Nonlinear Functional Analysis

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This is the manuscript for the corresponding lecture course given at the University of Vienna in the Summer Semester 2017. It is originally based on a similiar lecture courses given in the Summer Semester 1993 and 2006.

I want to use this opportunity to thank all those who attended these lectures and inspired me with the feedback I got from them. In particular I want to thank Cornelia Vizman who posed well selected and highly relevant questions after reading parts of my manuscript. My special thanks go to Konni Rietsch, who not only strongly influenced the selection of the covered topics but also sacrificed a huge amount of time during her holidays and lots of energy in order to make sense out of a preliminary version of these lecture notes. This way she supplied me with an extensive list of misprints, Germanisms, and imprecise or even incorrect mathematical formulations.

In the second edition from September 1994 an extensive list of misprints and corrections provided by Eva Adam has been taken gratefully into account.

All the remaining (and newly inserted) faux pas are of course all my own responsibility. And, as always, I explicitly ask the readers not only to pardon them but also to inform me about anything which sounds weird including possibly missing definitions and explanations.

These notes have been incorperated into the book [75]. And for the lecture course 2006 I ported its source from AMSTeX to LaTeX.

This version for the course in 2017 consists mainly of slightly modified excerpts from the book [75] and all reference numbers from there are preserved.

Andreas Kriegl, June 2017

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0. Motivation

0.1 Equations on function spaces

It should be unnecessary to convince the reader, that differential calculus is an important tool in mathematics. But probably some motivation is necessary why one should extend it to infinite dimensional spaces. This poses no big problem as long as one stays inside the realm of Banach spaces. However, I will sketch now, that we are quickly forced to go beyond. One of our main tasks as mathematicians is, like it or not, to solve equations like

$$f(u) = 0.$$

However quite often one has to consider functions f which don't take (real) numbers as arguments u but functions. Let us just mention DIFFERENTIAL EQUATIONS, where f is of the following form

$$f(u)(t) := F(t, u(t), u'(t), \dots, u^{(n)}(t)).$$

Note that this is not the most general form of a differential equation, consider for example the function f given by $f(u) := u' - u \circ u$, which is not treated by the standard theory.

If the arguments t of u are (real) numbers, then this is the general form of an ORDINARY DIFFERENTIAL EQUATION, and in the generic case one can solve this implicit equation $F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0$ with respect to $u^{(n)}(t)$ and obtains an equation of the form

$$u^{(n)}(t) = g(t, u(t), u'(t), \dots, u^{(n-1)}(t)).$$

By substituting $u_0(t) := u(t), u_1(t) := u^{(1)}(t), \dots, u_{n-1}(t) := u^{(n-1)}(t)$ one obtains a (vector valued) equation

$$\begin{aligned} u_0'(t) &= u_1(t) \\ u_1'(t) &= u_2(t) \\ &\vdots \\ u_{n-2}'(t) &= u_{n-1}(t) \\ u_{n-1}'(t) &= g(t, u_0(t), \dots, u_{n-1}(t)) \end{aligned}$$

And if we write $\mathbf{u} := (u_0, \dots, u_{n-1})$ and

$$\mathbf{g}(t, \mathbf{u}) := (u_1(t), \dots, u_{n-1}(t), g(t, u_0(t), \dots, u_{n-1}(t))),$$

we arrive at the ORDINARY DIFFERENTIAL EQUATION OF ORDER 1

$$\mathbf{u}'(t) = \mathbf{g}(t, \mathbf{u}(t)).$$

So we are searching for a solution $\mathbf{u}$ of the equation $\mathbf{u}' = G(\mathbf{u})$, where $G(\mathbf{u})(t) := \mathbf{g}(t, \mathbf{u}(t))$. The general existence and uniqueness results for equations usually depend on some fixed-point theorem and so the domain and the range space have to be equal or at least to be isomorphic. So we need that $u \mapsto u' - G(u)$ is a selfmapping. In order to apply it to a function u, we need that u is 1-times differentiable, but in order that the image $u' - G(u)$ is 1-times differentiable, we need that u is twice differentiable. Inductively we come to the conclusion that u should be smooth. So are there spaces of smooth functions, to which we can apply some fixed point theorem?

0.2 Spaces of continuous and differentiable functions

In [**68**, 3.2.5] we have shown that the space $C(X, \mathbb{R})$ of continuous real-valued functions on X is a Banach-space with respect to the supremum-norm, provided X is compact. Recall that the proof goes as follows: If f_n is a Cauchy-sequence, then it converges pointwise (since the point-evaluations $\mathrm{ev}_x = \delta_x : C(X, \mathbb{R}) \to \mathbb{R}$ are continuous linear functionals), by the triangle inequality the convergence is uniform and by elementary analysis (e.g. see [**64**, 4.2.8]) a uniform limit of continuous functions is continuous.

If X is not compact, one can nevertheless consider the linear restriction maps $C(X, \mathbb{R}) \to C(K, \mathbb{R})$ for compact subsets $K \subseteq X$ and then use the initial structure on $C(X, \mathbb{R})$, given by the seminorms $f \mapsto \|f|_K\|_\infty$, where K runs through some basis of the compact sets, see [**68**, 3.2.8]. If X has a countable basis of compact sets, then we obtain a locally convex space $C(X, \mathbb{R})$ with a countable base of seminorms. If we try to show completeness, we get as candidate for the limit a function f, which is on compact sets the uniform limit of the Cauchy-sequence f_n, and hence is continuous on these sets. If X is KELLEY (= COMPACTLY GENERATED, i.e. a set is open if its trace to all compact subsets is open, or equivalently if X carries the final topology with respect to all the inclusions of compact subsets, see [**72**, 2.3.1]) then we can conclude that f is continuous and hence $C(X, \mathbb{R})$ is complete. So under these assumptions (and in particular if X is locally compact) the space $C(X, \mathbb{R})$ is a Fréchet-space.

Is it really necessary to use countably many seminorms for non-compact X? – There is no norm which defines an equivalent structure on $C(X, \mathbb{R})$: Otherwise some seminorm $p_K := \|\_|_K\|_\infty$ must dominate it. However, this is not possible, since p_K is not a norm. In fact, since X is not compact there is some point $a \in X \setminus K$ and hence the function f defined by $f|_K = 0$ and $f(a) = 1$ is continuous on $K \cup \{a\}$. By Tietze-Urysohn [**72**, 1.3.2] it can be extended to a continuous function on X, which is obviously in the kernel of p_K but not zero.
Is there some other reasonable norm turning $C(X, \mathbb{R})$ into a Banach space E? – By reasonable we mean that at least the point-evaluations should be continuous (i.e. the topology should be finer than that of pointwise convergence). Then the identity mapping $E \to C(X, \mathbb{R})$ would be continuous by the application in [**68**, 5.3.8] of the closed graph theorem. Hence by the open mapping theorem [**68**, 5.3.5] for Fréchet spaces the identity would be an isomorphism, and thus $E \cong C(X, \mathbb{R})$ is not Banach. Note that this shows that, in a certain sense, the Fréchet structure of $C(X, \mathbb{R})$ is unique.

Now what can be said about spaces of differentiable functions? – Of course the space $D^1(X, \mathbb{R})$ of differentiable functions on some interval X is contained in $C(X, \mathbb{R})$. However it is not closed in $C(X, \mathbb{R})$ and hence not complete in the supremum-norm, since a uniform limit of differentiable functions need not be differentiable anymore, see the example in [**64**, 4.2.11]. We need some control on the derivative. So we consider the space $C^1(X, \mathbb{R})$ of continuously differentiable functions with the initial topology induced by the inclusion in $C(X,\mathbb{R})$ and by the map $d : C^1(X,\mathbb{R}) \to C(X,\mathbb{R})$ given by $f \mapsto f'$. If X is compact we can consider instead of the corresponding two seminorms $f \mapsto \|f\|_\infty$ and $f \mapsto \|f'\|_\infty$ equally well their maximum (or sum) and obtain a norm $f \mapsto \max\{\|f\|_\infty, \|f'\|_\infty\}$ on $C^1(X,\mathbb{R})$. Again elementary analysis gives completeness, since for a Cauchy-sequence f_n we have a uniform limit f_∞ of f_n and a uniform limit f^1_∞ of f'_n, and hence (e.g. see [**64**, 4.2.11] or [**71**, 2.40]) f_∞ is differentiable with derivative f^1_∞. Inductively, we obtain that for compact intervals X and natural numbers n the spaces $C^n(X,\mathbb{R})$ can be made canonically into Banach-spaces, see [**68**, 4.2.5].

0.3 Spaces of smooth functions

What about the space $C^\infty(X,\mathbb{R})$ of infinite differentiable maps on a compact interval X? – Here we have countably many seminorms $f \mapsto \|f^{(n)}\|_\infty$, and as before we obtain completeness. So we have again a Fréchet space.

Again the question arises: Is it really necessary to use countably many seminorms? Since X is assumed to be compact we have a continuous norm, the supremum norm, and we cannot argue as before. So let us assume that there is some norm on $C^\infty(X,\mathbb{R})$ defining an equivalent structure. In particular it has to be continuous and hence has to be dominated by the maximum of the suprema of finitely many derivatives. Let us consider an even higher derivative. Then the supremum of this derivative must be dominated by the norm. However, this is not possible, since there exist smooth functions f, for which all derivatives of order less than n are globally bounded by 1, but which have arbitrarily large n-th derivative at a given point, say 0. In fact, without loss of generality, we may assume assume that n is even and let $b \geq 1$. Take $f(x) := a \cos bx$ with $a := 1/b^{n-1}$. Then $|f^{(k)}(x)| = b^{k+1-n} \leq 1$ for $k < n$, but $f^{(n)}(0) = \pm b \cos 0$.
Is there some reasonable (nonequivalent) norm which turns $C^\infty(X,\mathbb{R})$ into a Banach-space? – Well, the same arguments as before show that any reasonable Fréchet-structure on $C^\infty(X,\mathbb{R})$ is identical to the standard one and hence not normable.

0.4 ODE's

By what we have said in 0.1 the straight forward formulation of a fixed point equation for a general ordinary differential equation, does not lead to Banach spaces but to Fréchet spaces. There is however a classical way around this difficulty. The idea can be seen from the simplest differential equation, namely when G doesn't depend on u, i.e. $u'(t) = G(t)$. Then the (initial value) problem can be solved by integration: $u(t) = u(0) + \int_0^t G(s)\,ds$ and in fact similar methods work in the case of separated variables, i.e. $u'(t) = G_1(t)\,G_2(u)$, since then $H_2(u) := \int \frac{1}{G_2(u)}\,du = c + \int G_1(t)\,dt =: H_1(t)$ and hence $u(t) = H_2^{-1}(H_1(t))$.
In [**68**, 1.3.2] of [**65**, 6.2.14] we have seen how to prove an existence and uniqueness result for differential equations $u'(t) = g(t,u(t))$ with initial value conditions $u(0) = a$. Namely, by integration one transforms it into the INTEGRAL EQUATION

$$u(t) = a + \int_0^t g(s,u(s))\,ds.$$

Thus one has to find a fixed point u of $u = G(u)$, where G is the integral operator given by
$$G(u)(t) := a + \int_0^t g(s, u(s))\, ds.$$
As space of possible solutions u one can now take the space $C(I, \mathbb{R})$ for some interval I around 0. If one takes I sufficiently small then it is easily seen that G is a contraction provided g is sufficiently smooth, e.g. locally Lipschitz. Hence the existence of a fixed point follows from Banach's fixed point theorem [**68**, 1.2.2] (or [**72**, 3.1.7], or [**64**, 3.4.12]).

A more natural approach was taken in [**65**, 6.2.10]: The idea there is to solve the equation $0 = u' - f \circ u =: (d - f_*)(u)$ on a space of differentiable functions u. However, since we cannot expect global existence of u but only on some interval $[-a, a]$ we transform the $u \in C^1([-a, a], \mathbb{R})$ into $u_a \in C^1([-1, 1], \mathbb{R})$, via $u_a(t) = u(ta)$ and the differential equation then becomes $u_a'(t) = au'(ta) = af(u(ta)) = a\, f(u_a(t))$, an implicit equation $0 = g(a, u_a)$, where $g : \mathbb{R} \times C^1([-a, a], \mathbb{R}) \to C([-a, a])$ is given by $g(a, u)(t) = u'(t) - a\, f(u(t)) = (d - a\, f_*)(u)(t)$. In order to apply the implicit function theorem we need that g is C^1 and $\partial_2 g(0, 0) : C^1([-1, 1], \mathbb{R}) \to C^0([-1, 1], \mathbb{R})$ is invertible. Since $d : C^1([-1, 1], \mathbb{R}) \to C([-1, 1], \mathbb{R})$ is linear and continuous we only have to show that f_* is C^1. Since $\mathrm{ev}_x : C([-1, 1], \mathbb{R}) \to \mathbb{R}$ is continuous and linear a possible (directional) derivative $(f_*)'(g)(h)$ should satisfy:
$$\begin{aligned}(f_*)'(g)(h)(x) &= \mathrm{ev}_x\big((f_*)'(g)(h)\big) = \mathrm{ev}_x\Big(\frac{d}{dt}\Big|_{t=0} f_*(g + t\, h)\Big)\\ &= \frac{d}{dt}\Big|_{t=0} (\mathrm{ev}_x \circ f_*)(g + t\, h)(x)\\ &= \frac{d}{dt}\Big|_{t=0} f(g(x) + t\, h(x)) = f'(g(x))(h(x)),\end{aligned}$$
In fact if ℓ is continuous and linear and c is a differentiable curve then $\ell \circ c$ is differentiable with derivative $\ell(c'(t))$ at t:
$$\begin{aligned}\lim_{s\to 0} \frac{\ell(c(t+s)) - \ell(c(t))}{s} &= \lim_{s\to 0} \ell\left(\frac{c(t+s) - c(t)}{s}\right) =\\ &= \ell\left(\lim_{s\to 0} \frac{c(t+s) - c(t)}{s}\right) = \ell(c'(t)).\end{aligned}$$
So for f_* to be differentiable we need that f is C^1 and then one can show that $f_* : C([-1, 1], \mathbb{R}) \to C([-1, 1], \mathbb{R})$ is in fact C^1 with derivative $(f_*)' = (f')_*$, see [**65**, 6.2.10] for the details (in a more general situation). Then $\partial_2 g(0, 0) = d$ is an isomorphism if we replace $C^1([-1, 1], \mathbb{R})$ by the closed hyperplane $\{u \in C^1([-1, 1], \mathbb{R}) : u(0) = 0\}$ involving the initial condition.

In the particular case of linear differential equation with constant coefficients $u' = Au$ we have seen in [**68**, 3.5.1] the (global) solution u with initial condition $u(0) = u_0$ is given by $u(t) := e^{tA}\, u_0$. Furthermore, the solution of a general initial value problem of a linear differential equation of order n
$$u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)\, u^{(i)}(t) = s(t), \quad u(0) = u_0, \dots, u^{(n-1)}(0) = u_{n-1}.$$
is given by an integral operator $G : f \mapsto u$ defined by $(Gf)(t) := f(t) + \int_0^1 g(t, \tau)\, d\tau$, with a certain continuous integral kernel g. We have also seen in [**68**, 3.5.5] that a boundary value problem of second order
$$u''(t) + a_1(t)\, u'(t) + a_0(t)\, u(t) = s(t), \quad R_a(u) = 0 = R_b(u),$$

where the boundary conditions are $R_a(u) := r_{a,0}\, u(a) + r_{a,1}\, u'(a)$ and $R_b(u) := r_{b,0}\, u(b) + r_{b,1}\, u'(b)$ is also solved in the generic case by an integral operator

$$u(t) = \int_a^b g(t,\tau)\, f(\tau)\, d\tau,$$

with continuous integral kernel obtained from the solutions of corresponding initial value problems.

0.5 PDE's

Now what happens, if the u in the differential equation are functions of several numerical variables. Then the derivatives $u^{(k)}$ are given by the corresponding Jacobi-matrices of partial derivatives, and our differential equation $F(t, u(t), \dots u^{(n)}(t)) = 0$ of $\boxed{0.1}$ is a PARTIAL DIFFERENTIAL EQUATION, see [**68**, 4.7.1].

Even if we have a LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS as in [**68**, 4.7.2]

$$F(u)(x) := p(\partial)(u)(x) := \sum_{|\alpha|\le n} a_\alpha \cdot \partial^\alpha u(x) = s(x),$$

where p is the polynomial $p(z) = \sum_{|\alpha|\le n} a_\alpha\, z^\alpha$, we cannot apply the trick from above for ODE's. The first problem is, that we no longer have a natural candidate, with respect to which we could pass to an explicit equation. In some special cases one can do. An example is the EQUATION OF HEAT-CONDUCTION

$$\frac{\partial}{\partial t} u = \Delta u,$$

where $u : \mathbb{R} \times X \to \mathbb{R}$ is the heat-distribution at the time t in the point x and Δ denotes the LAPLACE-OPERATOR given on $X = \mathbb{R}^n$ by $\Delta := \sum_{k=1}^n \left(\frac{\partial}{\partial x^k}\right)^2$. So this is an "ordinary" linear differential equation in an infinite dimensional space of functions on X. If we want Δ to be a self-mapping, we need smooth functions. But if we want to solve the equation as $u(t) = e^{t\Delta} u_0$ we need the functional calculus (i.e. applicability of the analytic function $e \mapsto e^t$ to the Operator Δ) and hence a Hilbert space of functions. But then Δ becomes an unbounded (symmetric) operator. This we treated in [**69**, 12.47].

Another example of such a situation is the SCHRÖDINGER EQUATION

$$i\hbar \frac{d}{dt} u = S\, u.$$

where the SCHRÖDINGER-OPERATOR is given by $S = -\frac{\hbar^2}{2m}\Delta + U(x)$ for some potential U.

A third important equation is the WAVE-EQUATION $(\frac{\partial}{\partial t})^2 u = \Delta u$, see [**66**, 9.3.1] or [**68**, 5.4]. If one makes an Ansatz of separated variables $u(t,x) = u_1(t)\, u_2(x)$ one obtains an Eigen-value equation $\Delta u(x) = \lambda u(x)$ for Δ and after having obtained the Eigen-functions $u_n : X \to \mathbb{R}$, one is lead to the problem of finding coefficients a_k and b_k such that

$$u(t,x) := \sum_k \Big(a_k \cos(\sqrt{\lambda_k}\, t) + b_k \sin(\sqrt{\lambda_k}\, t) \Big)\, u_k(x)$$

solves the initial conditions

$$u(0,x) = \sum_k a_k u_k(x) \qquad \text{and} \qquad \partial_1 u(0,x) = \sum_k \sqrt{\lambda_k}\, b_k\, u_k(x)$$

If we would have an inner-product, for which the u_k are orthonormal, then we could easily calculate the coefficients a_k and b_k. The space $C_{2\pi}$ of 2π-periodic functions is however not a Hilbert space. Otherwise it would be isomorphic to its dual, by the Riesz Representation theorem [68, 6.2.9]: However for $t \neq s$ we have that $\|\operatorname{ev}_t - \operatorname{ev}_s\| = \sup\{|f(t) - f(s) : \|f\|_\infty \leq 1\} = 1$ if we chose $f(t) = 1$ and $f(s) = 0$. Thus $C(X,\mathbb{R})'$ is not separable, since otherwise for every t there would be an ℓ_t in a fixed dense countable subset with $\|\operatorname{ev}_t - \ell_t\| < \frac{1}{2}$. Since the t are uncountable there have to be $t \neq s$ for which $\ell_t = \ell_s$, a contradiction. Another method to see this is to use Krein-Milman [68, 7.5.1]: If $C(X)$ were a dual-space, then its unit-ball would have to be contained in the closed convex hull of its extremal points. A function f in the unit-ball, which is not everywhere of absolute value 1, is not extremal. In fact, take a t_0 with $|f(t_0)| < 1$ and a function v with support in a neighborhood of t_0. Then $f + s\,v$ lies in the unit ball for all values of s near 0. Hence we have by far too few extremal points, since those real-valued functions have to be constant on connectivity components.

In analogy to the inner product on $\mathbb{R}^n$ we can consider the continuous positive definite hermitian bilinear map $(f,g) \mapsto \int_X f(x)\,\overline{g(x)}\,dx$. By what we said above, it cannot yield a complete norm on $C(X,\mathbb{R})$. But we can take the completion of $C(X,\mathbb{R})$ with respect to this norm and arrive by [68, 4.12.5] at $L^2(X)$, a space not consisting of functions, but equivalence classes thereof. Now for the one-dimensional wave-equation, i.e. the equation of a vibrating string, we can solve the Eigenvalue-problem directly (it is given by an ordinary differential equation). And Fourier-series solves the problem, see [68, 5.4] and [68, 6.3.8].

For general compact oriented manifolds X the Laplace operator will be symmetric with respect to that inner product, see [67, 49.1]. If it were bounded, then it would be selfadjoint and one could apply geometry in order to find Eigen-values and Eigen-vectors by minimizing the angle between x and Tx, or equivalently by maximizing $|\langle Tx, x\rangle|$, see [68, 6.5.3]. It is quite obvious that for a selfadjoint bounded operator the supremum of $|\langle Tx, x\rangle|$ is its norm, and that a point were it is attained is an Eigen-vector with maximal absolute Eigen-value. So one needs compactness of T in order to show the existence of such a point. Since Eigen-vectors to different Eigen-values are orthogonal to each other, one can then proceed recursively, provided the operator is compact.

Again the idea is that, although the linear differential-operator F is not bounded, its inverse should be an integral operator G (the Green-operator) with continuous kernel ε and hence compact. And instead of solving $Fu = \lambda u$ we can equally well solve $\frac{1}{\lambda}u = Gu$, see [67, 49.6].

In order to find the Green operator, we have seen in [68, 4.7.7] that a possible solution operator $G : s \mapsto u$ would be given by convolution of s with a Green-function ε, i.e. a solution of $F(\varepsilon) = \delta$, where δ is the neutral element with respect to convolution. In fact, since $u := \varepsilon \star s$ should be a solution of $F(u) = s$, we conclude that $s = F(u) = F(\varepsilon \star s) = F(\varepsilon) \star s$. However such an element doesn't exist in the algebra of smooth functions, and one has to extend the notion of function to include so called generalized functions or distributions. These are the continuous linear functionals on the space $\mathcal{D}$ of smooth functions with compact support.

As we have seen in [68, 4.8.2] the space $\mathcal{D}$ is no longer a Fréchet space, but a strict inductive limit of the Fréchet spaces $C^\infty_K(X) := \{f \in C^\infty : \operatorname{supp} f \subseteq K\}$: If it were Fréchet, then it would be Baire. However the closed linear subspaces C^∞_K have as union $\mathcal{D}$ and have empty interior, since non-empty open sets are absorbing. A contradiction to the Baire-property.

Assume that there is some reasonable Fréchet structure on C_c^∞. Then by the same arguments as before the identity from $\mathcal{D}$ to C_c^∞ would be continuous, hence closed, and hence the inverse to the webbed space $\mathcal{D}$ would be continuous too, i.e. a homeomorphism.

By passing to the transposed, we have seen in [68, 4.9.1] that every linear partial differential operator F can be extended to a continuous linear map $\tilde{F} : \mathcal{D}' \to \mathcal{D}'$, and so one can consider distributional solutions of such differential equations. In [68, 8.3.1] we have proven the Malgrange Ehrenpreis theorem on the existence of distributional fundamental solutions using the generalization of Fourier-series, namely the Fourier-transform $\mathcal{F}$. The idea is that $1 = \mathcal{F}(\delta) = \mathcal{F}(F(\varepsilon)) = \mathcal{F}(p(\partial)(\varepsilon)) = p \cdot \mathcal{F}(\varepsilon)$ and hence $\varepsilon = \mathcal{F}^{-1}(1/p)$. For this we have to consider the Schwartz-space $\mathcal{S}$ of rapidly decreasing smooth functions, which is a Fréchet space, and its dual $\mathcal{S}'$. In order that the poles of $1/p$ make no trouble we had to show that the Fourier-transform of smooth functions with compact support and even of distributions with compact support are entire functions.

If we want to solve *linear partial differential equations with non-constant coefficients* or even NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS, we have to consider not only the linear theory of $\mathcal{D}$ but the non-linear one. See [34] for an approach to this.

0.6 Differentiation and integration commute

Let us consider a much more elementary result. In fact, even in the introductory courses in analysis one considers infinite dimensional results, but usually disguised. Recall the result about DIFFERENTIATION UNDER THE INTEGRAL SIGN. There one considers a function f of two real variables and takes the integral $\int_0^1 f(t,s)\,ds$ with respect to one variable, and then one asks the question: Which assumptions guarantee that the resulting function is differentiable with respect to the remaining variable t and what is its derivative? Before we try to remember the correct answer let us reformulate this result without being afraid of infinite dimensions. We are given the function $f : \mathbb{R} \times I \to \mathbb{R}$, $(t,s) \mapsto f(t,s)$. What do we actually mean by writing down $\int_0^1 f(t,s)\,ds$? – Well we keep t fixed and consider the function $f_t : I \to \mathbb{R}$ given by $s \mapsto f(t,s)$ and integrate it, i.e. $\int_0^1 f(t,s)\,ds := \int(f_t)$, where $\int$ denotes the integration operator $\int : C[0,1] \to \mathbb{R}$, $g \mapsto \int_0^1 g(s)\,ds$. But now we want to vary t, so we have to consider the result as a function $t \mapsto \int(f_t)$, so we have to consider $t \mapsto f_t$ and we denote this function by $\check{f}$. It is given by the formula $\check{f}(t)(s) = f_t(s) = f(t,s)$. Then $\int(f_t) = (\int \circ \check{f})(t)$. Thus what we actually are interested in is, whether the composition $\int \circ \check{f}$ is differentiable and what its derivative is. This problem is usually solved by the CHAIN-RULE, but the situation here is much easier. In fact, recall that integration is linear and continuous with respect to the supremum norm (or even the 1-norm) and $\check{f}$ is a curve (into some function space). So it remains to show that $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$ is differentiable and to find its derivative. Let us assume it is differentiable and try to determine the derivative. On $C(I,\mathbb{R})$ we have nice functionals, namely the POINT EVALUATIONS $\mathrm{ev}_s : g \mapsto g(s)$. These are continuous and linear and separate points (they are far from being all continuous linear functionals, see Riesz's Representation theorem [68, 7.3.3] and [68, 7.3.4]). Applying what we said before in 0.4 to $\ell := \mathrm{ev}_s$ and $c := \check{f}$ we obtain $\mathrm{ev}_s(\check{f}'(t)) = (\mathrm{ev}_s \circ \check{f})'(t)$, and $(\mathrm{ev}_s \circ \check{f})(t) = \mathrm{ev}_s(\check{f}(t)) = \check{f}(t)(s) = f(t,s)$. Hence $\mathrm{ev}_s(\check{f}'(t))$ is nothing else but the first partial derivative $\frac{\partial}{\partial t} f(t,s)$. Conversely, assume that the first partial derivative of f exists on $\mathbb{R} \times I$ and is continuous, then we want to show, that $\check{f}$ is differentiable, and $(\check{f})'(t)(s) = \frac{\partial}{\partial t} f(t,s) = \partial_1 f(t,s)$, or in other words $(\check{f})' = (\partial_1 f)^\vee$.

For this we first consider the corresponding topological problem: Are the continuous mappings $f : \mathbb{R} \times I \to \mathbb{R}$ exactly the continuous maps $\check{f} : \mathbb{R} \to C(I, \mathbb{R})$? This has been solved in the calculus courses. In fact, a mapping $\check{f}$ is well-defined iff $f(x, \_)$ is continuous for all x and $\check{f}$ is continuous iff $f(\_, y)$ is equi-continuous with respect to y, i.e.

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in \mathbb{R}\ \forall y \in I : |x' - x| < \delta \Rightarrow |f(x,y) - f(x',y)| < \varepsilon.$$

However, these two conditions together are equivalent to the continuity of f, as can be seen for example in [**64**, 3.2.8].

Now to the differentiability question. We assume that $\partial_1 f$ exists and is continuous. Hence $(\partial_1 f)^\vee : \mathbb{R} \to C(I, \mathbb{R})$ is continuous. We want to show that $\check{f} : \mathbb{R} \to C(I, \mathbb{R})$ is differentiable (say at 0) with $(\partial_1 f)^\vee$ (at 0) as derivative. So we have to show that the mapping $t \mapsto \frac{\check{f}(t) - \check{f}(0)}{t}$ is continuously extendable to $\mathbb{R}$ by defining its value at 0 as $(\partial_1 f)^\vee(0)$. Or equivalently, by what we have shown for continuous maps before, that the map

$$(t,s) \mapsto \begin{cases} \frac{f(t,s) - f(0,s)}{t} & \text{for } t \neq 0 \\ \partial_1 f(0,s) & \text{otherwise} \end{cases}$$

is continuous. This follows immediately from the continuity of ∂_1 and that of $\int_0^1 \_\, dr$, since it can be written as $\int_0^1 \partial_1 f(r\, t, s)\, dr$ by the fundamental theorem.

So we arrive under this assumption at the conclusion, that $\int_0^1 f(t,s)\, ds$ is differentiable with derivative

$$\frac{d}{dt} \int_0^1 f(t,s)\, ds = \int((\check{f})'(t)) = \int_0^1 \frac{\partial}{\partial t} f(t,s)\, ds$$

and we have proved the

Proposition. *For a continuous map $f : \mathbb{R} \times I \to \mathbb{R}$ the partial derivative $\partial_1 f$ exists and is continuous iff $\check{f} : \mathbb{R} \to C(I, \mathbb{R})$ is continuously differentiable. And in this situation $\int((\check{f})'(t)) = \frac{d}{dt} \int_0^1 f(t,s)\, ds = \int_0^1 \frac{\partial}{\partial t} f(t,s)\, ds$.* □

And we see, it is much more natural to formulate and prove this result with the help of the infinite dimensional space $C([0,1], \mathbb{R})$. But this not only clarifies the proof, but is of importance for its own sake, as we will see in 0.8.

0.7 Exponential law for continuous mappings

Let us try to generalize this result. We will write Y^X for the function spaces $C(X, Y)$ for reasons of cardinality. So the question is whether the continuous mappings $f : X \times Y \to Z$ correspond exactly to the continuous maps $\check{f} : X \to C(Y, Z)$?

For this we need a topology on $C(X, Y)$. If Y is a locally convex space (or a uniform space) we can use the topology of uniform convergence on compact subsets of X, given by the seminorms $f \mapsto \sup\{q(f(x)) : x \in K\}$, where $K \subseteq X$ runs through the compact subsets and q through the seminorms of Y, see [**68**, 3.2.8]. For general Y we consider the compact-open topology, which has as subbasis the sets $N_{K,U} := \{f : f(K) \subseteq U\}$ where K runs through (a basis of) the compact subsets of X and Y through (a basis of) the open subsets of Y, see [**72**, 2.4.2].

Let us show first that for locally convex spaces F and topological spaces X the compact-open topology is the locally convex topology of uniform convergence on compact subsets:

So let $K \subseteq X$ be compact, $V \subseteq F$ be open, and $f \in N_{K,V}$, i.e. $f(K) \subseteq V$. Then for each $x \in K$ there exists a seminorm q on F and an $\varepsilon > 0$ such that $V_{f(x)} := \{y \in F : q(y - f(x)) < \varepsilon\} \subseteq V$. The sets $U_x := \{z' \in X : q(f(z') - f(x)) < \frac{\varepsilon}{2}\}$ with $x \in K$ form an open converging, so there are finitely many $x_1, \dots, x_n$ with $K \subseteq \bigcup_{i=1}^n U_i$, where $U_i := U_{x_i}$. Let q_i be the seminorm and ε_i the radius corresponding to x_i and $K_i := \{z' \in K : q_i(f(z') - f(x_i)) \leq \frac{\varepsilon_i}{2}\}$. We claim, that $q_i(g(x) - f(x)) < \frac{\varepsilon_i}{2}$ for all i and $x \in K_i$ implies $g \in N_{K,U}$. In fact, let $x \in K$, then there exists an i with $x \in U_i \cap K \subseteq K_i$ and hence $q_i(g(x) - f(x_i)) \leq q_i(g(x) - f(x)) + q_i(f(x) - f(x_i)) < \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} = \varepsilon_i$, i.e. $g(x) \in V_{f(x_i)} \subseteq V$.

Conversely, let a compact $K \subseteq X$, a seminorm q on F, an $\varepsilon > 0$, and $f \in C(X, F)$ be given. Note that $g \in C(X, F)$ is a subset of $W := \{(x, y) : x \in K \Rightarrow q(y - f(x)) < \varepsilon\}$ iff $q(g(x) - f(x)) < \varepsilon$ for all $x \in K$. For $x \in K$ let $U_x := \{x' : q(f(x') - f(x)) < \frac{\varepsilon}{3}\}$ and take finitely many $x_1, \dots, x_n$ such that the $U_i := U_{x_i}$ cover K. Let $K_i := \{x \in K : q(f(x) - f(x_i)) \leq \frac{\varepsilon}{3}\}$ and $V_i := \{y : q(y - f(x_i)) < \frac{\varepsilon}{2}\}$ then $f(K_i) \subseteq V_i$. If $g \in \bigcap_i N_{K_i,V_i}$ then for each $x \in K$ there exists an i with $x \in U_i \cap K \subseteq K_i$ and thus $q(g(x) - f(x)) \leq q(g(x) - f(x_i)) + q(f(x_i) - f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon$, i.e. $g \subseteq W$.

How is $\hat{g} : X \times Y \to Z$ constructed from a continuous $g : X \to Z^Y$? Well, one can consider $g \times Y : X \times Y \to Z^Y \times Y$ and compose it with the evaluation map $\mathrm{ev} : Z^Y \times Y \to Z$. Since the product of continuous maps is continuous, it remains to show that the evaluation map is continuous in order to obtain that $\hat{g}$ is continuous. So let $f \in Z^Y$ and $y \in Y$ and let U be a neighborhood of $f(y)$. If Y is locally compact, we can find a compact neighborhood $W \subseteq f^{-1}(U)$ of y and then $f \in N_{W,U} := \{g : g(W) \subset U\}$ and $\mathrm{ev}(N_{W,U} \times W) \subseteq U$.

Conversely let a continuous $f : X \times Y \to Z$ be given. We consider $f_* := f^Y : (X \times Y)^Y \to Z^Y$ and compose it from the right with the insertion map $\mathrm{ins} : X \to (X \times Y)^Y$ given by $x \mapsto (y \mapsto (x, y))$. Then we arrive at $\check{f}$. Obviously f_* is continuous since $(f_*)^{-1} N_{K,U} = N_{K,f^{-1}U}$. The insertion map is continuous, since $\mathrm{ins}^{-1}(N_{K,U\times V}) = U$ if $K \subseteq V$ and is empty otherwise, so $\check{f}$ is continuous. Thus the only difficult part was the continuity of the evaluation map.

Moreover we have the

Proposition. *Let X, Y and Z be topological spaces with Y being locally compact. Then we have a homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$, given by $f \mapsto \check{f}$, where the function spaces carry the compact open topology.*

Proof. We have already proved that we have a bijection. That this gives a homeomorphism follows, since the corresponding subbases $N_{K_1\times K_2,U}$ and $N_{K_1,N_{K_2,U}}$ correspond to each other. □

In general the compact open topology on Z^Y will not be locally compact even for locally compact spaces Y and Z (e.g. $C([0,1], \mathbb{R})$ is an infinite dimensional and hence not locally compact Banach space). So in order to get an intrinsic exponential law, one can modify the notion of continuity and call a mapping $f : X \to Y$ between Hausdorff topological spaces compactly-continuous iff its restriction to every compact subset $K \subseteq X$ is continuous. Thus $f : X \times Y \to Z$ is continuous iff $f|_{K\times L} : K \times L \to Z$ is continuous for all compact subsets $K \subseteq X$ and $L \subseteq Y$. By the exponential law for compact sets this is equivalent to $\check{f} : K \to Z^L$ being continuous. Since Z^Y carries the initial structure with respect to $\mathrm{inkl}^*: Z^Y \to Z^L$, this is furthermore equivalent to the continuity of $\check{f}: K \to Z^Y$, and thus to $\check{f}: X \to Z^Y$ being compactly-continuous, but for this we have to denote with Z^Y the space of compactly continuous maps from $Y \to Z$.

Instead of the category of compactly continuous maps between Hausdorff topological spaces, one can use the EQUIVALENT CATEGORY (see [70, 1.22]) of continuous mappings between compactly generated spaces. The equivalence between these two categories is given by the identity functor on one side, and on the other side by the Kelley-fication, i.e. by replacing the topology by the final topology with respect to the compact subsets. Note that the identity is compactly continuous in both directions. However, the natural topology on the products in this category is the Kelley-fication of the product topology and also on the function spaces one has to consider the Kelly-fication of the compact open topology, see [72, 2.4].

0.8 Variational calculus

In physics one is not a priori given an equation $f(x) = 0$, but often some OPTIMIZATION PROBLEM, i.e. the search for those x, for which the values $f(x)$ of some real-valued function (like the LAGRANGE FUNCTION in classical mechanics, which is given by the difference of kinematic energy and the potential) attain an extremum (i.e. are minimal or maximal), see for example [67, 45]. Again x is often not a finite dimensional vector but a function and $f(x)$ is given by some integral (like the action (german: Wirkungsintegral) in classical mechanics)

$$f(x) := \int_0^1 F(t, x(t), x'(t))\, dt.$$

For finite dimensional vectors x one finds solutions of the problem $f(x) \to \min$ by applying differential calculus and searching for solutions of $f'(x) = 0$. In infinite dimensions one proceeds similarly in the calculus of variations (see [66, 9.4.3]), by finding those points x, where the directional derivatives $f'(x)(v)$ vanish for all directions v. Since the boundary values of x are usually given, the variation v has to vanish on the boundary $\{0, 1\}$. One can calculate the directional derivative by what we have shown before as follows:

$$\begin{aligned}
f'(x)(v) &:= \frac{d}{dt}\Big|_{t=0} f(x + tv) \\
&= \frac{d}{dt}\Big|_{t=0} \int_0^1 F\Big(s, (x+tv)(s), (x+tv)'(s)\Big)\, ds \\
&= \int_0^1 \frac{\partial}{\partial t}\Big|_{t=0} F\Big(s, (x+tv)(s), (x+tv)'(s)\Big)\, ds \\
&= \int_0^1 \Big(\partial_2 F(s, x(s), x'(s)) \cdot v(s) + \partial_3 F(s, x(s), x'(s)) \cdot v'(s)\Big)\, ds \\
&= \int_0^1 \Big(\partial_2 F(s, x(s), x'(s)) - \frac{d}{ds}\partial_3 F(s, x(s), x'(s))\Big) \cdot v(s)\, ds
\end{aligned}$$

We have used partial integration and that the variation v has to vanish at the boundary points 0 and 1. Since $f'(x)(v)$ has to be 0 for all such v we arrive at the EULER-LAGRANGE PARTIAL DIFFERENTIAL EQUATION

$$\partial_2 F(s, x(s), x'(s)) = \frac{d}{ds}\partial_3 F(s, x(s), x'(s)),$$

or with slight abuse of notation:

$$\frac{\partial}{\partial x}F = \left(\frac{\partial}{\partial \dot{x}}F\right)^{\cdot},$$

where $(\_)^{\cdot}$ denotes the derivative with respect to time s.

Warning: Abuse may lead to disaster! In physics for example one has the GAS-EQUATION $p \cdot V \cdot t = 1$, where p is pressure, V the volume and t the temperature scaled appropriately. So we obtain the following partial derivatives:

$$\begin{aligned}
\frac{\partial p}{\partial V} &= \frac{\partial}{\partial V}\frac{1}{V\,t} = -\frac{1}{t\,V^2}\\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{t\,p} = -\frac{1}{p\,t^2}\\
\frac{\partial t}{\partial p} &= \frac{\partial}{\partial p}\frac{1}{p\,V} = -\frac{1}{V\,p^2}
\end{aligned}$$

And hence cancellation yields

$$1 = \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial p} = (-1)^3\frac{1}{t\,V^2}\cdot\frac{1}{p\,t^2}\cdot\frac{1}{V\,p^2} = -\frac{1}{(pVt)^3} = -1.$$

Try to find the mistake!

0.9 Flows as 1-parameter subgroups of diffeomorphisms

Another situation, where it is natural to consider differentiable curves into function spaces, occurs, when considering time-independent ordinary differential equations, i.e. equations of the form $\dot{u} = f(u)$. For given initial value $u(0) = x$ we can consider the solution u_x and obtain a mapping $u : \mathbb{R} \times X \to X$ given by $(t, x) \mapsto u_x(t)$. Obviously $u(0, x) = x$ and by uniqueness we have $u(t + s, x) = u(t, u(s, x))$, i.e. u is a FLOW on X, see [**67**, 28.3]. Conversely, we can reconstruct the differential equation by differentiating the flow with respect to t at $t = 0$, i.e. $\frac{\partial}{\partial t}|_{t=0}u(t, x) = f(u(t, x))|_{t=0} = f(x)$. It would be more natural to consider the associate mapping $\check{u}$ with values in some space of mappings from $X \to X$, since then the flow property translates into the assumption that $t \mapsto \check{u}(t)$ is a group-homomorphism from $\mathbb{R}$ into the group of invertible maps on X. The VECTOR FIELD f can thus be interpreted as the tangent vector $\check{u}'(0)$ at 0 of the curve $\check{u}$. So $\check{u}$ should be differentiable into a group DIFF(X) OF DIFFEOMORPHISMS on X where this group should carry some smooth structure, analogously to classical Lie-groups. In particular the composition $\mathrm{Diff}(X) \times \mathrm{Diff}(X) \to \mathrm{Diff}(X)$ map should be differentiable. Since $(f, g) \mapsto f \circ g$ is linear in the first variable (if we consider the range space X as submanifold of some $\mathbb{R}^n$), the difficult part is the differentiability in the second variable, i.e. that of the map $f_* : g \mapsto f \circ g$. We have noted at the end of 0.4 that for f_* to be differentiable we need that f is differentiable since $(f_*)' = (f')_*$. Thus in order that the composition map is differentiable, we need that its first variable f is differentiable, hence Diff should mean at least 1-times differentiable. But then in order that the derivative of the composition map has 1-time differentiable values we need that f' is 1-times differentiable, i.e. f is twice differentiable. Inductively we arrive at the smoothness of f, i.e. infinite often differentiability. But as we have mentioned before, even in the simplest case $C^\infty([0, 1], \mathbb{R})$ or $C^\infty(S^1, \mathbb{R})$, these function spaces are not Banach-spaces anymore, but Fréchet-spaces.

More generally, let M and N be finite dimensional manifolds with M being compact. We would like to idenitfy the space $\mathcal{F} := C^\infty(M, N)$ of smooth maps $M \to N$ as an (infinite dimensional) manifold. In order to find a candidate for its charts the tangent space $T_f\mathcal{F}$ should be formed by velocity vectors $c'(0)$ to curves $c : \mathbb{R} \to \mathcal{F}$ with $c(0) = f \in \mathcal{F}$. As before $c'(0)(x) = \frac{\partial}{\partial t}|_{t=0}\hat{c}(t,x) \in T_{f(x)}N$, i.e. these velocity vectors are given by vector fields $\bar{s} \in C^\infty(M, TN)$ along f (i.e. $\pi_N \circ \bar{s} = f$).

$$\begin{tikzcd}
M \times TN \arrow[rrd, bend left, "\mathrm{pr}_2"] \arrow[rdd, bend right, "\mathrm{pr}_1"'] & & \\
& f^*(TN) \arrow[lu, hook'] \arrow[r, "\pi_N^*(f)"'] \arrow[d, "f^*(\pi_N)"] & TN \arrow[d, "\pi_n"] \\
& M \arrow[u, dotted, bend left, "s"] \arrow[r, "f"'] \arrow[ru, dotted, bend right, "\bar{s}"] & N
\end{tikzcd}$$

If we supply N with a Riemannian metric, then we have the locally defined exponential mapping $\exp : TN \to N$, which gives a local diffeomorphism $(\pi, \exp) : TN \to N \times N$ along the 0-section $0 : N \hookrightarrow TN$. We can use exp to define **charts** φ_f centered at $f \in C^\infty(M, N)$ by

$$\varphi_f : C^\infty(M \leftarrow f^*TN) \cong \Big\{\bar{s} \in C^\infty(M, TN) : \pi_N \circ \bar{s} = f\Big\} \to C^\infty(M, N),$$
$$\varphi_f(s) := \exp \circ \pi_N^* f \circ s = \exp \circ \bar{s} : M \to f^*TN \to TN \to N,$$
$$\varphi_f^{-1}(g)(x) = \Big(x, \exp_{f(x)}^{-1}(g(x))\Big) = \Big(\mathrm{id}_M, (\pi_N, \exp)^{-1} \circ (f, g)\Big)(x) \in f^*TN$$

Their **transition mappings** are locally given by

$$\varphi_{f_2}{}^{-1} \circ \varphi_{f_1} : s \mapsto \Big(\mathrm{id}_M, (\pi_N, \exp)^{-1} \circ (f_2, \exp \circ \pi_N^* f_1 \circ s)\Big) = (\tau_{f_2}^{-1} \circ \tau_{f_1})_*(s),$$

where $\tau_f := \mathrm{id} \times \exp : M \times TN \supseteq f^*TN \to M \times N$

$$\tau_f(x, Y_{f(x)}) = \big(x, \exp_{f(x)}(Y_{f(x)})\big)$$
$$\tau_f^{-1} = \big(\mathrm{pr}_1, (\pi, \exp)^{-1} \circ (f \times \mathrm{id})\big) : M \times N \to M \times (N \times N) \to f^*TN \subseteq M \times TN$$

is locally a smooth fiber respecting diffeomorphism over M.

For the diffeomorphism group we even need that the **composition** is **smooth** in both variables jointly. This will follow easily from the exponential law.

0.10 Exponential law for differentiable mappings

A similar thing happens when searching for an exponential law for differentiable functions. If we want a nice correspondence between differentiable functions on a product and differentiable functions into a function space, we have seen in 0.6 that a curve $c : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ is C^1 if and only if $\partial_1\hat{c} : \mathbb{R}^2 \to \mathbb{R}$ exists and is continuous. If we want a (differentiability-)property which is invariant under base-change in $\mathbb{R}^2$, then also $\partial_2\hat{c} : \mathbb{R}^2 \to \mathbb{R}$ should exist and be continuous, and hence $c : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ should have values in $C^1(\mathbb{R}, \mathbb{R})$ and $d \circ c : \mathbb{R} \to C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ should be continuous. Summarizing $\hat{c} : \mathbb{R}^2 \to \mathbb{R}$ is C^1 if and only if $c : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ is C^1 (with derivative $c'(t)^\wedge = \partial_1\hat{c}$) and is C^0 into $C^1(\mathbb{R}, \mathbb{R})$ (with $(d \circ c)^\wedge = \partial_2\hat{c}$). So if we want to use just a single functions space (instead of $C^0(\mathbb{R}, \mathbb{R})$ and $C^1(\mathbb{R}, \mathbb{R})$ at the same time) we should assume $c : \mathbb{R} \to C^1(\mathbb{R}, \mathbb{R})$ to be C^1. But then $c' : \mathbb{R} \to C^1(\mathbb{R}, \mathbb{R})$ has to be continuous, and thus $d \circ c' : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ has to be continuous, i.e. $(d \circ c')^\wedge = \partial_2\partial_1\hat{c} : \mathbb{R}^2 \to \mathbb{R}$ should be continuous. Assumed invariance under base-change yields that $\hat{c} : \mathbb{R}^2 \to \mathbb{R}$ should be C^2 and then $\hat{c} : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ has to be C^2, $\hat{c} : \mathbb{R} \to C^1(\mathbb{R}, \mathbb{R})$ has to be C^1, and $\hat{c} : \mathbb{R} \to C^2(\mathbb{R}, \mathbb{R})$ has to be C^0.

Inductively we get that the exponential law for differentiable functions can only be valid for C^∞-functions (for details see 3.2).

0.11 Continuity of the derivative

Well, as has been discovered around 1900, the derivative should be a linear (more precisely, an affine) approximation to the function. Assume we have already defined the concept of DERIVATIVE $f'(x) \in L(E,F)$ for functions $f : E \supseteq U \to F$ at a given point $x \in U$. By collecting for all x in the open domain U of f these derivatives $f'(x)$, we obtain a mapping $x \mapsto f'(x)$, the derivative $f' : E \supseteq U \to L(E,F)$ with values in the space of continuous linear mappings. In order to speak about continuous differentiable (short: C^1) mappings, we need some topology on $L(E,F)$ and then this amounts to the assumption, that $f' : U \to L(E,F)$ is continuous. For C^1-maps we should have a CHAIN-RULE, which guarantees that the composite $f \circ g$ of C^1-maps is again C^1 and the derivative should be $(f \circ g)'(x) = f'(g(x)) \circ g'(x)$. This map is thus given by the following description: For any given x, first calculate $g(x)$ and then $f'(g(x)) \in L(F,G)$ and $g'(x) \in L(E,F)$, and finally apply the composition map $L(F,G) \times L(E,F) \to L(E,G)$ to obtain $f'(g(x)) \circ g'(x)$. Since f and g are assumed to be C^1 the components $f' \circ g$ and g' are continuous. So it remains to show the continuity of the composition mapping. Let us consider the simplified case where $G = E = \mathbb{R}$. Then composition reduces to the evaluation map $\mathrm{ev} : F' \times F \to \mathbb{R}$ and we are looking for a topology on F' such that this map is continuous. Assume we have found such a topology. Then there exists 0-neighborhoods V in F' and U in F such that $\mathrm{ev}(V \times U) \subseteq [-1,1]$. Since scalar-multiplication on F' should be continuous, we can find for every $\ell \in F'$ a number $K > 0$, such that $\ell \in K V$. Thus for $x \in U$ we have $\ell(x) = \mathrm{ev}(K \frac{1}{K}\ell, x) = K\,\mathrm{ev}(\frac{1}{K}\ell, x) \in K\,\mathrm{ev}(V \times U) \subseteq [-K,K]$. This shows that U is scalarly bounded, and hence is bounded by the corollary in [**68**, 5.2.7]. However, a locally convex space, which has a bounded 0-neighborhood has to be normed, by Kolmogoroff's theorem [**68**, 2.6.2].

So it seems that there is no reasonable notion of C^1, which applies to more than just functions between Banach spaces. However, we have assumed that continuity is meant with respect to topologies. In fact, there have been several (more or less successful) attempts in the past to remedy this situation by considering convergence structures on $L(E,F)$. If one defines that a net (or a filter) f_α should converge to f in $L(E,F)$ iff for nets (or filters) x_β converging to some x in E the net (or filter) $f_\alpha(x_\beta)$ should converge to $f(x)$, then the evaluation map, and more generally the composition map becomes continuous. A second way to come around this problem, is to assume for C^1 the continuity of $\widehat{f'} : U \times E \to F$ instead. Then the chain-rule becomes easy. However this notion is bad, since we cannot prove the INVERSE FUNCTION THEOREM for C^1 even for Banach spaces, see [**65**, 6.2.1] and [**65**, 6.3.15]. See [**65**, 6.1.19] for an example of a differentiable function f on a Hilbert space for which $\widehat{f'}$ is continuous, but f' is not. This examples shows in particular that the exponential law is wrong for continuous functions $\ell^2 \times \ell^2 \to \mathbb{R}$ which are linear in the second variable if one uses the operator norm on $L(\ell^2, \mathbb{R}) = (\ell^2)' \cong \ell^2$.

0.12 Derivatives of higher order

If we want to define higher derivatives - as needed in conditions for local extrema and the like - we would call a function f by recursion $(n+1)$-times differentiable iff f' exists and is n-times differentiable (D^n for short). In order to show that the composite $f \circ g$ of two D^2-maps is again D^2, we have to show that $(f \circ g)' : x \mapsto f'(g(x)) \circ g'(x)$ is again D^1. By the chain-rule for D^1-mappings, we would obtain that $f' \circ g \in D^1$ and by assumption $g' \in D^1$. So it remains to differentiate the bilinear composition map. Since it is linear in both entries separately, its partial derivatives should obviously exist and the derivative also. But recall that it is not even continuous.

0.13 Résumé

We have learned a few things from these introductory words:

1. Problems in finite dimensions often have a more natural formulation (and proof) involving infinite dimensional function-spaces, which are quite often not Banach spaces, but Fréchet spaces like $C(\mathbb{R},\mathbb{R})$ and $C^\infty(I,\mathbb{R})$ or even more general ones like $\mathcal{D}$ and $\mathcal{D}'$.
2. Mappings of two variables $f : X \times Y \to Z$, should often be considered as mappings $\check{f}$ from X to a space of mappings from Y to Z and properties such as continuity or differentiability should translate nicely. For differentiability this can only be true for C^∞.
3. It is not clear, how to obtain the basic ingredient to calculus, the chain-rule. For this the composition map, or at least the evaluation map, should be smooth, although it is not continuous in the topological setting.
4. There is no reasonable notion of C^1 generalizing classical (Fréchet-)calculus to mappings between spaces beyond Banach spaces.

After having found lots of, at first view devastating, difficulties, let's look what can be done easily:

1. It is obvious what differentiability for a curve c into any locally convex space means, since limits of difference quotients make sens. Hence we have also the notion of continuous differentiable, of n-times differentiable, and of smoothness for such curves.
2. Continuous (multi-)linear mappings preserve smoothness of curves, and satisfy the chain-rule.
3. Directional derivatives can be easily defined for mappings f between arbitrary locally convex spaces, since they are just derivatives of the curves $c : t \mapsto f(x + t\,v)$ obtained by composing f with an affine line $t \mapsto x + t\,v$
4. Candidates for derivatives $f'(x)$ of mappings f can be obtained by reduction to 1-dimensional analysis via affine mappings: $\ell(f'(x)(v)) = \frac{d}{dt}|_{t=0}\ell(f(x + t\,v))$.

Chapter I
Calculus of Smooth Mappings

This chapter is devoted to calculus of smooth mappings in infinite dimensions. The leading idea of our approach is to base everything on smooth curves in locally convex spaces, which is a notion without problems, and a mapping between locally convex spaces will be called smooth if it maps smooth curves to smooth curves.

We start by looking at the set of smooth curves $C^\infty(\mathbb{R}, E)$ with values in a locally convex space E, and note that it does not depend on the topology of E, only on the underlying system of bounded sets, its bornology. This is due to the fact, that for a smooth curve difference quotients converge to the derivative much better 2.1 than arbitrary converging nets or filters: we may multiply it by some unbounded sequences of scalars without disturbing convergence (or, even better, boundedness).

Then the basic results are proved, like existence, smoothness, and linearity of derivatives, the chain rule 3.18, and also the most important feature, the 'exponential law' 3.12 and 3.13: We have

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

without any restriction, for a natural structure on $C^\infty(F, G)$.

Smooth curves have integrals in E if and only if a weak completeness condition is satisfied: it appeared as bornological completeness, Mackey completeness, or local completeness in the literature, we call it c^∞-complete. This is equivalent to the condition that weakly smooth curves are smooth 2.14. All calculus in later chapters in this book will be done on CONVENIENT VECTOR SPACES: These are locally convex vector spaces which are c^∞-complete; note that the locally convex topology on a convenient vector space can vary in some range, only the system of bounded sets must remain the same.

Linear or more generally multilinear mappings are smooth if and only if they are bounded 5.5, and one has corresponding exponential laws 5.2 for them as well. Furthermore, there is an appropriate tensor product, the bornological tensor product 5.7, satisfying

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G)).$$

An important tool for convenient vector spaces are uniform boundedness principles as given in 5.18, 5.24 and 5.26.

It is very natural to consider on E the final topology with respect to all smooth curves, which we call the c^∞-topology, since all smooth mappings are continuous for it: the vector space E, equipped with this topology is denoted by $c^\infty E$, with lower case c in analogy to kE for the Kelley-fication and in order to avoid any confusion with any space of smooth functions or sections. The special curve lemma 2.8 shows that the c^∞-topology coincides with the usual Mackey closure topology. The space $c^\infty E$ is not a topological vector space in general. This is related to the fact that the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous only for normable E, but it is always smooth and hence continuous on $c^\infty(E \times E')$. The c^∞-open subsets are the natural domains of definitions of locally defined functions. For nice spaces (e.g. Fréchet and strong duals of Fréchet-Schwartz spaces, see 4.11) the c^∞-topology coincides with the given locally convex topology. In general, the c^∞-topology is finer than any locally convex topology with the same bounded sets.

In the last section of this chapter we discuss the structure of spaces of smooth functions on finite dimensional manifolds and, more generally, of smooth sections of finite dimensional vector bundles. They will become important in chapter IX as modeling spaces for manifolds of mappings. Furthermore, we give a short account of reflexivity of convenient vector spaces and on (various) approximation properties for them.

1. Smooth Curves

1.1. Notation

Since we want to have unique derivatives all locally convex spaces E will be assumed Hausdorff. The family of all bounded sets in E plays an important rôle. It is called the *bornology* of E. A linear mapping is called *bounded*, sometimes also called bornological, if it maps bounded sets to bounded sets. A bounded linear bijection with bounded inverse is called *bornological isomorphism*. The *space of all continuous linear functionals on* E will be denoted by E^* and the *space of all bounded linear functionals on* E by E'. The *adjoint* or *dual mapping* of a linear mapping ℓ, however, will be always denoted by ℓ^*, because of differentiation.

1.2. Differentiable curves

The concept of a smooth curve with values in a locally convex vector space is easy and without problems. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *differentiable* if the *derivative* $c'(t) := \lim_{s\to 0} \frac{1}{s}(c(t+s) - c(t))$ at t exists for all t. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all iterated derivatives exist. It is called C^n for some finite n if its iterated derivatives up to order n exist and are continuous.

A curve $c : \mathbb{R} \to E$ is called *locally Lipschitzian* if every point $r \in \mathbb{R}$ has a neighborhood U such that the *Lipschitz condition* is satisfied on U, i.e., the set

$$\Big\{\tfrac{1}{t-s}\Big(c(t) - c(s)\Big) : t \neq s; t, s \in U\Big\}$$

is bounded. Note that this implies that the curve satisfies the Lipschitz condition on each bounded interval, since for (t_i) increasing

$$\frac{c(t_n)-c(t_0)}{t_n-t_0}=\sum\frac{t_{i+1}-t_i}{t_n-t_0}\,\frac{c(t_{i+1})-c(t_i)}{t_{i+1}-t_i}$$

is in the absolutely convex hull of a finite union of bounded sets.

A curve $c:\mathbb{R}\to E$ is called $\mathcal{L}ip^k$ or $C^{(k+1)-}$ if all derivatives up to order k exist and are locally Lipschitzian.

1.3. Lemma. Continuous linear mappings are smooth. *A continuous linear mapping $\ell: E\to F$ between locally convex vector spaces maps $\mathcal{L}\mathrm{ip}^k$-curves in E to $\mathcal{L}\mathrm{ip}^k$-curves in F, for all $0\le k\le\infty$, and for $k>0$ one has $(\ell\circ c)'(t)=\ell(c'(t))$.*

Proof. As a linear map ℓ commutes with the formation of difference quotients, hence the image of a Lipschitz curve is Lipschitz since ℓ is bounded.

As a continuous map it commutes with the formation of the respective limits. Hence $(\ell\circ c)'(t)=\ell(c'(t))$.

Now the rest follows by induction. □

Note that a differentiable curve is continuous, and that a continuously differentiable curve is locally Lipschitzian: For $\ell\in E^*$ we have

$$\ell\left(\frac{c(t)-c(s)}{t-s}\right)=\frac{(\ell\circ c)(t)-(\ell\circ c)(s)}{t-s}=\int_0^1(\ell\circ c)'(s+(t-s)r)dr,$$

which is bounded, since $(\ell\circ c)'=\ell\circ c'$ is locally bounded. Since boundedness can be tested by continuous linear functionals (see [68, 5.2.7]) we conclude that c is locally Lipschitzian.

More general, we have by induction the following implications:

$$C^{n+1}\implies\mathcal{L}\mathrm{ip}^n\implies C^n,$$
$$\text{differentiable}\implies C.$$

1.4. The mean value theorem

In classical analysis the basic tool for using the derivative to get statements on the original curve is the mean value theorem. So we try to generalize it to infinite dimensions. For this let $c:\mathbb{R}\to E$ be a differentiable curve. If $E=\mathbb{R}$ the classical mean value theorem states, that the difference quotient $(c(a)-c(b))/(a-b)$ equals some intermediate value of c'. Already if E is two dimensional this is no longer true. Take for example a parameterization of the circle by arclength. However, we will show that $(c(a)-c(b))/(a-b)$ lies still in the closed convex hull of $\{c'(r):r\}$. Having weakened the conclusion, we can try to weaken the assumption. And in fact c may be not differentiable in at most countably many points. Recall however, that there exist strictly monotone functions $f:\mathbb{R}\to\mathbb{R}$, which have vanishing derivative outside a Cantor set (which is uncountable, but has still measure 0).

Sometimes one uses in one dimensional analysis a generalized version of the mean value theorem: For an additional differentiable function h with non-vanishing derivative the quotient $(c(a)-c(b))/(h(a)-h(b))$ equals some intermediate value of c'/h'. A version for vector valued c (for real valued h) is that $(c(a)-c(b))/(h(a)-h(b))$ lies in the closed convex hull of $\{c'(r)/h'(r):r\}$. One can replace the assumption that h' vanishes nowhere by the assumption that h' has constant sign, or, more generally, that h is monotone. But then we cannot form the quotients, so we should assume that $c'(t) \in h'(t) \cdot A$, where A is some closed convex set, and we should be able to conclude that $c(b) - c(a) \in (h(b) - h(a)) \cdot A$. This is the version of the *mean value theorem* that we are going to prove now. However, we will make use of it only in the case where $h = \mathrm{Id}$ and c is everywhere differentiable in the interior.

Proposition. Mean value theorem. *Let $c : [a, b] =: I \to E$ be a continuous curve, which is differentiable except at points in a countable subset $D \subseteq I$. Let h be a continuous monotone function $h : I \to \mathbb{R}$, which is differentiable on $I \setminus D$. Let A be a convex closed subset of E, such that $c'(t) \in h'(t) \cdot A$ for all $t \notin D$.*

Then $c(b) - c(a) \in (h(b) - h(a)) \cdot A$.

Proof. Assume that this is not the case. By the theorem of Hahn Banach [**68**, 7.2.1] there exists a continuous linear functional ℓ with $\ell(c(b)-c(a)) \notin \overline{\ell((h(b) - h(a)) \cdot A)}$. But then $\ell \circ c$ and $\overline{\ell(A)}$ satisfy the same assumptions as c and A, and hence we may assume that c is real valued and A is just a closed interval $[\alpha, \beta]$. We may furthermore assume that h is monotonely increasing. Then $h'(t) \geq 0$, and $h(b) - h(a) \geq 0$. Thus the assumption says that $\alpha h'(t) \leq c'(t) \leq \beta h'(t)$, and we want to conclude that $\alpha(h(b) - h(a)) \leq c(b) - c(a) \leq \beta(h(b) - h(a))$. If we replace c by $c - \beta h$ or by $\alpha h - c$ it is enough to show that $c'(t) \leq 0$ implies that $c(b) - c(a) \leq 0$. For given $\varepsilon > 0$ we will show that $c(b) - c(a) \leq \varepsilon(b - a + 1)$. For this let J be the set $\{t \in [a, b] : c(s) - c(a) \leq \varepsilon\left((s - a) + \sum_{t_n<s} 2^{-n}\right) \text{ for } a \leq s < t\}$, where $D =: \{t_n : n \in \mathbb{N}\}$. Obviously, J is a closed interval containing a, say $[a, b']$. By continuity of c we obtain that $c(b') - c(a) \leq \varepsilon\left((b' - a) + \sum_{t_n<b'} 2^{-n}\right)$. Suppose $b' < b$. If $b' \notin D$, then there exists a subinterval $[b', b' + \delta]$ of $[a, b]$ such that for $b' \leq s < b' + \delta$ we have $c(s) - c(b') - c'(b')(s - b') \leq \varepsilon(s - b')$. Hence we get

$$c(s) - c(b') \leq c'(b')(s - b') + \varepsilon(s - b') \leq \varepsilon(s - b'),$$

and consequently

$$\begin{aligned} c(s) - c(a) &\leq c(s) - c(b') + c(b') - c(a) \\ &\leq \varepsilon(s - b') + \varepsilon\Big(b' - a + \sum_{t_n<b'} 2^{-n}\Big) \leq \varepsilon\Big(s - a + \sum_{t_n<s} 2^{-n}\Big). \end{aligned}$$

On the other hand if $b' \in D$, i.e., $b' = t_m$ for some m, then by continuity of c we can find an interval $[b', b' + \delta]$ contained in $[a, b]$ such that for all $b' \leq s < b' + \delta$ we have

$$c(s) - c(b') \leq \varepsilon 2^{-m}.$$

Again we deduce that

$$c(s) - c(a) \leq \varepsilon 2^{-m} + \varepsilon\Big(b' - a + \sum_{t_n<b'} 2^{-n}\Big) \leq \varepsilon\Big(s - a + \sum_{t_n<s} 2^{-n}\Big).$$

So we reach in both cases a contradiction to the maximality of b'. □

Warning: One cannot drop the monotonicity assumption. In fact take $h(t) := t^2$, $c(t) := t^3$ and $[a, b] = [-1, 1]$. Then $c'(t) \in h'(t)[-2, 2]$, but $c(1) - c(-1) = 2 \notin \{0\} = (h(1) - h(-1))[-2, 2]$.

1.5. Testing with functionals

Recall that in classical analysis vector valued curves $c : \mathbb{R} \to \mathbb{R}^n$ are often treated by considering their components $c_k := \mathrm{pr}_k \circ c$, where $\mathrm{pr}_k : \mathbb{R}^n \to \mathbb{R}$ denotes the canonical projection onto the k-th factor $\mathbb{R}$. Since in general locally convex spaces do not have appropriate bases, we use all continuous linear functionals instead of the projections pr_k. We will say that a property of a curve $c : \mathbb{R} \to E$ is *scalarly true*, if $\ell \circ c : \mathbb{R} \to E \to \mathbb{R}$ has this property for each continuous linear functionals ℓ on E.

We want to compare scalar differentiability with differentiability. For finite dimensional spaces it is a trivial fact that these two notions coincide. For infinite dimensions we first consider $\mathcal{L}$ip-curves $c : \mathbb{R} \to E$. Since by [**68**, 5.2.7] boundedness can be tested by the continuous linear functionals we see, that c is $\mathcal{L}$ip if and only if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$ip for all $\ell \in E^*$. Moreover, if for a bounded interval $J \subset \mathbb{R}$ we take B as the absolutely convex hull of the bounded set $c(J) \cup \{\frac{c(t)-c(s)}{t-s} : t \neq s; t, s \in J\}$, then we see that $c|_J : J \to E_B$ is a well defined $\mathcal{L}$ip-curve into E_B, E_B the linear span of B in E, equipped with the *Minkowski functional* $p_B(v) := \inf\{\lambda > 0 : v \in \lambda \cdot B\}$. This is a normed space. Thus we have the following equivalent characterizations of $\mathcal{L}$ip-curves:

(1) locally c factors over a $\mathcal{L}$ip-curve into some E_B;
(2) c is $\mathcal{L}$ip;
(3) $\ell \circ c$ is $\mathcal{L}$ip for all $\ell \in E^*$.

For continuous instead of Lipschitz curves we obviously have the analogous implications ($1 \Rightarrow 2 \Rightarrow 3$). However, if we take a non-convergent sequence $(x_n)_n$, which converges weakly to 0 (e.g. take an orthonormal base in a separable Hilbert space), and consider an infinite polygon c through these points x_n, say with $c(\frac{1}{n}) = x_n$ and $c(0) = 0$. Then this curve is obviously not continuous but $\ell \circ c$ is continuous for all $\ell \in E^*$.

Furthermore, the “worst” continuous curve - i.e. $c : \mathbb{R} \to \prod_{C(\mathbb{R},\mathbb{R})} \mathbb{R} =: E$ given by $(c(t))_f := f(t)$ for all $t \in \mathbb{R}$ and $f \in C(\mathbb{R},\mathbb{R})$ - cannot be factored locally as a continuous curve over some E_B. Otherwise, $c(t_n)$ would converge into some E_B to $c(0)$, where t_n is a given sequence converging to 0, say $t_n := \frac{1}{n}$. So $c(t_n)$ would converge *Mackey* to $c(0)$, i.e., there have to be $\mu_n \to \infty$ with $\{\mu_n(c(t_n) - c(0)) : n \in \mathbb{N}\}$ bounded in E (e.g. $\mu_n := \min\{\frac{1}{p_B(c(t_n)-c(0))}, n+1\}$). Since a set is bounded in the product if and only if its coordinates are bounded, we conclude that for all $f \in C(\mathbb{R},\mathbb{R})$ the sequence $\mu_n(f(t_n) - f(0))$ has to be bounded. But we can choose a continuous function f with $f(0) = 0$ and $f(t_n) = \frac{1}{\sqrt{\mu_n}}$ and conclude that $\mu_n(f(t_n) - f(0)) = \sqrt{\mu_n}$ is unbounded.

Similarly, one shows that the reverse implications do not hold for differentiable curves, for C^1-curves and for C^n-curves. However, if we put instead some Lipschitz condition on the derivatives, there should be some chance, since this is a bornological concept. In order to obtain this result, we should study convergence of sequences in E_B.

1.6. Lemma. Mackey-convergence. *Let B be a bounded and absolutely convex subset of E and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in E_B. Then the following two conditions are equivalent:*

(1) *x_γ converges to 0 in the normed space E_B;*

(2) *There exists a net $\mu_\gamma \to 0$ in $\mathbb{R}$, such that $x_\gamma \in \mu_\gamma \cdot B$.*

In (2) we may assume that $\mu_\gamma \geq 0$ and is decreasing with respect to γ, at least for large γ. In the particular case of a sequence (or where we have a confinal infinite countable subset of Γ) we can choose $\mu_\gamma > 0$ for all large γ and hence we may divide.

A net (x_γ) for which a bounded absolutely convex $B \subseteq E$ exists, such that x_γ converges to x in E_B is called Mackey convergent to x or short *M-convergent*.

Proof. ($\Uparrow$) Let $x_\gamma = \mu_\gamma \cdot b_\gamma$ with $b_\gamma \in B$ and $\mu_\gamma \to 0$. Then $p_B(x_\gamma) = |\mu_\gamma|\, p_B(b_\gamma) \leq |\mu_\gamma| \to 0$, i.e. $x_\gamma \to 0$ in E_B.

($\Downarrow$) Set $\mu_\gamma := 2\, p_B(x_\gamma)$ and $b_\gamma := \frac{x_\gamma}{\mu_\gamma}$ if $\mu_\gamma \neq 0$ and $b_\gamma := 0$ otherwise. Then $p_B(b_\gamma) = \frac{1}{2}$ or $p_B(b_\gamma) = 0$, so $b_\gamma \in B$. By assumption, $\mu_\gamma \to 0$ and $x_\gamma = \mu_\gamma\, b_\gamma$.

For the final assertions, choose γ_1 such that $|\mu_\gamma| \leq 1$ for $\gamma \geq \gamma_1$, and for those γ we replace μ_γ by $\sup\{|\mu_{\gamma'}| : \gamma' \geq \gamma\} \geq |\mu_\gamma| \geq 0$ which is decreasing with respect to γ.

If we have a strictly increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ which is confinal in Γ, i.e. for every $\gamma \in \Gamma$ there exists an $n \in \mathbb{N}$ with $\gamma \leq \gamma_n$, then $\gamma \mapsto \nu_\gamma := 1/\min\{n : \gamma \leq \gamma_n\} > 0$ converges to 0, and we can replace μ_γ by $\max\{\mu_\gamma, \nu_\gamma\} > 0$. □

If Γ is the ordered set of all countable ordinals, then it is not possible to find a net $(\mu_\gamma)_{\gamma\in\Gamma}$, which is positive everywhere and converges to 0, since any converging net is finally constant.

1.7. The difference quotient converges Mackey

Now we show how to describe the quality of convergence of the difference quotient.

Corollary. *Let $c : \mathbb{R} \to E$ be a $\mathcal{L}\mathrm{ip}^1$-curve. Then the curve*

$$t \mapsto \frac{1}{t}\Big(\frac{c(t) - c(0)}{t} - c'(0)\Big)$$

is bounded on bounded subsets of $\mathbb{R} \setminus \{0\}$.

Proof. We apply 1.4 to c and obtain:

$$\begin{aligned}\frac{c(t) - c(0)}{t} - c'(0) &\in \Big\langle c'(r) : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}} - c'(0)\\ &= \Big\langle c'(r) - c'(0) : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}}\\ &= \Big\langle r\, \frac{c'(r) - c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}}\end{aligned}$$

Let $a > 0$. Since $\{\frac{c'(r)-c'(0)}{r} : 0 < |r| < a\}$ is bounded and hence contained in a closed absolutely convex and bounded set B, we can conclude for $|t| \leq a$ that

$$\frac{1}{t}\,\left(\frac{c(t) - c(0)}{t} - c'(0)\right) \in \Big\langle \frac{r}{t}\,\frac{c'(r) - c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}} \subseteq B. \quad \square$$

1.8. Corollary. Smoothness of curves is a bornological concept. *For $0 \leq k < \infty$ a curve c in a locally convex vector space E is $\mathcal{L}\mathrm{ip}^k$ if and only if for* *each bounded open interval $J \subset \mathbb{R}$ there exists an absolutely convex bounded set $B \subseteq E$ such that $c|_J$ is a $\mathcal{L}\mathrm{ip}^k$-curve in the normed space E_B.*

Attention: A smooth curve factors locally into some E_B as a $\mathcal{L}\mathrm{ip}^k$-curve for each finite k only, in general. Take the "worst" smooth curve $c : \mathbb{R} \to \prod_{C^\infty(\mathbb{R},\mathbb{R})} \mathbb{R}$, analogously to 1.5, and, using Borel's theorem, deduce from $c^{(k)}(0) \in E_B$ for all $k \in \mathbb{N}$ a contradiction.

Proof. ($\Uparrow$) This follows from lemma 1.3, since the inclusion $E_B \to E$ is continuous.

($\Downarrow$) For $k = 0$ this was shown in 1.5. For $k \geq 1$ take a closed absolutely convex bounded set $B \subseteq E$ containing all derivatives $c^{(i)}$ on J up to order k as well as their difference quotients on $\{(t,s) : t \neq s, t, s \in J\}$. We show first that c is differentiable in E_B, say at 0, with derivative $c'(0)$. By the proof of the previous corollary 1.7 we have that the expression $\frac{1}{t}(\frac{c(t)-c(0)}{t} - c'(0))$ lies in B. So $\frac{c(t)-c(0)}{t} - c'(0)$ converges to 0 in E_B. For the higher order derivatives we can now proceed by induction. □

A consequence of this is, that smoothness does not depend on the topology but only on the bounded sets, i.e. the bornology, and, in particular, it depends only on the dual (so all topologies with the same dual have the same smooth curves). Since on $L(E,F)$ there is essentially only one bornology (by the uniform boundedness principle, see [**68**, 5.2.2]) there is only one notion of $\mathcal{L}\mathrm{ip}^n$-curves into $L(E,F)$. Furthermore, the class of $\mathcal{L}\mathrm{ip}^n$-curves doesn't change if we pass from a given locally convex topology to its *bornologification*, see 4.2, which by definition is the finest locally convex topology having the same bounded sets.

Let us now return to scalar differentiability. Corollary 1.7 gives us $\mathcal{L}\mathrm{ip}^n$-ness provided we have appropriate candidates for the derivatives.

1.9. Corollary. Scalar testing of curves. *Let $c^k : \mathbb{R} \to E$ for $k \leq n$ be curves such that $\ell \circ c^0$ is $\mathcal{L}\mathrm{ip}^n$ and $(\ell \circ c^0)^{(k)} = \ell \circ c^k$ for all $k \leq n$ and all $\ell \in E^*$. Then c^0 is $\mathcal{L}\mathrm{ip}^n$ and $(c^0)^{(k)} = c^k$.*

Proof. For $n = 0$ this was shown in 1.5. For $n \geq 1$, by 1.7 applied to $\ell \circ c^0$ we have that

$$\ell\left(\frac{1}{t}\left(\frac{c^0(t) - c^0(0)}{t} - c^1(0)\right)\right)$$

is locally bounded, and hence by [**68**, 5.2.7] the set

$$\left\{\frac{1}{t}\left(\frac{c^0(t) - c^0(0)}{t} - c^1(0)\right) : t \in I\right\}$$

is bounded. Thus $\frac{c^0(t)-c^0(0)}{t}$ converges even Mackey to $c^1(0)$. Now the general statement follows by induction. □

2. Completeness

Do we really need the knowledge of a candidate for the derivative, as in 1.9? In finite dimensional analysis one often uses the Cauchy condition to prove convergence. Here we will replace the Cauchy condition again by a stronger condition, which provides information about the quality of being Cauchy:

A net $(x_\gamma)_{\gamma\in\Gamma}$ in E is called *Mackey-Cauchy* provided that there exist a bounded (absolutely convex) set B and a net $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbb{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$. As in 1.6 one shows that for a net x_γ in E_B this is equivalent to the condition that x_γ is Cauchy in the normed space E_B. In particular, every Mackey-convergent net is Mackey-Cauchy.

2.1. Lemma. The difference quotient is Mackey-Cauchy. *Let $c : \mathbb{R} \to E$ be scalarly a $\mathcal{L}\mathrm{ip}^1$-curve. Then $t \mapsto \frac{c(t)-c(0)}{t}$ is a Mackey-Cauchy net for $t \to 0$.*

Proof. For $\mathcal{L}\mathrm{ip}^1$-curves this is a immediate consequence of 1.7, but here we only assume it to be scalarly $\mathcal{L}\mathrm{ip}^1$. It is enough to show that $\frac{1}{t-s}\Big(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\Big)$ is bounded on bounded subsets in $\mathbb{R}\setminus\{0\}$. We may test this with continuous linear functionals, and hence may assume that $E = \mathbb{R}$. Then by the fundamental theorem of calculus we have

$$\begin{aligned}\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right) &= \int_0^1 \frac{c'(tr)-c'(sr)}{t-s}\,dr\\ &= \int_0^1 \frac{c'(tr)-c'(sr)}{tr-sr}\,r\,dr.\end{aligned}$$

Since $\frac{c'(tr)-c'(sr)}{tr-sr}$ is locally bounded by assumption, the same is true for the integral, and we are done. □

2.2. Lemma. Mackey Completeness. *For a space E the following conditions are equivalent:*

(1) *Every Mackey-Cauchy net converges (Mackey) in E;*

(2) *Every Mackey-Cauchy sequence converges (Mackey) in E;*

(3) *For every absolutely convex closed bounded set B the space E_B is complete;*

(4) *For every bounded set B there exists an absolutely convex bounded set $B' \supseteq B$ such that $E_{B'}$ is complete.*

A space satisfying the equivalent conditions is called Mackey complete. Note that any sequentially complete space is Mackey complete.

Proof. (1) ⇒ (2), and (3) ⇒ (4) are trivial.

(2) ⇒ (3) Since E_B is normed, it is enough to show sequential completeness. So let (x_n) be a Cauchy sequence in E_B. Then (x_n) is Mackey-Cauchy in E and hence converges in E to some point x. Since $p_B(x_n - x_m) \to 0$ there exists for every $\varepsilon > 0$ an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $p_B(x_n - x_m) < \varepsilon$, and hence $x_n - x_m \in \varepsilon B$. Taking the limit for $m \to \infty$, and using closedness of B we conclude that $x_n - x \in \varepsilon B$ for all $n > N$. In particular $x \in E_B$ and $x_n \to x$ in E_B.

(4) ⇒ (1) Let $(x_\gamma)_{\gamma\in\Gamma}$ be a Mackey-Cauchy net in E. So there is some net $\mu_{\gamma,\gamma'} \to 0$, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$ for some bounded set B. Let γ_0 be arbitrary. By (4) we may assume that B is absolutely convex and contains x_{γ_0}, and that E_B is complete. For $\gamma \in \Gamma$ we have that $x_\gamma = x_{\gamma_0} + x_\gamma - x_{\gamma_0} \in x_{\gamma_0} + \mu_{\gamma,\gamma_0} B \in E_B$, and $p_B(x_\gamma - x_{\gamma'}) \leq \mu_{\gamma,\gamma'} \to 0$. So (x_γ) is a Cauchy net in E_B, hence converges in E_B, and thus also in E (even Mackey). □

2.3. Corollary. Scalar testing of differentiable curves. *Let E be Mackey complete and $c : \mathbb{R} \to E$ be a curve for which $\ell \circ c$ is $\mathcal{L}\mathrm{ip}^n$ for all $\ell \in E^*$. Then c is $\mathcal{L}\mathrm{ip}^n$.*

Proof. For $n = 0$ this was shown in 1.5 without using any completeness, so let $n \geq 1$. Since we have shown in 2.1 that the difference quotient is a Mackey-Cauchy net we conclude that the derivative c' exists, and hence $(\ell \circ c)' = \ell \circ c'$. So we may apply the induction hypothesis to conclude that c' is $\mathcal{L}\mathrm{ip}^{n-1}$, and consequently c is $\mathcal{L}\mathrm{ip}^n$. □

Next we turn to integration. For continuous curves $c : [0, 1] \to E$ one can show completely analogously to 1-dimensional analysis that the *Riemann sums* $R(c, \mathcal{Z}, \xi)$, defined by $\sum_k (t_k - t_{k-1}) c(\xi_k)$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition $\mathcal{Z}$ of $[0, 1]$ and $\xi_k \in [t_{k-1}, t_k]$, form a Cauchy net with respect to the partial strict ordering given by the size of the *mesh* $\max\{|t_k - t_{k-1}| : 0 < k < n\}$. So under the assumption of sequential completeness we have a *Riemann integral* of curves. A second way to see this is the following reduction to the 1-dimensional case.

2.4. Lemma. *Let $L(E^*_{equi}, \mathbb{R})$ be the space of all linear functionals on E^* which are bounded on equicontinuous sets, equipped with the complete locally convex topology of uniform convergence on these sets. There is a natural topological embedding $\delta : E \to L(E^*_{equi}, \mathbb{R})$ given by $\delta(x)(\ell) := \ell(x)$.*

Proof. The space $L(E^*_{\mathrm{equi}}, \mathbb{R})$ is complete, since this is true for the space of all bounded mappings (see 2.15) in which it is obviously closed.

