, since $A_1 \cap A_2 = \emptyset$. Thus, $f := \frac{f_0}{f_0+f_1}$ is a continuous function in $C(X, [0, 1])$ with $f|_{A_j} = j$ for $j = 0, 1$.
Now we reason as in ((2) $\Rightarrow$ (1)): By (4) there exists a $g_0 \in \mathcal{S}$ with $A_0 \subseteq f^{-1}(0) \subseteq g_0^{-1}(0) \subseteq f^{-1}(\mathbb{R} \setminus \{1\}) = X \setminus f^{-1}(1) \subseteq X \setminus A_1$. By replacing g_0 by g_0^2 we may assume that $g_0 \geq 0$.
Applying the same argument to the zero-sets A_1 and $g_0^{-1}(0)$ we obtain a $g_1 \in \mathcal{S}$ with $A_1 \subseteq g_1^{-1}(0) \subseteq X \setminus g_0^{-1}(0)$. Thus, $(g_0 + g_1)(x) > 0$, and hence $g := \frac{g_0}{g_0+g_1} \in \mathcal{S}$ satisfies $g|_{A_j} = j$ for $j = 0, 1$ and $g(X) \subseteq [0, 1]$.

(4) $\Rightarrow$ (6) Let f be continuous. Without loss of generality we may assume $f \geq 0$ (decompose $f = f_+ - f_-$). Let $\varepsilon > 0$. Then choose $g_k \in \mathcal{S}$ with image in $[0, 1]$, and $g_k(x) = 0$ for all x with $f(x) \leq k\,\varepsilon$, and $g_k(x) = 1$ for all x with $f(x) \geq (k + 1)\,\varepsilon$. Let k be the largest integer less or equal to $\frac{f(x)}{\varepsilon}$. Then $g_j(x) = 1$ for all $j < k$, and $g_j(x) = 0$ for all $j > k$. Hence, the sum $g := \varepsilon \sum_{k\in\mathbb{N}} g_k \in \mathcal{S}$ is locally finite, and $|f(x) - g(x)| < 2\,\varepsilon$.

(6) $\Rightarrow$ (7) This is obvious, since for any given bounded continuous f and for any $\varepsilon > 0$, by (6) there exists $g \in \mathcal{S}$ with $|f(x) - g(x)| < \varepsilon$ for all $x \in X$, hence $\|f - g\|_\infty \leq \varepsilon$ and $\|g\|_\infty \leq \|f\|_\infty + \|f - g\|_\infty < \infty$.

(7) $\Leftrightarrow$ (8) This follows from the Stone-Weierstraß theorem, since obviously the bounded functions in $\mathcal{S}$ form a subalgebra in $C_b(X) = C(\beta X)$. Hence, it is dense if and only if it separates points in the compact space βX.

(7) ⇒ (4) By cutting off f at 0 and at 1, we may assume that f is bounded. By (7) there exists a bounded $g_0 \in \mathcal{S}$ with $\|f - g_0\|_\infty < \frac{1}{2}$. Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $h(t) = 0 \Leftrightarrow t \le \frac{1}{2}$. Then $g := h \circ g_0 \in \mathcal{S}$, and $f(x) = 0 \Rightarrow g_0(x) \le |g_0(x)| \le |f(x)| + \|f - g_0\|_\infty \le \frac{1}{2} \Rightarrow g(x) = h(g_0(x)) = 0$ and also $f(x) = 1 \Rightarrow g_0(x) \ge f(x) - \|f - g_0\|_\infty > 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow g(x) \ne 0$.

If X is metrizable and $A \subseteq X$ is closed, then $\operatorname{dist}(\quad, A) : x \mapsto \sup\{\operatorname{dist}(x, a) : a \in A\}$ is a continuous function with $f^{-1}(0) = A$. Thus, (1) and (4) are equivalent.

Let every open subset be the carrier of a smooth mapping, and let A_0 and A_1 be closed disjoint subsets of X. By assumption, there is a smooth function f with $\operatorname{carr}(f) = X \setminus A_0$.

Obviously, every $\mathcal{S}$-normal space is $\mathcal{S}$-regular. Take as second closed set in (2) a single point. If we take instead the other closed set as single point, then we have what has been called small zero-sets in 19.8.

That a space is $\mathcal{S}$-paracompact if and only if it is paracompact and $\mathcal{S}$-normal can be shown as in the proof that a paracompact space admits continuous partitions of unity, see [Engelking, 1989, 5.1.9]. □

In [Kriegl, Michor, Schachermayer, 1989] it is remarked that in an uncountable product of real lines there are open subsets, which are not carrier sets of continuous functions.

Corollary. Denseness of smooth functions. *Let X be $\mathcal{S}$-paracompact, let F be a convenient vector space, and let $U \subseteq X \times F$ be open such that for all $x \in X$ the set $\iota_x^{-1}(U) \subseteq F$ is convex and non-empty, where $\iota_x : F \to X \times F$ is given by $y \mapsto (x, y)$. Then there exists an $f \in \mathcal{S}$ whose graph is contained in U.*

Under the following assumption this result is due to [Bonic, Frampton, 1966]: For $U := \{(x, y) : p(y - g(x)) < \varepsilon(x)\}$, where $g : X \to F$, $\varepsilon : X \to \mathbb{R}^+$ are continuous and p is a continuous seminorm on F.

Proof. For every $x \in X$ let y_x be chosen such that $(x, y_x) \in U$. Next choose open neighborhoods U_x of x such that $U_x \times \{y_x\} \subseteq U$. Since X is $\mathcal{S}$-paracompact there exists a $\mathcal{S}$-partition of unity $\mathcal{F}$ subordinated to the covering $\{U_x : x \in X\}$. In particular, for every $\varphi \in \mathcal{F}$ there exists an $x_\varphi \in X$ with $\operatorname{carr} \varphi \subseteq U_{x_\varphi}$. Now define $f := \sum_{\varphi \in \mathcal{F}} y_{x_\varphi} \varphi$. Then $f \in \mathcal{S}$ and for every $x \in X$ we have

$$f(x) = \sum_{\varphi \in \mathcal{F}} y_{x_\varphi} \varphi(x) = \sum_{x \in \operatorname{carr} \varphi} y_{x_\varphi} \varphi(x) \in \iota_x^{-1}(U),$$

since $\iota_x^{-1}(U)$ is convex, contains y_{x_φ} for $x \in \operatorname{carr}(\varphi) \subseteq U_{x_\varphi}$, and $\varphi(x) \ge 0$ with $1 = \sum_\varphi \varphi(x) = \sum_{x \in \operatorname{carr} \varphi} \varphi(x)$. □

16.3. Lemma. $\mathcal{L}\mathrm{ip}^2$-functions on $\mathbb{R}^n$. [Wells, 1973]. *Let $B \in \mathbb{N}$ and $A := \{x \in \mathbb{R}^N : x_i \le 0$ for all i and $\|x\| \le 1\}$. Suppose that $f \in C_B^3(\mathbb{R}^N, \mathbb{R})$ with $f|_A = 0$ and $f(x) \ge 1$ for all x with $\operatorname{dist}(x, A) \ge 1$. Then $N < B^2 + 36\, B^4$.*

Proof. Suppose $N \ge B^2 + 36\, B^4$. We may assume that f is symmetric by replacing f with $x \mapsto \frac{1}{N!} \sum_\sigma f(\sigma^* x)$, where σ runs through all permutations, and σ^* just permutes the coordinates. Consider points $x^j \in \mathbb{R}^N$ for $j = 0, \dots, B^2$ of the form

$$x^j = \Big(\underbrace{\tfrac{1}{B}, \dots, \tfrac{1}{B}}_{j}, \underbrace{-\tfrac{1}{B}, \dots, -\tfrac{1}{B}}_{B^2 - j}, \underbrace{0, \dots, 0}_{>36\, B^4}\Big).$$

Then $\|x^j\| = 1$, $x^0 \in A$ and $d(x^{B^2}, A) \geq 1$. Since f is symmetric and $y^j := \frac{1}{2}(x^j + x^{j+1})$ has vanishing $j, B^2+1, \dots, N$ coordinates, we have for the partial derivatives $\partial_j f(y^j) = \partial_k f(y^j)$ for $k = B^2+1, \dots, N$. Thus

$$|\partial_j f(y^j)|^2 = \frac{1}{N - B^2} \sum_{k=B^2+1}^{N} |\partial_k f(y^j)|^2 \leq \frac{\|f'(y^j)\|_2^2}{36\, B^4} = \frac{\|f'(y^j)\|^2}{36\, B^4} \leq \frac{1}{36\, B^2},$$

since from $f|_A = 0$ we conclude that $f(0) = f'(0) = f''(0) = f'''(0)$ and hence $\|f^{(j)}(h)\| \leq B\, \|h\|^{3-j}$ for $j \leq 3$, see 15.6.

From $|f(x+h) - f(x) - f'(x)(h) - \frac{1}{2} f''(x)(h^2)| \leq B \frac{1}{3!} \|h\|^3$ we conclude that

$$\begin{aligned} |f(x+h) - f(x-h)| &\leq |f(x+h) - f(x) - f'(x)(h) - \tfrac{1}{2} f''(x)(h^2)| \\ &\quad + |f(x-h) - f(x) + f'(x)(h) - \tfrac{1}{2} f''(x)(h^2)| \\ &\quad + 2|f'(x)(h)| \\ &\leq \tfrac{2}{3!}\, B \|h\|^3 + 2|f'(x)(h)|. \end{aligned}$$

If we apply this to $x = y^j$ and $h = \frac{1}{B} e_j$, where e_j denotes the j-th unit vector, then we obtain

$$|f(x^{j+1}) - f(x^j)| \leq \frac{2}{3!}\, B \frac{1}{B^3} + 2|\partial_j f(y^j)| \frac{1}{B} \leq \frac{2}{3B^2}.$$

Summing up yields $1 \leq |f(x^{B^2})| = |f(x^{B^2}) - f(x^0)| \leq \frac{2}{3} < 1$, a contradiction. □

16.4. Corollary. ℓ^2 is not $\mathcal{L}\mathrm{ip}^2_{\mathbf{glob}}$-normal . [Wells, 1973]. *Let $A_0 := \{x \in \ell^2 : x_j \leq 0$ for all j and $\|x\| \leq 1\}$ and $A_1 := \{x \in \ell^2 : d(x, A) \geq 1\}$ and $f \in C^3(\ell^2, \mathbb{R})$ with $f|_{A_j} = j$ for $j = 0, 1$. Then $f^{(3)}$ is not bounded.*

Proof. By the preceding lemma a bound B of $f^{(3)}$ must satisfy for f restricted to $\mathbb{R}^N$, that $N < B^2 + 36B^4$. This is not for all N possible. □

16.5. Corollary. Whitney's extension theorem is false on ℓ^2. [Wells, 1973]. *Let $E := \mathbb{R} \times \ell^2 \cong \ell^2$ and $\pi : E \to \mathbb{R}$ be the projection onto the first factor. For subsets $A \subseteq \ell^2$ consider the cone $CA := \{(t, ta) : t \geq 0, a \in A\} \subseteq E$. Let $A := C(A_0 \cup A_1)$ with A_0 and A_1 as in 16.4. Let a jet (f^j) on A be defined by $f^j = 0$ on the cone CA_1 and $f^j(x)(v^1, \dots, v^j) = h^{(j)}(\pi(x))(\pi(v^1), \dots, \pi(v^j))$ for all x in the cone of CA_0, where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ is infinite flat at 0 but with $h(t) \neq 0$ for all $t \neq 0$. This jet has no C^3-prolongation to E.*

Proof. Suppose that such a prolongation f exists. Then $f^{(3)}$ would be bounded locally around 0, hence $f_a(x) := 1 - \frac{1}{h(a)} f(a, ax)$ would be a C^3_B function on ℓ^2 for small a, which is 1 on A_1 and vanishes on A_0. This is a contradiction to 16.4.

So it remains to show that the following condition of Whitney 22.2 is satisfied:

$$\Big\| f^j(y) - \sum_{i=0}^{k-j} \frac{1}{i!} f^{j+i}(x)(y-x)^j \Big\| = o(\|x-y\|^{k-j}) \text{ for } A \ni x, y \to a.$$

Let $f_1^j := 0$ and $f_0^j(x) := h^{(j)}(\pi(x)) \circ (\pi \times \ldots \times \pi)$. Then both are smooth on $\mathbb{R} \oplus \ell^2$, and thus Whitney's condition is satisfied on each cone separately. It remains to show this when x is in one cone and y in the other and both tend to 0. Thus, we have to replace f at some places by f_1 and at others by f_0. Since h is infinite flat at 0 we have $\|f_0^j(z)\| = o(\|z\|^n)$ for every n. Furthermore for $x_i \in CA_i$ for $i = 0, 1$ we have that $\|x_1 - x_0\| \geq \sin(\arctan 2 - \arctan 1) \max\{\|x_0\|, \|x_1\|\}$. Thus, we may replace $f_0^j(y)$ by $f_1^j(y)$ and vice versa. So the condition is reduced to the case, where y and z are in the same cone CA_i. □

16.6. Lemma. Smoothly regular strict inductive limits. *Let E be the strict inductive limit of a sequence of C^∞-normal convenient vector spaces E_n such that $E_n \hookrightarrow E_{n+1}$ is closed and has the extension property for smooth functions. Then E is C^∞-regular.*

Proof. Let U be open in E and $0 \in U$. Then $U_n := U \cap E_n$ is open in E_n. We choose inductively a sequence of functions $f_n \in C^\infty(E_n, \mathbb{R})$ such that $\operatorname{supp}(f_n) \subseteq U_n$, $f_n(0) = 1$, and $f_n|E_{n-1} = f_{n-1}$. If f_n is already constructed, we may choose by C^∞-normality a smooth $g : E_{n+1} \to \mathbb{R}$ with $\operatorname{supp}(g) \subseteq U_{n+1}$ and $g|_{\operatorname{supp}(f_n)} = 1$. By assumption, f_n extends to a function $\widetilde{f_n} \in C^\infty(E_{n+1}, \mathbb{R})$. The function $f_{n+1} := g \cdot \widetilde{f_n}$ has the required properties.

Now we define $f : E \to \mathbb{R}$ by $f|E_n := f_n$ for all n. It is smooth since any $c \in C^\infty(\mathbb{R}, E)$ locally factors to a smooth curve into some E_n by 1.8 since a strict inductive limit is regular by 52.8, so $f \circ c$ is smooth. Finally, $f(0) = 1$, and if $f(x) \neq 0$ then $x \in E_n$ for some n, and we have $f_n(x) = f(x) \neq 0$, thus $x \in U_n \subseteq U$. □

For counter-examples for the extension property see 21.7 and 21.11. However, for complemented subspaces the extension property obviously holds.

16.7. Proposition. C_c^∞ is C^∞-regular. *The space $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ of smooth functions on $\mathbb{R}^m$ with compact support satisfies the assumptions of 16.6.*

Let $K_n := \{x \in \mathbb{R}^m : |x| \le n\}$. Then $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ is the strict inductive limit of the closed subspaces $C_{K_n}^\infty(\mathbb{R}^m, \mathbb{R}) := \{f : \operatorname{supp}(f) \subseteq K_n\}$, which carry the topology of uniform convergence in all partial derivatives separately. They are nuclear Fréchet spaces and hence separable, see 52.27. Thus they are C^∞-normal by 16.10 below.