Let $\mathcal{U}$ be a basis of absolutely convex closed 0-neighborhoods in E. Then the family of *polars* $U^o := \{\ell \in E^* : |\ell(x)| \leq 1 \text{ for all } x \in U\}$, with $U \in \mathcal{U}$ form a basis for the equicontinuous sets, and hence the *bipolars* $U^{oo} := \{\ell^* \in L(E^*_{\mathrm{equi}}, \mathbb{R}) : |\ell^*(\ell)| \leq 1 \text{ for all } \ell \in U^o\}$ form a basis of 0-neighborhoods in $L(E^*_{\mathrm{equi}}, \mathbb{R})$. By the bipolar theorem [**68**, 7.4.7] we have $U = \delta^{-1}(U^{oo})$ for all $U \in \mathcal{U}$. This shows that δ is a homeomorphism onto its image. □

2.5. Lemma. Anti-derivative of continuous curves. *Let $c : \mathbb{R} \to E$ be a continuous curve in a locally convex vector space. Then there is a unique differentiable curve $\int c : \mathbb{R} \to \widehat{E}$ in the completion $\widehat{E}$ of E such that $(\int c)(0) = 0$ and $(\int c)' = c$.*

Proof. We show uniqueness first. Let $c_1 : \mathbb{R} \to \widehat{E}$ be a curve with derivative c and $c_1(0) = 0$. For every $\ell \in E^*$ the composite $\ell \circ c_1$ is an anti-derivative of $\ell \circ c$ with initial value 0, so it is uniquely determined, and since E^* separates points c_1 is also uniquely determined.

Now we show the existence. By the previous lemma 2.4 we have that $\widehat{E}$ is (isomorphic to) the closure of E in the complete space $L(E^*_{\mathrm{equi}}, \mathbb{R})$. We define $(\int c)(t) : E^* \to \mathbb{R}$ by $\ell \mapsto \int_0^t (\ell \circ c)(s) ds$. It is a bounded linear functional on E^*_{equi} since for each equicontinuous and hence bounded subset $\mathcal{E} \subseteq E^*$ the set $\{(\ell \circ c)(s) : \ell \in \mathcal{E}, s \in [0, t]\}$ is bounded. So $\int c : \mathbb{R} \to L(E^*_{\mathrm{equi}}, \mathbb{R})$.

Now we show that $\int c$ is differentiable with derivative $\delta \circ c$.

$$\Big(\frac{(\int c)(t+r)-(\int c)(r)}{t} - (\delta\circ c)(r)\Big)(\ell) =$$
$$= \frac{1}{t}\Big(\int_0^{t+r}(\ell\circ c)(s)ds - \int_0^r (\ell\circ c)(s)ds - t(\ell\circ c)(r)\Big) =$$
$$= \frac{1}{t}\int_r^{r+t}\Big((\ell\circ c)(s)-(\ell\circ c)(r)\Big)ds = \int_0^1 \ell\Big(c(r+ts)-c(r)\Big)ds.$$

Let $\mathcal{E}\subseteq E^*$ be equicontinuous, and let $\varepsilon>0$. Then there exists a neighborhood U of 0 such that $|\ell(U)|<\varepsilon$ for all $\ell\in\mathcal{E}$. For sufficiently small t, all $s\in[0,1]$ and fixed r we have $c(r+ts)-c(r)\in U$. So $\big|\int_0^1 \ell(c(r+ts)-c(r))ds\big|\le\varepsilon$. This shows that the difference quotient of $\int c$ at r converges to $\delta(c(r))$ uniformly on equicontinuous subsets.

It remains to show that $(\int c)(t)\in\widehat{E}$. By the mean value theorem 1.4 the difference quotient $\frac{1}{t}\big((\int c)(t)-(\int c)(0)\big)$ is contained in the closed convex hull in $L(E^*_{\text{equi}},\mathbb{R})$ of the subset $\big\{c(s)=(\int c)'(s):0<s<t\big\}$ of E. So it lies in $\widehat{E}$. □

Definition of the integral. For continuous curves $c:\mathbb{R}\to E$ the *definite integral* $\int_a^b c\in\widehat{E}$ is given by $\int_a^b c=(\int c)(b)-(\int c)(a)$.

2.6. Corollary. Basics on the integral. *For a continuous curve $c:\mathbb{R}\to E$ we have:*

1. $\ell(\int_a^b c)=\int_a^b(\ell\circ c)$ *for all $\ell\in E^*$.*
2. $\int_a^b c+\int_b^d c=\int_a^d c$.
3. $\int_a^b(c\circ\varphi)\varphi'=\int_{\varphi(a)}^{\varphi(b)} c$ *for $\varphi\in C^1(\mathbb{R},\mathbb{R})$.*
4. $\int_a^b c$ *lies in the closed convex hull in $\widehat{E}$ of the set $\{(b-a)c(t):a<t<b\}$ in E.*
5. $\int_a^b : C(\mathbb{R},E)\to\widehat{E}$ *is linear.*
6. *(Fundamental theorem of calculus.) For each C^1-curve $c:\mathbb{R}\to E$ we have $c(s)-c(t)=\int_t^s c'$.* □

2.7. We are mainly interested in smooth curves and we can test for this by applying linear functionals if the space is Mackey complete, see 2.3. So let us try to show that the integral for such curves lies in E if E is Mackey-complete. For this let $c:[0,1]\to E$ be a smooth or just a $\mathcal{L}$ip-curve, and take a partition $\mathcal{Z}_1$ with mesh $\mu(\mathcal{Z}_1)$ at most δ. If we have a second partition $\mathcal{Z}_2$, then we can take the common refinement $\mathcal{Z}'$. Let $[a,b]$ be one interval of the original partition with intermediate point t, and let $a=t_0<t_1<\dots<t_n=b$ be the refinement. Note that $|b-a|\le\delta$ and hence $|t-t_k|\le\delta$. Then we can estimate as follows.

$$(b-a)\,c(t)-\sum_k(t_k-t_{k-1})c(t_k)=\sum_k(t_k-t_{k-1})\,(c(t)-c(t_k))=\sum_k\mu_k b_k,$$

where $b_k:=\frac{c(t)-c(t_k)}{\delta}$ is contained in the *absolutely convex Lipschitz bound*

$$B:=\Big\langle\Big\{\frac{c(t)-c(s)}{t-s}:t,s\in[0,1]\Big\}\Big\rangle_{abs.conv}$$

of c and $\mu_k:=(t_k-t_{k-1})\delta\ge0$ and satisfies $\sum_k\mu_k=(b-a)\delta$. Hence we have for the Riemann sums with respect to the original partition $\mathcal{Z}_1$ and the refinement $\mathcal{Z}'$ that

$R(c,\mathcal{Z}_1)-R(c,\mathcal{Z}')$ lies in $\delta\cdot B$. So $R(c,\mathcal{Z}_1)-R(c,\mathcal{Z}_2)\in 2\delta B$ for any two partitions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of mesh at most δ, i.e. the Riemann sums form a Mackey-Cauchy net with coefficients $\mu_{\mathcal{Z}_1,\mathcal{Z}_2} := 2\max\{\mu(\mathcal{Z}_1),\mu(\mathcal{Z}_2)\}$. Since continuous linear functionals ℓ map the Riemann-sums of c to those of $\ell\circ c$ we have $\ell(\lim_{\mu(\mathcal{Z})\to 0} R(c,\mathcal{Z})) = \lim_{\mu(\mathcal{Z})\to 0} R(\ell\circ c,\mathcal{Z}) = \int_a^b(\ell\circ c)$, thus $\lim_{\mu(\mathcal{Z})\to 0} R(c,\mathcal{Z}) = \int_a^b c$ and we have proved:

Proposition. Integral of Lipschitz curves. *Let $c:[0,1]\to E$ be a Lipschitz curve into a Mackey complete space. Then the Riemann integral exists in E as (Mackey)-limit of the Riemann sums and coincides with the integral as defined in 2.5.* □

2.8. Now we have to discuss the relationship between differentiable curves and Mackey convergent sequences. Recall that a sequence (x_n) converges if and only if there exists a continuous curve c (e.g. a reparameterization of the infinite polygon) and $t_n \searrow 0$ with $c(t_n)=x_n$. The corresponding result for smooth curves uses the following notion.

Definition. We say that a sequence x_n in a locally convex space E *converges fast* to x in E, or *falls fast* towards x, if for each $k\in\mathbb{N}$ the sequence $n^k(x_n-x)$ is bounded.

Special curve lemma. *Let x_n be a sequence which converges fast to x in E. Then the infinite polygon through the x_n can be parameterized as a smooth curve $c:\mathbb{R}\to E$ such that $c(\frac{1}{n})=x_n$ and $c(0)=x$.*

Proof. Let $\varphi:\mathbb{R}\to[0,1]$ be a smooth function, which is 0 on $\{t: t\le 0\}$ and 1 on $\{t:t\ge 1\}$. The parameterization c is defined as follows:

$$c(t) := \begin{cases} x & \text{for } t\le 0,\\ x_{n+1}+\varphi\left(\frac{t-\frac{1}{n+1}}{\frac{1}{n}-\frac{1}{n+1}}\right)(x_n-x_{n+1}) & \text{for } \frac{1}{n+1}\le t\le \frac{1}{n},\\ x_1 & \text{for } t\ge 1\end{cases}.$$

Obviously, c is smooth on $\mathbb{R}\setminus\{0\}$, and the p-th derivative of c for $\frac{1}{n+1}\le t\le\frac{1}{n}$ is given by

$$c^{(p)}(t)=\varphi^{(p)}\left(\frac{t-\frac{1}{n+1}}{\frac{1}{n}-\frac{1}{n+1}}\right)(n(n+1))^p(x_n-x_{n+1}).$$

Since x_n converges fast to x, we have that $c^{(p)}(t)\to 0$ for $t\to 0$, because the first factor is bounded and the second goes to zero. Hence c is smooth on $\mathbb{R}$, by the following lemma. □

2.9. Lemma. Differentiable extension to an isolated point. *Let $c:\mathbb{R}\to E$ be continuous and differentiable on $\mathbb{R}\setminus\{0\}$, and assume that the derivative $c':\mathbb{R}\setminus\{0\}\to E$ has a continuous extension to $\mathbb{R}$. Then c is differentiable at 0 and $c'(0)=\lim_{t\to 0}c'(t)$.*

Proof. Let $a:=\lim_{t\to 0}c'(t)$. By the mean value theorem 1.4 we have $\frac{c(t)-c(0)}{t}\in\langle c'(s): 0\ne|s|\le|t|\rangle_{\text{closed, convex}}$. Since c' is assumed to be continuously extendable to 0 we have that for any closed convex 0-neighborhood U there exists a $\delta>0$ such that $c'(t)\in a+U$ for all $0<|t|\le\delta$. Hence $\frac{c(t)-c(0)}{t}-a\in U$, i.e. $c'(0)=a$. □

The next result shows that we can pass through certain sequences $x_n\to x$ even with given velocities $v_n\to 0$.

2.10. Corollary. *If $x_n \to x$ fast and $v_n \to 0$ fast in E, then there exist a smoothly parameterized polygon $c : \mathbb{R} \to E$ and $t_n \to 0$ in $\mathbb{R}$ such that $c(t_n + t) = x_n + tv_n$ for t in a neighborhood of 0 depending on n.*

Proof. Consider the sequence y_n defined by

$$y_{2n} := x_n + \tfrac{1}{4n(2n+1)} v_n \quad \text{and} \quad y_{2n+1} := x_n - \tfrac{1}{4n(2n+1)} v_n.$$

It is easy to show that y_n converges fast to x, and the parameterization c of the polygon through the y_n (using a function φ which satisfies $\varphi(t) = t$ for t near 1/2) has the claimed properties, where

$$t_n := \frac{4n+1}{4n(2n+1)} = \frac{1}{2}\left(\frac{1}{2n} + \frac{1}{2n+1}\right). \quad \square$$

As first application 2.10 we can give the following sharpening of 1.3.

2.11. Corollary. Bounded linear maps. *A linear mapping $f : E \to F$ between locally convex vector spaces is bounded (or bornological), i.e. it maps bounded sets to bounded ones, if and only if it maps smooth curves in E to smooth curves in F.*

Proof. As in the proof of 1.3 one shows using 1.7 that a bounded linear map preserves $\mathcal{L}\mathrm{ip}^k$-curves. Conversely, assume that a linear map $f : E \to F$ carries smooth curves to (at least) locally bounded curves. Take a bounded set B, and assume that $f(B)$ is unbounded. Then there is some $\ell \in F^*$ and a sequence (b_n) in B such that $|(\ell \circ f)(b_n)| \geq n^{n+1}$. The sequence $(n^{-n} b_n)$ converges fast to 0, hence lies on some compact part of a smooth curve by 2.8. Consequently, $(\ell \circ f)(n^{-n} b_n) = n^{-n}(\ell \circ f)(b_n)$ is bounded, a contradiction. $\square$

2.12. Definition. The *c^∞-topology* on a locally convex space E is the final topology with respect to all smooth curves $\mathbb{R} \to E$. Its open sets will be called *c^∞-open.* We will treat this topology in more detail in section 4: In general it describes neither a topological vector space 4.20 and 4.26, nor a uniform structure 4.27. However, by 4.4 and 4.6 the finest locally convex topology coarser than the c^∞-topology is the bornologification of the locally convex topology.

Let (μ_n) be a sequence of real numbers converging to ∞. Then a sequence (x_n) in E is called *μ-converging* to x_∞ if the sequence $(\mu_n(x_n - x_\infty))$ is bounded in E.

2.13. Theorem. $\mathbf{c^\infty}$-open subsets. *Let $\mu_n \to \infty$ be a real valued sequence and $k \in \mathbb{N}_\infty$. Then a subset $U \subseteq E$ is open for the c^∞-topology if it satisfies any of the following equivalent conditions:*

1. *All inverse images under $\mathcal{L}\mathrm{ip}^k$-curves are open in $\mathbb{R}$;*
2. *All inverse images under μ-converging sequences are open in $\mathbb{N}_\infty$;*
3. *The traces to E_B are open in E_B for all absolutely convex bounded subsets $B \subseteq E$.*

Note that for closed subsets an equivalent statement reads as follows: A set A is c^∞-closed if and only if for every sequence $x_n \in A$, which is μ-converging (respectively M-converging, resp. fast falling) towards x, the point x belongs to A.

With $\mathbb{N}_\infty$ we denote the one-point compactification $\mathbb{N} \cup \{\infty\}$ of the discrete space $\mathbb{N}$ and the converging sequences $x_n \to x_\infty$ can be considered as the continuous mappings on $\mathbb{N}_\infty$.

The topology described in (2) is also called *Mackey-closure topology.* It is not the Mackey topology discussed in duality theory.

Proof. (1) ⇒ (2) Suppose (x_n) is μ-converging to $x_\infty \in U$, but $x_n \notin U$ for infinitely many n. Then we may choose a subsequence again denoted by (x_n), which is fast falling to x_∞, hence lies on some compact part of a smooth curve c as described in 2.8. Then $c(\frac{1}{n}) = x_n \notin U$ but $c(0) = x_\infty \in U$. This is a contradiction to (1).

(2) ⇒ (3) A sequence (x_n), which converges in E_B to x_∞ with respect to p_B, is Mackey convergent, hence has a μ-converging subsequence. Note that E_B is normed, and hence it is enough to consider sequences.

(3) ⇒ (1) Let $c : \mathbb{R} \to E$ be $\mathcal{L}$ip. By 1.5 c factors locally as continuous curve over some E_B, hence $c^{-1}(U)$ is open. □

Let us show next that the c^∞-topology and c^∞-completeness are intimately related.

2.14. Theorem. Convenient vector spaces. *Let E be a locally convex vector space. E is said to be c^∞-*complete *or* convenient *if one of the following equivalent (completeness) conditions is satisfied:*

(1) *Any Lipschitz curve in E is locally Riemann integrable in the sense of 2.7.*
(2) *For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_2' = c_1$ (existence of an* anti-derivative*).*
(3) *E is c^∞-closed in any locally convex space it is embedded into.*
(4) *If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.*
(5) *Any Mackey-Cauchy sequence converges; i.e. E is Mackey complete.*
(6) *If B is bounded closed absolutely convex, then E_B is a Banach space. This property is called* locally complete *in* [**53**, p196].
(7) *Any continuous linear mapping from a normed space into E has a continuous extension to the completion of the normed space.*

Condition (4) says that in a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals. Condition (5) and (6) say via 2.2.4 that c^∞-completeness is a bornological concept. In [**40**] a convenient vector space is always considered with its bornological topology — an equivalent but not isomorphic category.

Proof.

(1) ⇒ (2) A smooth curve is Lipschitz, thus locally Riemann integrable by (1). By 2.7 the indefinite Riemann integral equals the “weakly defined” integral of lemma 2.5, hence is an anti-derivative.

(2) ⇒ (3) Let E be a topological linear subspace of F. To show that E is c^∞-closed we use 2.13. Let $x_n \to x_\infty$ be fast falling, $x_n \in E$ but $x_\infty \in F$. By 2.8 the polygon c through (x_n) can be smoothly symmetrically parameterized in F. Hence c' is smooth and has values in the vector space generated by $\{x_n : n \neq \infty\}$, which is contained in E. Its anti-derivative c_2 is up to an additive constant equal to c, and by (2) $x_1 - x_\infty = c(1) - c(0) = c_2(1) - c_2(0)$ lies in E. So $x_\infty \in E$.

([3]) ⇒ ([5]) Let F be the completion $\widehat{E}$ of E. Any Mackey Cauchy sequence in E has a limit in F, and since E is by assumption c^∞-closed in F the limit lies in E. Hence, the sequence converges in E.

([5]) ⇒ ([6]) was shown in [2.2].

([6]) ⇒ ([7]) Let $f : F \to E$ be a continuous linear mapping on a normed space F. Since the image of the unit ball is bounded, it is a bounded mapping into E_B for some closed absolutely convex B. But into E_B it can be extended to the completion, since E_B is complete.

([7]) ⇒ ([1]) Let $c : \mathbb{R} \to E$ be a Lipschitz curve. Then c is locally a continuous curve into E_B for some absolutely convex bounded set B by [1.5]. The inclusion of E_B into E has a continuous extension to the completion of E_B, and c is Riemann integrable in this Banach space, so also in E.

([5]) ⇒ ([4]) was shown in [2.3].

([4]) ⇒ ([3]) Let E be a topological linear subspace of F. We use again [2.13] in order to show that E is c^∞-closed in F. So let $x_n \to x_\infty \in F$ be fast falling with $x_n \in E$ for $n \neq \infty$. By [2.8] the polygon c through (x_n) can be smoothly symmetrically parameterized in F, and $c(t) \in E$ for $t \neq 0$. We consider $\tilde{c}(t) := tc(t)$. This is a curve in E which is smooth in F, so it is scalarwise smooth in E, thus smooth in E by ([4]). Then $x_\infty = \tilde{c}'(0) \in E$. □

2.15. Theorem. Inheritance of c^∞-completeness. *The following constructions preserve c^∞-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings, as well as formation of $\ell^\infty(X, \_)$, where X is a set together with a family $\mathcal{B}$ of subsets of X containing the finite ones, which are called bounded and $\ell^\infty(X, F)$ denotes the space of all functions $f : X \to F$, which are bounded on all $B \in \mathcal{B}$, supplied with the topology of uniform convergence on the sets in $\mathcal{B}$.*

Note that the definition of the topology of uniform convergence as initial topology shows, that adding all subsets of finite unions of elements in $\mathcal{B}$ to $\mathcal{B}$ does not change this topology. Hence, we may always assume that $\mathcal{B}$ has this stability property; this is the concept of a *bornology on a set.*

Proof. The limit [**68**, 4.8.1] of a functor $\mathcal{F}$ into the category of locally convex spaces is the c^∞-closed linear subspace

$$\Big\{(x_\alpha) \in \prod \mathcal{F}(\alpha) : \mathcal{F}(f)x_\alpha = x_\beta \text{ for all } f : \alpha \to \beta\Big\},$$

hence is c^∞-complete, since the product of c^∞-complete factors is obviously c^∞-complete.

Since the coproduct [**68**, 4.6.1] of spaces X_α is the topological direct sum, and has as bounded sets those which are contained and bounded in some finite subproduct, it is c^∞-complete if all factors are.

For colimits this is in general not true. For strict inductive limits of sequences of closed embeddings it is true, since bounded sets are contained and bounded in some step, see [**68**, 4.8.1].

For the result on $\ell^\infty(X, F)$ we consider first the case, where X itself is bounded. Then c^∞-completeness can be proved as in [**68**, 3.2.3] or reduced to this result.

In fact let $\mathcal{F}$ be bounded in $\ell^\infty(X,F)$. Then $\mathcal{F}(X)$ is bounded in F and hence contained in some absolutely convex bounded set B, for which F_B is a Banach space. So we may assume that $\mathcal{F} := \{f \in \ell^\infty(X,F) : f(X) \subseteq B\}$. The space $\ell^\infty(X,F)_{\mathcal{F}}$ is just the space $\ell^\infty(X,F_B)$ with the supremum norm, which is a Banach space by [68, 3.2.3]. In fact, we have the implications

$$\|f\|_\infty := \sup\{p_B(f(x)) : x \in X\} < \lambda \Rightarrow \forall x \in X : p_B\left(\frac{f(x)}{\lambda}\right) < 1 \Rightarrow \frac{f(X)}{\lambda} \subseteq B$$
$$\Rightarrow \forall x \in X : p_B\left(\frac{f(x)}{\lambda}\right) \le 1 \Rightarrow \|f\|_\infty \le \lambda,$$

i.e.

$$\{\lambda : \|f\|_\infty < \lambda\} \subseteq \{\lambda : f \in \lambda\mathcal{F}\} \subseteq \{\lambda : \|f\|_\infty \le \lambda\}$$

and hence

$$\underbrace{\inf\{\lambda : \|f\|_\infty < \lambda\}}_{=\|f\|_\infty} \ge \underbrace{\inf\{\lambda : f \in \lambda\mathcal{F}\}}_{=p_{\mathcal{F}}(f)} \ge \underbrace{\inf\{\lambda : \|f\|_\infty \le \lambda\}}_{=\|f\|_\infty}.$$

Let now X and $\mathcal{B}$ be arbitrary. Then the restriction maps $\ell^\infty(X,F) \to \ell^\infty(B,F)$ give an embedding ι of $\ell^\infty(X,F)$ into the product $\prod_{B\in\mathcal{B}} \ell^\infty(B,F)$. Since this product is c^∞-complete, by what we have shown above, it is enough to show that this embedding has a closed image. So let $f_\alpha|_B$ converge to some f_B in $\ell^\infty(B,F)$. Define $f(x) := f_{\{x\}}(x)$. For any $B \in \mathcal{B}$ containing x we have that $f_B(x) = (\lim_\alpha f_\alpha|_B)(x) = \lim_\alpha(f_\alpha(x)) = \lim_\alpha f_\alpha|_{\{x\}}(x) = f_{\{x\}}(x) = f(x)$, and $f(B)$ is bounded for all $B \in \mathcal{B}$, since $f|_B = f_B \in \ell^\infty(B,F)$. □

Example. In general, a quotient and an inductive limit of c^∞-complete spaces need not be c^∞-complete. In fact, let $E_D := \{x \in \mathbb{R}^{\mathbb{N}} : \operatorname{supp} x \subseteq D\}$ for any subset $D \subseteq \mathbb{N}$ of *density* $\operatorname{dens} D := \limsup\{\frac{|D\cap[1,n]|}{n}\} = 0$. It can be shown that $E := \bigcup_{\operatorname{dens} D=0} E_D \subset \mathbb{R}^{\mathbb{N}}$ is the inductive limit of the Fréchet subspaces $E_D \cong \mathbb{R}^D$. It cannot be c^∞-complete, since finite sequences are contained in E and are dense in $\mathbb{R}^{\mathbb{N}} \supset E$.

3. Smooth Mappings and the Exponential Law

A particular case of the exponential law 0.7 for continuous mappings is the following:

3.1. Lemma. *A map $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous if and only if the associated mapping $f^\vee : \mathbb{R} \to C(\mathbb{R},\mathbb{R})$ is continuous, where $C(\mathbb{R},\mathbb{R})$ carries the usual Fréchet-topology of uniform convergence on compact subsets of $\mathbb{R}$.*

Proof. ($\Rightarrow$) Obviously $f^\vee$ has values $f^\vee(t) : s \mapsto f(t,s)$ in $C(\mathbb{R},\mathbb{R})$. It is continuous, since for $t_0 \in \mathbb{R}$, compact $J \subseteq \mathbb{R}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(t,s) - f(t_0,s)| < \varepsilon$ for all $|t - t_0| < \delta$ and $s \in I$, i.e. $\|(f^\vee(t) - f^\vee(t_0))|_J\|_\infty \le \varepsilon$ for $|t - t_0| < \delta$.

($\Leftarrow$) Let $(t_0,s_0) \in \mathbb{R}^2$ and $\varepsilon > 0$ and choose a compact neighborhood J of s_0 such that $|f^\vee(t_0)(s) - f^\vee(t_0)(s_0)| < \varepsilon$ for all $s \in J$. Since $f^\vee$ is assumed to be continuous there exists a $\delta > 0$ auch that $\|(f^\vee(t) - f^\vee(t_0))|_J\|_\infty \le \varepsilon$ for $|t - t_0| < \delta$, and hence

$$|f(t,s) - f(t_0,s_0)| \le |f^\vee(t)(s) - f^\vee(t_0)(s)| + |f^\vee(t_0)(s) - f^\vee(t_0)(s_0)| \le 2\varepsilon$$

for all $|t - t_0| < \delta$ and all $s \in J$. □

Now let us start proving the exponential law $C^\infty(U \times V, F) \cong C^\infty(U, C^\infty(V, F))$ first for $U = V = F = \mathbb{R}$ as it has been sketched in 0.10.

3.2. Theorem. Simplest case of exponential law for C^∞. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping. Then all iterated partial derivatives exist and are continuous if and only if the associated mapping $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ exists as a smooth curve, where $C^\infty(\mathbb{R}, \mathbb{R})$ is considered as the Fréchet space with the topology of uniform convergence on compact sets of each derivative separately. Furthermore, we have*

$$(\partial_1 f)^\vee = d(f^\vee) \text{ and } (\partial_2 f)^\vee = d \circ f^\vee = d_*(f^\vee).$$

Proof. We have several possibilities to prove this result. Either we show Mackey convergence of the difference quotients, via the boundedness of $\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right)$, and then use the trivial exponential law $\ell^\infty(X \times Y, \mathbb{R}) \cong \ell^\infty(X, \ell^\infty(Y, \mathbb{R}))$; or we use exponential law $C(\mathbb{R}^2, \mathbb{R}) \cong C(\mathbb{R}, C(\mathbb{R}, \mathbb{R}))$ of 3.1. We choose the latter method.

($\Leftarrow$) Let $g := f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ be smooth. Then both curves dg and $d \circ g = d_* g$ are smooth (apply 1.3 to the continuous and linear mapping d). The following easy calculation shows that the partial derivatives of $f = g^\wedge$ exist and are given by $\partial_1 g^\wedge = (dg)^\wedge$ and $\partial_2 g^\wedge = (d \circ g)^\wedge$:

$$\begin{aligned}
\partial_1 g^\wedge(t, s) &= (\mathrm{ev}_s \circ g)'(t) = (\mathrm{ev}_s \circ dg)(t) = dg(t)(s) = (dg)^\wedge(t, s) \\
\partial_2 g^\wedge(t, s) &= g(t)'(s) = d(g(t))(s) = (d \circ g)(t)(s) = (d \circ g)^\wedge(t, s).
\end{aligned}$$

So one obtains inductively that all iterated derivatives of f exist. They are continuous by 3.1, since they are associated to smooth curves $\mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$.

($\Rightarrow$) First observe that $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ makes sense and that for all $t \in \mathbb{R}$ we have

$$d^p(f^\vee(t)) = (\partial_2^p f)^\vee(t). \tag{1}$$

Next we claim that $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is differentiable, with derivative $d(f^\vee) = (\partial_1 f)^\vee$, or equivalently that for all a the curve

$$c : t \mapsto \begin{cases} \frac{f^\vee(t+a) - f^\vee(a)}{t} & \text{for } t \neq 0 \\ (\partial_1 f)^\vee(a) & \text{otherwise} \end{cases}$$

is continuous (at 0) as curve $\mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$. Without loss of generality we may assume that $a = 0$. Since $C^\infty(\mathbb{R}, \mathbb{R})$ carries the initial structure with respect to the linear mappings $d^p : C^\infty(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ we have to show that $d^p \circ c : \mathbb{R} \to C(\mathbb{R}, \mathbb{R})$ is continuous, or equivalently by the exponential law 3.1 for continuous maps, that $(d^p \circ c)^\wedge : \mathbb{R}^2 \to \mathbb{R}$ is continuous. For $t \neq 0$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}
(d^p \circ c)^\wedge(t, s) &= d^p(c(t))(s) = d^p\left(\frac{f^\vee(t) - f^\vee(0)}{t}\right)(s) \\
&= \frac{\partial_2^p f(t, s) - \partial_2^p f(0, s)}{t} && \text{by (1)} \\
&= \int_0^1 \partial_1 \partial_2^p f(t\,\tau, s)\, d\tau && \text{by the fundamental theorem.}
\end{aligned}$$

For $t = 0$ we have

$$\begin{aligned}(d^p \circ c)^\wedge(0,s) &= d^p(c(0))(s) = d^p((\partial_1 f)^\vee(0))(s)\\ &= (\partial_2^p(\partial_1 f))^\vee(0)(s) \qquad \text{by (1)}\\ &= \partial_2^p\partial_1 f(0,s)\\ &= \partial_1\partial_2^p f(0,s) \qquad \text{by the theorem of Schwarz.}\end{aligned}$$

So we see that $(d^p \circ c)^\wedge(t,s) = \int_0^1 \partial_1\partial_2^p f(t\,\tau, s)\,d\tau$ for all (t,s). This function is continuous in (t,s), since $\partial_1\partial_2^p f : \mathbb{R}^2 \to \mathbb{R}$ is assumed to be continuous, hence $(t,s,\tau) \mapsto \partial_1\partial_2^p f(t\,\tau, s)$ is continuous, and therefore also $(t,s) \mapsto (\tau \mapsto \partial_1\partial_2^p f(t\,\tau,s))$, $\mathbb{R}^2 \to C([0,1],\mathbb{R})$, by 3.1. Composition with the continuous linear mapping $\int_0^1 : C([0,1],\mathbb{R}) \to \mathbb{R}$ gives the continuity of $(d^p \circ c)^\wedge$.

Now we proceed by induction. By the induction hypothesis applied to $\partial_1 f$, we obtain that $(\partial_1 f)^\vee : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ is n-times differentiable, and so $f^\vee$ is $(n+1)$-times differentiable since $d(f^\vee) = (\partial_1 f)^\vee$. $\square$

In order to proceed to more general cases of the exponential law we need a definition of C^∞-maps defined on infinite dimensional spaces. This definition should at least guarantee the chain rule, and so one could take the weakest notion that satisfies the chain rule. However, consider the following

3.3. Example. We consider the following 3-fold "singular covering" $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in polar coordinates by $(r,\varphi) \mapsto (r, 3\varphi)$. In cartesian coordinates we obtain the following formula for the values of f:

$$\begin{aligned}(r\,\cos(3\varphi), r\sin(3\varphi)) &= r\Big((\cos\varphi)^3 - 3\cos\varphi(\sin\varphi)^2, 3\sin\varphi(\cos\varphi)^2 - (\sin\varphi)^3\Big)\\ &= \left(\frac{x^3 - 3xy^2}{x^2+y^2}, \frac{3x^2y - y^3}{x^2+y^2}\right).\end{aligned}$$

Obviously, the map f is smooth on $\mathbb{R}^2 \setminus \{0\}$ and continuous also at 0. It is homogeneous of degree 1, and hence the directional derivative is $f'(0)(v) = \frac{\partial}{\partial t}|_{t=0} f(tv) = f(v)$. However, both components of f are nonlinear with respect to v and thus are *not differentiable* at $(0,0)$.

We claim that f *is differentiable along differentiable curves*, i.e. $(f \circ c)'(0)$ exists, provided $c'(0)$ exists.
Only the case $c(0) = 0$ is not trivial. Since c is differentiable at 0 the curve c_1 defined by

$$c_1(t) := \begin{cases} \frac{c(t)}{t} & \text{for } t \neq 0\\ c'(0) & \text{for } t = 0\end{cases}$$

is continuous at 0. Hence $\frac{f(c(t)) - f(c(0))}{t} = \frac{f(t\,c_1(t)) - 0}{t} = f(c_1(t))$ converges to $f(c_1(0)) = f(c'(0))$, since f is continuous.

Furthermore, if $f'(x)(v)$ denotes the directional derivative, which exists everywhere, then $(f \circ c)'(t) = f'(c(t))(c'(t))$. Indeed for $c(t) \neq 0$ this is clear and for $c(t) = 0$ it follows from $\frac{f(c(t+s)) - f(c(t))}{s} = f(\frac{c(t+s) - c(t)}{s}) \to f(c'(t)) = f'(0)(c'(t))$.

Each directional derivative $f'(\_)(v)$ of the 1-homogeneous mapping f is 0-homogeneous: In fact, for $s \neq 0$ we have

$$f'(sx)(v) = \frac{\partial}{\partial t}\Big|_{t=0} f(s\,x + tv) = s\ \frac{\partial}{\partial t}\Big|_{t=0} f\Big(x + \frac{t}{s}v\Big) = s\,f'(x)\left(\frac{1}{s}v\right) = f'(x)(v).$$

For any $s \in \mathbb{R}$ we have $f'(s\,v)(v) = \frac{\partial}{\partial t}|_{t=0} f(s\,v + tv) = \frac{\partial}{\partial t}|_{t=s} t\,f(v) = f(v)$.

Using this homogeneity we show next, that f *is also continuously differentiable along continuously differentiable curves.* So we have to show that $(f\circ c)'(t) \to (f\circ c)'(0)$ for $t\to 0$. Again only the case $c(0)=0$ is interesting. As before we factor c as $c(t)=t\,c_1(t)$. In the case, where $c'(0)=c_1(0)\neq 0$ we have for $t\neq 0$ that

$$\begin{aligned}(f\circ c)'(t)-(f\circ c)'(0) &= f'(t\,c_1(t))(c'(t)) - f'(0\cdot c_1(0))(c_1(0))\\ &= f'(c_1(t))(c'(t)) - f'(c_1(0))(c'(0)),\end{aligned}$$

which converges to 0 for $t\to 0$, since $(f')^\wedge$ is continuous (and even smooth) on $(\mathbb{R}^2\setminus\{0\})\times\mathbb{R}^2$.
In the other case, where $c'(0)=c_1(0)=0$ we consider first the values of t, for which $c(t)=0$. Then

$$\begin{aligned}(f\circ c)'(t)-(f\circ c)'(0) &= f'(0)(c'(t)) - f'(0)(c'(0))\\ &= f(c'(t)) - f(c'(0)) \to 0,\end{aligned}$$

since f is continuous. For the remaining values of t, where $c(t)\neq 0$, we factor $c(t)=\|c(t)\|\,e(t)$, with $e(t)\in\{x:\|x\|=1\}$. Then

$$(f\circ c)'(t)-(f\circ c)'(0) = f'(e(t))(c'(t)) - 0 \to 0,$$

since $f'(x)(c'(t))\to 0$ for $t\to 0$ uniformly for $\|x\|=1$, since $c'(t)\to 0$.

Furthermore, $f\circ c$ *is smooth for all* c *which are smooth and nowhere infinitely flat.* In fact, a smooth curve c with $c^{(k)}(0)=0$ for $k<n$ can be factorized as $c(t)=t^n c_n(t)$ with smooth c_n, by Taylor's formula with integral remainder. Since $c^{(n)}(0)=n!\,c_n(0)$, we may assume that n is chosen maximal and hence $c_n(0)\neq 0$. But then $(f\circ c)(t)=t^n\cdot(f\circ c_n)(t)$, and $f\circ c_n$ is smooth.

The same argument shows also that $f\circ c$ *is real analytic for all real analytic curves* $c:\mathbb{R}\to\mathbb{R}^2$.

However, let us show that $f\circ c$ *is not Lipschitz differentiable even for smooth curves* c. For $x\neq 0$ we have

$$\begin{aligned}(\partial_2)^2 f(x,0) = \left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(x,s) = x\left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(1,\tfrac{1}{x}s) =\\ = \tfrac{1}{x}\left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(1,s) =: \tfrac{a}{x}\neq 0.\end{aligned}$$

In fact, $f_1(s):=\mathrm{pr}_1(f(1,s))$ satisfies $(1+s^2)\,f_1(s)=1-3s^2$, and thus $2\,f_1(0)+f_1''(0)=-6$, i.e. $a_1:=f_1''(0)=-8\neq 0$. Now we choose a smooth curve c which passes for each n in finite time t_n through $(\frac{1}{n^{2n+1}},0)$ with locally constant velocity vector $(0,\frac{1}{n^n})$, by 2.10. Then for small t we get

$$(f\circ c)'(t_n+t) = \partial_1 f(c(t_n+t))\,\underbrace{\mathrm{pr}_1(c'(t_n+t))}_{=0} + \partial_2 f(c(t_n+t))\;\mathrm{pr}_2(c'(t_n+t))$$

$$(f\circ c)''(t_n) = 0 + (\partial_2)^2 f(c(t_n))\,(\mathrm{pr}_2(c'(t_n)))^2 = a\,\frac{n^{2n+1}}{n^{2n}} = n\,a,$$

which is unbounded.

So although preservation of (continuous) differentiability of curves is not enough to ensure differentiability of a function $\mathbb{R}^2\to\mathbb{R}$, we now prove that smoothness can be tested with smooth curves.

3.4. Boman's theorem. [14] *For a mapping* $f:\mathbb{R}^2\to\mathbb{R}$ *the following assertions are equivalent:*

(1) *All iterated partial derivatives exist and are continuous.*

(1') *All iterated partial derivatives exist and are locally bounded.*

(2) *For $v \in \mathbb{R}^2$ the iterated directional derivatives*

$$d_v^n f(x) := (\tfrac{\partial}{\partial t})^n|_{t=0}(f(x+tv))$$

exist and are continuous with respect to x.

(3) *For $v \in \mathbb{R}^2$ the iterated directional derivatives*

$$d_v^n f(x) := (\tfrac{\partial}{\partial t})^n|_{t=0}(f(x+tv))$$

exist and are locally bounded with respect to x.

(4) *For all smooth curves $c : \mathbb{R} \to \mathbb{R}^2$ the composite $f \circ c$ is smooth.*

Proof.

(1) $\Rightarrow$ (4) is a direct consequence of the classical chain rule, namely that $(f \circ c)'(t) = \partial_1 f(c(t)) \cdot x'(t) + \partial_2 f(c(t)) \cdot y'(t)$, where $c = (x, y)$.

(4) $\Rightarrow$ (3) Obviously, $d_v^p f(x) := (\frac{d}{dt})^p|_{t=0} f(x+tv)$ exists, since $t \mapsto x + tv$ is a smooth curve. Suppose $d_v^p f$ is not locally bounded. So we may find a sequence x_n which converges fast to x, and such that $|d_v^p f(x_n)| \geq 2^{n^2}$. Let c be a smooth curve with $c(t + t_n) = x_n + \frac{t}{2^n} v$ locally for some sequence $t_n \to 0$, by 2.8. Then $(f \circ c)^{(p)}(t_n) = d_v^p f(x_n) \frac{1}{2^{np}}$ is unbounded, which is a contradiction.

(3) $\Rightarrow$ (2) We prove this by induction on p: Note that

$$d_v^p f(\_ + tv) - d_v^p f(\_) = t \int_0^1 d_v^{p+1} f(\_ + t\tau v) d\tau \to 0$$

for $t \to 0$ uniformly on bounded sets. Suppose that $|d_v^p f(x_n) - d_v^p f(x)| \geq \varepsilon$ for some sequence $x_n \to x$. Without loss of generality we may assume that $d_v^p f(x_n) - d_v^p f(x) \geq \varepsilon$. Then by the uniform convergence there exists a $\delta > 0$ such that $d_v^p f(x_n + tv) - d_v^p f(x + tv) \geq \frac{\varepsilon}{2}$ for $|t| \leq \delta$. Integration $\int_0^\delta dt$ yields

$$\Big(d_v^{p-1} f(x_n + \delta v) - d_v^{p-1} f(x_n)\Big) - \Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) \geq \tfrac{\varepsilon\delta}{2},$$

but by induction hypothesis the left hand side converges towards

$$\Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) - \Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) = 0.$$

(2) $\Rightarrow$ (1) Note that for a smooth map $g : \mathbb{R}^2 \to \mathbb{R}$ we have by the chain rule

$$d_v g(x + tv) = \frac{d}{dt} g(x + tv) = \partial_1 g(x + tv) \cdot v_1 + \partial_2 g(x + tv) \cdot v_2$$

and by induction that

$$d_v^p g(x) = \sum_{i=0}^p \binom{p}{i} \partial_1^i \partial_2^{p-i} g(x) v_1^i v_2^{p-i}.$$

Hence, we can calculate the iterated derivatives $\partial_1^i \partial_2^{p-i} g(x)$ for $0 \leq i \leq p$ from $p+1$ many derivatives $d_{v^j}^p g(x)$ provided the v^j are chosen in such a way, that the *Vandermonde's determinant* $\det((v_1^j)^i (v_2^j)^{p-i})_{ij} \neq 0$. For this choose $v_2 = 1$ and all the v_1 pairwise distinct, then $\det((v_1^j)^i (v_2^j)^{p-i})_{ij} = \prod_{j>k}(v_1^j - v_1^k) \neq 0$.

To complete the proof we use convolution with an *approximation of unity*. So let $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ have compact support, $\int_{\mathbb{R}^2} \varphi = 1$, and $\varphi(y) \geq 0$ for all y. Define

$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2}\varphi(\frac{1}{\varepsilon}x)$, and let

$$f_\varepsilon(x) := (f\star\varphi_\varepsilon)(x) = \int_{\mathbb{R}^2} f(x-y)\,\varphi_\varepsilon(y)\,dy = \int_{\mathbb{R}^2} f(x-\varepsilon y)\varphi(y)dy.$$

Since the *convolution* $f_\varepsilon := f\star\varphi_\varepsilon$ of a continuous function f with a smooth function φ_ε with compact support is differentiable with directional derivative $d_v(f\star\varphi_\varepsilon)(x) = (f\star d_v\varphi_\varepsilon)(x)$, we obtain that f_ε is smooth. And since $f\star\varphi_\varepsilon \to f$ in $C(\mathbb{R}^2,\mathbb{R})$ for $\varepsilon\to 0$ and any continuous function f, we conclude $d_v^p f_\varepsilon = d_v^p f\star\varphi_\varepsilon \to d_v^p f$ uniformly on compact sets.

By what we said above for smooth g, the iterated partial derivatives of f_ε are linear combinations of the derivatives $d_v^p f_\varepsilon$ for $p+1$ many vectors v, where the coefficients depend only on the v's. So we conclude that the iterated partial derivatives of f_ε form a Cauchy sequence in $C(\mathbb{R}^2,\mathbb{R})$, and hence converge to continuous functions f^α. Thus, all iterated derivatives $\partial^\alpha f$ of f exist and are equal to these continuous functions f^α, by the following lemma 3.5 recursively applied to $c_\varepsilon(s) := \partial^\alpha f_\varepsilon(x + s\,v)$.

(1) $\Leftrightarrow$ (1') Local boundedness of all iterated partial derivatives is equivalent to their continuity, since if for a function $g:\mathbb{R}^2\to\mathbb{R}$ the partial derivatives $\partial_1 g$ and $\partial_2 g$ exist and are locally bounded then g is continuous:

$$\begin{aligned} g(x,y) - g(0,0) &= g(x,y) - g(x,0) + g(x,0) - g(0,0)\\ &= y\partial_2 g(x, r_2 y) + x\partial_1 g(r_1 x, 0)\end{aligned}$$

for suitable r_1, $r_2 \in [0,1]$, which goes to 0 with (x,y). □

3.5. Lemma. *Let $c_\varepsilon:\mathbb{R}\to E$ be C^1 into a locally convex space E such that $c_\varepsilon \to c$ and $c'_\varepsilon \to c^1$ uniformly on bounded subsets of $\mathbb{R}$ for $\varepsilon\to 0$. Then $c:\mathbb{R}\to E$ is C^1 and $c' = c^1$. With other words, the injection $c\mapsto (c,c')$, $C^1(\mathbb{R},E)\to \ell^\infty(\mathbb{R},E)^2$ has closed image.*

Proof. Since $C(\mathbb{R},E)$ is closed in $\ell^\infty(\mathbb{R},E)$ the curves c and c^1 are continuous, Remains to show that for fixed $s\in\mathbb{R}$ the curve

$$\gamma : t\mapsto \begin{cases} \frac{c(s+t)-c(s)}{t} & \text{for } t\neq 0\\ c^1(s) & \text{otherwise}\end{cases}$$

is continuous (at 0). The corresponding curve γ_ε for c_ε can be rewritten as $\gamma_\varepsilon(t) = \int_0^1 c'_\varepsilon(s+\tau\,t)\,d\tau$, which converges by assumption for $\varepsilon\to 0$ uniformly on compact sets to the continuous curve $t\mapsto \int_0^1 c^1(s+\tau\,t)\,d\tau$. Pointwise it converges to $\gamma(t)$, hence γ is continuous. □

For the vector valued case of the exponential law we need a locally convex structure on $C^\infty(\mathbb{R},E)$.

3.6. Definition. Space of curves

Let $C^\infty(\mathbb{R},E)$ be the locally convex vector space of all smooth curves in E, with the pointwise vector operations, and with the topology of uniform convergence on compact sets of each derivative separately. This is the initial topology with respect to the linear mappings $C^\infty(\mathbb{R},E)\xrightarrow{d^k} C^\infty(\mathbb{R},E)\to \ell^\infty(K,E)$, where k runs through $\mathbb{N}$, where K runs through all compact subsets of $\mathbb{R}$, and where $\ell^\infty(K,E)$ carries the topology of uniform convergence, see also 2.15.

Note that the derivatives $d^k : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$, the point evaluations $\mathrm{ev}_t : C^\infty(\mathbb{R}, E) \to E$ and the pull backs $g^* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$ for all $g \in C^\infty(\mathbb{R}, \mathbb{R})$ are continuous and linear. For the later one uses that obviously $g^* : \ell^\infty(Y, E) \to \ell^\infty(X, E)$ is continuous for bounded mappings $g : X \to Y$ as well as $g \cdot (\_) : \ell^\infty(X, E) \to \ell^\infty(X, E)$ for bounded mappings $g : X \to \mathbb{R}$.

3.7. Lemma. *A space E is c^∞-complete if and only if $C^\infty(\mathbb{R}, E)$ is so.*

Proof. ($\Rightarrow$) The mapping $c \mapsto (c^{(n)})_{n\in\mathbb{N}}$ is by definition an embedding of $C^\infty(\mathbb{R}, E)$ into the c^∞-complete product $\prod_{n\in\mathbb{N}} \ell^\infty(\mathbb{R}, E)$. Its image is a closed subspace by lemma 3.5.

($\Leftarrow$) Consider the continuous linear mapping $\mathrm{const} : E \to C^\infty(\mathbb{R}, E)$ given by $x \mapsto (t \mapsto x)$. It has as continuous left inverse the evaluation at any point (e.g. $\mathrm{ev}_0 : C^\infty(\mathbb{R}, E) \to E$, $c \mapsto c(0)$). Hence, E can be identified with the closed subspace of $C^\infty(\mathbb{R}, E)$ given by the constant curves, and is thereby itself c^∞-complete. □

3.8. Lemma. Curves into limits. *A curve into a c^∞-closed subspace of a space is smooth if and only if it is smooth into the total space. In particular, a curve is smooth into a projective limit if and only if all its components are smooth.*

Proof. Since the derivative of a smooth curve is the Mackey limit of the difference quotient, the c^∞-closedness implies that this limit belongs to the subspace. Thus, we deduce inductively that all derivatives belong to the subspace, and hence the curve is smooth into the subspace.

The result on projective limits now follows, since obviously a curve is smooth into a product, if all its components are smooth. □

We show now that the bornology, but obviously not the topology, on function spaces can be tested with the linear functionals on the range space.

3.9. Lemma. Bornology of $C^\infty(\mathbb{R}, E)$. *The family*

$$\{\ell_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, \mathbb{R}) : \ell \in E^*\}$$

generates the bornology of $C^\infty(\mathbb{R}, E)$. This also holds for E^* replaced by E'.

A set in $C^\infty(\mathbb{R}, E)$ is bounded if and only if each derivative is uniformly bounded on compact subsets.

Proof. A set $\mathcal{B} \subseteq C^\infty(\mathbb{R}, E)$ is bounded if and only if the sets $\{d^n c(t) : t \in I, c \in \mathcal{B}\}$ are bounded in E for all $n \in \mathbb{N}$ and compact subsets $I \subset \mathbb{R}$.

This is furthermore equivalent to the condition that the set $\{\ell(d^n c(t)) = d^n(\ell \circ c)(t) : t \in I, c \in \mathcal{B}\}$ is bounded in $\mathbb{R}$ for all $\ell \in E^*$ (or even all $\ell \in E'$), $n \in \mathbb{N}$, and compact subsets $I \subset \mathbb{R}$ and in turn equivalent to: $\ell_*(\mathcal{B}) = \{\ell \circ c : c \in \mathcal{B}\}$ is bounded in $C^\infty(\mathbb{R}, \mathbb{R})$. □

3.10. Proposition. Vector valued simplest exponential law. *For a mapping $f : \mathbb{R}^2 \to E$ into a locally convex space (which need not be c^∞-complete) the following assertions are equivalent:*

1. *f is smooth along smooth curves.*
2. *All iterated directional derivatives $d_v^p f$ exist and are locally bounded.*
3. *All iterated partial derivatives $\partial^\alpha f$ exist and are locally bounded.*

(4) $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ *exists as a smooth curve.*

Proof. We prove this result first for c^∞-complete spaces E.

We could do this either by carrying over the proofs of 3.2 and 3.4 to the vector valued situation, or we reduce the vector valued one by linear functionals to the scalar valued situation. We choose here the second way.

By 2.3 each of the statements (1)-(4) is valid if and only if the corresponding statement with f replaced by $\ell\circ f$ is valid for all $\ell \in E^*$. Only (4) needs some arguments: In fact, $f^\vee(t) \in C^\infty(\mathbb{R}, E)$ if and only if $\ell_*(f^\vee(t)) = (\ell \circ f)^\vee(t) \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E^*$ by 2.14. Since $C^\infty(\mathbb{R}, E)$ is c^∞-complete, its bornologically isomorphic image in $\prod_{\ell\in E^*} C^\infty(\mathbb{R}, \mathbb{R})$ is c^∞-closed. So $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ is smooth if and only if $\ell_* \circ f^\vee = (\ell \circ f)^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is smooth for all $\ell \in E^*$.

So the proof is reduced to the scalar valid case, which was proved in 3.2 and 3.4.

Now the general case. For the existence of certain derivatives we know by 1.9 that it is enough that we have some candidate in the space, which is the corresponding derivative of the map considered as map into the c^∞-completion (or even some larger space). Since the derivatives required in (1)-(4) depend linearly on each other, this is true. In more detail this means:

(1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) is the fact that ∂^α is a universal linear combination of $d_v^{|\alpha|} f$.

(3) $\Rightarrow$ (1) follows from the chain rule which says that $(f \circ c)^{(p)}(t)$ is a universal linear combination of $\partial_{i_1} \dots \partial_{i_q} f(c(t))c_{i_1}^{(p_1)}(t) \dots c_{i_q}^{(p_q)}(t)$ for $i_j \in \{1, 2\}$ and $\sum p_j = p$, see also 10.4.

(3) $\Leftrightarrow$ (4) holds by 1.9 since $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d\circ f^\vee = d_*(f^\vee)$. $\square$

3.11

For the general case of the exponential law we need a notion of smooth mappings and a locally convex topology on the corresponding function spaces. Of course, it would be also handy to have a notion of smoothness for locally defined mappings. Since the idea is to test smoothness with smooth curves, such curves should have locally values in the domains of definition, and hence these domains should be c^∞-open.

Definition. Smooth mappings and spaces of them. A mapping $f : E \supseteq U \to F$ defined on a c^∞-open subset U is called *smooth* (or C^∞) if it maps smooth curves in U to smooth curves in F.

Let $C^\infty(U, F)$ denote the locally convex *space of all smooth mappings* $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$.

For $U = E = \mathbb{R}$ this coincides with our old definition. Obviously, any composition of smooth mappings is also smooth.

Lemma. *The space $C^\infty(U,F)$ is the (inverse) limit of spaces $C^\infty(\mathbb{R},F)$, one for each $c \in C^\infty(\mathbb{R},U)$, where the connecting mappings are pull backs g^* along reparameterizations $g \in C^\infty(\mathbb{R},\mathbb{R})$.*

Note that this limit is the closed linear subspace in the product

$$\prod_{c\in C^\infty(\mathbb{R},U)} C^\infty(\mathbb{R},F)$$

consisting of all (f_c) with $f_{c\circ g} = f_c \circ g$ for all c and all $g \in C^\infty(\mathbb{R},\mathbb{R})$.

Proof. The mappings $c^* : C^\infty(U,F) \to C^\infty(\mathbb{R},F)$ define a continuous linear embedding $C^\infty(U,F) \to \lim_c C^\infty(\mathbb{R},F)$, since for the connecting mappings g^* we have $c^*(f)\circ g = f\circ c\circ g = (c\circ g)^*(f)$. It is surjective since for any $(f_c) \in \lim_c C^\infty(\mathbb{R},F)$ one has $f_c = f\circ c$ where f is defined as $x \mapsto f_{\mathrm{const}_x}(0)$. □

3.12. Theorem. Cartesian closedness. *Let $U_i \subseteq E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to C^\infty(U_2,F)$ exists and is smooth.*

Proof. We have the following implications:

$f^\vee : U_1 \to C^\infty(U_2,F)$ is smooth.
$\Leftrightarrow$ $f^\vee \circ c_1 : \mathbb{R} \to C^\infty(U_2,F)$ is smooth for all smooth curves c_1 in U_1, by 3.11.
$\Leftrightarrow$ $c_2^* \circ f^\vee \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R},F)$ is smooth for all smooth curves c_i in U_i, by 3.11 and 3.8.
$\Leftrightarrow$ $f\circ(c_1\times c_2) = (c_2^*\circ f^\vee\circ c_1)^\wedge : \mathbb{R}^2 \to F$ is smooth for all smooth curves c_i in U_i, by 3.10.
$\Leftrightarrow$ $f : U_1\times U_2 \to F$ is smooth.

Here the last equivalence is seen as follows: Each curve into $U_1 \times U_2$ is of the form $(c_1,c_2) = (c_1\times c_2)\circ\Delta$, where Δ is the diagonal mapping. Conversely, $f\circ(c_1 \times c_2) : \mathbb{R}^2 \to F$ is smooth for all smooth curves c_i in U_i, since the product and the composite of smooth mappings is smooth by definition 3.11 (and by 3.4). □

3.13. Corollary. Consequences of cartesian closedness. *Let E, F, G, etc. be locally convex spaces, and let U, V be c^∞-open subsets of such. Then the following canonical mappings are smooth.*

(1) $\mathrm{ev} : C^\infty(U,F)\times U \to F$, $(f,x)\mapsto f(x)$;
(2) $\mathrm{ins} : E \to C^\infty(F,E\times F)$, $x \mapsto (y\mapsto (x,y))$;
(3) $(\_)^\wedge : C^\infty(U,C^\infty(V,G)) \to C^\infty(U\times V,G)$;
(4) $(\_)^\vee : C^\infty(U\times V,G) \to C^\infty(U,C^\infty(V,G))$;
(5) $\mathrm{comp} : C^\infty(F,G)\times C^\infty(U,F) \to C^\infty(U,G)$, $(f,g)\mapsto f\circ g$;
(6) $C^\infty(\_,\_) : C^\infty(E_2,E_1)\times C^\infty(F_1,F_2) \to$
$\to C^\infty(C^\infty(E_1,F_1),C^\infty(E_2,F_2))$, $(f,g)\mapsto (h\mapsto g\circ h\circ f)$;
(7) $\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i,\prod F_i)$, *for any index set.*

Proof. (1) The mapping associated to *ev* via cartesian closedness is the identity on $C^\infty(U,F)$, which is C^∞, thus *ev* is also C^∞.

(2) The mapping associated to *ins* via cartesian closedness is the identity on $E\times F$, hence *ins* is C^∞.

(3) The mapping associated to $(\_)^\wedge$ via cartesian closedness is the smooth composition of evaluations $\mathrm{ev}\circ(\mathrm{ev}\times\mathrm{Id}) : (f;x,y)\mapsto f(x)(y)$.

(4) We apply cartesian closedness twice to get the associated mapping $(f;x;y)\mapsto f(x,y)$, which is just a smooth evaluation mapping.

(5) The mapping associated to *comp* via cartesian closedness is $(f,g;x)\mapsto f(g(x))$, which is the smooth mapping $\mathrm{ev}\circ(\mathrm{Id}\times\mathrm{ev})$.

(6) The mapping associated to the one in question by applying cartesian closed is $(f,g,h)\mapsto g\circ h\circ f$, which is appart permutation of the variables the C^∞-mapping $\mathrm{comp}\circ(\mathrm{Id}\times\mathrm{comp})$.

(7) Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $C^\infty(E_i,F_i)\times E_i\to F_i$. □

Next we generalize 3.4 to finite dimensions.

3.14. Corollary. [**14**]. *The smooth mappings on open subsets of $\mathbb{R}^n$ in the sense of definition 3.11 are exactly the usual smooth mappings.*

Proof. ($\Leftarrow$) is obvious by the usual chain rule.

($\Rightarrow$) Both conditions are of local nature, so we may assume that the open subset of $\mathbb{R}^n$ is an open box and (by reparametrizing with a diffeomorphism in usual sense) even $\mathbb{R}^n$ itself.

If $f:\mathbb{R}^n\to F$ is smooth along smooth curves then by cartesian closedness 3.12, for each coordinate the respective associated mapping $f^{\vee_i}:\mathbb{R}^{n-1}\to C^\infty(\mathbb{R},F)$ is smooth along smooth curves. Moreover the first partial derivative $\partial_i f$ exists and we have $\partial_i f=(d\circ f^{\vee_i})^\wedge$ (c.f. 3.2), so all first partial derivatives exist and are smooth along smooth curves. Inductively, all iterated partial derivatives exist and are smooth along smooth curves, thus continuous, so f is smooth in the usual sense. □

3.15. Differentiation of an integral

We return to the question of differentiating an integral. So let $f:E\times\mathbb{R}\to F$ be smooth, and let $\widehat{F}$ be the completion of the locally convex space F. Then we may form the function $f_0:E\to\widehat{F}$ defined by $x\mapsto\int_0^1 f(x,t)\,dt$. We claim that it is smooth, and that the directional derivative is given by $d_v f_0(x)=\int_0^1 d_v(f(\_,t))(x)\,dt$. By cartesian closedness 3.12 the associated mapping $f^\vee:E\to C^\infty(\mathbb{R},F)$ is smooth, so the mapping $f_0:=\int_0^1\circ f^\vee:E\to\widehat{F}$ is smooth since integration is a bounded linear operator, and

$$\begin{aligned}
d_v f_0(x) &= \tfrac{\partial}{\partial s}\big|_{s=0} f_0(x+sv) = \tfrac{\partial}{\partial s}\big|_{s=0} \Big(\int_0^1 \circ f^\vee\Big)(x+sv)\\
&= \int_0^1 \Big(\tfrac{\partial}{\partial s}\big|_{s=0} f^\vee(x+sv)\Big)(t)\,dt = \int_0^1 \mathrm{ev}_t\Big(\tfrac{\partial}{\partial s}\big|_{s=0} f^\vee(x+sv)\Big)\,dt\\
&= \int_0^1 \tfrac{\partial}{\partial s}\big|_{s=0} \Big(\mathrm{ev}_t\Big(f^\vee(x+sv)\Big)\Big)\,dt = \int_0^1 \tfrac{\partial}{\partial s}\big|_{s=0} f(x+sv,t)dt\\
&= \int_0^1 d_v(f(_,t))(x)\,dt.
\end{aligned}$$

We want to generalize this to functions f defined only on some c^∞-open subset $U \subseteq E\times\mathbb{R}$, so we have to show that the natural domain $U_0 := \{x \in E : \{x\}\times[0,1] \subseteq U\}$ of f_0 is c^∞-open in E. We will do this in lemma 4.15. From then on the proof runs exactly the same way as for globally defined functions, since for $x_0 \in U_0$ there exists a bounded open interval $J \supseteq [0,1]$ such that $\{x_0\}\times \overline{J} \subseteq U$ and hence $f^\vee$ is defined on a c^∞-neighborhood of x_0 and smooth into $C^\infty(J,F) \to C([0,1],F)$. So we obtain the

Proposition. *Let $f : E\times\mathbb{R} \supseteq U \to F$ be smooth with c^∞-open $U \subseteq E\times\mathbb{R}$. Then $x \mapsto \int_0^1 f(x,t)\,dt$ is smooth on the c^∞-open set $U_0 := \{x \in E : \{x\}\times[0,1] \subseteq U\}$ with values in the completion $\widehat{F}$ and $d_v f_0(x) = \int_0^1 d_v(f(\_,t))(x)\,dt$ for all $x \in U_0$ and $v \in E$.* □

Now we want to define the derivative of a general smooth map and prove the chain rule for them.

3.16. Corollary. Smoothness of the difference quotient. *For a smooth curve $c : \mathbb{R} \to E$ the difference quotient*

$$(t,s) \mapsto \begin{cases} \dfrac{c(t)-c(s)}{t-s} & \text{for } t \neq s\\ c'(t) & \text{for } t = s\end{cases}$$

is a smooth mapping $\mathbb{R}^2 \to E$. Cf. 1.7 and 2.1.

Proof. By 2.5 we have $f : (t,s) \mapsto \frac{c(t)-c(s)}{t-s} = \int_0^1 c'(s+r(t-s))\,dr$, and by 3.15 it is smooth $\mathbb{R}^2 \to \widehat{E}$. The left hand side has values in E, and for $t \neq s$ this is also true for all iterated directional derivatives. It remains to consider the derivatives for $t = s$. The iterated directional derivatives of f in $\hat{E}$ are given by 3.15 as

$$\begin{aligned}
d^p_{(v,w)} f(t,s) = d^p_{(v,w)} &\int_0^1 c'(\underbrace{s+r(t-s)}_{rt+(1-r)s})\,dr\\
&= \int_0^1 \big(\tfrac{d}{du}\big)^p\Big|_{u=0} c'(\underbrace{r\,(t+u\,v)+(1-r)\,(s+u\,w)}_{u\,(r\,v+(1-r)\,w)+(r\,t+(1-r)\,s})\,dr\\
&= \int_0^1 (r\,v+(1-r)\,w)^p\, c^{(p+1)}(r\,t+(1-r)\,s)\,dr
\end{aligned}$$

For $t = s$ the later integrand is just $\int_0^1 (r\,v+(1-r)\,w)^p\,dr \cdot c^{(p+1)}(t) \in E$. Hence $d^p_{(v,w)} f(t,s) \in E$. By 3.10 the mapping f is smooth into E. □

3.17. Definition. Spaces of linear mappings

Let $L(E,F)$ denote the *space of all bounded* (equivalently smooth by 2.11) *linear mappings* from E to F. It is a closed linear subspace of $C^\infty(E,F)$ since f is linear if and only if for all x, $y \in E$ and $\lambda \in \mathbb{R}$ we have $(\operatorname{ev}_x + \lambda \operatorname{ev}_y - \operatorname{ev}_{x+\lambda y})f = 0$. We equip it with this topology and linear structure.

So a mapping $f : U \to L(E,F)$ is smooth if and only if the composite mapping $U \xrightarrow{f} L(E,F) \to C^\infty(E,F)$ is smooth.

3.18. Theorem. Chain rule. *Let E and F be locally convex spaces, and let $U \subseteq E$ be c^∞-open. Then the differentiation operator*

$$d : C^\infty(U,F) \to C^\infty(U, L(E,F)),$$
$$df(x)v := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t},$$

exists, is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x)\cdot v = df(g(x))\cdot dg(x)\cdot v.$$

Proof. Since $t \mapsto x+tv$ is a smooth curve we know that $d^{\wedge\wedge} : C^\infty(U,F)\times U\times E \to F$ exists. We want to show that it is smooth, so let $(f,x,v) : \mathbb{R} \to C^\infty(U,F)\times U\times E$ be a smooth curve. Then

$$d^{\wedge\wedge}(f(t),x(t),v(t)) = \lim_{s\to 0} \frac{f(t)(x(t)+sv(t)) - f(t)(x(t))}{s} = \partial_2 h(t,0),$$

which is smooth in t, where the smooth mapping $h : \mathbb{R}^2 \supseteq \{(t,s) : x(t)+sv(t) \in U\} \to F$ is given by $(t,s) \mapsto f^\wedge(t, x(t)+sv(t))$. By cartesian closedness 3.12 the mapping $d^\wedge : C^\infty(U,F)\times U \to C^\infty(E,F)$ is smooth.

Now we show that this mapping has values in the subspace $L(E,F)$: $d^\wedge(f,x)$ is obviously homogeneous. It is additive, because we may consider the smooth mapping $(t,s)\mapsto f(x+tv+sw)$ and compute as follows, using 3.14.

$$\begin{aligned} df(x)(v+w) &= \tfrac{\partial}{\partial t}\big|_0 f(x+t(v+w)) \\ &= \tfrac{\partial}{\partial t}\big|_0 f(x+tv+0w) + \tfrac{\partial}{\partial t}\big|_0 f(x+0v+tw) = df(x)v + df(x)w. \end{aligned}$$

So we see that $d^\wedge : C^\infty(U,F)\times U \to L(E,F)$ is smooth, and the mapping $d : C^\infty(U,F) \to C^\infty(U,L(E,F))$ is smooth by 3.12 and obviously linear.

We first prove the chain rule for a smooth curve c instead of g. We have to show that the curve

$$t \mapsto \begin{cases} \frac{f(c(t))-f(c(0))}{t} & \text{for } t\neq 0 \\ df(c(0))\cdot c'(0) & \text{for } t = 0 \end{cases}$$

is continuous at 0. It can be rewritten as $t \mapsto \int_0^1 df(c(0)+s(c(t)-c(0)))\cdot c_1(t)\,ds$, where c_1 is the (by 3.16) smooth curve given by

$$t \mapsto \begin{cases} \frac{c(t)-c(0)}{t} & \text{for } t\neq 0 \\ c'(0) & \text{for } t = 0 \end{cases}.$$

Since $h : \mathbb{R}^2 \to E\times E$ given by

$$(t,s) \mapsto \Big(c(0) + s\big(c(t)-c(0)\big), c_1(t)\Big)$$

is smooth, there exist open neighborhoods I of $[0,1]$ and J of 0 in $\mathbb{R}$ such that map $t \mapsto \Big(s \mapsto df\big(c(0) + s(c(t) - c(0))\big) \cdot c_1(t)\Big)$ is smooth $J \to C^\infty(I,F)$, and hence $t \mapsto \int_0^1 df\big(c(0) + s(c(t) - c(0))\big) \cdot c_1(t)\, ds$ is smooth as in 3.15, and hence continuous.

For general g we have

$$\begin{aligned} d(f \circ g)(x)(v) &= \tfrac{\partial}{\partial t}\big|_0 (f \circ g)(x + tv) = (df)\big(g(x + 0v)\big)\big(\tfrac{\partial}{\partial t}\big|_0 (g(x + tv))\big) \\ &= (df)\big(g(x)\big)\big(dg(x)(v)\big). \quad \square \end{aligned}$$

3.19. Lemma. *Two locally convex spaces are locally diffeomorphic if and only if they are linearly diffeomorphic.*
Any smooth and 1-homogeneous mapping is linear.

Proof. By the chain rule the derivatives at corresponding points give the linear diffeomorphisms.

For a 1-homogeneous mapping f one has $df(0)v = \frac{\partial}{\partial t}\big|_0 f(tv) = f(v)$, and this is linear in v. $\square$

4. The c^∞-Topology

4.1. Definition. *A locally convex vector space E is called bornological if and only if the following equivalent conditions are satisfied:*

1. *Any bounded linear mapping $T : E \to F$ in any locally convex space F is continuous; It is sufficient to know this for all Banach spaces F.*
2. *Every bounded seminorm on E is continuous.*
3. *Every absolutely convex bornivorous subset is a 0-neighborhood.*

A *radial subset* U (i.e. $[0,1]U \subseteq U$) of a locally convex space E is called *bornivorous* if it *absorbs* each bounded set, i.e. for every bounded B there exists $r > 0$ such that $rU \supseteq B$.

We will make use of the following simple fact: Let $A, B \subseteq E$ be subsets of a vector space E with A absolutely convex. Then A absorbs B if and only if the Minkowski-funktional p_A is bounded on B.

Proof.

($1 \Rightarrow 2$) Let p be a bounded seminorm. Then the canonical projection $T : E \to E/\ker p \subseteq \widehat{E/\ker p}$ is bounded and hence continuous by (1). Thus, $p = \tilde{p} \circ T$ is continuous, where $\tilde{p}$ denotes the canonical norm on the completion $\widehat{E/\ker p}$ induced from p.

($2 \Rightarrow 3$), since the Minkowski-functional p generated by an absolutely convex bornivorous subset is a bounded seminorm.

($3 \Rightarrow 1$) Let $T : E \to F$ be bounded linear and $V \subseteq F$ be a absolutely convex 0-neighborhood. Then $T^{-1}(V)$ is absolutely convex and bornivorous, thus by (3) a 0-neighborhood, i.e. T is continuous. $\square$

4.2. Lemma. Bornologification. *The bornologification E_{born} of a locally convex space can be described in the following equivalent ways:*

(1) *It is the finest locally convex structure having the same bounded sets;*
(2) *It is the final locally convex structure with respect to the inclusions $E_B \to E$, where B runs through all bounded (closed) absolutely convex subsets.*

Moreover, E_{born} is bornological. For any locally convex vector space F the continuous linear mappings $E_{born} \to F$ are exactly the bounded linear mappings $E \to F$. The continuous seminorms on E_{born} are exactly the bounded seminorms of E. An absolutely convex set is a 0-neighborhood in E_{born} if and only if it is bornivorous, i.e. absorbs bounded sets.

Proof. Let E_{born} be the vector space E supplied with the topology described in (1) and E_{fin} be E supplied with the final locally convex topology described in (2).

($E_{\text{fin}} \to E_{\text{born}}$ is continuous), since all bounded absolutely convex sets B in E are bounded in E_{born}, thus the inclusions $E_B \to E_{\text{born}}$ are bounded and hence continuous since E_B is normed. Thus, the final structure on E induced by the inclusions $E_B \to E$ is finer than the structure of E_{born}.

($E_{\text{born}} \to E_{\text{fin}}$ is continuous). It is obviously bounded, since the construction the bounded subsets B of E_{born} are bounded in E, hence contained in bounded absolutely convex $B \subseteq E$ and hence bounded in $E_B \to E_{\text{fin}}$.
Hence, E_{fin} has exactly the same bounded sets as E, and so E_{born} is by definition finer than E_{fin}.

$E_{\text{born}} = E_{\text{fin}}$ is bornological by (1) in 4.1: Let $T : E \to F$ be bounded linear, then $T|_{E_B} : E_B \to E \to F$ is bounded and hence $T : E_{\text{fin}} \to F$ is continuous.

The remaining statements now follow, since E_{born} and E have the same bounded seminorms, the same bounded linear mappings with values in locally convex spaces and the same bornivorous absolutely convex subsets. And on the bornological space E_{born} these are by 4.1 exactly the continuous seminorms, the continuous linear mappings and the absolutely convex 0-neighborhoods. □

4.3. Corollary. Bounded seminorms. *For a seminorm p and a sequence $\mu_n \to \infty$ the following statements are equivalent:*

(1) *p is bounded;*
(2) *p is bounded on compact sets;*
(3) *p is bounded on M-converging sequences;*
(4) *p is bounded on μ-converging sequences;*
(5) *p is bounded on images of bounded intervals under $\mathcal{L}\mathrm{ip}^k$-curves (for fixed $0 \le k \le \infty$).*

The corresponding statement for subsets of E is the following. For a RADIAL *subset $U \subseteq E$ (i.e., $[0,1] \cdot U \subseteq U$) the following properties are equivalent:*

(1) *U is bornivorous.*
(1') *For all absolutely convex bounded sets B, the trace $U \cap E_B$ is a 0-neighborhood in E_B.*
(2) *U absorbs all compact subsets in E.*
(3) *U absorbs all Mackey convergent sequences.*

(3') *U absorbs all sequences converging Mackey to 0.*

(4) *U absorbs all μ-convergent sequences (for a fixed μ).*

(4') *U absorbs all sequences which are μ-converging to 0.*

(5) *U absorbs the images of bounded sets under $\mathcal{L}\mathrm{ip}^k$-curves (for a fixed $0 \le k \le \infty$).*

Proof. We prove the statement on radial subsets, for seminorms p it then follows since p is bounded on a subset $A \subseteq E$ if and only if the radial set $U := \{x \in E : p(x) \le 1\}$ absorbs A (using the equality $K \cdot U = \{x \in E : p(x) \le K\}$).

(1') $\Leftrightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (4'), (3) $\Rightarrow$ (3') $\Rightarrow$ (4'), (2) $\Rightarrow$ (5), are trivial.

(5) $\Rightarrow$ (4') Suppose that (x_n) is μ-converging to 0 but is not absorbed by U. Then for each $m \in \mathbb{N}$ there is an $n_m \in \mathbb{N}$ with $x_{n_m} \notin mU$ and by passing to a subsequence $(n_{m_k})_k$ of $(n_m)_m$ we may assume that $k \mapsto 1/\mu_{n_{m_k}}$ is fast falling. The sequence $(x_{n_{m_k}} = \frac{1}{\mu_{n_{m_k}}} \mu_{n_{m_k}} x_{n_{m_k}})_k$ is then fast falling and lies on some compact part of a smooth curve by the special curve lemma 2.8. The set U absorbs this by (5), a contradiction to $x_{m_{m_k}} \notin m_k U$ with $m_k \ge k \to \infty$.

(4') $\Rightarrow$ (1) Suppose U does not absorb some bounded B. Hence, there are $b_n \in B$ with $b_n \notin \mu_n^2 U$. However, $\frac{b_n}{\mu_n}$ is μ-convergent to 0, so it is contained in KU for some $K > 0$. Equivalently, $b_n \in \mu_n K U \subseteq \mu_n^2 U$ for all $\mu_n \ge K$, which gives a contradiction. □

4.4. Corollary. Bornologification as locally-convex-ification.
The bornologification of E is the finest locally convex topology with one (hence all) of the following properties:

(1) *It has the same bounded sets as E.*
(2) *It has the same Mackey converging sequences as E.*
(3) *It has the same μ-converging sequences as E (for some fixed μ).*
(4) *It has the same $\mathcal{L}\mathrm{ip}^k$-curves as E (for some fixed $0 \le k \le \infty$).*
(5) *It has the same bounded linear mappings from E into arbitrary locally convex spaces.*
(6) *It has the same continuous linear mappings from normed spaces into E.*

Proof. Since the bornologification has the same bounded sets as the original topology, the other objects are also the same: they depend only on the bornology – this would not be true for compact sets, e.g. the bornologification of the topology of pointwise convergence on the dual of any infinite dimensional Banach space is (by the unform boundedness theorem) that of uniform convergence on the unit ball, but the dual unit ball is only compact for the former.

Conversely, we consider a topology τ which has for one of the above mentioned types the same objects as the original one. Then the identity $E_{\mathrm{born}} \to (E, \tau)$ is bounded and hence continuous by 4.1.1. □

4.5. Lemma. *Let E be a bornological locally convex vector space, $U \subseteq E$ a convex subset. Then U is open for the locally convex topology of E if and only if U is open for the c^∞-topology.*

Furthermore, an absolutely convex subset U of E is a 0-neighborhood for the locally convex topology if and only if it is so for the c^∞-topology.

Proof. ($\Rightarrow$) The c^∞-topology is finer than the locally convex topology, cf. 4.2.

($\Leftarrow$) Let first U be an absolutely convex 0-neighborhood for the c^∞-topology. Hence, U absorbs Mackey-0-sequences by 2.13. By 4.1.3 we have to show that U is bornivorous, in order to obtain that U is a 0-neighborhood for the locally convex topology. But this follows immediately from 4.3.

Let now U be convex and c^∞-open, let $x \in U$ be arbitrary. We consider the c^∞-open absolutely convex set $W := (U - x) \cap (x - U)$ which is a 0-neighborhood of the locally convex topology by the argument above. Then $x \in W + x \subseteq U$. So U is open in the locally convex topology. □

4.6. Corollary. *The bornologification of a locally convex space E is the finest locally convex topology coarser than the c^∞-topology on E.* □

4.7. Definition. In 2.12 we defined the c^∞-topology on an arbitrary locally convex space E as the final topology with respect to the smooth curves $c : \mathbb{R} \to E$, see also 2.13. Now we will compare the c^∞-topology with other refinements of a given locally convex topology. We first specify those refinements.

Let E be a locally convex vector space.

(i) We denote by kE the *Kelley-fication* of the locally convex topology of E, i.e. the vector space E together with the final topology induced by the inclusions of the subsets being compact for the locally convex topology.

(ii) We denote by sE the vector space E with the final topology induced by the curves being continuous for the locally convex topology, or equivalently the sequences $\mathbb{N}_\infty \to E$ converging in the locally convex topology. The equivalence holds since the infinite polygon through a converging sequence can be continuously parameterized by a compact interval.

(iii) We recall that by $c^\infty E$ we denote the vector space E with its c^∞-topology, i.e. the final topology induced by the smooth curves.

Using that smooth curves are continuous and that converging sequences $\mathbb{N}_\infty \to E$ have compact images, the following identities are continuous: $c^\infty E \to sE \to kE \to E$.

If the locally convex topology of E coincides with the topology of $c^\infty E$, resp. sE, resp. kE then we call E smoothly generated, resp. sequentially generated, resp. compactly generated.

4.8. Example. *On $E = \mathbb{R}^J$ all the refinements of the locally convex topology described in 4.7 above are different, i.e. $c^\infty E \neq sE \neq kE \neq E$, provided the cardinality of the index set J is at least that of the continuum.*

Proof. It is enough to show this for J equipotent to the continuum, since $\mathbb{R}^{J_1}$ is a direct summand in $\mathbb{R}^{J_2}$ for $J_1 \subseteq J_2$.

($c^\infty E \neq sE$) We may take as index set J the set c_0 of all real sequences converging to 0. Define a sequence (x^n) in E by $(x^n)_j := j_n$. Since every $j \in J$ is a 0-sequence we conclude that the x^n converge to 0 in the locally convex topology of the product, hence also in sE. Assume now that the x^n converge towards 0 in $c^\infty E$. Then by 4.9 some subsequence converges Mackey to 0. Thus, there exists an unbounded sequence of reals λ_n with $\{\lambda_n x^n : n \in \mathbb{N}\}$ bounded. Let j be a 0-sequence with $\{j_n \lambda_n : n \in \mathbb{N}\}$ unbounded (e.g. $(j_n)^{-2} := 1 + \max\{|\lambda_k| : k \leq n\}$). Then the j-th coordinate $j_n \lambda_n$ of $\lambda_n x^n$ is not bounded with respect to n, a contradiction.

($sE \neq kE$) Consider in E the subset

$$A := \Big\{x \in \{0,1\}^J : x_j = 1 \text{ for at most countably many } j \in J\Big\}.$$

It is clearly closed with respect to the converging sequences, hence closed in sE. But it is not closed in kE since it is dense in the compact set $\{0,1\}^J$.

($kE \neq E$) Consider in E the subsets

$$A_n := \big\{x \in E : |x_j| < n \text{ for at most } n \text{ many } j \in J\big\}.$$

Each A_n is closed in E since its complement is the union of the open sets $\{x \in E : |x_j| < n \text{ for all } j \in J_o\}$ where J_o runs through all subsets of J with $n+1$ elements. We show that the union $A := \bigcup_{n\in\mathbb{N}} A_n$ is closed in kE. So let K be a compact subset of E; then $K \subseteq \prod \operatorname{pr}_j(K)$, and each $\operatorname{pr}_j(K)$ is compact, hence bounded in $\mathbb{R}$. Since the family $(\{j \in J : \operatorname{pr}_j(K) \subseteq [-n,n]\})_{n\in\mathbb{N}}$ covers J, there has to exist an $N \in \mathbb{N}$ and infinitely many $j \in J$ with $\operatorname{pr}_j(K) \subseteq [-N,N]$. Thus $K \cap A_n = \emptyset$ for all $n > N$, and hence, $A \cap K = \bigcup_{n\leq N} A_n \cap K$ is closed. Nevertheless, A is not closed in E, since 0 is in $\bar{A}$ but not in A. □

4.9. $\mathbf{c^\infty}$-convergent sequences

By 2.13 every M-convergent sequence gives a continuous mapping $\mathbb{N}_\infty \to c^\infty E$ and hence converges in $c^\infty E$. Conversely, a sequence converging in $c^\infty E$ is not necessarily Mackey convergent, see [**39**, Proposition 15.a]. However, one has the following result.

Lemma. *A sequence (x_n) is convergent to x in the c^∞-topology if and only if every subsequence has a subsequence which is Mackey convergent to x.*

Proof. ($\Leftarrow$) is true for any topological convergence: In fact, if x_n would not converge to x, then there would be a neighborhood U of x and a subsequence of x_n which lies outside of U and hence cannot have a subsequence converging to x.

($\Rightarrow$) It is enough to show that (x_n) has a subsequence which converges Mackey to x, since every subsequence of a c^∞-convergent sequence is clearly c^∞-convergent to the same limit. Without loss of generality we may assume that $x \notin A := \{x_n : n \in \mathbb{N}\}$. Hence, A cannot be c^∞-closed, and thus there is a sequence $n_k \in \mathbb{N}$ such that (x_{n_k}) converges Mackey to some point $x' \notin A$. The set $\{n_k : k \in \mathbb{N}\}$ cannot be bounded, and hence we may assume that the n_k are strictly increasing by passing to a subsequence. But then (x_{n_k}) is a subsequence of (x_n) which converges in $c^\infty E$ to x and Mackey to x' hence also in $c^\infty E$. Thus $x' = x$. □

Remark. A consequence of this lemma is, that there is no topology in general having as convergent sequences exactly the M-convergent ones, since this topology obviously would have to be coarser than the c^∞-topology.

One can use this lemma also to show that the c^∞-topology on a locally convex vector space gives a so called *arc-generated vector space*. See [41, 2.3.9,2.3.13] for a discussion of this.

Let us now describe several important situations where at least some of these topologies coincide. For the proof we will need the following

4.10. Lemma. [6] *For any locally convex space E the following statements are equivalent:*

(1) *The sequential closure of any subset is formed by all limits of sequences in the subset.*
(2) *For any given double sequence $(x_{n,k})$ in E with $x_{n,k}$ convergent to some x_k for $n \to \infty$ and k fixed and x_k convergent to some x, there are strictly increasing sequences $i \mapsto n(i)$ and $i \mapsto k(i)$ with $x_{n(i),k(i)} \to x$ for $i \to \infty$.*

Proof. (1⇒2) Take an $a_0 \in E$ different from $k\cdot(x_{n+k,k} - x)$ and from $k\cdot(x_k - x)$ for all k and n. Define $A := \{a_{n,k} := x_{n+k,k} - \frac{1}{k}\cdot a_0 : n, k \in \mathbb{N}\}$. Then x is in the sequential closure of A, since $x_{n+k,k} - \frac{1}{k}\cdot a_0$ converges to $x_k - \frac{1}{k}\cdot a_0$ as $n \to \infty$, and $x_k - \frac{1}{k}\cdot a_0$ converges to $x - 0 = x$ as $k \to \infty$. Hence, by (1) there has to exist a sequence $i \mapsto (n_i, k_i)$ with a_{n_i,k_i} convergent to x. By passing to a subsequence we may suppose that $i \mapsto k_i$ and $i \mapsto n_i$ are increasing. Assume that $i \mapsto k_i$ is bounded, hence finally constant. Then a subsequence $x_{n_i+k_i,k_i} - \frac{1}{k_i}\cdot a_0$ is converging to $x_k - \frac{1}{k}\cdot a_0 \neq x$ if $i \mapsto n_i$ is unbounded, and to $x_{n+k,k} - \frac{1}{k}\cdot a_0 \neq x$ if $i \mapsto n_i$ is bounded, which both yield a contradiction. Thus, $i \mapsto k_i$ can be chosen strictly increasing. But then

$$x_{n_i+k_i,k_i} = a_{n_i,k_i} + \tfrac{1}{k_i}a_0 \to x.$$

(1) ⇐ (2) is obvious. □

4.11. Theorem. *For any bornological vector space E the following implications hold:*

(1) *$c^\infty E = E$ provided the closure of subsets in E is formed by all limits of sequences in the subset; hence in particular if E is metrizable.*
(2) *$c^\infty E = E$ provided E is the strong dual of a Fréchet Schwartz space;*
(3) *$c^\infty E = kE$ provided E is the strict inductive limit of a sequence of Fréchet spaces.*
(4) *$c^\infty E = sE$ provided E satisfies the M-convergence condition, i.e. every sequence converging in the locally convex topology is M-convergent.*
(5) *$sE = E$ provided E is the strong dual of a Fréchet Montel space;*

Proof. (1) Using the lemma 4.10 above one obtains that the closure and the sequential closure coincide, hence $sE = E$. It remains to show that $sE \to c^\infty E$ is (sequentially) continuous. So suppose a sequence converging to x is given, and let (x_n) be an arbitrary subsequence. Then $x_{n,k} := k(x_n - x) \to k\cdot 0 = 0$ for $n \to \infty$, and hence by lemma 4.10 there are subsequences k_i, n_i with $k_i\cdot(x_{n_i} - x) \to 0$, i.e. $i \mapsto x_{n_i}$ is M-convergent to x. Thus, the original sequence converges in $c^\infty E$ by 4.9.

(3) Let E be the strict inductive limit of the Fréchet spaces E_n. By [68, 4.8.1] every E_n carries the trace topology of E, hence is closed in E, and every bounded subset of E is contained in some E_n. Thus, every compact subset of E is contained

as compact subset in some E_n. Since E_n is a Fréchet space such a subset is even compact in $c^\infty E_n$ by (1) and hence compact in $c^\infty E$. Thus, the identity $kE \to c^\infty E$ is continuous.

(4) is valid, since the M-closure topology is the final one induced by the M-converging sequences.

(5) Let E be the dual of any Fréchet Montel space F. By [**75**, 52.29] E is bornological. First we show that $kE = sE$. Let $K \subseteq E = F'$ be compact for the locally convex topology. Then K is bounded, hence equicontinuous since F is barrelled by [**68**, 5.2.2]. Since F is separable by [**53**, 11.6.2, p231] the set K is metrizable in the weak topology $\sigma(E, F)$ by [**75**, 52.21]. By [**68**, 7.4.12] this weak topology coincides with the topology of uniform convergence on precompact subsets of F. Since F is a Montel space, this latter topology is the strong one, and even the bornological one, as remarked at the beginning. Thus, the (metrizable) topology on K is the initial one induced by the converging sequences. Hence, the identity $kE \to sE$ is continuous, and therefore $sE = kE$.

It remains to show $kE = E$. Since F is Montel the locally convex topology of the strong dual coincides with the topology of uniform convergence on precompact subsets of F. Since F is metrizable this topology coincides with the so-called equicontinuous weak*-topology, cf. [**75**, 52.22], which is the final topology induced by the inclusions of the equicontinuous subsets. These subsets are by the Alaoğlu-Bourbaki theorem [**68**, 7.4.12] relatively compact in the topology of uniform convergence on precompact subsets. Thus, the locally convex topology of E is compactly generated.

(2) By (5), and since Fréchet Schwartz spaces are Montel by [**75**, 52.24], we have $sE = E$ and it remains to show that $c^\infty E = sE$. So let (x_n) be a sequence converging to 0 in E. Then the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact, and by [**41**, 4.4.39] it is relatively compact in some Banach space E_B. Hence, at least a subsequence has to be convergent in E_B. Clearly its Mackey limit has to be 0. This shows that (x_n) converges to 0 in $c^\infty E$, and hence $c^\infty E = sE$. One can even show that E satisfies the Mackey convergence condition, see [**75**, 52.28]. □

4.12. Example

We give now a non-metrizable example to which 4.11.1 applies. Let E denote the subspace of $\mathbb{R}^J$ of all sequences with countable support. Then the closure of subsets of E is given by all limits of sequences in the subset, but for non-countable J the space E is not metrizable. This was proved in [**7**].

4.13. Remark . The conditions 4.11.1 and 4.11.2 are rather disjoint since every locally convex space, that has a countable basis of its bornology and for which the *sequential adherence* of subsets (the set of all limits of sequences in it) is sequentially closed, is normable as the following proposition shows:

Proposition. *Let E be a non-normable bornological locally convex space that has a countable basis of its bornology. Then there exists a subset of E whose sequential adherence is not sequentially closed.*

Proof. Let $\{B_k : k \in \mathbb{N}_0\}$ be an increasing basis of the von Neumann bornology with $B_0 = \{0\}$. Since E is non-normable we may assume that B_k does not absorb

B_{k+1} for all k. Now choose $b_{n,k} \in \frac{1}{n}B_{k+1}$ with $b_{n,k} \notin B_k$. We consider the double sequence $\{b_{k,0} - b_{n,k} : n,k \geq 1\}$. For fixed k the sequence $b_{n,k}$ converges by construction (in $E_{B_{k+1}}$) to 0 for $n \to \infty$. Thus, $b_{k,0} - 0$ is the limit of the sequence $b_{k,0} - b_{n,k}$ for $n \to \infty$, and $b_{k,0}$ converges to 0 for $k \to \infty$. Suppose $b_{k(i),0} - b_{n(i),k(i)}$ converges to 0. So it has to be bounded, thus there must be an $N \in \mathbb{N}$ with $B_1 - \{b_{k(i),0} - b_{n(i),k(i)} : i \in \mathbb{N}\} \subseteq B_N$. Hence, $b_{n(i),k(i)} = b_{k(i),0} - (b_{k(i),0} - b_{n(i),k(i)}) \in B_N$, i.e. $k(i) < N$. This contradicts 4.10.2. □

4.14. Lemma. *Let U be a c^∞-open subset of a locally convex space, let $\mu_n \to \infty$ be a sequence of reals, and let $f : U \to F$ be a mapping which is bounded on each μ-converging sequence in U. Then f is bounded on every* BORNOLOGICALLY COMPACT SUBSET *(i.e. compact in some E_B) of U.*

Proof. Let $K \subseteq E_B \cap U$ be compact in E_B for some bounded absolutely convex set B. Assume that $f(K)$ is not bounded. By composing with linear functionals we may assume that $F = \mathbb{R}$. So there is a sequence (x_n) in K with $|f(x_n)| \to \infty$. Since K is compact in the normed space E_B we may assume that (x_n) converges to $x \in K$. By passing to a subsequence we may even assume that (x_n) is μ-converging. Contradiction. □

4.15. Lemma. *Let U be c^∞-open in $E \times \mathbb{R}$ and $K \subseteq \mathbb{R}$ be compact. Then $U_0 := \{x \in E : \{x\} \times K \subseteq U\}$ is c^∞-open in E.*

Proof. Let $x : \mathbb{R} \to E$ be a smooth curve in E with $x(0) \in U_0$, i.e. $(x(0), t) \in U$ for all $t \in K$. We have to show that $x(s) \in U_0$ for all s near 0. So consider the smooth map $x \times \mathbb{R} : \mathbb{R} \times \mathbb{R} \to E \times \mathbb{R}$. By assumption $(x \times \mathbb{R})^{-1}(U)$ is open in $c^\infty(\mathbb{R}^2) = \mathbb{R}^2$ (by 4.11.1). It contains the compact set $\{0\} \times K$ and hence also a $W \times K$ for some neighborhood W of 0 in $\mathbb{R}$. But this amounts in saying that $x(W) \subseteq U_0$. □

4.16. The $\mathbf{c^\infty}$-topology of a product. Consider the product $E \times F$ of two locally convex vector spaces. Since the projections onto the factors are linear and continuous, hence smooth, we always have that $\mathrm{id} : c^\infty(E \times F) \to c^\infty(E) \times c^\infty(F)$ is continuous. We will show in 4.20 that it is not always a homeomorphism. However, if one of the factors is finite dimensional the product is well behaved:

Corollary. *For any locally convex space E the c^∞-topology of $E \times \mathbb{R}^n$ is the product topology of the c^∞-topologies of the two factors, so that we have $c^\infty(E \times \mathbb{R}^n) = c^\infty(E) \times \mathbb{R}^n$.*

Proof. This follows recursively from the special case $E \times \mathbb{R}$, for which we can proceed as follows. Take a c^∞-open neighborhood U of some point $(x,t) \in E \times \mathbb{R}$. Since the inclusion map $s \mapsto (x,s)$ from $\mathbb{R}$ into $E \times \mathbb{R}$ is continuous and affine, the inverse image of U in $\mathbb{R}$ is an open neighborhood of t. Let's take a smaller compact neighborhood K of t. Then by the previous lemma 4.15 $U_0 := \{y \in E : \{y\} \times K \subseteq U\}$ is a c^∞-open neighborhood of x, and hence $U_0 \times K \subseteq U$ is a neighborhood of (x,t) in $c^\infty(E) \times \mathbb{R}$, what was to be shown. □

4.17. Lemma. *Let U be c^∞-open in a locally convex space and $x \in U$. Then the star $\mathrm{st}_x(U) := \{x + v : x + \lambda v \in U$ for all $|\lambda| \leq 1\}$ with center x in U is again c^∞-open.*

Proof. Let $c : \mathbb{R} \to E$ be a smooth curve with $c(0) \in \mathrm{st}_x(U)$. The smooth mapping $f : (t, s) \mapsto (1 - s)x + sc(t)$ maps $\{0\} \times \{s : |s| \le 1\}$ into U. So there exists $\delta > 0$ with $f\big(\{(t, s) : |t| < \delta, |s| \le 1\}\big) \subseteq U$. Thus, $c(t) \in \mathrm{st}_x(U)$ for $|t| < \delta$. □

4.18. Lemma. *The (absolutely) convex hull of a c^∞-open set is again c^∞-open.*

Proof. Let U be c^∞-open in a locally convex vector space E.
For each $x \in U$ the set

$$U_x := \{x + t(y - x) : t \in [0, 1], y \in U\} = U \cup \bigcup_{0<t\le 1} (x + t(U - x))$$

is c^∞-open. The convex hull can be constructed by applying n times the operation $U \mapsto \bigcup_{x\in U} U_x$ and taking the union over all $n \in \mathbb{N}$, which respects c^∞-openness.

The absolutely convex hull can be obtained by forming first $\{\lambda : |\lambda| = 1\} \cdot U = \bigcup_{|\lambda|=1} \lambda U$ which is c^∞-open, and then forming the convex hull. □

4.19. Corollary. *Let E be a bornological convenient vector space containing a nonempty c^∞-open subset which is either locally compact or metrizable in the c^∞-topology. Then the c^∞-topology on E is locally convex. In the first case E is finite dimensional, in the second case E is a Fréchet space.*

Proof. Let $U \subseteq E$ be a c^∞-open metrizable subset. We may assume that $0 \in U$. Then there exists a countable neighborhood basis of 0 in U consisting of c^∞-open sets. This is also a neighborhood basis of 0 for the c^∞-topology of E. We take the absolutely convex hulls of these open sets, which are again c^∞-open by 4.18, and obtain by 4.5 a countable neighborhood basis for the bornologification of the locally convex topology, so the latter is metrizable and Fréchet, and by 4.11.1 it equals the c^∞-topology.

If U is locally compact in the c^∞-topology we may find a c^∞-open neighborhood V of 0 with compact closure $\overline{V}$ in the c^∞-topology. By lemma 4.18 the absolutely convex hull of V is also c^∞-open, and by 4.5 it is also open in the bornologification E_{born} of E. The set $\overline{V}$ is also compact in E_{born}, hence precompact there. So the absolutely convex hull of $\overline{V}$ is also precompact by [**68**, 6.4.3]. Therefore, the absolutely convex hull of V is a precompact neighborhood of 0 in E_{born}, thus E is finite dimensional by [**68**, 4.4.5]. So $E_{\mathrm{born}} = c^\infty(E)$. □

Now we describe classes of spaces where $c^\infty E \ne E$ or where $c^\infty E$ is not even a topological vector space. Finally, we give an example where the c^∞-topology is not completely regular.

4.20. Proposition. *Let E and F be bornological locally convex vector spaces. If there exists a bilinear smooth mapping $m : E \times F \to \mathbb{R}$ that is not continuous with respect to the locally convex topologies, then $c^\infty(E \times F)$ is not a topological vector space and $c^\infty(E \times F) \ne c^\infty E \times c^\infty F$.*

We shall show in lemma 5.5 below that multilinear mappings are smooth if and only if they are bounded.

Proof. Suppose that the addition $c^\infty(E \times F) \times c^\infty(E \times F) \to c^\infty(E \times F)$ is continuous with respect to the product topology. Using the continuous inclusions

$c^\infty E \to c^\infty(E\times F)$ and $c^\infty F \to c^\infty(E\times F)$ we can factor the identity as $c^\infty E \times c^\infty F \to c^\infty(E\times F)\times c^\infty(E\times F) \xrightarrow{+} c^\infty(E\times F)$ and hence $c^\infty E\times c^\infty F = c^\infty(E\times F)$.

In particular, $m : c^\infty E \times c^\infty F = c^\infty(E\times F) \to \mathbb{R}$ is continuous. Thus, for every $\varepsilon > 0$ there are 0-neighborhoods U and V with respect to the c^∞-topology such that $m(U\times V) \subseteq (-\varepsilon,\varepsilon)$. Then also $m(\langle U\rangle \times \langle V\rangle) \subseteq (-\varepsilon,\varepsilon)$ where $\langle \_ \rangle$ denotes the absolutely convex hull. By 4.5 one concludes that m is continuous with respect to the locally convex topology, a contradiction. □

4.21. Corollary. *Let E be a non-normable bornological locally convex space. Then $c^\infty(E\times E')$ is not a topological vector space.*

Proof. By 4.20 it is enough to show that $\mathrm{ev} : E\times E' \to \mathbb{R}$ is not continuous for the bornological topologies on E and E'; if it were so there was be a neighborhood U of 0 in E and a neighborhood U' of 0 in E' such that $\mathrm{ev}(U\times U') \subseteq [-1,1]$. Since U' is absorbing, U is scalarwise bounded, hence a bounded neighborhood. Thus, E is normable. □

4.22. Remark. In particular, for a Fréchet Schwartz space E (e.g. $\mathbb{R}^{\mathbb{N}}$) and its dual E' we have $c^\infty(E\times E') \neq c^\infty E\times c^\infty E'$, since by 4.11 we have $c^\infty E = E$ and $c^\infty E' = E'$, so equality would contradict corollary 4.21.

In order to get a large variety of spaces where the c^∞-topology is not a topological vector space topology the next three technical lemmas will be useful.

4.23. Lemma. *Let E be a locally convex vector space. Suppose a double sequence $b_{n,k}$ in E exists which satisfies the following two conditions:*

(b') *For every sequence $k\mapsto n_k$ the sequence $k\mapsto b_{n_k,k}$ has no accumulation point in $c^\infty E$.*

(b") *For all k the sequence $n\mapsto b_{n,k}$ converges to 0 in $c^\infty E$.*

Suppose furthermore that a double sequence $c_{n,k}$ in E exists that satisfies the following two conditions:

(c') *For every 0-neighborhood U in $c^\infty E$ there exists some k_0 such that $c_{n,k}\in U$ for all $k\geq k_0$ and all n.*

(c") *For all k the sequence $n\mapsto c_{n,k}$ has no accumulation point in $c^\infty E$.*

Then $c^\infty E$ is not a topological vector space.

Proof. Assume that the addition $c^\infty E\times c^\infty E \to c^\infty E$ is continuous. In this proof convergence is meant always with respect to $c^\infty E$. We may without loss of generality assume that $c_{n,k}\neq 0$ for all n, k, since by (c") we may delete for each n all those (finitely many) $c_{n,k}$ which are equal to 0. Then we consider $A := \{b_{n,k}+\varepsilon_{n,k}c_{n,k} : n,k\in\mathbb{N}\}$ where the $\varepsilon_{n,k}\in\{-1,1\}$ are chosen in such a way that $0\notin A$.

We first show that A is closed in the sequentially generated topology $c^\infty E$: Let $b_{n_i,k_i}+\varepsilon_{n_i,k_i}c_{n_i,k_i}\to x$, and assume first that (k_i) is unbounded. By passing if necessary to a subsequence we may even assume that $i\mapsto k_i$ is strictly increasing. Then $c_{n_i,k_i}\to 0$ by (c'), hence $b_{n_i,k_i}\to x$ by the assumption that addition is continuous, which is a contradiction to (b'). Thus, (k_i) is bounded, and we may assume it to be constant. Now suppose that (n_i) is unbounded. Then $b_{n_i,k}\to 0$ by (b”), and hence $\varepsilon_{n_i,k}c_{n_i,k} \to x$, and for a subsequence where ε is constant one has $c_{n_i,k} \to \pm x$, which is a contradiction to (c”). Thus, n_i is bounded as well, and we may assume it to be constant. Hence, $x = b_{n,k} + \varepsilon_{n,k}c_{n,k} \in A$.

By the assumed continuity of the addition there exists an open and symmetric 0-neighborhood U in $c^\infty E$ with $U + U \subseteq E \setminus A$. For K sufficiently large and n arbitrary one has $c_{n,K} \in U$ by (c’). For such a fixed K and N sufficiently large $b_{N,K} \in U$ by (b”). Thus, $b_{N,K} + \varepsilon_{N,K}c_{N,K} \notin A$, which is a contradiction. □

Let us now show that many spaces have a double sequence $c_{n,k}$ as in the above lemma.

4.24. Lemma. *Let E be an infinite dimensional metrizable locally convex space. Then a double sequence $c_{n,k}$ subject to the conditions (c’) and (c”) of 4.23 exists.*

Proof. If E is normable we choose a sequence (c_n) in the unit ball without accumulation point and define $c_{n,k} := \frac{1}{k}c_n$. If E is not normable we take a countable increasing family of non-equivalent seminorms p_k generating the locally convex topology, and we choose $c_{n,k}$ with $p_k(c_{n,k}) = \frac{1}{k}$ and $p_{k+1}(c_{n,k}) > n$. □

Next we show that many spaces have a double sequence $b_{n,k}$ as in lemma 4.23.

4.25. Lemma. *Let E be a non-normable bornological locally convex space having a countable basis of its bornology. Then a double sequence $b_{n,k}$ subject to the conditions (b’) and (b”) of 2.11 exists.*

Proof. Let B_n ($n \in \mathbb{N}$) be absolutely convex sets forming an increasing basis of the bornology. Since E is not normable the sets B_n can be chosen such that B_n does not absorb B_{n+1}. Now choose $b_{n,k} \in \frac{1}{n}B_{k+1}$ with $b_{n,k} \notin B_k$. □

Using these there lemmas one obtains the

4.26. Proposition. *For the following bornological locally convex spaces the c^∞-topology is not a vector space topology:*

(i) *Every bornological locally convex space that contains as c^∞-closed subspaces an infinite dimensional Fréchet space and a space which is non-normable in the bornological topology and having a countable basis of its bornology.*
(ii) *Every strict inductive limit of a strictly increasing sequence of infinite dimensional Fréchet spaces.*
(iii) *Every product for which at least $2^{\aleph_0}$ many factors are non-zero.*
(iv) *Every coproduct for which at least $2^{\aleph_0}$ many summands are non-zero.*

Proof. (i) follows directly from the last 3 lemmas.

(ii) Let E be the strict inductive limit of the spaces E_n ($n \in \mathbb{N}$). Then E contains the infinite dimensional Fréchet space E_1 as subspace. The subspace generated by points $x_n \in E_{n+1} \setminus E_n$ ($n \in \mathbb{N}$) is bornologically isomorphic to $\mathbb{R}^{(\mathbb{N})}$, hence its bornology has a countable basis. Thus, by (i) we are done.

(iii) Such a product E contains the Fréchet space $\mathbb{R}^\mathbb{N}$ as complemented subspace. We want to show that $\mathbb{R}^{(\mathbb{N})}$ is also a subspace of E. For this we may assume that the index set J is $\mathbb{R}^\mathbb{N}$ and all factors are equal to $\mathbb{R}$. Now consider the linear subspace E_1 of the product generated by the elements $x^n \in E = \mathbb{R}^J$, where $(x^n)_j := j(n)$

for every $j \in J = \mathbb{R}^{\mathbb{N}}$. The linear map $\mathbb{R}^{(\mathbb{N})} \to E_1 \subseteq E$ that maps the n-th unit vector to x^n is injective, since for a given finite linear combination $\sum t_n x^n = 0$ the j-th coordinate for $j(n) := \operatorname{sign}(t_n)$ equals $\sum |t_n|$. It is continuous since $\mathbb{R}^{(\mathbb{N})}$ carries the finest locally convex structure. So it remains to show that it is a bornological embedding. We have to show that any bounded $B \subseteq E_1$ is contained in a subspace generated by finitely many x^n. Otherwise, there would exist a strictly increasing sequence (n_k) and $b^k = \sum_{n \le n_k} t_n^k x^n \in B$ with $t_{n_k}^k \neq 0$. Define an index j recursively by $j(n) := n|t_n^k|^{-1} \cdot \operatorname{sign}\big(\sum_{m<n} t_m^k j(m)\big)$ if $n = n_k$ and $j(n) := 0$ if $n \neq n_k$ for all k. Then the absolute value of the j-th coordinate of b^k evaluates as follows:

$$\begin{aligned}|(b^k)_j| &= \Big|\sum_{n \le n_k} t_n^k j(n)\Big| = \Big|\sum_{n<n_k} t_n^k j(n) + t_{n_k}^k j(n_k)\Big| \\ &= \Big|\sum_{n<n_k} t_n^k j(n)\Big| + |t_{n_k}^k j(n_k)| \ge |t_{n_k}^k j(n_k)| = n_k.\end{aligned}$$

Hence, the j-th coordinates of $\{b^k : k \in \mathbb{N}\}$ are unbounded with respect to $k \in \mathbb{N}$, thus B is unbounded.

(iv) We can not apply lemma 4.23 since every double sequence has countable support and hence is contained in the dual $\mathbb{R}^{(A)}$ of a Fréchet Schwartz space $\mathbb{R}^A$ for some countable subset $A \subset J$. It is enough to show (iv) for $\mathbb{R}^{(J)}$ where $J = \mathbb{N} \sqcup c_0$. Let $A := \{j_n(e_n + e_j) : n \in \mathbb{N}, j \in c_0, j_n \neq 0 \text{ for all } n\}$, where e_n and e_j denote the unit vectors in the corresponding summand. The set A is c^∞-closed, since its intersection with finite subsums is finite. Suppose there exists a symmetric c^∞-open 0-neighborhood U with $U + U \subseteq E \setminus A$. Then for each n there exists a $j_n \neq 0$ with $j_n e_n \in U$. We may assume that $n \mapsto j_n$ converges to 0 and hence defines an element $j \in c_0$. Furthermore, there has to be an $N \in \mathbb{N}$ with $j_N e_j \in U$, thus $j_N(e_N + e_j) \in (U + U) \cap A$, in contradiction to $U + U \subseteq E \setminus A$. □

Remark. A nice and simple example where one either uses (i) or (ii) is $\mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{(\mathbb{N})}$. The locally convex topology on both factors coincides with their c^∞-topology (the first being a Fréchet (Schwartz) space, cf. (i) of 4.11, the second as dual of the first, cf. (ii) of 4.11); but the c^∞-topology on their product is not even a vector space topology.

From (ii) it follows also that each space $C_c^\infty(M, \mathbb{R})$ of smooth functions with compact support on a non-compact separable finite dimensional manifold M has the property, that the c^∞-topology is not a vector space topology.

4.27

Although the c^∞-topology on a convenient vector space is always functionally separated, hence Hausdorff, it is not always completely regular as the following example shows.

Example. The c^∞-topology is not completely regular. *The c^∞-topology of $\mathbb{R}^J$ is not completely regular if the cardinality of J is at least $2^{\aleph_0}$.*

Proof. It is enough to show this for an index set J of cardinality $2^{\aleph_0}$, since the corresponding product is a complemented subspace in every product with larger index set. We prove the theorem by showing that every function $f : \mathbb{R}^J \to \mathbb{R}$

which is continuous for the c^∞-topology is also continuous with respect to the locally convex topology. Hence, the completely regular topology associated to the c^∞-topology is the locally convex topology of E. That these two topologies are different was shown in 4.8. We use the following theorem of [92]: Let $E_0 := \{x \in \mathbb{R}^J : \operatorname{supp}(x) \text{ is countable}\}$, and let $f : E_0 \to \mathbb{R}$ be sequentially continuous. Then there is some countable subset $A \subset J$ such that $f(x) = f(x_A)$, where in this proof x_A is defined as $x_A(j) := x(j)$ for $j \in A$ and $x_A(j) = 0$ for $j \notin A$. Every sequence which is converging in the locally convex topology of E_0 is contained in a metrizable complemented subspace $\mathbb{R}^A$ for some countable A and therefore is even M-convergent. Thus, this theorem of Mazur remains true if f is assumed to be continuous for the M-closure topology. This generalization follows also from the fact that $c^\infty E_0 = E_0$, cf. 4.12. Now let $f : \mathbb{R}^J \to \mathbb{R}$ be continuous for the c^∞-topology. Then $f|E_0 : E_0 \to \mathbb{R}$ is continuous for the c^∞-topology, and hence there exists a countable set $A_0 \subset J$ such that $f(x) = f(x_{A_0})$ for any $x \in E_0$. We want to show that the same is true for arbitrary $x \in \mathbb{R}^J$. In order to show this we consider for $x \in \mathbb{R}^J$ the map $\varphi_x : 2^J \to \mathbb{R}$ defined by $\varphi_x(A) := f(x_A) - f(x_{A\cap A_0})$ for any $A \subseteq J$, i.e. $A \in 2^J$. For countable A one has $x_A \in E_0$, hence $\varphi_x(A) = 0$. Furthermore, φ_x is sequentially continuous, where one considers on 2^J the product topology of the discrete factors $2 = \{0, 1\}$: In order to see this, consider a converging sequence of subsets $A_n \to A$, i.e. for every $j \in J$ one has for the characteristic functions $\chi_{A_n}(j) = \chi_A(j)$ for n sufficiently large. Then $\{n(x_{A_n} - x_A) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^J$ since for fixed $j \in J$ the j-th coordinate equals 0 for n sufficiently large. Thus, x_{A_n} converges Mackey to x_A, and since f is continuous for the c^∞-topology $\varphi_x(A_n) \to \varphi_x(A)$. Now we can apply another theorem of [92]: Any function $f : 2^J \to \mathbb{R}$ that is sequentially continuous and is zero on all countable subsets of J is identically 0, provided the cardinality of J is smaller than the first inaccessible cardinal. Thus, we conclude that $0 = \varphi_x(J) = f(x) - f(x_{A_0})$ for all $x \in \mathbb{R}^J$. Hence, f factors over the metrizable space $\mathbb{R}^{A_0}$ and is therefore continuous for the locally convex topology. □

In general, the trace of the c^∞-topology on a linear subspace is not its c^∞-topology. However, for c^∞-closed subspaces this is true:

4.28. Lemma. Closed embedding lemma. *Let E be a linear c^∞-closed subspace of F. Then the trace of the c^∞-topology of F on E is the c^∞-topology on E*

Proof. Since the inclusion is continuous and hence bounded it is c^∞-continuous. Therefore, it is enough to show that it is closed for the c^∞-topologies. So let $A \subseteq E$ be $c^\infty E$-closed. And let $x_n \in A$ converge Mackey towards x in F. Then $x \in E$, since E is assumed to be c^∞-closed, and hence x_n converges Mackey to x in E. Since A is c^∞-closed in E, we have that $x \in A$. □

We will give an example in 4.33 below which shows that c^∞-closedness of the subspace is essential for this result. Another example will be given in 4.36.

4.29. Theorem. The $\mathbf{c^\infty}$-completion. *For any locally convex space E there exists a unique (up to a bornological isomorphism) convenient vector space $\tilde{E}$ and a bounded linear injection $i : E \to \tilde{E}$ with the following universal property:*

> *Each bounded linear mapping $\ell : E \to F$ into a convenient vector space F has a unique bounded extension $\tilde{\ell} : \tilde{E} \to F$ such that $\tilde{\ell} \circ i = \ell$.*

Furthermore, $i(E)$ is dense for the c^∞-topology in $\tilde{E}$.

Proof. Let $\tilde{E}$ be the c^∞-closure of E in the locally convex completion $\widehat{E_{\text{born}}}$ of the bornologification E_{born} of E. This is a linear subspace, since the affine translations $x \mapsto x+y$ are bounded. The inclusion $i : E \to \tilde{E}$ is bounded (but not continuous in general). By 4.28 the c^∞-topology on $\tilde{E}$ is the trace of the c^∞-topology on $\widehat{E_{\text{born}}}$. Hence, $i(E)$ is dense also for the c^∞-topology in $\tilde{E}$.

Using the universal property of the locally convex completion the mapping ℓ has a unique continuous extension $\hat{\ell} : \widehat{E_{\text{born}}} \to \widehat{F}$ into the locally convex completion of F, whose restriction to $\tilde{E}$ has values in F, since F is c^∞-closed in $\widehat{F}$, so it is the desired $\tilde{\ell}$. Uniqueness follows, since $i(E)$ is dense for the c^∞-topology in $\tilde{E}$. □

4.30. Proposition. The c^∞-completion via c^∞-dense embeddings. *Let E be c^∞-dense and bornologically embedded into a c^∞-complete locally convex space F. If $E \to F$ has the extension property for bounded linear functionals, then F is bornologically isomorphic to the c^∞-completion of E.*

Example 4.36.6 shows, that the extension property cannot be dropped.

Proof. We have to show that $E \to F$ has the universal property for extending bounded linear maps T into c^∞-complete locally convex spaces G. Since we are only interested in bounded mappings, we may take the bornologification of G and hence may assume that G is bornological. Consider the following diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[dd, "T"'] & F \arrow[d, dotted] \arrow[ddl, dotted] \arrow[ddr, dotted, "\widetilde{\lambda\circ T}"] & \\
& \prod_{G'}\mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"'] & \\
G \arrow[ur, "\delta"] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

The arrow δ, given by $\delta(x)_\lambda := \lambda(x)$, is a *bornological embedding*, i.e. the image of a set is bounded if and only if the set is bounded, since $B \subseteq G$ is bounded if and only if $\lambda(B) \subseteq \mathbb{R}$ is bounded for all $\lambda \in G'$, i.e. $\delta(B) \subseteq \prod_{G'} \mathbb{R}$ is bounded.

By assumption, the dashed arrow on the right hand side exists, hence by the universal property of the product the dashed vertical arrow (denoted $\tilde{T}$) exists. It remains to show that it has values in the image of δ. Since $\tilde{T}$ is bounded we have

$$\tilde{T}(F) = \tilde{T}(\overline{E}^{c^\infty}) \subseteq \overline{\tilde{T}(E)}^{c^\infty} \subseteq \overline{\delta(G)}^{c^\infty} = \delta(G),$$

since G is c^∞-complete and hence also $\delta(G)$, which is thus c^∞-closed.

The uniqueness follows, since as a bounded linear map $\tilde{T}$ has to be continuous for the c^∞-topology (since it preserves the smooth curves by 2.11 which in turn generate the c^∞-topology), and E lies dense in F with respect to this topology. □

4.31. Proposition. Inductive representation of bornological locally convex spaces. *For a locally convex space E the bornologification E_{born} is by 4.2 the colimit of all the normed spaces E_B for the absolutely convex bounded sets B. The colimit of the respective completions $\tilde{E}_B$ is the linear subspace of the c^∞-completion $\tilde{E}$ consisting of all limits in $\tilde{E}$ of Mackey Cauchy sequences in E.*

Proof. Let $E^{(1)}$ be the *Mackey adherence* of E in the c^∞-completion $\tilde{E}$, by which we mean the limits in $\tilde{E}$ of all sequences in E which converge Mackey in $\tilde{E}$. Then $E^{(1)}$ is a subspace of the locally convex completion $\widehat{E_{\text{born}}}$. For every absolutely convex bounded set $B \subseteq E$ we have the continuous inclusion $E_B \to E_{\text{born}}$, and by passing to the c^∞-completion we get mappings $\iota_B : \widetilde{E_B} = \widehat{E_B} \to \tilde{E}$. These mappings commute with the connecting morphisms $\widehat{E_B} \to \widehat{E_{B'}}$ for $B \subseteq B'$ and have values in the Mackey adherence of E, since every point in $\widehat{E_B}$ is the limit of a sequence in E_B, and hence its image is the limit of this Mackey Cauchy sequence in E. Moreover, $E^{(1)} = \bigcup_B \iota_B(\widehat{E_B})$, since any $x \in E^{(1)}$ is the Mackey limit of a sequence (x_n) in E. This sequence is a Cauchy-sequence in some E_B and hence converge to some y in $\widehat{E_B}$. Then $\iota_B(y) = x$, since the mapping $\iota_B : \widehat{E_B} \to E^{(1)}$ is continuous.

We claim that the Mackey adherence $E^{(1)}$ together with the mappings $\iota_B : \widehat{E_B} \to E^{(1)}$ has the universal property of the colimit $\varinjlim_B \widehat{E_B}$. In fact, let $T_B : \widehat{E_B} \to F$ be linear mappings, which commute with the connecting morphisms $\widehat{E_B} \to \widehat{E_{B'}}$ for $B \subseteq B'$. In particular, the $T_B|_{E_B} : E_B \to F$ are continuous, hence define a unique continuous linear mapping $T : E_{\text{born}} = \varinjlim_B E_B \to F$, which in turn extends to a continuous linear mapping $\widehat{T} : \widehat{E_{\text{born}}} \supseteq E^{(1)} \to \widehat{F}$. Since $E^{(1)} = \bigcup_B \iota_B(\widehat{E_B})$ and $\widehat{T}|_{E_B} = T_B|_{E_B}$ we get $\widehat{T}|_{\widehat{E_B}} = T_B$ for all B. □

In spite of 4.36.1 we can use the Mackey adherence M-Adh : $A \mapsto A^{(1)}$ to describe the c^∞-closure in the following inductive way:

4.32. Proposition. Mackey adherences. *For ordinal numbers α the Mackey adherence $A^{(\alpha)}$ of order α is defined recursively by:*

$$A^{(\alpha)} := \begin{cases} \text{M-Adh}(A^{(\beta)}) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} A^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal number.} \end{cases}$$

Then the closure $\overline{A}$ of A in the c^∞-topology coincides with $A^{(\omega_1)}$, where ω_1 denotes the first uncountable ordinal number, i.e. the set of all countable ordinal numbers.

Proof. Let us first show that $A^{(\omega_1)}$ is c^∞-closed. So take a sequence $x_n \in A^{(\omega_1)} = \bigcup_{\alpha<\omega_1} A^{(\alpha)}$, which converges Mackey to some x. Then there are $\alpha_n < \omega_1$ with $x_n \in A^{(\alpha_n)}$. Let $\alpha := \sup_n \alpha_n$. Then α is a again countable and hence less than ω_1. Thus, $x_n \in A^{(\alpha_n)} \subseteq A^{(\alpha)}$, and therefore $x \in \text{M-Adh}(A^{(\alpha)}) = A^{(\alpha+1)} \subseteq A^{(\omega_1)}$ since $\alpha + 1 \leq \omega_1$.

It remains to show that $A^{(\alpha)}$ is contained in $\overline{A}$ for all α. We prove this by transfinite induction. So assume that for all $\beta < \alpha$ we have $A^{(\beta)} \subseteq \overline{A}$. If α is a limit ordinal number then $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)} \subseteq \overline{A}$. If $\alpha = \beta + 1$ then every point in $A^{(\alpha)} = \text{M-Adh}(A^{(\beta)})$ is the Mackey-limit of some sequence in $A^{(\beta)} \subseteq \overline{A}$, and since $\overline{A}$ is c^∞-closed, this limit has to belong to it. So $A^{(\alpha)} \subseteq \overline{A}$ in all cases. □

4.33. Example. *The trace of the c^∞-topology is not the c^∞-topology and the Mackey-adherence is not the c^∞-closure, in general.*

Proof. Consider

$$A := \Big\{ a_{n,k} := \big(\tfrac{1}{n}\chi_{\{1,\dots,k\}}, \tfrac{1}{k}\chi_{\{n\}}\big) : 1 \leq n, k \in \mathbb{N} \Big\} \subseteq E := \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{(\mathbb{N})}.$$

Let F be the linear subspace of E generated by A. We show that the closure of A with respect to the c^∞-topology of F is strictly smaller than that with respect to the trace topology of the c^∞-topology of E.

The set A is closed in the c^∞-topology of F: Assume that a sequence (a_{n_j,k_j}) is M-converging to (x,y). Then the second component of a_{n_j,k_j} has to be bounded. Thus, $j\mapsto n_j$ has to be bounded and may be assumed to have constant value n_∞. If $j\mapsto k_j$ were unbounded, then $(x,y)=(\frac{1}{n_\infty}\chi_{\mathbb{N}},0)$, which is not an element of F. Thus, $j\mapsto k_j$ has to be bounded too and may be assumed to have constant value k_∞. Thus, $(x,y)=a_{n_\infty,k_\infty}\in A$.

The set A is not closed in the trace topology since (0,0) is contained in the closure of A with respect to the c^∞-topology of E: For $k\to\infty$ and fixed n the sequence $a_{n,k}$ is M-converging to $(\frac{1}{n}\chi_{\mathbb{N}},0)$, and $\frac{1}{n}\chi_{\mathbb{N}}$ is M-converging to 0 for $n\to\infty$. □

4.34. Example. We consider the space $\ell^\infty_c(X):=\{f\in\ell^\infty(X):\operatorname{supp} f \text{ is finite}\}$ as subspace of $\ell^\infty(X):=\ell^\infty(X,\mathbb{R})$ as defined in 2.15 for a set X together with a family $\mathcal{B}$ of subsets called bounded.
Claim: The c^∞-closure of $\ell^\infty_c(X)$ in $\ell^\infty(X)$ equals

$$c_0(X):=\{f\in\ell^\infty(X):f|_B\in c_0(B)\text{ for all }B\in B\},$$

provided that X is countable.

Proof. The right hand side is just the intersection $c_0(X):=\bigcap_{B\in\mathcal{B}}\iota_B^{-1}(c_0(B))$, where $\iota_B:\ell^\infty(X)\to\ell^\infty(B)$ denotes the restriction map. We use the notation $c_0(X)$, since in the case where X is bounded this is exactly the space $\{f\in\ell^\infty(X):\{x:|f(x)|\ge\varepsilon\}$ is finite for all $\varepsilon>0\}$. In particular, this applies to the bounded space $\mathbb{N}$, where $c_0(\mathbb{N})=c_0$. Since $\ell^\infty(X)$ carries the initial structure with respect to these maps $c_0(X)$ is closed. It remains to show that $\ell^\infty_c(X)$ is c^∞-dense in $c_0(X)$. So take $f\in c_0(X)$. Let X be countable and $\{x_1,x_2,\dots\}:=\{x\in X:f(x)\ne0\}$.

We consider first the case, where there exists some $\delta>0$ such that $|f(x_n)|\ge\delta$ for all n. Then we consider the functions $f_n:=f\cdot\chi_{x_1,\dots,x_n}\in\ell^\infty_c(X)$. We claim that $n(f-f_n)$ is bounded in $\ell^\infty(X,\mathbb{R})$. In fact, let $B\in\mathcal{B}$. Then $\{n:x_n\in B\}=\{n: x_n\in B$ and $|f(x_n)|\ge\delta\}$ is finite. Hence, $\{n(f-f_n)(x):x\in B\}$ is finite and thus bounded, i.e. f_n converges Mackey to f.

Now the general case. We set $X_n:=\{x\in X:|f(x)|\ge\frac1n\}$ and define $f_n:=f\cdot\chi_{X_n}$. Then each f_n satisfies the assumption of the particular case with $\delta=\frac1n$ and hence is a Mackey limit of a sequence in $\ell^\infty_c(X)$. Furthermore, $n(f-f_n)$ is uniformly bounded by 1, since for $x\in X_n$ it is 0 and otherwise $|n(f-f_n)(x)|=n|f(x)|<1$. So after forming the Mackey adherence (i.e. adding the limits of all Mackey convergent sequences contained in the set, see 4.32 for a formal definition) twice, we obtain $c_0(X)$. □

Now we want to show that $c_0(X)$ is in fact the c^∞-completion of $\ell^\infty_c(X)$.

4.35. Example. $\mathbf{c_0(X)}$. We claim that $c_0(X)$ is the c^∞-completion of the subspace $\ell^\infty_c(X)$ in $\ell^\infty(X)$ formed by the finite sequences.
We may assume that the bounded sets of X are formed by those subsets B, for which $f(B)$ is bounded for all $f\in\ell^\infty(X)$: Obviously, any bounded set has this property, and the space $\ell^\infty(X)$ is not changed by adding these sets. Furthermore, the restriction map $\iota_B:\ell^\infty(X)\to\ell^\infty(B)$ is also bounded for such a B, since using the closed graph theorem [**68**, 5.3.3] for the c^∞-complete space $\ell^\infty(X)$ and the

Banach space $\ell^\infty(B)$ we only have to show that $\mathrm{ev}_b \circ \iota_B = \iota_{\{b\}}$ is bounded for every $b \in B$, which is obviously the case.

By proposition 4.30 it is enough to show the universal property for bounded linear functionals. We only have to show that in analogy to Banach-theory the dual $\ell^\infty_c(X)'$ is just

$$\ell^1(X) := \{g : X \to \mathbb{R} : \operatorname{supp} g \text{ is bounded and } g \text{ is absolutely summable}\}.$$

In fact, any such g acts even as bounded linear functional on $\ell^\infty(X, \mathbb{R})$ by $\langle g, f\rangle := \sum_x g(x)\, f(x)$, since a subset is bounded in $\ell^\infty(X)$ if and only if it is uniformly bounded on all bounded sets $B \subseteq X$. Conversely, let $\ell : \ell^\infty_c(X) \to \mathbb{R}$ be bounded and linear and define $g : X \to \mathbb{R}$, by $g(x) := \ell(e_x)$, where e_x denotes the function given by $e_x(y) := 1$ for $x = y$ and 0 otherwise. Obviously $\ell(f) = \langle g, f\rangle$ for all $f \in \ell^\infty_c(X)$. Suppose indirectly that $\operatorname{supp} g = \{x : \ell(e_x) \neq 0\}$ is not bounded. Then there exists a sequence $x_n \in \operatorname{supp} g$ and a function $f \in \ell^\infty(X)$ such that $|f(x_n)| \geq n$. In particular, the only bounded subsets of $\{x_n : n \in \mathbb{N}\}$ are the finite ones. Hence $\{\frac{n}{|g(x_n)|} e_{x_n} : n \in \mathbb{N}\}$ is bounded in $\ell^\infty_c(X)$, but the image under ℓ is not. Furthermore, g has to be absolutely summable since the set of finite subsums of $\sum_x \operatorname{sign}(g(x))\, e_x$ is uniformly bounded and hence bounded in $\ell^\infty_c(X)$ and its image under ℓ are the subsums of $\sum_x |g(x)|$.

4.36. Corollary. Counter-examples on $\mathbf{c^\infty}$-topology. *The following statements are* **false***:*

1. *The c^∞-closure of a subset (or even a linear subspace) is given by the Mackey adherence, i.e. the set formed by all limits of sequences in this subset which are Mackey convergent in the total space.*
2. *A subset U of E that contains a point x and has the property, that every sequence which M-converges to x belongs to it finally, is a c^∞-neighborhood of x.*
3. *A c^∞-dense subspace of a c^∞-complete space has this space as c^∞-completion.*
4. *If a subspace E is c^∞-dense in the total space, then it is also c^∞-dense in each linear subspace lying in between.*
5. *The c^∞-topology of a linear subspace is the trace of the c^∞-topology of the whole space.*
6. *Every bounded linear functional on a linear subspace can be extended to such a functional on the whole space.*
7. *A linear subspace of a bornological locally convex space is bornological.*
8. *The c^∞-completion preserves embeddings.*

Proof. (1) For this we give an example, where the Mackey adherence of $\ell^\infty_c(X)$ is not all of $c_0(X)$.
Let $X = \mathbb{N} \times \mathbb{N}$, and take as bounded sets all sets of the form $B_\mu := \{(n, k) : n \leq \mu(k)\}$, where μ runs through all functions $\mathbb{N} \to \mathbb{N}$. Let $f : X \to \mathbb{R}$ be defined by $f(n, k) := \frac{1}{k}$. Obviously, $f \in c_0(X)$, since for given $j \in \mathbb{N}$ and function μ the set of points $(n, k) \in B_\mu$ for which $f(n, k) = \frac{1}{k} \geq \frac{1}{j}$ is the finite set $\{(n, k) : k \leq j \text{ and } n \leq \mu(k)\}$.
Assume there is a sequence $f_n \in \ell^\infty_c(X)$ Mackey convergent to f. By passing to a subsequence we may assume that $n^2(f - f_n)$ is bounded. Now choose $\mu(k)$ to be larger than all of the finitely many n, with $f_k(n, k) \neq 0$. If $k^2(f - f_k)$ is bounded

on B_μ, then in particular $\{k^2(f - f_k)(\mu(k), k) : k \in \mathbb{N}\}$ has to be bounded, but $k^2(f - f_k)(\mu(k), k) = k^2\frac{1}{k} - 0 = k$.

($\boxed{2}$) Let A be a set for which ($\boxed{1}$) fails, and choose x in the c^∞-closure of A but not in the M-adherence of A. Then $U := E \setminus A$ satisfies the assumptions of ($\boxed{2}$). In fact, let x_n be a sequence which converges Mackey to x, and assume that it is not finally in U. So we may assume without loss of generality that $x_n \notin U$ for all n, but then $A \ni x_n \to x$ would imply that x is in the Mackey adherence of A. However, U cannot be a c^∞-neighborhood of x. In fact, such a neighborhood must meet A since x is assumed to be in the c^∞-closure of A.

($\boxed{3}$) Let F be a locally convex vector space whose Mackey adherence in its c^∞-completion E is not all of E, e.g. $\ell^\infty_c(X) \subseteq c_0(X)$ as in ($\boxed{1}$). Choose a $y \in E$ that is not contained in the Mackey adherence of F, and let F_1 be the subspace of E generated by $F \cup \{y\}$. We claim that $F_1 \subseteq E$ cannot be the c^∞-completion although F_1 is obviously c^∞-dense in the convenient vector space E. In order to see this we consider the linear map $\ell : F_1 \to \mathbb{R}$ characterized by $\ell(F) = 0$ and $\ell(y) = 1$. Clearly ℓ is well defined.

$\ell : F_1 \to \mathbb{R}$ is bornological: For any bounded $B \subseteq F_1$ there exists an N such that $B \subseteq F + [-N, N]y$. Otherwise, $b_n = x_n + t_n y \in B$ would exist with $t_n \to \infty$ and $x_n \in F$. This would imply that $b_n = t_n(\frac{x_n}{t_n} + y)$, and thus $-\frac{x_n}{t_n}$ would converge Mackey to y; a contradiction.

Now assume that a bornological extension $\bar{\ell}$ to E exists. Then $F \subseteq \ker(\bar{\ell})$ and $\ker(\bar{\ell})$ is c^∞-closed, which is a contradiction to the c^∞-denseness of F in E. So $F_1 \subseteq E$ does not have the universal property of a c^∞-completion.

This shows also that ($\boxed{6}$) fails.

($\boxed{4}$) Furthermore, it follows that F is $c^\infty F_1$-closed in F_1, although F (and hence F_1) is c^∞-dense in E.

($\boxed{5}$) The trace of the c^∞-topology of E to F_1 cannot be the c^∞-topology of F_1, since for the first one F is obviously dense.

($\boxed{7}$) Obviously, the trace topology of the bornological topology on E cannot be bornological on F_1, since otherwise the bounded linear functionals on F_1 would be continuous and hence extendable to E.

($\boxed{8}$) Furthermore, the extension of the inclusion $\iota : F \oplus \mathbb{R} \cong F_1 \to E$ to the completion is given by $(x, t) \in E \oplus \mathbb{R} \cong \tilde{F} \oplus \mathbb{R} = \tilde{F}_1 \mapsto x + ty \in E$ and has as kernel the linear subspace generated by $(y, -1)$. Hence, the extension of an embedding to the c^∞-completions need not be an embedding anymore, in particular the c^∞-completion functor does not preserve injectivity of morphisms. □

5. Uniform Boundedness Principles and Multilinearity

5.1. The category of locally convex spaces and smooth mappings

The category of all smooth mappings between bornological vector spaces is a subcategory of the category of all smooth mappings between locally convex spaces which is equivalent to it, since a locally convex space and its bornologification

4.4 have the same bounded sets and smoothness depends only on the bornology by 1.8. So it is also cartesian closed, but the topology on $C^\infty(E,F)$ from 3.11 has to be bornologized. For an example showing the necessity see [**74**, p. 297] or [**41**, 5.4.19]: The topology on $C^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})})$ is not bornological, in fact $\{c=(c_n)_n \in C^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})}) : |c_n^{(n)}(0)|<1\}$ is absolutely convex, bornivorous but not a 0-neighborhood.

We will in general, however, work in the category of locally convex spaces and smooth mappings, so function spaces carry the topology of 3.11.

The category of bounded (equivalently continuous) linear mappings between bornological vector spaces is in the same way equivalent to the category of all bounded linear mappings between all locally convex spaces, since a linear mapping is smooth if and only if it is bounded, by 2.11. It is closed under formation of colimits and under quotients (this is an easy consequence of 4.1.1). The Mackey-Ulam theorem [**53**, 13.5.4] tells us that a product of non trivial bornological vector spaces is bornological if and only if the index set does not admit a Ulam measure, i.e. a non trivial $\{0,1\}$-valued measure on the whole power set. A cardinal admitting a Ulam measure has to be strongly inaccessible, so we can restrict set theory to exclude measurable cardinals.

Let $L(E_1,\dots,E_n;F)$ denote the *space of all bounded n-linear mappings* from $E_1\times\dots\times E_n \to F$ with the topology of uniform convergence on bounded sets in $E_1\times\dots\times E_n$.

5.2. Proposition. Exponential law for L. *There are natural bornological isomorphisms*

$$L(E_1,\dots,E_{n+k};F)\cong L(E_1,\dots,E_n;L(E_{n+1},\dots,E_{n+k};F)).$$

Proof. We proof this for bilinear maps, the general case is completely analogous. We already know that bilinearity translates into linearity into the space of linear functions. Remains to prove boundedness. So let $\mathcal{B}\subseteq L(E_1,E_2;F)$ be given. Then $\mathcal{B}$ is bounded if and only if $\mathcal{B}(B_1\times B_2)\subseteq F$ is bounded for all bounded $B_i\subseteq E_i$. This however is equivalent to $\mathcal{B}^\vee(B_1)$ is contained and bounded in $L(E_2,F)$ for all bounded $B_1\subseteq E_1$, i.e. $\mathcal{B}^\vee$ is contained and bounded in $L(E_1,L(E_2,F))$. □

Recall that we have already put a structure on $L(E,F)$ in 3.17, namely the initial one with respect to the inclusion in $C^\infty(E,F)$. Let us now show that bornologically these definitions agree:

5.3. Lemma. Structure on L. *A subset is bounded in $L(E,F)\subseteq C^\infty(E,F)$ if and only if it is uniformly bounded on bounded subsets of E, i.e. $L(E,F)\to C^\infty(E,F)$ is initial.*

Proof. Let $\mathcal{B}\subseteq L(E,F)$ be bounded in $C^\infty(E,F)$, and assume that it is not uniformly bounded on some bounded set $B\subseteq E$. So there are $f_n\in\mathcal{B}$, $b_n\in B$, and $\ell\in F^*$ with $|\ell(f_n(b_n))|\ge n^n$. Then the sequence $n^{1-n}b_n$ converges fast to 0, and hence lies on some compact part of a smooth curve c by the special curve lemma 2.8. So $\mathcal{B}$ cannot be bounded, since otherwise $C^\infty(\ell,c)=\ell_*\circ c^*: C^\infty(E,F)\to C^\infty(\mathbb{R},\mathbb{R})\to \ell^\infty(\mathbb{R},\mathbb{R})$ would have bounded image, i.e. $\{\ell\circ f_n\circ c: n\in\mathbb{N}\}$ would be uniformly bounded on any compact interval.

Conversely, let $\mathcal{B} \subseteq L(E,F)$ be uniformly bounded on bounded sets and hence in particular on compact parts of smooth curves. We have to show that $d^n \circ c^* : L(E,F) \to C^\infty(\mathbb{R},F) \to \ell^\infty(\mathbb{R},F)$ has bounded image. But for $f \in L(E,F)$ we have by the chain rule 3.18, recursively applied, that $(d^n \circ c^*)(f)(t) = d^n(f \circ c)(t) = f(c^{(n)}(t))$, and since $c^{(n)}$ is still a smooth curve we are done. $\square$

Let us now generalize this result to multilinear mappings. For this we first characterize bounded multilinear mappings in the following two ways:

5.4. Lemma. *A multilinear mapping is bounded if and only if it is bounded on each sequence which converges Mackey to 0.*

Proof. Suppose that $f : E_1 \times \ldots \times E_k \to F$ is not bounded on some bounded set $B \subseteq E_1 \times \ldots \times E_k$. By composing with a linear functional we may assume that $F = \mathbb{R}$. So there are $b_n \in B$ with $\lambda_n^{k+1} := |f(b_n)| \to \infty$. Then $|f(\frac{1}{\lambda_n} b_n)| = \lambda_n \to \infty$, but $(\frac{1}{\lambda_n} b_n)$ is Mackey convergent to 0. $\square$

5.5. Lemma. Bounded multilinear mappings are smooth. *Let $f : E_1 \times \ldots \times E_n \to F$ be a multilinear mapping. Then f is bounded if and only if it is smooth. For the derivative we have the product rule:*

$$df(x_1,\ldots,x_n)(v_1,\ldots,v_n) = \sum_{i=1}^{n} f(x_1,\ldots,x_{i-1},v_i,x_{i+1},\ldots,x_n).$$

In particular, we get for $f : E \supseteq U \to \mathbb{R}$, $g : E \supseteq U \to F$ and $x \in U$, $v \in E$ the Leibniz formula

$$(f \cdot g)'(x)(v) = f'(x)(v) \cdot g(x) + f(x) \cdot g'(x)(v).$$

Proof. We use induction on n. The case $n = 1$ is corollary 2.11. The induction goes as follows:

f is bounded

$\iff$ $f(B_1 \times \ldots \times B_n) = f^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded sets B_i in E_i;

$\iff$ $f^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n,F) \subseteq C^\infty(E_n,F)$ is bounded, by 5.3;

$\iff$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n,F)$ is bounded;

$\iff$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n,F)$ is smooth by the inductive assumption;

$\iff$ $f : E_1 \times \ldots \times E_n \to F$ is smooth by cartesian closedness 3.13.

The formula for the derivative follows by direct evaluation of the directional difference quotient.

The particular case follows by application to the scalar multiplication $\mathbb{R} \times F \to F$. $\square$

Now let us show that also the structures coincide:

5.6. Proposition. Structure on space of multilinear maps. *The injection of $L(E_1,\ldots,E_n;F) \to C^\infty(E_1 \times \ldots \times E_n, F)$ is a bornological embedding.*

Proof. We can show this by induction. In fact, let $\mathcal{B} \subseteq L(E_1,\ldots,E_n;F)$. Then

$\mathcal{B}$ is bounded

$\iff$ $\mathcal{B}(B_1 \times \ldots \times B_n) = \mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded B_i in E_i;

$\iff$ $\mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n, F) \subseteq C^\infty(E_n, F)$ is bounded, by 5.3;

$\iff$ $\mathcal{B}^\vee \subseteq C^\infty(E_1 \times \ldots \times E_{n-1}, C^\infty(E_n, F))$ is bounded by the inductive assumption;

$\iff$ $\mathcal{B} \subseteq C^\infty(E_1 \times \ldots \times E_n, F)$ is bounded by cartesian closedness 3.13. $\square$

Algebraic Tensor Product

Remark. The importance of the tensor product is twofold. First it allows linearizing of multi-linear mappings and secondly it allows to calculate function spaces.

We will consider the spaces of linear and multi-linear mappings between vector spaces. If we supply all vector spaces E, $E_1, \ldots, E_n$, F with the finest locally convex topology (i.e. the final locally convex topology with respect to the inclusions of all finite dimensional subspaces - on which the topology is unique) then all linear mappings are continuous and all multi-linear mappings are bounded (but not necessarily continuous as the evaluation map $\mathrm{ev} : E^* \times E \to \mathbb{K}$ on an infinite dimensional vector space E shows) and hence it is consistent to denote the corresponding function spaces by $L(E, F) = \mathcal{L}(E, F)$ and $L(E_1, \ldots E_n; F)$.

In more detail the first feature is:

a3.1 Proposition. Linearization. *Given two linear spaces E and F, then there exists a solution $\otimes : E \times F \to E \otimes F$ – called the* ALGEBRAIC TENSOR PRODUCT *of E and F – to the following universal problem:*

$$\begin{tikzcd} E \times F \arrow[rr, "\otimes"] \arrow[dr, "T"'] & & E \otimes F \arrow[dl, dotted, "!\ \tilde{T}"] \\ & G & \end{tikzcd}$$

Here $\otimes : E \times F \to E \otimes F$ and $T : E \times F \to G$ are bilinear and $\tilde{T}$ is linear.

Proof. In order to find $E \otimes F$ one considers first the case, where $G = \mathbb{R}$. Then we have that $\otimes^* : (E \otimes F)^* \to L(E, F; \mathbb{R})$ should be an isomorphism. Hence $E \otimes F$ could be realized as subspace of $(E \otimes F)^{**} \cong L(E, F; \mathbb{R})^*$. Obviously to each bilinear functional $T : E \times F \to \mathbb{R}$ corresponds the linear map $\mathrm{ev}_T : L(E, F; \mathbb{R})^* \to \mathbb{R}$. The map $\otimes : E \times F \to E \otimes F \subseteq L(E, F; \mathbb{R})^*$ has to be such that $\mathrm{ev}_T \circ \otimes = T$ for all bilinear functionals $T : E \times F \to \mathbb{R}$, i.e. $\otimes(x, y)(T) = (\mathrm{ev}_T \circ \otimes)(x, y) = T(x, y)$. Thus we have proved the existence of $\tilde{T} := \mathrm{ev}_T$ for $G = \mathbb{R}$. But uniqueness can be true only on the linear subspace generated by the image of $\otimes$, and hence we denote this subspace $E \otimes F$.

For bilinear mappings $T : E \times F \to G$ into an arbitrary vector space G, we consider the following diagram, which has quite some similarities with that used in the construction of the c^∞-completion in [**71**, 2.31]:

$$\begin{tikzcd}
E \times F \arrow[r, "\otimes"] \arrow[dd, "T"'] & E \otimes F \arrow[r, hook] \arrow[d, dotted, "(2)"] \arrow[ddl, dotted, "\tilde{T}"'] & L(E,F;\mathbb{R})^* \arrow[dl, dotted, "(1)"] \arrow[dd, "\mathrm{ev}_{\lambda \circ T}"] \\
& \prod_{G'} \mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, dotted, "(3)\ \delta"] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

The right dashed arrow (1) and δ exist uniquely by the universal property of the product in the center. The arrow (2) exists uniquely as restriction of (1) to the subspace $E \otimes F$. Finally (3) exists, since the generating subset $\otimes(E \times F)$ in $E \otimes F$ is mapped to $T(E \times F) \subseteq G$ and since δ is injective. $\square$

Note that $\otimes$ extends to a functor, by defining $T \otimes S$ via the following diagram:

$$\begin{tikzcd}
E_1 \times F_1 \arrow[r, "\otimes"] \arrow[d, "T \times S"'] & E_1 \otimes F_1 \arrow[d, dotted, "!\ T \otimes S"] \\
E_2 \times F_2 \arrow[r, "\otimes"'] & E_2 \otimes F_2
\end{tikzcd}$$

Furthermore one easily proves the existence of the following natural isomorphisms:

$$E \otimes \mathbb{R} \cong E$$
$$E \otimes F \cong F \otimes E$$
$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$$

Note that if both spaces E and F are finite dimensional, then so is $L(E, F; \mathbb{R})$, hence also the dual $L(E, F; \mathbb{R})^*$ and thus the subspace $E \otimes F$ is finite dimensional too (in fact $\dim(E \otimes F) = \dim E \cdot \dim F$, as we will see in [**71**, 3.30]), and hence $E \otimes F = (E \otimes F)^{**} = L(E, F; \mathbb{R})^*$.
If one factor is infinite dimensional and the other one is not 0, then this is not true. In fact take $F = \mathbb{R}$, then $E \otimes \mathbb{R} \cong E$ whereas $L(E, \mathbb{R}; \mathbb{R})^* \cong L(E, L(\mathbb{R}, \mathbb{R}))^* \cong L(E, \mathbb{R})^* = E^{**}$.

Projective Tensor Product

We turn first to the property of making bilinear continuous mappings into linear ones. We call the corresponding solution the PROJECTIVE TENSOR PRODUCT of E and F and denote it by $E \otimes_\pi F$. Obviously it can be obtained by taking the algebraic tensor product and supplying it with the finest locally convex topology such that $E \times F \to E \otimes F$ is continuous: *This topology exists* since the union of locally convex topologies is locally convex and $E \times F \to E \otimes F$ is continuous for the weak topology on $E \otimes F$ generated by those linear functionals which correspond to continuous bilinear functionals on $E \times F$. *It has the universal property*, since the inverse image of a locally convex topology under a linear mapping $\tilde{T}$ is again a locally convex topology, such that $\otimes$ is continuous, provided the associated bilinear mapping T is continuous. However, it is not obvious that this topology is separated, and we prove that now. We will denote the SPACE OF CONTINUOUS LINEAR MAPPINGS from E to F by $\mathcal{L}(E, F)$, and the SPACE OF CONTINUOUS MULTI-LINEAR MAPPINGS by $\mathcal{L}(E_1,\dots,E_n;F)$. If all $E_1,\dots,E_n$ are the same space E, we will also write $\mathcal{L}^n(E;F)$.

a3.3 Lemma. *$E\otimes_\pi F$ is Hausdorff provided E and F are.*

Proof. It is enough to show that the set $E^*\times F^*$ separates points in $E\otimes F$. So let $0\neq z=\sum_k x_k\otimes y_k$ be given. By replacing linear dependent x_k by the corresponding linear combinations and using bilinearity of $\otimes$, we may assume that the x_k are linearly independent. Now choose $x^*\in E^*$ and $y^*\in F^*$ be such that $x^*(x_k)=\delta_{1,k}$ and $y^*(y_1)=1$. Then $(x^*\otimes y^*)(z)=1\neq 0$. □

Since a bilinear mapping (like $\otimes$) is continuous iff it is so at 0, a 0-neighborhood basis in $E\otimes_\pi F$ is given by all those absolutely convex sets, for which the inverse image under $\otimes$ is a 0-neighborhood in $E\times F$. A 0-neighborhod basis is thus given by the absolutely convex hulls, denoted $U\otimes V$, of the images of $U\times V$ under $\otimes$, where U resp. V runs through a 0-neighborhood basis of E resp. F: We only have to show that these sets $U\otimes V$ are absorbing (see [**54**, 6.5.3]). So let $z=\sum_k x_k\otimes y_k\in E\otimes F$ be arbitrary. Then there are $a_k>0$ and $b_k>0$ such that $x_k\in a_kU$ and $y_k\in b_kV$ and hence $z=\sum_{k\le K}a_k\,b_k\frac{x_k}{a_k}\otimes\frac{y_k}{b_k}\in(\sum_k a_k\,b_k)\cdot\langle U\otimes V\rangle_{\text{abs.conv.}}$. The Minkowski-functionals $p_{U\otimes V}$ of these hulls form a base of the seminorms of $E\otimes_\pi F$ and we will denote them by $\pi_{U,V}$. In terms of the Minkowski-functionals p_U and p_V of closed 0-neighborhoods U and V we obtain that $z\in\big(\sum_k p_U(x_k)\,p_V(y_k)\big)\,U\otimes V$ for any $z=\sum_k x_k\otimes y_k$, since $x_k\in p_U(x_k)\cdot U$ for closed U, and thus $p_{U\otimes V}(z)\le\inf\{\sum_k p_U(x_k)\,p_V(y_k):z=\sum_k x_k\otimes y_k\}$. We now show the converse:

a3.4 Proposition. Seminorms of the projective tensor product.

$$p_{U\otimes V}(z)=\inf\Big\{\sum_k p_U(x_k)\cdot p_V(y_k):z=\sum_k x_k\otimes y_k\Big\}.$$

Proof. Let $z\in\lambda\cdot U\otimes V$ with $\lambda>0$. Then $z=\lambda\sum_k\lambda_k\cdot u_k\otimes v_k$ with $u_k\in U$, $v_k\in V$ and $\sum_k|\lambda_k|=1$. Hence $z=\sum x_k\otimes v_k$, where $x_k=\lambda\lambda_k u_k$, and $\sum_k p_U(x_k)\cdot p_V(v_k)\le\sum\lambda|\lambda_k|=\lambda$. Taking the infimum of all those λ shows that $p_{U\otimes V}(z)$ is greater or equal to the infimum on the right side. □

a3.5 Corollary. *$E\otimes_\pi F$ is normable (metrizable) provided E and F are.* □

a3.6 Lemma. The semi-norms of decomposable tensors.

$$p_{U\otimes V}(x\otimes y)=p_U(x)\cdot p_V(y).$$

Proof. According to [**68**, 7.1.8] there are $x^*\in E^*$ and $y^*\in F^*$ such that $|x^*|\le p_U$ and $|y^*|\le p_V$ and $x^*(x)=p_U(x)$ and $y^*(y)=p_V(y)$. If $x\otimes y=\sum_k x_k\otimes y_k$, then

$$\begin{aligned}p_{U\otimes V}(x\otimes y)\le p_U(x)\cdot p_V(y)&=x^*(x)\cdot y^*(y)=(x^*\otimes y^*)(x\otimes y)=\\&=\sum_k x^*(x_k)\cdot y^*(y_k)\le\sum_k p_U(x_k)\cdot p_V(y_k),\end{aligned}$$

and taking the infimum gives the desired result. □

a3.7 Remark. Functorality. Given two continuous linear maps $T_1:E_1\to F_1$ and $T_2:E_2\to F_2$ we can consider the bilinear continuous map given by composing $T_1\times T_2:E_1\times E_2\to F_1\times F_2$ with $\otimes_\pi:F_1\times F_2\to F_1\otimes_\pi F_2$. By the universal property of $E_1 \times E_2 \to E_1 \otimes_\pi E_2$ we obtain a continuous linear map denoted by $T_1 \otimes_\pi T_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2$.

$$\begin{tikzcd} E_1 \times E_2 \arrow[r, "\otimes_\pi"] \arrow[d, "T_1 \times T_2"'] & E_1 \otimes_\pi E_2 \arrow[d, dotted, "T_1 \otimes_\pi T_2"] \\ F_1 \times F_2 \arrow[r, "\otimes_\pi"'] & F_1 \otimes_\pi F_2 \end{tikzcd}$$

By the uniqueness of the linearization one obtains immediately that $\otimes_\pi$ is a functor. Because of the uniqueness of universal solutions one sees easily that one has natural isomorphisms $\mathbb{R} \otimes_\pi E \cong E$, $E \otimes_\pi F \cong F \otimes_\pi E$ and $(E \otimes_\pi F) \otimes_\pi G \cong E \otimes_\pi (F \otimes_\pi G)$.

a3.14 Example. $\otimes_\pi$ does not preserve embeddings.
In fact consider the isometric embedding $\ell^2 \to C(K)$, where K is the closed unit-ball of $(\ell^2)^*$ supplied with its compact topology of pointwise convergence, see the corollary to the Alaoğlu-Bourbaki-theorem in [**68**, 7.4.12]. This subspace has however no topological complement, since $C(K)$ has the Dunford-Pettis property (see [**54**, 20.7.8 S.472], i.e. $x_n^*(x_n) \to 0$ for every two sequences $x_n \to 0$ in $\sigma(E, E^*)$ and $x_n^* \to 0$ in $\sigma(E^*, E^{**})$), but no infinite dimensional reflexive Banach space like ℓ^2 has it (e.g. $x_n := e_n$, $x_n^* := e_n$) and hence cannot be a complemented subspace of $C(K)$, see [**54**, 20.7 S.472].
Suppose now that $\ell^2 \otimes_\pi (\ell^2)^* \to C(K) \otimes_\pi (\ell^2)^*$ were an embedding. The duality mapping $\mathrm{ev} : \ell^2 \times (\ell^2)^* \to \mathbb{R}$ yields a continuous linear mapping $s : \ell^2 \otimes_\pi (\ell^2)^* \to \mathbb{R}$ and would hence have a continuous linear extension $\tilde{s} : C(K) \otimes_\pi (\ell^2)^* \to \mathbb{R}$. The corresponding bilinear map would give a continuous linear map $\tilde{s}^\vee : C(K) \to (\ell^2)^{**} \cong \ell^2$, which is a left inverse to the embedding $\ell^2 \to C(K)$, a contradiction.

5.7. Bornological tensor product

It is natural to consider the universal problem of linearizing bounded bilinear mappings. The solution is given by the *bornological tensor product* $E \otimes_\beta F$, i.e. the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded. A 0-neighborhood basis of this topology is given by those absolutely convex sets, which absorb $B_1 \otimes B_2$ for all bounded $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$. Note that this topology is bornological since it is the finest locally convex topology with given bounded linear mappings on it.

Theorem. *The bornological tensor product is the left adjoint functor to the Hom-functor $L(E, \_)$ on the category of bounded linear mappings between locally convex spaces, and one has the following bornological isomorphisms:*

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G))$$
$$E \otimes_\beta \mathbb{R} \cong E$$
$$E \otimes_\beta F \cong F \otimes_\beta E$$
$$(E \otimes_\beta F) \otimes_\beta G \cong E \otimes_\beta (F \otimes_\beta G)$$

Furthermore, the bornological tensor product preserves colimits. It neither preserves embeddings nor countable products.

Proof. We show first that this topology has the universal property for bounded bilinear mappings $f : E_1 \times E_2 \to F$. Let U be an absolutely convex zero neighborhood in F, and let B_1, B_2 be bounded sets. Then $f(B_1 \times B_2)$ is bounded, hence it is absorbed by U. Then $\tilde{f}^{-1}(U)$ absorbs $\otimes(B_1\times B_2)$, where $\tilde{f}: E_1\otimes E_2 \to F$ is the canonically associated linear mapping. So $\tilde{f}^{-1}(U)$ is in the zero neighborhood basis of $E_1 \otimes_\beta E_2$ described above. Therefore, $\tilde{f}$ is continuous.

An analogous argument for sets of mappings shows that the first isomorphism $L(E\otimes_\beta F, G) \cong L(E, F; G)$ is bornological.

The topology on $E_1 \otimes_\beta E_2$ is finer than the projective tensor product topology, and so it is Hausdorff. The rest of the positive results is clear.

The counter-example for embeddings given for the projective tensor product works also, since all spaces involved are Banach.

Since the bornological tensor-product preserves coproducts it cannot preserve products. In fact $(\mathbb{R}\otimes_\beta \mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ whereas $\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$. □

5.8. Proposition. Projective versus bornological tensor product. *If every bounded bilinear mapping on $E\times F$ is continuous then $E\otimes_\pi F = E\otimes_\beta F$. In particular, we have $E\otimes_\pi F = E\otimes_\beta F$ for any two metrizable spaces, and for a normable space F we have $E_{born}\otimes_\pi F = E\otimes_\beta F$.*

Proof. Recall that $E\otimes_\pi F$ carries the finest locally convex topology such that $\otimes : E\times F\to E\otimes F$ is continuous, whereas $E\otimes_\beta F$ carries the finest locally convex topology such that $\otimes : E\times F \to E\otimes F$ is bounded. By assumption the bounded bilinear map $\otimes : E\times F\to E\otimes_\beta F$ is continuous, and thus by the universal property the topology of $E\otimes_\pi F$ is finer than that of $E\otimes_\beta F$. Since the converse is true in general, we have equality.

In [**68**, 3.1.6] it is shown that in metrizable locally convex spaces the convergent sequences coincide with the Mackey-convergent ones. Now let $T: E\times F\to G$ be bounded and bilinear. We have to show that T is continuous. So let (x_n, y_n) be a convergent sequence in $E\times F$. Without loss of generality we may assume that its limit is $(0,0)$. So there are $\mu_n\to\infty$ such that $\{\mu_n(x_n,y_n) : n\in\mathbb{N}\}$ is bounded and hence also $T\Big(\{\mu_n(x_n,y_n) : n\in\mathbb{N}\}\Big) = \Big\{\mu_n^2 T(x_n,y_n) : n\in\mathbb{N}\Big\}$, i.e. $T(x_n,y_n)$ converges even Mackey to 0.

If F is normable and $T: E_{born}\times F\to G$ is bounded bilinear then $T^\vee : E_{born}\to L(F,G)$ is bounded, and since E_{born} is bornological it is even continuous. Clearly, for normed spaces F the evaluation map $\mathrm{ev} : L(F,G)\times F\to G$ is continuous, and hence $T = \mathrm{ev}\circ(T^\vee\times F) : E_{born}\times F\to G$ is continuous. Thus, $E_{born}\otimes_\pi F = E\otimes_\beta F$. □

Note that the bornological tensor product is invariant under bornologification, i.e. $E_{born}\otimes_\beta F_{born} = E\otimes_\beta F$. So it is no loss of generality to assume that both factors are bornological. Keep however in mind that the corresponding identity for the projective tensor product does not hold.

Another possibility to obtain the identity $E\otimes_\pi F = E\otimes_\beta F$ is to assume that E and F are bornological and every separately continuous bilinear mapping on $E\times F$ is continuous: In fact, every bounded bilinear mapping is obviously separately bounded, and since E and F are assumed to be bornological, it has to be separately continuous. We want to find another class beside the Fréchet spaces (see [**68**, 5.2.8]) which satisfies these assumptions.

a3.47 Theorem. Continuity versus separately continuity. *Let E and F be two barreled spaces with a countable base of bornology. Then every separately continuous bilinear map $E \times F \to G$ is continuous.*

Proof. Let A_n and B_n be a basis of the bornologies of E and F. Let $T : E \times F \to G$ be separately continuous. Then $T^\vee : E \to \mathcal{L}(F, G)$ is continuous for the topology of pointwise convergence on $\mathcal{L}(F, G)$. Thus $T^\vee(A_k)$ is bounded for this topology, and since F is barreled it is equi-continuous. Thus for every 0-neighborhood W in G there exists a 0-neighborhood V_k in F with $T(A_k \times V_k) \subseteq W$. By symmetry there exists a 0-neighborhood U_k in E with $T(U_k \times B_k) \subseteq W$. This implies for ($gDF$-spaces) E and F the continuity of T, see [**54**, 15.6.1 S.335]. $\square$

a3.48 Corollary. Projective versus bornological tensor product for LB-spaces. *Let E and F be regular inductive limits of sequences of Banach spaces (e.g. the duals of metrizable spaces with their bornological topology, i.e. the bornologification of the strong topology). Then $E \otimes_\pi F \cong E \otimes_\beta F$.*

Proof. Let $T : E \times F \to G$ be bounded. Since both spaces are bornological, T is separately continuous and, since both spaces are barreled and have a countable base of bornology, it is continuous by a3.47. This is enough to guarantee the equality of the two tensor products by 5.8. $\square$

5.9. Corollary. *The following mappings are bounded multilinear.*

(1) $\lim : \operatorname{Nat}(\mathcal{F}, \mathcal{G}) \to L(\lim \mathcal{F}, \lim \mathcal{G})$, *where $\mathcal{F}$ and $\mathcal{G}$ are two functors on the same index category, and where* $\operatorname{Nat}(\mathcal{F}, \mathcal{G})$ *denotes the space of all natural transformations with the structure induced by the embedding into* $\prod_i L(\mathcal{F}(i), \mathcal{G}(i))$.

(2) $\operatorname{colim} : \operatorname{Nat}(\mathcal{F}, \mathcal{G}) \to L(\operatorname{colim} \mathcal{F}, \operatorname{colim} \mathcal{G})$.

(3)
$$\begin{aligned} L : L(E_1, F_1) \times \ldots \times L(E_n, F_n) \times L(F, E) &\to \\ &\to L(L(F_1, \ldots, F_n; F), L(E_1, \ldots, E_n; E)) \\ (T_1, \ldots, T_n, T) &\mapsto (S \mapsto T \circ S \circ (T_1 \times \ldots \times T_n)); \end{aligned}$$

(4) $\bigotimes^n_\beta : L(E_1, F_1) \times \ldots \times L(E_n, F_n) \to L(E_1 \otimes_\beta \cdots \otimes_\beta E_n, F_1 \otimes_\beta \cdots \otimes_\beta F_n)$.

(5) $\bigwedge^n : L(E, F) \to L(\bigwedge^n E, \bigwedge^n F)$, *where $\bigwedge^n E$ is the linear subspace of all alternating tensors in $\bigotimes^n_\beta E$. It is the universal solution of*
$$L\Big(\bigwedge^n E, F\Big) \cong L^n_{alt}(E; F),$$
where $L^n_{alt}(E; F)$ is the space of all bounded n-linear alternating mappings $E \times \ldots \times E \to F$. This space is a direct summand of $L^n(E; F) := L(E, \ldots, E; F)$.

(6) $\bigvee^n : L(E, F) \to L(\bigvee^n E, \bigvee^n F)$, *where $\bigvee^n E$ is the linear subspace of all symmetric tensors in $\bigotimes^n_\beta E$. It is the universal solution of*
$$L\Big(\bigvee^n E, F\Big) \cong L^n_{sym}(E; F),$$
where $L^n_{sym}(E; F)$ is the space of all bounded n-linear symmetric mappings $E \times \ldots \times E \to F$. This space is also a direct summand of $L^n(E; F)$.

(7) $\bigotimes_\beta : L(E,F) \to L(\bigotimes_\beta E, \bigotimes_\beta F)$, *where* $\bigotimes_\beta E := \coprod_{n=0}^\infty \bigotimes_\beta^n E$ *is the tensor algebra of* E. *Note that it has the universal property of prolonging bounded linear mappings with values in locally convex spaces, which are algebras with bounded operations, to continuous algebra homomorphisms:*

$$L(E,F) \cong \mathrm{Alg}\Big(\bigotimes_\beta E, F\Big).$$

(8) $\bigwedge : L(E,F) \to L(\bigwedge E, \bigwedge F)$, *where* $\bigwedge E := \coprod_{n=0}^\infty \bigwedge^n E$ *is the exterior algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into graded-commutative algebras, i.e. algebras in the sense above, which are as vector spaces a coproduct* $\coprod_{n\in\mathbb{N}} E_n$ *and the multiplication maps* $E_k \times E_l \to E_{k+l}$ *and for* $x \in E_k$ *and* $y \in E_l$ *one has* $x \cdot y = (-1)^{kl} y \cdot x$.

(9) $\bigvee : L(E,F) \to L(\bigvee E, \bigvee F)$, *where* $\bigvee E := \coprod_{n=0}^\infty \bigvee^n E$ *is the symmetric algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into commutative algebras.*

Recall that the symmetric product is given as the image of the *symmetrizer* $\mathrm{sym} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$ defined by

$$x_1 \otimes \cdots \otimes x_n \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Similarly the wedge product is given as the image of the *alternator*

$$\mathrm{alt} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$$

$$\text{defined by } x_1 \otimes \cdots \otimes x_n \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Symmetrizer and alternator are bounded projections, so both subspaces are complemented in the tensor product.

Proof. All results follow easily by flipping coordinates until only a composition of products of evaluation maps remains.