In order to show the extension property for smooth functions we proof more general that for certain sets A the subspace $\{f \in C^\infty(E, \mathbb{R}) : f|_A = 0\}$ is a complemented subspace of $C^\infty(E, \mathbb{R})$. The first result in this direction is:

16.8. Lemma. [Seeley, 1964] *The subspace $\{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f(t) = 0$ for $t \le 0\}$ of the Fréchet space $C^\infty(\mathbb{R}, \mathbb{R})$ is a direct summand.*

Proof. We claim that the following map is a bounded linear mapping being left inverse to the inclusion: $s(g)(t) := g(t) - \sum_{k\in\mathbb{N}} a_k h(-t2^k) g(-t2^k)$ for $t > 0$ and $s(g)(t) = 0$ for $t \le 0$. Where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support satisfying $h(t) = 1$ for $t \in [-1, 1]$ and (a_k) is a solution of the infinite system of linear equations $\sum_{k\in\mathbb{N}} a_k(-2^k)^n = 1$ $(n \in \mathbb{N})$ (the series is assumed to converge absolutely). The existence of such a solution is shown in [Seeley, 1964] by taking the limit of solutions of the finite subsystems. Let us first show that $s(g)$ is smooth. For $t > 0$ the series is locally around t finite, since $-t2^k$ lies outside the support of h for k sufficiently large. Its derivative $(sg)^{(n)}(t)$ is

$$g^{(n)}(t) - \sum_{k\in\mathbb{N}} a_k(-2^k)^n \sum_{j=0}^{n} h^{(j)}(-t2^k) g^{(n-j)}(-t2^k)$$

and this converges for $t \to 0$ towards $g^{(n)}(0) - \sum_{k\in\mathbb{N}} a_k(-2^k)^n g^{(n)}(0) = 0$. Thus $s(g)$ is infinitely flat at 0 and hence smooth on $\mathbb{R}$. It remains to show that $g \mapsto s(g)$ is a bounded linear mapping. By the uniform boundedness principle 5.26 it is enough to show that $g \mapsto (sg)(t)$ is bounded. For $t \le 0$ this map is 0 and hence bounded. For $t > 0$ it is a finite linear combination of evaluations and thus bounded. □

Now the general result:

16.9. Proposition. *Let E be a convenient vector space, and let p be a smooth seminorm on E. Let $A := \{x : p(x) \ge 1\}$. Then the closed subspace $\{f : f|_A = 0\}$ in $C^\infty(E, \mathbb{R})$ is complemented.*

Proof. Let $g \in C^\infty(E, \mathbb{R})$ be a smooth reparameterization of p with support in $E \setminus A$ equal to 1 near $p^{-1}(0)$. By lemma 16.8, there is a bounded projection $P : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty_{(-\infty,0]}(\mathbb{R}, \mathbb{R})$. The following mappings are smooth in turn by the properties of the cartesian closed smooth calculus, see 3.12:

$$\begin{aligned}
E \times \mathbb{R} \ni (x, t) &\mapsto f(e^t, x) \in \mathbb{R}\\
E \ni x &\mapsto f(e^{(\ \ )} x) \in C^\infty(\mathbb{R}, \mathbb{R})\\
E \ni x &\mapsto P(f(e^{(\ \ )} x)) \in C^\infty_{(-\infty,0]}(\mathbb{R}, \mathbb{R})\\
E \times \mathbb{R} \ni (x, r) &\mapsto P(f(e^{(\ \ )} x))(r) \in \mathbb{R}\\
\operatorname{carr} p \ni x &\mapsto \left(\frac{x}{p(x)}, \ln(p(x))\right) \mapsto P\left(f(e^{(\ \ )} \tfrac{x}{p(x)})\right)(\ln(p(x))) \in \mathbb{R}.
\end{aligned}$$

So we get the desired bounded linear projection

$$\bar{P} : C^\infty(E, \mathbb{R}) \to \{f \in C^\infty(E, \mathbb{R}) : f|_A = 0\},$$
$$(\bar{P}(f))(x) := g(x)\, f(x) + (1 - g(x))\, P(f(e^{(\ \ )} \tfrac{x}{p(x)}))(\ln(p(x))). \quad \square$$

16.10. Theorem. Smoothly paracompact Lindelöf. [Wells, 1973]. *If X is Lindelöf and $\mathcal{S}$-regular, then X is $\mathcal{S}$-paracompact. In particular, all nuclear Fréchet spaces and strict inductive limits of sequences of such spaces are C^∞-paracompact. Furthermore, nuclear* Silva spaces*, see 52.37, are C^∞-paracompact.*

The first part was proved by [Bonic, Frampton, 1966] under stronger assumptions. The importance of the proof presented here lies in the fact that we need not assume that $\mathcal{S}$ is local and that $\frac{1}{f} \in \mathcal{S}$ for $f \in \mathcal{S}$. The only things used are that $\mathcal{S}$ is an algebra and for each $g \in \mathcal{S}$ there exists an $h : \mathbb{R} \to [0, 1]$ with $h \circ g \in \mathcal{S}$ and $h(t) = 0$ for $t \le 0$ and $h(t) = 1$ for $t \ge 1$. In particular, this applies to $\mathcal{S} = \mathcal{L}\mathrm{ip}^p_{\mathrm{global}}$ and X a separable Banach space.

Proof. Let $\mathcal{U}$ be an open covering of X.

Claim. There exists a sequence of functions $g_n \in \mathcal{S}(X, [0, 1])$ such that $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is a locally finite family subordinated to $\mathcal{U}$ and $\{g_n^{-1}(1) : n \in \mathbb{N}\}$ is a covering of X.

For every $x \in X$ there exists a neighborhood $U \in \mathcal{U}$ (since $\mathcal{U}$ is a covering) and hence an $h_x \in \mathcal{S}(X, [0, 2])$ with $h_x(x) = 2$ and $\operatorname{carr}(h_x) \subseteq U$ (since X is $\mathcal{S}$-regular). Since X is Lindelöf we find a sequence x_n such that $\{x : h_n(x) > 1 : n \in \mathbb{N}\}$ is a covering of X (we denote $h_n := h_{x_n}$). Now choose an $h \in C^\infty(\mathbb{R}, [0, 1])$ with $h(t) = 0$ for $t \le 0$ and $h(t) = 1$ for $t \ge 1$. Set

$$g_n(x) := h(n\,(h_n(x) - 1) + 1) \prod_{j<n} h(n\,(1 - h_j(x)) + 1).$$

Note that

$$h(n\,(h_n(x)-1)+1) = \begin{cases} 0 & \text{for } h_n(x) \le 1-\frac{1}{n} \\ 1 & \text{for } h_n(x) \ge 1 \end{cases}$$

$$h(n\,(1-h_j(x))+1) = \begin{cases} 0 & \text{for } h_j(x) \ge 1+\frac{1}{n} \\ 1 & \text{for } h_j(x) \le 1 \end{cases}$$

Then $g_n \in \mathcal{S}(X,[0,1])$ and $\operatorname{carr} g_n \subseteq \operatorname{carr} h_n$. Thus, the family $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is subordinated to $\mathcal{U}$.

The family $\{g_n^{-1}(1) : n \in \mathbb{N}\}$ covers X since for each $x \in X$ there exists a minimal n with $h_n(x) \ge 1$, and thus $g_n(x) = 1$.

If we could divide in $\mathcal{S}$, then $f_n := g_n/\sum_j g_j$ would be the required partition of unity (and we do not need the last claim in this strong from).

Instead we proceed as follows. The family $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is locally finite: Let n be such that $h_n(x) > 1$, and take $k > n$ so large that $1 + \frac{1}{k} < h_n(x)$, and let $U_x := \{y : h_n(y) > 1 + \frac{1}{k}\}$, which is a neighborhood of x. For $m \ge k$ and $y \in U_x$ we have that $h_n(y) > 1 + \frac{1}{k} \ge 1 + \frac{1}{m}$, hence the $(n+1)$-st factor of g_m vanishes at y, i.e. $\{j : \operatorname{carr} g_j \cap U_x \ne \emptyset\} \subseteq \{1, \dots, m-1\}$.

Now define $f_n := g_n \prod_{j<n}(1-g_j) \in \mathcal{S}$. Then $\operatorname{carr} f_n \subseteq \operatorname{carr} g_n$, hence $\{\operatorname{carr} f_n : n \in \mathbb{N}\}$ is a locally finite family subordinated to $\mathcal{U}$. By induction, one shows that $\sum_{j\le n} f_j = 1 - \prod_{j\le n}(1-g_j)$. In fact $\sum_{j\le n} f_j = f_n + \sum_{j<n} f_j = g_n \prod_{j<n}(1-g_j) + 1 - \prod_{j<n}(1-g_j) = 1 + (g_n - 1)\prod_{j<n}(1-g_j)$. For every $x \in U$ there exists an n with $g_n(x) = 1$, hence $f_k(x) = 0$ for $k > n$ and $\sum_{j=0}^\infty f_j(x) = \sum_{j\le n} f_j(x) = 1 - \prod_{j\le n}(1-g_j(x)) = 1$.

Let us consider a nuclear Silva space. By 52.37 its dual is a nuclear Fréchet space. By 4.11.2 on the strong dual of a nuclear Fréchet space the c^∞-topology coincides with the locally convex one. Hence, it is C^∞-regular since it is nuclear, so it has a base of (smooth) Hilbert seminorms. A Silva space is an inductive limit of a sequence of Banach spaces with compact connecting mappings (see 52.37), and we may assume that the Banach spaces are separable by replacing them by the closures of the images of the connecting mappings, so the topology of the inductive limit is Lindelöf. Therefore, by the first assertion we conclude that the space is C^∞-paracompact.

In order to obtain the statement on nuclear Fréchet spaces we note that these are separable, see 52.27, and thus Lindelöf. A strict inductive limit of a sequence of nuclear Fréchet spaces is C^∞-regular by 16.6, and it is also Lindelöf for its c^∞-topology, since this is the inductive limit of topological spaces (not locally convex spaces). □

Remark. In particular, every separable Hilbert space has $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$-partitions of unity, thus there is such a $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$-partition of functions φ subordinated to $\ell^2 \setminus A_0$ and $\ell^2 \setminus A_1$, with A_0 and A_1 mentioned in 16.4. Hence, $f := \sum_{\operatorname{carr}\varphi \cap A_0 = \emptyset} \varphi \in C^2$ satisfies $f|_{A_j} = j$ for $j = 0, 1$. However, $f \notin \mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$. The reason behind this is that $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$ is not a sheaf.

Open problem. *Classically, one proves the existence of continuous partitions of unity from the paracompactness of the space. So the question arises whether* *theorem 16.10 can be strengthened to: If the initial topology with respect to $\mathcal{S}$ is paracompact, do there exist $\mathcal{S}$-partitions of unity? Or equivalently: Is every paracompact $\mathcal{S}$-regular space $\mathcal{S}$-paracompact?*

16.11. Theorem. Smoothness of separable Banach spaces. *Let E be a separable Banach space. Then the following conditions are equivalent.*

(1) *E has a C^1-norm;*
(2) *E has C^1-bump functions, i.e., E is C^1-regular;*
(3) *The C^1-functions separate closed sets, i.e., E is C^1-normal;*
(4) *E has C^1-partitions of unity, i.e., E is C^1-paracompact;*
(5) *E has no rough norm, i.e. E is Asplund;*
(6) *E' is separable.*

Proof. The implications (1) ⇒ (2) and (4) ⇒ (3) ⇒ (2) are obviously true. The implication (2) ⇒ (4) is 16.10. (2) ⇒ (5) holds by 14.9. (5) ⇒ (6) follows from 14.10 since E is separable. (6) ⇒ (1) is 13.22 together with 13.20. □

A more general result is:

16.12. Result. [John, Zizler, 1976] *Let E be a WCG Banach space. Then the following statements are equivalent:*

(1) *E is C^1-normable;*
(2) *E is C^1-regular;*
(3) *E is C^1-paracompact;*
(4) *E has norm, whose dual norm is LUR;*
(5) *E has shrinking Markuševič basis, i.e. vectors $x_i \in E$ and $x_i^* \in E'$ with $x_i^*(x_j) = \delta_{i,j}$ and the span of the x_i is dense in E and the span of x_i^* is dense in E'.*

16.13. Results.

(1) [Godefroy, Pelant, et. al., 1983] *(*[Vanderwerff, 1992]*) Let E' is WCG Banach space (or even WCD, see 53.8). Then E is C^1-regular.*
(2) [Vanderwerff, 1992] *Let K be compact with $K^{(\omega_1)} = \emptyset$. Then $C(K)$ is C^1-paracompact. Compare with 13.18.2 and 13.17.5.*
(3) [Godefroy, Troyanski, et. al., 1983] *Let E be a subspace of a WCG Banach space. If E is C^k-regular then it is C^k-paracompact. This will be proved in 16.18.*
(4) [MacLaughlin, 1992] *Let E' be a WCG Banach space. If E is C^k-regular then it is C^k-paracompact.*

16.14. Lemma. Smooth functions on $c_0(\Gamma)$. [Toruńczyk, 1973]. *The norm-topology of $c_0(\Gamma)$ has a basis which is a countable union of locally finite families of carriers of smooth functions, each of which depends locally only on finitely many coordinates.*

Proof. The open balls $B_r := \{x : \|x\|_\infty < r\}$ are carriers of such functions: In fact, similarly to 13.16 we choose a $h \in C^\infty(\mathbb{R}, \mathbb{R})$ with $h = 1$ locally around 0 and $\operatorname{carr} h = (-1, 1)$, and define $f(x) := \prod_{\gamma\in\Gamma} h(x_\gamma)$. Let

$$\mathcal{U}_{n,r,q} = \{B_r + q_1 e_{\gamma_1} + \cdots + q_n e_{\gamma_n} : \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma\}$$

where $n \in \mathbb{N}$, $r \in \mathbb{Q}$, $q \in \mathbb{Q}^n$ with $|q_i| > 2r$ for $1 \leq i \leq n$. This is the required countable family.

Claim. The union $\bigcup_{n,r,q} \mathcal{U}_{n,r,q}$ is a basis for the topology.

Let $x \in c_0(\Gamma)$ and $\varepsilon > 0$. Choose $0 < r < \frac{\varepsilon}{2}$ such that $r \neq |x_\gamma|$ for all γ (note that $|x_\gamma| \geq \varepsilon/4$ only for finitely many γ). Let $\{\gamma_1, \dots, \gamma_n\} := \{\gamma : |x_\gamma| > r\}$. For q_i with $|q_i - x_{\gamma_i}| < r$ and $|q_i| > 2r$ we have

$$x - \sum_i q_i \, e_{\gamma_i} \in B_r,$$

and hence

$$x \in B_r + \sum_{i=1}^n q_i e_{\gamma_i} \subseteq x + B_{2r} \subseteq \{y : \|y - x\|_\infty \leq \varepsilon\}.$$

Claim. Each family $\mathcal{U}_{n,r,q}$ is locally finite.

For given $x \in c_0(\Gamma)$, let $\{\gamma_1, \dots, \gamma_m\} := \{\gamma : |x_\gamma| > \frac{r}{2}\}$ and assume there exists a $y \in (x + B_{\frac{r}{2}}) \cap (B_r + \sum_{i=1}^n q_i e_{\beta_i}) \neq \emptyset$. For $y \in x + B_{\frac{r}{2}}$ we have $|y_a| < r$ for all $\gamma \notin \{\gamma_1, \dots, \gamma_m\}$ and for $y \in B_r + \sum_{i=1}^n q_i \, e_{\beta_i}$ we have $|y_\gamma| > r$ for all $\gamma \in \{\beta_1, \dots, \beta_n\}$. Hence, $\{\beta_1, \dots, \beta_n\} \subseteq \{\gamma_1, \dots, \gamma_m\}$ and $\mathcal{U}_{n,r,q}$ is locally finite. □

16.15. Theorem, Smoothly paracompact metrizable spaces . [Toruńczyk, 1973]. *Let X be a metrizable smooth space. Then the following are equivalent:*

1. *X is $\mathcal{S}$-paracompact, i.e. admits $\mathcal{S}$-partitions of unity.*
2. *X is $\mathcal{S}$-normal.*
3. *The topology of X has a basis which is a countable union of locally finite families of carriers of smooth functions.*
4. *There is a homeomorphic embedding $i : X \to c_0(A)$ for some A (with image in the unit ball) such that $ev_a \circ i$ is smooth for all $a \in A$.*

Proof. (1) ⇒ (3) Let $\mathcal{U}_n$ be the cover formed by all open balls of radius $1/n$. By (1) there exists a partition of unity subordinated to it. The carriers of these smooth functions form a locally finite refinement $\mathcal{V}_n$. The union of all $\mathcal{V}_n$ is clearly a base of the topology since that of all $\mathcal{U}_n$ is one.

(3) ⇒ (2) Let A_1 and A_2 be two disjoint closed subsets of X. Let furthermore $\mathcal{U}_n$ be a locally finite family of carriers of smooth functions such that $\bigcup_n \mathcal{U}_n$ is a basis. Let $W_n^i := \bigcup\{U \in \mathcal{U}_n : U \cap A_i = \emptyset\}$. This is the carrier of the smooth locally finite sum of the carrying functions of the U's. The family $\{W_n^i : i \in \{0,1\}, n \in \mathbb{N}\}$ forms a countable cover of X. By the argument used in the proof of 16.10 we may shrink the W_n^i to a locally finite cover of X. Then $W^1 = \bigcup_n W_n^1$ is a carrier containing A_2 and avoiding A_1. Now use 16.2.2.

(2) ⇒ (1) is lemma 16.2, since metrizable spaces are paracompact.

(3) ⇒ (4) Let $\mathcal{U}_n$ be a locally finite family of carriers of smooth functions such that $\mathcal{U} := \bigcup_n \mathcal{U}_n$ is a basis. For every $U \in \mathcal{U}_n$ let $f_U : X \to [0, \frac{1}{n}]$ be a smooth function with carrier U. We define a mapping $i : X \to c_0(\mathcal{U})$, by $i(x) = (f_U(x))_{U \in \mathcal{U}}$. It is continuous at $x_0 \in X$, since for $n \in \mathbb{N}$ there exists a neighborhood V of x_0 which meets only finitely many sets $U \in \bigcup_{k \leq 2n} \mathcal{U}_k$, and so $\|i(x) - i(x_0)\| \leq \frac{1}{n}$ for those $x \in V$ with $|f_U(x) - f_U(x_0)| < \frac{1}{n}$ for all $U \in \bigcup_{k \leq n} \mathcal{U}_k$ meeting V. The mapping i is even an embedding, since for $x_0 \in U \in \mathcal{U}$ and $x \notin U$ we have $\|i(x) - i(x_0)\| = f_U(x_0) > 0$.

(4) ⇒ (3) By 16.14 the Banach space $c_0(A)$ has a basis which is a countable union of locally finite families of carriers of smooth functions, all of which depend locally only on finitely many coordinates. The pullbacks of all these functions via i are smooth on X, and their carriers furnish the required basis. □

16.16. Corollary. Hilbert spaces are C^∞-paracompact. [Toruńczyk, 1973]. *Every space $c_0(\Gamma)$ (for arbitrary index set Γ) and every Hilbert space (not necessarily separable) is C^∞-paracompact.*

Proof. The assertion for $c_0(\Gamma)$ is immediate from 16.15. For a Hilbert space $\ell^2(\Gamma)$ we use the embedding $i : \ell^2(\Gamma) \to c_0(\Gamma \cup \{*\})$ given by

$$i(x)_\gamma = \begin{cases} x_\gamma & \text{for } \gamma \in \Gamma \\ \|x\|^2 & \text{for } \gamma = * \end{cases}$$

This is an embedding: From $\|x^n - x\|_\infty \to 0$ we conclude by Hölder's inequality that $\langle y, x^n - x\rangle \to 0$ for all $y \in \ell^2$ and hence $\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\langle x, x_n\rangle \to 2\|x\|^2 - 2\|x\|^2 = 0$. □

16.17. Corollary. *A countable product of $\mathcal{S}$-paracompact metrizable spaces is again $\mathcal{S}$-paracompact.*

Proof. By theorem 16.15 we have certain embeddings $i_n : X_n \to c_0(A_n)$ with images contained in the unit balls. We consider the embedding $i : \prod_n X_n \to c_0(\bigsqcup_n A_n)$ given by $i(x)_a = \frac{1}{n} i_n(x_n)$ for $a \in A_n$ which has the required properties for theorem 16.15. It is an embedding, since $i(x^n) \to i(x)$ if and only if $x^n_k \to x_k$ for all k (all but finitely many coordinates are small anyhow). □

16.18. Corollary. [Godefroy, Troyanski, et. al., 1983]
Let E be a Banach space with a separable projective resolution of identity, see 53.13. If E is C^k-regular, then it is C^k-paracompact.

Proof. By 53.20 there exists a linear, injective, norm 1 operator $T : E \to c_0(\Gamma_1)$ for some Γ_1 and by 53.13 projections P_α for $\omega \le \alpha \le \operatorname{dens} E$. Let $\Gamma_2 := \{\Delta : \Delta \subseteq [\omega, \operatorname{dens} E),\ \text{finite}\}$. For $\Delta \in \Gamma_2$ choose a dense sequence $(x^\Delta_n)_n$ in the unit sphere of $P_\omega(E) \oplus \bigoplus_{\alpha\in\Delta}(P_{\alpha+1} - P_\alpha)(E)$ and let $y^\Delta_n \in E'$ be such that $\|y^\Delta_n\| = 1$ and $y^\Delta_n(x^\Delta_n) = 1$. For $n \in \mathbb{N}$ let $\pi^\Delta_n : x \mapsto x - y^\Delta_n(x) x^\Delta_n$. Choose a smooth function $h \in C^\infty(E, [0, 1])$ with $h(x) = 0$ for $\|x\| \le 1$ and $h(x) = 1$ for $\|x\| \ge 2$. Let $R_\alpha := (P_{\alpha+1} - P_\alpha)/\|P_{\alpha+1} - P_\alpha\|$.

Now define an embedding as follows: Let $\Gamma := \mathbb{N}^3 \times \Gamma_2 \sqcup \mathbb{N} \times [\omega, \operatorname{dens} E) \sqcup \mathbb{N} \sqcup \Gamma_1$ and let $u : E \to c_0(\Gamma)$ be given by

$$u(x)_\gamma := \begin{cases} \frac{1}{2^{n+m+l}}\, h(m\pi^\Delta_n x) \prod_{\alpha\in\Delta} h(lR_\alpha x) & \text{for } \gamma = (m, n, l, \Delta) \in \mathbb{N}^3 \times \Gamma_2, \\ \frac{1}{2^m}\, h(mR_\alpha x) & \text{for } \gamma = (m, \alpha) \in \mathbb{N} \times [\omega, \operatorname{dens} E), \\ \frac{1}{2}\, h(\frac{x}{m}) & \text{for } \gamma = m \in \mathbb{N}, \\ T(x)_\alpha & \text{for } \gamma = \alpha \in \Gamma_1. \end{cases}$$

Let us first show that u is well-defined and continuous. We do this only for the coordinates in the first row (for the others it is easier, the third has locally even finite support).
Let $x_0 \in E$ and $0 < \varepsilon < 1$. Choose n_0 with $1/2^{n_0} < \varepsilon$. Then $|u(x)_\gamma| < \varepsilon$ for all $x \in X$ and all $\alpha = (m, n, l, \Delta)$ with $m + n + l \ge n_0$.
For the remaining coordinates we proceed as follows: We first choose $\delta < 1/n_0$. By

53.13.8 there is a finite set $\Delta_0 \in \Gamma_2$ such that $\|R_\alpha x_0\| < \delta/2$ for all $\alpha \notin \Delta_0$. For those α and $\|x - x_0\| < \delta/2$ we get

$$\|R_\alpha(x)\| \le \|R_\alpha(x_0)\| + \|R_\alpha(x - x_0)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

hence $u(x)_\gamma = 0$ for all $\gamma = (m, n, l, \Delta)$ with $m + n + l < n_0$ and $\Delta \cap ([\omega, \operatorname{dens} E \setminus \Delta_0) \ne \emptyset$.
For the remaining finitely many coordinates $\gamma = (m, n, l, \Delta)$ with $m+n+l < n_0$ and $\Delta \subseteq \Delta_0$ we may choose a $\delta_1 > 0$ such that $|u(x)_\gamma - u(x_0)_\gamma| < \varepsilon$ for all $\|x - x_0\| < \delta_1$. Thus for $\|x - x_0\| < \min\{\delta/2, \delta_1\}$ we have $|u(x)_\gamma - u(x_0)_\gamma| < 2\varepsilon$ for all $\gamma \in \mathbb{N}^3 \times \Gamma_2$ and $|u(x_0)_\gamma| \ge \varepsilon$ only for $\alpha = (m, n, l, \Delta)$ with $m + n + l < n_0$ and $\Delta \subseteq \Delta_0$.

Since T is injective, so is u. In order to show that u is an embedding let $x_\infty, x_p \in E$ with $u(x_p) \to u(x_\infty)$. Then x_p is bounded, since for $n_0 > \|x_\infty\|$ implies that $h(x_\infty/n_0) = 0$ and from $h(x_p/n_0) \to h(x_\infty/n_0)$ we conclude that $\|x_p/n_0\| \le 2$ for large p.

Now we show that for any $\varepsilon > 0$ there is a finite ε-net for $\{x_p : p \in \mathbb{N}\}$: For this we choose $m_0 > 2/\varepsilon$. By 53.13.8 there is a finite set $\Delta_0 \subseteq \Lambda(x_\infty) := \bigcup_{\varepsilon>0}\{\alpha < \operatorname{dens} E : \|R_\alpha(x_\infty)\| \ge \varepsilon\}$ and an $n_0 := n \in \mathbb{N}$ such that $\|m_0 \pi_n^{\Delta_0}(x_\infty)\| \le 1$ and hence $h(m_0\pi_n^{\Delta_0}(x_\infty)) = 0$. In fact by 53.13.9 there is a finite linear combination of vectors $R_\alpha(x_\infty)$, which has distance less than ε from x_∞, let $\delta := \min\{\|R_\alpha(x)\| :$ for those $\alpha\} > 0$. Since the $y_n^{\Delta_0}$ are dense in the unit sphere of $P_\omega \oplus \bigoplus_{\alpha\in\Delta_0} R_\alpha E$ we may choose an n such that $\|x_\infty - \|x_\infty\| x_n^{\Delta_0}\| < \frac{1}{2m_0}$ and hence

$$\begin{aligned}\|\pi_n^{\Delta_0}(x_\infty)\| &= \|x_\infty - y_n^{\Delta_0}(x_\infty) x_n^{\Delta_0}\| \\ &\le \Big\| x_\infty - \|x_\infty\| x_n^{\Delta_0}\Big\| + \|x_\infty\| \Big\| x_n^{\Delta_0} - y_n^{\Delta_0}(x_n^{\Delta_0}) x_n^{\Delta_0}\Big\| \\ &\qquad + \|y_n^{\Delta_0}\| \Big\| \|x_\infty\| x_n^{\Delta_0} - x_\infty)\Big\| \, \|x_n^{\Delta_0}\| \\ &\le \frac{1}{2m_0} + 0 + \frac{1}{2m_0} = \frac{1}{m_0}\end{aligned}$$

Next choose $l_0 := l \in \mathbb{N}$ such that $l_0\delta_0 \ge 2$ and hence $\|l_0 R_\alpha x_\infty\| \ge 2$ for all $\alpha \in \Delta_0$. Then

$$h(m_0\pi_{n_0}^{\Delta_0} x_p) \prod_{\alpha\in\Delta_0} h(l_0\, R_\alpha x_p) \to h(m_0 \pi_{n_0}^{\Delta_0} x_\infty) \prod_{\alpha\in\Delta_0} h(l_0\, R_\alpha x_\infty)$$
$$\text{and} \qquad h(l_0 R_\alpha x_p) \to h(l_0 R_\alpha x_\infty) = 1 \text{ for } \alpha \in \Delta_0$$

Hence

$$h(m_0\pi_{n_0}^{\Delta_0} x_p) \to h(m_0\pi_{n_0}^{\Delta_0} x_\infty) = 0,$$

and so $\|\pi_{n_0}^{\Delta_0} x_p\| \le 2/m_0 < \varepsilon$ for all large p. Thus $d(x_p, \mathbb{R}\, x_{n_0}^{\Delta_0}) \le \varepsilon$, hence $\{x_p : p \in \mathbb{N}\}$ has a finite ε-net, since its projection onto the one dimensional subspace $\mathbb{R} x_{n_0}^{\Delta_0}$ is bounded.

Thus $\{x_\infty, x_p : p \in \mathbb{N}\}$ is relatively compact, and hence u restricted to its closure is a homeomorphism onto the image. So $x_p \to x_\infty$.

Now the result follows from 16.15. □

16.19. Corollary. [Deville, Godefroy, Zizler, 1990]. *Let $c_0(\Gamma) \to E \to F$ be a short exact sequence of Banach spaces and assume F admits C^p-partitions of unity. Then E admits C^p-partitions of unity.*

Proof. Without loss of generality we may assume that the norm of E restricted to $c_0(\Gamma)$ is the supremum norm. Furthermore there is a linear continuous splitting

$T : \ell^1(\Gamma) \to E'$ by 13.17.3 and a continuous splitting $S : F \to E$ by 53.22 with $S(0) = 0$. We put $T_\gamma := T(e_\gamma)$ for all $\gamma \in \Gamma$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be a C^p-partition of unity on F with $\operatorname{diam}(\operatorname{carr}(f)) \le 1/n$ for all $f \in \mathcal{F}_n$. Let $\mathcal{F} := \bigsqcup_n \mathcal{F}_n$ and let $\Gamma_2 := \{\Delta \subseteq \Gamma : \Delta \text{ is finite}\}$. For any $f \in \mathcal{F}$ choose $x_f \in S(\operatorname{carr}(f))$ and for any $\Delta \in \Gamma_2$ choose a dense sequence $\{y^\Delta_{f,m} : m \in \mathbb{N}\} \ni 0$ in the linear subspace generated by $\{x_f + e_\gamma : \gamma \in \Delta\}$. Let $\ell^\Delta_{f,m} \in E'$ be such that $\ell^\Delta_{f,m}(y^\Delta_{f,m}) = \|\ell^\Delta_{f,m}\| \cdot \|y^\Delta_{f,m}\| = 1$. Let $\pi^\Delta_{f,m} : E \to E$ be given by $\pi^\Delta_{f,m}(x) := x - \ell^\Delta_{f,m}(x)\, y^\Delta_{f,m}$. Let $h : E \to \mathbb{R}$ be C^p with $h(x) = 0$ for $\|x\| \le 1$ and $h(x) = 1$ for $\|x\| \ge 2$. Let $g : \mathbb{R} \to [-1, 1]$ be C^p with $g(t) = 0$ for $|t| \le 1$ and injective on $\{t : |t| > 1\}$. Now define a mapping $u : E \to c_0(\tilde\Gamma)$, where

$$\tilde\Gamma := (\mathcal{F} \times \Gamma_2 \times \mathbb{N}^2) \sqcup (\mathcal{F} \times \Gamma) \sqcup (\mathcal{F} \times \mathbb{N}) \sqcup \sqcup \mathbb{N} \sqcup \mathbb{N}$$

by

$$u(x)_{\tilde\gamma} := \frac{1}{2^{n+m+j}}\, f(\hat x)\, h(j\, \pi^\Delta_{f,m}(x)) \prod_{\gamma\in\Delta} g(n\, T_\gamma(x - x_f))$$

for $\tilde\gamma = (f, \Delta, j, m) \in \mathcal{F}_n \times \Gamma_2 \times \mathbb{N}^2$, and by

$$u(x)_{\tilde\gamma} := \begin{cases} \frac{1}{2^n}\, f(\hat x)\, g(n\, T_\gamma(x - x_f)) & \text{for } \tilde\gamma = (f,\gamma) \in \mathcal{F}_n \times \Gamma \\ \frac{1}{2^{n+j}}\, f(\hat x)\, h(j\,(x - x_f)) & \text{for } \tilde\gamma = (f,j) \in \mathcal{F}_n \times \mathbb{N} \\ \frac{1}{2^n}\, f(\hat x) & \text{for } \tilde\gamma = f \in \mathcal{F}_n \subseteq \mathcal{F} \\ \frac{1}{2^n}\, h(n\,x) & \text{for } \tilde\gamma = n \in \mathbb{N} \\ \frac{1}{2^n}\, h(x/n) & \text{for } \tilde\gamma = n \in \mathbb{N}. \end{cases}$$

We first claim that u is well-defined and continuous. Every coordinate $x \mapsto u(x)_\gamma$ is continuous, so it remains to show that for every $\varepsilon > 0$ locally in x the set of coordinates γ, where $|u(x)_\gamma| > \varepsilon$ is finite. We do this for the first type of coordinates. For this we may fix n, m and j (since the factors are bounded by 1). Since $\mathcal{F}_n$ is a partition of unity, locally $f(\hat x) \ne 0$ for only finitely many $f \in \mathcal{F}_n$, so we may also fix $f \in \mathcal{F}_n$. For such an f the set $\Delta_0 := \{\gamma : |T_\gamma(x - x_f)| \ge \pi(x - x_f) + \frac{1}{n}\}$ is finite by the proof of 13.17.3. Since $\|\hat x - x_f\| = \|\pi(x - x_f)\| \le 1/n$ be have $g(n\, T_\gamma(x - x_f)) = 0$ for $\gamma \notin \Delta_0$.

Thus only for those Δ contained in the finite set Δ_0, we have that the corresponding coordinate does not vanish.