In particular, consider the following diagrams:

(1)

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{F},\mathcal{G}) \times \lim \mathcal{F} \arrow[r] \arrow[d, "\mathrm{pr}_i \times \mathrm{pr}_i"] & \lim \mathcal{G} \arrow[d, "\mathrm{pr}_i"] \\
L(\mathcal{F}(i), \mathcal{G}(i)) \times \mathcal{F}(i) \arrow[r, "\mathrm{ev}"] & \mathcal{G}(i)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
\mathcal{F}(i) \arrow[r, "\mathrm{inj}_i"] & \operatorname{colim} \mathcal{F} \arrow[r, dotted] & L(\mathrm{Nat}(\mathcal{F},\mathcal{G}), \operatorname{colim} \mathcal{G})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(i) \times \mathrm{Nat}(\mathcal{F},\mathcal{G}) \arrow[r, dotted] \arrow[d, "\mathrm{id} \times \mathrm{pr}_i"'] & \operatorname{colim} \mathcal{G} \\
\mathcal{F}(i) \times L(\mathcal{F}(i), \mathcal{G}(i)) \arrow[r, "\mathrm{ev}"] & \mathcal{G}(i) \arrow[u, "\mathrm{inj}_i"]
\end{tikzcd}$$

([3])

$$\begin{tikzcd}
\Big(\prod_i L(E_i,F_i)\Big)\times L(F,E)\times L(F_1,\dots,F_n;F)\times\prod_i E_i \arrow[rr, dotted] \arrow[d, "\cong"] & & E \\
L(F,E)\times L(F_1,\dots,F_n;F)\times\prod_i(L(E_i,F_i)\times E_i) \arrow[d, "\mathrm{id}\times\mathrm{ev}\times\ldots\times\mathrm{ev}"] & & \\
L(F,E)\times L(F_1,\dots,F_n;F)\times\prod_i F_i \arrow[rr, "\mathrm{ev}"'] & & L(F,E)\times F \arrow[uu, "\mathrm{ev}"]
\end{tikzcd}$$

([4])

$$\begin{tikzcd}
E_1\times\ldots\times E_n \arrow[r, dotted] & L(L(E_1,F_1),\dots,L(E_n,F_n);F_1\otimes_\beta\cdots\otimes_\beta F_n)
\end{tikzcd}$$

$$\begin{tikzcd}
E_1\times\ldots\times E_n\times L(E_1,F_1)\times\ldots\times L(E_n,F_n) \arrow[r, dotted] \arrow[d, "\cong"] & F_1\otimes_\beta\cdots\otimes_\beta F_n \\
L(E_1,F_1)\times E_1\times\ldots\times L(E_n,F_n)\times E_n \arrow[r, "\mathrm{ev}\times\ldots\times\mathrm{ev}"] & F_1\times\ldots\times F_n \arrow[u]
\end{tikzcd}$$

([5])

$$\begin{tikzcd}
L(E,F) \arrow[r, dotted] \arrow[d, "\Delta"] & L(\bigwedge^n E,\bigwedge^n F) \\
L(E,F)\times\ldots\times L(E,F) \arrow[r, "\bigotimes^n"] & L(\bigotimes_\beta^n E,\bigotimes_\beta^n F) \arrow[u, "{L(\mathrm{incl},\mathrm{alt})}"']
\end{tikzcd}$$

The projection $L^n(E;F)\to L^n_{\mathrm{alt}}(E;F)$ is given by the alternator

$$T\mapsto\Big((v_1,\dots,v_n)\mapsto\frac{1}{n!}\sum_\sigma \mathrm{sign}(\sigma)\,T(v_{\sigma(1)},\dots,v_{\sigma(n)})\Big).$$

The universal property follows from the diagram:

$$\begin{tikzcd}
E\times\ldots\times E \arrow[r, "\otimes"] \arrow[dr, "f"'] & E\otimes_\beta\cdots\otimes_\beta E \arrow[r, "alt"] \arrow[d, "\tilde f"'] & \bigwedge^n E \arrow[dl, "\tilde f|_{\bigwedge^n E}"] \\
& F &
\end{tikzcd}$$

([6])

$$\begin{tikzcd}
L(E,F) \arrow[r, dotted] \arrow[d, "\Delta"] & L(\bigvee^n E,\bigvee^n F) \\
L(E,F)\times\ldots\times L(E,F) \arrow[r, "\bigotimes_\beta^n"] & L(\bigotimes_\beta^n E,\bigotimes_\beta^n F) \arrow[u, "{L(\mathrm{incl},\mathrm{sym})}"']
\end{tikzcd}$$

The projection $L^n(E;F)\to L^n_{\mathrm{sym}}(E;F)$ symmetrizer

$$T\mapsto\Big((v_1,\dots,v_n)\mapsto\frac{1}{n!}\sum_\sigma T(v_{\sigma(1)},\dots,v_{\sigma(n)})\Big).$$

The universal property follows from the diagram:

$$\begin{tikzcd} E\times\ldots\times E \arrow[r, "\otimes"] \arrow[dr, "f"'] & E\otimes_\beta\cdots\otimes_\beta E \arrow[r, "sym"] \arrow[d, "\tilde f"] & \bigvee^n E \arrow[dl, "\tilde f|_{\bigvee^n E}"] \\ & F & \end{tikzcd}$$

(7)

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigotimes"] \arrow[d, "(\bigotimes^n)_n"] & L(\bigotimes_\beta E, \bigotimes_\beta F) \\ \prod_n L(\bigotimes_\beta^n E, \bigotimes_\beta^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigotimes_\beta^n E, \coprod_k \bigotimes_\beta^k F) \arrow[u, "\cong"'] \end{tikzcd}$$

The universal property holds, since to $T\in L(E,F)$ we can associate $\sum_n \mu_n\circ\bigotimes^n T$, where $\mu_n:\bigotimes F\to F$ denotes the n-fold multiplication of the algebra F.

(8)

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigwedge"] \arrow[d, "(\bigwedge^n)_n"] & L(\bigwedge E, \bigwedge F) \\ \prod_n L(\bigwedge^n E, \bigwedge^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigwedge^n E, \coprod_k \bigwedge^k F) \arrow[u, "\cong"'] \end{tikzcd}$$

(9)

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigvee"] \arrow[d, "(\bigvee^n)_n"] & L(\bigvee E, \bigvee F) \\ \prod_n L(\bigvee^n E, \bigvee^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigvee^n E, \coprod_k \bigvee^k F) \arrow[u, "\cong"'] \end{tikzcd}$$

□

5.12. Theorem. Taylor formula. *Let $f:U\to F$ be smooth, where U is c^∞-open in E. Then for each segment $[x,x+y]=\{x+ty:0\le t\le 1\}\subseteq U$ we have*

$$f(x+y)=\sum_{k=0}^n \frac{1}{k!}d^kf(x)y^k+\int_0^1\frac{(1-t)^n}{n!}d^{n+1}f(x+ty)y^{n+1}dt,$$

where y^k stands for $(y,\ldots,y)\in E^k$.

Proof. Recall that we can form iterated derivatives as follows:

$$\begin{aligned} f&:E\supseteq U\to F\\ df&:E\supseteq U\to L(E,F),\\ df(x)(v)&:=\tfrac{\partial}{\partial t}|_{t=0}f(x+tv)\\ d(df)&:E\supseteq U\to L(E,L(E,F))\cong L(E,E;F),\\ d(df)(x)(v_1)(v_2)&:=\tfrac{\partial}{\partial t_1}|_{t_1=0}df(x+t_1v_1)(v_2)\\ &\vdots\\ d(\ldots(d(df))\ldots)&:E\supseteq U\to L(E,\ldots,L(E,L(E,F))\cdots)\cong L(E,\ldots,E;F) \end{aligned}$$

Thus, the iterated derivative $d^n f(x)(v_1, \dots, v_n)$ is given by

$$\tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x + t_1 v_1 + \dots + t_n v_n) = \partial_1 \dots \partial_n \tilde{f}(0, \dots, 0),$$

where $\tilde{f}(t_1, \dots, t_n) := f(x + t_1 v_1 + \dots + t_n v_n)$. In particular,

$$d^k f(x+tv)v^k = \tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x+tv+t_1 v + \dots + t_n v) = (\tfrac{d}{ds})^k|_{s=t} f(x+sv)$$

and the claimed Taylor formula is an assertion on the smooth curve $t \mapsto f(x+ty)$. Using functionals λ we can reduce it to the scalar valued case since $(\frac{d}{dt})^k|_{t=0}\lambda(f(x+ty)) = \lambda(d^k f(x) y^k)$.

Another method of proof is induction on n: The first step is 2.6.6, and the induction step is partial integration of the remainder integral. □

5.11. Proposition. Symmetry of higher derivatives. *Let $f : E \supseteq U \to F$ be smooth. The n-th derivative $f^{(n)}(x) = d^n f(x)$, considered as an element of $L^n(E;F)$, is symmetric, so lies in the space $L^n_{sym}(E;F) \cong L(\bigvee^n E; F)$*

Proof. The result follows from the finite dimensional property, since the iterated derivative $d^n f(x)(v_1, \dots, v_n)$ is given by

$$\tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x + t_1 v_1 + \dots + t_n v_n) = \partial_1 \dots \partial_n \tilde{f}(0, \dots, 0),$$

where $\tilde{f}(t_1, \dots, t_n) := f(x + t_1 v_1 + \dots + t_n v_n)$. □

5.13. Corollary. *The following two subspaces are direct summands:*

$$L(E_1, \dots, E_n; F) \subseteq C^\infty(E_1 \times \dots \times E_n, F),$$
$$L^n_{sym}(E;F) \xrightarrow{\Delta^*} C^\infty(E,F).$$

Note that direct summand is meant in the bornological category, i.e. the embedding admits a left-inverse in the category of bounded linear mappings, or, equivalently, with respect to the bornological topologies it is a topological direct summand.

Proof. The projection for $L(E,F) \subseteq C^\infty(E,F)$ is $f \mapsto df(0)$. The statement on L^n follows by induction using the exponential laws 3.13 and 5.2.

The second embedding is given by $\triangle^*$, which is bounded and linear $C^\infty(E \times \dots \times E, F) \to C^\infty(E,F)$. Here $\Delta : E \to E \times \dots \times E$ denotes the diagonal mapping $x \mapsto (x, \dots, x)$.

$$\begin{tikzcd}
L^k_{\mathrm{sym}}(E;F) \arrow[r, hook] \arrow[d, dotted] & L^k(E;F) \arrow[d, hook] \\
C^\infty(E,F) & C^\infty(E \times \dots \times E, F) \arrow[l, "\Delta^*"]
\end{tikzcd}$$

A bounded linear left inverse $C^\infty(E,F) \to L^k_{\mathrm{sym}}(E;F)$ is given by $f \mapsto \frac{1}{k!} d^k f(0)$, since each $f = \Delta^*(\tilde{f})$ in the image of $\Delta^*|_{L^k_{\mathrm{sym}}(E;F)}$ is k-homogeneous and so $d^k f(0) v^k = \left(\frac{d}{dt}\right)^k f(tv)|_{t=0} = \left((\frac{d}{dt})^k t^k\right)|_{t=0} f(v) = k!\, f(v) = k!\, \tilde{f}(v^k)$ and by the polarization formula (cf. 7.13) $\frac{1}{k!} d^k f(0) = \tilde{f}$.

5.15. Definition. A smooth mapping $f : E \to F$ is called a *polynomial* if some derivative $d^p f$ vanishes on E.
The largest p such that $d^p f \neq 0$ is called the *degree* of the polynomial.

The mapping f is called a *monomial of degree* p if it is of the form $f(x) = \tilde{f}(x, \dots, x)$ for some $\tilde{f} \in L^p_{\mathrm{sym}}(E; F)$.

5.16. Lemma. Polynomials versus monomials.

(1) *The smooth p-homogeneous maps are exactly the monomials of degree p.*

(2) *The symmetric multilinear mapping representing a monomial is unique.*

(3) *A smooth mapping is a polynomial of degree $\le p$ if and only if its restriction to each one dimensional subspace is a polynomial of degree $\le p$.*

(4) *The polynomials are exactly the finite sums of monomials.*

Proof. (1) Every monomial of degree p is clearly smooth and p-homogeneous. Conversely, if f is smooth and p-homogeneous, then

$$(d^p f)(0)(x, \dots, x) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} f(tx) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} t^p f(x) = p! f(x).$$

Thus f is a monomial.

(2) The symmetric multilinear mapping $g \in L^p_{\mathrm{sym}}(E; F)$ representing a monomial f is uniquely determined by the polarization formula 7.13.

(3) & (4) Let the restriction of f to each one dimensional subspace be a polynomial of degree $\le p$, i.e., we have $\ell(f(tx)) = \sum_{k=0}^p \frac{t^k}{k!} (\frac{\partial}{\partial t})^k\big|_{t=0} \ell(f(tx))$ for $x \in E$ and $\ell \in F'$. So $f(x) = \sum_{k=0}^p \frac{1}{k!} d^k f(0 \cdot x)(x, \dots, x)$ and hence is a finite sum of monomials.
For the derivatives of a monomial q of degree k we have $dq(x)(v) = \frac{d}{dt}|_{T=0} \tilde{q}(x + tv, \dots, x + tv) = k\, \tilde{q}(v, x, \dots, x)$ and hence $d^j q(x)(v_1, \dots, v_j) = k(k-1) \dots (k-j+1) \tilde{q}(v_1, \dots, v_j, x, \dots, x)$ for $j \le k$. Hence, any such finite sum is a polynomial in the sense of 5.15.
Finally, any such polynomial has obviously a polynomial as trace on each one dimensional subspace. $\square$

5.17. Lemma. Spaces of polynomials. *The space* $\mathrm{Poly}^p(E, F)$ *of polynomials of degree $\le p$ is isomorphic to $\bigoplus_{k \le p} L(\bigvee^k E; F)$ and is a direct summand in $C^\infty(E, F)$ with a complement given by the smooth functions which are p-flat at 0.*

Proof. By 5.16.3 the mapping $\bigoplus_{k \le p} L(\bigvee^k E; F) \to C^\infty(E, F)$ given on the summands by $L(\bigvee^k E; F) \cong L^k_{\mathrm{sym}}(E, F) \xrightarrow{\Delta^*} C^\infty(E, F)$ has $\mathrm{Poly}^p(E, F)$ as image. A retraction to it is given by $\bigoplus_{k \le p} \frac{1}{k!} d^k|_0$, since $\frac{1}{k!} d^k|_0$ is by 5.9.6 together with 5.13 a retraction to the inclusion of the summand $L(\bigvee^k E; F)$ which is 0 when composed with the inclusion of the summands $L(\bigvee^j E; F)$ for $j \ne k$ by the formula for $q^{(k)}(x)$ given in the proof of 5.16. $\square$

Remark. The corresponding statement is false for infinitely flat functions. E.g. the short exact sequence $E \to C^\infty(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ does not split, where E denotes the space of smooth functions which are *infinitely flat at 0* and where the projection is given by the Taylor-coefficients. Otherwise, $\mathbb{R}^{\mathbb{N}}$ would be a subspace of $C^\infty([0, 1], \mathbb{R})$ (compose the section with the restriction map from $C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty([0, 1], \mathbb{R})$) and hence would have the restriction of the supremum norm as continuous norm.

$$\begin{tikzcd} C^\infty(\mathbb{R},\mathbb{R}) \arrow[r, two heads] & C^\infty([0,1],\mathbb{R}) \arrow[d] \\ \mathbb{R}^{\mathbb{N}} \arrow[u, dotted] \arrow[r, "\mathrm{id}"] & \mathbb{R}^{\mathbb{N}} \end{tikzcd}$$

This is however easily seen to be not the case.

5.14. Remark. Recall that for finite dimensional spaces $E = \mathbb{R}^n$ a polynomial into a (locally convex) vector space F is just a finite sum

$$\sum_{k\in\mathbb{N}^n} a_k x^k,$$

where $a_k \in F$ and $x^k := \prod_{i=1}^n x_i^{k_i}$. Thus, it is just an element in the algebra generated by the coordinate projections pr_i tensorized with F. Since every (continuous) linear functional on $E = \mathbb{R}^n$ is a finite linear combination of coordinate projections, this algebra is also the algebra generated by E'. For a general locally convex space E we define the algebra $P_f(E)$ of *finite type polynomials* to be the subalgebra of $C^\infty(E,\mathbb{R}) \subseteq \mathbb{R}^E$ generated by E'.

This is not in general the algebra of polynomials as defined in 5.15. Take for example the square of the norm $\|\_\|^2 : E \to \mathbb{R}$ on some infinite dimensional Hilbert space E. This is a monomial of degree 2.
But it is not a finite type polynomial. Otherwise, it would be continuous for the weak topology $\sigma(E,E')$. Hence, the unit ball would be a 0-neighborhood for the weak topology, which is not true, since it is compact for it.

Note that for $E = \ell^2$ the space $\bigvee^2 E'$ is not even dense in $(\bigvee^2 E)' = L^2_{\mathrm{sym}}(E,\mathbb{R})$ and hence $P_f(\ell^2)$ is not dense in $\mathrm{Poly}(\ell^2,\mathbb{R})$: Otherwise $f := \langle\_,\_\rangle \in L^2_{\mathrm{sym}}(E,\mathbb{R}) \subseteq L^2(E,\mathbb{R}) \cong L(E,E')$ could be approximated by elements in $\bigvee^2 E' \subseteq \bigotimes^2 E'$. However $\check{f} : \ell^2 \to (\ell^2)' \cong \ell^2$ is the identity and elements in $\bigotimes^2 E'$ correspond to finite dimensional operators, so they approximate only compact operators.

5.10. Lemma. *Let E be a convenient vector space. Then $E' \hookrightarrow P_f(E) := \langle E'\rangle_{alg} \subseteq C^\infty(E,\mathbb{R})$ is the free commutative algebra over the vector space E', i.e. to every linear mapping $f : E' \to A$ into a commmutative algebra, there exists a unique algebra homomorphism $\tilde{f} : P_f(E) \to A$.*

Proof. The solution of this universal problem is given by the symmetric algebra $\bigvee E' := \coprod_{k=0}^\infty \bigvee^k E'$ described in 5.9.9. In particular we have an algebra homomorphism $\tilde{\iota} : \bigvee E' \to P_f(E)$, which is onto, since by definition $P_f(E)$ is generated by E'. It remains to show that it is injective. So let $\sum_{k=1}^N \alpha_k \in \bigvee E'$, i.e. $\alpha_k \in \bigvee^k E'$, with $\tilde{\iota}(\sum_{k=1}^N \alpha_k) = 0$. Thus all derivatives $\tilde{\iota}(\alpha_k)$ at 0 of this mapping in $P_f(E) \subseteq C^\infty(E,\mathbb{R})$ vanish. So it remains to show that $\bigotimes_\beta^k E' \to L(E,\dots,E;\mathbb{R})$ is injective, since then by 5.13 also $\bigvee^k E' \to P_f(E) \subseteq C^\infty(E,\mathbb{R})$ is injective.

$$\begin{tikzcd} \bigvee^k E' \arrow[r, tail] \arrow[d, hook] & L^k_{\mathrm{sym}}(E,\mathbb{R}) \arrow[r, hook, "\Delta^*"] \arrow[d, hook] & C^\infty(E,\mathbb{R}) \\ \bigotimes^k E' \arrow[r, tail] & L^k(E,\mathbb{R}) & \end{tikzcd}$$

We prove by induction that the mapping $E_1' \otimes_\beta \cdots \otimes_\beta E_n' \to L(E_1,\dots,E_n;\mathbb{R})$, $\alpha \mapsto \tilde\alpha$ is injective. For $n=0$ and $n=1$ this is obvious. So let $n \geq 2$ and let $\alpha = \sum_k \alpha_k \otimes x^k$, where $\alpha_k \in E_1' \otimes_\beta \cdots \otimes_\beta E_{n-1}'$ and $x^k \in E_n'$. We may assume that $(x^k)_k$ is linearly independent and hence may choose $x_j \in E_n$ with $x^k(x_j) = \delta_j^k$ and get $0 = \tilde\alpha(y^1,\dots,y^{n-1},x_j) = \tilde\alpha_j(y^1,\dots,y^{n-1})$ for all $y^1,\dots,y^{n-1}$. Hence $\tilde\alpha_j = 0$ and by induction hypothesis $\alpha_j = 0$ for all j and so $\alpha = 0$. $\square$

Note, however, that the injective mapping $\bigvee E' \to C^\infty(E,\mathbb{R})$ is not a bornological embedding in general:
Otherwise also $\bigvee^2 E' \to L^2_{\mathrm{sym}}(E,\mathbb{R})$ would be such an embedding. Take $E = \ell^2$ and consider $\mathcal{B} = \{z_n : n \in \mathbb{N}\} \subseteq \bigvee^2 \ell^2$ where $z_n := \sum_{k=1}^n e_k \otimes e_k$. The bilinear form $\tilde z_n \in L^2_{\mathrm{sym}}(\ell^2,\mathbb{R})$ associated to z_n ist given by $\tilde z_n(x,y) = \sum_{k\leq n} e_k(x)\cdot e_k(y) = \sum_{k\leq n} x^k\, y^k$. Thus the operator norm of $\tilde z_n$ is

$$\|\tilde z_n\| = \sup\Big\{\Big|\sum_{k\leq n} x^k\, y^k\Big| : \|x\|_2 \leq 1, \|y\|_2 \leq 1\Big\} = 1.$$

The projective tensor norm of z_n is

$$\|z_n\|_\pi = \inf\Big\{\sum_k \|a_k\|_2\, \|b_k\|_2 : z_n = \sum_k a_k \otimes b_k\Big\} \geq n,$$

since by Hölders inequality

$$\begin{aligned}\sum_k \|a_k\|_2\, \|b_k\|_2 &\geq \sum_k \|a_k \cdot b_k\|_1 = \sum_{k,j} |a_k^j \cdot b_k^j| \\ &\geq \sum_j \Big|\sum_k a_k^j \cdot b_k^j\Big| = \sum_j \Big|\Big(\sum_k a_k \otimes b_k\Big)^\sim (e_j,e_j)\Big| \\ &= \sum_j |\tilde z_n(e_j,e_j)| = \sum_{j\leq n} 1 = n.\end{aligned}$$

5.18. Theorem. Uniform boundedness principle. *If all E_i are convenient vector spaces, and if F is a locally convex space, then the bornology on the space $L(E_1,\dots,E_n;F)$ consists of all pointwise bounded sets.*

So a mapping into $L(E_1,\dots,E_n;F)$ is smooth if and only if all composites with evaluations at points in $E_1 \times \ldots \times E_n$ are smooth.

Proof. Let us first consider the case $n=1$. So let $\mathcal{B} \subseteq L(E,F)$ be a pointwise bounded subset. By lemma 5.3 we have to show that it is uniformly bounded on each bounded subset B of E. We may assume that B is closed absolutely convex, and thus E_B is a Banach space, since E is convenient. By the classical uniform boundedness principle, see [68, 5.2.2], the set $\mathcal{B}|_{E_B}$ is bounded in $L(E_B,F)$, and thus $\mathcal{B}$ is bounded on B.

The multilinear case follows from the exponential law 5.2 by induction on n: Let $\mathcal{B} \subseteq L(E_1,\dots,E_n;F)$ be pointwise bounded. Then $\mathcal{B}(x_1,\dots,x_{n-1},\_)$ is pointwise bounded in $L(E_n,F)$ for all $x_1,\dots,x_{n-1}$. So by the case $n=1$ it is bounded in the locally convex space $L(E_n,F)$ and by induction hypothesis $\check{\mathcal{B}}$ is bounded in $L(E_1,\dots,E_{n-1};L(E_n,F))$. By 5.2 $\mathcal{B}$ is bounded.

The subspace $L(E_1,\dots,E_n;F) \subseteq \prod_{E_1\times\ldots\times E_n} F$ is Mackey-closed: Let T_n converge to $T_\infty := (T_{(x_1,\dots,x_n)})_{(x_1,\dots,x_n)\in E_1\times\ldots\times E_n}$ Mackey in $\prod_{E_1\times\ldots\times E_n} F$. Obviously, T_∞ is n-linear, since T_n converges to T_∞ pointwise. Moreover $\{T_n - T_\infty : n \in \mathbb{N}\}$ is pointwise bounded and thus $\{T_n - T_m : n,m \in \mathbb{N}\}$ is bounded in $L(E_1,\dots,E_n;F)$.

Taking the pointwise limit for $m \to \infty$ shows that $T_n - T_\infty$ is uniformly bounded on bounded sets, and hence $T_\infty \in L(E_1, \dots, E_n; F)$.

From this the smoothness detection principle follows, since it clearly suffices to consider curves. $\square$

5.19. Theorem. Multilinear mappings on convenient vector spaces. *A multilinear mapping from convenient vector spaces to a locally convex space is bounded if and only if it is separately bounded.*

Proof. Let $f : E_1 \times \dots \times E_n \to F$ be n-linear and separately bounded, i.e. $x_i \mapsto f(x_1, \dots, x_n)$ is bounded for each i and all fixed x_j for $j \neq i$. Then $f^\vee : E_1 \times \dots \times E_{n-1} \to L(E_n, F)$ is $(n-1)$-linear. By 5.18 the bornology on $L(E_n, F)$ consists of the pointwise bounded sets, so $f^\vee$ is separately bounded. By induction on n it is bounded. The bornology on $L(E_n, F)$ consists also of the subsets which are uniformly bounded on bounded sets by lemma 5.3, so f is bounded. $\square$

We will now derive an infinite dimensional version of 3.4, which gives us minimal requirements for a mapping to be smooth.

5.20. Theorem. *Let E be a convenient vector space. An arbitrary mapping $f : E \supseteq U \to F$ is smooth if and only if all unidirectional iterated derivatives $d_v^p f(x) = (\frac{\partial}{\partial t})^p|_0 f(x + tv)$ exist, $x \mapsto d_v^p f(x)$ is bounded on sequences which are Mackey converging in U, and $v \mapsto d_v^p f(x)$ is bounded on fast falling sequences.*

Proof. A smooth mapping obviously satisfies this requirement. Conversely, from 3.4 we see that f is smooth restricted to each finite dimensional subspace, and the iterated directional derivatives $d_{v_1} \dots d_{v_n} f(x)$ exist and are bounded multilinear mappings in $v_1, \dots, v_n$ by 5.4, since they are universal linear combinations of the unidirectional iterated derivatives $d_v^p f(x)$ for $v = \sum_{i \leq n} \varepsilon_i v_i$ with $\varepsilon_i \in \{0, 1\}$ by the polarization formula 7.13. So $d^n f : U \to L^n(E; F)$ is bounded on Mackey converging sequences with respect to the pointwise bornology on $L^n(E; F)$. By the uniform boundedness principle 5.18 together with lemma 4.14 the mapping $d^n f : U \times E^n \to F$ is bounded on sets which are contained in a product of a BORNOLOGICALLY COMPACT SET in U - i.e. a set in U which is contained and compact in some E_B - and a bounded set in E^n.

Now let $c : \mathbb{R} \to U$ be a smooth curve. We have to show that $\frac{f(c(t)) - f(c(0))}{t}$ converges to $f'(c(0))(c'(0))$. It suffices to check that

$$\frac{1}{t}\left(\frac{f(c(t)) - f(c(0))}{t} - f'(c(0))(c'(0))\right)$$

is locally bounded with respect to t. Integrating along the segment from $c(0)$ to $c(t)$ we see that this expression equals

$$\frac{1}{t}\int_0^1 \left(f'\Big(c(0) + s(c(t) - c(0))\Big)\left(\frac{c(t) - c(0)}{t}\right) - f'(c(0))(c'(0))\right) ds =$$
$$= \int_0^1 f'\Big(c(0) + s(c(t) - c(0))\Big)\left(\frac{\frac{c(t)-c(0)}{t} - c'(0)}{t}\right) ds$$
$$+ \int_0^1 \int_0^1 f''\Big(c(0) + rs(c(t) - c(0))\Big)\left(s\frac{c(t) - c(0)}{t}, c'(0)\right) dr\, ds.$$

The first integral is bounded since $df : U \times E \to F$ is bounded on the product of the bornologically compact set $\{c(0) + s(c(t) - c(0)) : 0 \le s \le 1, t \text{ near } 0\}$ in U and the bounded set $\left\{\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right) : t \text{ near } 0\right\}$ in E (use 1.6).

The second integral is bounded since $d^2 f : U \times E^2 \to F$ is bounded on the product of the bornologically compact set $\{c(0) + rs(c(t) - c(0)) : 0 \le r, s \le 1, t \text{ near } 0\}$ in U and the bounded set $\left\{\left(s\frac{c(t)-c(0)}{t}, c'(0)\right) : 0 \le s \le 1, t \text{ near } 0\right\}$ in E^2.

Thus $f \circ c$ is differentiable in F with derivative $df \circ (c, c')$. Since $df((x, v) + t(y, w)) = df(x + ty, v) + t\, df(x + ty, w)$ the mapping $df : U \times E \to F$ satisfies again the conditions of the last part of the proof, so we may iterate. □

5.21. The following result shows that bounded multilinear mappings are the right ones for uses like homological algebra, where multilinear algebra is essential and where one wants a kind of 'continuity'. With continuity itself it does not work. The same results hold for convenient algebras and modules, one just may take c^∞-completions of the tensor products.

So by a *bounded algebra* A we mean a (real or complex) algebra which is also a locally convex vector space, such that the multiplication is a bounded bilinear mapping. Likewise, we consider *bounded modules* over bounded algebras, where the action is bounded bilinear.

Lemma. [Cap, Kriegl, Michor, Vanžura, 1993]. *Let A be a bounded algebra, M a bounded right A-module and N a bounded left A-module.*

(1) *There are a locally convex vector space $M \otimes_A N$ and a bounded bilinear map $b : M \times N \to M \otimes_A N$, $(m, n) \mapsto m \otimes_A n$ such that $b(ma, n) = b(m, an)$ for all $a \in A$, $m \in M$ and $n \in N$ which has the following universal property: If E is a locally convex vector space and $f : M \times N \to E$ is a bounded bilinear map such that $f(ma, n) = f(m, an)$ then there is a unique bounded linear map $\tilde f : M \otimes_A N \to E$ with $\tilde f \circ b = f$. The space of all such f is denoted by $L^A(M, N; E)$, a closed linear subspace of $L(M, N; E)$.*

(2) *We have a bornological isomorphism*
$$L^A(M, N; E) \cong L(M \otimes_A N, E).$$

(3) *Let B be another bounded algebra such that N is a bounded right B-module and such that the actions of A and B on N commute. Then $M \otimes_A N$ is in a canonical way a bounded right B-module.*

(4) *If in addition P is a bounded left B-module then there is a natural bornological isomorphism $M \otimes_A (N \otimes_B P) \cong (M \otimes_A N) \otimes_B P$.*

Proof. We construct $M \otimes_A N$ as follows: Let $M \otimes_\beta N$ be the algebraic tensor product of M and N equipped with the (bornological) topology mentioned in 5.7 and let V be the locally convex closure of the subspace generated by all elements of the form $ma \otimes n - m \otimes an$, and define $M \otimes_A N$ to be $M \otimes_A N := (M \otimes_\beta N)/V$. As $M \otimes_\beta N$ has the universal property that bounded bilinear maps from $M \times N$ into arbitrary locally convex spaces induce bounded and hence continuous linear maps on $M \otimes N$, (1) is clear.

(2) By (1) the bounded linear map $b^* : L(M \otimes_A N, E) \to L^A(M, N; E)$ is a bijection. Thus, it suffices to show that its inverse is bounded, too. From 5.7 we get a bounded linear map $\varphi : L(M, N; E) \to L(M \otimes_\beta N, E)$ which is inverse to the

map induced by the canonical bilinear map. Now let $L^{\mathrm{ann}(V)}(M \otimes_\beta N, E)$ be the closed linear subspace of $L(M \otimes_\beta N, E)$ consisting of all maps which annihilate V. Restricting φ to $L^A(M, N; E)$ we get a bounded linear map $\varphi : L^A(M, N; E) \to L^{\mathrm{ann}(V)}(M \otimes_\beta N, E)$.

Let $\psi : M \otimes_\beta N \to M \otimes_A N$ be the the canonical projection. Then ψ induces a well defined linear map $\hat{\psi} : L^{\mathrm{ann}(V)}(M \otimes_\beta N, E) \to L(M \otimes_A N, E)$, and $\hat{\psi} \circ \varphi$ is inverse to b^*. So it suffices to show that $\hat{\psi}$ is bounded.

This is the case if and only if the associated map $L^{\mathrm{ann}(V)}(M \otimes_\beta N, E) \times (M \otimes_A N) \to E$ is bounded. This in turn is equivalent to boundedness of the associated map $M \otimes_A N \to L(L^{\mathrm{ann}(V)}(M \otimes_\beta N, E), E)$ which sends x to the evaluation at x and is clearly bounded.

(3) Let $\rho : B^{op} \to L(N, N)$ be the right action of B on N and let $\Phi : L^A(M, N; M \otimes_A N) \cong L(M \otimes_A N, M \otimes_A N)$ be the isomorphism constructed in (2). We define the right module structure on $M \otimes_A N$ as:

$$\begin{aligned} B^{op} \xrightarrow{\rho} L(N, N) \xrightarrow{\mathrm{Id} \times \_} L(M \times N, M \times N) \xrightarrow{b_*} \\ \longrightarrow L^A(M, N; M \otimes_A N) \xrightarrow{\Phi} L(M \otimes_A N, M \otimes_A N). \end{aligned}$$

This map is obviously bounded and easily seen to be an algebra homomorphism.

(4) Straightforward computations show that both spaces have the following universal property: For a locally convex vector space E and a trilinear map $f : M \times N \times P \to E$ which satisfies $f(ma, n, p) = f(m, an, p)$ and $f(m, nb, p) = f(m, n, bp)$ there is a unique linear map prolonging f. $\square$

5.22. Lemma. Uniform $\mathcal{S}$-boundedness principle. *Let E be a locally convex space, and let $\mathcal{S}$ be a point separating set of bounded linear mappings with common domain E. Then the following conditions are equivalent.*

1. *If F is a Banach space (or even a c^∞-complete locally convex space) and $f : F \to E$ is a linear mapping with $\lambda \circ f$ bounded for all $\lambda \in \mathcal{S}$, then f is bounded.*
2. *If $B \subseteq E$ is absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space E_B generated by B is complete, then B is bounded in E.*
3. *Let (b_n) be an unbounded sequence in E with $\lambda(b_n)$ bounded for all $\lambda \in \mathcal{S}$, then there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the initial locally convex topology induced by $\mathcal{S}$.*

Definition. We say that E satisfies the *uniform $\mathcal{S}$-boundedness principle* if these equivalent conditions are satisfied.

Proof. (1) $\Rightarrow$ (3) : Suppose that (3) is not satisfied. So let (b_n) be an unbounded sequence in E such that $\lambda(b_n)$ is bounded for all $\lambda \in \mathcal{S}$, and such that for all $(t_n) \in \ell^1$ the series $\sum t_n b_n$ converges in E for the initial locally convex topology induced by $\mathcal{S}$. We define a linear mapping $f : \ell^1 \to E$ by $f((t_n)_n) = \sum t_n b_n$, i.e. $f(e_n) = b_n$. It is easily checked that $\lambda \circ f$ is bounded, hence by (1) the image of the closed unit ball, which contains all b_n, is bounded. Contradiction.

(3) $\Rightarrow$ (2): Let $B \subseteq E$ be absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and that the normed space E_B generated by B is complete. Suppose that B is unbounded. Then B contains an unbounded sequence (b_n), so by (3) there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the initial locally convex

topology induced by $\mathcal{S}$. But $\sum t_n\, b_n$ is a Cauchy sequence in E_B, since $\sum_{k=n}^m t_n b_n \in (\sum_{k=n}^m |t_n|)\cdot B$, and thus converges even bornologically, a contradiction.

(2) $\Rightarrow$ (1): Let F be convenient, and let $f : F \to E$ be linear such that $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$. It suffices to show that $f(B)$, the image of an absolutely convex bounded set B in F with F_B complete, is bounded. By assumption, $\lambda(f(B))$ is bounded for all $\lambda \in \mathcal{S}$ and $f : F_B \to E_{f(B)}$ is a quotient mapping of normed spaces:

$$\begin{aligned}\widetilde{q_B}(y) &= \inf\{q_B(x) : y = f(x)\} = \inf\{\lambda : y = f(x),\, x \in \lambda\, B\}\\ &= \inf\{\lambda : y \in \lambda\, f(B)\} = q_{f(B)}(y).\end{aligned}$$

Since F_B is complete, so is $E_{f(B)}$ and by (2) the set $f(B)$ is bounded. □

5.23. Lemma. *A convenient vector space E satisfies the uniform $\mathcal{S}$-boundedness principle for each point separating set $\mathcal{S}$ of bounded linear mappings on E if and only if there exists no strictly weaker ultrabornological topology than the bornological topology of E.*

Proof. ($\Rightarrow$) Let τ be an ultrabornological topology on E which is weaker than the natural bornological topology. Consider $\mathcal{S} := \{\mathrm{Id} : E \to (E,\tau)\}$ and the identity $(E,\tau) \to E$. Since every ultra-bornological space is an inductive limit of Banach spaces, cf. [**75**, 52.31], it is enough to show that for each of these Banach spaces F the mapping $F \to (E,\tau) \to E$ is continous. By 5.22.1 this is the case.

($\Leftarrow$) If $\mathcal{S}$ is a point separating set of bounded linear mappings, the ultrabornological topology given by the inductive limit of the spaces E_B with B satisfying the assumptions of 5.22.2 equals the natural bornological topology of E. Hence, 5.22.2 is satisfied. □

5.24. Theorem. Webbed spaces have the uniform boundedness property. *A locally convex space which is webbed satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear mappings.*

Proof. Since the bornologification of a webbed space is webbed, cf. [**68**, 5.3.3], we may assume that E is bornological, and hence that every bounded linear mapping on it is continuous, see 4.1.1. Now the closed graph principle [**68**, 5.3.3] applies to any mapping satisfying the assumptions of 5.22.1. □

5.25. Lemma. Stability of the uniform boundedness principle. *Let $\mathcal{F}$ be a set of bounded linear mappings $f : E \to E_f$ between locally convex spaces, let $\mathcal{S}_f$ be a point separating set of bounded linear mappings on E_f for every $f \in \mathcal{F}$, and let $\mathcal{S} := \bigcup_{f\in\mathcal{F}} f^*(\mathcal{S}_f) = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{S}_f\}$. If $\mathcal{F}$ generates the bornology and E_f satisfies the uniform $\mathcal{S}_f$-boundedness principle for all $f \in \mathcal{F}$, then E satisfies the uniform $\mathcal{S}$-boundedness principle.*

Proof. We check the condition 5.22.1. So assume $h : F \to E$ is a linear mapping for which $g \circ f \circ h$ is bounded for all $f \in \mathcal{F}$ and $g \in \mathcal{S}_f$. Then $f \circ h$ is bounded by the uniform $\mathcal{S}_f$-boundedness principle for E_f. Consequently, h is bounded since $\mathcal{F}$ generates the bornology of E. □

5.26. Theorem. Smooth uniform boundedness principle. *Let E and F be convenient vector spaces, and let U be c^∞-open in E. Then $C^\infty(U,F)$ satisfies the uniform $\mathcal{S}$-boundedness principle where $\mathcal{S} := \{\mathrm{ev}_x : x \in U\}$.*

Proof. For $U = E = F = \mathbb{R}$ this follows from 5.24, since $C^\infty(\mathbb{R},\mathbb{R})$ is a Fréchet space hence webbed. The general case then follows from 5.25. □

41. Jets and Whitney Topologies

Jet spaces or jet bundles consist of the invariant expressions of Taylor developments up to a certain order of smooth mappings between manifolds. Their invention goes back to [Ehresmann, 1951.]

41.1. Jets between convenient vector spaces

Let E and F be convenient vector spaces, and let $U \subseteq E$ and $V \subseteq F$ be c^∞-open subsets. For $0 \le k \le \infty$ the space of *k-jets* from U to V is defined by

$$J^k(U,V) := U \times V \times \mathrm{Poly}_0^k(E,F), \text{ where } \mathrm{Poly}_0^k(E,F) = \prod_{j=1}^k L^j_{\mathrm{sym}}(E;F).$$

We shall use the source and image projections $\alpha : J^k(U,V) \to U$ and $\beta : J^k(U,V) \to V$, and we shall consider $J^k(U,V) \to U \times V$ as a trivial bundle, with fibers $J^k_x(U,V)_y := \mathrm{Poly}_0^k(E,F)$ for $(x,y) \in U \times V$. Moreover, we have obvious projections $\pi^k_l : J^k(U,V) \to J^l(U,V)$ for $k > l$, given by truncation at order l. For a smooth mapping $f : U \to V$ the *k-jet extension* is defined by

$$j^k f(x) = j^k_x f := \Big(x, f(x), df(x), \frac{1}{2!}d^2 f(x), \dots, \frac{1}{j!}d^j f(x), \dots\Big),$$

the Taylor expansion of f at x of order k. If $k < \infty$ then $j^k_x : C^\infty(U,F) \to J^k(U,F)$ is smooth with a smooth right inverse $(x, p_0, \dots, p_k) \mapsto \big(u \mapsto \sum_{j\le k} p_j(u-x)^k\big)$, see 5.17. If $k = \infty$ then j^k need not be surjective for infinite dimensional E, see 15.4. For later use, we consider now the *(truncated) composition*

$$\bullet : \mathrm{Poly}_0^k(F,G) \times \mathrm{Poly}_0^k(E,F) \to \mathrm{Poly}_0^k(E,G),$$

where $p \bullet q$ is the composition $p \circ q$ of the polynomials p, q (formal power series in case $k = \infty$) without constant terms, and without all terms of order $> k$. Obviously, $\bullet$ is polynomial of degree $k+1$ for finite k and is real analytic for $k = \infty$ since then each component is polynomial. Now let $U \subset E$, $V \subset F$, and $W \subset G$ be open subsets, and consider the fibered product

$$\begin{aligned} J^k(U,V) \times_U J^k(W,U) &= \{\, (\sigma,\tau) \in J^k(U,V) \times J^k(W,U) : \alpha(\sigma) = \beta(\tau) \,\} \\ &= U \times V \times W \times \mathrm{Poly}_0^k(E,F) \times \mathrm{Poly}_0^k(G,E). \end{aligned}$$

Then the mapping

$$\bullet : J^k(U,V) \times_U J^k(W,U) \to J^k(W,V),$$
$$\sigma \bullet \tau = (\alpha(\sigma), \beta(\sigma), \bar\sigma) \bullet (\alpha(\tau), \beta(\tau), \bar\tau) := (\alpha(\tau), \beta(\sigma), \bar\sigma \bullet \bar\tau),$$

is a real analytic mapping, called the *fibered composition of jets.*

Let U, $W \subset E$ and $V \subset F$ be open subsets, and let $g : W \to U$ be a smooth diffeomorphism. We define a mapping $g^* := J^k(g,V) : J^k(U,V) \to J^k(W,V)$ by

$J^k(g,V)(\sigma) = \sigma \bullet j^k_{g^{-1}(\alpha(\sigma))}g$, which also satisfies $J^k(g,V)(j^k_x f) = j^k_{g^{-1}(\alpha(\sigma))}(f \circ g)$. If $g' : W' \to W$ is another diffeomorphism, then clearly $J^k(g',V) \circ J^k(g,V) = J^k(g \circ g', V)$, and $J^k(\_,V)$ is a contravariant functor acting on diffeomorphisms between open subsets of E. Since the truncated composition $\bar\sigma \mapsto \bar\sigma \bullet j^k_{g^{-1}(x)}g$ is linear, the mapping $J^k_x(g,F) := J^k(g,F)|_{J^k_x(U,F)} : J^k_x(U,F) \to J^k_{g^{-1}(x)}(W,F)$ is also linear.

Now let $W \subset E$, $U \subset F$, and $V \subset G$ be c^∞-open subsets, and let $h : U \to V$ be a smooth mapping. Then we define $h_* := J^k(W,h) : J^k(W,U) \to J^k(W,V)$ by $J^k(W,h)\sigma = j^k_{\beta(\sigma)}h \bullet \sigma$, which satisfies $J^k(W,h)(j^k_x f) = j^k_x(h \circ f)$. Clearly, $J^k(W,\_)$ is a covariant functor acting on smooth mappings between c^∞-open subsets of convenient vector spaces. The mapping $J^k_x(W,h)_y : J^k_x(W,U)_y \to J^k_x(W,V)_{h(y)}$ is linear if and only if h is affine or $k = 1$ or one of the spaces E, F, and G is 0.

41.3. Jets between manifolds

Now let M and N be smooth manifolds with smooth atlas (U_α, u_α) and (V_β, v_β), modeled on convenient vector spaces E and F, respectively. Then we may glue the open subsets $J^k(u_\alpha(U_\alpha), v_\beta(V_\beta))$ of convenient vector spaces via the chart change mappings

$$\begin{aligned} J^k(u_{\alpha'} \circ u_\alpha^{-1}, v_\beta \circ v_{\beta'}^{-1}) : J^k(u_{\alpha'}(U_\alpha \cap U_{\alpha'}), v_{\beta'}(V_\beta \cap V_{\beta'})) \to \\ \to J^k(u_\alpha(U_\alpha \cap U_{\alpha'}), v_\beta(V_\beta \cap V_{\beta'})), \end{aligned}$$

and we obtain a smooth fiber bundle $J^k(M,N) \to M \times N$ with standard fiber $\mathrm{Poly}^k_0(E,F)$. With the identification topology $J^k(M,N)$ is Hausdorff, since it is a fiber bundle and the usual argument for gluing fiber bundles applies which was given, e.g., in 28.12.

Theorem. *If M and N are smooth manifolds, modeled on convenient vector spaces E and F, respectively. Let $0 \le k \le \infty$. Then the following results hold.*

(1) *$(J^k(M,N), (\alpha,\beta), M \times N, \mathrm{Poly}^k_0(E,F))$ is a fiber bundle with standard fiber $\mathrm{Poly}^k_0(E,F)$, with the smooth group $GL^k(E) \times GL^k(F)$ as structure group, where $(\gamma,\chi) \in GL^k(E) \times GL^k(F)$ acts on $\sigma \in \mathrm{Poly}^k_0(E,F)$ by $(\gamma,\chi).\sigma = \chi \bullet \sigma \bullet \gamma^{-1}$.*

(2) *If $f : M \to N$ is a smooth mapping then $j^k f : M \to J^k(M,N)$ is also smooth, called the k-jet extension of f. We have $\alpha \circ j^k f = \mathrm{Id}_M$ and $\beta \circ j^k f = f$.*

(3) *If $g : M' \to M$ is a diffeomorphism then also the induced mapping $J^k(g,N) : J^k(M,N) \to J^k(M',N)$ is a diffeomorphism.*

(4) *If $h : N \to N'$ is a smooth mapping then $J^k(M,h) : J^k(M,N) \to J^k(M,N')$ is also smooth. Thus, $J^k(M,\_)$ is a covariant functor from the category of smooth manifolds and smooth mappings into itself which respects each of the following classes of mappings: initial mappings, embeddings, closed embeddings, splitting embeddings, fiber bundle projections. Furthermore, $J^k(\_,\_)$ is a contra-covariant bifunctor, where we have to restrict in the first variable to the category of diffeomorphisms.*

(5) *For $k \ge l$, the projections $\pi^k_l : J^k(M,N) \to J^l(M,N)$ are smooth and natural, i.e., they commute with the mappings from (3) and (4).*

(6) $(J^k(M,N), \pi^k_l, J^l(M,N), \prod_{i=l+1}^k L^i_{sym}(E;F))$ *are fiber bundles for all* $l \le k$*. For finite* k *the bundle* $(J^k(M,N), \pi^k_{k-1}, J^{k-1}(M,N), L^k_{sym}(E,F))$ *is an affine bundle. The first jet space* $J^1(M,N) \to M \times N$ *is a vector bundle. It is isomorphic to the bundle* $(L(TM,TN), (\pi_M, \pi_N), M \times N)$*, see* [**75**, 29.4] *and* [**75**, 29.5]*. Moreover, we have* $J^1_0(\mathbb{R}, N) = TN$ *and* $J^1(M,\mathbb{R})_0 = T^*M$.

(7) *Truncated composition is a smooth mapping*

$$\bullet : J^k(N,P) \times_N J^k(M,N) \to J^k(M,P).$$

Proof. (1) is already proved. (2), (3), (5), and (7) are obvious from 41.1, mainly by the functorial properties of $J^k(\_,\_)$.

(4) It is clear from 41.1 that $J^k(M,h)$ is a smooth mapping. The rest follows by looking at special chart representations of h and the induced chart representations for $J^k(M,h)$.

It remains to show (6), and here we concentrate on the affine bundle. Let $a_1 + a \in GL(E) \times \prod_{i=2}^k L^i_{\text{sym}}(F;F)$, $\sigma + \sigma_k \in \text{Poly}^{k-1}_0(E,F) \times L^k_{\text{sym}}(E;F)$, and $b_1 + b \in GL(E) \times \prod_{i=2}^k L^i_{\text{sym}}(E;E)$, then the only term of degree k containing σ_k in $(a_1 + a) \bullet (\sigma + \sigma_k) \bullet (b_1 + b)$ is $a_1 \circ \sigma_k \circ b_1^k$, which depends linearly on σ_k. To this the degree k-components of compositions of the lower order terms of σ with the higher order terms of a and b are added, and these may be quite arbitrary. So an affine bundle results.

We have $J^1(M,N) = L(TM,TN)$ since both bundles have the same transition functions. Finally,

$$J^1_0(\mathbb{R}, N) = L(T_0\mathbb{R}, TN) = TN \quad \text{and} \quad J^1(M,\mathbb{R})_0 = L(TM, T_0\mathbb{R}) = T^*M. \qquad \square$$

41.4. Jets of sections of fiber bundles

If $(p : E \to M, S)$ is a fiber bundle, let (U_α, u_α) be a smooth atlas of M such that $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times S)$ is a fiber bundle atlas. If we glue the smooth manifolds $J^k(U_\alpha, S)$ via $(\sigma \mapsto j^k(\psi_{\alpha\beta}(\alpha(\sigma), \_))) \bullet \sigma : J^k(U_\alpha \cap U_\beta, S) \to J^k(U_\alpha \cap U_\beta, S)$, we obtain the smooth manifold $J^k(E)$, which for finite k is the space of all k-jets of local sections of E.

Theorem. *In this situation we have:*

(1) $J^k(E)$ *is a splitting closed submanifold of* $J^k(M,E)$*, namely the set of all* $\sigma \in J^k_x(M,E)$ *with* $J^k(M,p)(\sigma) = j^k(\text{Id}_M)(x)$.

(2) $J^1(E)$ *of sections is an affine subbundle of the vector bundle* $J^1(M,E) = L(TM,TE)$*. In fact, we have*

$$J^1(E) = \{\, \sigma \in L(TM,TE) : Tp \circ \sigma = \text{Id}_{TM} \,\}.$$

(3) *For* k *finite* $(J^k(E), \pi^k_{k-1}, J^{k-1}(E))$ *is an affine bundle.*

(4) *If* $p : E \to M$ *is a vector bundle, then* $(J^k(E), \alpha, M)$ *is also a vector bundle. If* $\phi : E \to E'$ *is a homomorphism of vector bundles covering the identity, then* $J^k(\varphi)$ *is of the same kind.*

Proof. Locally $J^k(E)$ in $J^k(M,E)$ looks like $u_\alpha(U_\alpha) \times \text{Poly}^k_0(F_M, F_S)$ in $u_\alpha(U_\alpha) \times (u_\alpha(U_\alpha) \times v_\beta(V_\beta)) \times \text{Poly}^k_0(F_M, F_M \times F_S)$, where F_M and F_S are modeling spaces of M and S, respectively, and where (V_β, v_β) is a smooth atlas for S. The rest is clear. $\square$

6. Some Spaces of Smooth Functions

6.1. Proposition. *Let M be a smooth finite dimensional paracompact manifold. Then the space $C^\infty(M,\mathbb{R})$ of all smooth functions on M is a convenient vector space in any of the following (bornologically) isomorphic descriptions, and it satisfies the uniform boundedness principle for the point evaluations.*

(1) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R},\mathbb{R})$$
for all $c \in C^\infty(\mathbb{R}, M)$.

(2) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{(u_\alpha^{-1})^*} C^\infty(\mathbb{R}^n,\mathbb{R}),$$
where (U_α, u_α) is a smooth atlas with $u_\alpha(U_\alpha) = \mathbb{R}^n$.

(3) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{j^k} C(M \leftarrow J^k(M,\mathbb{R}))$$
for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M, where j^k is the jet prolongation, and where all the spaces of continuous sections are equipped with the compact open topology.

It is easy to see that the cones in (2) and (3) induce even the same locally convex topology which is sometimes called the *compact C^∞ topology*, if $C^\infty(\mathbb{R}^n,\mathbb{R})$ is equipped with its usual Fréchet topology. From (2) we see also that with the bornological topology $C^\infty(M,\mathbb{R})$ is nuclear by [**75**, 52.35], and is a Fréchet space if and only if M is separable.

Proof. For all three descriptions the initial locally convex topology is convenient, since the spaces are closed linear subspaces in the relevant products of the right hand sides:
(1) For this structure $C^\infty(M,\mathbb{R}) = \varprojlim_{c\in C^\infty(\mathbb{R},M)} C^\infty(\mathbb{R},\mathbb{R})$, where the connecting mappings are given by g^* for $g \in C^\infty(\mathbb{R},\mathbb{R})$. Obviously, $(c^*)_{c\in C^\infty(\mathbb{R},M)}$ has values in this inductive limit and induces the structure of (1) on $C^\infty(M,\mathbb{R})$. This mapping is bijective, since to $(f_c)_{c\in C^\infty(\mathbb{R},\mathbb{R})} \in \varprojlim_c C^\infty(\mathbb{R},\mathbb{R})$ we can associate $f : M \to \mathbb{R}$ given by $f(x) = f_{\mathrm{const}_x}(0)$. Then $c^*(f) = f_c$, since $\mathrm{const}_{c(t)} = c \circ \mathrm{const}_t$. Moreover $\mathrm{const}_x^*(f) = \mathrm{const}_{f(x)}$, so we found the inverse.
(2) For this structure $C^\infty(M,\mathbb{R}) = \varprojlim_u C^\infty(\mathbb{R}^n,\mathbb{R})$, where u run through all smooth open embeddings $\mathbb{R}^n \to M$ and where the connecting mappings are given by g^* for smooth embeddings $g \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Obviously, $(u^*)_u$ has values in this inductive limit and induces the structure of (2) on $C^\infty(M,\mathbb{R})$, since locally such u coincide with some $(u_\alpha)^{-1}$ and $C^\infty(\mathbb{R}^n,\mathbb{R})$ carries the initial structure with resperct to $\mathrm{incl}_V^* : C^\infty(\mathbb{R}^n,\mathbb{R}) \to C^\infty(V,\mathbb{R})$, where the V form some open covering of $\mathbb{R}^n$ This mapping $(u^*)_u$ is bijective, since to $(f_u)_u \in \varprojlim_u C^\infty(\mathbb{R}^n,\mathbb{R})$ we can associate $f : M \to \mathbb{R}$ given by $f(x) = f_u(t)$, where $u : \mathbb{R}^m \to M$ is some smooth open embedding with $u(t) = x$. This definition does not depend on the choice of (u,t) since two such embeddings can be locally reparametrized into each another.

As before this gives the required invese.

(3) First note for vector bundles $p : E \to M$ the compact open topology turns $C(M \leftarrow E)$ into a locally convex space. In fact for a neighborhood subbasis of this topology it is enough to consider the convex sets $N_{K,U} := \{\sigma \in C(M \leftarrow E) : \sigma(K) \subseteq U\}$ for compact subsets K contained in trivializing open subsets V of the basis and open sets $U \subseteq E$ of the form $\psi^{-1}(V \times W)$, where $\psi : p^{-1}(V) \to V \times \mathbb{R}^k$ is the trivialization and $W \subseteq \mathbb{R}^k$ is open and convex in the typical fiber. This shows also, that the topology is the initial one induced by the restriction maps $\operatorname{incl}_K^* : C(M \leftarrow E) \to C(K \leftarrow E|_K) \cong C(K, \mathbb{R}^k) \subseteq \ell^\infty(K, \mathbb{R}^k)$. So it is enough to show closednes of the image of $C^\infty(M, \mathbb{R}) \to \prod_{k,K} C(K, \prod_{j=0}^k L^j_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^k))$ where the K are assumed to be compact in some chart domain in M. This is clearly the case.

Thus, the uniform boundedness principle for the point evaluations holds for all structures since it holds for all right hand sides (for $C(M \leftarrow J^k(M, \mathbb{R}))$ we may reduce to a connected component of M, and we then have a Fréchet space). So the identity is bibounded between all structures. $\square$

6.2. Spaces of smooth functions with compact supports

For a smooth finite dimensional Lindelöf (equivalently, separable metrizable) Hausdorff manifold M we denote by $C_c^\infty(M, \mathbb{R})$ the vector space of all smooth functions with compact supports in M.

Corollary. *The following convenient structures on the space $C_c^\infty(M, \mathbb{R})$ are all isomorphic:*

(1) *Let $C_K^\infty(M, \mathbb{R})$ be the space of all smooth functions on M with supports contained in the fixed compact subset $K \subseteq M$, a closed linear subspace of $C^\infty(M, \mathbb{R})$. Let us consider the final convenient vector space structure on the space $C_c^\infty(M, \mathbb{R})$ induced by the cone*
$$C_K^\infty(M, \mathbb{R}) \hookrightarrow C_c^\infty(M, \mathbb{R})$$
where K runs through a basis for the compact subsets of M. Then the space $C_c^\infty(M, \mathbb{R})$ is even the strict inductive limit of a sequence of Fréchet spaces $C_K^\infty(M, \mathbb{R})$.

(2) *We equip $C_c^\infty(M, \mathbb{R})$ with the initial structure with respect to the inclusion $C_c^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ and the cone*
$$C_c^\infty(M, \mathbb{R}) \xrightarrow{x^*} \ell_c^\infty(\mathbb{N}, \mathbb{R}) = \coprod_{n \in \mathbb{N}} \mathbb{R}^n = \mathbb{R}^{(\mathbb{N})},$$
where $x = (x_n)_n$ runs through all sequences in M without accumulation point.

(3) *The initial structure with respect to the cone*
$$C_c^\infty(M, \mathbb{R}) \xrightarrow{j^k} \ell_c^\infty(M \leftarrow J^k(M, \mathbb{R}))$$
for all $k \in \mathbb{N}$, where $J^k(M, \mathbb{R})$ is the bundle of k-jets of smooth functions on M, where j^k is the jet prolongation, and where the spaces of continuous sections with compact support are equipped with the inductive limit topology with steps $C_K(M \leftarrow J^k(M, \mathbb{R})) \subseteq C(M \leftarrow J^k(M, \mathbb{R}))$.

For M with only finitely many connected components which are all non-compact, this is also true for

(4) *the convenient vector space structure induced by* $c^* : C^\infty_c(M, \mathbb{R}) \to C^\infty_c(\mathbb{R}, \mathbb{R})$, *where* $c : \mathbb{R} \to M$ *run through the proper smooth curves.*

The space $C^\infty_c(M, \mathbb{R})$ *satisfies the uniform boundedness principle for the point evaluations.*

First Proof. We show that in all four descriptions the space $C^\infty_c(M, \mathbb{R})$ is convenient and satisfies the uniform boundedness principle for point evaluations, hence the identity is bibounded for all structures:

In (1) we may assume that the basis of compact subsets of M is countable, since M is Lindelöf, hence has only countable many connected components and these are metrizable, so the inductive limit is a strict inductive limit of a sequence of Fréchet spaces, hence $C^\infty_c(M, \mathbb{R})$ is convenient and webbed by [**68**, 5.3.3] and [**68**, 5.3.3] and satisfies the uniform boundedness principle by 5.24.

In (2)–(4) the space is a closed subspace of the product of $C^\infty(M, \mathbb{R})$ and spaces on the right hand side which are strict inductive limits of Fréchet spaces, hence convenient and satisfy the uniform boundedness principle:
In (2) closedness follows, since for smoothness of $f : M \to \mathbb{R}$ follows from the inclusion into $C^\infty(M, \mathbb{R})$, and compactness of the support follows because this can be tested along sequences without accumulation point.
In (3) closedness follows, since $C^\infty(M, \mathbb{R})$ is closed in $\prod_k C(M \leftarrow J^k(M, \mathbb{R}))$ by the proof of 6.1 and the support is that of $f = f^0 \in \ell^\infty_c(M \leftarrow J^0(M, \mathbb{R})) = \ell^\infty_c(M, \mathbb{R})$.
In (4) this follows from (2), since every smooth curve in M coincides locally with a proper smooth curve and if $A \subseteq M$ is closed and not compact then there exists some end $e \in \varprojlim_U \pi(U)$ (where $\pi(U)$ denotes the finite set of (non-compact) connected components of $M \setminus \overline{U}$ for open relative compact $U \subseteq M$) which is in the closure of A in the compact topology of the Freudenthal-compactification $M \cup \varprojlim_U \pi(U)$ with the sets $e_K \cup \{e' \in \varprojlim_U \pi(U) : e'_K = e_K\}$ for the open relative compact sets $U \subseteq M$ as neighborhoodbasis of e. See [H.Freudenthal: Über die Enden topologischer Räume und Gruppen, Math. Zeitschrift 33 (1931) 692-713] und [Frank Reymond: the end point compactification of manifolds, Pacific J. Math. 10 (1960) 947-963]. Thus for every compact $K_n \subseteq M$ there exists a point $a_n \in e_{K_n} \cap A$. Since $e_{K_{n+1}} \subseteq e_{K_n}$ there is a curve in the connected component $e_{K_n} \subseteq M \setminus K_n$ connecting a_n with a_{n+1} we may piece these curves smoothly together to obtain a proper smooth curve $c : \mathbb{R} \to M$ with $c(\pm n) = a_n$. □

Second Proof.

(1→2) For this we consider for sequences $x = (x_n)_n$ without accumulation point the diagram

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "x^*"'] & C^\infty_K(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "x^*"] \arrow[dr, dotted] & C^\infty_c(M,\mathbb{R}) \arrow[d, "x^*"', "(2)"] \\
\prod_{n\in\mathbb{N}} \mathbb{R} & \mathbb{R}^{x^{-1}(K)} \arrow[l, hook'] \arrow[r, hook] & \coprod_{n\in\mathbb{N}} \mathbb{R},
\end{tikzcd}$$

where $x^{-1}(K) := \{n : x_n \in K\}$ is by assumption finite. Then obviously the identity on $C^\infty_c(M, \mathbb{R})$ is bounded from the structure (1) to the structure (2).

(1→3) We consider the diagram:

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "j^k"'] & C^\infty_K(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "j^k"'] \arrow[dr, dotted] & C^\infty_c(M,\mathbb{R}) \arrow[d, "j^k"', "(3)"] \\
C(M \leftarrow J^k(M,\mathbb{R})) & C_K(M \leftarrow J^k(M,\mathbb{R})) \arrow[l, hook'] \arrow[r, hook] & \ell^\infty_c(M \leftarrow J^k(M,\mathbb{R}))
\end{tikzcd}$$

Obviously, the identity on $C^\infty_c(M,\mathbb{R})$ is bounded from the structure (1) into the structure (3).

(1→4) follows from the diagram

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "e^*"'] & C^\infty_K(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "e^*"] \arrow[dr, dotted] & C^\infty_c(M,\mathbb{R}) \arrow[d, "e^*"', "(2)"] \\
C^\infty(\mathbb{R},\mathbb{R}) & C^\infty_{e^{-1}(K)}(\mathbb{R},\mathbb{R}) \arrow[l, hook'] \arrow[r, hook] & C^\infty_c(\mathbb{R},\mathbb{R})
\end{tikzcd}$$

with proper $e:\mathbb{R}\to M$.

(2→1) Now let $\mathcal{B}\subseteq C^\infty_c(M,\mathbb{R})$ be bounded in the structure of (2). We claim that $\mathcal{B}$ is contained in some $C^\infty_{K_n}(M,\mathbb{R})$, where K_n form an exhaustion of M by compact subsets such that K_n is contained in the interior of K_{n+1}. Otherwise there would be $x_n\notin K_n$ and $f_n\in\mathcal{B}$ with $f_n(x_n)\neq 0$. Then $x^*(\mathcal{B})$ ist not bounded in $\coprod_{\mathbb{N}}\mathbb{R}=\varinjlim_n \mathbb{R}^n$, since this limit is regular, but $x^*(f_n)(n)=f_n(x_n)\neq 0$. Since $C^\infty_c(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ is bounded, $\mathcal{B}$ is also bounded in $C^\infty_{K_n}(M,\mathbb{R})$ and hence in the structure (1).

(3→1) Now let $\mathcal{B}\subseteq C^\infty_c(M,\mathbb{R})$ be bounded in the structure of (3). Then $\mathcal{B}=j^0(\mathcal{B})$ is bounded in $\ell^\infty_c(M\leftarrow J^0(M,\mathbb{R}))=\ell^\infty_c(M,\mathbb{R})=\varinjlim_K C_K(M,\mathbb{R})$ and since this limit is regular there exists a compact $K\subseteq M$ such that $\mathcal{B}\subseteq C_K(M,\mathbb{R})$. But then also $\mathcal{B}\subseteq C^\infty_K(M,\mathbb{R})$. Since $j^k(\mathcal{B})\subseteq C_K(M\leftarrow J^k(M,\mathbb{R}))\subseteq \ell^\infty_c(M\leftarrow J^k(M,\mathbb{R}))$ is bounded we get that $\mathcal{B}\subseteq C^\infty_c(M,\mathbb{R})$ is bounded in the structure (3).

(4→2) Let now M have only finitely many connected components which are all non-compact and let $\mathcal{B}\subseteq C^\infty_c(M,\mathbb{R})$ be bounded for the structure (4). Since every smooth curve in M coincides locally with a proper smooth curve the set $\mathcal{B}$ is bounded in $C^\infty(M,\mathbb{R})$. Suppose there were a sequence $x=(x_n)_n$ without accumulation point for which $x^*(\mathcal{B})$ is not bounded in $\coprod_{n\in\mathbb{N}}\mathbb{R}^n$. Since $\mathrm{ev}_{x_n}(\mathcal{B})$ is bounded there are infinitely many $n\in\mathbb{N}$ for which $f_n\in\mathcal{B}$ exists with $f_n(x_n)\neq 0$. Since we only have finitely many connected components we may assume that all x_n are in the same non-compact connected component. Now we may choose a proper smooth curve c passing through a subsequence of the x_n and hence $c^*(\mathcal{B})$ would not be bounded in $C^\infty_c(\mathbb{R},\mathbb{R})$.

For the uniform boundedness principle we refer to the first proof. □

Remark

Note that the locally convex topologies described in (1) and (3) are distinct: The continuous dual of $(C^\infty_c(\mathbb{R},\mathbb{R}),(1))$ is the space of all distributions (generalized functions), whereas the continuous dual of $(C_c^\infty(\mathbb{R},\mathbb{R}), (\boxed{3}))$ are all distributions of finite order, i.e., globally finite derivatives of continuous functions.

If M is only assumed to be a smooth paracompact Hausdorff manifold, then we can still consider the structure on $C_c^\infty(M,\mathbb{R})$ given in $\boxed{1}$. It will no longer be an inductive limit of a sequence of Fréchet spaces but will still satisfy the uniform boundedness principle for the point-evaluations, by [41, 3.4.4]. since

$$\begin{aligned} C_c^\infty(M,\mathbb{R}) = \varinjlim_K C_K^\infty(M,\mathbb{R}) = \varinjlim_K \bigoplus_i C_{K\cap M_i}^\infty(M_i,\mathbb{R}) \cong \\ \cong \coprod_i \varinjlim_K C_{K\cap M_i}^\infty(M_i,\mathbb{R}) = \coprod_i C_c^\infty(M_i,\mathbb{R}), \end{aligned}$$

where the M_i are the connected components and these are Lindelöf.

- $\boxed{12.13}$ Smolyanov's Example.
- $\boxed{16.21}$ Some radial subsets are diffeomorphic to the whole space.
- [75, 21.6]–$\boxed{21.11}$ Counter-examples for lifting and extension properties.

54. Differentiabilities discussed by Keller [61]

54.1 Remark. (e,g. [65, 6.1.4]) Recall that for Banach spaces E (and F) a mapping $f: E \supseteq U \to F$ defined on an open subset U of E is called (Fréchet-)differentiable at $x \in U$ iff there exists a continuous linear operator $\ell: E \to F$, such that

$$\frac{f(x+v) - f(x) - \ell(v)}{\|v\|} \to 0 \text{ for } v \to 0.$$

Existence of ℓ implies its unicity, and hence it is denoted $f'(x)$ and called the (Fréchet-)derivative of f at x.

In order to calculate $f'(x)$ we may consider the directional derivatives

$$d_v f(x) := \lim_{t\searrow 0} \frac{f(x+tv) - f(x)}{t}.$$

Note that this is $\mathbb{R}^+$-homogeneous with respect to v. If f is Fréchet differentiable at x with derivative $f'(x)$, then $d_v f(x)$ exists and equals $f'(x)(v)$, since

$$\begin{aligned} \frac{f(x+tv)-f(x)}{t} - f'(x)(v) &= \frac{f(x+tv) - f(x) - f'(x)(tv)}{t} \\ &= \frac{f(x+tv) - f(x) - \ell(tv)}{\|tv\|}\,\|v\| \to 0. \end{aligned}$$

The converse direction does not hold, but one has:

54.2 Lemma. *(e.g. [65, 6.1.6]) Let E and F be Banach spaces, $U \subseteq E$ be open, $x \in U$. Then $f: E \supseteq U \to F$ is Fréchet differentiable at x iff the following conditions are satisfied:*

1. $\forall v \in E\ \exists d_v f(x)$;
2. $v \mapsto d_v f(x)$ *is linear and continuous;*
3. $\frac{f(x+tv)-f(x)}{t} \to d_v f(x)$ *for* $t \searrow 0$ *uniformly for* v *in the unit-sphere.*

Proof. (⇒) was shown just before this lemma.

(⇐) We claim that $v \mapsto d_v f(x)$ is the Fréchet derivative of f. So consider an arbitrary $v \neq 0$ and put $t := \|v\|$, $w := \frac{1}{t} v$. Then

$$\begin{aligned}\frac{f(x+v) - f(x) - d_v f(x)}{\|v\|} &= \frac{f(x+t\,w) - f(x) - d_{tw} f(x)}{t}\\ &= \frac{f(x+t\,w) - f(x)}{t} - d_w f(x) \to 0\end{aligned}$$

for $t = \|v\| \to 0$ uniformly for $\|w\| = 1$. □

Definition. The straight forward generalization of this notion to mappings between locally convex spaces is the following:

A mapping $f : E \supseteq U \to F$ defined on an open subset U of a locally convex space E is called ($\mathcal{B}$-)DIFFERENTIABLE AT $x \in U$, iff for all $v \in E$ the directional derivative $d_v f(x) := \lim_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}$ exists, this convergence is uniformly for $v \in B$, for any $B \in \mathcal{B}$, where $\mathcal{B}$ is some given set of bounded subsets of E, and $v \mapsto d_v f(x)$ is linear and continuous. In [61] the following particular cases for $\mathcal{B}$ are treated:

- 's' the finite subsets (leading to so called simple (or pointwise) convergence).
- 'k' the compact subsets. These are in general not stable under formation of closed convex hulls.
- 'pk' the precompact subsets. These are in contrast stable under formation of closed convex hulls.
- 'b' the bounded sets.

It is called CONTINUOUSLY ($\mathcal{B}$-)DIFFERENTIABLE ($C^1_{\mathcal{B}}$ for short), iff it is differentiable at each point $x \in U$ and $x \mapsto (v \mapsto f'(x)(v))$ is continuous from U to $\mathcal{L}_{\mathcal{B}}(E,F) := \{\ell : E \to F | \ell \text{ is linear and continuous}\}$, where we put the topology of uniform convergence on sets $B \in \mathcal{B}$ on $\mathcal{L}(E,F)$.

A mapping $f : E \supseteq U \to F$ is called GÂTEAUX DIFFERENTIABLE at $x \in U$, iff for all $v \in E$ the directional derivative $d_v f(x)$ exists and is linear in v (and most often it is also required to be continuous).

Moreover, it is sufficient to assume the continuity of the directional derivative to get differentiability:

54.3 Lemma. *Let $f : E \supseteq U \to F$ be defined on an open subset U of a locally convex space E and assume that for all $x \in U$ and $v \in E$ the directional derivative $d_v f(x)$ exists and $x \mapsto d_v f(x)$ defines a continuous mapping $f' : E \supseteq U \to \mathcal{L}_{\mathcal{B}}(E,F)$.*

Then f is $\mathcal{B}$-differentiable on U and f' is its derivative.

Proof. By 54.2 we only have to show that $\frac{f(x+tv)-f(x)}{t} \to d_v f(x)$ for $t \searrow 0$ uniformly for v in $B \in \mathcal{B}$. So we consider the difference and get by the fundamental theorem of calculus:

$$\begin{aligned}\frac{f(x+t\,v) - f(x)}{t} - d_v f(x) &= \int_0^1 \frac{1}{t}\frac{d}{ds} f(x+s\,t\,v) - d_v f(x)\, ds\\ &= \int_0^1 \Big(d_v f(x+s\,t\,v) - d_v f(x)\Big)(v)\, ds,\end{aligned}$$

which converges as required, since $d_v f : U \to \mathcal{L}_{\mathcal{B}}(E,F)$ is assumed to be continuous. □

This observation has been used by [61] to compare various differentiability notions given in the literature.

However, the problem with this type of definition, is to show the chain-rule for $C^1_{\mathcal{B}}$: Let $f : E \to F$ and $g : F \to G$ be $C^1_{\mathcal{B}}$. We would like to have that $g \circ f : E \to G$ is $C^1_{\mathcal{B}}$ and its derivative should be $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$. Obviously $f' : E \to \mathcal{L}(E,F)$ is continuous and also $g' \circ f : E \to F \to \mathcal{L}(F,G)$. Thus we would need that the composition $\circ : \mathcal{L}(F,G) \times \mathcal{L}(E,F) \to \mathcal{L}(E,G)$ is continuous. We have seen that even for $E = G = \mathbb{R}$ this is only the case, iff F is normed.

For this reason limit structures where used instead of topology by several authors. The coarsest reasonable structure is that of continuous convergence (denoted c), i.e. one calls a filter $\mathcal{F}$ on $\mathcal{L}(E,F)$ to be convergent to $\ell \in \mathcal{L}(E,F)$, iff for each filter $\mathcal{E}$ in E converging to some $x \in E$ the image filter $\mathcal{F}(\mathcal{E})$ converges to $\ell(x)$ in F. This definition turns $\mathcal{L}(E,F)$ into a convergence vector space denoted $\mathcal{L}_c(E,F)$. This is (by definition) the weakest convergence structure on $\mathcal{L}(E,F)$ which makes $\mathrm{ev} : \mathcal{L}(E,F) \times E \to F$ continuous. Moreover, a mapping $f : X \to \mathcal{L}_c(E,F)$ on a topological space X is continuous, iff the associated mapping $\widehat{f} : X \times E \to F$ is continuous.

Using some convergence structure Λ on $\mathcal{L}(E,F)$ (like continuous convergence) one can define $f : E \supseteq U \to F$ to be C^1_Λ, iff it is Gâteaux-differentiable and the derivative $f' : E \supseteq U \to \mathcal{L}_\Lambda(E,F)$ is continuous. For C^1_c mappings one can easily show the chain-rule. However, in Banach spaces one does not recover classical Fréchet differentiability (for which the inverse and implicit function theorem can be shown) but something weaker, see the following example of Smolyanov (12.13).

According to [61] one has the following implications, where qb denotes the limit structure of quasi-bounded convergence, which I will not explain here.

$$\begin{tikzcd}
C^1_s & C^1_k \arrow[l] & C^1_{pk} \arrow[l] & C^1_b \arrow[l] \\
 & & C^1_c \arrow[u] & C^1_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

The two smaller frames indicate groups of definitions which are equivalent for mappings between Fréchet spaces. And the large frame indicates that all definitions are equivalent for Fréchet-Schwarz spaces.

54.4 Higher Order Differentiability. In order to define differentiability of higher order we need appropriate spaces of multi-linear mappings in which the higher derivatives should take values.

For the concepts of $C^n_{\mathcal{B}}$ the spaces $\mathcal{H}^n_{\mathcal{B}}(E,F)$ (for hyper-continuity) were defined recursively in [61] by

$$\mathcal{H}^0_{\mathcal{B}}(E,F) := F$$
$$\mathcal{H}^{n+1}_{\mathcal{B}}(E,F) := \mathcal{L}_{\mathcal{B}}(E, \mathcal{H}^n_{\mathcal{B}}(E,F))$$

For C^n_c he considers $\mathcal{L}^n_c(E,F)$ as space of all continuous n-linear mappings $E\times\ldots\times E\to F$ with the convergence structure c of continuous convergence.

54.5 Definition. Let $\mathcal{B}$ be some family of bounded sets on E. A mapping $f: E\supseteq U\to F$ is called $C^n_{\mathcal{B}}$ iff it is n-times Gâteaux differentiable, i.e. all the n-fold iterated directional derivatives $d_{v_n}\ldots d_{v_1}f(x)$ exist, and $(v_1,\ldots,v_n)\mapsto d_{v_n}\ldots d_{v_1}f(x)$ is n-linear and defines a continuous mapping $f^{(n)}: E\supseteq U\to\mathcal{H}^n_{\mathcal{B}}(E,F)$.

It is called $C^\infty_{\mathcal{B}}$, if it is $C^n_{\mathcal{B}}$ for all $n\in\mathbb{N}$.

Similarly, let Λ be a convergence structure on $\mathcal{L}^k(E,F)$ for all $k\le n$. Then f is called C^n_Λ, iff it is n-times Gâteaux-differentiable and the n-th derivative $f^{(n)}: E\supseteq U\to\mathcal{L}^n_\Lambda(E,F)$ is continuous. It is called C^∞_Λ, if it is C^n_Λ for all $n\in\mathbb{N}$.

Again one has the same implications for C^n instead of C^1.

$$\begin{tikzcd}
C^n_s & C^n_k \arrow[l] & C^n_{pk} \arrow[l] & C^n_b \arrow[l] \\
 & & C^n_c \arrow[u] & C^n_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

One gets the following dependencies by using that from the continuity of a higher derivative with respect to some convergence structure one can deduce continuity of lower derivatives with respect to certain stronger convergence structures:

$$\begin{tikzcd}
C^\infty_s & C^\infty_k \arrow[l] & C^\infty_{pk} \arrow[l] & C^\infty_b \arrow[l] \\
 & & C^\infty_c \arrow[u] & C^\infty_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

Where this time the definitions in the smaller frame are equivalent for all lcs's, and for Fréchet spaces all mentioned definitions are equivalent. This has become popular as "In Fréchet spaces all concepts of smoothness coincide" although strictly speaking this is not true: Gâteaux-smoothness is strictly weaker, whereas tame-smoothness and the concepts of C^∞_Δ and C^∞_Θ (see [61]) are strictly stronger.

54.6 Remark. In order to compare the concepts of smoothness to be found in [61] with our smoothness we first have to compare the spaces of (multi-)linear mappings. For the following results [80] is the appropriate reference.

54.7 Lemma. *Let $\mathcal{B}$ be some set of bounded subsets of a locally convex space E, containing the finite subsets and being stable under the formation of finite unions and subsets.*

We denote with $L_{\mathcal{B}}(E,F)$ the space of all bounded linear mappings with the topology of uniform convergence on each bounded subset $B\in\mathcal{B}$. A 0-neighborhood-basis of this locally convex topology is given by the sets $N_{B,V}:=\{f: f(B)\subseteq V\}$, where

$B \in \mathcal{B}$ and V runs through the 0-neighborhoods in F. Note that $\mathcal{L}_\mathcal{B}(E,F)$ is the topological subspace of this space formed by the continuous linear mappings.

A subset $\mathcal{F} \subseteq L_\mathcal{B}(E,F)$ is bounded, iff it is uniformly bounded on bounded subsets $B \in \mathcal{B}$. In fact, $N_{B,V}$ absorbs $\mathcal{F} \Leftrightarrow \exists k$: $N_{B,k\,V} = k\,N_{B,V} \supseteq \mathcal{F}$, i.e. $\mathcal{F}(B) \subseteq k\,V$.

54.8 Corollary. *The bornology of $L_\mathcal{B}(E,F)$ is that of $L(E,F)$ provided $\mathcal{B}$ is any of the families mentioned in 4.3. And if E is c^∞-complete then this is true for all $\mathcal{B}$ between s and b.*

Proof. By what we said just before, $\mathcal{F} \subseteq L_\mathcal{B}(E,F)$ is bounded, iff $\mathcal{F}(B)$ is bounded for all $B \in \mathcal{B}$, or equivalently, iff $\mathcal{F}(B)$ is absorbed by any 0-neighborhood V in F, i.e. the absolutely convex set $U := \bigcap_{f\in\mathcal{F}} f^{-1}(V)$ absorbs all B. Now we may apply 4.3 and, in the c^∞-complete case, 5.18. □

54.9 Corollary. *Let $\mathcal{B}$ be any of the bornologies in 54.8. Then the inclusion $\mathcal{H}^n_\mathcal{B}(E,F) \to L(E,\dots,E;F)$ is well-defined, bounded and linear.*

Proof. For $n=0$ nothing is to be shown.

For $n=1$ we have that $\mathcal{H}^1_\mathcal{B}(E,F) = \mathcal{L}_\mathcal{B}(E,F) \subseteq L_\mathcal{B}(E,F) \overset{b}{\cong} L(E,F)$ by 54.8. By induction we get for $n+1$ the following sequence of bounded mappings:

$$\begin{aligned}\mathcal{H}^{n+1}_\mathcal{B}(E,F) &\cong \mathcal{L}_\mathcal{B}(E,\mathcal{H}^n_\mathcal{B}(E,F)) \to \\ &\to L(E,\mathcal{H}^n_\mathcal{B}(E,F)) \to \\ &\to L(E,L(E,\dots,E;F)) \cong L(E,\dots,E;F)\end{aligned}$$

□

54.10 Theorem. *For a mapping $f : E \supseteq U \to F$ from a c^∞-open subset E of a lcs E with values in an lcs F the following statements are equivalent:*

1. *f is C^∞;*
2. *All the iterated directional derivatives $d^n f(x)(v_1,\dots,v_n)$ exist and are bounded on M-converging sequences in $U \times E^n$;*
3. *The iterated directional derivatives $d^n f(x)(v_1,\dots,v_n)$ exist and define a mapping $d^n f : E \supseteq U \to L(E,\dots,E;F)$ which is bounded on M-converging sequences (or bornologically compact subsets of U);*

If E is c^∞-complete then this is further equivalent to

(4) *The iterated unidirectional derivatives $d^n_v f(x)$ exist and are separately bounded in x and in v on M-converging sequences.*

Proof. ($4 \Rightarrow 3 \Rightarrow 1$) In the proof of 5.20 we have shown that for c^∞-complete lcs E a mappings satisfying (4) satisfies (3) as well.
Then we showed without using any completeness condition that from (3) the chain rule for curves $c : \mathbb{R} \to U$ follows and hence (1).

($1 \Rightarrow 3$) follows from the chain rule given in 3.18, since then $d^n f : E \supseteq U \to L(E,\dots,E;F)$ is C^∞ and hence continuous on bornologically compact sets $K \subseteq E_B \subseteq E$.

($3 \Rightarrow 4$) and ($3 \Rightarrow 2$) are trivial, since bounded subsets of $L(E,\dots,E;F)$ are bounded on M-converging sequences.

$(2 \Rightarrow 3)$ It was shown in 5.20 that from (2) we conclude that $d^n f : E \supseteq U \to L(E, \dots, E; F)$ exists and is bounded on M-converging sequences with respect to the pointwise topology on $L(E, \dots, E; F)$. But by assumption this is even true for the topology of uniform convergence on M-converging sequences, and this is induces the same bornology as that of uniform convergence on bounded sets by 54.8. □

54.11 Proposition. *Let η be some real sequence converging to ∞ and $f : E \supseteq U \to F$ be a mapping from an open subset U of a lcs E with value in an lcs F.*
If $f \in C^\infty_{\mathcal{B}}$, where $\mathcal{B}$ contains all η-sequences, then $f \in C^\infty$.
If E is c^∞-complete, then $f \in C^\infty_s$ implies $f \in C^\infty$.