Next we show that u is injective. Let $x \ne y \in E$.
If $\hat x \ne \hat y$, then there is some n and a $f \in \mathcal{F}_n$ such that $f(\hat x) \ne 0 = f(\hat y)$. Thus this is detected by the 4th row.
If $\hat x = \hat y$ then $S\hat x = S\hat y$ and since $x - S\hat x$, $y - S\hat y \in c_0(\Gamma)$ there is a $\gamma \in \Gamma$ with

$$T_\gamma(x - S\hat x) = (x - S\hat x)_\gamma \ne (y - S\hat y)_\gamma = T_\gamma(y - S\hat y).$$

We will make use of the following method repeatedly:
For every n there is a $f_n \in \mathcal{F}_n$ with $f_n(\hat x) \ne 0$ and hence $\|\hat x - \hat x_{f_n}\| \le 1/n$. Since S is continuous we get $x_{f_n} = S(\hat x_{f_n}) \to S(\hat x)$ and thus $\lim_n T_\gamma(x - x_{f_n}) = \lim_n T_\gamma(x - S(\hat x_{f_n})) = T_\gamma(x - S(\hat x))$.
So we get

$$\lim_n T_\gamma(x - x_{f_n}) = T_\gamma(x - S(\hat x)) \ne T_\gamma(y - S(\hat y)) = \lim_n T_\gamma(y - x_{f_n}).$$

If all coordinates for $u(x)$ and $u(y)$ in the second row would be equal, then

$$g(n\, T_\gamma(x - x_f)) = g(n\, T_\gamma(y - x_f))$$

since $f_\gamma(\hat x) \ne 0$, and hence $\|T_\gamma(x - x_f) - T_\gamma(y - x_f)\| \le 2/n$, a contradiction.

Now let us show that u is a homeomorphism onto its image. We have to show $x_k \to x$ provided $u(x_k) \to u(x)$.
We consider first the case, where $x = S\hat{x}$. As before we choose $f_n \in \mathcal{F}_n$ with $f_n(\hat{x}) \neq 0$ and get $x_{f_n} = S(\hat{x}_{f_n}) \to S(\hat{x}) = x$. Let $\varepsilon > 0$ and $j > 3/\varepsilon$. Choose an n such that $\|x_{f_n} - x\| < 1/j$. Then $h(j\,(x_{f_n} - x)) = 0$. From the coordinates in the third and fourth row we conclude

$$f(\hat{x}_k)\,h(j\,(x_k - x_{f_n})) \to f(\hat{x})\,h(j\,(x - x_{f_n})) \quad \text{and} \quad f(\hat{x}_k) \to f(\hat{x}) \neq 0.$$

Hence

$$h(j\,(x_k - x_{f_n})) \to h(j\,(x - x_{f_n})) = 0.$$

Thus $\|x_k - x_{f_n}\| < 2/j$ for all large k. But then

$$\|x_k - x\| \le \|x_k - x_{f_n}\| + \|x_{f_n} - x\| < \frac{3}{j} < \varepsilon,$$

i.e. $x_k \to x$.
Now the case, where $x \neq S\hat{x}$. We show first that $\{x_k : k \in \mathbb{N}\}$ is bounded. Pick $n > \|x\|$. From the coordinates in the last row we get that $\lim_k h(x_k/n) = 0$, i.e. $\|x_k\| \le 2n$ for all large k.
We claim that for $j \in \mathbb{N}$ there is an $n \in \mathbb{N}$ and an $f \in \mathcal{F}_n$ with $f(\hat{x}) \neq 0$, a finite set $\Delta \subseteq \Gamma$ with $\prod_{\gamma\in\Delta} g(n\,T_\gamma(x - x_f)) \neq 0$ and an $m \in \mathbb{N}$ with $h(j\,\pi^\Delta_{f,m}(x)) = 0$.
From $0 \neq (x - S\hat{x}) \in c_0(\Gamma)$ we deduce that there is a finite set $\Delta \subseteq \Gamma$ with $T_\gamma(x - S\hat{x}) = (x - S\hat{x})_\gamma \neq 0$ for all $\gamma \in \Delta$ and $\operatorname{dist}(x - S\hat{x}, \langle e_\gamma : \gamma \in \Delta\rangle) < 1/(3j)$, i.e. $|(x - S\hat{x})_\gamma| \le 1/(3j)$ for all $\gamma \notin \Delta$. As before we choose $f_n \in \mathcal{F}_n$ with $f_n(\hat{x}) \neq 0$ and get $x_{f_n} = S(\hat{x}_{f_n}) \to S(\hat{x})$ and

$$\lim_n T_\gamma(x - x_{f_n}) = (x - S\hat{x})_\gamma \neq 0 \text{ for } \gamma \in \Delta.$$

Thus $g(n\,(T_\gamma(x - x_{f_n}))) \neq 0$ for all large n and $\gamma \in \Delta$. Furthermore, $\operatorname{dist}(x, x_{f_n} + \langle e_\gamma : \gamma \in \Delta\rangle) = \operatorname{dist}(x - x_{f_n}, \langle e_\gamma : \gamma \in \Delta\rangle) < 1/(2j)$. Since $\{y^\Delta_{f_n,m} : m \in \mathbb{N}\}$ is dense in $\langle x_{f_n} + e_\gamma : \gamma \in \Delta\rangle$ there is an m such that $\|x - y^\Delta_{f_n,m}\| < 1/(2j)$. Since $\|\pi^\Delta_{f_n,m}\| \le 2$ we get

$$\begin{aligned}\|\pi^\Delta_{f_n,m}(x)\| &\le \|x - y^\Delta_{f_n,m}\| + |1 - \ell^\Delta_{f_n,m}(x)|\,\|y^\Delta_{f_n,m}\| \\ &\le \frac{1}{2j} + \|\ell^\Delta_{f_n,m}\|\,\|x - y^\Delta_{f_n,m}\|\,\|y^\Delta_{f_n,m}\| \le \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j},\end{aligned}$$

hence $h(j\,\pi^\Delta_{f_n,m}(x)) = 0$.

We claim that for every $\varepsilon > 0$ there is a finite ε-net of $\{x_k : k \in \mathbb{N}\}$. Let $\varepsilon > 0$. We choose $j > 4/\varepsilon$ and we pick $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $\Delta \subseteq \Gamma$ finite, and $m \in \mathbb{N}$ satisfying the previous claim. From $u(x_k) \to u(x)$ we deduce from the coordinates in the first row, that

$$\begin{aligned}&f(\hat{x}_k)\,h(j\,\pi^\Delta_{f,m}(x_k)) \prod_{\gamma\in\Delta} g(n\,T_\gamma(x_k - x_f)) \to \\ &\qquad \to f(\hat{x})\,h(j\,\pi^\Delta_{f,m}(x)) \prod_{\gamma\in\Delta} g(n\,T_\gamma(x - x_f)) \text{ for } k \to \infty\end{aligned}$$

and since by the coordinates in the fourth row $f(\hat{x}_k) \to f(\hat{x}) \neq 0$ we obtain from the coordinates in the second row, that

$$g(n\,T_\gamma(x_k - x_f)) \to g(n\,T_\gamma(x - x_f)) \neq 0 \text{ for } \gamma \in \Delta.$$

Hence

$$h(j\,\pi^\Delta_{f,m}(x_k)) \to h(j\,\pi^\Delta_{f,m}(x)) = 0.$$

Therefore

$$\|x_k - \ell^{\Delta}_{f,m}(x_k)\, y^{\Delta}_{f,n}\| = \|\pi^{\Delta}_{f,m}(x_k)\| < \frac{1}{j} < \frac{\varepsilon}{4} \text{ for all large } k.$$

Thus there is a finite dimensional subspace in E spanned by $y^{\Delta}_{f,n}$ and finitely many x_k, such that all x_k have distance $\le \varepsilon/4$ from it. Since $\{x_k : k \in \mathbb{N}\}$ are bounded, the compactness of the finite dimensional balls implies that $\{x_k : k \in \mathbb{N}\}$ has an ε-net, hence $\{x_k : k \in \mathbb{N}\}$ is relatively compact, and since u is injective we have $\lim_k x_k = x$.

Now the result follows from 16.15. □

Remark. In general, the existence of C^∞-partitions of unity is not inherited by the middle term of short exact sequences: Take a short exact sequence of Banach spaces with Hilbert ends and non-Hilbertizable E in the middle, as in 13.18.6. If both E and E^* admitted C^2-partitions of unity, then they would admit C^2-bump functions, hence E was isomorphic to a Hilbert space by [Meshkov, 1978], a contradiction.

16.20. Results on $C(K)$. *Let K be compact. Then for the Banach space $C(K)$ we have:*

1. (1) [Deville, Godefroy, Zizler, 1990]. *If $K^{(\omega)} = \emptyset$ then $C(K)$ is C^∞-paracompact.*
2. (2) [Vanderwerff, 1992] *If $K^{(\omega_1)} = \emptyset$ then $C(K)$ is C^1-paracompact.*
3. (3) [Haydon, 1990] *In contrast to (2) there exists a compact space K with $K^{(\omega_1)} = \{*\}$, but such that $C(K)$ has no Gâteaux-differentiable norm. Nevertheless $C(K)$ is C^1-regular by* [Haydon, 1991]. *Compare with 13.18.2.*
4. (4) [Namioka, Phelps, 1975]. *If there exists an ordinal number α with $K^{(\alpha)} = \emptyset$ then the Banach space $C(K)$ is Asplund (and conversely), hence it does not admit a rough norm, by 13.8.*
5. (5) [Ciesielski, Pol, 1984] *There exists a compact K with $K^{(3)} = \emptyset$. Consequently, there is a short exact sequence $c_0(\Gamma_1) \to C(K) \to c_0(\Gamma_2)$, and the space $C(K)$ is Lipschitz homeomorphic to some $c_0(\Gamma)$. However, there is no continuous linear injection of $C(K)$ into some $c_0(\Gamma)$.*

Notes. (1) Applying theorem 16.19 recursively we get the result as in 13.17.5.

16.21. Some radial subsets are diffeomorphic to the whole space. We are now going to show that certain subsets of convenient vector spaces are diffeomorphic to the whole space. So if these subsets form a base of the c^∞-topology of the modeling space of a manifold, then we may choose charts defined on the whole modeling space. The basic idea is to 'blow up' subsets $U \subseteq E$ along all rays starting at a common center. Without loss of generality assume that the center is 0. In order for this technique to work, we need a positive function $\rho : U \to \mathbb{R}$, which should give a diffeomorphism $f : U \to E$, defined by $f(x) := \frac{1}{\rho(x)}x$. For this we need that ρ is smooth, and since the restriction of f to $U \cap \mathbb{R}^+ x \to \mathbb{R}^+ x$ has to be a diffeomorphism as well, and since the image set is connected, we need that the domain $U \cap \mathbb{R}^+ x$ is connected as well, i.e., U has to be radial. Let $U_x := \{t > 0 : tx \in U\}$, and let $f_x : U_x \to \mathbb{R}$ be given by $f(tx) = \frac{t}{\rho(tx)}x =: f_x(t)x$. Since up to diffeomorphisms this is just the restriction of the diffeomorphism f, we need that $0 < f_x'(t) = \frac{\partial}{\partial t}\frac{t}{\rho(tx)} = \frac{\rho(tx) - t\rho'(tx)(x)}{\rho(tx)^2}$ for all $x \in U$ and $0 < t \le 1$. This means that $\rho(y) > \rho'(y)(y)$ for all $y \in U$, which is quite a restrictive condition, and so we want to construct out of an arbitrary smooth function $\rho : U \to \mathbb{R}$, which tends to 0 towards the boundary, a new smooth function ρ satisfying the additional assumption.

Theorem. *Let $U \subseteq E$ be c^∞-open with $0 \in U$ and let $\rho : U \to \mathbb{R}^+$ be smooth, such that for all $x \notin U$ with $tx \in U$ for $0 \le t < 1$ we have $\rho(tx) \to 0$ for $t \nearrow 1$. Then $\operatorname{star} U := \{x \in U : tx \in U$ for all $t \in [0,1]\}$ is diffeomorphic to E.*

Proof. First remark that $\operatorname{star} U$ is c^∞-open. In fact, let $c : \mathbb{R} \to E$ be smooth with $c(0) \in \operatorname{star} U$. Then $\varphi : \mathbb{R}^2 \to E$, defined by $\varphi(t,s) := t\,c(s)$ is smooth and maps $[0,1] \times \{0\}$ into U. Since U is c^∞-open and $\mathbb{R}^2$ carries the c^∞-topology there exists a neighborhood of $[0,1] \times \{0\}$, which is mapped into U, and in particular there exists some $\varepsilon > 0$ such that $c(s) \in \operatorname{star} U$ for all $|s| < \varepsilon$. Thus $c^{-1}(\operatorname{star} U)$ is open, i.e., $\operatorname{star} U$ is c^∞-open. Note that ρ satisfies on $\operatorname{star} U$ the same boundary condition as on U. So we may assume without loss of generality that U is radial. Furthermore, we may assume that $\rho = 1$ locally around 0 and $0 < \rho \le 1$ everywhere, by composing with some function which is constantly 1 locally around $[\rho(0), +\infty)$.

Now we are going to replace ρ by a new function $\tilde\rho$, and we consider first the case, where $E = \mathbb{R}$. We want that $\tilde\rho$ satisfies $\tilde\rho'(t)t < \tilde\rho(t)$ (which says that the tangent to $\tilde\rho$ at t intersects the $\tilde\rho$-axis in the positive part) and that $\tilde\rho(t) \le \rho(t)$, i.e., $\log \circ \tilde\rho \le \log \circ \rho$, and since we will choose $\tilde\rho(0) = 1 = \rho(0)$ it is sufficient to have $\frac{\tilde\rho'}{\tilde\rho} = (\log \circ \tilde\rho)' \le (\log \circ \rho)' = \frac{\rho'}{\rho}$ or equivalently $\frac{\tilde\rho'(t)t}{\tilde\rho(t)} \le \frac{\rho'(t)t}{\rho(t)}$ for $t > 0$. In order to obtain this we choose a smooth function $h : \mathbb{R} \to \mathbb{R}$ which satisfies $h(t) < 1$, and $h(t) \le t$ for all t, and $h(t) = t$ for t near 0, and we take $\tilde\rho$ as solution of the following ordinary differential equation

$$\tilde\rho'(t) = \frac{\tilde\rho(t)}{t} \cdot h\left(\frac{\rho'(t)t}{\rho(t)}\right) \text{ with } \tilde\rho(0) = 1.$$

Note that for t near 0, we have $\frac{1}{t}h\left(\frac{\rho'(t)t}{\rho(t)}\right) = \frac{\rho'(t)}{\rho(t)}$, and hence locally a unique smooth solution $\tilde\rho$ exists. In fact, we can solve the equation explicitly, since $(\log \circ \tilde\rho)'(t) = \frac{\tilde\rho'(t)}{\tilde\rho(t)} = \frac{1}{t} \cdot h\left(\frac{\rho'(t)t}{\rho(t)}\right)$, and hence $\tilde\rho(s) = \exp(\int_0^s \frac{1}{t} \cdot h(\frac{\rho'(t)t}{\rho(t)})\,dt)$, which is smooth on the same interval as ρ is.

Note that if ρ is replaced by $\rho_s : t \mapsto \rho(ts)$, then the corresponding solution $\widetilde{\rho_s}$ satisfies $\widetilde{\rho_s} = \tilde\rho_s$. In fact,

$$(\log \circ \tilde\rho_s)'(t) = \frac{(\tilde\rho_s)'(t)}{\tilde\rho_s(t)} = \frac{s\tilde\rho'(st)}{\tilde\rho(st)} = \frac{1}{t} \cdot \frac{st\tilde\rho'(st)}{\tilde\rho(st)} = \frac{1}{t}h\Big(\frac{st\rho'(st)}{\rho(st)}\Big) = \frac{1}{t}h\Big(\frac{t(\rho_s)'(t)}{\rho_s(t)}\Big).$$

For arbitrary E and $x \in E$ let $\rho_x : U_x \to \mathbb{R}^+$ be given by $\rho_x(t) := \rho(tx)$, and let $\tilde\rho : U \to \mathbb{R}^+$ be given by $\tilde\rho(x) := \widetilde{\rho_x}(1)$, where $\widetilde{\rho_x}$ is the solution of the differential equation above with ρ_x in place of ρ.

Let us now show that $\tilde\rho$ is smooth. Since U is c^∞-open, it is enough to consider a smooth curve $x : \mathbb{R} \to U$ and show that $t \mapsto \tilde\rho(x(t)) = \tilde\rho_{(x(t))}(1)$ is smooth. This is the case, since $(t,s) \mapsto \frac{1}{s}h\left(\frac{\rho'_{x(t)}(s)s}{\rho_{x(t)}(s)}\right) = \frac{1}{s}h\left(\frac{\rho'(s\,x(t))(s\,x(t))}{\rho(s\,x(t))}\right)$ is smooth, since $\varphi(t,s) := \frac{\rho'(s\,x(t))(s\,x(t))}{\rho(s\,x(t))}$ satisfies $\varphi(t,0) = 0$, and hence $\frac{1}{s}h(\varphi(t,s)) = \frac{\varphi(t,s)}{s} = \frac{\rho'(s\,x(t))(x(t))}{\rho(s\,x(t))}$ locally.

From $\rho_{sx}(t) = \rho(tsx) = \rho_x(ts)$ we conclude that $\widetilde{\rho_{sx}}(t) = \widetilde{\rho_x}(ts)$, and hence $\tilde\rho(sx) = \widetilde{\rho_x}(s)$. Thus, $\tilde\rho'(x)(x) = \frac{\partial}{\partial t}|_{t=1}\tilde\rho(tx) = \frac{\partial}{\partial t}|_{t=1}\tilde\rho_x(t) = \tilde\rho'_x(1) < \tilde\rho_x(1) = \tilde\rho(x)$. This shows that we may assume without loss of generality that $\rho : U \to (0,1]$ satisfies the additional assumption $\rho'(x)(x) < \rho(x)$.

Note that $f_x : t \mapsto \frac{t}{\rho(tx)}$ is bijective from $U_x := \{t > 0 : tx \in U\}$ to $\mathbb{R}^+$, since 0 is mapped to 0, the derivative is positive, and $\frac{t}{\rho(tx)} \to \infty$ if either $\rho(tx) \to 0$ or $t \to \infty$ since $\rho(tx) \leq 1$.

It remains to show that the bijection $x \mapsto \frac{1}{\rho(x)}x$ is a diffeomorphism. Obviously, its inverse is of the form $y \mapsto \sigma(y)y$ for some $\sigma : E \to \mathbb{R}^+$. They are inverse to each other so $\frac{1}{\rho(\sigma(y)y)}\sigma(y)y = y$, i.e., $\sigma(y) = \rho(\sigma(y)y)$ for $y \neq 0$. This is an implicit equation for σ. Note that $\sigma(y) = 1$ for y near 0, since ρ has this property. In order to show smoothness, let $t \mapsto y(t)$ be a smooth curve in E. Then it suffices to show that the implicit equation $(\sigma \circ y)(t) = \rho((\sigma \circ y)(t) \cdot y(t))$ satisfies the assumptions of the 2-dimensional implicit function theorem, i.e., $0 \neq \frac{\partial}{\partial \sigma}(\sigma - \rho(\sigma \cdot y(t))) = 1 - \rho'(\sigma \cdot y(t))(y(t))$, which is true, since multiplied with $\sigma > 0$ it equals $\sigma - \rho'(\sigma \cdot y(t))(\sigma \cdot y(t)) < \sigma - \rho(\sigma \cdot y(t)) = 0$. □

Bibliography

[1] Ralph Abraham. *Lectures of Smale on differential topology*. Columbia University, New York, 1962. Lecture Notes.

[2] E. Adam, P. Biström, and A. Kriegl. Countably evaluating homomorphisms on real function algebras. was article, Preprint, 1995.

[3] Eva Adam. Smoothly real compact spaces. Master's thesis, Universität Wien, 1993.

[4] Eva Adam. *Approximation properties and spaces of smooth functions*. PhD thesis, Universität Wien, Vienna, 1995.

[5] M. Adams, T. Ratiu, and R. Schmid. *The Lie group structure of diffeomorphism groups and invertible Fourier Integral operators*, pages 1–69. Springer-Verlag, New York, 1985.

[6] M. Adams, T. Ratiu, and R. Schmid. A Lie group structure for Fourier integral operators. *Math. Ann.*, 276:19–41, 1986.

[7] M. Adams, T. Ratiu, and R. Schmid. A Lie group structure for pseudo differential operators. *Math. Ann.*, 273:529–551, 1986.

[8] I. Aharoni and J. Lindenstrauss. Uniform equivalence between Banach spaces. *Bull. Am. Math. Soc.*, 84:281–283, 1978.

[9] Dimitri Alekseevsky and Peter W. Michor. Differential geometry of Cartan connections. *Publ. Math.*, 47:349–375, 1995.

[10] Dimitri Alekseevsky and Peter W. Michor. Differential geometry of $\mathfrak{g}$-manifolds. *Differ. Geom. Appl.*, 5:371–403, 1995.

[11] Dmitri Alekseevsky, Andreas Kriegl, Mark Losik, and Peter W. Michor. Choosing roots of polynomials smoothly. was article, ESI preprint 314

[12] W Ambrose and Singer I.M. Palais, R.S. Sprays. *An. Acad. Bras. Cienc.*, 32:163–178, 1960.

[13] D. Amir and J. Lindenstrauss. The structure of weakly compact sets in Banach spaces. *Ann. Math.*, 88:35–46, 1968.

[14] Mercourakis S. Negrepontis S. Argyros, S. Functional-analytic properties of Corson-compact spaces. *Stud. Math.*, 89:197–229, 1988.

[15] Juan Arias-de Reyna. A real valued homomorphism on algebras of differentiable functions. *Proc. Am. Math. Soc.*, 104, No. 4:1054–1058, 1988.

[16] V.I. Arnold. An a priori estimate in the theory of hydrodynamic stability. *Izv. Vyssh. Uchebn. Zaved. Mat.*, 54,5:3–5, 1966. Russian.

[17] V.I. Arnold. Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:319–361, 1966.

[18] V.I. Arnold. *Mathematical methods of classical mechanics*. Springer-Verlag, New York, Heidelberg, 1978. Graduate Texts in Math. 60.

[19] R.M. Aron. *Compact polynomials and compact differentiable mappings between Banach spaces*, pages 231–222. Springer-Verlag, 1976. Lecture notes in Math. 524.

[20] N. Aronszajn. Differentiability of Lipschitzian mappings between Banach spaces. *Stud. Math.*, 57:147–190, 1976.

[21] E. Asplund. Fréchet differentiability of convex functions. *Acta Math.*, 121:31–48, 1968.

[22] M. Atiyah. Complex analytic connections on fiber bundles. *Trans. Am. Math. Soc.*, 85:181–207, 1957.

[23] V.I. Averbukh and O.G. Smolyanov. The various definitions of the derivative in linear topological spaces. *Russ. Math. Surv.*, 23:67–113, 1968.

[24] M. Balanzat. La différentielle d'hadamard-fréchet dans les espaces vectoriels topologiques. *C. R. Acad. Sci. Paris*, 251:2459–2461, 1960.

[25] M.Manuel Balanzat. La différential en les espacios métricas affines. *Math. Notae*, 9:29–51, 1949.

[26] S. Banach. Théorie des opérations linéaires. was book,, 1932.

[27] W. Banaszczyk. On the existence of exotic Banach Lie groups. *Math. Ann.*, 264:485–493, 1983.

[28] W. Banaszczyk. Characterization of nuclear spaces by means of additive subgroups. *Math. Z.*, 186:125–133, 1984.

[29] W. Banaszczyk. Pontryagin duality for subgroups and quotients of nuclear spaces. *Math. Ann.*, 273:653–664, 1986.

[30] W. Banaszczyk. *Additive subgroups of topological vector spaces*. Springer-Verlag, 1991. Lecture Notes in Math. 1466.

[31] Augustin Banyaga. Sur la structure du groupe des diffeomorphismes qui preservent une forme symplectique. *Comment. Math. Helv.*, 53:174–227, 1978.

[32] Augustin Banyaga. On fixed points of symplectic maps. *Invent. Math.*, 56:215–229, 1980.

[33] R.G. Bartle and L.M. Graves. Mappings between function spaces. *Trans. Am. Math. Soc.*, 72:400–413, 1952.

[34] Andrée Bastiani. Applications différentiables et variétés différentiables de dimension infinie. *J. Anal. Math.*, 13:1–114, 1964.

[35] Hellmut Baumgärtel. *Endlichdimensionale analytische Störungstheorie*. Akademie-Verlag, Berlin, 1972.

[36] J. Bemelmans, St. Hildebrandt, and W. von Wahl. *Partielle Differentialgleichungen und Variantionsrechnung*, pages 149–230. Vieweg, Braunschweig, Wiesbaden, 1990. Festschrift zum Jubiläum der DMV.

[37] Czesłav Bessaga. Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere. *Bull. Pol. Acad. Sci. Math.*, 14:27–31, 1966.

[38] Czesłav Bessaga and Aleksander Pełczyński. *Selected topics in infinite dimensional topology*. Polish Scientific Publishers, Warszawa, 1975. Monografie Matematyczne, Tom 58.

[39] E. Binz. Two natural metrics and their covariant derivatives on a manifold of embeddings. *Monatsh. Math.*, 89:275–288, 1980.

[40] Ernst Binz. Ein Differenzierbakeitsbegriff in limitierten Vektorräumen. *Comment. Math. Helv.*, 41:137–156, 1966.

[41] Ernst Binz and Hans R. Fischer. The manifold of embeddings of a closed manifold. was inbook,, 1981.

[42] P. Biström, S. Bjon, and M. Lindström. Function algebras on which homomorphisms are point evaluations on sequences. *Manuscr. Math.*, 73:179–185, 1991.

[43] P. Biström, S. Bjon, and M. Lindström. Remarks on homomorphisms on certain subalgebras of $c(x)$. *Math. Jap.*, 37:105–109, 1992.

[44] P. Biström, S. Bjon, and M. Lindström. On c^m-bounding sets. *J. Aust. Math. Soc.*, 54:20–28, 1993.

[45] P. Biström and J.A. Jaramillo. c^∞-bounding sets and compactness. *Math. Scand.*, 75:82–86, 1994.

[46] P. Biström, J.A. Jaramillo, and M. Lindström. Algebras of real analytic functions; Homomorphisms and bounding sets. *Stud. Math.*, 115:23–37, 1995.

[47] P. Biström and M. Lindström. Characterization of the spectra of certain function algebras. *Arch Math.*, 60:177–181, 1993.

[48] P. Biström and M. Lindström. Homomorphisms on $c^\infty(e)$ and c^∞-bounding sets. *Monatsh. Math.*, 115:257–266, 1993.

[49] Peter Biström. *The homomorphisms on algebras of functions defined on locally convex spaces and bounding sets*. PhD thesis, University of Åbo, 1993.

[50] Peter Biström, Sten Bjon, and Mikael Lindström. Homomorphisms on some function algebras. *Monatsh. Math.*, 111:5, 1990.

[51] Jacek Bochnak and Józef Siciak. Analytic functions in topological vector spaces. *Stud. Math.*, 39:77–112, 1971.

[52] Bockstein. Une théorème de séparabilité pour les produits topologiques Math. was article,, 1948.

[53] O. Bolza. Vorlesungen über Variationsrechnung. was book, transl. Second edition, 1909 1962.

[54] Jan Boman. Differentiability of a function and of its compositions with functions of one variable. *Math. Scand.*, 20:249–268, 1967.

[55] R. Bonic and J. Frampton. Differentiable functions on certain Banach spaces. *Bull. Am. Math. Soc.*, 71:393–395, 1965.

[56] R. Bonic and J. Frampton. Smooth functions on Banach manifolds. *J. Math. Mech.*, 15:877–898, 1966.

[57] R. Bonic and F. Reis. A characterization of Hilbert space. *An. Acad. Bras. Cienc.*, 38:239–241, 1966.

[58] W.M. Boothby. The transitivity of the automorphisms of certain geometric structures. *Trans. Am. Math. Soc.*, 137:93–100, 1969.

[59] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. Math.*, 75:485–535, 1962.

[60] A. Borel and J. P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973.

[61] J. Borwein, S. Fitzpatrick, and P. Kenderov. Minimal convex uscos and monotone operators on small sets. *Can. J. Math.*, 43:461–476, 1991.

[62] N. Bourbaki. *General Topology.* Hermann, Paris, 1966.

[63] Jean-Pierre Bourguignon. Une stratification de l'espace des structures Riemanniens. *Compos. Math.*, 30:1–41, 1975.

[64] Theodor Bröcker and Klaus Jänich. *Einführung in die Differentialtopologie.* Springer-Verlag, Heidelberg, 1973. Heidelberger Taschenbücher 143.

[65] W. Browder. Diffeomorphisms of 1-connected manifolds. *Trans. Am. Math. Soc.*, 128:155–163, 1967.

[66] Ronald Brown. Some problems of algebraic topology. was book, Ph. D. Thesis, 1961.

[67] Ronald Brown. Ten topologies for $x\times y$. *Q. J. Math. Oxf.*, 14:303–319, 1963.

[68] Ronald Brown. Function spaces and product topologies. *Q. J. Math. Oxf.*, 15:238–250, 1964.

[69] Henri Buchwalter. Espaces vectoriels bornologiques. *Publ. Dep. Math. Lyon*, 2:1–53, 1965.

[70] Andreas Cap, Andreas Kriegl, Peter W. Michor, and Ji˘
ri Van˘
zura. The Frölicher-nijenhuis bracket in non commutative differential geometry. *Acta Math. Univ. Comenianae*, 62:17–49

[71] T.K. Carne, B. Cole, and T.W. Gamelin. A uniform algebra of analytic functions on a Banach space. *Trans. Am. Math. Soc.*, 314:639–659, 1989.

[72] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton University Press, Princeton, 1956.

[73] B. Cascales and J. Orihuela. On compactness in locally convex spaces. *Math. Z.*, 195:365–381, 1987.

[74] J. Cerf. La stratification naturelle des espaces de fonctions différentiables reélles et le théoréme de la pseudoisotopie. *Publ. Math. Inst. Hautes Etud. Sci.*, 39, 1970.

[75] Vicente Cervera, Francisca Mascaro, and Peter W. Michor. The orbit structure of the action of the diffeomorphism group on the space of immersions. *Differ. Geom. Appl.*, 1:391–401, 1991.

[76] K. Ciesielski and R. Pol. A weakly Lindelöf function space $c(k)$ without continuous injection into $c_0(\gamma)$. *Bull. Pol. Acad. Sci. Math.*, 32:681–688, 1984.

[77] Johann Cigler, Viktor Losert, and Peter W. Michor. *Banach modules and functors on categories of Banach spaces.* Marcel Dekker Inc., 1979. Lecture Notes in Pure and Applied Mathematics 46.

[78] J.B. Collier. A class of strong differentiability spaces. *Proc. Am. Math. Soc.*, 53:420–422, 1975.

[79] Jean Fran¸
cois Colombeau. *Différentiation et Bornologie.* PhD thesis, University of Bordeaux, 1973.

[80] Jean Fran¸
cois Colombeau. Infinite dimensional c^∞-mappings with a given sequence of derivatives at a given point. *J. Math. Anal. Appl.*, 71:95–104, 1979.

[81] Jean François Colombeau. *Differential Calculus and Holomorphy.* North Holland, Amsterdam, 1982. Math. Studies 64.

[82] Jean-Francois Colombeau and Reinhold Meise. *C^∞-functions on locally convex and on bornological vector spaces*, pages 195–216. Springer-Verlag, 1981. Lect. Notes Math. 843.

[83] J.F. Colombeau and B. Perrot. The (partial d) equation in Dfn spaces. *J. Math. Anal. Appl.*, 78:466–487, 1980.

[84] H.H. Corson. The weak topology of a Banach space. *Trans. Am. Math. Soc.*, 101:1–15, 1961.

[85] D. Cudia. The geometry of Banach spaces. Smoothness. *Trans. Am. Math. Soc.*, 110:284–314, 1964.

[86] P.J. Daniell. The derivative of a functional. *Bull. Am. Math. Soc.*, 25:414–416, 1919.

[87] M.M. Day. Strict convexity and smoothness of normed spaces. *Trans. Am. Math. Soc.*, 78:516–528, 1955.

[88] P. de la Harpe. *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space.* Springer-Verlag, 1972. Lecture Notes in Math. 285.

[89] Marc De Wilde. *Closed graph theorems and webbed spaces.* Pitman, London, 1978.

[90] R. Deville, G. Godefroy, and V.E. Zizler. The three space problem for smooth partitions of unity and $c(k)$ spaces. *Math. Ann.*, 288:613–625, 1990.

[91] R. Deville, G. Godefroy, and V.E. Zizler. *Smoothness and renormings in Banach spaces.* Longman, John Wiley, London, New York, 1993. Pitman Monographs and Surveys in Pure and Applied Mathematics 64.

[92] B. S. DeWitt. Quantum theory of gravity. I. The canonical theory. *Phys. Rev.*, 160 (5):1113–1148, 1967.

[93] J. Diestel. *Geometry of Banach spaces, selected topics.* Springer-Verlag, 1975. Lecture Notes in Math. 485.

[94] J. Diestel. *Sequences and series in Banach spaces.* Springer Verlag, 1984.