Proof. By assumption we have that f is infinite often Gâteaux differentiable and $f^{(n)} : E \supseteq U \to \mathcal{H}^n_{\mathcal{B}}(E, F)$ is continuous. Since $\mathcal{H}^n_{\mathcal{B}}(E, F) \to L(E, \dots, E; F)$ is well-defined and bounded by 54.9 the result follows from 54.10. □

54.12 Theorem. *Let $f : E \supseteq U \to F$ with U open in an lcs E with $c^\infty(E) = E$ and let F an lcs. Then $f \in C^\infty \Leftrightarrow f \in C^\infty_{\mathcal{B}}$, with any $\mathcal{B}$ as in 54.11.*

Proof. Because of 54.11 we only have to show $(\Rightarrow)$. By 3.18 we have the directional derivative $df : E \supseteq U \to L(E, F)$ which is C^∞ as well. So f is infinitely often Gâteaux differentiable and it remains to show that $d^n f : E \supseteq U \to L(E, \dots, E; F)$ is well-defined and continuous into $\mathcal{H}^n_b(E, F)$. Since $d^n f$ is smooth, we have that $d^n f : c^\infty(U) = c^\infty(E)|_U \to c^\infty(L(E, \dots, E; F)) \to L(E, \dots, E; F)_{\mathrm{born}}$ is continuous, and since $c^\infty E = E$, we get that $d^n f : E \supseteq U \to L(E, \dots, E; F)_{\mathrm{born}}$ is continuous. Since $c^\infty E = E$ we have that E is bornological, so

$$\mathcal{H}^1_{\mathcal{B}}(E, F)_{\mathrm{born}} = \mathcal{L}_{\mathcal{B}}(E, F)_{\mathrm{born}} = L(E, F)_{\mathrm{born}} = L(E, F_{\mathrm{born}})_{\mathrm{born}}$$

and we get $\mathcal{H}^n_{\mathcal{B}}(E, F)_{\mathrm{born}} = L(E, \dots, E; F)_{\mathrm{born}}$ by induction:

$$\begin{aligned} L(E, \dots, E, E; F)_{\mathrm{born}} &\cong L(E, L(E, \dots, E; F)_{\mathrm{born}})_{\mathrm{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E, L(E, \dots, E; F)_{\mathrm{born}})_{\mathrm{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E, \mathcal{H}^{n-1}_{\mathcal{B}}(E, F)_{\mathrm{born}})_{\mathrm{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E, \mathcal{H}^{n-1}_{\mathcal{B}}(E, F))_{\mathrm{born}} \\ &= \mathcal{H}^{n+1}_{\mathcal{B}}(E, F)_{\mathrm{born}}. \end{aligned}$$

So the derivatives are continuous into $\mathcal{H}^n_{\mathcal{B}}(E, F)$. □

55. Silva-Differentiability

See [24]. The idea here is to use the normed spaces E_B with B bounded in E and $F_q := F/q^{-1}(0)$ for continuous seminorms q on F associated with each locally convex space, and in fact for E we only need a CONVEX BORNOLOGICAL SPACE (CBS, for short) E (i.e. a vector space together with a bornology which is invariant under addition, homotheties and formation of convex hulls).

55.1 Definition. Let E and F be cbs's. A mapping $f : E \to F$ is called SILVA DIFFERENTIABLE AT $x \in E \Leftrightarrow \forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $f(\_ + x) - f(x) : E_A \to F_B$ is locally around 0 defined and Fréchet differentiable at 0.

Equivalently, $\forall A \subseteq E$ absolutely convex bounded with $x \in A$ $\exists B \subseteq F$ absolutely convex bounded such that $f : E_A \to F_B$ is locally around x defined and Fréchet differentiable at x.

In fact, $f(\_ + x) - f(x) = (\_ - f(x)) \circ f \circ (\_ + x) : E_A \to F_{B_{f(x)}}$ has some local property at 0 provided $f : E_{A_x} \to F_B$ has the same property at x, where $A_x := \langle \{x\} \cup A \rangle_{\text{abs.conv.}}$ and analogously $B_{f(x)} := \langle \{f(x)\} \cup B \rangle_{\text{abs.conv.}}$, since $\_ + x : E_A \to E_{A_x}$ is affine and bounded because

$$\rho A + x = (1+\rho)\left(\frac{\rho}{1+\rho} A_x + \frac{1}{1+\rho} x\right) \subseteq (1+\rho) A_x.$$

Conversely, $f : E_A \to F_{B_{-f(x)}}$ has some local property at $x \in A$ provided $f(\_ + x) : E_{A_{-x}} \to F_B$ has the same property at 0, since $f(\_) = (\_ + f(x)) \circ (f(\_ + x) - f(x)) \circ (\_ - x)$.

Note that in this situation the derivatives at x of the restrictions of $f : E \to F$ to locally defined mappings $E_A \to F_B$ fit together to define a bounded linear mapping $f'(x) : E \to F$. Thus the definition of Silva-differentiability of f at x can be rephrased as in [**24**, 1.1.1]: $\exists \ell : E \to F$ bounded and linear, such that for $r(h) := f(h+x) - f(x) - \ell \cdot h$ one has:
$\forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $\exists \varepsilon > 0 : r(\varepsilon A) \subseteq B$ and $p_B(r(h))/p_A(h) \to 0$ for $p_A(h) \to 0$.

55.2 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called SILVA DIFFERENTIABLE iff it is Silva-differentiable at each point $x \in E$. Note that the B in the definition 55.1 of differentiability at x may depend not only on the given A but also on x. Thus a Silva differentiable f need not have a locally differentiable restriction $E_A \to F_B$ for some B.

55.3 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called M-CONTINUOUS at $x \in E$ iff $\forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $f(\_ + x) - f(x) : E_A \to F_B$ is defined locally around 0 and continuous at 0, i.e. $\exists \varepsilon > 0$ with $f(\varepsilon A + x) \subseteq B$ and $p_B(f(h+x) - f(x)) \to 0$ for $p_A(h) \to 0$.

Equivalently, $\forall A \subseteq E$ absolutely convex bounded with $x \in A$ $\exists B \subseteq F$ absolutely convex bounded such that $f : E_A \to F_B$ is locally around x defined and continuous at x.

The mapping f is called M-CONTINUOUS, iff it is so at every point $x \in E$.

55.4 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called CONTINUOUSLY SILVA DIFFERENTIABLE (S^1 for short) iff it is Silva differentiable and $f' : E \to L(E, F)$ is M-continuous, where $L(E, F)$ denotes the cbs of bounded linear mappings from E to F with the bornology formed by the subsets being uniformly bounded in F on each bounded subset of E.

55.5 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called $n+1$-TIMES CONTINUOUSLY SILVA DIFFERENTIABLE (S^{n+1} for short) if it is S^n and the n-th derivative $f^{(n)} : E \to L(E, \dots, E; F)$ is S^1, or equivalently, avoiding the higher derivative, if f is S^1 and its derivative $f' : E \to L(E, F)$ is S^n.

The mapping f is called S^∞ iff f is n-times Silva-differentiable for all $n \in \mathbb{N}$.

55.6 Definition. Let E be a cbs, F an lcs and $f : E \to F$. Then f is called Silva differentiable in the enlarged sense, iff $\forall x \in E$ there exists a bounded linear $\ell : E \to F$ such that $r_x(h) := f(h + x) - f(x) - \ell \cdot h$ is a remainder in the following sense: For every absolutely convex bounded $A \subseteq E$ and every continuous seminorm q on F we have

$$q(r_x(h))/p_A(h) \to 0 \text{ for all } h \to 0 \text{ in } E_A.$$

For complete F this condition is equivalent to $E_A \to E \to F \to F/\operatorname{Ker}(q)$ being differentiable between normed spaces since then F is embedded as closed subspace of $\prod_q \widehat{F/\operatorname{Ker}(q)}$ and hence the directional derivative of f exists in F.

55.7 Definition. Analogously, for $n \in \mathbb{N} \cup \{\infty\}$, one may define n-times (continuously) Silva differentiable in the enlarged sense (S_e^n for short) and this is for complete (and in case $n = \infty$ even for c^∞-complete) F equivalent to $E_B \to E \to F \to F/\operatorname{Ker}(q)$ being n-times (continuously) differentiable between normed spaces.

Thus for locally convex spaces E and convenient vector spaces F a mapping $f : E \to F$ is S_e^∞ for the von Neumann bornology on E if and only if it is C^∞.

55.8 Remark. This definition makes problems with the chain-rule $E \to F \to G$ even if the space F in the middle is a locally convex space, since for $F \to G$ we only have properties on F_B but the restriction of $E \to F$ to E_A need not have values in F_B for some B.

55.9 Example. Note that S^n implies S_e^n (see [**24**, 1.4.8]), but not conversely even for $f : E \to \mathbb{R}$ and $n = \infty$, see [**24**, 2.5.2].

55.10 Definition. Let $p \in \mathbb{N} \cup \{\infty\}$ and E and F be cbs's. A mapping $f : E \to F$ is called locally p-times differentiable between normed spaces at a point $x \in E$ iff $\forall A \subseteq E$ absolutely convex bounded $\exists \varepsilon > 0$ $\exists B \subseteq F$ absolutely convex bounded such that $f(\varepsilon A + x) \subseteq B$ and $f : \{z \in E_A : \|z\|_A < \varepsilon\} + x \to F_B$ is p-times differentiable. Note that here in contrast to definitions 55.1–55.4 the bounded set B is locally independent on x and on the order of the derivative.

55.11 Proposition. [**24**, 1.5.2]. *Let E and F be cbs's and F be polar, i.e. the lcs-closure of bounded sets is bounded. Then S^{p+1} (i.e. $p+1$-times continuously Silva differentiable) implies locally p-times continuously differentiable between normed spaces.*

Example. There exist scalar valued mappings which are locally C^∞ between normed spaces but are not S^∞, see [**24**, 2.5].

55.12 Corollary. *Let $f : E \to F$ be smooth and $K \subseteq E$ be bornologically-compact. Then the image $f(K)$ in F is bornologically compact. Moreover, if $K \subseteq E_B$ is compact we find a bounded absolutely convex set $A \subseteq F$ such that $f : E_B \supseteq K \to F_A$ is a contraction.*

Proof. Since $f : E \to F$ is smooth, we have that $g := \ell \circ f : E_B \to \mathbb{R}$ is C^∞. In particular it is continuous, and from continuity of $g' : E_B \to L(E_B, \mathbb{R})$ we deduce locally Lipschitzness of g, since

$$\begin{aligned}|g(y)-g(x)| &= \left|\int_0^1 g'(x+t(y-x))(y-x)\,dt\right| \\ &\leq \int_0^1 |g'(x+t(y-x))(y-x)|\,dt \\ &\leq \sup\Big\{\|g'(x+t(y-x))\| : t\in[0,1]\Big\}\cdot\|y-x\|\end{aligned}$$

Since $K \subseteq E_B$ is compact we get a Lipschitz bound of $\ell\circ f$ on K for each $\ell\in E'$ (see the paragraph below) and hence $\{\frac{f(x)-f(y)}{\|x-y\|_B} : x,y\in K\}$ is bounded in F. Let A be the absolutely convex hull of this set, then $f : E_B \supseteq K \to F_A$ is a contraction, and hence continuous and thus $f(K)$ is compact in F_A.

A locally Lipschitzian mapping on a normed space is Lipschitzian on each compact subset: Otherwise we would find x_n and y_n with $|f(x_n)-f(y_n)|/\|x_n-y_n\|$ unbounded. Without loss of generality we may assume that $x_n\to x_\infty$ and $y_n\to y_\infty$. If $x_\infty\neq y_\infty$ then by continuity of f we get boundedness of the difference quotient. And if $x_\infty = y_\infty$ this contradicts the local Lipschitzness of f at x_∞. □

55.13 Proposition. [80] *Let E and F be convenient vector spaces and $f : E\to F$. Then f is C^∞ $\Leftrightarrow$ $\forall K\subseteq E$, absolutely convex, bornologically compact, $\forall x\in K$ $\forall n\in\mathbb{N}$ $(n\neq\infty)$ $\exists J\subseteq F$, absolutely convex, bornologically compact such that $f : E_K\to F_J$ is C^n locally around x, i.e. f is locally n-times continuously differentiable between normed spaces for the bornologies of bornologically compact sets.*

Proof. ($\Leftarrow$) Let $c:\mathbb{R}\to E$ be C^∞, let $I\subseteq\mathbb{R}$ be a bounded open interval, $t_0\in I$ and $n\in\mathbb{N}$. Since $\delta c:\mathbb{R}^2\to E$ given by $\delta c(t,s) := \int_0^1 c'(t+r(s-t))\,dr$ is smooth the image of $I\times I$ is bornologically-compact by 55.12. And inductively we get that $K := \delta c(I\times I)\cup\dots\cup\delta c^{(n)}(I\times I)\cup\{c(t_0),\dots,c^{(n)}(t_0)\}$ is bornologically-compact and hence compact in some E_B.

Thus there exists a sequence $x_n\to 0$ in E_B such that K is in the closed absolutely convex hull of $\{x_n : n\in\mathbb{N}\}$. The closed convex hull B' of this sequence is compact in E_B, so K is in the unit-sphere of $E_{B'}$ with bornologically compact B'.

Now we can deduce recursively that $c: I\to E_{B'}$ is C^n and hence the composite $f\circ c : I\to F$ is C^n.

($\Rightarrow$) Let K be bornologically compact and $n\in\mathbb{N}$. It suffices to show the existence of a bornologically compact $K_n\subseteq F$ such that $f: E_K\supseteq o(E_K)\to F_{K_n}$ is C^n_s, i.e. $x\mapsto f^{(k)}(x)(v_1,\dots v_k)$, $oE_K\to F_{K_n}$ is continuous for all $k\leq n$.

Since these derivatives are smooth $E_B\to F$ there exists some bornologically compact $K_n\subseteq F$, such that they are Lipschitz $E_B\supseteq K\to E_{K_n}$ by what we proved in 55.12. Hence they are continuous $K\subseteq E_K\to E_B\to E_{K_n}$. □

55.14 Remark. Ulrich Seip defined f to be smooth iff it is smooth along all smooth mappings $c:\mathbb{R}^n\to U$ (by Boman [15] $n=1$ suffices) and all derivatives are continuous on compact subsets $U\times E^n$. This is weaker than C^∞_c, since continuity $f^{(n)} : U\to\mathcal{L}^n_c(E,F)$ is required only on compact subsets of U.

However, it is not clear, whether all compact subsets are bounding (i.e. all smooth mappings in the sense of convenient calculus are bounded on them), hence the smoothness notion of Seip might be strictly stronger.

Chapter II
Calculus of Holomorphic and Real Analytic Mappings

This chapter starts with an investigation of holomorphic mappings between infinite dimensional vector spaces along the same lines as we investigated smooth mappings in chapter I. This theory is rather easy if we restrict to convenient vector spaces.

The basic tool is the set of all holomorphic mappings from the unit disk $\mathbb{D} \subset \mathbb{C}$ into a complex convenient vector space E, where all possible definitions of being holomorphic coincide, see 7.4. This replaces the set of all smooth curves in the smooth theory. A mapping between c^∞-open sets of complex convenient vector spaces is then said to be holomorphic if it maps holomorphic curves to holomorphic curves. This can be tested by many equivalent descriptions (see 7.19), the most important are that f is smooth and $df(x)$ is complex linear for each x (i.e. f satisfies the Cauchy-Riemann differential equation); or that f is holomorphic along each affine complex line and is c^∞-continuous (generalized Hartog's theorem). Again (multi-) linear mappings are holomorphic if and only if they are bounded 7.12.

The space $\mathcal{H}(U, F)$ of all holomorphic mappings from a c^∞-open set $U \subseteq E$ into a convenient vector space F carries a natural structure of a complex convenient vector space 7.21, and satisfies the holomorphic uniform boundedness principle 8.10. Of course our general aim of cartesian closedness 7.22, 7.23 is valid also in this setting: $\mathcal{H}(U, \mathcal{H}(V, F)) \cong \mathcal{H}(U \times V, F)$.

As in the smooth case we have to pay a price for cartesian closedness: holomorphic mappings can be expanded into power series, but these converge only on a c^∞-open subset in general, and not on open subsets.

The second part of this chapter is devoted to real analytic mappings in infinite dimensions. The ideas are similar as in the case of smooth and holomorphic mappings, but our wish to obtain cartesian closedness forces us to some modifications: In 9.1 we shall see that for the real analytic mapping $f : \mathbb{R}^2 \ni (s,t) \mapsto \frac{1}{(st)^2+1} \in \mathbb{R}$ there is no reasonable topology on $C^\omega(\mathbb{R}, \mathbb{R})$, such that the mapping $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is locally given by its convergent Taylor series, which looks like a counterexample to cartesian closedness. Recall that smoothness (holomorphy) of curves can be tested by applying bounded linear functionals (see 2.14, 7.4). The example above shows at the same time that this is not true in the real analytic case in general; if E' carries a Baire topology then it is true 9.6.

So we are forced to take as basic tool the space $C^\omega(\mathbb{R}, E)$ of all curves c such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for each bounded linear functional, and we call these the real analytic curves. In order to proceed we have to show that real analyticity of a curve can be tested with any set of bounded linear functionals which generates the bornology. This is done in 9.4 with the help of an unusual bornological description of real analytic functions $\mathbb{R} \to \mathbb{R}$ 9.3.

Now a mapping $f : U \to F$ is called real analytic if $f \circ c$ is smooth for smooth c and is real analytic for real analytic $c : \mathbb{R} \to U$. The second condition alone is not sufficient, even for $f : \mathbb{R}^2 \to \mathbb{R}$. Then a version of Hartog's theorem is true: f is real analytic if and only if it is smooth and real analytic along each affine line 10.4. In order to get to the aim of cartesian closedness we need a natural structure of a convenient vector space on $C^\omega(U, F)$. We start with $C^\omega(\mathbb{R}, \mathbb{R})$ which we consider as real part of the space of germs along $\mathbb{R}$ of holomorphic functions. The latter spaces of holomorphic germs are investigated in detail in section 8. At this stage of the theory we can prove the real analytic uniform boundedness theorem 11.6 and 11.12, but unlike in the smooth and holomorphic case for the general exponential law 11.18 we still have to investigate mixing of smooth and real analytic variables in 11.17. The rest of the development of section 11 then follows more or less standard (categorical) arguments.

7. Calculus of Holomorphic Mappings

7.1. Basic notions in the complex setting

In this section all locally convex spaces E will be complex ones, which we can view as real ones $E_\mathbb{R}$ together with continuous linear mapping J with $J^2 = -\operatorname{Id}$ (the complex structure). So all concepts for real locally convex spaces from sections 1 to 5 make sense also for complex locally convex spaces.

A set which is absolutely convex in the real sense need not be absolutely convex in the complex sense. However, the $\mathbb{C}$-absolutely convex hull of a bounded subset is still bounded, since there is a neighborhood basis of 0 consisting of $\mathbb{C}$-absolutely convex sets. So in this section absolutely convex will refer always to the complex notion. For absolutely convex bounded sets B the real normed spaces E_B (see 1.5) inherit the complex structure.

In this section all considered locally convex spaces will assumed to be convenient.

A complex linear functional ℓ on a convex vector space is uniquely determined by its real part $\operatorname{Re} \circ \ell$, by $\ell(x) = (\operatorname{Re} \circ \ell)(x) - \sqrt{-1}(\operatorname{Re} \circ \ell)(Jx)$. So for the respective spaces of bounded linear functionals we have

$$E_\mathbb{R}{}' = L_\mathbb{R}(E_\mathbb{R}, \mathbb{R}) \cong L_\mathbb{C}(E, \mathbb{C}) =: E^\star,$$

where the complex structure on the left hand side is given by $\lambda \mapsto \lambda \circ J$.

7.2. Definition

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. A mapping $c : \mathbb{D} \to E$ into a locally convex space E is called *complex differentiable*, if the complex derivative

$$c'(z) = \lim_{\mathbb{C} \ni w \to 0} \frac{c(z+w) - c(z)}{w}$$

exists for all $z \in \mathbb{D}$.

7.3. Lemma (Power series with values in convenient vector spaces). *Let E be convenient and $a_n \in E$. Then the following statements are equivalent:*

(1) *$\{r^n a_n : n \in \mathbb{N}\}$ is bounded for all $|r| < 1$.*
(2) *The power series $\sum_{n \geq 0} z^n a_n$ is Mackey convergent in E, uniformly on each compact subset of $\mathbb{D}$, i.e., the Mackey coefficient sequence and the bounded set can be chosen valid in the whole compact subset.*
(3) *The power series converges weakly for each $z \in \mathbb{D}$.*

Proof. (1) $\Rightarrow$ (2) Any compact set is contained in $r\mathbb{D}$ for some $0 < r < 1$ and for any $r < R < 1$ the set $\{R^n a_n : n \in \mathbb{N}\}$ is contained in some absolutely convex bounded B. So the partial sums of the series form a Mackey Cauchy sequence uniformly on $r\mathbb{D}$ since

$$\frac{1}{(r/R)^N - (r/R)^{M+1}} \sum_{n=N}^{M} z^n a_n \in \frac{1}{1-(r/R)} B.$$

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) The summands are weakly bounded, thus bounded. $\square$

7.4. Theorem. *If E is convenient then the following statements for a curve $c : \mathbb{D} \to E$ are equivalent:*

(1) *c is complex differentiable.*
(2) *$\ell \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for all $\ell \in E^\star$*
(3) *c is continuous and $\int_\gamma c = 0$ in the completion of E for all closed smooth ($\mathcal{L}\mathrm{ip}^0$) curves γ in $\mathbb{D}$.*
(4) *All $c^{(n)}(0)$ exist and $c(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} c^{(n)}(0)$ is Mackey convergent, uniformly on each compact subset of $\mathbb{D}$.*
(5) *For each $z \in \mathbb{D}$ all $c^{(n)}(z)$ exist and $c(w) = \sum_{n=0}^{\infty} \frac{(w-z)^n}{n!} c^{(n)}(z)$ is Mackey convergent, uniformly for w in each compact set in the largest disk with center z contained in $\mathbb{D}$.*
(6) *$c(z)\,dz$ is a closed smooth ($\mathcal{L}\mathrm{ip}^1$) 1-form with values in $E_\mathbb{R}$.*
(7) *c is the complex derivative of some complex differentiable curve in E.*
(8) *c is smooth ($\mathcal{L}\mathrm{ip}^1$) with complex linear real derivative $dc(z)$ for all z.*

A curve $c : \mathbb{D} \to E$ satisfying these equivalent conditions will be called a *holomorphic curve*.

Proof. Composition with a continuous $\mathbb{C}$-linear functional obviously translates all statements to one dimensional versions which in turn are all equivalent by complex analysis.

Conversely, let $\ell \circ c$ satisfy these equivalent conditions for each continuous linear $\ell : E \to \mathbb{C}$. Then we get the conditions for c as follows:

(8) By 2.14 c is smooth with real derivative satisfying $\ell \circ dc(z) = d(\ell \circ c)(z)$, hence being complex linear.

(1) For any $z \in \mathbb{D}$ the difference quotient of $\ell \circ c$ at z extends from $\mathbb{C} \setminus \{0\}$ to a holomorphic function $\mathbb{C} \to \mathbb{C}$ and hence is locally Lipschitz. Thus the difference quotient of c is Lipschitz on $U \setminus \{0\}$ for some 0-neighborhood U, hence forms a Cauchy net, cf. 2.1. Hence c is complex differentiable at z and by induction the complex derivatives $c^{(n)}(z)$ exist for all n.

(3) Since c is continuous the integral $\int_\gamma c$ exists in the completioon of E and satisfies $\ell(\int_\gamma c) = \int_\gamma (\ell \circ c) = 0$.

(5,4) By 7.3 (3⇒2) the series $\sum_n \frac{(w-z)^n}{n!} c^{(n)}(z)$ is Mackey convergent, uniformly for w in each compact set in the largest disk with center z contained in $\mathbb{D}$. Moreover, the image of its sum under ℓ equals $\ell(c(w))$, hence its sum is $c(w)$.

(6) $\ell(\frac{\partial c}{\partial \bar z}) = \frac{\partial (\ell \circ c)}{\partial \bar z} = 0$.

(7) The anti-derivative of c is given by $z \mapsto \int_0^1 c(tz)\, z\, dt$.

(1) ⇒ (2) Suppose that ℓ is $\mathbb{C}$-linear and only bounded. Let $c : \mathbb{D} \to E$ be a complex differentiable curve. Then $c_1 : z \mapsto \frac{1}{z}\Big(\frac{c(z)-c(0)}{z} - c'(0)\Big)$ is a complex differentiable curve by (1⇐2) proved above, hence

$$(\ell \circ c_1)(z) = \frac{1}{z}\Big(\frac{\ell(c(z)) - \ell(c(0))}{z} - \ell(c'(0))\Big)$$

is locally bounded in z. So $\ell \circ c$ is complex differentiable with derivative $\ell \circ c'$. □

7.5. Remarks

In the holomorphic case the equivalence of 7.4 (1⇔2) does not characterize c^∞-completeness as it does in the smooth case. The complex differentiable curves do not determine the bornology of the space, as do the smooth ones. See [**79**, 1.4]. For a discussion of the holomorphic analogues of smooth characterizations for c^∞-completeness (see 2.14) we refer to [**79**, pp. 2.16].

7.6. Lemma. *Let $c : \mathbb{D} \to E$ be a holomorphic curve in a convenient space. Then locally in $\mathbb{D}$ the curve factors to a holomorphic curve into E_B for some bounded absolutely convex set B.*

First Proof. By the obvious extension of lemma 1.8 for smooth mappings $\mathbb{R}^2 \supset \mathbb{D} \to E$ the curve c factors locally to a $\mathcal{L}\text{ip}^1$-mapping into some complete E_B. Since it has complex linear derivative, by theorem 7.4 it is holomorphic. □

Second direct proof. Let W be a relatively compact neighborhood of some point in $\mathbb{D}$. Then $c(W)$ is bounded in E. It suffices to show that for the absolutely convex closed hull B of $c(W)$ the Taylor series of c at each $z \in W$ converges in E_B, i.e. that $c|_W : W \to E_B$ is holomorphic. This follows from the

Vector valued Cauchy inequalities. If $r > 0$ is smaller than the radius of convergence at z of c then

$$\tfrac{r^k}{k!} c^{(k)}(z) \in B$$

where B is the closed absolutely convex hull of $\{\, c(w) : |w - z| = r\}$. (By the Hahn-Banach theorem this follows directly from the scalar valued case.)

Thus, we get

$$\textstyle\sum_{k=n}^m (\frac{w-z}{r})^k \cdot \frac{r^k}{k!} c^{(k)}(z) \in \sum_{k=n}^m (\frac{w-z}{r})^k \cdot B$$

and so $\sum_k \frac{c^{(k)}(z)}{k!}(w-z)^k$ is convergent in E_B for $|w - z| < r$. $\square$

This proof also shows that holomorphic curves with values in complex convenient vector spaces are topologically and bornologically holomorphic in the sense analogous to 9.4.

7.7. Lemma. *Let E be a regular (i.e. every bounded set is contained and bounded in some step E_α) inductive limit of complex locally convex spaces $E_\alpha \subseteq E$, let $c : \mathbb{C} \supseteq U \to E$ be a holomorphic mapping, and let $W \subseteq \mathbb{C}$ be open and such that the closure $\overline{W}$ is compact and contained in U. Then there exists some α, such that $c|W : W \to E_\alpha$ is well defined and holomorphic.*

Proof. By lemma 7.6 the restriction of c to W factors to a holomorphic curve $c|W : W \to E_B$ for a suitable bounded absolutely convex set $B \subseteq E$. Since B is contained and bounded in some E_α one has $c|W : W \to E_B = (E_\alpha)_B \to E_\alpha$ is holomorphic. $\square$

7.8. Definition. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. A mapping $f : U \to F$ is called *holomorphic*, if it maps holomorphic curves in U to holomorphic curves in F.

It is remarkable that [35] already gave this definition. Connections to other concepts of holomorphy are discussed in [79, 2.19].

So by 7.4 f is holomorphic if and only if $\ell \circ f \circ c : \mathbb{D} \to \mathbb{C}$ is a holomorphic function for all $\ell \in F^\star$ and holomorphic curve c.

Clearly, any composition of holomorphic mappings is again holomorphic.

For finite dimensions this coincides with the usual notion of holomorphic mappings, by the finite dimensional Hartogs' theorem.

7.9. Hartogs' Theorem. *Let E_1, E_2, and F be convenient vector spaces with U c^∞-open in $E_1 \times E_2$. Then a mapping $f : U \to F$ is holomorphic if and only if it is separately holomorphic, i.e. $f(\_, y)$ and $f(x, \_)$ are holomorphic.*

Proof. If f is holomorphic then $f(\_, y)$ is holomorphic on the c^∞-open set $E_1 \times \{y\} \cap U = \mathrm{incl}_y^{-1}(U)$, likewise for $f(x, \_)$.

If f is separately holomorphic, for any holomorphic curve $(c_1, c_2) : \mathbb{D} \to U \subseteq E_1 \times E_2$ we consider the holomorphic mapping $c_1 \times c_2 : \mathbb{D}^2 \to E_1 \times E_2$. Since the c_k are smooth by 7.4.8 also $c_1 \times c_2$ is smooth and thus $(c_1 \times c_2)^{-1}(U)$ is open in $\mathbb{C}^2$. For each $\lambda \in F^*$ the mapping $\lambda \circ f \circ (c_1 \times c_2) : (c_1 \times c_2)^{-1}(U) \to \mathbb{C}$ is separately holomorphic and so holomorphic by the usual Hartogs' theorem. By composing with the diagonal mapping we see that $\lambda \circ f \circ (c_1, c_2)$ is holomorphic, thus f is holomorphic. $\square$

7.10. Lemma. *Let $f : E \supseteq U \to F$ be holomorphic from a c^∞-open subset in a convenient vector space to another convenient vector space. Then the derivative $(df)^\wedge : U \times E \to F$ is again holomorphic and complex linear in the second variable.*

Proof. $(z, v, w) \mapsto f(v + zw)$ is holomorphic. We test with all holomorphic curves and linear functionals and see that $(v, w) \mapsto \frac{\partial}{\partial z}|_{z=0} f(v + zw) =: df(v)w$ is again holomorphic, $\mathbb{C}$-homogeneous in w by 7.4.

Now $w \mapsto df(v)w$ is a holomorphic and $\mathbb{C}$-homogeneous mapping $E \to F$. But any such mapping is automatically $\mathbb{C}$-linear: Composed with a bounded linear functional on F and restricted to any two dimensional subspace of E this is a finite dimensional assertion. □

7.11. Remark. In the definition of holomorphy 7.8 one could also have admitted subsets U which are only open in the final topology with respect to holomorphic curves. But then there is a counterexample to 7.10, see [**79**, 2.5].

7.12. Theorem. *A multilinear mapping between convenient vector spaces is holomorphic if and only if it is bounded.*

This result is false for locally convex spaces being not c^∞-complete, see [**79**, 1.4].

Proof. Since both conditions can be tested in each factor separately by Hartogs' theorem 7.9 and by 5.19, and by testing with linear functionals, we may restrict our attention to linear mappings $f : E \to \mathbb{C}$ only.

By theorem 7.4.2 a bounded linear mapping is holomorphic. Conversely, suppose that $f : E \to \mathbb{C}$ is a holomorphic but unbounded linear functional. So there exists a sequence (a_n) in E with $|f(a_n)| > 1$ and $\{2^n a_n\}$ bounded. Consider the power series $\sum_{n=0}^\infty (a_n - a_{n-1})(2z)^n$. This describes a holomorphic curve c in E, by 7.3 and 7.4.2. Then $f \circ c$ is holomorphic and thus has a power series expansion $f(c(z)) = \sum_{n=0}^\infty b_n z^n$. On the other hand

$$f(c(z)) = \sum_{n=0}^{N} \big(f(a_n) - f(a_{n-1})\big)(2z)^n + (2z)^N f\Big(\sum_{n>N} (a_n - a_{n-1})(2z)^{n-N}\Big).$$

So $b_n = 2^n \big(f(a_n) - f(a_{n-1})\big)$ and we get the contradiction

$$0 = f(0) = f(c(1/2)) = \sum_{n=0}^{\infty} \big(f(a_n) - f(a_{n-1})\big) = \lim_{n\to\infty} f(a_n). \qquad \square$$

Parts of the following results 7.13 to 10.2 can be found in [**13**]. For x in any vector space E let x^k denote the element $(x, \dots, x) \in E^k$.

7.13. Lemma. Polarization formulas. *Let $f : E \times \cdots \times E \to F$ be an k-linear symmetric mapping between vector spaces. Then we have:*

$$f(x_1,\dots,x_k) = \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} (-1)^{k-\Sigma\varepsilon_j}\, f\Big(\big(x_0 + \sum \varepsilon_j x_j\big)^k\Big).$$

$$f(x^k) = \tfrac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \tbinom{k}{j} f\big((a+jx)^k\big).$$

$$f(x^k) = \tfrac{k^k}{k!} \sum_{j=0}^{k} (-1)^{k-j} \tbinom{k}{j} f\big((a+\tfrac{j}{k}x)^k\big).$$

$$f(x_1^0 + \lambda x_1^1, \dots, x_k^0 + \lambda x_k^1) = \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} \lambda^{\Sigma\varepsilon_j} f(x_1^{\varepsilon_1},\dots,x_k^{\varepsilon_k}).$$

Formula $\boxed{4}$ will mainly be used for $\lambda = \sqrt{-1}$ in the passage to the complexification.

Proof. $\boxed{1}$. (see [**93**]). By multilinearity and symmetry the right hand side expands to

$$\sum_{j_0+\dots+j_k=k} \frac{A_{j_0,\dots,j_k}}{j_0!\cdots j_k!} f(\underbrace{x_0,\dots,x_0}_{j_0},\dots,\underbrace{x_k,\dots,x_k}_{j_k}),$$

where the coefficients are given by

$$A_{j_0,\dots,j_k} = \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} (-1)^{k-\Sigma\varepsilon_j} \varepsilon_1^{j_1}\cdots\varepsilon_k^{j_k}.$$

The only nonzero coefficient is $A_{0,1,\dots,1} = 1$: If $j_i = 0$ for some i (in particular this is the case when some $j_{i'} > 1$), then the summands with $\varepsilon_i = 0$ and $\varepsilon_i = 1$ cancel.

$\boxed{2}$. In formula $\boxed{1}$ we put $x_0 = a$ and all $x_j = x$.

$\boxed{3}$. In formula $\boxed{2}$ we replace a by ka and pull k out of the k-linear expression $f((ka+jx)^k)$.

$\boxed{4}$ is obvious. □

7.14. Lemma. Power series on Fréchet spaces. *Let E be a real or complex Fréchet space and let f_k be a k-linear symmetric scalar valued bounded functional on E, for each $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *$\sum_k f_k(x^k)$ converges pointwise on an absorbing subset of E.*
(2) *$\sum_k f_k(x^k)$ converges uniformly and absolutely on some neighborhood of 0.*
(3) *$\{f_k(x^k) : k \in \mathbb{N}, x \in U\}$ is bounded for some neighborhood U of 0.*
(4) *$\{f_k(x_1,\dots,x_k) : k \in \mathbb{N}, x_j \in U\}$ is bounded for some neighborhood U of 0.*

If any of these statements are satisfied over the reals, then also for the complexification of the functionals f_k.

Proof. ($\boxed{1}$) ⇒ ($\boxed{3}$) The set $A_{K,r} := \{x \in E : |f_k(x^k)| \le Kr^k \text{ for all } k\}$ is closed in E since every bounded multilinear mapping on Fréchet spaces is continuous. The countable union $\bigcup_{K,r} A_{K,r}$ is E, since the series converges pointwise on an absorbing subset. Since E is Baire there are $K > 0$ and $r > 0$ such that the interior

U of $A_{K,r}$ is non void. Let $x_0 \in U$ and let V be an absolutely convex neighborhood of 0 contained in $U - x_0$

From 7.13.3 we get for all $x \in V$ the following estimate using Stirlings formula:

$$|f_k(x^k)| \le \tfrac{k^k}{k!} \sum_{j=0}^{k} \tbinom{k}{j} |f_k((x_0 + \tfrac{j}{k}x)^k)| \le \tfrac{k^k}{k!} 2^k K r^k \le K(2re)^k.$$

Now we replace V by $\frac{1}{2re}V$ and get the result.

(3) $\Rightarrow$ (4) From 7.13.1 we get for all $x_j \in U$ the estimate:

$$\begin{aligned} |f_k(x_1, \dots, x_k)| &\le \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} \Big| f_k\Big(\big(\textstyle\sum \varepsilon_j x_j\big)^k\Big)\Big| \\ &= \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} \big(\textstyle\sum \varepsilon_j\big)^k \Big| f_k\Big(\Big(\frac{\sum \varepsilon_j x_j}{\sum \varepsilon_j}\Big)^k\Big)\Big| \le \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} \big(\textstyle\sum \varepsilon_j\big)^k C \\ &\le \tfrac{C}{k!} \sum_{l=0}^{k} \tbinom{k}{l} l^k \le C\, \tfrac{k^k}{k!} \sum_{l=0}^{k} \tbinom{k}{l} \le C(2e)^k. \end{aligned}$$

Now we replace U by $\frac{1}{2e}U$ and get (4).

(4) $\Rightarrow$ (2) The series converges on rU uniformly and absolutely for any $0 < r < 1$.

(2) $\Rightarrow$ (1) is clear.

(4, real case), $\Rightarrow$ (4, complex case), by 7.13.4 for $\lambda := \sqrt{-1}$. □

7.15. Lemma. *Let E be a complex convenient vector space and let f_k be a k-linear symmetric scalar valued bounded functional on E, for each $k \in \mathbb{N}$. Then the power series $\sum_k f_k(x^k)$ converges uniformly on bounded sets if and only if it converges pointwise on E and $x \mapsto f(x) := \sum_{k=0}^\infty f_k(x^k)$ is bounded on bounded sets,*

Proof. ($\Rightarrow$) If the power series converges uniformly on the bounded set B, then the remainder $\sum_{k\ge n} f_k(x^k)$ converges to 0 uniformly for $x \in B$ and hence $\sum_{k=0}^\infty f_k(x^k) = \sum_{k<n} f_k(x^k) + \sum_{k\ge n} f_k(x^k)$ is uniformly bounded on B.

($\Leftarrow$) Let B be an absolutely convex bounded set in E. For $x \in 2B$ we apply the vector valued Cauchy inequalities from 7.6 to the holomorphic curve $z \mapsto f(zx)$ at $z = 0$ for $r = 1$ and get that $f_k(x^k)$ is contained in the closed absolutely convex hull of $\{f(zx) : |z| = 1\}$. So $\{f_k(x^k) : x \in 2B, k \in \mathbb{N}\}$ is bounded and the series converges uniformly on B. □

7.16. Example. We consider the power series $\sum_k k(x_k)^k$ on the Hilbert space $\ell^2 = \{x = (x_k) : \sum_k |x_k|^2 < \infty\}$. This series converges pointwise everywhere, it yields a holomorphic function f on ℓ^2 by 7.19.5 which however is unbounded on the unit sphere, so convergence cannot be uniform on the unit sphere.

The function $g : \mathbb{C}^{(\mathbb{N})} \times \ell^2 \to \mathbb{C}$ given by $g(x,y) := \sum_{k>1} x_k f(kx_1 y)$ is holomorphic since it is a finite sum locally along each holomorphic curve by 7.7, but its Taylor series at 0 does not converge uniformly on any neighborhood of 0 in the locally convex topology since it is not locally bounded: A typical neighborhood is of the form $\{(x,y) : |x_k| \le \varepsilon_k \text{ for all } k, \|y\|_2 \le \varepsilon\}$ with $\varepsilon, \varepsilon_1, \dots > 0$ and for any $k > 1/\varepsilon_1$

we may choose z with $\|z\|_2 \le 1$ and $|f(z)| \ge k/\varepsilon_k$ and then $|g(x,y)| = |\varepsilon_k f(k\varepsilon_1 y)| \ge k$ for $x := \varepsilon_1 e_1 + \varepsilon_k e_k$ and $y := z/k\varepsilon_1$. This shows that lemma 7.14 is not true for arbitrary convenient vector spaces.

7.17. Corollary. *Let E be a real or complex Fréchet space and let f_k be a k-linear symmetric scalar valued bounded functional on E, for each $k \in \mathbb{N}$ such that the power series $\sum f_k(x^k)$ converges to $f(x)$ for x near 0 in E. Let $\sum_{k\ge1} a_k z^k$ be a power series in E which converges to $a(z) \in E$ for z near 0 in $\mathbb{C}$.*

Then the composite

$$\sum_{k\ge0}\sum_{n\ge0}\sum_{\substack{k_1,\dots,k_n\in\mathbb{N}\\ k_1+\dots+k_n=k}} f_n(a_{k_1},\dots,a_{k_n})\, z^k$$

of the power series converges to $f\circ a$ near 0.

Proof. By 7.14 there exists a 0-neighborhood U in E such that $\{f_k(x_1,\dots,x_k) : k \in \mathbb{N}, x_j \in U\}$ is bounded. Since the series for a converges there is $r > 0$ such that $a_k r^k \in U$ for all k. For $|z| < \frac{r}{2}$ we have

$$\begin{aligned} f(a(z)) &= \sum_{n\ge0} f_n\Big(\sum_{k_1\ge1} a_{k_1} z^{k_1}, \dots, \sum_{k_n\ge1} a_{k_n} z^{k_n}\Big)\\ &= \sum_{n\ge0}\sum_{k_1\ge1}\cdots\sum_{k_n\ge1} f_n\big(a_{k_1},\dots,a_{k_n}\big) z^{k_1+\dots+k_n}\\ &= \sum_{k\ge0}\sum_{k\ge n\ge0}\sum_{\substack{k_1,\dots,k_n\ge1\\ k_1+\dots+k_n=k}} f_n(a_{k_1},\dots,a_{k_n})\, z^k, \end{aligned}$$

since the last complex series converges absolutely: the coefficient of z^k is a sum of 2^{k-1} terms which are bounded when multiplied by r^k. The second equality follows from boundedness and multilinearity of all f_n. □

7.18. Almost continuous functions

In the proof of the next theorem we will need the following notion: A (real valued) function on a topological space is called *almost continuous* if removal of a meager set yields a continuous function on the remainder.

Lemma. [48, p. 221] *A pointwise limit of a sequence of almost continuous functions on a Baire space is almost continuous.*

Proof. Let (f_k) be a sequence of almost continuous real valued functions on a Baire space X which converges pointwise to f. Since the complement of a meager set in a Baire space is dense and hence again Baire we may assume that each function f_k is continuous on X. We denote by X_n the set of all $x \in X$ such that there exists $N \in \mathbb{N}$ and a neighborhood U of x with $|f_k(y) - f(y)| < \frac{1}{n}$ for all $k \ge N$ and all $y \in U$. The set X_n is clearly open.

We claim that each X_n is dense: Let V be a nonempty open subset of X. For $N \in \mathbb{N}$ the set $V_N := \{x \in V : |f_k(x) - f_\ell(x)| \le \frac{1}{2n}$ for all $k, \ell \ge N\}$ is closed in V and $V = \bigcup_N V_N$ since the sequence (f_k) converges pointwise. Since V is a Baire space, some V_N contains a nonempty open set W. For each $y \in W$ we have

$|f_k(y) - f_\ell(y)| \le \frac{1}{2n}$ for all k, $\ell \ge N$. We take the pointwise limit for $\ell \to \infty$ and see that $W \subseteq V \cap X_n$.

Since X is Baire, the set $\bigcap_n X_n$ has a meager complement and obviously the restriction of f on this set is continuous. $\square$

7.19. Theorem. *Let $f : E \supseteq U \to F$ be a mapping from a c^∞-open subset in a convenient vector space to another convenient vector space. Then the following assertions are equivalent:*

(1) *f is holomorphic.*
(2) *For all $\ell \in F^\star$ and absolutely convex closed bounded sets B the mapping $\ell \circ f : E_B \to \mathbb{C}$ is holomorphic.*
(3) *f is holomorphic along all affine complex lines and is c^∞-continuous.*
(4) *f is holomorphic along all affine complex lines and is bounded on* BORNOLOGICALLY COMPACT SET*s (i.e. those compact in some E_B).*
(5) *f is holomorphic along all affine complex lines and at each point $z \in U$ the directional derivative $f'(z)$ is a bounded linear mapping.*
(6) *For each $z \in U$ the mapping $f(z + \_)$ is c^∞-locally at 0 a convergent series of bounded homogeneous complex polynomials.*
(7) *f is holomorphic along all affine complex lines and in every connected component for the c^∞-topology there is at least one $z \in U$, where each derivative $f^{(k)}(z)$ is a bounded multi-linear mapping.*
(8) *f is smooth and for each $z \in U$ the derivative $f'(z)$ is complex linear.*
(9) *f is $\mathcal{L}\mathrm{ip}^1$ in the sense of 12.1 and the derivative is complex linear at every point.*

Proof. (1) $\Leftrightarrow$ (2) By 7.6 every holomorphic curve factors locally over some E_B and we may test with linear functionals on F by 7.4.

We prove the equivalence of the remaining statements first for the case where E is a Banach space and $F = \mathbb{C}$.

(1) $\Rightarrow$ (5) Obviously, f is holomorphic along affine lines. By lemma 7.10 the derivative $U \times E \to F$ of f is holomorphic and $\mathbb{C}$-linear in the second variable and by 7.12 $f'(z)$ is bounded.

(5) $\Rightarrow$ (6) Choose a fixed point $z \in U$. Since f is holomorphic along each affine complex line through z it is given there by a pointwise convergent power series. By the classical Hartogs' theorem f is holomorphic along each finite dimensional linear subspace and $f(z+v) = \sum_{k=0}^\infty \frac{f^{(k)}(z)(v^k)}{k!}$ for all $z+v$ in the open set $\{z+v : z+\lambda v \in U \text{ for all } |\lambda| \le 1\}$. We only have to show the boundedness of the symmetric k-linear mapping $f^{(k)}(z)$. For $k = 1$ this is true by assumption, i.e. $f' : E \supseteq U \to L(E,F)$ is well defined. Moreover, f' satisfies the same assumption (5) as f, since by 5.18 $f' \circ c$ is C^∞ into $L(E,F)$ for every affine line $c : \lambda \mapsto z + \lambda v$ with real derivative $d(f' \circ c)(\lambda,\mu)(v_1) = d^2 f(z+\lambda v; \mu v, v_1)$ being $\mathbb{C}$-linear in μ and bounded in v_1, hence $f' \circ c$ is holomorphic by 7.4.8, and by symmetry $(f')'(z)(v, v_1)$ is bounded in v as well, hence $(f')'(z)$ is jointly bounded by 5.19. Thus the general case follows by induction on k.

(6) $\Rightarrow$ (1) follows by using 7.4, 7.3, and composing the two locally uniformly converging power series via 7.17.

([6]) $\Rightarrow$ ([3]) By lemma [7.14] the series converges uniformly and hence f is continuous.

([3]) $\Rightarrow$ ([4]) is obvious.

([4]) $\Rightarrow$ ([5]) By the (1-dimensional) Cauchy integral formula we have

$$f'(z)v = \frac{1}{2\pi\sqrt{-1}} \int_{|\lambda|=1} \frac{f(z+\lambda v)}{\lambda^2} d\lambda.$$

So $f'(z)$ is a linear functional which is bounded on compact sets K for which $\{z + \lambda v : |\lambda| \le 1, v \in K\} \subseteq U$, thus it is bounded, by lemma [5.4].

Sublemma. *Let E be a Fréchet space and let $U \subseteq E$ be open. Let $f : U \to \mathbb{C}$ be holomorphic along affine lines and be also the pointwise limit on U of continuous polynomials. Then f is holomorphic on U.*

Proof. By assumption, and the lemma in [7.18] the function f is almost continuous, since it is the pointwise limit of continuous polynomials. For each $z \in U$ the (directional) derivative $f'(z) : E \to \mathbb{C}$ (as pointwise limit of difference quotients) is also almost continuous on the open set $\{v : z+\lambda v \in U \text{ for } |\lambda| \le 1\}$, thus continuous on E since it is linear and by the Baire property. By ([5]) $\Rightarrow$ ([1]) the function f is holomorphic on U. □

([6]) $\Rightarrow$ ([7]) is obvious.

([7]) $\Rightarrow$ ([1]) [**134**]
We treat each connected component of U separately and assume thus that U is connected. The set $U_0 := \{z \in U : f \text{ is holomorphic in a neighborhood of } z\}$ is open. Let z_0 be a point in U, where each derivative $f^{(k)}(z_0)$ is bounded. On the open star $U_1 := \{z_0 + v : z_0 + \lambda v \in U \text{ for all } |\lambda| \le 1\}$ the restriction of f is holomorphic along affine lines and thus $f : z \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0)((z - z_0)^k)$ is the pointwise limit on U_1 of continuous polynomials, hence $z_0 \in U_0$ by the sublemma. The same argument shows, that with $z \in U_0$ the whole star $\{z + v : z + \lambda v \in U \text{ for all } |\lambda| \le 1\}$ is contained in U_0. Since U is in particular polygonally connected, we have $U_0 = U$.

([1]) $\Rightarrow$ ([8]) All derivatives $d^k f$ are again holomorphic by [7.10] and thus locally bounded. So f is smooth by [5.20].

([8]) $\Rightarrow$ ([9]) is trivial.

([9]) $\Rightarrow$ ([3]) Obviously, f is c^∞-continuous and it is holomorphic along affine lines by [7.4].

Now we generalize to arbitrary E and F: If f satisfies any of the statements, then $\ell \circ f|_{E_B}$ satisfies all the (then equivalent) statements for each $\ell \in F^*$ and absolutely convex closed bounded $B \subseteq E$. Conversely, by ([1])$\Leftrightarrow$([2]) we get that f is holomorphic, C^∞ (by [2.3]) and its derivative $f'(z)$ is $\mathbb{C}$-linear (since $\ell(f'(z)) = (\ell \circ f)'(z)$). So we get all statements for f, which is obvious except for ([6]) and ([7]):

For ([6]) we argue as follows. It remains to show that the Taylor series at z converges pointwise on a c^∞-open neighborhood of z. The star $\{z+v : z+\lambda v \in U \text{ for all } |\lambda| \le$

$1\}$ with center z in U is again c^∞-open by 4.17 and on it the Taylor series of f at z converges pointwise by 7.3.

For (7) we may replace the condition "at least one point" by "for all points". □

7.20. Chain rule. *The composition of holomorphic mappings is holomorphic and the usual formula for the derivative of the composite holds.*

Proof. Use 7.19 (1⇔8), and the real chain rule 3.18. □

7.21. Definition. For convenient vector spaces E and F and for a c^∞-open subset $U \subseteq E$ we denote by $\mathcal{H}(U, F)$ the *space of all holomorphic mappings* $U \to F$. It is a closed linear subspace of $C^\infty(U, F)$ by 7.19.8 and we give it the induced convenient vector space structure.

7.22. Theorem. Cartesian closedness. *For convenient vector spaces E_1, E_2, and F, and for c^∞-open subsets $U_j \subseteq E_j$ a mapping $f : U_1 \times U_2 \to F$ is holomorphic if and only if the canonically associated mapping $f^\vee : U_1 \to \mathcal{H}(U_2, F)$ is holomorphic.*

Proof. (⇒) Obviously, $f^\vee$ has values in $\mathcal{H}(U_2, F)$ and is smooth by smooth cartesian closedness 3.12. Since its derivative is canonically associated to the first partial derivative of f, it is complex linear. So $f^\vee$ is holomorphic by 7.19.8.

(⇐) If conversely $f^\vee$ is holomorphic, then it is smooth into $\mathcal{H}(U_2, F)$ by 7.19, thus also smooth into $C^\infty(U_2, F)$. Hence $f : U_1 \times U_2 \to F$ is smooth by smooth cartesian closedness. The derivative $df(x, y) : (v, w) \mapsto d(f^\vee)(x)(v)(y) + d(f^\vee(x))(y)(w)$ is obviously complex linear, so f is holomorphic. □

7.23. Corollary. *Let E etc. be convenient vector spaces and let U etc. be c^∞-open subsets of such. Then the following canonical mappings are holomorphic.*

$$\begin{aligned}
&\operatorname{ev} : \mathcal{H}(U, F) \times U \to F, \quad \operatorname{ev}(f, x) = f(x)\\
&\operatorname{ins} : E \to \mathcal{H}(F, E \times F), \quad \operatorname{ins}(x)(y) = (x, y)\\
&(_)^\wedge : \mathcal{H}(U, \mathcal{H}(V, G)) \to \mathcal{H}(U \times V, G)\\
&(_)^\vee : \mathcal{H}(U \times V, G) \to \mathcal{H}(U, \mathcal{H}(V, G))\\
&\operatorname{comp} : \mathcal{H}(F, G) \times \mathcal{H}(U, F) \to \mathcal{H}(U, G)\\
&\mathcal{H}(_, _) : \mathcal{H}(F, F') \times \mathcal{H}(U', E) \to \mathcal{H}(\mathcal{H}(E, F), \mathcal{H}(U', F'))\\
&\qquad (f, g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod \mathcal{H}(E_i, F_i) \to \mathcal{H}(\prod E_i, \prod F_i)
\end{aligned}$$

Proof. Just consider the canonically associated holomorphic mappings on multiple products as in the proof of 3.13. □

In contrast to 7.16 we have:

7.24. Theorem (Holomorphic functions on Fréchet spaces). *Let $U \subseteq E$ be open in a complex Fréchet space E. The following statements on $f : U \to \mathbb{C}$ are equivalent:*

(1) *f is holomorphic.*
(2) *f is smooth and is locally given by its uniformly and absolutely converging Taylor series.*
(3) *f is locally given by a uniformly and absolutely converging power series.*

Proof. (1) $\Rightarrow$ (2) follows from 7.19 (1$\Rightarrow$6) and 7.14 (1$\Rightarrow$2).

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) is the chain rule for converging power series 7.17 and 7.4.4. $\square$

8. Spaces of Holomorphic Mappings and Germs

8.1. Spaces of holomorphic functions

For a complex manifold N (always assumed to be separable) let $\mathcal{H}(N,\mathbb{C})$ be the *space of all holomorphic functions* on N with the topology of uniform convergence on compact subsets of N.

Let $\mathcal{H}_b(N,\mathbb{C})$ denote the Banach space of bounded holomorphic functions on N equipped with the supremum norm.

For any open subset W of N let $\mathcal{H}_{bc}(N \supseteq W,\mathbb{C})$ be the closed subspace of $\mathcal{H}_b(W,\mathbb{C})$ of all holomorphic functions on W which extend to continuous functions on the closure $\overline{W}$.

For a poly-radius $r = (r_1,\dots,r_n)$ with $r_i > 0$ and for $1 \le p \le \infty$ let ℓ^p_r denote the real Banach space $\{\, x \in \mathbb{R}^{\mathbb{N}^n} : \|(x_\alpha r^\alpha)_{\alpha\in\mathbb{N}^n}\|_p < \infty \,\}$.

8.2. Theorem (Structure of $\mathcal{H}(N,\mathbb{C})$ for complex manifolds N).
The space $\mathcal{H}(N,\mathbb{C})$ of all holomorphic functions on N with the topology of uniform convergence on compact subsets of N is a (strongly) nuclear Fréchet space and embeds bornologically as a closed subspace into $C^\infty(N,\mathbb{R})^2$ considered with its Fréchet topology.

Proof. By taking a countable covering of N with compact sets, one obtains a countable neighborhood basis of 0 in $\mathcal{H}(N,\mathbb{C})$. Hence, $\mathcal{H}(N,\mathbb{C})$ is metrizable.
That $\mathcal{H}(N,\mathbb{C})$ is complete, and hence a Fréchet space, follows since the limit of a sequence of holomorphic functions with respect to the topology of uniform convergence on compact sets is again holomorphic.

The vector space $\mathcal{H}(N,\mathbb{C})$ is a closed subspace of $C^\infty(N,\mathbb{R}^2) = C^\infty(N,\mathbb{R})^2$ since a function $N \to \mathbb{C}$ is holomorphic if and only if it is smooth and the derivative at every point is $\mathbb{C}$-linear. Obviously, the identity from $\mathcal{H}(N,\mathbb{C})$ with the subspace topology to $\mathcal{H}(N,\mathbb{C})$ is continuous, hence by the open mapping theorem [**68**, 5.3.5] for Fréchet spaces it is an isomorphism.

That $\mathcal{H}(N,\mathbb{C})$ is nuclear and unlike $C^\infty(N,\mathbb{R})$ even strongly nuclear can be shown as follows. For N equal to the open polycylinder $\mathbb{D}^n \subseteq \mathbb{C}^n$ this result can be found in [**53**, 21.8.3.b]. For an arbitrary N the space $\mathcal{H}(N,\mathbb{C})$ carries the initial topology induced by the linear mappings $(u^{-1})^* : \mathcal{H}(N,\mathbb{C}) \to \mathcal{H}(u(U),\mathbb{C})$ for all charts (u,U) of N, for which we may assume $u(U) = \mathbb{D}^n$, and hence by the stability properties of strongly nuclear spaces, cf. [**53**, 21.1.7], $\mathcal{H}(N,\mathbb{C})$ is strongly nuclear. $\square$

8.3. Spaces of germs of holomorphic functions

For a subset $A \subseteq N$ let $\mathcal{H}(N\supseteq A, \mathbb{C})$ be the space of *germs along A of holomorphic functions $W \to \mathbb{C}$* for open sets W in N containing A. We equip $\mathcal{H}(N\supseteq A, \mathbb{C})$ with the locally convex topology induced by the inductive cone $\mathcal{H}(W, \mathbb{C}) \to \mathcal{H}(N\supseteq A, \mathbb{C})$ for all W. This is Hausdorff, since iterated derivatives at points in A are continuous functionals and separate points. In particular, $\mathcal{H}(N\supseteq W, \mathbb{C}) = \mathcal{H}(W, \mathbb{C})$ for W open in N. For $A_1 \subseteq A_2 \subseteq N$ the "restriction" mappings $\mathcal{H}(N\supseteq A_2, \mathbb{C}) \to \mathcal{H}(N\supseteq A_1, \mathbb{C})$ are continuous.

The structure of $\mathcal{H}(S^2\supseteq A, \mathbb{C})$, where $A \subseteq S^2$ is a subset of the Riemannian sphere, has been studied by [**120**], [Sebastião e Silva, 1950b,] [**128**], [**63**], and [**46**].

8.4. Theorem (Structure of $\mathcal{H}(N\supseteq K, \mathbb{C})$ for compact subsets K of complex manifolds N). *The following inductive cones are cofinal to each other.*

$$\big\{\mathcal{H}(N\supseteq K, \mathbb{C}) \leftarrow \mathcal{H}(W, \mathbb{C}) : W \supseteq K,\ W \text{ open in } N\big\}$$
$$\big\{\mathcal{H}(N\supseteq K, \mathbb{C}) \leftarrow \mathcal{H}_b(W, \mathbb{C}) : W \supseteq K,\ W \text{ open in } N\big\}$$
$$\big\{\mathcal{H}(N\supseteq K, \mathbb{C}) \leftarrow \mathcal{H}_{bc}(N\supseteq W, \mathbb{C}) : W \supseteq K,\ W \text{ open in } N\big\}$$

If $K = \{z\}$ these inductive cones and the following ones for $1 \le p \le \infty$ are cofinal to each other.

$$\big\{\mathcal{H}(N\supseteq\{z\}, \mathbb{C}) \leftarrow \ell^p_r \otimes \mathbb{C} :\ r \in \mathbb{R}^n_+\big\}$$

So all their induced inductive limit topologies coincide. Furthermore, the space $\mathcal{H}(N\supseteq K, \mathbb{C})$ is a Silva space, i.e. a countable inductive limit of Banach spaces, where the connecting mappings between the steps are compact, i.e. mapping bounded sets to relatively compact ones. The connecting mappings are even strongly nuclear. In particular, the limit is regular, i.e. every bounded subset is contained and bounded in some step, and $\mathcal{H}(N\supseteq K, \mathbb{C})$ is complete and (ultra-)bornological (hence a convenient vector space), webbed, strongly nuclear and thus reflexive, and its dual is a nuclear Fréchet space. The space $\mathcal{H}(N\supseteq K, \mathbb{C})$ is smoothly paracompact. It is however not a Baire space.

Proof. Let $K \subseteq V \subseteq \overline{V} \subseteq W \subseteq N$, where W and V are open and $\overline{V}$ is compact. Then the obvious mappings

$$\mathcal{H}_{bc}(N\supseteq W, \mathbb{C}) \to \mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}(W, \mathbb{C}) \to \mathcal{H}_{bc}(N\supseteq V, \mathbb{C})$$

are continuous. This implies the first cofinality assertion. For $q \le p$ and multiradii $s < r$ the obvious maps $\ell^q_r \to \ell^p_r$, $\ell^\infty_r \to \ell^1_s$, and $\ell^1_r \otimes \mathbb{C} \to \mathcal{H}_b(\{w \in \mathbb{C}^n : |w_i - z_i| < r_i\}, \mathbb{C}) \to \ell^\infty_s \otimes \mathbb{C}$ are continuous, by the Cauchy inequalities from the proof of 7.6. So the remaining cofinality assertion follows.

Let us show next that the connecting mapping $\mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}_b(V, \mathbb{C})$ is strongly nuclear (hence nuclear and compact). Since the restriction mapping from $E := \mathcal{H}(W, \mathbb{C})$ to $\mathcal{H}_b(V, \mathbb{C})$ is continuous, it factors over $E \to \widetilde{E_{(U)}}$ for some zero neighborhood U in E. Since E is strongly nuclear by 8.2, there exists by definition some larger 0-neighborhood U' in E such that the natural mapping $\widetilde{E_{(U')}} \to \widetilde{E_{(U)}}$ is strongly nuclear. So the claimed connecting mapping is strongly nuclear, since it can be factorized as

$$\mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}(W, \mathbb{C}) = E \to \widetilde{E_{(U')}} \to \widetilde{E_{(U)}} \to \mathcal{H}_b(V, \mathbb{C}).$$

So $\mathcal{H}(N \supseteq K, \mathbb{C})$ is a Silva space. It is strongly nuclear by the permanence properties of strongly nuclear spaces [**53**, 21.1.7]. By 16.10 this also shows that $\mathcal{H}(N \supseteq K, \mathbb{C})$ is smoothly paracompact. The remaining properties follow from [**75**, 52.37]. □

Completeness of $\mathcal{H}(\mathbb{C}^n \supseteq K, \mathbb{C})$ was shown in [**128**, théorème II], and for regularity of the inductive limit $\mathcal{H}(\mathbb{C} \supseteq K, \mathbb{C})$ see e.g. [**63**, Satz 12].

8.5. Lemma. *For a closed subset $A \subseteq \mathbb{C}$ the spaces $\mathcal{H}(A \subseteq S^2, \mathbb{C})$ and the space $\mathcal{H}_\infty(S^2 \supseteq S^2 \setminus A, \mathbb{C})$ of all germs vanishing at ∞ are strongly dual to each other.*

Proof. This is due to [**63**, Satz 12] and has been generalized by [Grothendieck, 1953, théorème 2 bis], to arbitrary subsets $A \subseteq S^2$. □

Compare this also with the modern theory of hyperfunctions, cf. [**60**].

8.6. Theorem (Structure of $\mathcal{H}(N \supseteq A, \mathbb{C})$ for closed subsets A of complex manifolds N). *The inductive cone*

$$\{\mathcal{H}(N \supseteq A, \mathbb{C}) \leftarrow \mathcal{H}(W, \mathbb{C}) : W \supseteq A,\ W \textit{ open in } N\}$$

is regular, i.e. every bounded set is contained and bounded in some step.

The projective cone

$$\{\mathcal{H}(N \supseteq A, \mathbb{C}) \to \mathcal{H}(N \supseteq K, \mathbb{C}) :\ K \subseteq A,\ K \textit{ compact}\}$$

generates the bornology of $\mathcal{H}(N \supseteq A, \mathbb{C})$.

The space $\mathcal{H}(N \supseteq A, \mathbb{C})$ is Montel (hence quasi-complete and reflexive), and ultrabornological (hence a convenient vector space). Furthermore, it is webbed and conuclear.

Proof. Compare also with the proof of the more general theorem [**75**, 30.6].

We choose a continuous function $f : N \to \mathbb{R}$ which is positive and proper, i.e. inverse images of compact sets are compact. Then $\big(f^{-1}([n, n+1])\big)_{n \in \mathbb{N}_0}$ is an exhaustion of N by compact subsets and $\big(K_n := A \cap f^{-1}([n, n+1])\big)$ is a compact exhaustion of A.

Let $\mathcal{B} \subseteq \mathcal{H}(N \supseteq A, \mathbb{C})$ be bounded. Then $\mathcal{B}|_K$ is also bounded in $\mathcal{H}(N \supseteq K, \mathbb{C})$ for each compact $K \subseteq A$. Since the cone

$$\{\mathcal{H}(N \supseteq K, \mathbb{C}) \leftarrow \mathcal{H}(W, \mathbb{C}) : W \supseteq K,\ W \text{ open in } N\}$$

is regular by 8.4, there exist open subsets W_K of N containing K such that $\mathcal{B}|_K$ is contained (so that the extension of each germ is unique) and bounded in $\mathcal{H}(W_K, \mathbb{C})$. In particular, we choose $W_{K_n \cap K_{n+1}} \subseteq W_{K_n} \cap W_{K_{n+1}} \cap f^{-1}((n, n+2))$. Then we let W be the union of those connected components of

$$W' := \bigcup_n \Big(W_{K_n} \cap f^{-1}\big((n, n+1)\big)\Big) \cup \bigcup_n W_{K_n \cap K_{n+1}}$$

which meet A. Clearly, W is open and contains A. Each $f \in \mathcal{B}$ has an extension to W': Extend $f|_{K_n}$ uniquely to f_n on W_{K_n}. The function $f|_{K_n \cap K_{n+1}}$ has also a unique extension $f_{n,n+1}$ on $W_{K_n \cap K_{n+1}}$, so we have $f_n|_{W_{K_n \cap K_{n+1}}} = f_{n,n+1}$. This extension of $f \in \mathcal{B}$ has a unique restriction to W. The set $\mathcal{B}$ is bounded in $\mathcal{H}(W, \mathbb{C})$ if it is uniformly bounded on each compact subset K of W. Each K is covered by finitely many W_{K_n} and $\mathcal{B}|_{K_n}$ is bounded in $\mathcal{H}(W_{K_n}, \mathbb{C})$, so $\mathcal{B}$ is bounded as required. Which shows the first two paragraphs of the theorem.

The space $\mathcal{H}(N\supseteq A,\mathbb{C})$ is ultra-bornological, Montel and in particular quasi-complete, and conuclear, as regular inductive limit of the nuclear Fréchet spaces $\mathcal{H}(W,\mathbb{C})$.

And it is webbed because it is the (ultra-)bornologification of the countable projective limit of webbed spaces $\mathcal{H}(N\supseteq K,\mathbb{C})$, see [**68**, 5.3.3] and [**68**, 5.3.3]. □

8.7. Lemma. *Let A be closed in $\mathbb{C}$. Then the dual generated by the projective cone*

$$\big\{\mathcal{H}(\mathbb{C}\supseteq A,\mathbb{C})\to\mathcal{H}(\mathbb{C}\supseteq K,\mathbb{C}) : K\subseteq A,\ K \text{ compact}\big\}$$

is just the topological dual of $\mathcal{H}(\mathbb{C}\supseteq A,\mathbb{C})$.

Proof. The induced topology is obviously coarser than the given one. So let λ be a continuous linear functional on $\mathcal{H}(\mathbb{C}\supseteq A,\mathbb{C})$. Then we have $\lambda\in\mathcal{H}_\infty(S^2\supseteq S^2\setminus A,\mathbb{C})$ by 8.5. Hence, $\lambda\in\mathcal{H}_\infty(U,\mathbb{C})$ for some open neighborhood U of $S^2\setminus A$, so again by 8.5 λ is a continuous functional on $\mathcal{H}(S^2\supseteq K,\mathbb{C})$, where $K=S^2\setminus U$ is compact and contained in A. So λ is continuous for the induced topology. □

Remark. In [**90**, Proposition 1.9 and Théorèm 1.2] it ws shown that this cone generates even the topology of $\mathcal{H}(\mathbb{C}\supseteq A,\mathbb{C})$. This implies that the bornological topology on $H(\mathbb{C}\supseteq A,\mathbb{C})$ is complete and nuclear.

8.8. Lemma (Structure of $\mathcal{H}(N\supseteq A,\mathbb{C})$ for smooth closed submanifolds A of complex manifolds N). *The projective cone*

$$\big\{\mathcal{H}(N\supseteq A,\mathbb{C})\to\mathcal{H}(N\supseteq\{z\},\mathbb{C}) : z\in A\big\}$$

generates the bornology.

Proof. Let $\mathcal{B}\subseteq\mathcal{H}(N\supseteq A,\mathbb{C})$ be such that the set $\mathcal{B}$ is bounded in $\mathcal{H}(N\supseteq\{z\},\mathbb{C})$ for each $z\in A$. By the regularity of the inductive cone $\mathcal{H}(\mathbb{C}^n\supseteq\{0\},\mathbb{C})\leftarrow\mathcal{H}(W,\mathbb{C})$ we find arbitrary small open neighborhoods W_z such that the set $\mathcal{B}_z$ of the germs at z of all germs in $\mathcal{B}$ is contained and bounded in $\mathcal{H}(W_z,\mathbb{C})$.

Now choose a tubular neighborhood $p:U\to A$ of A in N. We may assume that all W_z are contained in U, have fibers which are star shaped with respect to the zero-section and the intersections $W_z\cap A$ are connected. The union W of all the W_z is therefore an open subset of U containing A. And it remains to show that the germs in $\mathcal{B}$ extend to W. For this it is enough to show that the extensions of the germs at z_1 and z_2 agree on the intersection of W_{z_1} with W_{z_2}. So let w be a point in the intersection. It can be radially connected with the base point $p(w)$, which itself can be connected by curves in A with z_1 and z_2. Hence, the extensions of both germs to $p(w)$ coincide with the original germ, and hence their extensions to w are equal.

That $\mathcal{B}$ is bounded in $\mathcal{H}(W,\mathbb{C})$, follows immediately since every compact subset $K\subseteq W$ can be covered by finitely many W_z. □

8.9. The following example shows that 8.8 fails to be true for general closed subsets $A\subseteq N$.

Example. *Let $A:=\{\frac{1}{n}:n\in\mathbb{N}\}\cup\{0\}$. Then A is compact in $\mathbb{C}$ but the projective cone $\big\{\mathcal{H}(\mathbb{C}\supseteq A,\mathbb{C})\to\mathcal{H}(\mathbb{C}\supseteq\{z\},\mathbb{C}) : z\in A\big\}$ does not generate the bornology.*

Proof. Let $\mathcal{B} \subseteq \mathcal{H}(\mathbb{C}\supseteq A, \mathbb{C})$ be the set of germs of the following locally constant functions $f_n : \{x + iy \in \mathbb{C} : x \neq r_n\} \to \mathbb{C}$, with $f_n(x + iy)$ equal to 0 for $x < r_n$ and equal to 1 for $x > r_n$, where $r_n := \frac{2}{2n+1}$, for $n \in \mathbb{N}$. Then $\mathcal{B} \subseteq \mathcal{H}(\mathbb{C}\supseteq A, \mathbb{C})$ is not bounded, otherwise, by 8.6, there would exist a neighborhood W of A such that the germ of f_n extends to a holomorphic mapping on W for all n. Since every f_n is 0 on some neighborhood of 0, these extensions have to be zero on the component of W containing 0, which is not possible, since $f_n(\frac{1}{n}) = 1$.

But on the other hand the set $\mathcal{B}_z \subseteq \mathcal{H}(\mathbb{C}\supseteq\{z\}, \mathbb{C})$ of germs at $z \in A$ of all germs in $\mathcal{B}$ is bounded, since it contains only the germs of the constant functions 0 and 1. □

8.10. Theorem (Holomorphic uniform boundedness principle).
Let E and F be complex convenient vector spaces, and let $U \subseteq E$ be a c^∞-open subset. Then $\mathcal{H}(U, F)$ satisfies the uniform boundedness principle for the point evaluations ev_x, $x \in U$.

For any closed subset $A \subseteq N$ of a complex manifold N the locally convex space $\mathcal{H}(N\supseteq A, \mathbb{C})$ satisfies the uniform $\mathcal{S}$-boundedness principle for every point separating set $\mathcal{S}$ of bounded linear functionals.

Proof. By definition 7.21 $\mathcal{H}(U, F)$ carries the structure induced from the embedding into $C^\infty(U, F)$ and hence satisfies the uniform boundedness principle 5.26 and 5.25.

The second part is an immediate consequence of 5.24 and 8.6. □

Direct proof of a particular case of the second part. We prove the theorem for a closed smooth submanifold $A \subseteq \mathbb{C}$ and the set $\mathcal{S}$ of all iterated derivatives at points in A.

Let us suppose first that A is the point 0. We will show that condition 5.22.3 is satisfied. Let (b_n) be an unbounded sequence in $\mathcal{H}(\mathbb{C}\supseteq\{0\}, \mathbb{C})$ such that each Taylor coefficient $b_{n,k} = \frac{1}{k!} b_n^{(k)}(0)$ is bounded with respect to n:

$$\sup\{\, |b_{n,k}| : n \in \mathbb{N} \,\} < \infty.$$

We have to find $(t_n) \in \ell^1$ such that $\sum_n t_n\, b_n$ is no longer the germ of a holomorphic function at 0.

Each b_n has positive radius of convergence, in particular there is an $r_n > 0$ such that

$$\sup\{\, |b_{n,k}\, r_n^k| : k \in \mathbb{N} \,\} < \infty.$$

By theorem 8.4 the space $\mathcal{H}(\mathbb{C}\supseteq\{0\}, \mathbb{C})$ is a regular inductive limit of spaces ℓ_r^∞. Hence, a subset $\mathcal{B}$ is bounded in $\mathcal{H}(\mathbb{C}\supseteq\{0\}, \mathbb{C})$ if and only if there exists an $r > 0$ such that $\{\, \frac{1}{k!} b^{(k)}(0)\, r^k : b \in \mathcal{B}, k \in \mathbb{N} \,\}$ is bounded. That the sequence (b_n) is unbounded thus means that for all $r > 0$ there are n and k such that $|b_{n,k}| > (\frac{1}{r})^k$. We can even choose $k > 0$ for otherwise the set $\{\, b_{n,k} r^k : n, k \in \mathbb{N}, k > 0 \,\}$ is bounded, so only $\{\, b_{n,0} : n \in \mathbb{N} \,\}$ can be unbounded. This contradicts 1.

Hence, for each m there are $k_m > 0$ such that $\mathcal{N}_m := \{\, n \in \mathbb{N} : |b_{n,k_m}| > m^{k_m} \,\}$ is not empty. We can choose (k_m) strictly increasing, for if they were bounded, $|b_{n,k_m}| < C$ for some C and all n by 1, but $|b_{n_m,k_m}| > m^{k_m} \to \infty$ for some n_m.

Since by 1 the set $\{ b_{n,k_m} : n \in \mathbb{N} \}$ is bounded, we can choose $n_m \in \mathcal{N}_m$ such that

$$|b_{n_m,k_m}| \geq \tfrac{1}{2}|b_{j,k_m}| \quad \text{ for } j > n_m$$
$$|b_{n_m,k_m}| > m^{k_m}$$

We can choose also (n_m) strictly increasing, for if they were bounded we would get $|b_{n_m,k_m} r^{k_m}| < C$ for some $r > 0$ and C by (2). But $(\frac{1}{m})^{k_m} \to 0$.

We pass now to the subsequence (b_{n_m}) which we denote again by (b_m). We put

$$t_m := \operatorname{sign}\left(\frac{1}{b_{m,k_m}} \sum_{j<m} t_j\, b_{j,k_m}\right) \cdot \frac{1}{4^m}.$$

Assume now that $b_\infty = \sum_m t_m\, b_m$ converges weakly with respect to $\mathcal{S}$ to a holomorphic germ. Then its Taylor series is $b_\infty(z) = \sum_{k\geq 0} b_{\infty,k}\, z^k$, where the coefficients are given by $b_{\infty,k} = \sum_{m\geq 0} t_m\, b_{m,k}$. But we may compute as follows, using 3 and 4 :

$$\begin{aligned}
|b_{\infty,k_m}| &\geq \left|\sum_{j\leq m} t_j\, b_{j,k_m}\right| - \sum_{j>m} |t_j\, b_{j,k_m}| \\
&= \left|\sum_{j<m} t_j\, b_{j,k_m}\right| + |t_m\, b_{m,k_m}| \qquad \text{(same sign)} \\
&\qquad\qquad - \sum_{j>m} |t_j\, b_{j,k_m}| \geq \\
&\geq 0 + |b_{m,k_m}| \cdot \left(|t_m| - 2\sum_{j>m} |t_j|\right) \\
&= |b_{m,k_m}| \cdot \frac{1}{3\cdot 4^m} \geq \frac{m^{k_m}}{3\cdot 4^m}.
\end{aligned}$$

So $|b_{\infty,k_m}|^{1/k_m}$ goes to ∞, hence b_∞ cannot have a positive radius of convergence, a contradiction. So the theorem follows for the space $H(\{t\}, \mathbb{C})$.

Let us consider now an arbitrary closed smooth submanifold $A \subseteq \mathbb{C}$. By 8.8 the projective cone $\{\mathcal{H}(N{\supseteq}A, \mathbb{C}) \to \mathcal{H}(N{\supseteq}\{z\}, \mathbb{C}) : z \in A\}$ generates the bornology. Hence, the result follows from the case where $A = \{0\}$ by 5.25. □

9. Real Analytic Curves

9.1. As for smoothness and holomorphy we would like to obtain cartesian closedness for real analytic mappings. Thus, one should have at least the following:

A mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is real analytic in the classical sense if and only if $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is real analytic in some appropriate sense.

The following example shows that there are some subtleties involved.

Example. *The mapping $f : \mathbb{R}^2 \ni (s,t) \mapsto \frac{1}{1+(st)^2} \in \mathbb{R}$ is real analytic, whereas there is no reasonable convenient vector space topology on $C^\omega(\mathbb{R}, \mathbb{R})$, such that the mapping $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is locally given by its convergent Taylor series.*

Proof. For a topology on $C^\omega(\mathbb{R},\mathbb{R})$ to be reasonable we require only that all evaluations $\mathrm{ev}_t : C^\omega(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ are bounded linear functionals. Now suppose that $f^\vee(s) = \sum_{k=0}^\infty f_k s^k$ converges in $C^\omega(\mathbb{R},\mathbb{R})$ for all small s, where $f_k \in C^\omega(\mathbb{R},\mathbb{R})$. Since the point evaluations are assumed to be continuous $f(s,t) = (\mathrm{ev}_t \circ f^\vee)(s) = \sum_{k=0}^\infty f_k(t)\, s^k$ for all t and small s. On the other hand $f(s,t) = \sum_{k=0}^\infty (-1)^k (st)^{2k}$ for $|st| < 1$. Comparing coefficients of the real analytic function $\mathrm{ev}_t \circ f^\vee$ for each t gives $f_k(t) = (-1)^m t^k$ for $k = 2m$, and 0 otherwise. Moreover, by 9.5 there has to exist a $\delta > 0$ such that series $\sum f_k z^k$ converges in $C^\omega(\mathbb{R},\mathbb{R}) \otimes \mathbb{C}$ for each $|z| \le \delta$. But this is not the case for $z := \sqrt{-1}\,\delta$, as composing with $\mathrm{ev}_{1/\delta}$ shows. □

There is, however, another notion of real analytic curves:

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a real analytic function with finite radius of convergence at 0, e.g. $f(t) := \frac{1}{1+t^2}$. Now consider the curve $c : \mathbb{R} \to \mathbb{R}^\mathbb{N}$ defined by $c(t) := (f(k \cdot t))_{k\in\mathbb{N}}$. Clearly, the composite of c with any continuous linear functional is real analytic, since these functionals depend only on finitely many coordinates. But the Taylor series of c at 0 does not converge on any neighborhood of 0, since the radii of convergence of the coordinate functions go to 0. For an even more natural example see 11.8.

9.2. Lemma. *For a formal power series $\sum_{k\ge 0} a_k t^k$ with real coefficients the following conditions are equivalent.*

(1) *The series has positive radius of convergence.*
(2) *The series $\sum a_k r_k$ converges absolutely for each sequence $(r_k)_{k\in\mathbb{N}}$ with $r_k\, t^k \to 0$ for all $t > 0$.*
(3) *The sequence $(a_k r_k)$ is bounded for all (r_k) with $r_k\, t^k \to 0$ for all $t > 0$.*
(4) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$ there exists an $\varepsilon > 0$ such that $(a_k\, r_k\, \varepsilon^k)$ is bounded.*

This bornological description of real analytic curves will be rather important for the theory presented here, since condition (3) and (4) are linear boundedness conditions on the coefficients of a formal power series enforcing local convergence.

Proof. (1) ⇒ (2) If $\sum_k a_k t^k$ converges for some $t > 0$, then $\{a_k t^k : k \in \mathbb{N}\}$ is bounded. Since $\{r_k (2/t)^k : k \in \mathbb{N}\}$ is bounded by assumption, the series $\sum a_k r_k = \sum a_k t^k\, r_k (\frac{2}{t})^k\, \frac{1}{2^k}$ converges absolutely.

(2) ⇒ (3) ⇒ (4) is clear.

(4) ⇒ (1) If the radius of convergence is 0, then $\sum_k |a_k|\, (\frac{1}{n^2})^k = \infty$ for each n. Thus there are $k_n \nearrow \infty$ with

$$\sum_{k=k_{n-1}}^{k_n - 1} |a_k|\, (\tfrac{1}{n^2})^k \ge 1.$$

We put $r_k := (\frac{1}{n})^k$ for $k_{n-1} \le k < k_n$, then for each m we have

$$\begin{aligned}\sum_k |a_k|\, r_k (\tfrac{1}{m})^k &= \sum_n \sum_{k_{n-1}\le k<k_n} |a_k|\, (\tfrac{1}{nm})^k \\ &\ge \sum_{n\ge m} \sum_{k_{n-1}\le k<k_n} |a_k|\, (\tfrac{1}{n^2})^k \ge \sum_{n\ge m} 1 = \infty,\end{aligned}$$

so $\{a_k r_k (\frac{2}{m})^k : k \in \mathbb{N}\}$ is not bounded, but $r_k\, t^k$, which equals $(\frac{t}{n})^k$ for $k_{n-1} \le k < k_n$, converges to 0 for all $t > 0$, and the sequence (r_k) is subadditive (since $r_{n-1} \le k + l < r_n \Rightarrow r_{k+l} = \frac{1}{n^k}\frac{1}{n^l} \le r_k\, r_l$) as required. □

9.3. Theorem (Description of real analytic functions). *For a smooth function $c : \mathbb{R} \to \mathbb{R}$ the following statements are equivalent.*

(1) *The function c is real analytic.*

(2) *For each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded.*

(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, there exists an $\varepsilon > 0$ such that $\{\frac{1}{k!}\, c^{(k)}(a)\, r_k\, \varepsilon^k : a \in K, k \in \mathbb{N}\}$ is bounded.*

(4) *For each compact set $K \subset \mathbb{R}$ there exist constants $M, \rho > 0$ with the property that $|\frac{1}{k!} c^{(k)}(a)| < M\rho^k$ for all $k \in \mathbb{N}$ and $a \in K$.*

Proof. ($\boxed{1}$) $\Rightarrow$ ($\boxed{4}$) Since the Taylor series of c converges at a there are constants $M_a, \rho_a > 0$ satisfying the claimed inequality for fixed a. For a' with $|a' - a| \le \frac{1}{2\rho_a}$ we obtain by differentiating $c : a' \mapsto \sum_{\ell=0}^{\infty} \frac{c^{(\ell)}(a)}{\ell!}(a' - a)^\ell$ the estimate

$$\left|\frac{c^{(k)}(a')}{k!}\right| \le \frac{1}{k!}\sum_{\ell \ge k}\left|\frac{c^{(\ell)}(a)}{\ell!}\right| (\ell)_k\, |a' - a|^{\ell-k} \le \frac{1}{k!}\sum_{\ell\ge k} M_a \rho_a^\ell\, (\ell)_k\, (2\rho_a)^{k-\ell}$$
$$= \frac{M_a\rho_a^k}{k!}\sum_{\ell\ge k}(\ell)_k \left(\frac{1}{2}\right)^{\ell-k} = M_a\rho_a^k\, \frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k\Big|_{t=\frac12}\frac{1}{1-t},$$

hence the condition is satisfied locally with some new constants M_a', ρ_a' incorporating the estimates for the Taylor coefficients of $t \mapsto \frac{1}{1-t}$ at $t := 1/2$. Since K is compact the claim follows.