[95] J. Dieudonné. Sur un theoreme de Glaeser. *J. Anal. Math.*, 23:85–88, 1970.

[96] J. A. Dieudonné. *Foundations of modern analysis, I.* Academic Press, New York – London, 1960.

[97] Adrien Douady. Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné. *Ann. Inst. Fourier*, 16:1–95, 1966.

[98] Michel Dubois-Violette and Peter W. Michor. A common generalization of the Frölicher-nijenhuis bracket and the Schouten bracket for symmetric multi vector fields. *Indag. Math. New Ser.*, 6:51–66, 1995.

[99] D. Ebin. *The manifold of Riemannian metrics*, pages 11–40. Am. Math. Soc., 1970.

[100] D.G. Ebin and J.E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.*, 92:102–163, 1970.

[101] G.A. Edgar. Measurability in Banach spaces. *Indiana Univ. Math. J.*, 26:663–677, 1977.

[102] G.A. Edgar. Measurability in a Banach space, II. *Indiana Univ. Math. J.*, 28:559–579, 1979.

[103] J. Eells and K. D. Elworthy. Open embeddings of certain Banach manifolds. *Ann. Math.*, 91:465–485, 1970.

[104] James Eells. A setting for global analysis. *Bull. Am. Math. Soc.*, 72:571–807, 1966.

[105] C. Ehresmann. Les prolongements d'une variété différentiable, I: Calcul des jets, prolongement principal, Ii: L'espace des jets d'ordre r de v_n dans v_m. *C. R. Acad. Sci. Paris*, 233:598–600, 777–779, 1951.

[106] Jürgen Eichhorn. Gauge theory on open manifolds of bounded geometry. *Int. J. Mod. Phys. A*, 7:3927–3977, 1992.

[107] Jürgen Eichhorn. The manifold structure of maps between open manifolds. *Ann. Global Anal. Geom.*, 11:253–300, 1993.

[108] I. Ekeland and G. Lebourg. Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces. *Trans. Am. Math. Soc.*, 224:193–216, 1976.

[109] P. Enflo. Uniform structures and square roots in topological groups Ii. *Isr. J. Math.*, 9:253–272, 1970.

[110] P. Enflo. On Banach spaces which can be given an equivalent uniformly convex norm. *Isr. J. Math.*, 13:114–146, 1972.

[111] P. Enflo, J. Lindenstrauss, and G. Pisier. On the 'three space problem'. *Math. Scand.*, 36:199–210, 1975.

[112] Ryszard Engelking. *Outline of General Topology.* Polish Scientific Publishers, Warszawa, 1968.

[113] Ryszard Engelking. *General Topology.* Heldermann-Verlag, Berlin, 1989. revised and completed edition.

[114] D.B.A. Epstein. The simplicity of certain groups of homeomorphisms. *Compos. Math.*, 22:165–173, 1970.

[115] M. Fabian and G. Godefroy. The dual of every Asplund space admits a projectional resolution of identity. *Stud. Math.*, 91:141–151, 1988.

[116] L. Fantappié. I functionali analitici. *Atti Accad. Naz. Lincei*, Mem. 3-11:453–683, 1930.

[117] L. Fantappié. überblick über die Theorie der analytischen Funktionale und ihre Anwendungen. *Jahresber. Dtsch. Math. Ver.*, 43:1–25, 1933.

[118] Claude-Alain Faure. Sur un théorème de Boman. *C. R. Acad. Sci. Paris*, 309:1003–1006, 1989.

[119] Claude-Alain Faure. *Théorie de la différentiation dans les espaces convenables.* PhD thesis, Université de Genéve, 1991.

[120] S. Fitzpatrick. Metric projections and the differentiability of distance functions. *Bull. Aust. Math. Soc.*, 22:291–312, 1980.

[121] Klaus Floret. Lokalkonvexe Sequenzen mit kompakten Abbildungen. *J. Reine Angew. Math.*, 247:155–195, 1971.

[122] Maurice Fréchet. Sur la notion de différentielle. *C. R. Acad. Sci. Paris*, 152:845–847, 1050–1051, 1911.

[123] Maurice Fréchet. La notion de différentielle dans l'analyse générale. *Ann. Sci. Ec. Norm. Super.*, XLII:293–323, 1925.

[124] Maurice Fréchet. Sur la notion de différentielle dans l'analyse générale. *C. R. Acad. Sci. Paris*, 180:806–809, 1925.

[125] Maurice Fréchet. Sur la notion de différentielle. *J. Math. Pures Appl.*, 16:233–250, 1937.

[126] D. Freed and K. Uhlenbeck. *Instantons and 4-manifolds.* Springer-Verlag, New York, 1984. MSRI Publications 1.

[127] D. S. Freed and D. Groisser. The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group. *Mich. Math. J.*, 36:323–344, 1989.
[128] Michael H. Freedman. The topology of four dimensional manifolds. *J. Differ. Geo.*, 17:357–454, 1982.
[129] C. Freifeld. *One-parameter subgroups do not fill a neighborhood of the identity in an infinite dimensional Lie (pseudo)-group*, pages 538–543. Benjamin, New York, 1967. Lectures in Mathematics and Physics.
[130] A. Frölicher and W. Bucher. *Calculus in vector spaces without norm.* Springer-Verlag, Heidelberg, 1966. Lecture Notes in Math. 30.
[131] A. Frölicher and A. Kriegl. Differentiable extensions of functions. *Differ. Geom. Appl.*, 3:71–90, 1993.
[132] Alfred Frölicher. Categories cartésiennement fermées engendrées par des monoides. *Cah. Topologie Géom. Differ.*, 21:367–375, 1980.
[133] Alfred Frölicher. Applications lisses entre espaces et variétés de Fréchet. *C. R. Acad. Sci. Paris*, 293:125–127, 1981.
[134] Alfred Frölicher, Bernard Gisin, and Andreas Kriegl. *General differentiation theory*, pages 125–153. Aarhus Universitet, 1983.
[135] Alfred Frölicher and Andreas Kriegl. *Convergence vector spaces for Analysis*, pages 115–126. Akademie Verlag, Berlin, 1985.
[136] Alfred Frölicher and Andreas Kriegl. *Linear spaces and differentiation theory.* J. Wiley, Chichester, 1988. Pure and Applied Mathematics.
[137] D. B. Fuks. *Cohomology of infinite dimensional Lie algebras.* Nauka Consultants Bureau (Plenum Press), Moscow New York, 1984 1986. (Russian) transl. English Contemporary Soviet Mathematics.
[138] Ronald O. Fulp. Connections on the path bundle of a principal fiber bundle. *Differ. Geom. Appl.*, 4:201–237, 1994.
[139] Peter Gabriel and M. Zisman. *Fondements de la topologie simpliciale.* Université de Strasbourg, 1963/64. Séminaire homotopique.
[140] W. Gähler. *Grundstrukturen der Analysis I, II.* Birkhäuser/Akademie Verlag, Basel/Berlin, 1977, 1978.
[141] I. Garrido, J. Gómez, and J.A. Jaramillo. Homomorphisms on function algebras. *Can. J. Math.*, 46:734–745, 1994.
[142] M.R. Gâteaux. Sur les fonctionnelles continues et les fonctionnelles analytiques. *C. R. Acad. Sci. Paris*, 157:325–327, 1913.
[143] M.R. Gâteaux. Fonctions d'une infinité de variables independantes. *Bull. Soc. Math. Fr.*, 47:70–96, 1919.
[144] M.R. Gâteaux. Sur les fonctionnelles continues et les fonctionnelles analytiques. *Bull. Soc. Math. Fr.*, 50:1–21, 1922.
[145] I.M. Gelfand and I.Y. Dorfman. Hamiltonian operators and the algebraic structures connected with them. *Funct. Anal. Appl.*, 13:13–30, 1979.
[146] M. Gieraltowska-Kedzierska and F.S. Van Vleck. Fréchet differentiability of regular locally Lipschitzian functions. *J. Math. Anal. Appl.*, 159:147–157, 1991.
[147] Olga Gil-Medrano and Peter W. Michor. Pseudoriemannian metrics on spaces of almost hermitian structures. *Isr. J. Math.*, page 11, 1994. to appear.
[148] Olga Gil-Medrano, Peter W. Michor, and Martin Neuwirther. Pseudoriemannian metrics on spaces of bilinear structures. *Q. J. Math. Oxf.*, 43:201–221, 1992.
[149] Olga Gil-Medrano and W. Michor, Peter. The Riemannian manifold of all Riemannian metrics. *Q. J. Math. Oxf.*, 42:183–202, 1991.
[150] Leonard Gillman and Meyer Jerison. *Rings of continuous functions.* Van Nostrand, Princeton, 1960.
[151] G. Glaeser. Racine carré d'une fonction différentiable. *Ann. Inst. Fourier*, 13,2:203–210, 1963.
[152] G. Godefroy, J. Pelant, J.H.M. Whitefield, and V.E. Zizler. Banach space properties of Ciesielski-pol's $c(k)$-space. *Proc. Am. Math. Soc.*, 103:1087–1093, 1988.
[153] G Godefroy, S.L Troyanski, J.H.M Whitfield, and V.E. Zizler. Smoothness in weakly compactly generated spaces. *J. Funct. Anal.*, 52:344–352, 1983.
[154] G Godefroy, S.L. Troyanski, J.H.M Whitfield, and V.E. Zizler. Locally uniformly rotund renorming and injection into $c_0(\gamma)$. *Can. Math. Bull.*, 27:494–500, 1984.
[155] R. Godement. *Topologie algébrique et théorie des fausceaux.* Hermann, Paris, 1958.
[156] J. Gómez and J.G. Llavona. Multiplicative functionals on function algebras. *Rev. Mat. Univ. Complutense Madrid*, 1:19–22, 1988.
[157] Janusz Grabowski. Free subgroups of diffeomorphism groups. *Fundam. Math.*, 131:103–121, 1988.

[158] Janusz Grabowski. Derivative of the exponential mapping for infinite dimensional Lie groups. *Ann. Global Anal. Geom.*, 11:213–220, 1993.

[159] Hans Grauert. On Levi's problem and the embedding of real analytic manifolds. *Ann. Math.*, 68:460–472, 1958.

[160] Werner Greub, Steve Halperin, and Ray Vanstone. *Connections, Curvature, and Cohomology I, II, III.* Academic Press, New York and London, 1972, 1973, 1976.

[161] P. Griffiths. On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry. *Duke Math. J.*, 41:775–814, 1974.

[162] Mikhael Gromov. *Partial differential relations.* Springer-Verlag, Berlin, Heidelberg, New York, 1986. Ergebnisse 3. Folge, Band 9.

[163] A. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires.* Am. Math. Soc., 1955. Mem. Am. Math. Soc. 16.

[164] Alexandre Grothendieck. Sur certains espaces de fonctions holomorphes. I. *J. Reine Angew. Math.*, 192:35–64, 1953.

[165] Rober C. Gunning and Hugo Rossi. *Analytic functions in several complex variables.* Prentice-Hall, Englewood Cliffs, N.J., 1965.

[166] Matthias Günther. *Beiträge zur lokalen Lösbarkeit nichtlinearer partieller Differentialgleichungen.* PhD thesis, Karl-Marx-Universität, Leipzig, 1988.

[167] Matthias Günther. On the perturbation problem associated to isometric embeddings of Riemannian manifolds. *Ann. Global Anal. Geom.*, 7,1:69–77, 1989.

[168] Matthias Günther. Isometric embeddings of Riemannian manifolds. was inbook,, 1991.

[169] Federer H. *Geometric Measure Theory.* Springer, Berlin, 1969.

[170] J. Hadamard. *Le¸*
cons sur le calcul des variations. Hermann, Paris, 1910.

[171] J. Hadamard. *La notion de différentielle dans l'enseignement.* Bib Hierosolymitanarum, Jérusalem, 1923. Scripta Univ. Ab. 3.

[172] Hans Hahn. über lineare Gleichungssysteme in linearen Räumen. *J. Reine Angew. Math.*, 157:214–229, 1926.

[173] Hans Hahn. Reelle Funktionen I. was article,, 1932.

[174] Richard S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Am. Math. Soc.*, 7:65–222, 1982.

[175] Y. Hatakeyama. Some notes on the groups of automorphisms of contact and symplectic structures. *Tôhoku Math. J.*, 18:338–347, 1966.

[176] A. Hatcher. A proof of a Smale conjecture. *Ann. Math.*, 117:553–607, 1983.

[177] A. Hatcher and J. Wagoner. Pseudo-isotopies of compact manifolds. *Asterisque*, 6, 1973.

[178] R. Haydon. A counterexample to several questions about scattered compact spaces. *Bull. Lond. Math. Soc.*, 22:261–268, 1990.

[179] R. Haydon. Trees and renormings *Publ. Math. Univ. Pierre Marie Curie*, 104, No.8, 1991.

[180] Gerd Heber. *Die Topologie des Konfigurationsraumes der Yang-Mills-Theorie offenen Mannigfaltigkeiten beschränkter Geometrie.* PhD thesis, Universität Greifswald, 1994.

[181] Edwin Hewitt. Rings of real valued continuous functions. I. *Trans. Am. Math. Soc.*, 64:45–99, 1948.

[182] Edwin Hewitt. Linear functionals on spaces of continuous functions. *Fundam. Math.*, 37:161–189, 1950.

[183] Takeshi Hirai. Irreducible unitary representations of the group of diffeomorphisms of a non-compact manifold. *J. Math. Kyoto Univ.*, 33, 3:827–864, 1993.

[184] Morris W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1976. GTM 33.

[185] F. Hirzebruch. *Neue topologische Methoden in der algebrischen Topologie.* Springer-Verlag, Berlin, Göttingen, Heidelberg, 1962. Ergebnisse der Math. 9, 2. edition.

[186] Henri Hogbe-Nlend. Complétions, tenseurs, nucléarité en bornologie. *J. Math. Pures Appl.*, 1970.

[187] Henri Hogbe-Nlend. *Théorie des Bornologies et Applications.* Springer-Verlag, Berlin, 1971. Lecture Notes in Math. 213.

[188] Henri Hogbe-Nlend. *Bornologies and Functional Analysis.* North Holland, Amsterdam, 1977. Math. Studies 26.

[189] Lars Hörmander. *The Analysis of linear partial differential operators I.* Springer-Verlag, Berlin, 1983. Grundlehren 256.

[190] John Horváth. *Topological vector spaces and distributions.* Addison Wesley, Reading, Mass., 1966.

[191] K. Igusa. *What happens to Hatcher and Wagoner's formula for $\pi_0(C(M))$ when the first Postnikoc invariant of M is nontrivial*, pages 104–172. Springer-Verlag, New York, Heidelberg, 1984.

[192] Yoichi Imayoshi. Holomorphic maps of compact Riemannian surfaces into 2-dimensional compact c-hyperbolic manifolds. *Math. Ann.*, 270:403–416, 1985.

[193] Yoichi Imayoshi. Holomorphic maps of projective algebraic manifolds into compact c-hyperbolic manifolds. *J. Math. Soc. Japan*, 46:289–307, 1994.

[194] Dieudonné J. Sur les espaces uniformes complets. *Ann. Sci. Éc. Norm. Super.*, 56:277–291, 1939.

[195] H. Jacobowitz. Implicit function theorems and isometric embeddings. *Ann. Math.*, 95:191–225, 1972.

[196] R. C. James. A non-reflexive Banach space isometric with its second conjugate space. *Proc. Natl. Acad. Sci. USA*, 37:174–177, 1951.

[197] J.A. Jaramillo. Multiplicative functionals on algebras of differentiable functions. *Arch Math.*, 58:384–387, 1992. To appear in.

[198] Hans Jarchow. *Locally convex spaces.* Teubner, Stuttgart, 1981.

[199] J.E. Jayne and C.A. Rogers. Borel selectors for upper semicontinuous set valued maps. *Acta Math.*, 155:41–79, 1985.

[200] K. John, H. Torunczyk, and V. Zizler. Uniformly smooth partitions of unity on superreflexive Banach spaces. *Stud. Math.*, 70:129–137, 1981.

[201] K. John and V. Zizler. A note on strong differentiability spaces. *Commentat. Math. Univ. Carol.*, 17:127–134, 1976.

[202] K. John and V. Zizler. On rough norms on Banach spaces. *Commentat. Math. Univ. Carol.*, 19:335–349, 1978.

[203] W.B. Johnson and J. Lindenstrauss. Some remarks on weakly compactly generated Banach spaces. *Isr. J. Math.*, 17:219–230, 1974.

[204] H Joris. Une c^∞-application non-immersive qui possède la propriété universelle des immersions. *Arch Math.*, 39:269–277, 1982.

[205] B. Josefson. Bounding subsets of $\ell^\infty(a)$. *J. Math. Pures. Appl.*, 57:397–421, 1978.

[206] M.I. Kadec. Spaces isomorphic to a locally uniformly convex space. *Izv. Vyssh. U ˇ
cebn. Zaved. Mat.*, 13:51–57, 1959.

[207] M.I. Kadec. Letters to the editors. *Izv. Vyssh. U ˇ
cebn. Zaved. Mat.*, 15:139–141, 1961.

[208] M.I. Kadec. Some conditions of the differentiability of the norm of Banach spaces. *Uspechi Mat. Nauk*, 20:183–187, 1965.

[209] Gerd Kainz. A note on the manifold of immersions and its Riemannian curvature. *Monatsh. Math.*, 98:211–217, 1984.

[210] Gerd Kainz, Andreas Kriegl, and Peter W. Michor. c^∞-algebras from the functional analytic viewpoint. *J. Pure Appl. Algebra*, 46:89–107, 1987.

[211] Gerd Kainz and Peter W. Michor. Natural transformations in differential geometry. *Czech. Math. J.*, 37:584–607, 1987.

[212] Masaki Kashiwara, Takahiro Kawai, and Tatsuo Kimura. *Foundations of algebraic analysis.* Princeton Univ. Press, Princeton, 1986.

[213] Kat ˇ
etov. Measures in fully normal spaces. *Fundam. Math.*, 38:73–84, 1951.

[214] Tosio Kato. *Perturbation theory for linear operators.* Springer-Verlag, Berlin, 1976. Grundlehren 132.

[215] Hans Heinrich Keller. Räume stetiger multilinerarer Abbildungen als Limesräume. *Math. Ann.*, 159:259–270, 1965.

[216] Hans Heinrich Keller. *Differential calculus in locally convex spaces.* Springer-Verlag, 1974. Lecture Notes in Math. 417.

[217] P.S. Kenderov. The quasi-differentiable functionals are almost everywhere differentiable. *Math. Educ.*, 2:123–126, 1974.

[218] M Kervaire and J. Milnor. Groups of homotopy spheres, I. *Ann. Math.*, 77:504–537, 1963.

[219] A. A. Kirillov. *Elements of the theory of representations.* Springer-Verlag, Berlin, 1976.

[220] A. A. Kirillov. The orbits of the group of diffeomorphisms of the circle, and local Lie superalgebras. *Funct. Anal. Appl.*, 15:135–136, 1981.

[221] A. A. Kirillov. *The orbit method I: geometric quantization; II: Infinite dimensional Lie groups and Lie algebras.* Department of Math., University of Maryland, 1991. Notes by J. Adams, A. Kleppner, R. Lipsman, J.-Sh. Li, and J. Rosenberg.

[222] W. Knapp, Anthony. *Representation theory of semisimple Lie groups.* Princeton University Press, Princeton, 1986.

[223] A. Kneser. *Lehrbuch der Variationsrechnung.* Vieweg, Braunschweig, 1900.

[224] K Kodaira, L Nirenberg, and D.C. Spencer. On the existence of deformations of complex analytic structures. *Ann. Math.*, 68:450–459, 1958.

[225] K Kodaira and D.C. Spencer. On the deformations of complex analytic structures. *Ann. Math.*, 67:328–460, 1958.

[226] Ivan Koláˇ r and Peter W. Michor. Determination of all natural concomitants of the type of the Frölicher Nijenhuis bracket. *Rend. Circ. Mat. Palermo, Suppl.*, 16:101–108

[227] Ivan Koláˇ r, Peter W. Michor, and Jan Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin Heidelberg New York, 1993.

[228] Nancy Koppell. Commuting diffeomorphisms. *Proc. Symp. Pure Math. 14*, pages 165–184, 1970.

[229] J. L. Koszul. Homologie et Cohomologie des algèbres de Lie. *Bull. Soc. Math. Fr.*, 78:65–127, 1950.

[230] Gottfried Köthe. Dualität in der Funktionentheorie. *J. Reine Angew. Math.*, 191:30–49, 1953.

[231] Andreas Kriegl. *Eine Theorie glatter Mannigfaltigkeiten und Vektorbündel*. PhD thesis, Universität Wien, 1980.

[232] Andreas Kriegl. Die richtigen Räume für Analysis im Unendlich - Dimensionalen. *Monatsh. Math.*, 94:109–124, 1982.

[233] Andreas Kriegl. Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen. *Monatsh. Math.*, 95:287–309, 1983.

[234] Andreas Kriegl. *A cartesian closed extension of the category of smooth Banach manifolds*, pages 323–336. Heldermann, Berlin, 1984.

[235] Andreas Kriegl. Remarks on germs in infinite dimensions. *Acta Math. Univ. Comenianae*, 66:1–18, 1997.

[236] Andreas Kriegl and Peter W. Michor. A convenient setting for real analytic mappings. *Acta Math.*, 165:105–159, 1990.

[237] Andreas Kriegl and Peter W. Michor. Aspects of the theory of infinite dimensional manifolds. *Differ. Geom. Appl.*, 1:159–176, 1991.

[238] Andreas Kriegl and Peter W. Michor. More smoothly real compact spaces. *Proc. Am. Math. Soc.*, 117:467–471, 1993.

[239] Andreas Kriegl and Peter W. Michor. Regular infinite dimensional Lie groups. *J. Lie Theory*, 7,1:61–99, 1997. to appear ESI Preprint 200.

[240] Andreas Kriegl, Peter W. Michor, and Walter Schachermayer. Characters on algebras of smooth functions. *Ann. Global Anal. Geom.*, 7, No. 2:85–92, 1989.

[241] Andreas Kriegl and D. Nel, Louis. A convenient setting for holomorphy. *Cah. Topologie Géom. Differ.*, 26:273–309, 1985.

[242] Andreas Kriegl and Louis D. Nel. Convenient vector spaces of smooth functions. *Math. Nachr.*, 147:7, 1990.

[243] M. Kuranishi. On the locally complete families of complex analytic structures. *Ann. Math.*, 75:536–577, 1962.

[244] J. Kurzweil. On approximation in real Banach spaces. *Stud. Math.*, 14:214–231, 1954.

[245] M. Ky Fan. Sur quelques notions fondamentales de l'analyse générale. *J. Math. Pures Appl.*, 21:289–368, 1942.

[246] Serge Lang. *Differential manifolds*. Addison Wesley, 1972.

[247] F. W. Lawvere. Categorical dynamics. was inbook, Lectures given 1967 at the University of Chicago, reprinted in Aarhus Math. Inst. Var. Publ. Series 30, 1979.

[248] F. William Lawvere, S.H. Schanuel, and W.R. Zame. On c^∞-function spaces. was article, Preprint, State University of New York, Buffalo, 1981.

[249] D. Lazet. Sur la différentiabilité des applications analytiques en dimension infinie. *C. R. Acad. Sci. Paris*, 273:155–157, 1971.

[250] E.B. Leach and J.H.M. Whitfield. Differentiable functions and rough norms on Banach spaces. *Proc. Am. Math. Soc.*, 33:120–126, 1972.

[251] M. Leduc. Densité de certaines familles d'hyperplans tangents. *C. R. Acad. Sci. Paris*, 270:326–328, 1970.

[252] Joshua Leslie. On a differential structure for the group of diffeomorphisms. *Topology*, 6:264–271, 1967.

[253] Joshua Leslie. On the group of real analytic diffeomorphisms of a compact real analytic manifold. *Trans. Am. Math. Soc.*, 274 (2):651–669, 1982.

[254] M. Paul Lévy. *Le¸ cons d'analyse fonctionnelle*. Gauthier-Villars, Paris, 1922.

[255] Paulette Libermann and C. M. Marle. *Symplectic geometry and analytical mechanics*. D. Reidel, Dordrecht, 1987. Mathematics and its applications.

[256] Joram Lindenstrauss and L. Tzafriri. On the complemented subspace problem,. *Isr. J. Math.*, 9:263–269, 1971.

[257] J.G. Llavona. *Approximation of continuously differentiable functions*. North-Holland, 1986.

[258] Susana Fernandez Long de Foglio. La différentielle au sens d'hadamard dans les espaces l vectorielles. *Port. Math.*, 19:165–184, 1960.

[259] Anthony R. Lovaglia. Locally uniformly convex Banach spaces. *Trans. Am. Math. Soc.*, 78:225–238, 1955.

[260] Valentin Lychagin. On sufficient orbits of a group of contact diffeomorphisms. *Math. USSR Sb.*, 33:223–242, 1977.

[261] George W. Mackey. Equivalence of a problem in measure theory to a problem in the theory of vector lattices. *Bull. Am. Math. Soc.*, 50:719–722, 1944.

[262] George W. Mackey. On infinite dimensional spaces. *Trans. Am. Math. Soc.*, 57:155–207, 1945.

[263] D.P. MacLaughlin. Smooth partitions of unity in preduals of Wcg spaces. *Math. Z.*, 211:189–194, 1992.

[264] B. Maissen. Lie Gruppen mit Banach Räumen als Parameterräumen. *Acta Math.*, 108:229–270, 1962.

[265] B. Maissen. über Topologien im Endomorphismenraum eines topologischen Vektorraums. *Math. Ann.*, 151:283–285, 1963.

[266] Bernard Malgrange. *Ideals of differentiable functions*. Oxford Univ. Press, London, 1966.

[267] Juan Margalef Roig and Enrique Outerelo Dominguez. Une variedad diferenciable de dimension infinita, seperada y no regular. *Rev. Mat. Hisp.-Amer.*, 42:51–55, 1982.

[268] G. Marinescu. *Espaces vectoriels pseudo topologiques et le theorie des distributions*. Deutsche Verlag d. Wiss., Berlin, 1963.

[269] J.N. Mather. Commutators of diffeomorphisms I. *Comment. Math. Helv.*, 49:512–528, 1974.

[270] J.N. Mather. Commutators of diffeomorphisms Ii. *Comment. Math. Helv.*, 50:33–40, 1975.

[271] J.N. Mather. A curious remark concerning the geometric transfer map. *Comment. Math. Helv.*, 59:86–110, 1984.

[272] J.N. Mather. Commutators of diffeomorphisms Iii. *Comment. Math. Helv.*, 60:122–124, 1985.

[273] Josef Mattes. On the convenient setting for real analytic mappings. *Monatsh. Math.*, 116:127–141, 1993.

[274] Markus Mauhart and Peter W. Michor. Commutators of flows and fields. *Arch. Math. Brno*, 28:228–236

[275] S. Mazur. über konvexe Mengen in linearen normierten Räumen. *Stud. Math.*, 4:70–84, 1933.

[276] S. Mazur. On continuous mappings on cartesian products. *Fundam. Math.*, 39:229–238, 1952.

[277] S. Mazur and W. Orlicz. Grundlegende Eigenschaften der polynomischen Operationen. *Stud. Math.*, 5:50–68, 1935.

[278] V.Z. Meshkov. Smoothness properties in Banach spaces. *Stud. Math.*, 63:111–123, 1978.

[279] A.D. Michal. Differential calculus in linear topological spaces. *Proc. Natl. Acad. Sci. USA*, 24:340–342, 1938.

[280] Peter W. Michor. Manifolds of smooth maps. *Cah. Topologie Géom. Differ.*, 19:47–78, 1978.

[281] Peter W. Michor. *Manifolds of differentiable mappings*. Shiva, Orpington, 1980.

[282] Peter W. Michor. Manifolds of smooth maps Ii: The Lie group of diffeomorphisms of a non compact smooth manifold. *Cah. Topologie Géom. Differ.*, 21:63–86, 1980.

[283] Peter W. Michor. Manifolds of smooth maps Iii: The principal bundle of embeddings of a non compact smooth manifold. *Cah. Topologie Géom. Differ.*, 21:325–337, 1980.

[284] Peter W. Michor. The manifold of embeddings of a non compact manifold. was inbook, Appendix to: E. Binz, H.R. Fischer: The manifold of embeddings of a closed manifold, 1981.

[285] Peter W. Michor. Manifolds of smooth mappings Iv: Theorem of De rham. *Cah. Topologie Géom. Differ.*, 24:57–86, 1983.

[286] Peter W. Michor. *Applications of Hamilton's inverse function theorem to manifolds of mappings*, pages 159–162 Akademie Verlag, Berlin, 1984.

[287] Peter W. Michor. A convenient setting for differential geometry and global analysis I, Ii. *Cah. Topologie Géom. Differ.*, 25:63–109, 113–178., 1984.

[288] Peter W. Michor. The cohomology of the diffeomorphism group is a Gelfand-fuks cohomology. *Rend. Circ. Mat. Palermo, Suppl.*, 14:235– 246

[289] Peter W. Michor. *Remarks on the Frölicher-Nijenhuis bracket*, pages 197–220 D. Reidel, 1987.

[290] Peter W. Michor. *Gauge theory for the diffeomorphism group*, pages 345–371. Kluwer, Dordrecht, 1988.

[291] Peter W. Michor. Knit products of graded Lie algebras and groups. *Rend. Circ. Mat. Palermo, Suppl.*, 22:171–175

[292] Peter W. Michor. The moment mapping for a unitary representation. *Ann. Global Anal. Geom.*, 8, No 3:299–313, 1990.

[293] Peter W Michor. *Gauge theory for fiber bundles.* Bibliopolis, Napoli, 1991. Monographs and Textbooks in Physical Sciences, Lecture Notes 19.

[294] Peter W. Michor. *All unitary representations admit moment mappings*, pages 477–489 János Bolyai Math. Soc. and Elsevier, Budapest and Amsterdam, 1992.

[295] Peter W. Michor and Tudor Ratiu. Curvature of the Virasoro-bott group. was article, Preprint

[296] Peter W. Michor and Hermann Schichl. No slices on the space of generalized connections. was article, Preprint

[297] Peter W. Michor and Ji ˇ
ri Van ˇ
zura. Characterizing algebras of smooth functions on manifolds. *Commentat. Math. Univ. Carol.*, 37,3:519–521, 1996.

[298] Peter W. Michor and Cornelia Vizman. n-transitivity of certain diffeomorphism groups. *Acta Math. Univ. Comenianae*, 63:4, 1994.

[299] J. Mikusinski. Distributions a valeurs dans le reúnions d'espaces de Banach. *Stud. Math.*, 19:251–285, 1960.

[300] J. Milnor. *Topology from the differentiable viewpoint.* University Press of Virginia, Charlottesville, 1965.

[301] John Milnor. Remarks on infinite dimensional Lie groups. was inbook,, 1984.

[302] John Milnor and James Stasheff. *Characteristic classes.* Princeton University Press, Princeton, 1974. Annals of Math. Studies 76.

[303] I. Moerdijk and G.E. Reyes. *Models for smooth infinitesimal analysis.* Springer-Verlag, Heidelberg Berlin, 1991.

[304] J.J. Moreau. Proximité et dualité dans un espace hilbertien. *Bull. Soc. Math. Fr.*, 93:273–299, 1965.

[305] J. Morrow. The denseness of complete Riemannian metrics. *J. Differ. Geo.*, 4:225–226, 1970.

[306] Jürgen Moser. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Natl. Acad. Sci. USA*, 47:1824–1831, 1961.

[307] Jürgen Moser. A rapidly cenvergent iteration method and non-linear partial differential equations I, Ii. *Ann. Sc. Norm. Super. Pisa*, 20:265–315, 499–535, 1966.

[308] I. Namioka and R.R Phelps. Banach spaces which are Asplund spaces. *Duke Math. J.*, 42:735–749, 1975.

[309] John Nash. The embedding problem for Riemannian manifolds. *Ann. Math.*, 63:20–63, 1956.

[310] L Natarajan, E. Rodrǵuez-Carrington, and J. A. Wolf. Locally convex Lie groups. was article, Preprint

[311] L Natarajan, E. Rodrǵuez-Carrington, and J. A. Wolf. New classes of infinite dimensional Lie groups. was inbook, to appear

[312] L Natarajan, E. Rodrǵuez-Carrington, and J. A. Wolf. Differentiable structures for direct limit groups. *Lett. Math. Phys.*, 23:99–109, 1991.

[313] S. Negrepontis. Banach spaces and topology,. was inbook,, 1984.

[314] Martin Neuwirther. A family of pseudo metrics on the space of all pseudo metrics. Master's thesis, Universität Wien, 1990.

[315] Martin Neuwirther. *Submanifold geometry and Hessians on the Pseudoriemanninan manifold of metrics.* PhD thesis, Universität Wien, 1992. Dr. rer. nat.