($\boxed{4}$) $\Rightarrow$ ($\boxed{2}$) We have $|\frac{1}{k!}\, c^{(k)}(a)\, r_k| \le M r_k \rho^k$ which is bounded since $r_k\rho^k \to 0$, as required.

($\boxed{2}$) $\Rightarrow$ ($\boxed{3}$) follows by choosing $\varepsilon = 1$.

($\boxed{3}$) $\Rightarrow$ ($\boxed{1}$) For $a \in \mathbb{R}$ let K be a compact neighborhood and $a_k := \sup_{a'\in K} |\frac{1}{k!}\, c^{(k)}(a')|$. Using ($\boxed{4}\Rightarrow\boxed{1}$) in $\boxed{9.2}$. These are the coefficients of a power series with positive radius ρ of convergence. Hence, the remainder $\frac{1}{(k+1)!}c^{(k+1)}(a+\theta(a'-a))(a'-a)^{k+1}$ of the Taylor series of c at a goes to zero for $a' inK$ with $|a' - a| < \rho$. □

9.4. Corollary. Real analytic curves. *For a curve $c : \mathbb{R} \to E$ in a convenient vector space E are equivalent:*

(1) *$\ell \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for all ℓ in some family of bounded linear functionals, which generates the bornology of E.*

(2) *$\ell \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for all $\ell \in E'$*

A curve satisfying these equivalent conditions will be called real analytic.

Proof. The non-trivial implication is ($\boxed{1}\Rightarrow\boxed{2}$). So assume ($\boxed{1}$). By the arguments in $\boxed{2.3}$ the curve c is smooth and hence $\ell \circ c$ is smooth for all bounded linear

$\ell : E \to \mathbb{R}$ and satisfies $(\ell \circ c)^{(k)}(t) = \ell(c^{(k)}(t))$. In order to show that $\ell \circ c$ is real analytic, we have to prove boundedness of

$$\ell\Big(\Big\{\frac{1}{k!}c^{(k)}(a)r_k : a \in K, k \in \mathbb{N}\Big\}\Big) = \Big\{\frac{1}{k!}(\ell \circ c)^{(k)}(a)r_k : a \in K, k \in \mathbb{N}\Big\}$$

for all compact $K \subset \mathbb{R}$ and all appropriate r_k, by 9.3. Since ℓ is bounded it suffices to show that $\{\frac{1}{k!}c^{(k)}(a)r_k : a \in K, k \in \mathbb{N}\}$ is bounded, which follows since its image under each ℓ mentioned in (1) is bounded, again by 9.3. □

9.5. Lemma. *Let E be a convenient vector space and let $c : \mathbb{R} \to E$ be a curve. Then the following conditions are equivalent.*

(1) *The curve c is topologically real analytic, i.e. it is locally given by a power series converging with respect to the locally convex topology.*
(2) *The curve c is bornologically real analytic, i.e. it factors locally over a topologically real analytic curve into E_B for some bounded absolutely convex set $B \subseteq E$.*
(3) *The curve c extends to a holomorphic curve from some open $U \supseteq \mathbb{R}$ in $\mathbb{C}$ into the complexification $E_{\mathbb{C}}$.*

Proof. (1) $\Rightarrow$ (3) For every $t \in \mathbb{R}$ one has for some $\delta > 0$ and all $|s| < \delta$ a converging power series representation $c(t+s) = \sum_{k=1}^{\infty} x_k s^k$. For any complex number z with $|z| < \delta$ the series converges for $z = s$ in $E_{\mathbb{C}}$, hence c can be locally extended to a holomorphic curve into $E_{\mathbb{C}}$. By the 1-dimensional uniqueness theorem for holomorphic maps, these local extensions fit together to give a holomorphic extension as required.

(3) $\Rightarrow$ (2) A holomorphic curve factors locally over $(E_{\mathbb{C}})_B$ by 7.6, where B can be chosen of the form $B \times \sqrt{-1}B$. Hence, the restriction of this factorization to $\mathbb{R}$ is topologically real analytic into E_B by 7.4.

(2) $\Rightarrow$ (1) Let c be bornologically real analytic, i.e. c is locally topologically real analytic into some E_B and hence also into E. □

Although topological real analyticity is a strictly stronger than real analyticity, cf. 9.4, sometimes the converse is true as the following slight generalization of [**13**, Lemma 7.1] shows.

9.6. Theorem. *Let E be a convenient vector space and assume that a Baire vector space topology on E^* exists for which the point evaluations ev_x for $x \in E$ are continuous. Then any real analytic curve $c : \mathbb{R} \to E$ is locally given by its Mackey convergent Taylor series, and hence is bornologically real analytic and topologically real analytic for every locally convex topology compatible with the bornology.*

Proof. Since c is real analytic, it is smooth and all derivatives exist in E, since E is convenient, by 2.14.6.

Let us fix $t_0 \in \mathbb{R}$, let $a_n := \frac{1}{n!}c^{(n)}(t_0)$. It suffices to find some $r > 0$ for which $\{r^n a_n : n \in \mathbb{N}\}$ is bounded; because then $\sum t^n a_n$ is Mackey-convergent for $|t| < r$, and its limit is $c(t_0 + t)$ since we can test this with functionals.

Consider the sets $A_r := \{\lambda \in E^* : |\lambda(a_n)| \le r^{n+1} \text{ for all } n\}$. These A_r are closed in the Baire topology, since the point evaluations at a_n are assumed to be continuous. Since c is real analytic, $\bigcup_{r>0} A_r = E^*$, and by the Baire property there is an $r > 0$

such that the interior U of A_r is not empty. Let $\lambda_0 \in U$, then for all λ in the open neighborhood $U - \lambda_0$ of 0 we have $|\lambda(a_n)| \le |(\lambda + \lambda_0)(a_n)| + |\lambda_0(a_n)| \le 2r^{n+1}$ for all n. The set $U - \lambda_0$ is absorbing, thus for every $\lambda \in E^*$ some multiple $\varepsilon\lambda$ is in $U - \lambda_0$ and so $\lambda(a_n) \le \frac{2}{\varepsilon} r^{n+1}$ as required. □

9.7. Theorem. Linear real analytic mappings. *Let E and F be convenient vector spaces. For any linear mapping $\lambda : E \to F$ the following assertions are equivalent.*

1. *λ is bounded.*
2. *$\lambda \circ c : \mathbb{R} \to F$ is real analytic for all real analytic $c : \mathbb{R} \to E$.*
3. *$\lambda \circ c : \mathbb{R} \to F$ is bornologically real analytic for all bornologically real analytic curves $c : \mathbb{R} \to E$*
4. *$\lambda \circ c : \mathbb{R} \to F$ is real analytic for all bornologically real analytic curves $c : \mathbb{R} \to E$*

This will be generalized in 10.4 to non-linear mappings.

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4), and (2) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let λ satisfy (4) and suppose that λ is unbounded. By composing with an $\ell \in E'$ we may assume that $\lambda : E \to \mathbb{R}$ and there is a bounded sequence (x_k) such that $\lambda(x_k)$ is unbounded. By passing to a subsequence we may suppose that $|\lambda(x_k)| > k^{2k}$. Let $a_k := k^{-k} x_k$, then $(r^k a_k)$ is bounded and $(r^k \lambda(a_k))$ is unbounded for any $r > 0$. Hence, the curve $c(t) := \sum_{k=0}^\infty t^k a_k$ is given by a Mackey convergent power series by 7.3. So $\lambda \circ c$ is real analytic and near 0 we have $\lambda(c(t)) = \sum_{k=0}^\infty b_k t^k$ for some $b_k \in \mathbb{R}$. But $\lambda(c(t)) = \sum_{k=0}^N \lambda(a_k) t^k + t^N \lambda(\sum_{k>N} a_k t^{k-N})$ and $t \mapsto \sum_{k>N} a_k t^{k-N}$ is still a Mackey converging power series in E. Comparing coefficients we see that $b_k = \lambda(a_k)$ and consequently $\lambda(a_k) r^k$ is bounded for some $r > 0$, a contradiction.

(1) $\Rightarrow$ (2) Let $c : \mathbb{R} \to E$ be real analytic. By theorem 9.3 the set $\{\frac{1}{k!} c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$ is bounded for all compact sets $K \subset \mathbb{R}$ and for all sequences (r_k) with $r_k\, t^k \to 0$ for all $t > 0$. Since c is smooth and bounded linear mappings are smooth by 2.11, the function $\lambda \circ c$ is smooth and $(\lambda \circ c)^{(k)}(a) = \lambda(c^{(k)}(a))$. By applying 9.3 we obtain that $\lambda \circ c$ is real analytic. □

9.8. Corollary. *For two convenient vector space structures on a vector space E the following statements are equivalent:*

1. *They have the same bounded sets.*
2. *They have the same smooth curves.*
3. *They have the same real analytic curves.*

Proof. (1) $\Leftrightarrow$ (2) was shown in 2.11. The implication (1) $\Rightarrow$ (3) follows from 9.4, which shows that real analyticity is a bornological concept, whereas the implication (1) $\Leftarrow$ (3) follows from 9.7. □

9.9. Corollary. *If a cone of linear maps $T_\alpha : E \to E_\alpha$ between convenient vector spaces generates the bornology on E, then a curve $c : \mathbb{R} \to E$ is C^ω resp. C^∞ provided all the composites $T_\alpha \circ c : \mathbb{R} \to E_\alpha$ are.*

Proof. The statement on the smooth curves is shown in 3.8. That on the real analytic curves follows again from the bornological condition of 9.3. □

10. Real Analytic Mappings

10.1. Theorem (Real analytic functions on Fréchet spaces). *Let $U \subseteq E$ be open in a real Fréchet space E. The following statements on $f : U \to \mathbb{R}$ are equivalent:*

(1) *f is smooth and is real analytic along topologically real analytic curves.*
(2) *f is smooth and is real analytic along affine lines.*
(3) *f is smooth and is locally given by its pointwise converging Taylor series.*
(4) *f is smooth and is locally given by its uniformly and absolutely converging Taylor series.*
(5) *f is locally given by a uniformly and absolutely converging power series.*
(6) *f extends to a holomorphic mapping $\tilde{f} : \tilde{U} \to \mathbb{C}$ for an open subset $\tilde{U}$ in the complexification $E_{\mathbb{C}}$ with $\tilde{U} \cap E = U$.*

Proof. (1) ⇒ (2) is obvious. The implication (2) ⇒ (3) follows from 7.14, (1) ⇒ (2), whereas (3) ⇒ (4) follows from 7.14, (2) ⇒ (3), and (4) ⇒ (5) is obvious.

(5) ⇒ (6) Locally we can extend converging power series into the complexification by 7.14. Then we take the union $\tilde{U}$ of their domains of definition and use uniqueness to glue $\tilde{f}$ which is holomorphic by 7.24.

(6) ⇒ (1) By 7.19, f is smooth. Any topologically real analytic curve c in E can locally be extended to a holomorphic curve in $E_{\mathbb{C}}$ by 9.5. The composite of the extensions is holomorphic and hence $f \circ c$ is real analytic. □

10.2. Bemerkung. The assumptions 'f is smooth' cannot be dropped in 10.1.1 even in finite dimensions, as shown by the following example, due to [**14**].

Example. *The mapping $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) := \frac{xy^{n+2}}{x^2+y^2}$ is real analytic along real analytic curves, is n-times continuous differentiable but is not smooth and hence not real analytic.*

Proof. Take a real analytic curve $t \mapsto (x(t), y(t))$ into $\mathbb{R}^2$. The components can be factored as $x(t) = t^k u(t)$, $y(t) = t^k v(t)$ for some k and real analytic curves u, v with $u(0)^2 + v(0)^2 \neq 0$. The composite $f \circ (x, y)$ is then the function $t \mapsto t^{k(n+1)} \frac{uv^{n+2}}{u^2+v^2}(t)$, which is obviously real analytic near 0. The mapping f is n-times continuous differentiable, since it is real analytic on $\mathbb{R}^2 \setminus \{0\}$ and the directional derivatives of order i are $(n + 1 - i)$-homogeneous, hence continuously extendable to $\mathbb{R}^2$. But f cannot be $(n + 1)$-times continuous differentiable, otherwise $v \mapsto f(v) = \frac{1}{(n+1)!} f^{(n+1)}(0)(v, \dots, v)$ would be and hence f would be a homogeneous polynomial of degree $n + 1$. □

10.3. Definition (Real analytic mappings). Let E be a convenient vector space. Let us denote by $C^\omega(\mathbb{R}, E)$ the *space of all real analytic curves.*

Let $U \subseteq E$ be c^∞-open, and let F be a second convenient vector space. A mapping $f : U \to F$ will be called *real analytic* or C^ω for short, if f is real analytic along real analytic curves and is smooth (i.e. is smooth along smooth curves); so $f \circ c \in C^\omega(\mathbb{R}, F)$ for all $c \in C^\omega(\mathbb{R}, E)$ with $c(\mathbb{R}) \subseteq U$ and $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$ with $c(\mathbb{R}) \subseteq U$. Let us denote by $C^\omega(U, F)$ the *space of all real analytic mappings* from U to F.

10.4. Analogue of Hartogs' Theorem for real analytic mappings. *Let E and F be convenient vector spaces, let $U \subseteq E$ be c^∞-open, and let $f : U \to F$. Then f is real analytic if and only if f is smooth and $\lambda \circ f$ is real analytic along each affine line in E, for all $\lambda \in F'$.*

Proof. ($\Rightarrow$) is clear. For the converse we may assume that $F = \mathbb{R}$, by definition 10.3 and 2.3. Let $c : \mathbb{R} \to U$ be real analytic. We show that $f \circ c$ is real analytic by using theorem 9.3. So let (r_k) be a sequence such that $r_k r_\ell \ge r_{k+\ell}$ and $r_k\, t^k \to 0$ for all $t > 0$ and let $K \subset \mathbb{R}$ be compact. We have to show, that there is an $\varepsilon > 0$ such that the set $\{\frac{1}{\ell!}(f \circ c)^{(\ell)}(a)\, r_l\, (\frac{\varepsilon}{2})^\ell : a \in K, \ell \in \mathbb{N}\}$ is bounded.

By theorem 9.3 the set $\{\frac{1}{n!}c^{(n)}(a)\, r_n : n \ge 1, a \in K\}$ is contained in some bounded absolutely convex subset $B \subseteq E$, such that E_B is a Banach space. Clearly, for the inclusion $i_B : E_B \to E$ the function $f \circ i_B$ is smooth and real analytic along affine lines. Since E_B is a Banach space, by 10.1 (2 $\Rightarrow$ 4) $f \circ i_B$ is locally given by its uniformly and absolutely converging Taylor series. Then for each $a \in K$ by 7.14 (2 $\Rightarrow$ 4) there is an $1 > \varepsilon > 0$ such that the set $\{\frac{1}{k!}d^k f(c(a))(x_1, \dots, x_k) : k \in \mathbb{N}, x_j \in \varepsilon B\}$ is bounded. For each $y \in \frac{1}{2}\varepsilon B$ termwise differentiation of the Taylor series of $f \circ \iota_B$ gives

$$d^k f(c(a) + y)(x_1, \dots, x_k) = \sum_{\ell \ge k} \frac{1}{(\ell - k)!} d^\ell f(c(a))(x_1, \dots, x_k, y, \dots, y),$$

so we may assume that $\{d^k f(c(a))(x_1, \dots, x_k)/k! : k \in \mathbb{N}, x_j \in \varepsilon B, a \in K\}$ is contained in $[-C, C]$ for some $C > 0$ and some uniform $\varepsilon > 0$.

The Taylor coefficient of $f \circ c$ at a is given by

$$\frac{(f \circ c)^{(\ell)}(a)}{\ell!} = \sum_{k \ge 0} \sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} \frac{d^k f(c(a))}{k!} \Big(\prod_n (\tfrac{1}{n!} c^{(n)}(a))^{m_n} \Big),$$

$$\text{where } \prod_n x_n^{m_n} := (\underbrace{x_1, \dots, x_1}_{m_1}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}, \dots).$$

Furthermore, we have

$$\sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} = \binom{\ell-1}{k-1}$$

by the following argument: It is the ℓ-th Taylor coefficient at 0 of the function $(\sum_{n \ge 0} t^n - 1)^k = (\frac{t}{1-t})^k = t^k \sum_{j=0}^\infty \binom{-k}{j}(-t)^j$, which turns out to be the binomial coefficient in question.

By the foregoing considerations we may estimate as follows.

$$\begin{aligned}
&\tfrac{|(f\circ c)^{(\ell)}(a)|}{\ell!}\, r_l \left(\tfrac{\varepsilon}{2}\right)^\ell \le \\
&\le \sum_{k\ge 0}\Big|\tfrac{1}{k!}\sum_{\substack{(m_n)\in\mathbb{N}_0^{\mathbb{N}}\\ \sum_n m_n = k\\ \sum_n m_n\, n=\ell}} \frac{k!}{\prod_n m_n!} d^k f(c(a))\Big(\prod_n (\tfrac{1}{n!}c^{(n)}(a))^{m_n}\Big)\Big|\, r_\ell \left(\tfrac{\varepsilon}{2}\right)^\ell \\
&\le \sum_{k\ge 0}\Big|\tfrac{1}{k!}\sum_{\substack{(m_n)\in\mathbb{N}_0^{\mathbb{N}}\\ \sum_n m_n = k\\ \sum_n m_n\, n=\ell}} \frac{k!}{\prod_n m_n!} d^k f(c(a))\Big(\prod_n (\tfrac{1}{n!}c^{(n)}(a)\, r_n\, \varepsilon^n)^{m_n}\Big)\Big|\tfrac{1}{2^\ell} \\
&\le \sum_{k\ge 0}\tbinom{\ell-1}{k-1}\, C\, \tfrac{1}{2^\ell} = \tfrac{1}{2}C,
\end{aligned}$$

because

$$\sum_{\substack{(m_n)\in\mathbb{N}_0^{\mathbb{N}}\\ \sum_n m_n = k\\ \sum_n m_n\, n=\ell}} \frac{k!}{\prod_n m_n!}\prod_n (\underbrace{\tfrac{1}{n!}c^{(n)}(a)\,\varepsilon^n\, r_n}_{\in\varepsilon^n B\subseteq \varepsilon B})^{m_n} \in \tbinom{\ell-1}{k-1}(\varepsilon B)^k \subseteq (E_B)^k. \qquad \square$$

10.5. Corollary. *Let E and F be convenient vector spaces, let $U \subseteq E$ be c^∞-open, and let $f : U \to F$. Then f is real analytic if and only if f is smooth and $\lambda\circ f\circ c$ is real analytic for every periodic (topologically) real analytic curve $c : \mathbb{R} \to U \subseteq E$ and all $\lambda \in F'$.*

Proof. By 10.4 f is real analytic if and only if f is smooth and $\lambda\circ f$ is real analytic along topologically real analytic curves $c : \mathbb{R} \to E$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = t_0 + \varepsilon\cdot\sin t$. Then $c\circ h : \mathbb{R} \to \mathbb{R} \to U$ is a (topologically) real analytic, periodic function with period 2π, provided c is (topologically) real analytic. If $c(t_0) \in U$ we can choose $\varepsilon > 0$ such that $h(\mathbb{R}) \subseteq c^{-1}(U)$. Since *sin* is locally around 0 invertible, real analyticity of $\lambda\circ f\circ c\circ h$ implies that $\lambda\circ f\circ c$ is real analytic near t_0. Hence, the proof is completed. □

10.6. Corollary. Reduction to Banach spaces. *Let E be a convenient vector space, let $U \subseteq E$ be c^∞-open, and let $f : U \to \mathbb{R}$ be a mapping. Then f is real analytic if and only if the restriction $f : E_B \supseteq U\cap E_B \to \mathbb{R}$ is real analytic for all bounded absolutely convex subsets B of E.*

So any result valid on Banach spaces can be translated into a result valid on convenient vector spaces.

Proof. By theorem 10.4 it suffices to check f along bornologically real analytic curves. These factor by definition 9.4 locally to real analytic curves into some E_B. □

10.7. Corollary. *Let U be a c^∞-open subset in a convenient vector space E and let $f : U \to \mathbb{R}$ be real analytic. Then for every bounded B there is some $r_B > 0$ such that the Taylor series $y \mapsto \sum \frac{1}{k!}d^k f(x)(y^k)$ converges to $f(x + y)$ uniformly and absolutely on $r_B B$.*

Proof. Use 10.6 and 10.1.4. □

10.8. Scalar valued analytic functions on convenient vector spaces E are in general not germs of holomorphic functions from $E_{\mathbb{C}}$ to $\mathbb{C}$:

Example. *Let $f_k : \mathbb{R} \to \mathbb{R}$ be real analytic functions with radius of convergence at zero converging to 0 for $k \to \infty$. Let $f : \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}$ be the mapping defined on the countable sum $\mathbb{R}^{(\mathbb{N})}$ of the reals by $f(x_0, x_1, \dots) := \sum_{k=1}^{\infty} x_k f_k(x_0)$. Then f is real analytic, but there is no complex valued holomorphic mapping $\tilde{f}$ on some neighborhood of 0 in $\mathbb{C}^{(\mathbb{N})}$ which extends f, and the Taylor series of f is not pointwise convergent on any c^∞-open neighborhood of 0.*

Proof. Claim. f is real analytic.
Since the limit $\mathbb{R}^{(\mathbb{N})} = \varinjlim_n \mathbb{R}^n$ is regular, every smooth curve (and hence every real analytic curve) in $\mathbb{R}^{(\mathbb{N})}$ is locally smooth (resp. real analytic) into $\mathbb{R}^n$ for some n. Hence, $f \circ c$ is locally just a finite sum of smooth (resp. real analytic) functions and is therefore smooth (resp. real analytic).

Claim. f has no holomorphic extension.
Suppose there exists some holomorphic extension $\tilde{f} : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^{(\mathbb{N})}$ is c^∞-open neighborhood of 0, and is therefore open in the locally convex Silva topology by 4.11.2. Then U is even open in the box-topology [**68**, 4.6.1], i.e., there exist $\varepsilon_k > 0$ for all k, such that $\{(z_k) \in \mathbb{C}^{(\mathbb{N})} : |z_k| \le \varepsilon_k \text{ for all } k\} \subseteq U$. Let U_0 be the open disk in $\mathbb{C}$ with radius ε_0 and let $\tilde{f}_k : U_0 \to \mathbb{C}$ be defined by $\tilde{f}_k(z) := \tilde{f}(z, 0, \dots, 0, \varepsilon_k, 0, \dots)\frac{1}{\varepsilon_k}$, where ε_k is inserted instead of the variable x_k. Obviously, $\tilde{f}_k$ is an extension of f_k, which is impossible, since the radius of convergence of f_k is less than ε_0 for k sufficiently large.

Claim. The Taylor series does not converge.
If the Taylor series would be pointwise convergent on some U, then the previous arguments for $\mathbb{R}^{(\mathbb{N})}$ instead of $\mathbb{C}^{(\mathbb{N})}$ would show that the radii of convergence of the f_k were bounded from below. □

11. The Real Analytic Exponential Law

11.1. Spaces of germs of real-analytic functions

Let M be a real analytic finite dimensional manifold. If $f : M \to N$ is a mapping between two such manifolds, then f is real analytic if and only if f maps smooth curves into smooth ones and real analytic curves into real analytic ones, by 10.1.

For each real analytic manifold M of real dimension m there is a *complex manifold* $M_{\mathbb{C}}$ of complex dimension m containing M as a real analytic closed submanifold, whose germ along M is unique ([**133**, Prop. 1]), and which can be chosen even to be a Stein manifold, see [**45**, section 3]. The complex charts are just extensions of the real analytic charts of an atlas of M into the complexification of the modeling real vector space.

Real analytic mappings $f : M \to N$ are the germs along M of holomorphic mappings $W \to N_{\mathbb{C}}$ for open neighborhoods W of M in $M_{\mathbb{C}}$.

Definition. Let $C^\omega(M, F)$ be the *space of real analytic functions* $f : M \to F$ into a convenient vector space F, and let $\mathcal{H}(M_{\mathbb{C}} \supseteq M, \mathbb{C})$ be the space of germs along

M of holomorphic functions as in 8.3. Furthermore, for any subset $A \subseteq M$ let $C^\omega(M \supseteq A, \mathbb{R})$ denotes the *space of germs of real analytic functions* along A, defined on some neighborhood of A.

We will topologize $C^\omega(M \supseteq A, \mathbb{R})$ as subspace of $\mathcal{H}(M_\mathbb{C} \supseteq A, \mathbb{C})$, in fact as the real part of it, as the following lemma shows.

11.2. Lemma. *For any subset A of M the complexification of the real vector space $C^\omega(M \supseteq A, \mathbb{R})$ is the complex vector space $\mathcal{H}(M_\mathbb{C} \supseteq A, \mathbb{C})$.*

Proof. Let $f, g \in C^\omega(M \supseteq A, \mathbb{R})$. These are germs of real analytic mappings defined on some open neighborhood of A in M. Inserting complex numbers into the locally convergent Taylor series in local coordinates shows, that f and g can be considered as holomorphic mappings from some neighborhood W of A in $M_\mathbb{C}$ to $\mathbb{C}$, which have real values if restricted to $W \cap M$. The mapping $h := f + \sqrt{-1}g : W \to \mathbb{C}$ gives then an element of $\mathcal{H}(M_\mathbb{C} \supseteq A, \mathbb{C})$.

Conversely, let $h \in \mathcal{H}(M_\mathbb{C} \supseteq A, \mathbb{C})$. Then h is the germ of a holomorphic function $\widetilde{h} : W \to \mathbb{C}$ for some open neighborhood W of A in $M_\mathbb{C}$. The decomposition of h into real and imaginary part $f = \frac{1}{2}(h + \bar{h})$ and $g = \frac{1}{2\sqrt{-1}}(h - \bar{h})$, which are real analytic functions if restricted to $W \cap M$, gives elements of $C^\omega(M \supseteq A, \mathbb{R})$.

These correspondences are inverse to each other since a holomorphic germ is determined by its restriction to a germ of mappings $M \supseteq A \to \mathbb{C}$. □

11.3. Lemma. *For a finite dimensional real analytic manifold M the inclusion $C^\omega(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is continuous.*

Proof. Consider the following diagram, where W is an open neighborhood of M in its complexification $M_\mathbb{C}$.

$$\begin{tikzcd}
C^\omega(M,\mathbb{R}) \arrow[r, tail, "\text{inclusion}"] \arrow[d, "\text{direct summand}"', "\boxed{11.2}"] & C^\infty(M,\mathbb{R}) \arrow[d, "\text{direct summand}"] \\
\mathcal{H}(M_\mathbb{C} \supseteq M,\mathbb{C}) \arrow[r, tail, "\text{inclusion}"] & C^\infty(M,\mathbb{R}^2) \\
\mathcal{H}(W,\mathbb{C}) \arrow[r, hook, "\text{inclusion}", "\boxed{8.2}"'] \arrow[u, "\text{restriction}", "\boxed{8.4}"'] & C^\infty(W,\mathbb{R}^2) \arrow[u, "\text{restriction}"']
\end{tikzcd}$$

□

11.4. Theorem (Structure of $C^\omega(M \supseteq A, \mathbb{R})$ for closed subsets A of real analytic manifolds M). *The inductive cone*

$$C^\omega(M \supseteq A, \mathbb{R}) \leftarrow \{ C^\omega(W, \mathbb{R}) : A \subseteq W \underset{open}{\subseteq} M\}$$

is regular, i.e. every bounded set is contained and bounded in some step.

The projective cone

$$C^\omega(M \supseteq A, \mathbb{R}) \to \{ C^\omega(M \supseteq K, \mathbb{R}) : K \text{ compact in } A\}$$

generates the bornology of $C^\omega(M \supseteq A, \mathbb{R})$.

If A is even a smooth submanifold, then the following projective cone also generates the bornology.

$$C^\omega(M \supseteq A, \mathbb{R}) \to \{\, C^\omega(M \supseteq \{x\}, \mathbb{R}) : x \in A\}$$

The space $C^\omega(\mathbb{R}^m \supseteq \{0\}, \mathbb{R})$ is also the regular inductive limit of the spaces $\ell^p_r(r \in \mathbb{R}^m_+)$ for all $1 \le p \le \infty$, see 8.1.

For general closed $A \subseteq N$ the space $C^\omega(M \supseteq A, \mathbb{R})$ is Montel (hence quasi-complete and reflexive), and ultra-bornological (hence a convenient vector space). It is also webbed and conuclear. If A is compact then it is even a strongly nuclear Silva space and its dual is a nuclear Fréchet space and it is smoothly paracompact. It is however not a Baire space.

Proof. This follows from 8.4, 8.6, and 8.8 by passing to the real parts using 11.2 and from the fact that all mentioned properties are inherited by complemented subspaces. □

11.5. Corollary. *A subset $\mathcal{B} \subseteq C^\omega(\mathbb{R}^m \supseteq \{0\}, \mathbb{R})$ is bounded if and only if there exists an $r > 0$ such that $\{\frac{f^{(\alpha)}(0)}{\alpha!} r^{|\alpha|} : f \in \mathbb{B}, \alpha \in \mathbb{N}_0^m\}$ is bounded in $\mathbb{R}$.*

Proof. The space $C^\omega(\mathbb{R}^m \supseteq \{0\}, \mathbb{R})$ is the regular inductive limit of the spaces ℓ^∞_r for $r \in \mathbb{R}^m_+$ by 11.4. Hence, $\mathcal{B}$ is bounded if and only if it is contained and bounded in ℓ^∞_r for some $r \in \mathbb{R}^m_+$, which is the looked for condition. □

11.6. Theorem (Special real analytic uniform boundedness principle). *For any closed subset A of a real analytic manifold M, the space $C^\omega(M \supseteq A, \mathbb{R})$ satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear functionals.*

If A has no isolated points and M is 1-dimensional this applies to the set of all point evaluations ev_t, $t \in A$.

Proof. Again this follows from 5.24 using now 11.4. If A has no isolated points and M is 1-dimensional the point evaluations are separating, by the uniqueness theorem for holomorphic functions. □

Direct proof of a particular case. We show that $C^\omega(\mathbb{R}, \mathbb{R})$ satisfies the uniform S-boundedness principle for the set $\mathcal{S}$ of all point evaluations.

We check property 5.22.2. Let $\mathcal{B} \subseteq C^\omega(\mathbb{R}, \mathbb{R})$ be absolutely convex such that $\mathrm{ev}_t(\mathcal{B})$ is bounded for all t and such that $C^\omega(\mathbb{R}, \mathbb{R})_B$ is complete. We have to show that $\mathcal{B}$ is complete.

By lemma 11.3 the set $\mathcal{B}$ satisfies the conditions of 5.22.2 in the space $C^\infty(\mathbb{R}, \mathbb{R})$. Since $C^\infty(\mathbb{R}, \mathbb{R})$ satisfies the uniform S-boundedness principle, cf. [**40**], the set $\mathcal{B}$ is bounded in $C^\infty(\mathbb{R}, \mathbb{R})$. Hence, all iterated derivatives at points are bounded on $\mathcal{B}$, and a fortiori the conditions of 5.22.2 are satisfied for $\mathcal{B}$ in $\mathcal{H}(\mathbb{R}, \mathbb{C})$. By the particular case of theorem 8.10 the set $\mathcal{B}$ is bounded in $\mathcal{H}(\mathbb{R}, \mathbb{C})$ and hence also in the direct summand $C^\omega(\mathbb{R}, \mathbb{R})$. □

11.7. Theorem. *The real analytic curves $\mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ correspond exactly to the real analytic functions $\mathbb{R}^2 \to \mathbb{R}$.*

Proof. ($\Rightarrow$) Let $f : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ be a real analytic curve. Then $f : \mathbb{R} \to C^\omega(\mathbb{R} \supseteq \{t\}, \mathbb{R})$ is also real analytic. We use theorems 11.4 and 9.6 to conclude that f is even a topologically real analytic curve in $C^\omega(\mathbb{R} \supseteq \{t\}, \mathbb{R})$. By lemma 9.5 for every $s \in \mathbb{R}$ the curve f can be extended to a holomorphic mapping from an open neighborhood of s in $\mathbb{C}$ to the complexification 11.2 $\mathcal{H}(\mathbb{C} \supseteq \{t\}, \mathbb{C})$ of $C^\omega(\mathbb{R} \supseteq \{t\}, \mathbb{R})$.

By 8.4 the space $\mathcal{H}(\mathbb{C} \supseteq \{t\}, \mathbb{C})$ is the regular inductive limit of all spaces $\mathcal{H}(U, \mathbb{C})$, where U runs through some neighborhood basis of t in $\mathbb{C}$. Lemma 7.7 shows that f is a holomorphic mapping $V \to H(U, \mathbb{C})$ for some open neighborhoods U of t and V of s in $\mathbb{C}$.

By the exponential law 7.22 for holomorphic mappings the canonically associated mapping $f^\wedge : V \times U \to \mathbb{C}$ is holomorphic. So its restriction is a real analytic function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ near (s, t) which coincides with $f^\wedge$ for the original f.

($\Leftarrow$) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real analytic mapping. Then $f(t, \_)$ is real analytic, so the associated mapping $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ makes sense. It remains to show that it is real analytic. Since the mappings $C^\omega(\mathbb{R}, \mathbb{R}) \to C^\omega(\mathbb{R} \supseteq K, \mathbb{R})$ generate the bornology, by 11.4, it is by 9.9 enough to show that $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R} \supseteq K, \mathbb{R})$ is real analytic for each compact $K \subseteq \mathbb{R}$, which may be checked locally near each $s \in \mathbb{R}$.

The real analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ extends to a holomorphic function on an open neighborhood $V \times U$ of $\{s\} \times K$ in $\mathbb{C}^2$. By cartesian closedness for the holomorphic setting the associated mapping $f^\vee : V \to \mathcal{H}(U, \mathbb{C})$ is holomorphic, so its restriction $V \cap \mathbb{R} \to C^\omega(U \cap \mathbb{R}, \mathbb{R}) \to C^\omega(K, \mathbb{R})$ is real analytic as required. □

11.8. Remark. From 11.7 it follows that the curve $c : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ defined in 9.1 is real analytic, but it is not topologically real analytic. In particular, it does not factor locally to a real analytic curve into some Banach space $C^\omega(\mathbb{R}, \mathbb{R})_B$ for a bounded subset B and it has no holomorphic extension to a mapping defined on a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ with values in the complexification $\mathcal{H}(\mathbb{R}, \mathbb{C})$ of $C^\omega(\mathbb{R}, \mathbb{R})$, cf. 9.5.

11.9. Lemma. *For a real analytic manifold M, the bornology on $C^\omega(M, \mathbb{R})$ is induced by the following cone:*
$C^\omega(M, \mathbb{R}) \xrightarrow{c^*} C^\alpha(\mathbb{R}, \mathbb{R})$ *for all $\alpha \in \{\infty, \omega\}$ and all C^α-curves $c : \mathbb{R} \to M$.*

Proof. The maps c^* are bornological, since $C^\omega(M, \mathbb{R})$ is convenient by 11.4, and by the uniform S-boundedness principle 11.6 for $C^\omega(\mathbb{R}, \mathbb{R})$ and by 5.26 for $C^\infty(\mathbb{R}, \mathbb{R})$ it suffices to check that $\mathrm{ev}_t \circ c^* = \mathrm{ev}_{c(t)}$ is bornological, which is obvious.

Conversely, we consider the identity mapping i from the space E into $C^\omega(M, \mathbb{R})$, where E is the vector space $C^\omega(M, \mathbb{R})$, but with the locally convex structure induced by the cone.
Claim. The bornology of E is complete.
The spaces $C^\omega(\mathbb{R}, \mathbb{R})$ and $C^\infty(\mathbb{R}, \mathbb{R})$ are convenient by 11.4 and 2.15, respectively. So their product

$$\prod_{c \in C^\omega(\mathbb{R}, M)} C^\omega(\mathbb{R}, \mathbb{R}) \times \prod_{c \in C^\infty(\mathbb{R}, M)} C^\infty(\mathbb{R}, \mathbb{R})$$

is also convenient. By theorem 10.1 (1⇔5) the embedding of E into this product has closed image, hence the bornology of E is complete.

Now we may apply the uniform S-boundedness principle 11.6 for $C^\omega(M,\mathbb{R})$, since obviously $\mathrm{ev}_p \circ i = \mathrm{ev}_0 \circ c_p^*$ is bounded, where c_p is the constant curve with value p, for all $p \in M$. □

11.10. Structure on $C^\omega(U,F)$. Let E be a real convenient vector space and let U be c^∞-open in E. We equip the space $C^\omega(U,\mathbb{R})$ of all real analytic functions (cf. 10.3) with the locally convex topology induced by the families of mappings

$$C^\omega(U,\mathbb{R}) \xrightarrow{c^*} C^\omega(\mathbb{R},\mathbb{R}), \text{ for all } c \in C^\omega(\mathbb{R},U)$$

$$C^\omega(U,\mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R},\mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R},U).$$

For a finite dimensional vector spaces E this definition gives the same bornology as the one defined in 11.1, by lemma 11.9.

If F is another convenient vector space, we equip the space $C^\omega(U,F)$ of all real analytic mappings (cf. 10.3) with the locally convex topology induced by the family of mappings

$$C^\omega(U,F) \xrightarrow{\lambda_*} C^\omega(U,\mathbb{R}), \text{ for all } \lambda \in F'.$$

Obviously, the injection $C^\omega(U,F) \to C^\infty(U,F)$ is bounded and linear.

11.11. Lemma. *Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then $C^\omega(U,F)$ is also convenient.*

Proof. This follows immediately from the fact that $C^\omega(U,F)$ can be considered as closed subspace of the product of factors $C^\omega(U,\mathbb{R})$ indexed by all $\lambda \in F'$ and these factors can in turn be considered as closed subspaces of the product of the factors $C^\omega(\mathbb{R},\mathbb{R})$ indexed by all $c \in C^\omega(\mathbb{R},U)$ and the factors $C^\infty(\mathbb{R},\mathbb{R})$ indexed by all $c \in C^\infty(\mathbb{R},U)$. Since all factors are convenient so are the closed subspaces. □

11.12. Theorem (General real analytic uniform boundedness principle). *Let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. Then $C^\omega(U,F)$ satisfies the uniform $\mathcal{S}$-boundedness principle, where $\mathcal{S} := \{ev_x : x \in U\}$.*

Proof. The convenient structure of $C^\omega(U,F)$ is induced by the cone of mappings $c^* : C^\omega(U,F) \to C^\omega(\mathbb{R},F)$ (for $c \in C^\omega(\mathbb{R},U)$) together with the maps $c^* : C^\omega(U,F) \to C^\infty(\mathbb{R},F)$ (for $c \in C^\infty(\mathbb{R},U)$). Both spaces $C^\omega(\mathbb{R},F)$ and $C^\infty(\mathbb{R},F)$ satisfy the uniform $\mathcal{T}$-boundedness principle, where $\mathcal{T} := \{ev_t : t \in \mathbb{R}\}$, by 11.6 and 5.26, respectively. Hence, $C^\omega(U,F)$ satisfies the uniform $\mathcal{S}$-boundedness principle by lemma 5.25, since $ev_t \circ c^* = ev_{c(t)}$. □

11.14. Theorem. *Let E_i for $i = 1,\dots n$ and F be convenient vector spaces. Then the bornology on $L(E_1,\dots,E_n;F)$ (described in 5.1, see also 5.6) is induced by the embedding $L(E_1,\dots,E_n;F) \to C^\omega(E_1 \times \dots \times E_n, F)$.*

Thus, any mapping f into $L(E_1,\dots,E_n;F)$ is real analytic if and only if the composites $ev_x \circ f$ are real analytic for all $x \in E_1 \times \dots \times E_n$, by 9.9.

Proof. Let $\mathcal{S} := \{\mathrm{ev}_x : x \in E_1 \times \dots \times E_n\}$. Since $C^\omega(E_1 \times \dots \times E_n, F)$ satisfies the uniform $\mathcal{S}$-boundedness principle 11.12, the inclusion is bounded. On the other hand $L(E_1,\dots,E_n;F)$ also satisfies the uniform $\mathcal{S}$-boundedness principle by 5.18, so the identity from $L(E_1,\dots,E_n;F)$ with the bornology induced from $C^\omega(E_1\times\dots\times E_n,F)$ into $L(E_1,\dots,E_n;F)$ is bounded as well.

Since to be real analytic depends only on the bornology by 9.4 and since the convenient vector space $L(E_1,\dots,E_n;F)$ satisfies the uniform $\mathcal{S}$-boundedness principle, the second assertion follows also. □

The following two results will be generalized in 11.20. At the moment we will make use of the following lemma only in case where $E=C^\infty(\mathbb{R},\mathbb{R})$.

11.15. Lemma. *For any convenient vector space E the flip of variables induces an isomorphism $L(E,C^\omega(\mathbb{R},\mathbb{R}))\cong C^\omega(\mathbb{R},L(E,\mathbb{R}))=C^\omega(\mathbb{R},E')$ as vector spaces.*

Proof. For $c\in C^\omega(\mathbb{R},E')$ consider $\tilde c(x):=\mathrm{ev}_x\circ c\in C^\omega(\mathbb{R},\mathbb{R})$ for $x\in E$. By the uniform $\mathcal{S}$-boundedness principle 11.6 for $\mathcal{S}=\{\mathrm{ev}_t:t\in\mathbb{R}\}$ the linear mapping $\tilde c$ is bounded, since $\mathrm{ev}_t\circ\tilde c=c(t)\in E'$.

If conversely $\ell\in L(E,C^\omega(\mathbb{R},\mathbb{R}))$, we consider $\tilde\ell(t)=\mathrm{ev}_t\circ\ell\in E'=L(E,\mathbb{R})$ for $t\in\mathbb{R}$. Since the bornology of E' is generated by $\mathcal{S}:=\{ev_x:x\in E\}$, $\tilde\ell:\mathbb{R}\to E'$ is real analytic, for $\mathrm{ev}_x\circ\tilde\ell=\ell(x)\in C^\omega(\mathbb{R},\mathbb{R})$, by 11.14. □

11.16. Corollary. *We have $C^\infty(\mathbb{R},C^\omega(\mathbb{R},\mathbb{R}))\cong C^\omega(\mathbb{R},C^\infty(\mathbb{R},\mathbb{R}))$ as vector spaces.*

Proof. The dual $C^\infty(\mathbb{R},\mathbb{R})'$ is the free convenient vector space (see [**75**, 23.6]) over $\mathbb{R}$ by [**75**, 23.11], and $C^\omega(\mathbb{R},\mathbb{R})$ is convenient by 11.4, so we have

$$\begin{aligned}C^\infty(\mathbb{R},C^\omega(\mathbb{R},\mathbb{R}))&\cong L(C^\infty(\mathbb{R},\mathbb{R})',C^\omega(\mathbb{R},\mathbb{R}))\\&\cong C^\omega(\mathbb{R},C^\infty(\mathbb{R},\mathbb{R})'')\qquad\text{by lemma 11.15}\\&\cong C^\omega(\mathbb{R},C^\infty(\mathbb{R},\mathbb{R})),\end{aligned}$$

by reflexivity of $C^\infty(\mathbb{R},\mathbb{R})$, see [**75**, 6.5.7]. □

11.17. Theorem. *Let E be a convenient vector space, let U be c^∞-open in E, let $f:\mathbb{R}\times U\to\mathbb{R}$ be a real analytic mapping and let $c\in C^\infty(\mathbb{R},U)$. Then $c^*\circ f^\vee:\mathbb{R}\to C^\omega(U,\mathbb{R})\to C^\infty(\mathbb{R},\mathbb{R})$ is real analytic.*

This result on the mixing of C^∞ and C^ω will become quite essential in the proof of cartesian closedness. It will be generalized in 11.21, see also 42.15.

Proof. Let $I\subseteq\mathbb{R}$ be open and relatively compact, let $t\in\mathbb{R}$ and $k\in\mathbb{N}$. Now choose an open and relatively compact $J\subseteq\mathbb{R}$ containing the closure $\bar I$ of I. There is a bounded subset $B\subseteq E$ such that $c\mid J:J\to E_B$ is a $\mathcal{L}\mathrm{ip}^k$-curve in the Banach space E_B generated by B, by 1.8. Let U_B denote the open subset $U\cap E_B$ of the Banach space E_B. Since the inclusion $E_B\to E$ is continuous, f is real analytic as a function $\mathbb{R}\times U_B\to\mathbb{R}\times U\to\mathbb{R}$. Thus, by 10.1 there is a holomorphic extension $f:V\times W\to\mathbb{C}$ of f to an open set $V\times W\subseteq\mathbb{C}\times(E_B)_{\mathbb{C}}$ containing the compact set $\{t\}\times c(\bar I)$. By cartesian closedness of the category of holomorphic mappings $f^\vee:V\to H(W,\mathbb{C})$ is holomorphic. By 8.2 the bornological structure of $\mathcal{H}(W,\mathbb{C})$ is induced by that of $C^\infty(W,\mathbb{C}):=C^\infty(W,\mathbb{R}^2)$. And $c^*:C^\infty(W,\mathbb{C})\to\mathcal{L}\mathrm{ip}^k(I,\mathbb{C})$ is a bounded $\mathbb{C}$-linear map, by the chain rule 12.9 for $\mathcal{L}\mathrm{ip}^k$-mappings and by the uniform boundedness principle 12.10 for the point evaluations on $\mathcal{L}\mathrm{ip}^k(I,\mathbb{C})$.

Thus, $c^* \circ f^\vee : V \to \mathcal{L}ip^k(I,\mathbb{C})$ is holomorphic, and hence its restriction to $\mathbb{R}\cap V$, which has values in $\mathcal{L}ip^k(I,\mathbb{R})$, is (even topologically) real analytic by 9.5. Since $t\in\mathbb{R}$ was arbitrary we conclude that $c^*\circ f^\vee : \mathbb{R}\to \mathcal{L}ip^k(I,\mathbb{R})$ is real analytic. But the bornology of $C^\infty(\mathbb{R},\mathbb{R})$ is generated by the inclusions into $\mathcal{L}ip^k(I,\mathbb{R})$, by the uniform boundedness principles 5.26 for $C^\infty(\mathbb{R},\mathbb{R})$ and 12.9 for $\mathcal{L}ip^k(\mathbb{R},\mathbb{R})$, and hence $c^*\circ f^\vee : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is real analytic. □

11.18. Theorem. Cartesian closedness. *The category of real analytic mappings between convenient vector spaces is cartesian closed. More precisely, for convenient vector spaces E, F and G and c^∞-open sets $U\subseteq E$ and $W\subseteq G$ a mapping $f : W\times U\to F$ is real analytic if and only if $f^\vee : W \to C^\omega(U,F)$ is real analytic.*

Proof.Step 1. The theorem is true for $W=G=F=\mathbb{R}$.

($\Leftarrow$) Let $f^\vee : \mathbb{R}\to C^\omega(U,\mathbb{R})$ be C^ω. We have to show that $f:\mathbb{R}\times U\to\mathbb{R}$ is C^ω. We consider a curve $c_1:\mathbb{R}\to\mathbb{R}$ and a curve $c_2:\mathbb{R}\to U$.

If the c_i are C^∞, then $c_2^*\circ f^\vee : \mathbb{R}\to C^\omega(U,\mathbb{R})\to C^\infty(\mathbb{R},\mathbb{R})$ is C^ω by assumption, hence is C^∞, so $c_2^*\circ f^\vee\circ c_1 : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is C^∞. By cartesian closedness of smooth mappings, $(c_2^*\circ f^\vee\circ c_1)^\wedge = f\circ(c_1\times c_2) : \mathbb{R}^2\to\mathbb{R}$ is C^∞. By composing with the diagonal mapping $\Delta:\mathbb{R}\to\mathbb{R}^2$ we obtain that $f\circ(c_1,c_2):\mathbb{R}\to\mathbb{R}$ is C^∞.

If the c_i are C^ω, then $c_2^*\circ f^\vee : \mathbb{R}\to C^\omega(U,\mathbb{R})\to C^\omega(\mathbb{R},\mathbb{R})$ is C^ω by assumption, so $c_2^*\circ f^\vee\circ c_1 : \mathbb{R}\to C^\omega(\mathbb{R},\mathbb{R})$ is C^ω. By theorem 11.7 the associated map $(c_2^*\circ f^\vee\circ c_1)^\wedge = f\circ(c_1\times c_2):\mathbb{R}^2\to\mathbb{R}$ is C^ω. So $f\circ(c_1,c_2):\mathbb{R}\to\mathbb{R}$ is C^ω.

($\Rightarrow$) Let $f:\mathbb{R}\times U\to\mathbb{R}$ be C^ω. We have to show that $f^\vee:\mathbb{R}\to C^\omega(U,\mathbb{R})$ is real analytic. Obviously, $f^\vee$ has values in this space. We consider a curve $c:\mathbb{R}\to U$.

If c is C^∞, then by theorem 11.17 the associated mapping $c^*\circ f^\vee:\mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is C^ω.

If c is C^ω, then $f\circ(\mathrm{id}\times c):\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times U\to\mathbb{R}$ is C^ω. By theorem 11.7 the associated mapping $(f\circ(\mathrm{id}\times c))^\vee = c^*\circ f^\vee : \mathbb{R}\to C^\omega(\mathbb{R},\mathbb{R})$ is C^ω.

Step 2. The theorem is true for $F=\mathbb{R}$.

($\Leftarrow$) Let $f^\vee : W\to C^\omega(U,\mathbb{R})$ be C^ω. We have to show that $f:W\times U\to\mathbb{R}$ is C^ω. We consider a curve $c_1:\mathbb{R}\to W$ and a curve $c_2:\mathbb{R}\to U$.

If the c_i are C^∞, then $c_2^*\circ f^\vee : W\to C^\omega(U,\mathbb{R})\to C^\infty(\mathbb{R},\mathbb{R})$ is C^ω by assumption, hence is C^∞, so $c_2^*\circ f^\vee\circ c_1:\mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is C^∞. By cartesian closedness of smooth mappings, the associated mapping $(c_2^*\circ f^\vee\circ c_1)^\wedge = f\circ(c_1\times c_2):\mathbb{R}^2\to\mathbb{R}$ is C^∞. So $f\circ(c_1,c_2):\mathbb{R}\to\mathbb{R}$ is C^∞.

If the c_i are C^ω, then $f^\vee\circ c_1:\mathbb{R}\to W\to C^\omega(U,\mathbb{R})$ is C^ω by assumption, so by step 1 the mapping $(f^\vee\circ c_1)^\wedge = f\circ(c_1\times \mathrm{id}_U):\mathbb{R}\times U\to\mathbb{R}$ is C^ω. Hence, $f\circ(c_1,c_2) = f\circ(c_1\times\mathrm{id}_U)\circ(\mathrm{id},c_2):\mathbb{R}\to\mathbb{R}$ is C^ω.

($\Rightarrow$) Let $f:W\times U\to\mathbb{R}$ be C^ω. We have to show that $f^\vee:W\to C^\omega(U,\mathbb{R})$ is real analytic. Obviously, $f^\vee$ has values in this space. We consider a curve $c_1:\mathbb{R}\to W$.

If c_1 is C^∞, we consider a second curve $c_2:\mathbb{R}\to U$. If c_2 is C^∞, then $f\circ(c_1\times c_2): \mathbb{R}\times\mathbb{R}\to W\times U\to\mathbb{R}$ is C^∞. By cartesian closedness the associated mapping $(f\circ(c_1\times c_2))^\vee = c_2^*\circ f^\vee\circ c_1:\mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is C^∞. If c_2 is C^ω, the mapping $f\circ(\mathrm{id}_W\times c_2):W\times\mathbb{R}\to\mathbb{R}$ and also the flipped one $(f\circ(\mathrm{id}_W\times c_2))^\sim:\mathbb{R}\times W\to\mathbb{R}$

are C^ω, hence by theorem 11.17 $c_1^* \circ ((f \circ (\mathrm{id}_W \times c_2))^\sim)^\vee : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ is C^ω. By corollary 11.16 the associated mapping $(c_1^* \circ ((f \circ (\mathrm{id}_W \times c_2))^\sim)^\vee)^\sim = c_2^* \circ f^\vee \circ c_1 : \mathbb{R} \to C^\omega(\mathbb{R},\mathbb{R})$ is C^∞. So for both families describing the structure of $C^\omega(U,\mathbb{R})$ we have shown that the composite with $\check{f} \circ c_1$ is C^∞, so $f^\vee \circ c_1$ is C^∞.

If c_1 is C^ω, then $f \circ (c_1 \times \mathrm{id}_U) : \mathbb{R} \times U \to W \times U \to \mathbb{R}$ is C^ω. By step 1 the associated mapping $(f \circ (c_1 \times \mathrm{id}_U))^\vee = f^\vee \circ c_1 : \mathbb{R} \to C^\omega(U,\mathbb{R})$ is C^ω.

Step 3. The general case.

$$\begin{aligned}
& f : W \times U \to F \text{ is } C^\omega \\
\Leftrightarrow\quad & \lambda \circ f : W \times U \to \mathbb{R} \text{ is } C^\omega \text{ for all } \lambda \in F' \\
\Leftrightarrow\quad & (\lambda \circ f)^\vee = \lambda_* \circ f^\vee : W \to C^\omega(U,\mathbb{R}) \text{ is } C^\omega, \text{ by step 2} \\
\Leftrightarrow\quad & f^\vee : W \to C^\omega(U,F) \text{ is } C^\omega, \text{ by } 11.10 \text{ and } 9.4 \qquad \square
\end{aligned}$$

11.19. Corollary. Canonical mappings are real analytic. *The following mappings are C^ω:*

(1) $\mathrm{ev} : C^\omega(U,F) \times U \to F$, $(f,x) \mapsto f(x)$,
(2) $\mathrm{ins} : E \to C^\omega(F, E \times F)$, $x \mapsto (y \mapsto (x,y))$,
(3) $(\_)^\wedge : C^\omega(U, C^\omega(V,G)) \to C^\omega(U \times V, G)$,
(4) $(\_)^\vee : C^\omega(U \times V, G) \to C^\omega(U, C^\omega(V,G))$,
(5) $\mathrm{comp} : C^\omega(F,G) \times C^\omega(U,F) \to C^\omega(U,G)$, $(f,g) \mapsto f \circ g$,
(6) $C^\omega(\_,\_) : C^\omega(E_2,E_1) \times C^\omega(F_1,F_2) \to$
$\to C^\omega(C^\omega(E_1,F_1), C^\omega(E_2,F_2))$, $(f,g) \mapsto (h \mapsto g \circ h \circ f)$.

Proof. Just consider the canonically associated smooth mappings on multiple products, as in 3.13. $\square$

11.20. Lemma. Canonical isomorphisms. *One has the following natural isomorphisms:*

(1) $C^\omega(W_1, C^\omega(W_2,F)) \cong C^\omega(W_2, C^\omega(W_1,F))$,
(2) $C^\omega(W_1, C^\infty(W_2,F)) \cong C^\infty(W_2, C^\omega(W_1,F))$.
(3) $C^\omega(W_1, L(E,F)) \cong L(E, C^\omega(W_1,F))$.
(4) $C^\omega(W_1, \ell^\infty(X,F)) \cong \ell^\infty(X, C^\omega(W_1,F))$.
(5) $C^\omega(W_1, \mathcal{L}\mathrm{ip}^k(X,F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^\omega(W_1,F))$.

In (4) the space X is a ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf. [**41**, 1.2.4]. *In (5) the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [**41**, 1.4.1]. *The spaces $\ell^\infty(X,F)$ and $\mathcal{L}\mathrm{ip}^k(W,F)$ are defined in* [**41**, 3.6.1,4.4.1].

Proof. All isomorphisms, as well as their inverse mappings, are given by the flip of coordinates: $f \mapsto \tilde{f}$, where $\tilde{f}(x)(y) := f(y)(x)$. Furthermore, all occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations, by 11.11, 11.14, 11.12, and by [**41**, 3.6.1,4.4.2,3.6.6,4.4.7].

That $\tilde{f}$ has values in the corresponding spaces follows from the equation $\tilde{f}(x) = ev_x \circ f$. One only has to check that $\tilde{f}$ itself is of the corresponding class, since it follows that $f \mapsto \tilde{f}$ is bounded as a consequence of the uniform boundedness principle:

$$(\mathrm{ev}_x \circ \tilde{(\_)})(f) = \mathrm{ev}_x(\tilde{f}) = \tilde{f}(x) = \mathrm{ev}_x \circ f = (\mathrm{ev}_x)_*(f).$$

That $\tilde{f}$ is of the appropriate class in (1) and (2) follows by composing with $c_1 \in C^{\beta_1}(\mathbb{R}, W_1)$ and $C^{\beta_2}(\lambda, c_2) : C^{\alpha_2}(W_2, F) \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ for all $\lambda \in F'$ and $c_2 \in C^{\beta_2}(\mathbb{R}, W_2)$, where β_k and α_k are in $\{\infty, \omega\}$ and $\beta_k \le \alpha_k$ for $k \in \{1, 2\}$. Then $C^{\beta_2}(\lambda, c_2) \circ \tilde{f} \circ c_1 = (C^{\beta_1}(\lambda, c_1) \circ f \circ c_2)^\sim : \mathbb{R} \to C^{\beta_2}(\mathbb{R}, \mathbb{R})$ is C^{β_1} by 11.7 and 11.16, since $C^{\beta_1}(\lambda, c_1) \circ f \circ c_2 : \mathbb{R} \to W_2 \to C^{\alpha_1}(W_1, F) \to C^{\beta_1}(\mathbb{R}, \mathbb{R})$ is C^{β_2}.

That $\tilde{f}$ is of the appropriate class in (3) follows, since $L(E, F)$ is the c^∞-closed subspace of $C^\omega(E, F)$ formed by the linear C^ω-mappings.

That $\tilde{f}$ is of the appropriate class in (4) or (5) follows from (3), using the free convenient vector spaces $\ell^1(X)$ or $\lambda^k(X)$ over the ℓ^∞-space X or the the $\mathcal{L}\mathrm{ip}^k$-space X, see [**41**, 5.1.24or5.2.3], satisfying $\ell^\infty(X, F) \cong L(\ell^1(X), F)$ or satisfying $\mathcal{L}\mathrm{ip}^k(X, F) \cong L(\lambda^k(X), F)$. Existence of these free convenient vector spaces can be proved in a similar way as [**75**, 23.6]. □

Definition. By a $C^{\infty,\omega}$-mapping $f : U \times V \to F$ we mean a mapping f for which $f^\vee \in C^\infty(U, C^\omega(V, F)) \cong C^\omega(V, C^\infty(U, F))$.

11.21. Theorem. Composition of $C^{\infty,\omega}$-mappings. *Let $f : U \times V \to F$ and $g : U_1 \times V_1 \to V \subseteq G$ be $C^{\infty,\omega}$, and $h : U_1 \to U$ be C^∞. Then $f \circ (h \circ \mathrm{pr}_1, g) : U_1 \times V_1 \to F$, $(x, y) \mapsto f(h(x), g(x, y))$ is $C^{\infty,\omega}$.*

Proof. We have to show that the mapping $x \mapsto (y \mapsto f(h(x), g(x, y)))$, $U_1 \to C^\omega(V_1, F)$ is C^∞. It is well-defined, since f and g are C^ω in the second variable. In order to show that it is C^∞ we compose with $\lambda_* : C^\omega(V_1, F) \to C^\omega(V_1, \mathbb{R})$, where $\lambda \in F'$ is arbitrary. Thus, it is enough to consider the case $F = \mathbb{R}$. Furthermore, we compose with $c^* : C^\omega(V_1, \mathbb{R}) \to C^\alpha(\mathbb{R}, \mathbb{R})$, where $c \in C^\alpha(\mathbb{R}, V_1)$ is arbitrary for α equal to ω and ∞.

In case $\alpha = \infty$ the composite with c^* is C^∞, since the associated mapping $U_1 \times \mathbb{R} \to \mathbb{R}$ is $f \circ (h \circ \mathrm{pr}_1, g \circ (\mathrm{id} \times c))$ which is C^∞.

Now the case $\alpha = \omega$. Let $I \subseteq \mathbb{R}$ be an arbitrary open bounded interval. Then $c^* \circ g^\vee : U_1 \to C^\omega(\mathbb{R}, G)$ is C^∞, where G is the convenient vector space containing V as an c^∞-open subset, and has values in $\{\gamma : \gamma(\bar{I}) \subseteq V\} \subseteq C^\omega(\mathbb{R}, G)$. This set is c^∞-open, since it is open for the topology of uniform convergence on compact sets which is coarser than the bornological topology on $C^\infty(\mathbb{R}, G)$ and hence than the c^∞-topology on $C^\omega(\mathbb{R}, G)$, see 11.10.

Thus, the composite of $(f \circ (h \circ \mathrm{pr}_1, g))^\vee$ with c^*, $\mathrm{comp} \circ (f^\vee \circ h, c^* \circ g^\vee)$ is C^∞, since $f^\vee \circ h : U_1 \to U \to C^\omega(V, F)$ is C^∞, $c^* \circ g^\vee : U_1 \to C^\omega(\mathbb{R}, G)$ is C^∞ and $\mathrm{comp} : C^\omega(V, \mathbb{R}) \times \{\gamma \in C^\omega(\mathbb{R}, G) : \gamma(\bar{I}) \subseteq V\} \to C^\omega(I, \mathbb{R})$ is C^ω, because it is associated to $\mathrm{ev} \circ (\mathrm{id} \times \mathrm{ev}) : C^\omega(V, \mathbb{R}) \times \{\gamma \in C^\omega(\mathbb{R}, G) : \gamma(\bar{I}) \subseteq V\} \times I \to V$. That $\mathrm{ev} : \{\gamma \in C^\omega(\mathbb{R}, G) : \gamma(\bar{I}) \subseteq V\} \times I \to V \subseteq G$ is C^ω follows, since the associated mapping is the restriction mapping $C^\omega(\mathbb{R}, G) \to C^\omega(I, G)$. □

11.22. Corollary. *Let $w : W_1 \to W$ be C^ω, let $u : U \to U_1$ be smooth, let $v : V \to V_1$ be C^ω, and let $f : U_1 \times V_1 \to W_1$ be $C^{\infty,\omega}$. Then $w \circ f \circ (u \times v) : U \times V \to W$ is again $C^{\infty,\omega}$.*

This is a generalization of theorem 11.17.

Proof. Use 11.21 twice: First $h := u$, $g := v \circ \mathrm{pr}_2$, $f := f$, and then $h := \mathrm{id}$, $f := w \circ \mathrm{pr}_1$, $g := f \circ (u \times v)$. $\square$

11.23. Corollary. *Let $f : E \supseteq U \to F$ be C^ω, let $I \subseteq \mathbb{R}$ be open and bounded, and α be ω or ∞. Then $f_* : C^\alpha(\mathbb{R}, E) \supseteq \{c : c(\bar{I}) \subseteq U\} \to C^\alpha(I, F)$ is C^ω.*

Proof. Obviously, $f_*(c) := f \circ c \in C^\alpha(I, F)$ is well-defined for all $c \in C^\alpha(\mathbb{R}, E)$ satisfying $c(\bar{I}) \subseteq U$.

Furthermore, $\{c : c(\bar{I}) \subseteq U\} \subseteq C^\alpha(\mathbb{R}, E)$ is c^∞-open, since it is open for the topology of uniform convergence on compact sets which is coarser than the bornological and hence than the c^∞-topology on $C^\alpha(\mathbb{R}, E)$.

Finally, the composite of f_* with any C^β-curve $\gamma : \mathbb{R} \to \{c : c(\bar{I}) \subseteq U\} \subseteq C^\alpha(\mathbb{R}, E)$ is a C^β-curve in $C^\alpha(I, F)$ for β equal to ω or ∞: For $\beta = \alpha$ this follows from cartesian closedness of the C^α-maps since $(f_* \circ \gamma)^\wedge = f \circ \gamma^\wedge$. For $\alpha \neq \beta$ this follows from 11.22: For $\beta < \alpha$ use $u := \mathrm{id}$, $v := \mathrm{id}$, $f := \gamma^\wedge$, $w := f$; And for $\alpha < \beta$ flip the variables in $\mathbb{R}$ and I. $\square$

11.24. Lemma. Derivatives. *The derivative d, where $df(x)(v) := \frac{d}{dt}|_{t=0} f(x + tv)$, is bounded and linear $d : C^\omega(U, F) \to C^\omega(U, L(E, F))$.*

Proof. The differential $df(x)(v)$ makes sense and is linear in v, because every real analytic mapping f is smooth. So it remains to show that $(f, x, v) \mapsto df(x)(v)$ is real analytic. For this let f, x, and v depend real analytically (resp. smoothly) on a real parameter s. Since $(t, s) \mapsto x(s) + tv(s)$ is real analytic (resp. smooth) into $U \subseteq E$, the mapping $r \mapsto ((t, s) \mapsto f(r)(x(s) + tv(s))$ is real analytic into $C^\omega(\mathbb{R}^2, F)$ (resp. smooth into $C^\infty(\mathbb{R}^2, F)$. Composing with the bounded linear map $\frac{\partial}{\partial t}|_{t=0}$: $C^\omega(\mathbb{R}^2, F) \to C^\omega(\mathbb{R}, F)$ (resp. : $C^\infty(\mathbb{R}^2, F) \to C^\infty(\mathbb{R}, F)$) shows that $r \mapsto (s \mapsto d(f(r))(x(s))(v(s))$ is real analytic into $C^\omega(\mathbb{R}, F)$ (resp. smooth into $C^\infty(\mathbb{R}, \mathbb{R})$). Considering the associated real analytic (resp. smooth) mapping on $\mathbb{R}^2$ composed with the diagonal map shows that $(f, x, v) \mapsto df(x)(v)$ is real analytic. $\square$

The following examples as well as several others can be found in [41, 5.3.6].

11.25. Example. *Let $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ be given by $T(f) = f'$. Then the continuous linear differential equation $x'(t) = T(x(t))$ with initial value $x(0) = x_0$ has a unique smooth solution $x(t)(s) = x_0(t+s)$ which is however not real analytic.*

Note the curious form $x'(t) = x(t)'$ of this differential equation. Beware of careless notation!

Proof. A smooth curve $x : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is a solution of the differential equation $x'(t) = T(x(t))$ if and only if $\frac{\partial}{\partial t}\hat{x}(t, s) = \frac{\partial}{\partial s}\hat{x}(t, s)$. Hence, we have $\frac{d}{dt}\hat{x}(t, r-t) = 0$, i.e. $t \mapsto \hat{x}(t, r - t)$ is constant and hence equal to $\hat{x}(0, r) = x_0(r)$. Thus, $\hat{x}(t, s) = x_0(t + s)$.

Suppose $x : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ were real analytic. Then the composite with $ev_0 : C^\infty(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ were a real analytic function. But this composite is just $x_0 = ev_0 \circ x$, which is not in general real analytic. $\square$

11.26. Example. *Let E be either $C^\infty(\mathbb{R},\mathbb{R})$ or $C^\omega(\mathbb{R},\mathbb{R})$. Then the mapping $\exp_* : E \to E$ is C^ω, has invertible derivative at every point, but the image does not contain an open neighborhood of $\exp_*(0)$.*

Proof. The mapping $\exp_*$ is real analytic by 11.23. Its derivative is given by $(\exp_*)'(f)(g) : t \mapsto g(t)e^{f(t)}$ and hence is invertible with $g \mapsto (t \mapsto g(t)e^{-f(t)})$ as inverse mapping. Now consider the real analytic curve $c : \mathbb{R} \to E$ given by $c(t)(s) = 1 - (ts)^2$. One has $c(0) = 1 = \exp_*(0)$, but $c(t)$ is not in the image of $\exp_*$ for any $t \neq 0$, since $c(t)(\frac{1}{t}) = 0$ but $\exp_*(g)(t) = e^{g(t)} > 0$ for all g and t. $\square$

Historical Remarks on Holomorphic and Real Analytic Calculus

The notion of holomorphic mappings used in section 15 was first defined by the Italian Luigi Fantappié in the papers [**35**] and [**36**]:

S.1: "Wenn jeder Funktion $y(t)$ einer Funktionenmenge H eine bestimmte Zahl f entspricht, d.h. die Zahl f von der Funktion $y(t)$ (unabhängige Veränderliche in der Menge H) abhängt, werden wir sagen, daß ein Funktional von $y(t)$:

$$f = F[y(t)]$$

ist; H heißt das Definitionsfeld des Funktionals F.
[...] gemischtes Funktional [...]

$$f = F[y_1(t_1, \dots), \dots, y_n(t_1, \dots); z_1, \dots, z_m]\text{"}$$

He also considered the 'functional transform' and noticed the relation

$$f = F[y(t); z] \text{ corresponds to } y \mapsto f(z)$$

S.4: "Sei jetzt $F(y(t))$ ein Funktional, das in einem Funktionenbereich H (von analytischen Funktionen) definiert ist, und $y_0(t)$ ein Funktion von H, die mit einer Umgebung (r) oder (r, σ) zu H angehört. Wenn für jede analytische Mannigfaltigkeit $y(t; \alpha_1, \dots, \alpha_m)$, die in diese Umgebung eindringt (d.h. eine solche, die für alle Wertesysteme $\alpha_1, \dots, \alpha_m$) eines Bereichs Γ eine Funktion von t der Umgebung liefert), der Wert des Funktionals

$$F_t[y(t; \alpha_1, \dots, \alpha_m)] = f(\alpha_1, \dots, \alpha_m)$$

immer eine Funktion der Parameter $\alpha_1, \dots, \alpha_m$ ist, die nicht nur in Γ definiert, sondern dort noch eine analytische Funktion ist, werden wir sagen, daß das Funktional F regulär ist in der betrachteten Umgebung $y_0(t)$. Wenn ein Funktional F regulär ist in einer Umgebung jeder Funktion seines Definitionsbereiches, so heißt F analytisch."

The development in the complex case was much faster than in the smooth case since one did not have to explain the concept of higher derivatives.

The Portuguese José Sebastião e Silva showed that analyticity in the sense of Fantappié coincides with other concepts, in his dissertation [**111**], published as [**112**], and in [**113**]. An overview over various notions of holomorphicity was given by the Brasilian Domingos Pisanelli in [**103**] and [**104**].

Chapter III
Partitions of Unity

The main aim of this chapter is to discuss the abundance or scarcity of smooth functions on a convenient vector space: E.g. existence of bump functions and partitions of unity. This question is intimately related to differentiability of seminorms and norms, and in many examples these are, if at all, only finitely often differentiable. So we start this chapter with a short (but complete) account of finite order differentiability, based on Lipschitz conditions on higher derivatives, since with this notion we can get as close as possible to exponential laws. A more comprehensive exposition of finite order Lipschitz differentiability can be found in the monograph [**40**].

Then we treat differentiability of seminorms and convex functions, and we have tried to collect all relevant information from the literature. We give full proofs of all what will be needed later on or is of central interest. We also collect related results, mainly on 'generic differentiability', i.e. differentiability on a dense G_δ-set.

If enough smooth bump functions exist on a convenient vector space, we call it 'smoothly regular'. Although the smooth (i.e. bounded) linear functionals separate points on any convenient vector space, stronger separation properties depend very much on the geometry. In particular, we show that ℓ^1 and $C[0,1]$ are not even C^1-regular. We also treat more general 'smooth spaces' here since most results do not depend on a linear structure, and since we will later apply them to manifolds.

In many problems like E. Borel's theorem 15.4 that any power series appears as Taylor series of a smooth function, or the existence of smooth functions with given carrier 15.3, one uses in finite dimensions the existence of smooth functions with globally bounded derivatives. These do not exist in infinite dimensions in general; even for bump functions this need not be true globally. Extreme cases are Hilbert spaces where there are smooth bump functions with globally bounded derivatives, and c_0 which does not even admit C^2-bump functions with globally bounded derivatives.

In the final section of this chapter a space which admits smooth partitions of unity subordinated to any open cover is called smoothly paracompact. Fortunately, a wide class of convenient vector spaces has this property, among them all spaces of smooth sections of finite dimensional vector bundles which we shall need later as modeling spaces for manifolds of mappings. The theorem 16.15 of [**121**] characterizes smoothly paracompact metrizable spaces, and we will give a full proof. It is the only tool for investigating whether non-separable spaces are smoothly paracompact and we give its main applications.

12. Differentiability of Finite Order

12.1. Definition

A mapping $f : E \supseteq U \to F$, where E and F are convenient vector spaces, and $U \subseteq E$ is c^∞-open, is called $\mathcal{L}\mathrm{ip}^k$ if $f \circ c$ is a $\mathcal{L}\mathrm{ip}^k$-curve (see 1.2) for each $c \in C^\infty(\mathbb{R}, U)$.

This is equivalent to the property that $f \circ c$ is $\mathcal{L}\mathrm{ip}^k$ on $c^{-1}(U)$ for each $c \in C^\infty(\mathbb{R}, E)$. This can be seen by reparameterization.

12.2. General curve lemma. *Let E be a convenient vector space, and let $c_n \in C^\infty(\mathbb{R}, E)$ be a sequence of curves which converges fast to 0, i.e., for each $k \in \mathbb{N}$ the sequence $n^k c_n$ is bounded. Let $s_n \geq 0$ be reals with $\sum_n s_n < \infty$.*

Then there exists a smooth curve $c \in C^\infty(\mathbb{R}, E)$ and a converging sequence of reals t_n such that $c(t + t_n) = c_n(t)$ for $|t| \leq s_n$, for all n.

Proof. Let $r_n := \sum_{k<n}(\frac{2}{k^2} + 2s_k)$ and $t_n := \frac{r_n + r_{n+1}}{2}$. Let $h : \mathbb{R} \to [0, 1]$ be smooth with $h(t) = 1$ for $t \geq 0$ and $h(t) = 0$ for $t \leq -1$, and put $h_n(t) := h(n^2(s_n + t)) \cdot h(n^2(s_n - t))$. Then we have $h_n(t) = 0$ for $|t| \geq \frac{1}{n^2} + s_n$ and $h_n(t) = 1$ for $|t| \leq s_n$, and for the derivatives we have $|h_n^{(j)}(t)| \leq n^{2j} \cdot H_j$, where $H_j := \max\{|h^{(i)}| : t \in \mathbb{R}, i \leq j\}^2$. Thus, in the sum

$$c(t) := \sum_n h_n(t - t_n) \cdot c_n(t - t_n)$$

at most one summand is non-zero for each $t \in \mathbb{R}$, and c is a smooth curve since for each $\ell \in E'$ we have

$$(\ell \circ c)(t) = \sum_n f_n(t), \qquad \text{where } f_n(t + t_n) := h_n(t) \cdot \ell(c_n(t)),$$

$$\begin{aligned}
n^2 \cdot \sup_t |f_n^{(k)}(t)| &= n^2 \cdot \sup\Big\{|f_n^{(k)}(s + t_n)| : |s| \leq \tfrac{1}{n^2} + s_n\Big\}\\
&\leq n^2 \sum_{j=0}^{k} \tbinom{k}{j} n^{2j} H_j \cdot \sup\Big\{|(\ell \circ c_n)^{(k-j)}(s)| : |s| \leq \tfrac{1}{n^2} + s_n\Big\}\\
&\leq \Big(\sum_{j=0}^{k} \tbinom{k}{j} n^{2j+2} H_j\Big) \cdot \sup\Big\{|(\ell \circ c_n)^{(i)}(s)| : |s| \leq \max_n(\tfrac{1}{n^2} + s_n) \text{ and } i \leq k\Big\},
\end{aligned}$$

which is uniformly bounded with respect to n, since c_n converges to 0 fast. □

12.3. Corollary. *Let $c_n : \mathbb{R} \to E$ be polynomials of bounded degree with values in a convenient vector space E. If for each $\ell \in E'$ the sequence $n \mapsto \sup\{|(\ell \circ c_n)(t) : |t| \leq 1\}$ converges to 0 fast, then the sequence c_n converges to 0 fast in $C^\infty(\mathbb{R}, E)$, so the conclusion of 12.2 holds.*

Proof. The structure on $C^\infty(\mathbb{R}, E)$ is the initial one with respect to the cone $\ell_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E'$, by 3.9. So we only have to show the result for $E = \mathbb{R}$. On the finite dimensional space of all polynomials of degree at most d the expression in the assumption is a norm, and the inclusion into $C^\infty(\mathbb{R},\mathbb{R})$ is bounded. $\square$

12.4. Difference quotients

For a curve $c:\mathbb{R}\to E$ with values in a vector space E the *difference quotient* $\delta^k c$ of order k is given recursively by

$$\delta^0 c := c,$$
$$\delta^k c(t_0,\dots,t_k) := k\,\frac{\delta^{k-1}c(t_0,\dots,t_{k-1})-\delta^{k-1}c(t_1,\dots,t_k)}{t_0-t_k},$$

for pairwise different t_i. The constant factor k in the definition of δ^k is chosen in such a way that δ^k approximates the k-th derivative. By induction, one can easily see that

$$\delta^k c(t_0,\dots,t_k) = k!\sum_{i=0}^{k} c(t_i)\prod_{\substack{0\le j\le k\\ j\ne i}}\tfrac{1}{t_i-t_j}$$

and, in particular, $\delta^k c$ is symmetric. We shall mainly need the *equidistant difference quotient* $\delta^k_{\mathrm{eq}}c$ of order k, which is given by

$$\delta^k_{\mathrm{eq}}c(t;v) := \delta^k c(t,t+v,\dots,t+kv) = \frac{k!}{v^k}\sum_{i=0}^{k} c(t+iv)\prod_{\substack{0\le j\le k\\ j\ne i}}\tfrac{1}{i-j}.$$

Lemma. *For a convenient vector space E and a curve $c:\mathbb{R}\to E$ the following conditions are equivalent:*

(1) *c is $\mathcal{L}\mathrm{ip}^{k-1}$.*
(2) *The difference quotient $\delta^k c$ of order k is bounded on bounded sets.*
(3) *$\ell\circ c$ is continuous for each $\ell\in E'$, and the equidistant difference quotient $\delta^k_{eq}c$ of order k is bounded on bounded sets in $\mathbb{R}\times(\mathbb{R}\setminus\{0\})$.*

The continuity assumption in 3 is necessary, see [**41**, 1.3.12].

Proof. All statements can be tested by composing with bounded linear functionals $\ell\in E'$, so we may assume that $E=\mathbb{R}$ and $k>1$.

(3) $\Rightarrow$ (2) Let $I\subset\mathbb{R}$ be a bounded interval. Then there is some $K>0$ such that $|\delta^k_{\mathrm{eq}}c(x;v)|\le K$ for all $x\in I$ and $kv\in I$. Let $t_i\in I$ be pairwise different points. We claim that $|\delta^k c(t_0,\dots,t_k)|\le K$. Since $\delta^k c$ is symmetric we may assume that $t_0<t_1<\dots<t_k$, and since it is continuous (c is continuous) we may assume that all $\frac{t_i-t_0}{t_k-t_0}$ are of the form $\frac{n_i}{N}$ for $n_i,N\in\mathbb{N}$. Put $v:=\frac{t_k-t_0}{N}$, then $\delta^k c(t_0,\dots,t_k)=\delta^k c(t_0,t_0+n_1v,\dots,t_0+n_kv)$ is a universal convex combination of $\delta^k_{\mathrm{eq}}c(t_0+rv;v)$ for $0\le r\le\max_i n_i-k$: This follows by recursively inserting intermediate points of the form t_0+mv, and using

$$\begin{aligned}\delta^k(t_0+m_0v,\dots,\widehat{t_0+m_iv},\dots,t_0+m_{k+1}v) &=\\ =\frac{m_i-m_0}{m_{k+1}-m_0}&\delta^k(t_0+m_0v,\dots,t_0+m_kv)\\ &+\frac{m_{k+1}-m_i}{m_{k+1}-m_0}\delta^k(t_1+m_1v,\dots,t_0+m_{k+1}v)\end{aligned}$$

which itself may be proved by induction on k (see [41, 1.3.9]).

(2) $\Rightarrow$ (1) We have to show that c is k times differentiable and that $\delta^1 c^{(k)}$ is bounded on bounded sets. We use induction, $k=0$ is clear.

Let $T \neq S$ be two subsets of $\mathbb{R}$ of cardinality $j+1$. Then there exist enumerations $T = \{t_0, \dots, t_j\}$ and $S = \{s_0, \dots, s_j\}$ such that $t_i \neq s_j$ for $i \leq j$; then we have

$$\delta^j c(t_0, \dots, t_j) - \delta^j c(s_0, \dots, s_j) = \tfrac{1}{j+1} \sum_{i=0}^{j} (t_i - s_i) \delta^{j+1} c(t_0, \dots, t_i, s_i, \dots, s_j).$$

In fact, for the enumerations we put the elements of $T \cap S$ at the end in T and at the beginning in S. Using the recursive definition of $\delta^{j+1}c$ and its symmetry the right hand side becomes a telescoping sum, see [41, 1.3.13]

Since $\delta^k c$ is bounded we see from the last equation that all $\delta^j c$ are also bounded, in particular this is true for $\delta^2 c$. Then

$$\frac{c(t+s)-c(t)}{s} - \frac{c(t+s')-c(t)}{s'} = \tfrac{s-s'}{2}\, \delta^2 c(t, t+s, t+s')$$

shows that the difference quotient of c forms a Mackey Cauchy net, and hence the limit $c'(t)$ exists.

Using the easily checked formula (see [41, 1.3.6])

$$c(t_j) = \sum_{i=0}^{j} \tfrac{1}{i!} \prod_{l=0}^{i-1} (t_j - t_l)\, \delta^i c(t_0, \dots, t_i), \tag{7}$$

induction on j and differentiability of c one shows (see [41, 1.3.16.ii]) that

$$\delta^j c'(t_0, \dots, t_j) = \frac{1}{j+1} \sum_{i=0}^{j} \delta^{j+1} c(t_0, \dots, t_j, t_i), \tag{4}$$

where $\delta^{j+1} c(t_0, \dots, t_j, t_i) := \lim_{t \to t_i} \delta^{j+1} c(t_0, \dots, t_j, t)$. The right hand side of 4 is bounded, so c' is $\mathcal{L}\mathrm{ip}^{k-2}$ by induction on k.

(1) $\Rightarrow$ (2) For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and $t_0 < \dots < t_j$ there exist s_i with $t_i < s_i < t_{i+1}$ such that

$$\delta^j f(t_0, \dots, t_j) = \delta^{j-1} f'(s_0, \dots, s_{j-1}). \tag{5}$$

In fact, let p be the interpolation polynomial (use 7 to see this, cf. [41, 1.3.7.i])

$$p(t) := \sum_{i=0}^{j} \tfrac{1}{i!} \prod_{l=0}^{i-1} (t - t_l)\, \delta^i f(t_0, \dots, t_i). \tag{6}$$

Since f and p agree on all t_i, by Rolle's theorem the first derivatives of f and p agree on some intermediate points s_i. So p' is the interpolation polynomial for f' at these points s_i. Comparing the coefficient of highest order of p' and of the interpolation polynomial 6 for f' at the points s_i 5 follows. (see [41, 1.3.15])

Applying 5 recursively for $f = c^{(k-2)}, c^{(k-3)}, \dots, c$ shows that $\delta^k c$ is bounded on bounded sets, and (2) follows.