[316] Martin Neuwirther. Submanifold geometry and Hessians on the Pseudoriemanninan manifold of metrics. *Acta Math. Univ. Comenianae*, 62:51–85, 1993.

[317] A. Nijenhuis. On a class of common properties of some different types of algebras I, Ii. *Nieuw Arch. Wiskd.*, 17:17–46, 87–108, 1969.

[318] A. Nijenhuis and R. Richardson. Cohomology and deformations in graded Lie algebras. *Bull. Am. Math. Soc.*, 72:1–29, 1966.

[319] A. Nijenhuis and R. Richardson. Deformation of Lie algebra structures. *J. Math. Mech.*, 17:89–105, 1967.

[320] K. Nomizu and Hideki Ozeki. The existence of complete Riemannian metrics. *Proc. Am. Math. Soc.*, 12:889–891, 1961.

[321] H. Omori. *Infinite dimensional Lie transformation groups.* Springer-Verlag, Berlin, 1974. Lecture Notes in Math. 427.

[322] H. Omori. On Banach Lie groups acting on finite dimensional manifolds. *Tôhoku Math. J.*, 30:223–250, 1978.

[323] H. Omori. *Theory of infinite dimensional Lie groups.* Kinokuniya, 1978. lang. Japanese.

[324] H. Omori and P. de la Harpe. About interactions between Banach Lie groups and finite dimensional manifolds. *J. Math. Kyoto Univ.*, 12:543–570, 1972.

[325] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups I. Some differential geometric expressions of Fourier integral operators on a Riemannian manifold. *Tokyo J. Math.*, 3:353–390, 1980.

[326] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups Ii. Composition rules of Fourier Integral operators on a Riemannian manifold. *Tokyo J. Math.*, 4:221–253, 1981.

[327] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups Iii. *Tokyo J. Math.*, 4:255–277, 1981.

[328] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups Iv. Definitions and fundamental theorems. *Tokyo J. Math.*, 5:365–398, 1982.

[329] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups V. Several basic properties. *Tokyo J. Math.*, 6:39–64, 1983.

[330] H. Omori, Y. Maeda, A. Yoshioka, and O. Kobayashi. On regular Fréchet Lie groups Vi. Infinite dimensional Lie groups which appear in general relativity. *Tokyo J. Math.*, 6:217–246, 1983.

[331] A. L. Onishchik. On the classification of fiber spaces. *Sov. Math. Dokl.*, 2:1561–1564, 1961.

[332] A. L. Onishchik. Connections with zero curvature and the de Rham theorem. *Sov. Math. Dokl.*, 5:1654–1657, 1964.

[333] A. L. Onishchik. Some concepts and applications of non-abelian cohomology theory. *Trans. Moscow Math. Soc.*, 17:49–98, 1967.

[334] J. Orihuela and M. Valdivia. Projective generators and resolutions of identity in Banach spaces. *Rev. Mat. Univ. Complutense Madrid*, 2:179–199, 1989.

[335] V.Y Ovsienko and B.A. Khesin. Korteweg–de vrieß superequations as an Euler equation. *Funct. Anal. Appl.*, 21:329–331, 1987.

[336] R. Palais. *Foundations of global non-linear analysis.* Benjamin, New York, 1968.

[337] R.S. Palais and C.L. Terng. *Critical point theory and submanifold geometry.* Springer-Verlag, Berlin, 1988. Lecture Notes in Mathematics 1353.

[338] R.R. Phelps. *Convex functions, monotone operators and differentiability.* Springer-Verlag, New York, 1989. Lecture Notes in Math. 1364.

[339] J. Pierpont. Theory of functions of real variables. was book, Vol. 1

[340] Domingos Pisanelli. Applications analytiques en dimension infinie. *C. R. Acad. Sci. Paris*, 274:760–762, 1972.

[341] Domingos Pisanelli. Applications analytiques en dimension infinie. *Bull. Sci. Math.*, 96, 1972.

[342] D. Preiss. Gâteaux differentiable functions are somewhere Fréchet differentiable. *Rend. Circ. Mat. Pal.*, 33:122–133, 1984.

[343] D. Preiss. Differentiability of Lipschitz functions on Banach spaces. *J. Funct. Anal.*, 91:312–345, 1990.

[344] Andrew Pressley and Graeme Segal. *Loop groups.* Oxford University Press, 1986. Oxford Mathematical Monographs.

[345] Claudio Procesi. Positive symmetric functions. *Adv. Math.*, 29:219–225, 1978.

[346] Frank Quinn. Ends Iii. *J. Differ. Geo.*, 17:503–521, 1982.

[347] H. Rademacher. über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale. *Math. Ann.*, 79, 1919.

[348] T. R. Ramadas. On the space of maps inducing isomorphic connections. *Ann. Inst. Fourier*, 32:263–276, 1982.

[349] T. Ratiu and R. Schmid. The differentiable structure of three remarkable diffeomorphism groups. *Math. Z.*, 177:81–100, 1981.

[350] F. Rellich. Störungstheorie der Spektralzerlegung, I. *Math. Ann.*, 113:600–619, 1937.

[351] F. Rellich. Störungstheorie der Spektralzerlegung, V. *Math. Ann.*, 118:462–484, 1940.

[352] F. Rellich. *Perturbation theory of eigenvalue problems.* New York University Gordon and Breach, New York, London, Paris, 1953 1969. Lecture Notes.

[353] G. Restrepo. Differentiable norms in Banach spaces. *Bull. Am. Math. Soc.*, 70:413–414, 1964.

[354] G. Restrepo. Differentiable norms. *Bol. Soc. Mat. Mex.*, 10:47–55, 1965.

[355] Bernhard Riemann. Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. was inbook, Habilitationsvortrag 10. Juni 1854, 1876.

[356] H.H. Schaefer. *Topological vector spaces.* Springer-Verlag, New York, 1971. GTM 3.

[357] Hermann Schichl. *On the existence of slice theorems for moduli spaces on fiber bundles.* Doctoral dissertation, Univ. Wien, 1997.

[358] R Schlafli. Universal connections. *Invent. Math.*, 59:59–65, 1980.

[359] Hans-Jürgen Schmidt. *The metric in the superspace of Riemannian metrics and its relation to gravity*, pages 405–411. World Scientific, Singapur, 1990.

[360] José Sebastião e Silva. *As fun¸
cões anal*

*[361] José Sebastião e Silva. As fun¸
cões anal*

*[362] José Sebastião e Silva. Sobre a topologia dos espa¸
cos funcionais anal*

[363] José Sebastião e Silva. Sui fondamenti della teoria dei funzionali analitici. Port. Math., *12:1–46, 1953.*

[364] José Sebastião e Silva. Le calcul différentielle et intégral dans les espaces localement convexes, réels ou complexes, I. Atti. Acad. Naz. Lincei, *20:743–750, 1956.*

[365] José Sebastião e Silva. Le calcul différentielle et intégral dans les espaces localement convexes, réels ou complexes, Ii. Atti. Acad. Naz. Lincei, *21:40–46, 1956.*

*[366] José Sebastião e Silva. Conceitos de fun¸
cã o diferenciável em espa¸
cos localmente convexos.* Publ. do Centro Estudos Math. de Lisboa, *1957.*

[367] José Sebastião e Silva. Les espaces à bornés et la notion de fonction différentiable, *pages 57–63. CBRM, 1961.*

[368] R. T. Seeley. Extension of c^∞-functions defined in a half space. Proc. Am. Math. Soc., *15:625–626, 1964.*

[369] Graeme Segal. The geometry of the Kdv equation. Int. J. Mod. Phys. A, *6:2859–2869, 1991.*

[370] Ulrich Seip. Kompakt erzeugte Vektorräume und Analysis,. *Springer-Verlag, Heidelberg, 1972. Lecture Notes in Math. 273.*

[371] Ulrich Seip. Differential calculus and cartesian closedness, *pages 578–604. Springer-Verlag, Heidelberg, 1976. Lecture Notes in Math. 540.*

[372] Ulrich Seip. A convenient setting for differential calculus. J. Pure Appl. Algebra, *14:73–100, 1979.*

[373] Ulrich Seip. A convenient setting for smooth manifolds. J. Pure Appl. Algebra, *21:279–305, 1981.*

[374] Taira Shirota. A class of topological spaces. Osaka Math. J., *4:23–40, 1952.*

[375] Erhard Siegl. A free convenient vector space for holomorphic spaces. Monatsh. Math., *119:85–97, 1995.*

[376] Erhard Siegl. Free convenient vector spaces. *PhD thesis, Universität Wien, Vienna, 1997.*

[377] S. Smale. Diffeomorphisms of the 2-sphere. Proc. Am. Math. Soc., *10:621–626, 1959.*

[378] Miroslav Sova. Conditions of differentiability in topological vector spaces. Czech. Math. J., *16:339–362, 1966.*

[379] Miroslav Sova. General theory of differentiability in linear topological spaces,. Czech. Math. J., *14:485–508, 1966.*

[380] N. E. Steenrod. A convenient category for topological spaces. Mich. Math. J., *14:133–152, 1967.*

[381] Ch. Stegall. The Radon-nikodym property in conjugate Banach spaces. Trans. Am. Math. Soc., *206:213–223, 1975.*

[382] Ch. Stegall. The duality between Asplund spaces and spaces with the Radon-nikodym property. Isr. J. Math., *29:408–412, 1978.*

[383] Ch. Stegall. The Radon-nikodym property in conjugate Banach spaces Ii. Trans. Am. Math. Soc., *264:507–519, 1981.*

[384] Elias M. Stein. Singular integrals and differentiability properties of functions. *Princeton Univ. Press, Princeton, 1970.*

[385] O. Stolz. Grundz"uge der Differential- und Integralrechnung. was book, Bd. 1

[386] Dennis Sullivan. Infinitesimal computations in topology. Publ. Math. Inst. Hautes Etud. Sci., *47:269–331, 1978.*

[387] J. Sylvester. On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraic common measure Mathematical papers. Philos. Trans. R. Soc. Lond., *CXLIII, part III:407–548 511ff, 1853 1904. Vol. I.*

[388] M. Talagrand. Renormages de quelques $c(k)$. Isr. J. Math., *54:327–334, 1986.*

[389] W Thurston. Foliations and groups of diffeomorphisms. Bull. Am. Math. Soc., *80:304–307, 1974.*

[390] Otto Toeplitz. Die linearen vollkommenen Räume der Funktionentheorie. Comment. Math. Helv., *23:222–242, 1949.*

[391] H. Toruńczyk. A correction of two papers concerning Hilbert manifolds. Fundam. Math., *125:89–93, 1985.*

[392] *H. Toruńzcyk. Smooth partitions of unity on some non–separable Banach spaces.* Stud. Math.*, 46:43–51, 1973.*

[393] *Jean Claude Tougeron.* Idéaux de fonctions différentiables*. Springer-Verlag, 1972. Ergebnisse 71.*

[394] *Fran¸*
cois Treves. Topological vector spaces, distributions, and kernels*. Academic Press, New York, 1967.*

[395] *Georgia Triantafillou. Diffeomorphisms of manifolds with finite fundamental group.* Bull. Am. Math. Soc.*, 31:50–53, 1994.*

[396] *S.L. Troyanski. Equivalent norms in unseparable B-spaces with an unconditional basis.* Teor. Funkts. Funkts. Anal. Prilozh.*, 6:59–65, 1968.*

[397] *S.L. Troyanski. On locally convex and differentiable norm in certain non-separable Banach spaces.* Stud. Math.*, 37:173–180, 1971.*

[398] *E. Turner. Diffeomorphisms of a product of spheres.* Invent. Math.*, 8:69–82, 1969.*

[399] *L. Tzafriri.* Some directions of results in Banach space theory*, pages 2–18. North. Holl., 1980.*

[400] *M. Valdivia. Resolution of identity in certain metrizable locally convex spaces.* Rev. R. Acad. Cienc. Exactas Fis. Nat. Madr.*, 83:75–96, 1989.*

[401] *M. Valdivia. Projective resolution of identity in* c(*k*) *spaces.* Arch. Math.*, 54:493–498, 1990.*

[402] *Manuel Valdivia.* Topics in locally convex spaces*. North Holland, 1982.*

[403] *W. T. Van Est and T. J. Korthagen. Nonenlargable Lie algebras.* Indag. Math.*, 26:15–31, 1964.*

[404] *Léon Van Hove. Topologie des espaces fontionnels analytiques et de groupes infinis de transformations.* Bull. Cl. Sci., Acad. R. Belg.*, 38:333–351, 1952.*

[405] *J Vanderwerff. Smooth approximations in Banach spaces.* Proc. Am. Math. Soc.*, 115:113–120, 1992.*

[406] *V. S. Varadarajan.* Harmonic analysis on real reductive groups*. Springer-Verlag, Berlin-Heidelberg-New York, 1977. Lecture Notes in Mathematics 576.*

[407] *L. Va ˇ*
sák. On one generalization of weakly compactly generated Banach spaces. Stud. Math.*, 70:11–19, 1981.*

[408] *Oswald Veblen and J. H. C. Whitehead.* The foundations of differential geometry*. Cambridge University Press, Cambridge, 1932.*

[409] *Paul Ver Eecke.* Fondements du calcul différentiel*. Presses Universitaires de France, Paris, 1983.*

[410] *Paul Ver Eecke.* Applications du calcul différentiel*. Presses Universitaires de France, Paris, 1985.*

[411] *Luis Verde-Star. Interpolation and Combinatorial Functions.* Stud. Appl. Math.*, 79:65–92, 1988.*

[412] *R. Vogt. Convenient categories of topological spaces for Homotopie theory,.* Arch. Math.*, 22:545–555, 1971.*

[413] *Vito Volterra. Sopra le funzioni che dipendone de altre funzioni, Nota I,ii,iii.* Rend. Acad. Naz. Lincei*, 4 (3):97–105, 141–146, 153–158, 1887.*

[414] *L. Waelbroeck. études spectrale des algèbres complètes.* Mém. Cl. Sci., Collect., Acad. R. Belg.*, 31:142–, 1960.*

[415] *L. Waelbroeck. Differentiable Mappings in b-spaces.* J. Funct. Anal.*, 1:409–418, 1967.*

[416] *L. Waelbroeck. Some theorems about bounded structures.* J. Funct. Anal.*, 1:392–408, 1967.*

[417] *Frank W. Warner.* Foundations of differentiable manifolds and Lie groups*. Scott Foresman, 1971.*

[418] *Garth Warner.* Harmonic analysis on semisimple Lie groups, Volume I*. Springer-Verlag, New York, 1972.*

[419] *Klaus Wegenkittl. Topologien auf Räumen differenzierbarer Funktionen. Master's thesis, Universität Wien, 1987.*

[420] *Klaus Wegenkittl.* The space of isometric immersions into Euclidean space*. PhD thesis, Universität Wien, 1989.*

[421] *André Weil. Théorie des points proches sur les variétés differentielles. was inbook,, 1953.*

[422] *Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds.* Adv. Math.*, 6:329–345, 1971.*

[423] *Alan Weinstein. Lectures on symplectic manifolds.* Reg. Conf. Ser. Math.*, 29, 1977.*

[424] *C. Wells, John. Differentiable functions on Banach spaces with Lipschitz derivatives.* J. Differ. Geo.*, 8:135–152, 1973.*

[425] *H. Whitney. Differentiable Even Functions.* Duke Math. J.*, 10:159–166, 1943.*

[426] *Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets.* Trans. Am. Math. Soc.*, 36:63–89, 1934.*

[427] *Hassler Whitney and F. Bruhat. Quelques propriétés fondamentales des ensembles analytiques-réels.* Comment. Math. Helv.*, 33:132–160, 1959.*

[428] *Wojciech Wojtyński. One parameter subgroups and B-c-h formula.* Stud. Math.*, 111:163–185, 1994.*

[429] *S. Yamamuro. A theory of differentiation in locally convex spaces.* Mem. Am. Math. Soc.*, 17, Nr 212, 1979.*

[430] *S. Yamamuro. Notes on the inverse mapping theorem in locally convex spaces.* Bull. Austr. Math. Soc.*, 21:419–461, 1980.*

[431] *W.H. Young.* The fundamental theorems of differential calculus. *University Press, Cambridge, 1910.*

[432] *N.V. Zhivkov. Generic Gâteaux differentiability of directionally differentiable mappings.* Rev. Roum. Math. Pure. Appl.*, 32:179–188, 1987.*

[433] *N.V. ˇ
Zivkov. Generic Gâteaux differentiability of locally Lipschitzian functions. was inbook,, 1983.*

[434] *A. Zorn, Max. Gâteaux-differentiability and essential boundedness.* Duke Math. J.*, 12:579–583, 1945.*

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