(2) $\Rightarrow$ (3) is obvious. $\square$

12.5

Let $r_0, \dots, r_k$ be the unique rational solution of the linear equation

$$\sum_{i=0}^{k} i^j r_i = \begin{cases} 1 & \text{for } j = 1 \\ 0 & \text{for } j = 0, 2, 3, \dots, k. \end{cases}$$

Lemma. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}\mathrm{ip}^k$ for $k \geq 1$ and I is a compact interval then there exists M such that for all $t, v \in I$ we have*

$$\Big| \tfrac{\partial}{\partial s}|_0 f(t,s) \cdot v - \sum_{i=0}^{k} r_i f(t, iv) \Big| \leq M|v|^{k+1}.$$

Proof. We consider first the case $0 \notin I$ so that v stays away from 0. For this it suffices to show that the derivative $\frac{\partial}{\partial s}|_0 f(t,s)$ is locally bounded. If it is unbounded near some point x_∞, there are x_n with $|x_n - x_\infty| \leq \frac{1}{2^n}$ such that $\frac{\partial}{\partial s}|_0 f(x_n, s) \geq n \cdot 2^n$. We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := (x_n, \frac{t}{2^n})$ and to $s_n := \frac{1}{2^n}$ in order to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $|t| \leq s_n$. Then $(f \circ c)'(t_n) = \frac{1}{2^n} \frac{\partial}{\partial s}|_0 f(x_n, s) \geq n$, which contradicts that f is $\mathcal{L}\mathrm{ip}^1$.

Now we treat the case $0 \in I$. If the assertion does not hold there are $x_n, v_n \in I$, such that $\Big| \frac{\partial}{\partial s}|_0 f(x_n, s) \cdot v_n - \sum_{i=0}^k r_i f(x_n, iv_n) \Big| \geq n \cdot 2^{n(k+1)} |v_n|^{k+1}$. We may assume $x_n \to x_\infty$, and by the case $0 \notin I$ we may assume that $v_n \to 0$, even with $|x_n - x_\infty| \leq \frac{1}{2^n}$ and $|v_n| \leq \frac{1}{2^n}$. We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := (x_n, \frac{t}{2^n})$ and to $s_n := \frac{1}{2^n}$ to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $|t| \leq s_n$. Then we have

$$\begin{aligned} \Big| (f \circ c)'(t_n) 2^n v_n - \sum_{i=0}^{k} r_i (f \circ c)(t_n + i 2^n v_n) \Big| &= \\ &= \Big| (f \circ c_n)'(0) 2^n v_n - \sum_{i=0}^{k} r_i (f \circ c_n)(i 2^n v_n) \Big| \\ &= \Big| \tfrac{1}{2^n} \tfrac{\partial}{\partial s}|_0 f(x_n, s) 2^n v_n - \sum_{i=0}^{k} r_i f(x_n, i v_n) \Big| \geq n(2^n |v_n|)^{k+1}. \end{aligned}$$

This contradicts the next claim for $g = f \circ c$.

Claim. If $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}\mathrm{ip}^k$ for $k \geq 1$ and I is a compact interval then there is $M > 0$ such that for $t, v \in I$ we have $\Big| g'(t) \cdot v - \sum_{i=0}^k r_i g(t + iv) \Big| \leq M|v|^{k+1}$.

Consider $g_t(v) := g'(t) \cdot v - \sum_{i=0}^k r_i g(t + iv)$. Then the derivatives up to order k at $v = 0$ of g_t vanish by the choice of the r_i. Since $g^{(k)}$ is locally Lipschitzian there exists an M such that $|g_t^{(k)}(v)| \leq M|v|$ for all $t, v \in I$, which we may integrate in turn to obtain $|g_t(v)| \leq M \frac{|v|^{k+1}}{(k+1)!}$. □

12.6. Lemma. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{L}\mathrm{ip}^{k+1}$. Then $t \mapsto \frac{\partial}{\partial s}|_0 f(t,s)$ is $\mathcal{L}\mathrm{ip}^k$.*

Proof. Suppose that $g : t \mapsto \frac{\partial}{\partial s}|_0 f(t,s)$ is not $\mathcal{L}\mathrm{ip}^k$. Then by lemma 12.4 the equidistant difference quotient $\delta^{k+1}_{\mathrm{eq}} g$ is not locally bounded at some point which we may assume to be 0. Thus there are x_n and v_n with $|x_n| \le 1/4^n$ and $0 < v_n < 1/4^n$ such that

$$|\delta^{k+1}_{\mathrm{eq}} g(x_n; v_n)| > n \cdot 2^{n(k+2)}. \tag{1}$$

We apply the general curve lemma 12.2 to the curves $c_n : \mathbb{R} \to \mathbb{R}^2$ given by $c_n(t) := e_n(\frac{t}{2^n} + x_n) := (\frac{t}{2^n} + x_n - v_n, \frac{t}{2^n})$ and to $s_n := \frac{k+2}{2^n}$ in order to obtain a smooth curve $c : \mathbb{R} \to \mathbb{R}^2$ and scalars $t_n \to 0$ with $c(t + t_n) = c_n(t)$ for $0 \le t \le s_n$.

Put $f_0(t,s) := \sum_{i=0}^k r_i\, f(t, is)$ for r_i as in 12.5, put $f_1(t,s) := g(t)s$, finally put $f_2 := f_1 - f_0$. Then f_0 in $\mathcal{L}\mathrm{ip}^{k+1}$, so $f_0 \circ c$ is $\mathcal{L}\mathrm{ip}^{k+1}$, hence the equidistant difference quotient $\delta^{k+2}_{\mathrm{eq}}(f_0 \circ c)(x_n; 2^n v_n)$ is bounded.

By lemma 12.5 there exists $M > 0$ such that $|f_2(t,s)| \le M|s|^{k+2}$ for all $t, s \in [-(k+1), k+1]$, so we get

$$\begin{aligned}
|\delta^{k+2}_{\mathrm{eq}}(f_2 \circ c)(x_n; 2^n v_n)| &= |\delta^{k+2}_{\mathrm{eq}}(f_2 \circ c_n)(0; 2^n v_n)| \\
&= \tfrac{1}{2^{n(k+2)}} |\delta^{k+2}_{\mathrm{eq}}(f_2 \circ e_n)(x_n; v_n)| \\
&\le \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=1}^{k+2} \frac{|f_2((i-1)v_n + x_n, iv_n)|}{|iv_n|^{(k+2)}} \frac{i^{(k+2)}}{\prod_{j \ne i} |i-j|} \\
&\le \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=1}^{k+2} M \frac{i^{(k+2)}}{\prod_{j \ne i} |i-j|}.
\end{aligned}$$

This is bounded, and so for $f_1 = f_0 + f_2$ the expression $|\delta^{k+2}_{\mathrm{eq}}(f_1 \circ c)(x_n; 2^n v_n)|$ is also bounded, with respect to n. However, on the other hand we get

$$\begin{aligned}
\delta^{k+2}_{\mathrm{eq}}(f_1 \circ c)(x_n; 2^n v_n) &= \delta^{k+2}_{\mathrm{eq}}(f_1 \circ c_n)(0; 2^n v_n) \\
&= \tfrac{1}{2^{n(k+2)}} \delta^{k+2}_{\mathrm{eq}}(f_1 \circ e_n)(x_n; v_n) \\
&= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=0}^{k+2} \frac{f_1((i-1)v_n + x_n, iv_n)}{v_n^{(k+2)}} \prod_{\substack{0 \le j \le k+2 \\ j \ne i}} \tfrac{1}{i-j} \\
&= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{i=0}^{k+2} \frac{g((i-1)v_n + x_n) i v_n}{v_n^{(k+2)}} \prod_{\substack{0 \le j \le k+2 \\ j \ne i}} \tfrac{1}{i-j} \\
&= \tfrac{(k+2)!}{2^{n(k+2)}} \sum_{l=0}^{k+1} \frac{g(l v_n + x_n)}{v_n^{(k+1)}} \prod_{\substack{0 \le j \le k+1 \\ j \ne l}} \tfrac{1}{l-j} \\
&= \tfrac{k+2}{2^{n(k+2)}} \delta^{k+1}_{\mathrm{eq}} g(x_n; v_n),
\end{aligned}$$

which in absolute value is larger than $(k+2)n$ by 1, a contradiction. $\square$

12.7. Lemma. *Let $U \subseteq E$ be open in a normed space. Then, a mapping $f : U \to F$ into a convenient vector space is $\mathcal{L}\mathrm{ip}^0$ if and only if f is Lipschitz on compact subsets K of U, i.e., $\{\frac{f(x)-f(y)}{\|x-y\|} : x \ne y \in K\}$ is bounded.*

A mapping $f : U \to F$ into a Banach space is $\mathcal{L}\mathrm{ip}^0$ if and only if f is locally Lipschitz, i.e., for each $z \in U$ there exists a ball B_z around z such that $\{\frac{f(x)-f(y)}{\|x-y\|} : x \ne y \in B_z\}$ is bounded.

Proof. ($\Rightarrow$) If F is Banach and f is $\mathcal{L}\mathrm{ip}^0$ but not locally Lipschitz near $z \in U$, there are points $x_n \neq y_n$ in U with $\|x_n - z\| \leq 1/4^n$ and $\|y_n - z\| \leq 1/4^n$, such that $\|f(y_n) - f(x_n)\| \geq n \cdot 2^n \cdot \|y_n - x_n\|$. Now we apply the general curve lemma 12.2 with $s_n := 2^n \cdot \|y_n - x_n\|$ and $c_n(t) := x_n - z + t\frac{y_n - x_n}{2^n\|y_n - x_n\|}$ to get a smooth curve c with $c(t + t_n) - z = c_n(t)$ for $0 \leq t \leq s_n$. Then $\frac{1}{s_n}\|(f \circ c)(t_n + s_n) - (f \circ c)(t_n)\| = \frac{1}{2^n \cdot \|y_n - x_n\|}\|f(y_n) - f(x_n)\| \geq n$.

If F is convenient, f is $\mathcal{L}\mathrm{ip}^0$ but not Lipschitz on a compact K, there exist $\ell \in F'$ such that $\ell \circ f$ is not Lipschitz on K. By the first part of the proof, $\ell \circ f$ is locally Lipschitz, a contradiction.

($\Leftarrow$) This is obvious, since the composition of Lipschitz mappings is again Lipschitz. $\square$

12.8. Theorem. *Let $f : E \supseteq U \to F$ be a mapping, where E and F are convenient vector spaces, and $U \subseteq E$ is c^∞-open. Then the following assertions are equivalent for each $k \geq 0$:*

(1) *f is $\mathcal{L}\mathrm{ip}^{k+1}$.*
(2) *The directional derivative*
$$(d_v f)(x) := \tfrac{\partial}{\partial t}|_{t=0} f(x + tv)$$
exists for $x \in U$ and $v \in E$ and defines a $\mathcal{L}\mathrm{ip}^k$-mapping $U \times E \to F$.

Note that this result gives a different (more algebraic) proof of Boman's theorem 3.4 and 3.14.

Proof. (1) $\Rightarrow$ (2) Clearly, $t \mapsto f(x + tv)$ is $\mathcal{L}\mathrm{ip}^{k+1}$, and so the directional derivative exists and is the Mackey-limit of the difference quotients, by lemma 1.7. In order to show that $df : (x, v) \mapsto d_v f(x)$ is $\mathcal{L}\mathrm{ip}^k$ we take a smooth curve $(x, v) : \mathbb{R} \to U \times E$ and $\ell \in F'$, and we consider $g(t, s) := x(t) + s \cdot v(t)$, $g : \mathbb{R}^2 \to E$. Then $\ell \circ f \circ g : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{L}\mathrm{ip}^{k+1}$, so by lemma 12.6 the curve
$$t \mapsto \ell(df(x(t), v(t))) = \ell\left(\tfrac{\partial}{\partial s}|_0 f(g(t, s))\right) = \tfrac{\partial}{\partial s}|_0 \ell(f(g(t, s)))$$
is of class $\mathcal{L}\mathrm{ip}^k$.

(2) $\Rightarrow$ (1) If $c \in C^\infty(\mathbb{R}, U)$ then
$$\frac{f(c(t)) - f(c(0))}{t} - df(c(0), c'(0)) =$$
$$= \int_0^1 \underbrace{\Big(df\big(c(0) + s(c(t) - c(0)), \tfrac{c(t)-c(0)}{t}\big) - df\big(c(0), c'(0)\big)\Big)}_{=: g(t,s)} ds$$
converges to 0 for $t \to 0$: In fact, g is $\mathcal{L}\mathrm{ip}^k$, thus by lemma 12.7 g is locally Lipschitz, so the set of all $\frac{g(t_1,s) - g(t_2,s)}{t_1 - t_2}$ is locally bounded, and finally $t \mapsto \int_0^1 g(t, s) ds$ is locally Lipschitz and 0 at $t = 0$. Thus, $f \circ c$ is differentiable with derivative $(f \circ c)'(0) = df(c(0), c'(0))$.

Since df is $\mathcal{L}\mathrm{ip}^k$ and (c, c') is smooth we get that $(f \circ c)'$ is $\mathcal{L}\mathrm{ip}^k$, hence $f \circ c$ is $\mathcal{L}\mathrm{ip}^{k+1}$. $\square$

12.9. Corollary. Chain rule. *The composition of $\mathcal{L}\mathrm{ip}^k$-mappings is again $\mathcal{L}\mathrm{ip}^k$, and the usual formula for the derivative of the composite holds.*

Proof. We have to compose $f \circ g$ with a smooth curve c, but then $g \circ c$ is a $\mathcal{L}\mathrm{ip}^k$-curve, thus it is sufficient to show that the composition of a $\mathcal{L}\mathrm{ip}^k$ curve $c : \mathbb{R} \to U \subseteq E$ with a $\mathcal{L}\mathrm{ip}^k$-mapping $f : U \to F$ is again $\mathcal{L}\mathrm{ip}^k$, and that $(f\circ c)'(t) = df(c(t), c'(t))$.

This follows by induction on k for $k \geq 1$ by the proof of theorem 12.8 (2$\Rightarrow$1), where it is enough to assume c to be $\mathcal{L}\mathrm{ip}^{k+1}$. □

12.10. Definition and Proposition. *Let F be a convenient vector space. The space $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ of all $\mathcal{L}\mathrm{ip}^k$-curves in F is again a convenient vector space with the following equivalent structures:*

(1) *The initial structure with respect to the $k+2$ linear mappings (for $0 \leq j \leq k+1$) $c \mapsto \delta^j c$ from $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ into the space of all F-valued maps in $j+1$ pairwise different real variables $(t_0, \dots, t_j)$ which are bounded on bounded subsets, with the c^∞-complete locally convex topology of uniform convergence on bounded subsets. In fact, the mappings δ^0 and δ^{k+1} are sufficient.*

(2) *The initial structure with respect to the $k+2$ linear mappings (for $0 \leq j \leq k+1$) $c \mapsto \delta^j_{eq} c$ from $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ into the space of all maps from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ into F which are bounded on bounded subsets, with the c^∞-complete locally convex topology of uniform convergence on bounded subsets. In fact, the mappings δ^0_{eq} and δ^{k+1}_{eq} are sufficient.*

(3) *The initial structure with respect to the derivatives of order $j \leq k$ considered as linear mappings into the space of $\mathcal{L}\mathrm{ip}^0$-curves, with the locally convex topology of uniform convergence of the curve on bounded subsets of $\mathbb{R}$ and of the difference quotient on bounded subsets of $\{(t, s) \in \mathbb{R}^2 : t \neq s\}$.*

The convenient vector space $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ satisfies the uniform boundedness principle with respect to the point evaluations.

Proof. All three structures describe closed embeddings into finite products of spaces, which in (1) and (2) are obviously c^∞-complete. For (3) this follows, since by (1) the structure on $\mathcal{L}\mathrm{ip}^0(\mathbb{R}, E)$ is convenient.

All structures satisfy the uniform boundedness principle for the point evaluations by 5.25, and since spaces of all bounded mappings on some (bounded) set satisfy this principle. This can be seen by composing with ℓ_* for all $\ell \in E'$, since Banach spaces do this by 5.24.

By applying this uniform boundedness principle one sees that all these structures are indeed equivalent. □

12.11. Definition and Proposition. *Let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. The space $\mathcal{L}\mathrm{ip}^k(U, F)$ of all $\mathcal{L}\mathrm{ip}^k$-mappings from U to F is again a convenient vector space with the following equivalent structures:*

(1) *The initial structure with respect to the linear mappings $c^* : \mathcal{L}\mathrm{ip}^k(U, F) \to \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, F)$.*

(2) *The initial structure with respect to the linear mappings $c^* : \mathcal{L}\mathrm{ip}^k(U, F) \to \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$ for all $c \in \mathcal{L}\mathrm{ip}^k(\mathbb{R}, F)$.*

This space satisfies the uniform boundedness principle with respect to the evaluations $\mathrm{ev}_x : \mathcal{L}\mathrm{ip}^k(U, F) \to F$ for all $x \in U$.

Proof. The structure (1) is convenient since by 12.1 it is a closed subspace of the product space which is convenient by 12.10. The structure in (2) is convenient since it is closed by 12.9. The uniform boundedness principle for the point evaluations now follows from 5.25 and 12.10, and this in turn gives us the equivalence of the two structures. □

12.12. Remark

We want to call the attention of the reader to the fact that there is no general exponential law for $\mathcal{L}\text{ip}^k$-mappings. In fact, if $f \in \mathcal{L}\text{ip}^k(\mathbb{R}, \mathcal{L}\text{ip}^k(\mathbb{R}, F))$ then $(\frac{\partial}{\partial t})^p(\frac{\partial}{\partial s})^q f^\wedge(t,s)$ exists if $\max(p,q) \le k$. This describes a smaller space than $\mathcal{L}\text{ip}^k(\mathbb{R}^2, F)$, which is not invariantly describable.

However, some partial results still hold, namely for convenient vector spaces E, F, and G, and for c^∞-open sets $U \subseteq E$, $V \subseteq F$ we have

$$\mathcal{L}\text{ip}^k(U, L(F,G)) \cong L(F, \mathcal{L}\text{ip}^k(U,G)),$$
$$\mathcal{L}\text{ip}^k(U, \mathcal{L}\text{ip}^l(V,G)) \cong \mathcal{L}\text{ip}^l(V, \mathcal{L}\text{ip}^k(U,G)),$$

see [**41**, 4.4.5,4.5.1,4.5.2]. For a mapping $f : U \times F \to G$ which is linear in F we have: $f \in \mathcal{L}\text{ip}^k(U \times F, G)$ if and only if $f^\vee \in \mathcal{L}\text{ip}^k(U, L(E,F))$, see [**41**, 4.3.5]. The last property fails if we weaken Lipschitz to continuous, see the following example.

12.13. Smolyanov's Example

Let $f : \ell^2 \to \mathbb{R}$ be defined by $f := \sum_{k\ge1} \frac{1}{k^2} f_k$, where $f_k(x) := \varphi(k(kx_k - 1)) \cdot \prod_{j<k} \varphi(jx_j)$ and $\varphi : \mathbb{R} \to [0,1]$ is smooth with $\varphi(0) = 1$ and $\varphi(t) = 0$ for $|t| \ge \frac14$. We shall show that

(1) $f : \ell^2 \to \mathbb{R}$ is Fréchet differentiable.
(2) $f' : \ell^2 \to (\ell^2)'$ is not continuous.
(3) $f' : \ell^2 \times \ell^2 \to \mathbb{R}$ is continuous.

Proof. Let $A := \{x \in \ell^2 : |kx_k| \le \frac14 \text{ for all } k\}$. This is a closed subset of ℓ^2.

(1) Remark that for $x \in \ell^2$ at most one $f_k(x)$ can be unequal to 0. In fact $f_k(x) \ne 0$ implies that $|kx_k - 1| \le \frac{1}{4k} \le \frac14$, and hence $kx_k \ge \frac34$ and thus $f_j(x) = 0$ for $j > k$.

For $x \notin A$ there exists a $k > 0$ with $|kx_k| > \frac14$ and the set of points satisfying this condition is open. It follows that $\varphi(kx_k) = 0$ and hence $f = \sum_{j\le k} \frac{1}{j^2} f_j$ is smooth on this open set.

On the other hand let $x \in A$. Then $|kx_k - 1| \ge \frac34 > \frac14$ and hence $\varphi(k(kx_k - 1)) = 0$ for all k and thus $f(x) = 0$. Let $v \in \ell^2$ be such that $f(x+v) \ne 0$. Then there exists a unique k such that $f_k(x+v) \ne 0$ and therefore $|j(x_j + v_j)| < \frac14$ for $j < k$ and $|k(x_k+v_k)-1| < \frac{1}{4k} \le \frac14$. Since $|kx_k| \le \frac14$ we conclude $|kv_k| \ge 1-|k(x_k+v_k)-1| - |kx_k| \ge 1 - \frac14 - \frac14 = \frac12$. Hence $|f(x+v)| = \frac{1}{k^2}|f_k(x+v)| \le \frac{1}{k^2} \le (2|v_k|)^2 \le 4\|v\|^2$. Thus $\frac{\|f(x+v)-0-0\|}{\|v\|} \le 4\|v\| \to 0$ for $\|v\| \to 0$, i.e. f is Fréchet differentiable at x with derivative 0.

($\boxed{2}$) If fact take $a \in \mathbb{R}$ with $\varphi'(a) \neq 0$. Then $f'(t\,e^k)(e^k) = \frac{d}{dt}\frac{1}{k^2} f_k(t\,e^k) = \frac{d}{dt}\frac{1}{k^2}\varphi(k^2\,t - k) = \varphi'(k\,(k\,t-1)) = \varphi'(a)$ if $t = t_k := \frac{1}{k}\left(\frac{a}{k}+1\right)$, which goes to 0 for $k \to \infty$. However $f'(0)(e^k) = 0$ since $0 \in A$.

($\boxed{3}$) We have to show that $f'(x^n)(v^n) \to f'(x)(v)$ for $(x^n, v^n) \to (x, v)$. For $x \notin A$ this is obviously satisfied, since then there exists a k with $|kx_k| > \frac{1}{4}$ and hence $f = \sum_{j\le k} \frac{1}{j^2} f_j$ locally around x.

If $x \in A$ then $f'(x) = 0$ and thus it remains to consider the case, where $x^n \notin A$. Let $\varepsilon > 0$ be given. Locally around x^n at most one summand f_k does not vanish: If $x^n \notin A$ then there is some k with $|kx^k| > 1/4$ which we may choose minimal. Thus $|jx^j| \le 1/4$ for all $j < k$, so $|j(jx^j - 1)| \ge 3j/4$ and hence $f_j = 0$ locally since the first factor vanishes. For $j > k$ we get $f_j = 0$ locally since the second factor vanishes. Thus we can evaluate the derivative:

$$|f'(x^n)(v^n)| = \Big|\frac{1}{k^2} f_k'(x^n)(v^n)\Big| \le \frac{\|\varphi'\|_\infty}{k^2}\Big(k^2|v_k^n| + \sum_{j<k} j|v_j^n|\Big).$$

Since $v \in \ell^2$ we find a K_1 such that $(\sum_{j\ge K_1}|v_j|^2)^{1/2} \le \frac{\varepsilon}{2\|\varphi'\|_\infty}$. Thus we conclude from $\|v^n - v\|_2 \to 0$ that $|v_j^n| \le \frac{\varepsilon}{\|\varphi'\|_\infty}$ for $j \ge K_1$ and large n. For the finitely many small n we can increase K_1 such that for these n and $j \ge K_1$ also $|v_j^n| \le \frac{\varepsilon}{\|\varphi'\|_\infty}$. Furthermore there is a constant $K_2 \ge 1$ such that $\|v^n\|_\infty \le \|v^n\|_2 \le K_2$ for all n. Now choose $N \ge K_1$ so large that $N^2 \ge \frac{1}{\varepsilon}\|\varphi'\|_\infty K_2\,K_1^2$. Obviously $\sum_{n<N}\frac{1}{n^2} f_n$ is smooth. So it remains to consider those n for which the non-vanishing term has index $k \ge N$. For those terms we have

$$\begin{aligned}
|f'(x^n)(v^n)| &= \Big|\frac{1}{k^2} f_k'(x^n)(v^n)\Big| \le \|\varphi'\|_\infty\Big(|v_k^n| + \frac{1}{k^2}\sum_{j<k} j|v_j^n|\Big)\\
&\le |v_k^n|\|\varphi'\|_\infty + \|\varphi'\|_\infty \frac{1}{k^2}\sum_{j<K_1} j|v_j^n| + \frac{1}{k^2}\sum_{K_1\le j<k} j|v_j^n|\,\|\varphi'\|_\infty\\
&\le \varepsilon + \|\varphi'\|_\infty \frac{K_1^2}{N^2}\|v^n\|_\infty + \frac{1}{k^2}\sum_{K_1\le j<k} j\,\varepsilon \le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

This shows the continuity. □

13. Differentiability of Seminorms

A desired separation property is that the smooth functions generate the topology. Since a locally convex topology is generated by the continuous seminorms it is natural to look for smooth seminorms. Note that every seminorm $p : E \to \mathbb{R}$ on a vector space E factors over $E_p := E/\ker p$ and gives a norm on this space. Hence, it can be extended to a norm $\tilde{p} : \tilde{E}_p \to \mathbb{R}$ on the completion $\tilde{E}_p$ of the space E_p which is normed by this factorization. If E is a locally convex space and p is continuous, then the canonical quotient mapping $E \to E_p$ is continuous. Thus, smoothness of $\tilde{p}$ off 0 implies smoothness of p on its carrier, and so the case where E is a Banach space is of central importance.

Obviously, every seminorm is a convex function, and hence we can generalize our treatment slightly by considering convex functions instead. The question of their differentiability properties is exactly the topic of this section.

Note that since the smooth functions depend only on the bornology and not on the locally convex topology the same is true for the initial topology induced by all smooth functions. Hence, it is appropriate to make the following

Convention. In this chapter the locally convex topology on all convenient vector spaces is assumed to be the bornological one.

13.1. Remark. It can be easily seen that for a function $f : E \to \mathbb{R}$ on a vector space E the following statements are equivalent (see for example [**41**, p. 199]):

1. The function f is convex, i.e. $f(\sum_{i=1}^n \lambda_i x_i) \le \sum_{i=1}^n \lambda_i f(x_i)$ for $\lambda_i \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$;
2. The set $U_f := \{(x,t) \in E \times \mathbb{R} : f(x) < t\}$ is convex;
3. The set $A_f := \{(x,t) \in E \times \mathbb{R} : f(x) \le t\}$ is convex.

Proof. (1⇒2) Let $(x_i, t_i) \in U_f$ and consider $(x,t) := \sum_i r_i(x_i,t_i)$ with $r_i \ge 0$ and $\sum_i r_i = 1$. Then

$$f(x) = f\Big(\sum_i r_i\, x_i\Big) \le \sum_i r_i\, f(x_i) < \sum_i r_i\, t_i = t,$$

so $(x, f) \in U_f$.

(2 ⇒3) Let $(x_i, t_i) \in A_f$ and $r_i \ge 0$ with $\sum_i r_i = 1$. For every $\varepsilon > 0$ we have $(x_i, t_i + \varepsilon) \in U_f$, hence

$$(0,\varepsilon) + \sum_i r_i(x_i, t_i) = \sum r_i(x_i, t + i + \varepsilon) \in U_f \subseteq A_f,$$

so $\sum_i r_i(x_i, t_i) \in A_f$.

(3 ⇒1) Let $x_i \in E$ and $r_i \ge 0$ with $\sum_i r_i = 1$. Then $(x_i, f(x_i)) \in A_f$ and hence $\sum_i r_i(x_i, f(x_i)) \in A_f$, i.e. $f\Big(\sum_i r_i\, x_i\Big) \le \sum_i r_i\, f(x_i)$. □

Moreover, for any translation invariant topology on E (and hence in particular for the locally convex topology or the c^∞-topology on a convenient vector space) and any convex function $f : E \to \mathbb{R}$ the following statements are equivalent:

1. The function f is continuous;
2. The set U_f is open in $E \times \mathbb{R}$;

3. The set $f_{<t} := \{x \in E : f(x) < t\}$ is open in E for all $t \in \mathbb{R}$, i.e. f is upper semi-continuous.

Proof. ($1 \Rightarrow 2$) Let f be continuous. Then $f \times \mathbb{R}$ is continuous and hence $U_f = (f \times \mathbb{R})^{-1}(\{(t,s) \in \mathbb{R}^2 : t < s\})$ is open.

($2 \Rightarrow 3$) $\{x : f(x) < t\} = \mathrm{incl}_t^{-1}(U_f)$ is open, since $\mathrm{incl}_t : E \to E \times \mathbb{R}$, $x \mapsto (x,t)$ is continuous.

($3 \Rightarrow 1$) We have to show that $f^{-1}(\{t : r < t < s\}) = \{x : f(x) < s\} \cap \{x : f(x) > r\}$ is open. So let $V := \{x : f(x) > r\}$. Then $V = \bigcup_x (2x - \{y \in E : f(y) < 2\,f(x) - r\})$, since x with $f(x) > r$ can be written as $x = 2x - x$ and $f(x) < 2\,f(x) - r$ and conversely, for y with $f(y) < 2\,f(x) - r$ and $z = 2x - y$ we have

$$f(z) + f(y) \geq 2\,f\left(\frac{z+y}{2}\right) = 2\,f(x) > f(y) + r,$$

so $z \in V$. □

Moreover the following statements are equivalent:

1. The function f is lower semicontinuous, i.e. the set $f_{>t} := \{x \in E : f(x) > t\}$ is open in E for all $t \in \mathbb{R}$;
2. The set A_f is closed in $E \times \mathbb{R}$.

Proof. ($\Downarrow$) Let $(x_0, t_0) \notin A_f$, i.e. $f(x_0) > t_0$. Let $t_0 < \delta < f(x_0)$ then $\{(x,t) : f(x) > \delta, t < \delta\}$ is an open neighborhood of (x_0, t_0) contained in $E \times \mathbb{R} \setminus A_f$.

($\Uparrow$) $f_{>t} = \mathrm{incl}_t^{-1}(E \times \mathbb{R} \setminus A_f)$. □

13.2. Result. Convex Lipschitz functions. *Let $f : E \to \mathbb{R}$ be a convex function on a convenient vector space E. Then the following statements are equivalent:*

(1) *It is $\mathcal{L}\mathrm{ip}^0$;*

(2) *It is continuous for the bornological locally convex topology;*

(3) *It is continuous for the c^∞-topology;*

(4) *It is bounded on Mackey converging sequences;*

If f is a seminorm, then these further are equivalent to

(5) *It is bounded on bounded sets.*

If E is normed this further is equivalent to

(6) *It is locally bounded.*

The proof is due to [4] for Banach spaces and [41, p. 200], for convenient vector spaces.

13.3. Basic definitions. Let $f : E \supseteq U \to F$ be a mapping defined on a c^∞-open subset of a convenient vector space E with values in another one F. Let $x \in U$ and $v \in E$. Then the (one sided) *directional derivative* of f at x in direction v is defined as

$$f'(x)(v) = d_v f(x) := \lim_{t \searrow 0} \frac{f(x + t\,v) - f(x)}{t}.$$

Obviously, if $f'(x)(v)$ exists, then so does $f'(x)(s\,v)$ for $s > 0$ and equals $s\,f'(x)(v)$.

Even if $f'(x)(v)$ exists for all $v \in E$ the mapping $v \mapsto f'(x)(v)$ may not be linear in general, and if it is linear it will not be bounded in general. Hence, f is called *Gâteaux-differentiable* at x, if the directional derivatives $f'(x)(v)$ exist for all $v \in E$ and $v \mapsto f'(x)(v)$ is a bounded linear mapping from $E \to F$.

Even for Gâteaux-differentiable mappings the difference quotient $\frac{f(x+t\,v)-f(x)}{t}$ need not converge uniformly for v in bounded sets (or even in compact sets). Hence, one defines f to be *Fréchet-differentiable* at x if f is Gâteaux-differentiable at x and $\frac{f(x+t\,v)-f(x)}{t} - f'(x)(v) \to 0$ uniformly for v in any bounded set. For a Banach space E this is equivalent to the existence of a bounded linear mapping denoted $f'(x) : E \to F$ such that

$$\lim_{v\to 0} \frac{f(x+v)-f(x)-f'(x)(v)}{\|v\|} = 0.$$

If $f : E \supseteq U \to F$ is Gâteaux-differentiable and the derivative $f' : E \supseteq U \to L(E,F)$ is continuous, then f is Fréchet-differentiable, and we will call such a function C^1. In fact, the fundamental theorem applied to $t \mapsto f(x+t\,v)$ gives us

$$f(x+v)-f(x) = \int_0^1 f'(x+t\,v)(v)\,dt,$$

and hence

$$\frac{f(x+s\,v)-f(x)}{s} - f'(x)(v) = \int_0^1 \Big(f'(x+t\,s\,v)-f'(x)\Big)(v)\,dt \to 0,$$

which converges to 0 for $s \to 0$ uniformly for v in any bounded set, since $f'(x+t\,s\,v) \to f'(x)$ uniformly on bounded sets for $s \to 0$ and uniformly for $t \in [0,1]$ and v in any bounded set, since f' is assumed to be continuous.

Recall furthermore that a mapping $f : E \supseteq U \to F$ on a Banach space E is called *Lipschitz* if

$$\Big\{\frac{f(x_1)-f(x_2)}{\|x_1-x_2\|} : x_1, x_2 \in U, x_1 \neq x_2\Big\} \text{ is bounded in } F.$$

It is called *Hölder* of order $0 < \alpha \le 1$ if

$$\Big\{\frac{f(x_1)-f(x_2)}{\|x_1-x_2\|^\alpha} : x_1, x_2 \in U, x_1 \neq x_2\Big\} \text{ is bounded in } F.$$

13.4. Lemma. Gâteaux-differentiability of convex functions. *Every convex function $q : E \to \mathbb{R}$ has one sided directional derivatives. The derivative $q'(x)$ is sublinear and locally bounded (or continuous at 0) if q is locally bounded (or continuous). In particular, such a locally bounded function is Gâteaux-differentiable at x if and only if $q'(x)$ is an odd function, i.e. $q'(x)(-v) = -q'(x)(v)$.*

If E is not normed, then locally bounded-ness should mean bounded on bornologically compact sets.

Proof. For $0 < t < t'$ we have by convexity that

$$q(x+t\,v) = q\Big((1-\tfrac{t}{t'})x + \tfrac{t}{t'}\,(x+t'\,v)\Big) \le (1-\tfrac{t}{t'})\,q(x) + \tfrac{t}{t'}\,q(x+t'v).$$

Hence $\frac{q(x+t\,v)-q(x)}{t} \le \frac{q(x+t'\,v)-q(x)}{t'}$. Thus, the difference quotient is monotone falling for $t \searrow 0$. It is also bounded from below, since for $t' < 0 < t$ we have

$$\begin{aligned} q(x) &= q\Big(\tfrac{t}{t-t'}\,(x+t'\,v) + (1-\tfrac{t}{t-t'})\,(x+t\,v)\Big) \\ &\le \tfrac{t}{t-t'}\,q(x+t'\,v) + (1-\tfrac{t}{t-t'})\,q(x+t\,v), \end{aligned}$$

and hence $\frac{q(x+t'\,v)-q(x)}{t'} \leq \frac{q(x+t\,v)-q(x)}{t}$. Thus, the one sided derivative

$$q'(x)(v) := \lim_{t\searrow 0} \frac{q(x+t\,v)-q(x)}{t}$$

exists.

As a derivative $q'(x)$ automatically satisfies $q'(x)(t\,v) = t\,q'(x)(v)$ for all $t \geq 0$. The derivative $q'(x)$ is convex as limit of the convex functions $v \mapsto \frac{q(x+tv)-q(x)}{t}$. Hence it is sublinear.

The convexity of q implies that

$$q(x) - q(x-v) \leq q'(x)(v) \leq q(x+v) - q(x).$$

Therefore, the local boundedness of q at x implies that of $q'(x)$ at 0. Let $\ell := f'(x)$, then subadditivity and odd-ness implies $\ell(a) \leq \ell(a+b) + \ell(-b) = \ell(a+b) - \ell(b)$ and hence the converse triangle inequality. □

Remark. If q is a seminorm, then $\frac{q(x+tv)-q(x)}{t} \leq \frac{q(x)+t\,q(v)-q(x)}{t} = q(v)$, hence $q'(x)(v) \leq q(v)$, and furthermore $q'(x)(x) = \lim_{t\searrow 0} \frac{q(x+t\,x)-q(x)}{t} = \lim_{t\searrow 0} q(x) = q(x)$. Hence we have

$$\|q'(x)\| := \sup\{|q'(x)(v)| : q(v) \leq 1\} = \sup\{q'(x)(v) : q(v) \leq 1\} = 1.$$

Convention. Let $q \neq 0$ be a seminorm and let $q(x) = 0$. Then there exists a $v \in E$ with $q(v) \neq 0$, and we have $q(x+tv) = |t|\,q(v)$, hence $q'(x)(\pm v) = q(v)$. So q is not Gâteaux differentiable at x. Therefore, we call a seminorm *smooth* for some differentiability class, if and only if it is smooth on its carrier $\{x : q(x) > 0\}$.

13.5

Differentiability properties of convex functions f can be translated in geometric properties of A_f:

Lemma. Differentiability of convex functions. *Let $f : E \to \mathbb{R}$ be a continuous convex function on a Banach space E, and let $x_0 \in E$. Then the following statements are equivalent:*

(1) *The function f is Gâteaux differentiable at x_0;*
(2) *There exists a unique $\ell \in E'$ with*
$$\ell(v) \leq f(x_0+v) - f(x_0) \text{ for all } v \in E;$$
(3) *There exists a unique affine hyperplane tangent to A_f through $(x_0, f(x_0))$.*
(4) *The Minkowski functional of (some translate of) A_f is Gâteaux differentiable at $(x_0, f(x_0))$.*

Moreover, for a sublinear function f and $f(x_0) \neq 0$ the following statements are equivalent:

(5) *The function f is Gâteaux (Fréchet) differentiable at x_0;*
(6) *The point x_0 (strongly) exposes the polar of the set $\{x : f(x) \leq 1\}$.*

In particular, the following statements are equivalent for a convex function f:

(7) *The function f is Gâteaux (Fréchet) differentiable at x_0;*

(8) *The Minkowski functional of (some translate of) A_f is Gâteaux (Fréchet) differentiable at the point $(x_0, f(x_0))$;*

(9) *The point $(x_0, f(x_0))$ (strongly) exposes the polar of some translate of A_f.*

An element $x^* \in E^*$ is said to *expose a subset* $K \subseteq E$ if there exists a unique point $k_0 \in K$ with $x^*(k_0) = \sup\{x^*(k) : k \in K\}$, i.e. x^* takes it supremum on K on a unique point k_0. It is said to *strongly expose* K, if satisfies in addition that $x^*(x_n) \to x^*(k_0)$ implies $x_n \to k_0$.

By an affine hyperplane H *tangent* to a convex set K at a point $x \in K$ we mean that $x \in H$ and K lies on one side of H.

Proof. Let f be a convex function. By the proof of 13.4 we have $f'(x_0)(v) \leq f(x_0 + v) - f(x_0)$. For any $\ell \in E'$ with $\ell(v) \leq f(x_0 + v) - f(x_0)$ for all $v \in E$ we have $\ell(v) = \frac{1}{t}\ell(tv) \leq \frac{f(x_0+t\,v)-f(x_0)}{t}$ for all $t > 0$, and hence $\ell \leq f'(x_0)$.

(1) $\Rightarrow$ (2) Let f be continuous and Gâteaux-differentiable at x_0, so $f'(x_0)$ is linear (and continous) and thus minimal among all sub-linear mappings. By what we said before $f'(x_0)$ is the unique linear functional satisfying (2).

(2) $\Rightarrow$ (1) By what we said before the unique ℓ in (2) satisfied $\ell \leq f'(x_0)$. So $f'(x_0) - \ell \geq 0$. If this is not identical zero, then there exists a $\mu \in E^*$ with $0 \neq \mu \leq f'(x_0) - \ell$ by Hahn-Banach. Thus $\ell + \mu$ satisfies (2) also, a contradiction to the uniqueness of ℓ.

(2) $\Leftrightarrow$ (3) Any hyperplane tangent to A_f at $(x_0, f(x_0))$ is described by a functional $0 \neq (\ell, s) \in E' \times \mathbb{R}$ such that $\ell(x) + s\,t \geq \ell(x_0) + s\,f(x_0)$ for all $t \geq f(x)$. Note that the scalar s cannot be 0, since this would imply that $\ell(x) \geq \ell(x_0)$ for all x. It has to be positive, since otherwise the left side would go to $-\infty$ for $f(x) \leq t \to +\infty$. Without loss of generality we may thus assume that $s = 1$, so the hyperplane uniquely determines the linear functional ℓ with $\ell(x-x_0) \geq f(x_0)-f(x)$ for all x or, by replacing ℓ by $-\ell$ and x by $x_0 + v$, we have a unique ℓ with $\ell(v) \leq f(x_0 + v) - f(x_0)$ for all $v \in E$.

(3) $\Leftrightarrow$ (4) A sublinear functional $p \geq 0$ is Gâteaux-differentiable at x_0 with $p(x_0) \neq 0$ if and only if there is a unique affine hyperplane tangent to $\{x : p(x) \leq p(x_0)\}$ at x_0:
By (1) $\Leftrightarrow$ (2) p is differentiable at x_0 iff there exists a unique $\ell \in E'$ with $\ell(v) \leq p(x_0 + v) - p(x_0)$ for all v, or, equivalently, $\ell(x - x_0) \leq p(x) - p(x_0)$ for all x. Thus $\ell(x) \leq \ell(x_0)$ for all $p(x) \leq p(x_0)$. Conversely let $0 \neq \ell \in E'$ satisfy this condition and x be arbitary. Since $\{x : p(x) \leq p(x_0)\}$ is absorbing, $\ell(x_0) > 0$ and we may replace ℓ by $\frac{p(x_0)}{\ell(x_0)}\ell$. If $p(x) = 0$ then $p(r\,x) = 0 \leq p(x_0)$ for all r and hence $\ell(r\,x) \leq \ell(x_0)$ for all r, i.e. $\ell(x) = 0$ and hence $\ell(x - x_0) = -\ell(x_0) = -p(x_0) = p(x) - p(x_0)$. Otherwise we may consider $x' := \frac{p(x_0)}{p(x)}x$ which satisfies $p(x') = p(x_0)$ and hence $\ell(x_0) \geq \ell(x') = \frac{p(x_0)}{p(x)}\ell(x)$ so $\ell(x - x_0) = \ell(x) - \ell(x_0) \leq (p(x) - p(x_0))\frac{\ell(x_0)}{p(x_0)} = p(x) - p(x_0)$.

We translate A_f such that it becomes absorbing (e.g. by $-(0, f(0) + 1)$). The sublinear Minkowski functional p of this translated set A_f is by what we just showed Gâteaux-differentiable at $(x_0, f(x_0))$ with $p(x_0, f(x_0)) = 1$ iff there exists a unique affine hyperplane tangent to $\{(x, t) : p(x, t) \leq p(x_0, f(x_0))\} = f(x_0)\,A_f$ in $(x_0, f(x_0))$, since A_f is closed. Since $f(x_0) \neq 0$ this is equivalent with (3).

(5) $\Leftrightarrow$ (6) We show this for Gâteaux-differentiability. We have to show that there is a unique tangent hyperplane to $x_0 \in K := \{x : f(x) \leq 1\}$ if and only if x_0 exposes $K^o := \{\ell \in E^* : \ell(x) \leq 1 \text{ for all } x \in K\}$. Let us assume $0 \in K$ and $0 \neq x_0 \in \partial K$. Then a tangent hyperplane to K at x_0 is uniquely determined by a linear functional $\ell \in E^*$ with $\ell(x_0) = 1$ and $\ell(x) \leq 1$ for all $x \in K$. This is equivalent to $\ell \in K^o$ and $\ell(x_0) = 1$, since by Hahn-Banach there exists an $\ell \in K^o$ with $\ell(x_0) = 1$. From this the result follows.

This shows also (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) for Gâteaux-differentiability, since $\{(x,t) : p_{A_f}(x,t) \leq 1\} = A_f$.

In order to show the statements for Fréchet-differentiability one has to show that $\ell = f'(x)$ is a Fréchet derivative if and only if x_0 is a strongly exposing point. This is left to the reader, see also 13.19 for a more general result. □

13.6. Lemma. Duality for convex functions. The Legendre-Fenchel transform. [100].
Let $\langle \_, \_ \rangle : G \times F \to \mathbb{R}$ be a dual pairing.

(1) *For $f : F \to \mathbb{R} \cup \{+\infty\}$, $f \neq +\infty$ one defines the dual function*
$$f^* : G \to \mathbb{R} \cup \{+\infty\}, \quad f^*(z) := \sup\{\langle z, y\rangle - f(y) : y \in F\}.$$
(2) *The dual function f^* is convex and lower semi-continuous with respect to the weak topology. Since a function g is lower semi-continuous if and only if for all $a \in \mathbb{R}$ the set $\{x : g(x) > a\}$ is open, equivalently the convex set $\{x : g(x) \leq a\}$ is closed, this is for convex functions the same for every topology which is compatible with the duality.*
(3) $f_1 \leq f_2 \;\Rightarrow\; f_1^* \geq f_2^*$.
(4) $f^* \leq g \;\Leftrightarrow\; g^* \leq f$.
(5) *$f^{**} = f$ if and only if f is lower semi-continuous and convex.*
(6) *Suppose $z \in G$ satisfies $f(x+v) \geq f(x) + \langle z, v\rangle$ for all v (in particular, this is true if $z = f'(x)$). Then $f(x) + f^*(z) = \langle z, x\rangle$.*
(7) *If $f_1(y) = f(y-a)$ for all y, then $f_1^*(z) = \langle z, a\rangle + f^*(z)$ for all z.*
(8) *If $f_1(y) = f(y) + a$ for all y, then $f_1^*(z) = f^*(z) - a$ for all z.*
(9) *If $f_1(y) = f(y) + \langle b, y\rangle$ for all y, then $f_1^*(z) = f^*(z-b)$ for all z.*
(10) *If $E = F = \mathbb{R}$ and $f \geq 0$ with $f(0) = 0$, then $f^*(s) = \sup\{ts - f(t) : t \geq 0\}$ for $t \geq 0$.*
(11) *If $\gamma : \mathbb{R} \to \mathbb{R}^+$ is convex and $\frac{\gamma(t)}{t} \to 0$, then $\gamma^*(t) > 0$ for $t > 0$.*
(12) *Let (F, G) be a Banach space and its dual. If $\gamma \geq 0$ is convex and $\gamma(0) = 0$, and $f(y) := \gamma(\|y\|)$, then $f^*(z) = \gamma^*(\|z\|)$.*
(13) *A convex function f on a Banach space is Fréchet differentiable at a with derivative $b := f'(a)$ if and only if there exists a convex non-negative function γ, with $\gamma(0) = 0$ and $\lim_{t\to 0} \frac{\gamma(t)}{t} = 0$, such that*
$$f(a+h) \leq f(a) + \langle b, h\rangle + \gamma(\|h\|).$$

Proof. (1) Since $f \neq +\infty$, there is some y for which $\langle z, y\rangle - f(y)$ is finite, hence $f^*(z) > -\infty$.

(2) The function $z \mapsto \langle z, y\rangle - f(y)$ is continuous and linear, and hence the supremum $f^*(z)$ is lower semi-continuous and convex. One would like to show that f^* is not constant $+\infty$: This is not true. In fact, take $f(t) = -t^2$ then

$f^*(s) = \sup\{s\,t - f(t) : t \in \mathbb{R}\} = \sup\{s\,t + t^2 : t \in \mathbb{R}\} = +\infty$. More generally, $f^* \neq +\infty \Leftrightarrow f$ lies above some affine hyperplane, see (5).

(3) If $f_1 \le f_2$ then $\langle z, y\rangle - f_1(y) \ge \langle z, y\rangle - f_2(y)$, and hence $f_1^*(z) \ge f_2^*(z)$.

(4) One has

$$\begin{aligned}\forall z : f^*(z) \le g(z) &\Leftrightarrow \forall z, y : \langle z, y\rangle - f(y) \le g(z)\\ &\Leftrightarrow \forall z, y : \langle z, y\rangle - g(z) \le f(y)\\ &\Leftrightarrow \forall y : g^*(y) \le f(y).\end{aligned}$$

(5) Since $(f^*)^*$ is convex and lower semi-continuous, this is true for f provided $f = (f^*)^*$. Conversely, let $g(b) = -a$ and $g(z) = +\infty$ otherwise. Then $g^*(y) = \sup\{\langle z, y\rangle - g(z) : z \in G\} = \langle b, y\rangle + a$. Hence, $a + \langle b, \_\rangle \le f \Leftrightarrow f^*(b) \le -a$. If f is convex and lower semi-continuous, then A_f is closed and convex and hence f is the supremum of all continuous linear functionals $a + \langle b, \_\rangle$ below it by Hahn-Banach, and this is exactly the case if $f^*(b) \le -a$. Hence, $f^{**}(y) = \sup\{\langle z, y\rangle - f^*(z) : z \in G\} \ge \langle b, y\rangle + a$ and thus $f = f^{**}$.

(6) Let $f(a + y) \ge f(a) + \langle b, y\rangle$. Then $f^*(b) = \sup\{\langle b, y\rangle - f(y) : y \in F\} = \sup\{\langle b, a + v\rangle - f(a + v) : v \in F\} \le \sup\{\langle b, a\rangle + \langle b, v\rangle - f(a) - \langle b, v\rangle : v \in F\} = \langle b, a\rangle - f(a)$.

(7) Let $f_1(y) = f(y - a)$. Then

$$\begin{aligned}f_1^*(z) &= \sup\{\langle z, y\rangle - f(y - a) : y \in F\}\\ &= \sup\{\langle z, y + a\rangle - f(y) : y \in F\} = \langle z, a\rangle + f^*(z).\end{aligned}$$

(8) Let $f_1(y) = f(y) + a$. Then

$$f_1^*(z) = \sup\{\langle z, y\rangle - f(y) - a : y \in F\} = f^*(z) - a.$$

(9) Let $f_1(y) = f(y) + \langle b, y\rangle$. Then

$$\begin{aligned}f_1^*(z) &= \sup\{\langle z, y\rangle - f(y) - \langle b, y\rangle : y \in F\}\\ &= \sup\{\langle z - b, y\rangle - f(y) : y \in F\} = f^*(z - b).\end{aligned}$$

(10) Let $E = F = \mathbb{R}$ and $f \ge 0$ with $f(0) = 0$, and let $s \ge 0$. Using that $s\,t - f(t) \le 0$ for $t \le 0$ and that $s\,0 - f(0) = 0$ we obtain

$$f^*(s) = \sup\{s\,t - f(t) : t \in \mathbb{R}\} = \sup\{s\,t - f(t) : t \ge 0\}.$$

(11) Let $\gamma \ge 0$ with $\lim_{t\searrow 0} \frac{\gamma(t)}{t} = 0$, and let $s > 0$. Then there are t with $s > \frac{\gamma(t)}{t}$, and hence

$$\gamma^*(s) = \sup\{st - \gamma(t) : t \ge 0\} = \sup\{t(s - \frac{\gamma(t)}{t}) : t \ge 0\} > 0.$$

(12) Let $f(y) = \gamma(\|y\|)$. Then

$$\begin{aligned}f^*(z) &= \sup\{\langle z, y\rangle - \gamma(\|y\|) : y \in F\}\\ &= \sup\{t\langle z, y\rangle - \gamma(t) : \|y\| = 1, t \ge 0\}\\ &= \sup\{\sup\{t\langle z, y\rangle - \gamma(t) : \|y\| = 1\}, t \ge 0\}\\ &= \sup\{t\|z\| - \gamma(t) : t \ge 0\}\\ &= \gamma^*(\|z\|).\end{aligned}$$

(13) ($\Leftarrow$) If $f(a+h)\le f(a)+\langle b,h\rangle+\gamma(\|h\|)$ for all h, then we have for $t>0$

$$\frac{f(a+t\,h)-f(a)}{t}\le\langle b,h\rangle+\frac{\gamma(t\,\|h\|)}{t},$$

hence $f'(a)(h)\le\langle b,h\rangle$. Since $h\mapsto f'(a)(h)$ is sub-linear and the linear functionals are minimal among the sublinear ones, we have equality. By convexity we have

$$\frac{f(a+t\,h)-f(a)}{t}\ge\langle b,h\rangle=f'(a)(h).$$

So f is Fréchet-differentiable at a with derivative $f'(a)(h)=\langle b,h\rangle$, since the remainder is bounded by $\gamma(\|h\|)$ which satisfies $\frac{\gamma(\|h\|)}{\|h\|}\to 0$ for $\|h\|\to 0$.

($\Rightarrow$) Assume that f is Fréchet-differentiable at a with derivative b. Then

$$\frac{|f(a+h)-f(a)-\langle b,h\rangle|}{\|h\|}\to 0\text{ for }h\to 0,$$

and by convexity

$$g(h):=f(a+h)-f(a)-\langle b,h\rangle\ge 0.$$

Let $\gamma(t):=\sup\{g(u):\|u\|=|t|\}$. Since g is convex γ is convex, and obviously $\gamma(t)\in[0,+\infty]$, $\gamma(0)=0$ and $\frac{\gamma(t)}{t}\to 0$ for $t\to 0$. This is the required function. $\square$

13.7. Proposition. Continuity of the Fréchet derivative. [5]. *The differential f' of any continuous convex function f on a Banach space is continuous on the set of all points where f is Fréchet differentiable. In general, it is however neither uniformly continuous nor bounded, see 15.8.*

Proof. Let $f'(x)(h)$ denote the one sided derivative. From convexity we conclude that $f(x+v)\ge f(x)+f'(x)(v)$. Suppose $x_n\to x$ are points where f is Fréchet differentiable. Then we obtain $f'(x_n)(v)\le f(x_n+v)-f(x_n)$ which is bounded in n. Hence, the $f'(x_n)$ form a bounded sequence. We get

$$\begin{aligned}
f(x)&\ge\langle f'(x_n),x\rangle-f^*(f'(x_n)) && \text{since } f(y)+f^*(z)\ge\langle z,y\rangle\\
&=\langle f'(x_n),x\rangle+f(x_n)-\langle f'(x_n),x_n\rangle && \text{since } f^*(f'(z))+f(z)=f'(z)(z)\\
&\ge\langle f'(x_n),x-x_n\rangle+f(x)+\langle f'(x),x_n-x\rangle && \text{since } f(x+h)\ge f(x)+f'(x)(h)\\
&=\langle f'(x_n)-f'(x),x-x_n\rangle+f(x).
\end{aligned}$$

Since $x_n\to x$ and $f'(x_n)$ is bounded, both sides converge to $f(x)$, hence

$$\lim_{n\to\infty}\langle f'(x_n),x\rangle-f^*(f'(x_n))=f(x).$$

Since f is convex and Fréchet-differentiable at $a:=x$ with derivative $b:=f'(x)$, there exists by 13.6.13 a γ with

$$f(h)\le f(a)+\langle b,h-a\rangle+\gamma(\|h-a\|).$$

By duality we obtain using 13.6.3

$$f^*(z)\ge\langle z,a\rangle-f(a)+\gamma^*(\|z-b\|).$$

If we apply this to $z:=f'(x_n)$ we obtain

$$f^*(f'(x_n))\ge\langle f'(x_n),x\rangle-f(x)+\gamma^*(\|f'(x_n)-f'(x)\|).$$

Hence

$$\gamma^*(\|f'(x_n)-f'(x)\|)\le f^*(f'(x_n))-\langle f'(x_n),x\rangle+f(x),$$

and since the right side converges to 0, we have that $\gamma^*(\|f'(x_n)-f'(x)\|)\to 0$. Then $\|f'(x_n)-f'(x)\|\to 0$ where we use that γ is convex, $\gamma(0)=0$, and $\gamma(t)>0$ for $t>0$, thus γ is strictly monotone increasing. $\square$

13.8. Asplund spaces and generic Fréchet differentiability

From 13.4 it follows easily that a convex function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at all except countably many points. This has been generalized by [106] to: Every Lipschitz mapping from an open subset of $\mathbb{R}^n$ to $\mathbb{R}$ is differentiable almost everywhere. Recall that a locally bounded convex function is locally Lipschitz, see 13.2.

Proposition. *For a Banach space E the following statements are equivalent:*

(1) *Every continuous convex function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense G_δ-subset of E;*
(2) *Every continuous convex function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense subset of E;*
(3) *Every locally Lipschitz function $f : E \to \mathbb{R}$ is Fréchet-differentiable on a dense subset of E;*
(4) *Every equivalent norm is Fréchet-differentiable at least at one point;*
(5) *E has no equivalent rough norm;*
(6) *Every (closed) separable subspace has a separable dual;*
(7) *The dual E^* has the Radon-Nikodym property;*
(8) *Every linear mapping $E \to L^1(X, \Omega, \mu)$ which is integral is nuclear;*
(9) *Every closed convex bounded subset of E^* is the closed convex hull of its extremal points;*
(10) *Every bounded subset of E^* is dentable.*

A Banach space satisfying these equivalent conditions is called Asplund space.
Every Banach space with a Fréchet differentiable bump function is Asplund, [31, p. 203]. *It is an open question whether the converse is true.*
Every WCG-Banach-space (i.e. a Banach space for which a weakly compact subset K exists, whose linear hull is the whole space) is Asplund, [55].
The Asplund property is inherited by subspaces, quotients, and short exact sequences, [118].

About the proof. (1⇔2) [5]: If a convex function is Fréchet differentiable on a dense subset then it is so on a dense G_δ-subset, i.e. a dense countable intersection of open subsets.

(2) is in fact a local property, since in [19] it is mentioned that for a Lipschitz function $f : E \to \mathbb{R}$ with Lipschitz constant L defined on a convex open set U the function

$$\tilde{f}(x) := \inf\{f(y) + L\|x - y\| : y \in U\}$$

is a Lipschitz extension with constant L, and it is convex if f is.

(2) ⇒ (3) is due to [105], Every locally Lipschitz function on an Asplund space is Fréchet differentiable at points of a dense subset.

(3) ⇒ (2) follows from the fact that continuous convex functions are locally Lipschitz, see 13.2.

(2) ⇔ (4) is mentioned in [105] without any proof or reference.

(2) ⇔ (10) is due to [116]. A subset D of a Banach space is called *dentable*, if and only if for every $\varepsilon > 0$ there exists an $x \in D$ such that x is not in the closed convex hull of $\{y \in D : \|y - x\| \ge \varepsilon\}$.

(2) ⇔ (5) is due to [56]. A norm p is called *rough*, see also 13.23, if and only if there exists an $\varepsilon > 0$ such that arbitrary close to each $x \in X$ there are points x_i and u with $\|u\| = 1$ such that $|p'(x_2)(u) - p'(x_1)(u)| \geq \varepsilon$. The usual norms on $C[0,1]$ and on ℓ^1 are rough by 13.12 and 13.13.
A norm is not rough if and only if the dual ball is w^*-dentable. The unit ball is dentable if and only if the dual norm is not rough.

(2) ⇔ (6) is due to [116].

(2) ⇔ (7) is due to [117]. A closed bounded convex subset K of a Banach space E is said to have the *Radon-Nikodym property* if for any finite measure space (Ω, Σ, μ) every μ-continuous countably additive function $m : \Sigma \to E$ of finite variation with average range $\{\frac{m(S)}{\mu(S)} : S \in \Sigma, \mu(S) > 0\}$ contained in K is representable by a Bochner integrable function, i.e. there exists a Borel-measurable essentially separably valued function $f : \Omega \to E$ with $m(S) = \int_S f \, d\mu$. This function f is then called the Radon-Nikodym derivative of m. A Banach space is said to have the Radon-Nikodym property if every closed bounded convex subset has it. See also [29]. A subset K is a Radon-Nikodym set if and only if every closed convex subset of K is the closed convex hull of its strongly exposed points.

(7) ⇔ (8) can be found in [116] and is due to [47]. A linear mapping $E \to F$ is called *integral* if and only if it has a factorization

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[r] & F^{**} \\ C(K) \arrow[rr] & & L^1(K,\mu) \arrow[u] \end{tikzcd}$$

for some Radon-measure μ on a compact space K.
A linear mapping $E \to F$ is called *nuclear* if and only if there are $x_n^* \in E^*$ and $y_n \in F$ such that $\sum_n \|x_n^*\| \, \|y_n\| < \infty$ and $T = \sum_n x_n^* \otimes y_n$.

(2) ⇔ (9) is due to [118, p.516]. □

13.10. Lemma. Smoothness of $2n$-norm. *For $n \in \mathbb{N}$ the $2n$-norm is smooth on $L^{2n} \setminus \{0\}$.*

See also 13.13.

Proof. Since $t \mapsto t^{1/2n}$ is smooth on $\mathbb{R}^+$ it is enough to show that $x \mapsto (\|x\|_{2n})^{2n}$ is smooth. Let $p := 2n$. Since $(x_1, \dots, x_p) \mapsto x_1 \cdot \dots \cdot x_p$ is a p-linear contraction from $L^p \times \dots \times L^p \to L^1$ by the Hölder-inequality ($\sum_{i=1}^p \frac{1}{p} = 1$) and $\int : L^1 \to \mathbb{R}$ is a linear contraction the mapping $x \mapsto (x, \dots, x) \mapsto \int x^{2n}$ is smooth. Note that since we have a real Banach space and $p = 2n$ is even we can drop the absolute value in the formula of the norm. □

13.11. Derivative of the 1-norm

Let $x \in \ell^1$ and $j \in \mathbb{N}$ be such that $x_j = 0$. Let e_j be the characteristic function of $\{j\}$. Then $\|x + t\, e_j\|_1 = \|x\|_1 + |t|$ since the supports of x and e_j are disjoint. Hence, the directional derivative of the norm $p : v \mapsto \|v\|_1$ is given by $p'(x)(e_i) = 1$ and $p'(x)(-e_i) = 1$, and p is not differentiable at x. More generally we have:

Lemma. [**91**, p.79]. *Let Γ be some set, and let p be the* 1-norm *given by* $\|x\|_1 = p(x) := \sum_{\gamma\in\Gamma} |x_\gamma|$ *for* $x \in \ell^1(\Gamma)$. *Then* $p'(x)(h) = \sum_{x_\gamma=0} |h_\gamma| + \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma$.

The basic idea behind this result is, that the unit sphere of the 1-norm is a hyperoctahedra, and the points on the faces are those, for which no coordinate vanishes.

Proof. Without loss of generality we may assume that $p(x) = 1 = p(h)$, since for $r > 0$ and $s \geq 0$ we have $p'(r\,x)(s\,h) = \frac{d}{dt}|_{t=0} p(r\,x + t\,s\,h) = \frac{d}{dt}|_{t=0} r\,p(x + t\,(\frac{s}{r}h)) = r\,p'(x)(\frac{s}{r}h) = s\,p'(x)(h)$.

We have $|x_\gamma + h_\gamma| - |x_\gamma| = \big||x_\gamma| + h_\gamma \operatorname{sign} x_\gamma\big| - |x_\gamma| \geq |x_\gamma| + h_\gamma \operatorname{sign} x_\gamma - |x_\gamma| = h_\gamma \operatorname{sign} x_\gamma$, and is equal to $|h_\gamma|$ if $x_\gamma = 0$. Summing up these (in)equalities we obtain

$$p(x+h) - p(x) - \sum_{x_\gamma=0} |h_\gamma| - \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma \geq 0.$$

For $\varepsilon > 0$ choose a finite set $F \subset \Gamma$, such that $\sum_{\gamma\notin F} |h_\gamma| < \frac{\varepsilon}{2}$. Now choose t so small that

$$|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma \geq 0 \text{ for all } \gamma \in F \text{ with } x_\gamma \neq 0.$$

We claim that

$$\frac{p(x+t\,h) - p(x)}{t} - \sum_{x_\gamma=0} |h_\gamma| - \sum_{x_\gamma\neq 0} h_\gamma \operatorname{sign} x_\gamma \leq \varepsilon.$$

Let first γ be such that $x_\gamma = 0$. Then $\frac{|x_\gamma + t\,h_\gamma| - |x_\gamma|}{t} = |h_\gamma|$, hence these terms cancel with $-\sum_{x_\gamma=0} |h_\gamma|$.

Let now $x_\gamma \neq 0$. For $|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma \geq 0$ (hence in particular for $\gamma \in F$ with $x_\gamma \neq 0$) we have

$$\frac{|x_\gamma + t\,h_\gamma| - |x_\gamma|}{t} = \frac{|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma - |x_\gamma|}{t} = h_\gamma \operatorname{sign} x_\gamma.$$

Thus, these terms sum up to the corresponding sum $\sum_\gamma h_\gamma \operatorname{sign} x_\gamma$.

It remains to consider γ with $x_\gamma \neq 0$ and $|x_\gamma| + t\,h_\gamma \operatorname{sign} x_\gamma < 0$. Then $\gamma \notin F$ and

$$\begin{aligned}\frac{|x_\gamma + t\,h_\gamma| - |x_\gamma|}{t} - h_\gamma \operatorname{sign} x_\gamma &= \frac{-|x_\gamma| - t\,h_\gamma \operatorname{sign} x_\gamma - |x_\gamma| - t\,h_\gamma \operatorname{sign} x_\gamma}{t}\\ &\leq -2h_\gamma \operatorname{sign} x_\gamma,\end{aligned}$$

and since $\sum_{\gamma\notin F} |h_\gamma| < \frac{\varepsilon}{2}$ these remaining terms sum up to something smaller than ε. □

Remark. The 1-norm is rough. This result shows that the 1-norm is Gâteaux-differentiable exactly at those points, where all coordinates are non-zero. Thus, if Γ is uncountable, the 1-norm is nowhere Gâteaux-differentiable.

In contrast to what is claimed in [**91**, p.79], the 1-norm on ℓ^1 is nowhere Fréchet differentiable. In fact, take $0 \neq x \in \ell^1(\mathbb{N})$. For γ with $x_\gamma \neq 0$ and $t > 0$ we have that

$$\begin{aligned}&p(x + t\,(-\operatorname{sign} x_\gamma\, e_\gamma)) - p(x) - t\,p'(x)(-\operatorname{sign} x_\gamma\, e_\gamma) =\\ &\qquad = |x_\gamma - t \operatorname{sign} x_\gamma| - |x_\gamma| + t = \big||x_\gamma| - t\big| - |x_\gamma| + t \geq t \cdot 1,\end{aligned}$$

provided $t \geq 2\,|x_\gamma|$, since then $\big||x_\gamma| - t\big| = t - |x_\gamma| \geq |x_\gamma|$. Obviously, for every $t > 0$ there are γ satisfying this required condition; either $x_\gamma = 0$ then we have a corner, or $x_\gamma \neq 0$ then it gets arbitrarily small. Thus, the directional difference quotient does not converge uniformly on the unit-sphere.

The set of points x in ℓ^1 where at least for one n the coordinate x_n vanishes is dense, and one has

$$p(x + t\,e^n) = p(x) + |t|, \text{ hence } p'(x + t\,e^n)(e^n) = \begin{cases} +1 & \text{for } t \geq 0 \\ -1 & \text{for } t < 0 \end{cases}.$$

Hence the derivative of p is uniformly discontinuous, i.e., in every non-empty open set there are points x_1, x_2 for which there exists an $h \in \ell^1$ with $\|h\| = 1$ and $|p'(x_1)(h) - p'(x_2)(h)| \geq 2$.

13.12. Derivative of the ∞-norm. On c_0 the norm is not Gâteaux-differentiable at points x, where the norm is attained in at least two points: In fact, let $|x(a)| = \|x\| = |x(b)|$ with $a \neq b$ and let $h := \operatorname{sign} x(a)\, e_a$. Then $p(x+th) = |(x+th)(a)| = \|x\| + t$ for $t \geq 0$ and $p(x+th) = |(x+th)(b)| = \|x\|$ for $t \leq 0$. Thus, $t \mapsto p(x+th)$ is not differentiable at 0 and thus p not at x.

If the norm of x is attained at a single coordinate a, then p is Fréchet differentiable at x with derivative $p'(x) : h \mapsto h(a)\ \operatorname{sign}(x(a))$: In fact, for $|t|\,\|h\| \leq \|x\| - \sup\{|x(t)| : t \neq a\}$ we have $p(x+th) = |(x+th)(a)| = |\operatorname{sign}(x(a))\|x\| + th(a)\operatorname{sign}(x(a))^2| = |\|x\| + th(a)\operatorname{sign}(x(a))| = p(x) + th(a)\operatorname{sign}(x(a))$. Hence the directional difference-quotient converges uniformly for h in the unit-ball.

Let $x \in C[0,1]$ be such that $\|x\|_\infty = |x(a)| = |x(b)|$ for $a \neq b$. Choose a y with $y(s)$ between 0 and $x(s)$ for all s and $y(a) = x(a)$ but $y(b) = 0$. For $t \geq 0$ we have $|(x + t\,y)(s)| \leq (1+t)\,\|x\|_\infty = |x(a) + t\,y(a)|$ and hence $\|x + t\,y\|_\infty = (1+t)\,\|x\|_\infty$. For $-1 \leq t \leq 0$ we have $|(x + t\,y)(s)| \leq |x(s)| \leq \|x\|$ and $|(x+t\,y)(b)| = |x(b)| = \|x\|$ and hence $\|x + t\,y\|_\infty = \|x\|_\infty$. Thus the directional derivative is given by $p'(x)(y) = \|x\|_\infty$ and $p'(x)(-y) = 0$. More precisely we have the following results.

Lemma. [8, p. 168]. *Let T be a compact metric space. Let $x \in C(T,\mathbb{R}) \setminus \{0\}$ and $h \in C(T,\mathbb{R})$. By p we denote the ∞-NORM $\|x\|_\infty = p(x) := \sup\{|x(t)| : t \in T\}$. Then*

$$p'(x)(h) = \sup\{h(t)\ \operatorname{sign} x(t) : |x(t)| = p(x)\}.$$

The idea here is, that the unit-ball is a hyper-cube, and the points on the faces are exactly those for which the supremum is attained only in one point.

Proof. Without loss of generality we may assume that $p(x) = 1 = p(h)$, since for $r > 0$ and $s \geq 0$ we have $p'(r\,x)(s\,h) = \frac{d}{dt}|_{t=0} p(r\,x + t\,s\,h) = \frac{d}{dt}|_{t=0} r\,p(x + t\,(\frac{s}{r}h)) = r\,p'(x)(\frac{s}{r}h) = s\,p'(x)(h)$.

Let $A := \{t \in T : |x(t)| = p(x)\}$. For given $\varepsilon > 0$ we find by the uniform continuity of x and h a δ_1 such that $|x(t) - x(t')| < \frac{1}{2}$ and $|h(t) - h(t')| < \varepsilon$ for $\operatorname{dist}(t,t') < \delta_1$. Then $\{t : \operatorname{dist}(t,A) \geq \delta_1\}$ is closed, hence compact. Therefore $\delta := \|x\|_\infty - \sup\{|x(t)| : \operatorname{dist}(t,A) \geq \delta_1\} > 0$.

Now we **claim** that for $0 < 2t < \min\{\delta, 1\}$ we have

$$0 \leq \frac{p(x + t\,h) - p(x)}{t} - \sup\{h(a)\ \operatorname{sign} x(a) : a \in A\} \leq \varepsilon.$$

For each $s \in A$ we have

$$\begin{aligned} p(x + t\,h) &\geq |(x + t\,h)(s)| = \big||x(s)|\ \operatorname{sign} x(s) + t\,h(s)\ \operatorname{sign} x(s)^2\big| \\ &= \big|p(x) + t\,h(s)\ \operatorname{sign} x(s)\big| = p(x) + t\,h(s)\ \operatorname{sign} x(s) \end{aligned}$$

for $0 \le t \le 1$, since $|h(s)| \le p(h) = p(x)$. Hence

$$\frac{p(x+t\,h)-p(x)}{t} \ge \sup\big\{h(a)\,\operatorname{sign} x(a) : a \in A\big\}.$$

This shows the claimed left inequality.

Now let s be a point where the supremum $p(x+t\,h)$ is attained. From the just proved inequality it follows that $p(x+t\,h) \ge p(x) + t\,\sup\big\{h(a)\,\operatorname{sign} x(a) : a \in A\big\}$, and hence

$$\begin{aligned}
|x(s)| &\ge |(x+th)(s)| - t\,|h(s)| \ge p(x+t\,h) - t\,p(h)\\
&\ge p(x) - t\,\underbrace{\Big(p(h) - \sup\big\{h(r)\,\operatorname{sign} x(r) : r \in A\big\}\Big)}_{\le 2}\\
&> \|x\|_\infty - \delta = \sup\big\{|x(r)| : \operatorname{dist}(r,A) \ge \delta_1\big\}.
\end{aligned}$$

Therefore $\operatorname{dist}(s,A) < \delta_1$, and thus there exists an $a \in A$ with $\operatorname{dist}(s,a) < \delta_1$ and consequently $|x(s)-x(a)| < \frac{1}{2}$ and $|h(s)-h(a)| < \varepsilon$. In particular, $\operatorname{sign} x(s) = \operatorname{sign} x(a) \ne 0$ and $|x(s)| > \frac{1}{2}$. So we get

$$\begin{aligned}
\frac{p(x+t\,h)-p(x)}{t} &= \frac{|(x+t\,h)(s)| - p(x)}{t} = \frac{\big||x(s)| + t\,h(s)\,\operatorname{sign} x(s)\big| - p(x)}{t}\\
&= \frac{|x(s)| + t\,h(s)\,\operatorname{sign} x(s) - p(x)}{t} \le h(s)\,\operatorname{sign} x(s) = h(s)\,\operatorname{sign} x(a)\\
&\le |h(s)-h(a)| + h(a)\,\operatorname{sign} x(a)\\
&< \varepsilon + \sup\big\{h(a')\,\operatorname{sign} x(a') : a' \in A\big\}.
\end{aligned}$$

This proves the claim and thus

$$p'(x)(v) = \lim_{t\searrow 0}\frac{p(x+t\,h)-p(x)}{t} = \sup\big\{h(a)\,\operatorname{sign} x(a) : a \in A\big\}. \quad \square$$

Remark. Nowhere Fréchet differentiabilty of the ∞-norm. This result shows that the points where the ∞-norm is Gâteaux-differentiable are exactly those x where the supremum $p(x)$ is attained in a single point a. The Gâteaux-derivative is then given by $p'(x)(h) = h(a)\,\operatorname{sign} x(a)$. In general, this is however not the Fréchet derivative:

Let $x \ne 0$. Without loss we may assume (that $p(x)=1$ and) that there is a unique non-isolated point a, where $|x(a)| = p(x)$. Moreover, we may assume $x(a) > 0$. Let $a_n \to a$ be such that $0 < x(a_n) < x(a)$ and let $0 < \frac{s_n}{2} := x(a) - x(a_n) < x(a)$. Now choose $h_n \in C(T,\mathbb{R})$ with $p(h_n) \le 1$, $h_n(a) := 0$, and $h_n(a_n) := 1$. Then $p(x+s_n h_n) \ge (x+s_n h_n)(a_n) = x(a_n) + 2(x(a)-x(a_n)) = 2x(a) - x(a_n)$ and $p'(x)(h_n) = h_n(a)\operatorname{sign}(x(a)) = 0$ by the previous lemma. Therefore

$$\frac{p(x+s_n\,h_n)-p(x)}{s_n} - p'(x)(h_n) \ge \frac{2x(a)-x(a_n)-x(a)}{s_n} = \frac{1}{2}$$

Thus the limit of the difference quotinent is not uniform, i.e. p is not Fréchet differentiable at x.

Remark. The ∞-norm is rough. If T has no isolated points, then the set of vectors $x \in C(T,\mathbb{R})$ which attain their norm at least at two points a and b is dense, and for those x with $\|x\|_\infty = 1$ and h with $h(a) = -x(a)$, $h(b) = x(b)$, and $|h(t)| \le |x(t)|$ for all t we have

$$p(x+t\,h) = (1+|t|)\,p(x), \text{ hence } p'(x+t\,h)(h) = \begin{cases} +1 & \text{for } t \ge 0\\ -1 & \text{for } t < 0\end{cases}.$$

Therefore, the derivative of the norm is uniformly discontinuous, i.e., in every non-empty open set there are points x_1, x_2 for which there exists an $h \in C[0,1]$ with $\|h\| = 1$ and $|p'(x_1)(h) - p'(x_2)(h)| \geq 2$.

13.13. Results on the differentiability of p-norms. [17, p.887].
For $1 < p < \infty$ the function $t \mapsto |t|^p$ is differentiable of order n if $n < p$, and the highest derivative $(t \mapsto p\,(p-1)\dots(p-n+1)\,|t|^{p-n})$ satisfies a Hölder-condition with modulus $p-n$, one can show that the p-norm has exactly these differentiability properties, i.e.

(1) *It is $(p-1)$-times differentiable with Lipschitzian highest derivative if p is an odd integer.*
(2) *It is $[p]$-times differentiable with highest derivative being Hölderian of order $p-[p]$, if p is not an integer.*
(3) *The norm has no higher Hölder-differentiability properties.*

That the norm on L^p is C^1 for $1 < p < \infty$ was already shown by [91].

13.15. Theorem. Characterization of smooth seminorms. *Let E be a convenient vector space.*

(1) Let $p : E \to \mathbb{R}$ be a continuous convex function which is smooth on a neighborhood of $p^{-1}(1)$, and assume that $U := \{x \in E : p(x) < 1\}$ is not empty. Then U is open, and its boundary ∂U equals $\{x : p(x) = 1\}$, a smooth splitting submanifold of E.

(2) If U is a convex absorbing open subset of E whose boundary is a smooth submanifold of E then the Minkowski functional p_U is a smooth sublinear mapping, and $U = \{x \in E : p_U(x) < 1\}$.

Proof. (1) The set U is obviously convex and open by 4.5 and 13.1. Let $M := \{x : p(x) = 1\}$. We claim that $M = \partial U$. Let $x_0 \in U$ and $x_1 \in M$. Since $t \mapsto p(x_1 + t(x_0 - x_1))$ is convex it is strictly decreasing in a neighborhood of 0. Hence, there are points x close to $x_1 \notin U$ with $p(x) < p(x_1)$, i.e. x belongs to ∂U. Conversely, let $x_1 \in \partial U$. Since U is open we have $p(x_1) \geq 1$. Suppose $p(x_1) > 1$, then $p(x) > 1$ locally around x_1, a contradiction to $x_1 \in \partial U$.

Now we show that M is a smooth splitting submanifold of E, i.e. every point has a neighborhood, in which M is up to a diffeomorphism a complemented linear subspace. Let $x_1 \in M = \partial U$. We consider again the convex mapping $t \mapsto p(x_1 + t(x_0 - x_1))$. It is locally around 0 differentiable, and its value at 1 is strictly less than that at 0. Thus, $p'(x_1)(x_1 - x_0) \geq p(x_1) - p(x_0) > 0$, and hence we may replace x_0 by some point on the segment from x_0 to x_1 closer to x_1, such that $p'(x_0)(x_1 - x_0) > 0$. Without loss of generality we may assume that $x_0 = 0$. Let $U := \{x \in E : p'(0)(x) > 0$ and $p'(x_1)(x) > 0\}$ and $V := (U - x_1) \cap \ker p'(x_1) \subseteq \ker p'(x_1)$. A continuous mapping Ψ from the open set $U \subseteq E$ to the open set $V \times (p(0), +\infty) \subseteq \ker p'(x_1) \times \mathbb{R}$ is given by $x \mapsto (\frac{1}{t}x - x_1, p(x))$, where $t := \frac{p'(x_1)(x)}{p'(x_1)(x_1)} > 0$. This mapping is a bijection $U \to V \times (p(0), +\infty)$: For $(y, r) \in \ker p'(x_1) \times (p(0), +\infty)$ the inverse image is given as $t(y + x_1)$ where t can be calculated from $r = p(t\,(y + x_1))$. Since $t \mapsto p(t\,(y + x_1))$ is a bijection between the intervals $(0, +\infty) \to (p(0), +\infty)$ this s is uniquely determined.

We claim that $\Psi^{-1} : V \times (p(0), +\infty) \to U$ is continuous for the c^∞-topologies. So let $(y_n, r_n) := \Psi(u_n) \to \Psi(u_\infty) = (y_\infty, r_\infty)$ be M-convergent. Since $u_n =$

$t_n(y_n+x_1)$ with t_n being the unique solution of $r_n = p(t_n(y_n+x_1))$ and analogous for $(u_\infty, y_\infty, t_\infty, r_\infty)$, it suffices to show that $t_n \to t_\infty$. For an accumulation point $\tau \le +\infty$ of the sequence t_n we denote the corresponding subsequence again t_n. Then $u_n \to \tau(y_\infty + x_1)$ and $r_n = p(u_n)$ converges to $p(\tau(y_\infty + x_1))$ but also to $r_\infty = p(t_\infty(y_\infty + x_1))$. Since $t \mapsto p(t(y_\infty + x_1))$ is strictly increasing we get $\tau(y_\infty + x_1) = t_\infty(y_\infty + x_1)$ and hence $\tau = t_\infty$.

Thus we may restrict Ψ to a smooth local homeomorphism $E \supseteq U \supseteq U_1 \cong V_1 \subseteq V \times (p(0), +\infty) \subseteq \ker p'(x_1) \times \mathbb{R}$.

Furthermore, Ψ^{-1} is smooth, since t depends smoothly on $(y,r) \in V_1$:
Let $s \mapsto (y(s), r(s))$ be a smooth curve in V_1, then $t(s)$ is given by the implicit equation $p(t(s)\,(y(s)+x_1)) = r(s)$, and by the 2-dimensional implicit function theorem the solution $s \mapsto t(s)$ is smooth since $\frac{\partial}{\partial t}(p(t\,(y(s)+x_1)) - r(s)) = \frac{1}{t}p'(u(s))(u(s)) \ge \frac{1}{t}p'(0)(u(s)) > 0$.

(2) By general principles p_U is a sublinear mapping, and $U = \{x : p_U(x) < 1\}$ since U is open. Thus it remains to show that p_U is smooth on its open carrier. So let c be a smooth curve in the carrier. By assumption, there is a diffeomorphism v, locally defined on E near an intersection point a of the ray through $c(0)$ with the boundary $\partial U = \{x : p_U(x) = 1\}$, such that ∂U corresponds to a closed linear subspace $F \subseteq E$. Since U is convex there is a continuous linear functional $\lambda \in E'$ with $\lambda(a) = 1$ and $\bar{U} \subseteq \{x \in E : \lambda(x) \le 1\}$ by the theorem of Hahn-Banach. Then $\lambda(T_a(\partial U)) = 0$, since any smooth curve in ∂U through a stays inside $\{x : \lambda(x) \le 1\}$. Furthermore, $b := \frac{\partial}{\partial t}|_{t=1} v(ta) \notin F$, since otherwise $t \mapsto v^{-1}(tb) \in \partial U$ and hence $\frac{\partial}{\partial t}|_{t=0}\lambda(v^{-1}(tb)) = 0$ but $\frac{\partial}{\partial t}|_{t=0}\lambda(v^{-1}(tb)) = \lambda((v^{-1})'(0)(b)) = \lambda((v^{-1})'(0)(v'(a)(a))) = \lambda(a) = 1$.

It remains to show that $f := 1/p_U \circ c : \mathbb{R} \to \mathbb{R}$ is smooth. Since $v(f(t)c(t)) \in F$ and $\lambda((v^{-1})'(0)(F)) \subseteq \lambda(T_a\partial U) = 0$, we see that f is a solution of the implicit equation $(\lambda \circ (v^{-1})'(0) \circ v)(f(t)c(t)) = 0$, This solution is unique and smooth by the implicit function theorem in dimension 2:

$$\tfrac{\partial}{\partial s}|_{s=f(t)}\big(\lambda \circ (v^{-1})'(0) \circ v\big)(s\,c(t)) = \big(\lambda \circ (v^{-1})'(0) \circ v'(f(t)c(t))\big)(c(t)) \ne 0$$

for t near 0, since for $t = 0$ we have $f(0)c(0) = a$ and hence this partial derivative equals $\lambda(c(0)) = \frac{1}{f(0)}$. So p_U is smooth on its carrier. □

13.16. The space $c_0(\Gamma)$

For an arbitrary set Γ the space $c_0(\Gamma)$ is the closure of all functions on Γ with finite support in the Banach space $\ell^\infty(\Gamma)$ of globally bounded functions on Γ with the supremum norm. The supremum norm on $c_0(\Gamma)$ is not differentiable on its carrier, see 13.12. Nevertheless, it was shown in [16] that c_0 is C^∞-regular.

Proposition. Smooth norm on c_0. Due to Kuiper according to [17]. *There exists an equivalent norm on $c_0(\Gamma)$ which is smooth off 0.*

Proof. To prove this let $h : \mathbb{R} \to [0, +\infty)$ be an unbounded smooth symmetric convex function vanishing near 0. Let $f : c_0(\Gamma) \to \mathbb{R}$ be given by $f(x) := \sum_{\gamma\in\Gamma} h(x_\gamma)$. Locally on $c_0(\Gamma)$ the function f is just a finite sum, hence f is smooth: In fact let $h(t) = 0$ for $|t| \le \delta$. For $x \in c_0(\Gamma)$ the set $F := \{\gamma : |x_\gamma| \ge \delta/2\}$ is finite, and for $\|y - x\| < \delta/2$ we have that $f(y) = \sum_{\gamma\in F} h(y_\gamma)$.

The set $U := \{x : f(x) < 1\}$ is absolutely convex: Since h is convex and symmetric, so is f and hence also U.
Furthermore, U is open and bounded: Let $h(t) \geq 1$ for $|t| \geq \Delta$ and $f(x) < 1$, then $h(x_\gamma) < 1$ and thus $|x_\gamma| \leq \Delta$ for all γ.

The boundary $\partial U = f^{-1}(1)$ is a splitting submanifold of $c_0(\Gamma)$ by 13.15. So again by 13.15 the Minkowski functional p_U is smooth off 0. Obviously, it is an equivalent norm since U is open and bounded. □

13.17. Proposition. Inheritance properties for differentiable norms.

(1) *The product of two spaces with C^n-norm has again a C^n-norm given by $\|(x_1, x_2)\| := \sqrt{\|x_1\|^2 + \|x_2\|^2}$. More generally, the ℓ^2-sum of C^n-normable Banach spaces is C^n-normable.*

(2) *A subspace of a space with a C^n-norm has a C^n-norm.*

(3) [42]. *If $c_0(\Gamma) \to E \to F$ is a short exact sequence of Banach spaces, and F has a C^k-norm, then E has a C^k-norm. See also 14.12.1 and 16.19.*

(4) *For a compact space K let K' be the set of all accumulation points of K. The operation $K \mapsto K'$ has the following properties:*

$$
\begin{array}{ll}
(a) & A \subseteq B \Rightarrow A' \subseteq B' \\
(b) & (A \cup B)' = A' \cup B' \\
(c) & (A \times B)' = (A' \times B) \cup (A \times B') \\
(d) & \big(\{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{N}\}\big)' = \{0\} \\
(e) & K' = \emptyset \Leftrightarrow K \textit{ discrete.}
\end{array}
$$

(5) *If K is compact and $K^{(\omega)} = \emptyset$ then $C(K)$ has an equivalent C^∞-norm, see also 16.20.*

Proof. (1) and (2) are obvious.

(4) (a) is obvious, since if $\{x\}$ is open in B and $x \in A$, then it is also open in A in the trace topology, hence $A \cap (B \setminus B') \subseteq A \setminus A'$ and hence $A' = A \setminus (A \setminus A') \subseteq (A \setminus A \cap (B \setminus B')) = A \cap B' \subseteq B'$.

(b) By monotonicity we have '$\supseteq$'. Conversely let $x \in A' \cup B'$, w.l.o.g. $x \in A'$, suppose $x \notin (A \cup B)'$, then $\{x\}$ is open in $A \cup B$ and hence $\{x\} = \{x\} \cap A$ would be open in A, i.e. $x \notin A'$, a contradiction.

(c) is obvious, since $\{(x, y)\}$ is open in $A \times B \Leftrightarrow \{x\}$ is open in A and $\{y\}$ is open in B.

(d) and (e) are trivial.

For (3) a construction is used similar to that of Kuiper's smooth norm for c_0. Let $\pi : E \to F$ be the quotient mapping and $\|\_\|$ the quotient norm on F. The dual sequence $\ell^1(A) \leftarrow E^* \leftarrow F^*$ splits (just define $T : \ell^1(A) \to E^*$ by selection of $x_a^* := T(e_a) \in E^*$ with $\|x_a^*\| = 1$ and $x_a^*|_{c_0(A)} = \mathrm{ev}_a$ using Hahn Banach). Note that for every $x \in E$ and $\varepsilon > 0$ the set $\{\alpha : |x_\alpha^*(x)| \geq \|\pi(x)\| + \varepsilon\}$ is finite. In fact, by definition of the quotient norm $\|\pi(x)\| := \sup\{\|x + y\| : y \in c_0(\Gamma)\}$ there is a $y \in c_0(\Gamma)$ such that $\|x + y\| \leq \|\pi(x)\| + \varepsilon/2$. The set $\Gamma_0 := \{\alpha : |y_\alpha| \geq \varepsilon/2\}$ is finite. For all other α we have

$$|x^*_\alpha(x)| \le |x^*_\alpha(x+y)| + |x^*_\alpha(y)| \le \|x^*_\alpha\|\,\|x+y\| + |y_\alpha| < \\ < 1\,(\|\pi(x)\| + \varepsilon/2) + \varepsilon/2 = \|\pi(x)\| + \varepsilon.$$

Furthermore, we have

$$\|x\| \le 2\|\pi(x)\| + \sup\{|x^*_\alpha(x)| : \alpha\}.$$

In fact,

$$\begin{aligned}\|x\| &= \sup\{|\langle x^*, x\rangle| : \|x^*\| \le 1\} \\ &\le \sup\{|\langle T(\lambda) + y^* \circ \pi, x\rangle| : \|\lambda\|_1 \le 1, \|y^*\| \le 2\} \\ &= \sup\{|x^*_\alpha(x)| : \alpha\} + 2\|\pi(x)\|,\end{aligned}$$

since $x^* = T(\lambda) + x^* - T(\lambda)$, where $\lambda := x^*|_{c_0(\Gamma)}$ and hence $\|\lambda\|_1 \le \|x^*\| \le 1$, and $|T(\lambda)(x)| \le \|\lambda\|_1 \sup\{|x^*_\alpha(x)| : \alpha\} \le \|x\|$ hence $\|T(\lambda)\| \le \|\lambda\|_1$, and $y^* \circ \pi = x^* - T(\lambda)$. Let $\|\_\|$ denote a norm on F which is smooth and is larger than the quotient norm. Analogously to 13.16 we define

$$f(x) := h(4\|\pi(x)\|) \prod_{a\in A} h(x^*_a(x)),$$

where $h : \mathbb{R} \to [0,1]$ is smooth, even, 1 for $|t| \le 1$, 0 for $|t| \ge 2$ and concave on $\{t : h(t) \ge 1/2\}$. Then f is smooth, since if $\pi(x) > 1/2$ then the first factor vanishes locally, and if $\|\pi(x)\| < 1$ we have that $\Gamma_0 := \{\alpha : |x^*_\alpha(x)| \ge 1 - \varepsilon\}$ is finite, where $\varepsilon := (1 - \|\pi(x)\|)/2$, for $\|y - x\| < \varepsilon$ also $|x^*_\alpha(y) - x^*_\alpha(x)| < \varepsilon$ and hence $|x^*_\alpha(y)| < 1 - \varepsilon + \varepsilon = 1$ for all $\alpha \notin \Gamma_0$. So the product is locally finite. The set $\{x : f(x) > \frac{1}{2}\}$ is open, bounded and absolutely convex and has a smooth boundary $\{x : f(x) = \frac{1}{2}\}$. It is symmetric since f is symmetric. It is bounded, since $f(x) > 1/2$ implies $h(4\|\pi(x)\|) \ge 1/2$ and $h(x^*_a(x)) \ge 1/2$ for all a. Thus $4\|\pi(x)\| \le 2$ and $|x^*_a(x)| \le 2$ and thus $\|x\| \le 2 \cdot 1/2 + 2 = 3$. For the convexity note that $x_i \ge 0$, $y_i \ge 0$, $0 \le t \le 1$, $\prod_i x_i \ge 1/2$, $\prod_i y_i \ge 1/2$ imply $\prod_i (tx_i + (1-t)y_i) \ge 1/2$, since log is concave. Since all factors of f have to be $\ge 1/2$ and h is concave on this set, convexity follows. Since one factor of $f(x) = \prod_\alpha f_\alpha(x)$ has to be unequal to 1, the derivative $f'(x)(x) < 0$, since $f'_\alpha(x)(x) \le 0$ for all α by concavity and $f'_\alpha(x)(x) < 0$ for all x with $f_\alpha(x) < 1$. So its Minkowski-functional is an equivalent smooth norm on E.

Statement (5) follows from (3). First recall that K' is the set of accumulation points of K, i.e. those points x for which every neighborhood meets $K \setminus \{x\}$, i.e. $\{x\}$ is not open. Thus $K \setminus K'$ is discrete. For successor ordinals $\alpha = \beta + 1$ one defines $K^{(\alpha)} := (K^{(\beta)})'$ and for limit ordinals α as $\bigcap_{\beta<\alpha} K^{(\beta)}$. For a compact space K the equality $K^{(\omega)} = \emptyset$ implies $K^{(n)} = \emptyset$ for some $n \in \omega$, since $K^{(n)}$ is closed. Now one shows (5) by induction on n. Let $E := \{f \in C(K) : f|_{K'} = 0\}$. By the Tietze-Urysohn theorem one has a short exact sequence $c_0(K \setminus K') \cong E \to C(K) \to C(K')$. The equality $E = c_0(K \setminus K_0)$ can be seen as follows:
Let $f \in C(K)$ with $f|_{K'} = 0$. Suppose there is some $\varepsilon > 0$ such that $\{x : |f(x)| \ge \varepsilon\}$ is not finite. Then there is some accumulation point x_∞ of this set and hence $|f(x_\infty)| \ge \varepsilon$ but $x_\infty \in K'$ and so $f(x_\infty) = 0$. Conversely let $f \in c_0(K \setminus K')$ and define $\tilde f$ by $\tilde f|_{K'} := 0$ and $\tilde f|_{K\setminus K'} = f$. Then $\tilde f$ is continuous on $K \setminus K'$, since $K \setminus K'$ is discrete. For $x \in K'$ we have that $\tilde f(x) = 0$ and for each $\varepsilon > 0$ the set $\{y : |\tilde f(y)| \ge \varepsilon\}$ is finite, hence its complement is a neighborhood of x, and $\tilde f$ is continuous at x. So the result follows by induction. □

13.18. Results.

(1) *We do not know whether the quotient of a C^n-normable space is again C^n-normable. Compare however with* [37].

(2) *The statement* 13.17.5 *is quite sharp, since by* [49] *there is a compact space K with $K^{(\omega)} = \{\infty\}$ but without a Gâteaux-differentiable norm.*

(3) [119, Théorème 4] *proved that $C([0,\gamma])$ is C^1-normable for every ordinal number γ and the associated compact and scattered space $[0,\gamma]$ with the order topology.*

(4) *It was shown by* [122] *that two Banach spaces are homeomorphic if and only if their density number is the same. Hence, one can view Banach spaces as exotic (differentiable or linear) structures on Hilbert spaces. If two Banach spaces are even C^1-diffeomorphic then the differential (at 0) gives a linear homeomorphism. It was for some time unknown if also uniformly homeomorphic (or at least Lipschitz homeomorphic) Banach spaces are already linearly homeomorphic. By* [32] *a Banach space which is uniformly homeomorphic to a Hilbert space is linearly homeomorphic to it. A counter-example to the general statement was given by* [2], *and another one is due to* [22]: *There exists a short exact sequence $c_0(\Gamma_1) \to C(K) \to c_0(\Gamma_2)$ where $C(K)$ cannot be continuously linearly injected into some $c_0(\Gamma)$ but is Lipschitz equivalent to $c_0(\Gamma)$. For these and similar questions see* [126].

(5) *A Banach space all of whose closed subspaces are complemented is a Hilbert space,* [86].

(6) [33] *There exists a Banach space E not linearly homeomorphic to a Hilbert space and a short exact sequence $\ell^2 \to E \to \ell^2$.*

(7) [18]. *If the norm of a Banach space and its dual norm are C^2 then the space is a Hilbert space.*

(8) [28]. *This yields also an example that existence of smooth norms is not a* three-space property, *cf.* 14.12.

Notes. (2) Note that $K \setminus K'$ is discrete, open and dense in K. So we get for every $n \in \mathbb{N}$ by induction a space K_n with $K_n^{(n)} \neq \emptyset$ and $K_n^{(n+1)} = \emptyset$. In fact $(A \times B)^{(n)} = \bigcup_{i+j=n} A^{(i)} \times B^{(j)}$. Next consider the 1-point compactification K_∞ of the locally compact space $\bigsqcup_{n\in\mathbb{N}} K_n$. Then $K'_\infty = \{\infty\} \cup \bigsqcup_{n\in\mathbb{N}} K'_n$. In fact every neighborhood of $\{\infty\}$ contains all but finitely many of the K_n, thus we have $\supseteq$. The obvious relation is clear. Hence $K_\infty^{(n)} = \{\infty\} \cup \bigsqcup_{i\ge n} K_n^{(i)}$. And $K_\infty^{(\omega)} = \bigcap_{n<\omega} K_\infty^{(n)} = \{\infty\} \neq \emptyset$. The space of [49] is the one-point compactification of a locally compact space L given as follows: $L := \bigsqcup_{\alpha<\omega_1} \omega_1^\alpha$, i.e. the space of functions $\omega_1 \to \omega_1$, which are defined on some countable ordinal. It is ordered by restriction, i.e. $s \preceq t :\Leftrightarrow \operatorname{dom} s \subseteq \operatorname{dom} t$ and $t|_{\operatorname{dom} s} = s$.

(3) The order topology on $X := [0,\gamma]$ has the sets $\{x : x < a\}$ and $\{x : x > a\}$ as basis. In particular open intervals $(a,b) := \{x : a < x < b\}$ are open. It is compact, since every subset has a greatest lower bound. In fact let $\mathcal{U}$ on X be a covering. Consider $S := \{x \in X : [\inf X, x)$ is covered by finitely many $U \in \mathcal{U}\}$. Let $s_\infty := \sup S$. Note that $x \in S$ implies that $[\inf X, x]$ is covered by finitely many sets in $\mathcal{U}$. We have that $s_\infty \in S$, since there is an $U \in \mathcal{U}$ with $s_\infty \in U$. Then there is an x with $s_\infty \in (x, s_\infty] \subseteq U$, hence $[\inf X, x]$ is covered by finitely many sets in $\mathcal{U}$ since there is an $s \in S$ with $x < s$, so $[\inf X, s_\infty] = [\inf X, x] \cup (x, s_\infty]$ is covered by finitely many sets, i.e. $s_\infty \in S$.

The space X is *scattered*, i.e. $X^{(\alpha)} = \emptyset$ for some ordinal α. For this we have to show that every closed non-empty subset $K \subseteq X$ has open points. For every subset K

of X there is a minimum $\min K \in K$, hence $[\inf X, \min K + 1) \cap K = \{\min K\}$ is open in K.

For γ equal to the first infinite ordinal ω we have $[0, \gamma] = \mathbb{N}_\infty$, the one-point compactification of the discrete space $\mathbb{N}$. Thus $C([0, \gamma]) \cong c_0 \times \mathbb{R}$ and the result follows in this case from 13.16.

(5) For splitting short exact sequences the result analogous to 13.17.3 is by 13.17.1 obviously true. By (5) there are non-splitting exact sequences $0 \to F \to E \to E/F \to 0$ for every Banach space E which is not Hilbertizable.

(8) By (6) there is a sort exact sequence with hilbertizable ends, but with middle term E not hilbertizable. So neither the sequence nor the dualized sequence splits. Assuming the 3-space property provides E and E' with C^2-norms, hence E would be hilbertizable by (7).

13.19. Proposition. *Let E be a Banach space, $\|x\| = 1$. Then the following statements are equivalent:*

1. *The norm is Fréchet differentiable at x;*
2. $\lim_{h \to 0} \frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|} = 0$
   (or equivalently, $\lim_{t \to 0} \frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = 0$ uniformly in $\|h\| \leq 1$.)
3. $\|y_n^*\| = 1$, $\|z_n^*\| = 1$, $y_n^*(x) \to 1$, $z_n^*(x) \to 1 \Rightarrow y_n^* - z_n^* \to 0$.

Proof. (1)⇒(2) This is obvious, since for the derivative ℓ of the norm at x we have $\lim_{h \to 0} \frac{\|x \pm h\| - \|x\| - l(\pm h)}{\|h\|} = 0$ and adding these equations gives (2).

(2) ⇒ (1) Since $\ell(h) := \lim_{t \searrow 0} \frac{\|x+th\|-\|x\|}{t}$ always exists, and since

$$\begin{aligned}\frac{\|x+th\|+\|x-th\|-2\|x\|}{t} &= \frac{\|x+th\|-\|x\|}{t} + \frac{\|x+t(-h)\|-\|x\|}{t}\\ &\geq l(h) + l(-h) \geq 0\end{aligned}$$

we have $\ell(-h) = \ell(h)$, thus ℓ is linear. Moreover $\frac{\|x \pm th\| - \|x\|}{t} - \ell(\pm h) \geq 0$, so the limit is uniform for $\|h\| \leq 1$.

(2) ⇒ (3) By (2) we have that for $\varepsilon > 0$ there exists a δ such that

$$\|x+h\| + \|x-h\| \leq 2 + \varepsilon\|h\| \text{ for all } \|h\| < \delta.$$

For $\|y_n^*\| = 1$ and $\|z_n^*\| = 1$ we have

$$(y_n^* - z_n^*)(h) + (y_n^* + z_n^*)(x) = y_n^*(x+h) + z_n^*(x-h) \leq \|x+h\| + \|x-h\|.$$

Since $y_n^*(x) \to 1$ and $z_n^*(x) \to 1$ we get for large n that

$$(y_n^* - z_n^*)(h) \leq -y_n^*(x) - z_n^*(x) + 2 + \varepsilon\|h\| \leq 2\varepsilon\delta,$$

hence $\|y_n^* - z_n^*\| \leq 2\varepsilon$, i.e. $z_n^* - y_n^* \to 0$.

(3) ⇒ (2) Otherwise, there exists an $\varepsilon > 0$ and $0 \neq h_n \to 0$, such that

$$\|x+h_n\| + \|x-h_n\| \geq 2 + \varepsilon\|h_n\|.$$

Now choose $\|y_n^*\| = 1$ and $\|z_n^*\| = 1$ with

$$y_n^*(x+h_n) \geq \|x+h_n\| - \tfrac{1}{n}\|h_n\| \text{ and } z_n^*(x-h_n) \geq \|x-h_n\| - \tfrac{1}{n}\|h_n\|.$$

Then $y_n^*(x) = y_n^*(x+h_n) - y_n^*(h_n) \to \|x\| = 1$ and similarly $z_n^*(x) \to 1$. Furthermore

$$(y_n^* + z_n^*)(x) + (y_n^* - z_n^*)(h_n) = y_n^*(x+h_n) + z_n^*(x-h_n) \geq 2 + (\varepsilon - \tfrac{2}{n})\,\|h_n\|,$$

hence

$$(y_n^* - z_n^*)(h_n) \geq -(y_n^* + z_n^*)(x) + 2 + (\varepsilon - \tfrac{2}{n})\,\|h_n\| \geq (\varepsilon - \tfrac{2}{n})\,\|h_n\|,$$

thus $\|y_n^* - z_n^*\| \geq \varepsilon - \frac{2}{n}$, a contradiction. □

13.20. Proposition. Fréchet differentiable norms via locally uniformly rotund duals. [87] *If the dual norm of a Banach space E is locally uniformly rotund on E' then the norm is Fréchet differentiable on E.*

A norm is called *locally uniformly rotund* if $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$ implies $x_n \to x$. This is equivalent to $2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 \to 0$ implies $x_n \to x$, since

$$2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 \geq 2\|x\|^2 + 2\|x_n\|^2 - (\|x\| + \|x_n\|)^2 = (\|x\| - \|x_n\|)^2.$$

Proof. We verify 13.19.3: So let $\|x\| = 1$, $\|y_n^*\| = 1$, $\|z_n^*\| = 1$, $y_n^*(x) \to 1$, $z_n^*(x) \to 1$. Let $\|x^*\| = 1$ with $x^*(x) = 1$. Then $2 \geq \|x^* + y_n^*\| \geq (x^* + y_n^*)(x) \to 2$. Since $\|\_\|_{E'}$ is locally uniformly rotund we get $y_n^* \to x^*$ and similarly $z_n^* \to x^*$, hence $y_n^* - z_n^* \to 0$. □

13.21. Remarks on locally uniformly rotund spaces

By [57] and [58] every separable Banach space is isomorphic to a locally uniformly rotund Banach space. By [27] the space $\ell^\infty(\Gamma)$ is not isomorphic to a locally uniformly rotund Banach space. Every Banach space admitting a continuous linear injection into some $c_0(\Gamma)$ is locally uniformly rotund renormable, see [125]. By 53.21 every WCG-Banach space has such an injection, which is due to [3]. By [124] every Banach space with unconditional basis (see [53, 14.7]) is isomorphic to a locally uniformly rotund Banach space.

In particular, it follows from these results that every reflexive Banach space has an equivalent Fréchet differentiable norm. In particular L^p has a Fréchet differentiable norm for $1 < p < \infty$ and in fact the p-norm itself is Fréchet differentiable, see 13.13.

13.22. Proposition. *If the dual E' of a Banach space E is separable, then E admits an equivalent norm, whose dual norm is locally uniform rotund. Thus E is C^1-normable by 13.20.*

Proof. Let E' be separable. Then there exists a bounded linear operator $T : E \to \ell^2$ such that $T^*((\ell^2)')$ is dense in E' (and obviously T^* is weak*-continuous):
Take a dense subset $\{x_i^* : i \in \mathbb{N}\} \subseteq E'$ of $\{x^* \in E' : \|x^*\| \leq 1\}$ with $\|x_i^*\| \leq 1$. Define $T : E \to \ell^2$ by

$$T(x)_i := \frac{x_i^*(x)}{2^i}.$$

Then for the basic unit vector $e_i \in (\ell^2)'$ we have

$$T^*(e_i)(x) = e_i(T(x)) = T(x)_i = \frac{x_i^*(x)}{2^i},$$

i.e. $T^*(e_i) = 2^{-i}\,x_i^*$.

Note that the canonical norm on ℓ^2 is locally uniformly rotund. We now claim that E' has a dual locally uniform rotund norm. For $x^* \in E'$ and $n \in \mathbb{N}$ we define

$$\|x^*\|_n^2 := \inf\Big\{\|x^* - T^* y^*\|^2 + \tfrac{1}{n}\|y^*\|^2 : y^* \in (\ell^2)'\Big\} \text{ and}$$
$$\|x^*\|_\infty := \sum_{n=1}^\infty \frac{1}{2^n}\|x^*\|_n.$$

We claim that $\|\_\|_\infty$ is the required norm.

So we show first, that it is an equivalent norm. For $\|x^*\| = 1$ we have $\|x^*\|_n \geq \min\{1/(2\sqrt{n}\|T^*\|), 1/2\}$. In fact if $\|y^*\| \geq 1/(2\|T^*\|)$ then $\|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2 \geq 1/(2n^2\|T^*\|^2)$ and if $\|y^*\| \leq 1/(2\|T^*\|)$ then $\|x^* - T^*y^*\| \geq \|x\| - \|T^*y^*\| \geq 1 - \frac{1}{2} = \frac{1}{2}$. Furthermore if we take $y := 0$ then we see that $\|x^*\|_n \leq \|x\|$. Thus $\|\_\|_n$ and $\|\_\|$ are equivalent norms, and hence also $\|\_\|_\infty$.

Note first, that a dual norm is the supremum of the weak* (lower semi-)continuous functions $x^* \mapsto |x^*(x)|$ for $\|x\| \leq 1$. Conversely the unit ball B has to be weak* closed in E' since the norm is assumed to be weak* lower semi-continuous and B is convex. Let B_o be its polar in E. By the bipolar-theorem $(B_o)^o = B$, and thus the dual of the Minkowski functional of B_o is the given norm.

Next we show that the infimum defining $\|\_\|_n$ is in fact a minimum, i.e. for each n and x^* there exists a y^* with $\|x^*\|_2^n = \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2$. Since $f_x : y^* \mapsto \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y^*\|^2$ is weak* lower semi-continuous and satisfies $\lim_{y^* \to \infty} f_x(y^*) = +\infty$, hence it attains its minimum on some large (weak*-compact) ball.

We have that $\|x\|_n \to 0$ for $n \to \infty$.
In fact since the image of T^* is dense in E', there is for every $\varepsilon > 0$ a y^* with $\|x^* - T^*y^*\| < \varepsilon$, and so for large n we have $\|x^*\|_n^2 \leq \|x^* - T^*y^*\|^2 + \frac{1}{n}\|y\|^2 < \varepsilon^2$.

Let us next show that $\|\_\|_\infty$ is a dual norm. For this it is enough to show that $\|\_\|_n$ is a dual norm, i.e. is weak* lower semi-continuous. So let x_i^* be a net converging weak* to x^*. Then we may choose y_i^* with $\|x_i^*\|_n^2 = \|x_i^* - T^*y_i^*\|^2 + \frac{1}{n}\|y_i^*\|^2$. Then $\{x_i^* : i\}$ is bounded, and hence also $\|y_i^*\|^2$. Let thus y^* be a weak* cluster point of the (y_i^*). Without loss of generality we may assume that $y_i^* \to y^*$. Since the original norms are weak* lower semicontinuous we have

$$\|x^*\|_n^2 \leq \|x^* - T^*y^*\|^2 + \tfrac{1}{n}\|y^*\|^2 \leq \varliminf_i \Big(\|x_i^* - T^*y_i^*\|^2 + \tfrac{1}{n}\|y_i^*\|^2\Big) = \varliminf_i \|x_i^*\|_2^n.$$

So $\|\_\|_n$ is weak* lower semicontinuous.
Here we use that a function $f : E \to \mathbb{R}$ is lower semicontinuous if and only if $x_\infty = \lim_i x_i \Rightarrow f(x_\infty) \leq \varliminf_i f(x_i)$.
($\Rightarrow$) otherwise for some subnet (which we again denote by x_i) we have $f(x_\infty) > \lim_i f(x_i)$ and this contradicts the fact that $f^{-1}((a, \infty))$ has to be a neighborhood of x_∞ for $2a := f(x_\infty) + \lim_i f(x_i)$.
($\Leftarrow$) otherwise there exists some x_∞ and an $a < f(x_\infty)$ such that in every neighborhood U of x_∞ there is some x_U with $f(x_U) \leq a$. Hence $\lim_U x_U = x_\infty$ and $\varliminf_U f(x_U) \leq \varlimsup_U f(x_U) \leq a < f(x_\infty)$.

Let us finally show that $\|\_\|_\infty$ is locally uniform rotund.
So let $x^*, x_j^* \in E'$ with

$$2(\|x^*\|_\infty^2 + \|x_j^*\|_\infty^2) - \|x^* + x_j^*\|_\infty^2 \to 0,$$

or equivalently

$$\|x_j^*\|_\infty \to \|x^*\|_\infty \text{ and } \|x^* + x_j^*\|_\infty \to 2\|x^*\|_\infty.$$

Thus also
$$\|x_j^*\|_n \to \|x^*\|_n \text{ and } \|x^*+x_j^*\|_n \to 2\|x^*\|_n$$
and equivalently
$$2(\|x^*\|_n^2+\|x_j^*\|_n^2)-\|x^*+x_j^*\|_n^2 \to 0.$$
Now we may choose y^* and y_j^* such that
$$\|x^*\|_2^n = \|x^*-T^*y^*\|^2+\tfrac{1}{n}\|y^*\|^2 \text{ and } \|x_j^*\|_2^n = \|x_j^*-T^*y_j^*\|^2+\tfrac{1}{n}\|y_j^*\|^2.$$
We calculate as follows:
$$\begin{aligned}
2\big(\|x^*\|_n^2+\|x_j^*\|_n^2\big)-\|x^*+x_j^*\|^2 \geq&\\
&\geq 2\Big(\|x^*-T^*y^*\|^2+\tfrac{1}{n}\|y^*\|^2+\|x_j^*-T^*y_j^*\|^2+\tfrac{1}{n}\|y_j^*\|^2\Big)\\
&\quad-\|x^*+x_j^*-T^*(y^*+y_j^*)\|^2-\tfrac{1}{n}\|y^*+y_j^*\|^2\\
&\geq 2\Big(\|x^*-T^*y^*\|^2+\tfrac{1}{n}\|y^*\|^2+\|x_j^*-T^*y_j^*\|^2+\tfrac{1}{n}\|y_j^*\|^2\Big)\\
&\quad-\big(\|x^*-T^*(y^*)\|+\|x_j^*-T^*(y_j^*)\|\big)^2-\tfrac{1}{n}\|y^*+y_j^*\|^2\\
&\geq \big(\|x^*-T^*y^*\|-\|x_j^*-T^*y_j^*\|\big)^2+\\
&\quad+\tfrac{1}{n}\big(2\|y^*\|^2+2\|y_j^*\|^2-\|y^*+y_j^*\|^2\big)\geq 0,
\end{aligned}$$
hence
$$\|x_j^*-T^*y_j^*\| \to \|x^*-T^*y^*\| \text{ and } 2(\|y^*\|^2+\|y_j^*\|^2)-\|y^*+y_j^*\|^2 \to 0.$$
Since $\|\_\|$ is locally uniformly rotund on $(\ell^2)^*$ we get that $y_j^* \to y^*$. Hence
$$\begin{aligned}
\overline{\lim_j}\,\|x^*-x_j^*\| &\leq \overline{\lim_j}\Big(\|x^*-T^*y^*\|+\|T^*(y^*-y_j^*)\|+\|x_j^*-T^*y_j^*\|\Big)\\
&= 2\,\|x^*-T^*y^*\| \leq 2\|x^*\|_n.
\end{aligned}$$
Since $\|x^*\|_n \to 0$ for $n\to\infty$ we get $x_j^* \to x^*$. □

13.23. Proposition. [82]. *For the norm $\|\_\| = p$ on a Banach space E the following statements are equivalent:*

1. *The norm is rough, i.e. p' is uniformly discontinuous, see 13.8.5.*
2. *There exists an $\varepsilon>0$ such that for all $x\in E$ with $\|x\|=1$ and all y_n^*, $z_n^*\in E'$ with $\|y_n^*\|=1=\|z_n^*\|$ and $\lim_n y_n^*(x)=1=\lim_n z_n^*(x)$ we have:*
$$\overline{\lim_n}\,\|y_n^*-z_n^*\|\geq\varepsilon;$$
3. *There exists an $\varepsilon>0$ such that for all $x\in E$ with $\|x\|=1$ we have that*
$$\overline{\lim_{h\to 0}}\,\frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|}\geq\varepsilon;$$
4. *There exists an $\varepsilon>0$ such that for every $x\in E$ with $\|x\|=1$ and $\delta>0$ there is an $h\in E$ with $\|h\|\leq 1$ and $\|x+th\|\geq\|x\|+\varepsilon|t|-\delta$ for all $|t|\leq 1$.*

Note that we always have
$$0\leq\frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|}\leq 2,$$
hence ε in (3) satisfies $\varepsilon\leq 2$. For ℓ^1 and $C[0,1]$ the best choice is $\varepsilon=2$, see 13.11 and 13.12.

Proof. (3)⇒(2) is due to [26]. Let $\varepsilon>0$ such that for all $\|x\|=1$ there are $0\neq h_n\to 0$ with $\|x+h_n\|+\|x-h_n\|-2\geq\varepsilon\|h_n\|$. Now choose y_n^*, $z_n^*\in E'$

with $\|y_n^*\| = 1 = \|z_n\|^*$, $y_n^*(x+h_n) = \|x+h_n\|$ and $z_n^*(x-h_n) = \|x-h_n\|$. Then $\lim_n y_n^*(x) = \|x\| = 1$ and also $\lim_n z_n^*(x) = 1$. Moreover,

$$y_n^*(x+h_n) + z_n^*(x-h_n) \geq 2 + \varepsilon\|h_n\|$$

and hence

$$(y_n^* - z_n^*)(h_n) = \|x+h_n\| + \|x-h_n\| \geq -y_n^*(x) - z_n^*(x) + 2 + \varepsilon\|h_n\| \geq \varepsilon\|h_n\|,$$

thus (2) is satisfied.

(2)⇒(1) By (2) we have an $\varepsilon > 0$ such that for all $\|x\| = 1$ there are y_n^* and z_n^* with $\|y_n^*\| = 1 = \|z_n^*\|$, $\lim_n y_n^*(x) = 1 = \lim_n z_n^*(x)$ and h_n with $\|h_n\| = 1$ and $(y_n^* - z_n^*)(h_n) \geq \varepsilon$. Let $0 < \delta < \varepsilon/2$ and $t > 0$. Then

$$y_n^*(x) > 1 - \frac{\delta^2}{4} \quad \text{and} \quad z_n^*(x) > 1 - \frac{\delta^2}{4} \text{ for large } n.$$

Thus

$$\|x + th_n\| \geq y_n^*(x+th_n) \geq 1 - \frac{\delta^2}{4} + ty_n^*(h_n)$$

and hence by convexity of $t \mapsto p(x+th_n)$

$$\begin{aligned} t\,p'(x+th_n)(h_n) &\geq \|x+th_n\| - \|x\| \geq ty_n^*(h_n) - \frac{\delta^2}{4} \quad \Rightarrow \\ p'(x+th_n)(h_n) &\geq y_n^*(h_n) - \frac{\delta^2}{4t} \\ \text{and similarly } -p'(x-th_n)(h_n) &\geq -z_n^*(h_n) - \frac{\delta^2}{4t} \end{aligned}$$

If we choose $0 < t < \delta$ such that $\delta^2/(2t) < \delta$ we get

$$p'(x+th_n)(h_n) - p'(x-th_n)(h_n) \geq (y_n^* - z_n^*)(h_n) - \frac{\delta^2}{2t} > \varepsilon - \delta > \frac{\varepsilon}{2}.$$

(1)⇒(4) By the uniform discontinuity assumption of p' there exists an $\varepsilon > 0$ such that for each $x \in E$ and $\eta > 0$ there exist $x_j \in E$ with $p(x_j - x) \le \eta/4$ and $u \in E$ with $p(u) = 1$ such that $\big(p'(x_2) - p'(x_1)\big)(u) \ge \varepsilon$.
Let $\mu := (p'(x_1) + p'(x_2))(u)/(2p(x))$ and $v := u - \mu\, x$.

Since $p'(x_1)(u) \le p'(x_2)(u) - \varepsilon$ we get

$$(p'(x_1) + p'(x_2))(u))/2 \le p'(x_2)(u) - \varepsilon/2 \le p(u) - \varepsilon/2 < 1$$
$$\text{and } (p'(x_1) + p'(x_2))(u)/2 \ge p'(x_1)(u) + \varepsilon/2 \ge -p(u) + \varepsilon/2 > -1,$$

i.e. $|\mu| p(x) = |(p'(x_1) + p'(x_2))(u)/2| < 1$, so $0 < p(v) \le p(u) + \mu p(x) < 2$.
For $0 \le t \le p(x)$ and $s := 1 - t\,\mu$ we get

$$x + tv = sx + tu = s(x + \tfrac{t}{s}u) = s\Big((x_2 + \tfrac{t}{s}u) + (x - x_2)\Big).$$

Thus $0 < s < 2$ and

$$\begin{aligned} p(x+tv) &\ge s\Big(p(x_2 + \tfrac{t}{s}u) - p(x - x_2)\Big) \\ &\ge s\Big(p(x_2) + \tfrac{t}{s}p'(x_2)u - \tfrac{\eta}{4}\Big) \quad \text{since } p(y+w) \ge p(y) + p'(y)(w) \\ &\ge (1 - t\mu)\, p(x) + t\, p'(x_2)(u) - s\,\tfrac{\eta}{2} \quad \text{since } p(x) \le p(x_2) + p(x - x_2) \\ &= p(x) + \tfrac{t}{2}\,(p'(x_2) - p'(x_1))(u) - s\,\tfrac{\eta}{2} \\ &> p(x) + t\tfrac{\varepsilon}{2} - \eta. \end{aligned}$$

If $-p(x) \le t < 0$ we proceed with the role of x_1 and x_2 exchanged and obtain

$$\begin{aligned} p(x+tv) &> s\, p(x) + t\, p'(x_1)(u) - s\,\tfrac{\eta}{2} \\ &= p(x) + (-\tfrac{t}{2})\,(p'(x_2) - p'(x_1))(u) - s\,\tfrac{\eta}{2} \\ &> p(x) + |t|\,\tfrac{\varepsilon}{2} - \eta. \end{aligned}$$

Thus

$$p(x+tv) \ge p(x) + |t|\,\varepsilon/2 - \eta.$$

(4)⇒(3) By (4) there exists an $\varepsilon > 0$ such that for every $x \in E$ with $\|x\| = 1$ and $\delta > 0$ there is an $h \in E$ with $\|h\| \le 1$ and $\|x + th\| \ge \|x\| + \varepsilon|t| - \delta$ for all $|t| \le 1$. For $n \in \mathbb{N}$ choose $\delta := \frac{\varepsilon}{2n}$ and $t := \frac{1}{n}$ and obtain

$$\frac{\|x + h_n/n\| + \|x - h_n/n\| - 2}{\|h_n/n\|} \ge 2n\Big(\|x\| + \frac{\varepsilon}{n} - \frac{\varepsilon}{2n} - 1\Big) = \varepsilon. \quad \square$$

13.24. Results on the non-existence of C^1-norms on certain spaces.

(1) [108] *and* [109]. *A separable Banach space has an equivalent C^1-norm if and only if E^* is separable. This will be proved in* 16.11.

(2) [59]. *More generally, if for a Banach space* $\operatorname{dens} E < \operatorname{dens} E^*$ *then no C^1-norm exists. This will be proved by showing the existence of a rough norm in* 14.10 *and then using* 14.9. *The density number* $\operatorname{dens} X$ *of a topological space X is the minimum of the cardinalities of all dense subsets of X.*

(3) [49]. *There exists a compact space K, such that $K^{(\omega_1)} = \{*\}$, in particular $K^{(\omega_1+1)} = \emptyset$, but $C(K)$ has no equivalent Gâteaux differentiable norm, see also* 13.18.2.

One can interpret the results (2) and (3) by saying that in these spaces every convex body necessarily has corners.

14. Smooth Bump Functions

In this section we return to the original question whether the smooth functions generate the topology. Since we will use the results given here also for manifolds, and since the existence of charts is of no help here, we consider fairly general non-linear spaces. This allows us at the same time to treat all considered differentiability classes in a unified way.

14.1. Convention. We consider a Hausdorff topological space X with a subalgebra $\mathcal{S} \subseteq C(X,\mathbb{R})$, whose elements will be called the *smooth* or *$\mathcal{S}$-functions* on X. We assume that for functions $h \in C^\infty(\mathbb{R},\mathbb{R})$ (at least for those being constant off some compact set, in some cases) one has $h_*(\mathcal{S}) \subseteq \mathcal{S}$, and that $f \in \mathcal{S}$ provided it is locally in $\mathcal{S}$, i.e., there exists an open covering $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists an $f_U \in \mathcal{S}$ with $f = f_U$ on U. In particular, we will use for $\mathcal{S}$ the classes of C^∞- and of $\mathcal{L}\mathrm{ip}^k$-mappings on c^∞-open subsets X of convenient vector spaces with the c^∞-topology and the class of C^n-mappings on open subsets of Banach spaces, as well as subclasses formed by boundedness conditions on the derivatives or their difference quotients.

Under these assumptions on $\mathcal{S}$ one has that $\frac{1}{f} \in \mathcal{S}$ provided $f \in \mathcal{S}$ with $f(x) > 0$ for all $x \in X$: Just choose everywhere positive $h_n \in C^\infty(\mathbb{R},\mathbb{R})$ with $h_n(t) = \frac{1}{t}$ for $t \geq \frac{1}{n}$. Then $h_n \circ f \in \mathcal{S}$ and $\frac{1}{f} = h_n \circ f$ on the open set $\{x : f(x) > \frac{1}{n}\}$. Hence, $\frac{1}{f} \in \mathcal{S}$.

For a (convenient) vector space F the *carrier* $\operatorname{carr}(f)$ of a mapping $f : X \to F$ is the set $\{x \in X : f(x) \neq 0\}$. The *zero set* of f is the set where f vanishes, $\{x \in X : f(x) = 0\}$. The *support* of f $\operatorname{support}(f)$ is the closure of $\operatorname{carr}(f)$ in X.

We say that X is *smoothly regular* (with respect to $\mathcal{S}$) or *$\mathcal{S}$-regular* if for any neighborhood U of a point x there exists a smooth function $f \in \mathcal{S}$ such that $f(x) = 1$ and $\operatorname{carr}(f) \subseteq U$. Such a function f is called a *bump function*.

14.2. Proposition. Bump functions and regularity. [17]. *A Hausdorff space is $\mathcal{S}$-regular (i.e. the topology has a basis of carriers of functions in $\mathcal{S}$) if and only if its topology is initial with respect to $\mathcal{S}$.*

Proof. The initial topology with respect to $\mathcal{S}$ has as a subbasis the sets $f^{-1}(I)$, where $f \in \mathcal{S}$ and I is an open interval in $\mathbb{R}$. Let $x_0 \in U$, with U open for the initial topology. Then there exist finitely many open intervals $I_1, \dots, I_n$ and $f_1, \dots, f_n \in \mathcal{S}$ with $x_0 \in \bigcap_{i=1}^n f_i^{-1}(I_i)$. Without loss of generality we may assume that $I_i = \{t : |f_i(x_0) - t| < \varepsilon_i\}$ for certain $\varepsilon_i > 0$. Let $h \in C^\infty(\mathbb{R},\mathbb{R})$ be chosen such that $h(0) = 1$ and $h(t) = 0$ for $|t| \geq 1$. Set $f(x) := \prod_{i=1}^n h(\frac{f_i(x)}{\varepsilon_i})$. Then f is the required bump function. Thus this subbasis is a basis. $\square$

14.3. Corollary. Smooth regularity is inherited by products and subspaces. *Let X_i be topological spaces and $\mathcal{S}_i \subseteq C(X_i,\mathbb{R})$. On a space X we consider the initial topology with respect to mappings $f_i : X \to X_i$, and we assume that $\mathcal{S} \subseteq C(X,\mathbb{R})$ is given such that $f_i^*(\mathcal{S}_i) \subseteq \mathcal{S}$ for all i. If each X_i is $\mathcal{S}_i$-regular, then X is $\mathcal{S}$-regular.* $\square$

Note however that the c^∞-topology on a locally convex subspace is not the trace of the c^∞-topology in general, see 4.33 and 4.36.5. However, for c^∞-closed subspaces this is true, see 4.28.

14.4. Proposition. [17]. *Every Banach space with $\mathcal{S}$-norm is $\mathcal{S}$-regular.*

More general, a convenient vector space is smoothly regular if its c^∞-topology is generated by seminorms which are smooth on their respective carriers. For example, nuclear Fréchet spaces have this property.

Proof. Namely, $g \circ p$ is a smooth bump function with carrier contained in $\{x : p(x) < 1\}$ if g is a suitably chosen real function, i.e., $g(t) = 1$ for $t \le 0$ and $g(t) = 0$ for $t \ge 1$.

Nuclear spaces have a basis of Hilbert-seminorms [53, 21.1.7], and on Fréchet spaces the c^∞-topology coincides with the locally convex one 4.11.1, hence nuclear Fréchet spaces are c^∞-regular. □

14.5. Open problem. Has every non-separable $\mathcal{S}$-regular Banach space an equivalent $\mathcal{S}$-norm? Compare with 16.11.

A partial answer is given in:

14.6. Proposition. *Let E be a C^∞-regular Banach space. Then there exists a smooth function $h : E \to \mathbb{R}_+$, which is positively homogeneous and smooth on $E \setminus \{0\}$.*

Proof. Let $f : E\setminus\{0\} \to \{t \in \mathbb{R} : t \ge 0\}$ be a smooth function, such that $\operatorname{carr}(f)$ is bounded in E and $f(x) \ge 1$ for x near 0. Let $U := \{x : f(tx) \ne 0 \text{ for some } t \ge 1\}$. Then there exists a smooth function $Mf : E \setminus \{0\} \to \mathbb{R}$ with $(Mf)'(x)(x) < 0$ for $x \in U$, $\lim_{x\to 0} Mf(x) = +\infty$ and $\operatorname{carr} Mf \subseteq U$.
The idea is to construct out of the smooth function $f \ge 0$ another smooth function Mf with $(Mf)'(x)(x) = -f(x) \le 0$, i.e. $(Mf)'(tx)(tx) = -f(tx)$ and hence

$$\frac{d}{dt} Mf(tx) = (Mf)'(tx)(x) = -\frac{f(tx)}{t} \text{ for } t \ne 0.$$

Since we want bounded support for Mf, we get

$$Mf(x) = -\Big[Mf(tx)\Big]_{t=1}^{\infty} = -\int_1^\infty \frac{d}{dt} Mf(tx)\, dt = \int_1^\infty \frac{f(tx)}{t}\, dt,$$

and we take this as a definition of Mf. Since the support of f is bounded, we may replace the integral locally by $\int_1^N$ for some large N, hence Mf is smooth on $E\setminus\{0\}$ and $(Mf)'(x)(x) = -f(x)$.
Since $f(x) > \varepsilon$ for all $\|x\| < \delta$, we have that

$$Mf(x) \ge \int_1^N \frac{1}{t} f(tx)\, dt \ge \log(N)\varepsilon$$

for all $\|x\| < \frac{\delta}{N}$, i.e. $\lim_{x\to 0} Mf(x) = +\infty$.
Furthermore $\operatorname{carr}(Mf) \subseteq U$, since $f(tx) = 0$ for all $t \ge 1$ and $x \notin U$.

Now consider $M^2 f := M(Mf) : E \setminus \{0\} \to \mathbb{R}$. Since $(Mf)'(x)(x) \le 0$, we have $(M^2 f)'(x)(x) = \int_1^\infty (Mf)'(tx)(x)\, dt \le 0$ and it is < 0 if for some $t \ge 1$ we have $0 > (Mf)'(tx)(x) = -\frac{f(tx)}{t}$, in particular this is the case if $x \in U$.

Thus $U_\varepsilon := \{x : M^2 f(x) \geq \varepsilon\}$ is radial set with smooth boundary, and the Minkowski-functional is smooth on $E \setminus \{0\}$. Moreover $U_\varepsilon \cong E$ via $x \mapsto \frac{x}{M^2 f(x)}$. □

14.7. Lemma. Existence of smooth bump functions.
For a class $\mathcal{S}$ on a Banach space E in the sense of 14.1 the following statements are equivalent:

1. *E is not $\mathcal{S}$-regular;*
2. *For every $f \in \mathcal{S}$, every $0 < r_1 < r_2$ and $\varepsilon > 0$ there exists an x with $r_1 \leq \|x\| \leq r_2$ and $|f(x) - f(0)| < \varepsilon$;*
3. *For every $f \in \mathcal{S}$ with $f(0) = 0$ there exists an x with $1 \leq \|x\| \leq 2$ and $|f(x)| \leq \|x\|$*

Proof. (1) ⇒ (2) Assume that there exists an f and $0 < r_1 < r_2$ and $\varepsilon > 0$ such that $|f(x) - f(0)| \geq \varepsilon$ for all $r_1 \leq \|x\| \leq r_2$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth bump function on $\mathbb{R}$. Let $g(x) := h\big(\frac{1}{\varepsilon}(f(r_1 x) - f(0))\big)$. Then g is of the corresponding class, $g(0) = h(0) = 1$, and for all x with $1 \leq \|x\| \leq \frac{r_2}{r_1}$ we have $|f(r_1 x) - f(0)| \geq \varepsilon$, and hence $g(x) = 0$. By redefining g on $\{x : \|x\| \geq \frac{r_2}{r_1}\}$ as 0, we obtain the required bump function.

(2) ⇒ (3) Take $r_1 = 1$ and $r_2 = 2$ and $\varepsilon = 1$.

(3) ⇒ (1) Assume a bump function g exists, i.e., $g(0) = 1$ and $g(x) = 0$ for all $\|x\| \geq 1$. Take $f := 3(1 - g)$. Then $f(0) = 0$ and $f(x) = 3$ for $\|x\| \geq 1$, a contradiction to (3). □

14.8. Proposition. Boundary values for smooth mappings. [17] *Let E and F be convenient vector spaces, let F be $\mathcal{S}$-regular but E not $\mathcal{S}$-regular. Let $U \subseteq E$ be c^∞-open and $f \in C(\overline{U}, F)$ with $f^*(\mathcal{S}) \subseteq \mathcal{S}$. Then $\overline{f(\partial U)} \supseteq f(\overline{U})$. Hence, $f = 0$ on ∂U implies $f = 0$ on U.*

Proof. Since $f(\overline{U}) \subseteq \overline{f(U)}$ it is enough to show that $f(U) \subseteq \overline{f(\partial U)}$. Suppose $f(x) \notin \overline{f(\partial U)}$ for some $x \in U$. Since F is regular, we find disjoint open neighborhoods V of $f(x)$ and W of $\overline{f(\partial U)}$. Choose a smooth h on F such that $h(f(x)) = 1$ and $h|_W = 0$. Let $g = h \circ f$ on U and 0 on $f^{-1}(W) \cup (E \setminus \overline{U})$. Then g is a smooth bump function on E, a contradiction. □

14.9. Theorem. C^1-regular spaces admit no rough norm. [Leach, Whitfield, 1972]. *Let E be a Banach space whose norm $p = \|\_\|$ has uniformly discontinuous directional derivative. If f is Fréchet differentiable with $f(0) = 0$ then there exists an $x \in E$ with $1 \leq \|x\| < 2$ and $f(x) \leq \|x\|$.*

By 14.7 this result implies that on a Banach space with rough norm there exists no Fréchet differentiable bump function. In particular, $C([0, 1])$ and ℓ^1 are not C^1-regular by 13.11 and 13.12, which is due to [81].

Proof. We try to reach the exterior of the unit ball by a recursively defined sequence x_n in $\{x : f(x) \leq p(x)\}$ starting at 0 with large step-length ≤ 1 in directions, where

p' is large. For this let ε be given by 13.23.4. Given x_n we consider the set

$$\mathcal{M}_n := \left\{ y \in E : \begin{array}{l} (1)\ f(y) \le p(y), \\ (2)\ p(y-x_n) \le 1 \text{ and} \\ (3)\ p(y) - p(x_n) \ge (\varepsilon/8)\, p(y-x_n) \end{array} \right\}.$$

Since $x_n \in \mathcal{M}_n$, this set is not empty and hence $M_n := \sup\{p(y-x_n) : y \in \mathcal{M}_n\} \le 1$ is well-defined and it is possible to choose $x_{n+1} \in \mathcal{M}_n$ with

$$(4)\ p(x_{n+1} - x_n) \ge M_n/2.$$

We claim that $p(x_n) \ge 1$ for some n, since then $x := x_n$ for the minimal n satisfies the conclusion of the theorem:
Otherwise $p(x_n)$ is bounded by 1 and increasing by (3), hence a Cauchy-sequence. By (3) we then get that (x_n) is a Cauchy-sequence. So let x_∞ be its limit. If $x_\infty = 0$, then $x_n = 0$, hence $M_n = 0$ and $\mathcal{M}_n = \{0\}$. Thus $f(y) > p(y)$ for all $p(y) \le 1$ and so f would not be differentiable. So $0 < p(x_\infty) \le 1$ and $f(x_\infty) \le p(x_\infty)$. Since f is Fréchet-differentiable at x_∞ there exists a $\delta > 0$ such that

$$f(x_\infty + u) - f(x_\infty) - f'(x_\infty)(u) \le \varepsilon p(u)/8 \text{ for all } p(u) \le \delta.$$

Without loss of generality let $\delta \le 1$ and $\delta \le 2\,p(x_\infty)$. By 13.23.4 applied to $x := x_\infty/p(x_\infty)$ and $\delta := \varepsilon\delta/8p(x_\infty)$ there exists a h such that $p(h) \le 1$ and $p(x + th) \ge p(x) + \varepsilon|t| - \varepsilon\delta/8p(x_\infty)$ for all $t = s/p(x_\infty)$ with $|s| \le p(x_\infty)$, i.e. $p(x_\infty + sh) \ge p(x_\infty) + \varepsilon|s| - \varepsilon\delta/8$. Now let $s := -\operatorname{sign}(f'(x_\infty)(h))\ \delta/2$. Then $p(sh) \le |s| = \delta/2$ and hence

(1) $p(x_\infty + sh) > p(x_\infty) + \varepsilon\delta/8 \ge f(x_\infty) + \varepsilon p(sh)/8 \ge f(x_\infty + sh)$,

(2) $p(x_\infty + sh - x_\infty) < \delta \le 1$,

(3) $p(x_\infty + sh) - p(x_\infty) > \varepsilon\delta/8 > \varepsilon p(sh)/8$.

Since f and p are continuous, we have $x_\infty + sh \in \mathcal{M}_n$ for large n and hence $M_n \ge p(x_\infty + s\,h - x_n)$. From $p(x_\infty + sh - x_\infty) > \varepsilon\delta/8$ we get $M_n > \varepsilon\delta/8$ and so $p(x_{n+1} - x_n) > \varepsilon\delta/16$ by (4) contradicts the convergence of x_n. □

14.10. Proposition. *Let E be a Banach-space with* $\operatorname{dens} E < \operatorname{dens} E'$. *Then there is an equivalent rough norm on E.*

Proof. The idea is to describe the unit ball of a rough norm as intersection of half-spaces $\{x \in E : x^*(x) \le 1\}$ for certain functionals $x^* \in E'$. The fewer functionals we use the more 'corners' the unit ball will have, but we have to use sufficiently many in order that this ball is bounded and hence that its Minkowski-functional is an equivalent norm. We temporarily call a set X **large**, if and only if $|X| > \operatorname{dens}(E)$ and **small** otherwise. For $x \in E$ and $\varepsilon > 0$ let $B_\varepsilon(x) := \{y \in E : \|x - y\| \le \varepsilon\}$. Now we choose using Zorn's lemma a subset $D \subseteq E'$ maximal with respect to the following conditions:

(1) $0 \in D$;
(2) $x^* \in D \Rightarrow -x^* \in D$;
(3) $x^*, y^* \in D,\ x^* \ne y^* \Rightarrow \|x^* - y^*\| > 1$.

Note that D is then also maximal with respect to (3) alone, since otherwise, we could add a point x^* with $\|x^* - y^*\| > 1$ for all $y^* \in D$ and also add the point $-x^*$, and obtain a larger set satisfying all three conditions.

Claim. $D_\infty := \bigcup_{n\in\mathbb{N}} \frac{1}{n}D$ is dense in E', and hence $|D_\infty| \geq \operatorname{dens}(E')$:
Assume indirectly, that there is some $x^* \in E'$ and $n \in \mathbb{N}$ with $B_{1/n}(x^*) \cap D_\infty = \emptyset$. Then $B_1(n\,x^*) \cap D = \emptyset$ and hence we may add x^* to D, contradicting the maximality.

Without loss of generality we may assume that D is at least countable. Then $|D| = |\bigcup_{n\in\mathbb{N}} \frac{1}{n}D| \geq \operatorname{dens}(E') > \operatorname{dens}(E)$, i.e. D is large. Since $D = \bigcup_{n\in\mathbb{N}} D \cap B_n(0)$, we find some n such that $D\cap B_n(0)$ is large. Let $y^* \in E'$ be arbitrary and $w^* := \frac{1}{4n+2}y^*$. For every $x^* \in D$ there is a $z^* \in \frac{1}{2}D$ such that $\|x^* + w^* - z^*\| \leq \frac{1}{2}$ (otherwise we could add $2(x^* + w^*)$ to D). Thus we may define a mapping $D \to \frac{1}{2}D$ by $x^* \mapsto z^*$. This mapping is injective, since $\|x_j^* + w^* - z^*\| \leq \frac{1}{2}$ for $j \in \{1, 2\}$ implies $\|x_1^* - x_2^*\| \leq 1$ and hence $x_1^* = x_2^*$. If we restrict it to the large set $D \cap B_n(0)$ it has image in $\frac{1}{2}D \cap B_{n+1/2}(w^*)$, since $\|z^* - w^*\| \leq \|z^* + x^* - w^*\| + \|x^*\| \leq \frac{1}{2} + n$. Hence also $\frac{1}{4(2n+1)}D \cap B_{1/4}(y^*) = \frac{1}{4n+2}\big(\frac{1}{2}D \cap B_{n+1/2}(w^*)\big)$ is large.

In particular for $y^* := 0$ and $1/4$ replaced by 1 we get that $A := \frac{1}{4(2n+1)}D \cap B_1(0)$ is large. Now let

$$\begin{aligned} U &:= \Big\{x \in E : \exists A_0 \subseteq A \text{ small}, \forall x^* \in A \setminus A_0 : x^*(x) \leq 1\Big\}. \\ &= \Big\{x \in E : \{x^* \in A : x^*(x) > 1\} \text{ is small } \Big\}. \end{aligned}$$

Since A is symmetric, the set U is absolutely convex (use that the union of two small exception sets is small). It is a 0-neighborhood, since $\{x : \|x\| \leq 1\} \subseteq U$ ($x^*(x) \leq \|x^*\| \cdot \|x\| \leq \|x\| \leq 1$ for $x^* \in A$). It is bounded, since for $x \in E$ we may find by Hahn-Banach an $x^* \in E'$ with $x^*(x) = \|x\|$ and $\|x^*\| = 1$. For all y^* in the large set $A \cap B_{1/4}(\frac{3}{4}x^*)$ we have $y^*(x) = \frac{3}{4}x^*(x) + \big(y^* - \frac{3}{4}x^*\big)(x) \geq \frac{3}{4}\|x\| - \frac{1}{4}\|x\| \geq \frac{1}{2}\|x\|$. For $\|x\| > 2$ we thus get $x \notin U$.

Now let σ be the Minkowski-functional generated by U and σ^* the dual norm on E'. Let $\Delta \subseteq E$ be a small dense subset. Then

$$\big\{x^* \in A : \sigma^*(x^*) > 1\big\} = \bigcup_{x\in\Delta} \bigcup_{n\in\mathbb{N}} \big\{x^* \in A : x^*(x) > \sigma(x) + \tfrac{1}{n}\big\},$$

since $\sigma^*(x^*) := \sup_x \frac{x^*(x)}{\sigma(x)} > 1$ for $x^* \in A$ implies that there exists an $x \in \Delta$ with $x^*(x) > \sigma(x)$. So this set is small, since Δ is small and each set of the union is small by construction of $\sigma(x) = \inf\big\{\lambda > 0 : \{x^* \in A : x^*(x) > \lambda\} \text{ is small}\big\}$. Thus $A_1 := \{x^* \in A : \sigma^*(x^*) \leq 1\}$ is large.

Now let $\varepsilon := \frac{1}{8(2n+1)}$, let $x \in E$, and let $0 < \eta < \varepsilon$. We may choose two different $x_i^* \in A_1$ for $i \in \{1, 2\}$ with $x_i^*(x) > \sigma(x) - \eta^2/2 = \sup\{x^*(x) : \sigma^*(x^*) \leq 1\} - \eta^2/2$. This is possible, since this is true for all but a small set of $x^* \in A$. Thus $\sigma^*(x_1^* - x_2^*) \geq \|x_1^* - x_2^*\| > 2\varepsilon$, and hence there is an $h \in E$ with $\sigma(h) = 1$ and $(x_1^* - x_2^*)(h) > 2\varepsilon$. Let now $t > 0$. Then

$$\begin{aligned} \sigma(x + th) &\geq x_1^*(x + th) = x_1^*(x) + t x_1^*(h) > \sigma(x) - \frac{\eta^2}{2} + t x_1^*(h), \\ \sigma(x - th) &\geq x_2^*(x - th) > \sigma(x) - \frac{\eta^2}{2} - t x_2^*(h). \end{aligned}$$

Furthermore $\sigma(x) \ge \sigma(x+th) - t\sigma'(x+th)(h)$ implies

$$\sigma'(x+th)(h) \ge \frac{\sigma(x+th)-\sigma(x)}{t} > x_1^*(h) - \frac{\eta^2}{2t},$$
$$-\sigma'(x-th)(h) \ge -x_2^*(h) - \frac{\eta^2}{2t}.$$

Adding the last two inequalities gives

$$\sigma'(x+th)(h) - \sigma'(x-th)(h) \ge (x_2^* - x_1^*)(h) - \frac{\eta^2}{t} > \varepsilon,$$

since $(x_2^* - x_1^*)(h) > 2\varepsilon$ and we choose $t < \eta$ such that $\frac{\eta^2}{t} < \varepsilon$. □

14.11. Results. Spaces which are not smoothly regular. *For Banach spaces one has the following results:*

(1) [**16**]. *By 14.9 no Fréchet-differentiable bump function exists on $C[0,1]$ and on ℓ^1. Hence, most infinite dimensional C^*-algebras are not regular for 1-times Fréchet-differentiable functions, in particular those for which a normal operator exists whose spectrum contains an open interval, hence have $C[0,1]$ as subspace.*

(2) [**83**]. *If* $\operatorname{dens} E < \operatorname{dens} E^*$ *then no C^1-bump function exists. This follows from* 14.10, 14.9, and 14.7. *See also* 13.24.2.

(3) [**56**]. *A norm is called strongly rough if and only if there exists an $\varepsilon > 0$ such that for every x with $\|x\| = 1$ there exists a unit vector y with $\overline{\lim}_{t\searrow 0} \frac{\|x+ty\|+\|x-ty\|-2\|x\|}{t} \ge \varepsilon$. The usual norm on $\ell^1(\Gamma)$ is strongly rough, if Γ is uncountable. There is however an equivalent non-rough norm on $\ell^1(\Gamma)$ with no point of Gâteaux-differentiability. If a Banach space has Gâteaux differentiable bump functions then it does not admit a strongly rough norm.*

(4) [**27**]. *On $\ell^1(\Gamma)$ with uncountable Γ there is not even a Gâteaux differentiable continuous bump function.*

(5) [**16**]. *$E < \ell^p$, $\dim E = \infty$: If $p = 2n+1$ then E is not D^p-regular. If $p \notin \mathbb{N}$ then E is not $\mathcal{S}$-regular, where $\mathcal{S}$ denotes the $C^{[p]}$-functions whose highest derivative satisfies a Hölder like condition of order $p - [p]$ but with $o(\_)$ instead of $O(\_)$.*

14.12. Results.

(1) [**28**]. *If $c_0(\Gamma) \to E \to F$ is a short exact sequence of Banach spaces and F has C^k-bump functions then also E has them. Compare with 16.19.*

(2) [**94**] *If a Banach space E and its dual E^* admit C^2-bump functions, then E is linearly homeomorphic to a Hilbert space. Compare with 13.18.7.*

(3) *Smooth bump functions are not inherited by short exact sequences. Compare with 13.18.8.*

Notes. (1) As in 13.17.3 one chooses $x_a^* \in E^*$ with $x_a^*|_{c_0(\Gamma)} = \mathrm{ev}_a$. Let g be a smooth bump function on E/F and $h \in C^\infty(\mathbb{R},[0,1])$ with compact support and equal to 1 near 0. Then $f(x) := g(x+F)\prod_{a\in\Gamma} h(x_a^*(x))$ is the required bump function.

(3) Use the example mentioned in 13.18.6, and apply (2).

Open problems

Is the product of C^∞-regular convenient vector spaces again C^∞-regular? Beware of the topology on the product!

Is every quotient of any $\mathcal{S}$-regular space again $\mathcal{S}$-regular?

15. Functions with Globally Bounded Derivatives

In many problems (like Borel's theorem 15.4, or the existence of smooth functions with given carrier 15.3) one uses in finite dimensions the existence of smooth functions with bounded derivatives. In infinite dimensions C^k-functions have locally bounded k-th derivatives, but even for bump functions this need not be true globally.

15.1. Definitions

For normed spaces we use the following notation: $C^k_B := \{f \in C^k : \|f^{(k)}(x)\| \le B \text{ for all } x \in E\}$ and $C^k_b := \bigcup_{B>0} C^k_B$. For general convenient vector spaces we may still define C^∞_b as those smooth functions $f : U \to F$ for which the image $d^k f(U)$ of each derivative is bounded in the space $L^k_{\mathrm{sym}}(E, F)$ of bounded symmetric multilinear mappings.

Let $\mathcal{L}\mathrm{ip}^k_K$ denote the space of C^k-functions with global Lipschitz-constant K for the k-th derivatives and $\mathcal{L}\mathrm{ip}^k_{\mathrm{global}} := \bigcup_{K>0} \mathcal{L}\mathrm{ip}^k_K$. Note that $C^k_K = C^k \cap \mathcal{L}\mathrm{ip}^{k-1}_K$.

15.2. Lemma. Completeness of C^n. *Let f_j be C^n-functions on some Banach space such that $f_j^{(k)}$ converges uniformly on bounded sets to some function f^k for each $k \le n$. Then $f := f^0$ is C^n, and $f^{(k)} = f^k$ for all $k \le n$.*

Proof. It is enough to show this for $n = 1$. Since $f_n' \to f^1$ uniformly, we have that f^1 is continuous, and hence $\int_0^1 f^1(x + t\,h)(h)\,dt$ makes sense and

$$f_n(x+h) - f_n(x) = \int_0^1 f_n'(x + t\,h)(h)\,dt \to \int_0^1 f^1(x + t\,h)(h)\,dt$$

for x and h fixed. Since $f_n \to f$ pointwise, this limit has to be $f(x + h) - f(x)$. Thus we have

$$\begin{aligned}\frac{\|f(x+h) - f(x) - f^1(x)(h)\|}{\|h\|} &= \frac{1}{\|h\|}\left\|\int_0^1 (f^1(x + t\,h) - f^1(x))(h)\,dt\right\| \\ &\le \int_0^1 \|f^1(x + t\,h) - f^1(x))\|\,dt\end{aligned}$$

which goes to 0 for $h \to 0$ and fixed x, since f^1 is continuous. Thus, f is differentiable and $f' = f^1$. □

15.3. Proposition. When are closed sets zero-sets of smooth functions. [132]. *Let E be a separable Banach space and $n \in \mathbb{N}$. Then E has a C^n_b-bump function if and only if every closed subset of E is the zero-set of a C^n-function.*

For $n = \infty$ and E a convenient vector space we still have ($\Rightarrow$), provided all $L^k(E;\mathbb{R})$ satisfy the SECOND COUNTABILITY CONDITION OF MACKEY, *i.e. for every countable family of bounded sets B_k there exist $t_k > 0$ such that $\bigcup_k t_k B_k$ is bounded.*

Proof. ($\Rightarrow$) Suppose first that E has a C^n_b-bump function. Let $A \subseteq E$ be closed and $U := E \setminus A$ be the open complement. For every $x \in U$ there exists an $f_x \in C^n_b(E)$ with $f_x(x) = 1$ and $\mathrm{carr}(f_x) \subseteq U$. The family of carriers of the f_x is an open covering of U. Since E is separable, those points in a countable dense subset that lie in U are dense in the metrizable space U. Thus, U is Lindelöf, and consequently we can find a sequence of points x_n such that for the corresponding functions $f_n := f_{x_n}$ the carriers still cover U. Now choose constants $t_n > 0$ such that $t_n \cdot \sup\{\|f_n^{(j)}(x)\| : x \in E\} \leq \frac{1}{2^{n-j}}$ for all $j < n$. Then $f := \sum_n t_n f_n$ converges uniformly in all derivatives, hence represents by 15.2 a C^n-function on E that vanishes on A. Since the carriers of the f_n cover U, it is strictly positive on U, and hence the required function has as 0-set exactly A.

($\Leftarrow$) Consider a vector $a \neq 0$, and let $A := E \setminus \bigcup_{n\in\mathbb{N}}\{x : \|x - \frac{1}{2^n}a\| < \frac{1}{2^{n+1}}\}$. Since A is closed there exists by assumption a C^n-function $f : E \to \mathbb{R}$ with $f^{-1}(0) = A$ (without loss of generality we may assume $f(E) \subseteq [0,1]$). By continuity of the derivatives we may assume that $f^{(n)}$ is bounded on some neighborhood U of 0. Choose n so large that $D := \{x : \|x - \frac{1}{2^n}a\| < \frac{1}{2^n}\} \subseteq U$, and let $g := f$ on $A \cup D$ and 0 on $E \setminus D$. Then $f \in C^n$ and $f^{(n)}$ is bounded. Up to affine transformations this is the required bump function. □

15.4. Borel's theorem. [132]. *Suppose a Banach space E has C_b^∞-bump functions. Then every formal power series with coefficients in $L^n_{sym}(E;F)$ for another Banach space F is the Taylor-series of a smooth mapping $E \to F$.*

Moreover, if G is a second Banach space, and if for some open set $U \subseteq G$ we are given $b_k \in C_b^\infty(U, L^k_{\mathrm{sym}}(E,F))$, then there is a smooth $f \in C^\infty(E \times U, F)$ with $d^k(f(\_,y))(0) = b_k(y)$ for all $y \in U$ and $k \in \mathbb{N}$. In particular, smooth curves can be lifted along the mapping $C^\infty(E,F) \to \prod_k L^k_{sym}(E;F)$.

Proof. Let $\rho \in C_b^\infty(E,\mathbb{R})$ be a C_b^∞-bump function, which equals 1 locally at 0. We shall use the notation $b_k(x,y) := b_k(y)(x^k)$. Define

$$f_k(x,y) := \frac{1}{k!} b_k(x,y)\,\rho(x)$$

and

$$f(x,y) := \sum_{k\geq 0} \frac{1}{t_k^k} f_k(t_k \cdot x, y)$$

with appropriately chosen $t_k > 0$. Then $f_k \in C^\infty(E \times U, F)$ and f_k has carrier inside of $\mathrm{carr}(\rho) \times U$, i.e. inside $\{x : \|x\| < 1\} \times U$. For the derivatives of b_k we have

$$\partial_1^j \partial_2^i b_k(x,y)(\xi,\eta) = k\,(k-1)\,\dots\,(k-j)\,(d^i b_k(y)(\eta))(x^{k-j},\xi^j).$$

Hence, for $\|x\| \leq 1$ this derivative is bounded by

$$(k)_j \sup_{y\in U} \|d^i b_k(y)\|_{L(F,L^k_{\mathrm{sym}}(E;G))},$$

where $(k)_j := k(k-1)\dots(k-j)$. Using the product rule we see that for $j \geq k$ the derivative $\partial_1^j\partial_2^i f_k$ of f_k is globally bounded by

$$\sum_{l\leq k} \binom{j}{l} \sup\{\|\rho^{(j-l)}(x)\| : x \in E\}\,(k)_l \sup_{y\in U} \|d^i b_k(y)\| < \infty.$$

The partial derivatives of f would be

$$\partial_1^j \partial_2^i f_k(x,y) = \sum_k \frac{t_k^j}{t_k^k} \partial_1^j \partial_2^i f_k(t_k x, y).$$

We now choose the $t_k \geq 1$ such that these series converge uniformly. This is the case if,

$$\frac{1}{t_k^{k-j}} \sup\{\|\partial_1^j \partial_2^i f_k(x,y)\| : x \in E, y \in U\} \leq$$
$$\leq \frac{1}{t_k^{k-(j+i)}} \sup\{\|\partial_1^j \partial_2^i f_k(x,y)\| : x \in E, y \in U\} \leq \frac{1}{2^{k-(j+i)}},$$

and thus if

$$t_k \geq 2. \sup\{\|\partial_1^j \partial_2^i f_k(x,y)\|^{\frac{1}{k-(j+i)}} : x \in E, y \in U, j+i < k\}.$$

Since we have $\partial_1^j f_k(0,y)(\xi) = \frac{1}{k!}(k)_j b_k(y)(0^{k-j},\xi^j)\,\rho(0) = \delta_k^j\, b_k(y)$, we conclude the desired result $\partial_1^j f(0,y) = b_k(y)$. □

Remarks on Borel's theorem.

(1) [25]. Let E be a strict inductive limit of a non-trivial sequence of Fréchet spaces E_n. Then Borel's theorem is wrong for $f : \mathbb{R} \to E$. The idea is to choose $b_n = f^{(n)}(0) \in E_{n+1} \setminus E_n$ and to use that locally every smooth curve has to have values in some E_n.

(2) [25]. Let $E = \mathbb{R}^{\mathbb{N}}$. Then Borel's theorem is wrong for $f : E \to \mathbb{R}$. In fact, let $b_n : E \times \ldots \times E \to \mathbb{R}$ be given by $b_n := \mathrm{pr}_n \otimes \cdots \otimes \mathrm{pr}_n$. Assume $f \in C^\infty(E,\mathbb{R})$ exists with $f^{(n)}(0) = b_n$. Let f_n be the restriction of f to the n-th factor $\mathbb{R}$ in E. Then $f_n \in C^\infty(\mathbb{R},\mathbb{R})$ and $f_n^{(n)}(0) = 1$. Since $f' : \mathbb{R}^n \to (\mathbb{R}^n)' = \mathbb{R}^{(\mathbb{N})}$ is continuous, the image of $B := \{x : |x_n| \leq 1 \text{ for all } n\}$ in $\mathbb{R}^{(\mathbb{N})}$ is bounded, hence contained in some $\mathbb{R}^{N-1}$. Since f_N is not constant on the interval $(-1,1)$ there exists some $|t_N| < 1$ with $f_N'(t_N) \neq 0$. For $x_N := (0,\ldots,0,t_N,0,\ldots)$ we obtain

$$f'(x_N)(y) = f_N'(t_N)(y_N) + \sum_{i \neq N} a_i\, y_i,$$

a contradiction to $f'(x_n) \in \mathbb{R}^{N-1}$.

(3) [25] showed that Borel's theorem is true for mappings $f : E \to F$, where E has a basis of Hilbert-seminorms and for any countable family of 0-neighborhoods U_n there exist $t_n > 0$ such that $\bigcap_{n=1}^\infty t_n\, U_n$ is a 0-neighborhood.

(4) If theorem 15.4 would be true for $G = \prod_k L^k_{\mathrm{sym}}(E;F)$ and $b_k = \mathrm{pr}_k$, then the quotient mapping $C^\infty(E,F) \to G = \prod_k L^k_{\mathrm{sym}}(E;F)$ would admit a smooth and hence a linear section. This is well know to be wrong even for $E = F = \mathbb{R}$, see [75, 21.5].

15.5. Proposition. Hilbert spaces have C_b^∞-bump functions. [132] *If the norm is given by the n-th root of a homogeneous polynomial b of even degree n, then $x \mapsto \rho(b(x^n))$ is a C_b^∞-bump function, where $\rho : \mathbb{R} \to \mathbb{R}$ is smooth with $\rho(t) = 1$ for $t \leq 0$ and $\rho(t) = 0$ for $t \geq 1$.*

Proof. As before in the proof of 15.4 we see that the j-th derivative of $x \mapsto b(x^n)$ is bounded by $(n)_j$ on the closed unit ball. Hence, by the chain-rule and the global boundedness of all derivatives of ρ separately, the composite has bounded derivatives on the unit ball, and since it is zero outside, even everywhere. Obviously, $\rho(b(0)) = \rho(0) = 1$. □

In [17] it is shown that L^p is $\mathcal{L}\mathrm{ip}^n_{\mathrm{global}}$-smooth for all n if p is an even integer and is $\mathcal{L}\mathrm{ip}^{[p-1]}_{\mathrm{global}}$-smooth otherwise. This follows from the fact (see loc. cit., p. 140) that $d^{(p+1)}\|x\|^p = 0$ for even integers p and

$$\left\| d^k\|x+h\|^p - d^k\|x\|^p \right\| \le \frac{p!}{k!}\|h\|^{p-k}$$

otherwise, cf. 13.13.

15.6. Estimates for the remainder in the Taylor-expansion. The Taylor formula of order k of a C^{k+1}-function is given by

$$f(x+h) = \sum_{j=0}^{k} \frac{1}{j!} f^{(j)}(x)(h^j) + \int_0^1 \frac{(1-t)^k}{k!}\, f^{(k+1)}(x+th)(h^{k+1})\, dt,$$

which can easily be seen by repeated partial integration of $\int_0^1 f'(x+th)(h)\,dt = f(x+h) - f(x)$.

For a C^2_B function we have

$$|f(x+h) - f(x) - f'(x)(h)| \le \int_0^1 (1-t)\|f^{(2)}(x+th)\|\,\|h\|^2\,dt \le B\frac{1}{2!}\,\|h\|^2.$$

If we take the Taylor formula of f up to order 0 instead, we obtain

$$f(x+h) = f(x) + \int_0^1 f'(x+th)(h)\,dt$$

and usage of $f'(x)(h) = \int_0^1 f'(x)(h)\,dt$ gives

$$|f(x+h) - f(x) - f'(x)(h)| \le \int_0^1 \frac{\|f'(x+th) - f'(x)\|}{\|th\|}\,\|h\|^2\,dt \le B\frac{1}{2!}\,\|h\|^2,$$

so it is in fact enough to assume $f \in C^1$ with f' satisfying a Lipschitz-condition with constant B.

For a C^3_B function we have

$$\begin{aligned}|f(x+h) - f(x) - f'(x)(h) - \tfrac{1}{2}f''(x)(h^2)| &\le \\ \le \int_0^1 \frac{(1-t)^2}{2!}\,&\|f^{(3)}(x+th)\|\,\|h\|^3\,dt \le B\frac{1}{3!}\,\|h\|^3.\end{aligned}$$

If we take the Taylor formula of f up to order 1 instead, we obtain

$$f(x+h) = f(x) + f'(x)(h) + \int_0^1 (1-t)\, f''(x+th)(h^2)\,dt,$$

and using $\frac{1}{2}f''(x)(h^2) = \int_0^1 (1-t)\, f''(x)(h^2)\,dt$ we get

$$\begin{aligned}|f(x+h) - f(x) - f'(x)(h) - \tfrac{1}{2}f''(x)(h^2)| &\le \\ \le \int_0^1 (1-t)t\,\frac{\|f''(x+th) - f''(x)\|}{\|th\|}\,&\|h\|^3\,dt \le B\frac{1}{3!}\,\|h\|^3.\end{aligned}$$

Hence, it is in fact enough to assume $f \in C^2$ with f'' satisfying a Lipschitz-condition with constant B.

Let $f \in C^k_B$ be flat of order k at 0. Applying $\|f(h) - f(0)\| = \|\int_0^1 f'(th)(h)\,dt\| \le \sup\{\|f'(th)\| : t \in [0,1]\}\,\|h\|$ to $f^{(j)}(\_)(h_1,\dots,h_j)$ gives using $\|f^{(k)}(x)\| \le B$ inductively

$$\begin{aligned}
\|f^{(k-1)}(x)\| &\le B\cdot\|x\| \\
\|f^{(k-2)}(x)\| &\le \int_0^1 \|f^{(k-1)}(tx)(x,\dots)\|\,dt \le B\int_0^1 t\,dt\,\|x\|^2 = \frac{B}{2}\,\|x\|^2 \\
&\vdots \\
\|f^{(j)}(x)\| &\le \frac{B}{(k-j)!}\,\|x\|^{k-j}.
\end{aligned}$$

15.7. Lemma. $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-functions on $\mathbb{R}^n$. [132]. *Let $n := 2^N$ and $E = \mathbb{R}^n$ with the ∞-norm. Suppose $f \in \mathcal{L}\mathrm{ip}^1_M(E,\mathbb{R})$ with $f(0) = 0$ and $f(x) \ge 1$ for $\|x\| \ge 1$. Then $M \ge 2N$.*

The idea behind the proof is to construct recursively a sequence of points $x_k := \sum_{j<k}\sigma_j h_j$ of norm $\frac{k}{N}$ (starting at $x_0 = 0$), such that the increment along the segment is as small as possible. In order to evaluate this increment one uses the Taylor-formula and chooses the direction h_k such that the derivative at x_k vanishes.

Proof. Let A be the set of all edges of the unit-sphere, i.e.

$$A := \Big\{x : x_i = \pm 1 \text{ for all } i \text{ except one } i_0 \text{ and } |x_{i_0}| \le 1\Big\}.$$

Then A is symmetric. Let $x \in E$ be arbitrary. We want to find $h \in A$ with $f'(x)(h) = 0$. By permuting the coordinates we may assume that $i \mapsto |f'(x)(e_i)|$ is monotone increasing. For $1 \le i < n$ we choose recursively $h^i \in \{\pm 1\}$ such that $\sum_{j\le i} h^j\, f'(x)(e_j)$ is an alternating sum. Then $|\sum_{j\le i} f'(x)(e_j)h^j| \le |f'(x)(e_i)|$. Finally, we may choose $|h^n| \le 1$ such that $f'(x)(h) = \sum_{j=1}^n h^j\, f'(x)(e_j) = 0$.

Now we choose inductively $h_i \in \frac{1}{N}A$ and $\sigma_i \in \{\pm 1\}$ such that $f'(x_i)(h_i) = 0$ for $x_i := \sum_{j<i}\sigma_j h_j$ and x_i having at least 2^{N-i} coordinates equal to $\frac{i}{N}$: By induction hypothesis at least 2^{N-i} coordinates of x_i are $\frac{i}{N}$. Among those coordinates all but at most 1 of the h_i are $\pm\frac{1}{N}$. Now let σ_i be the sign which occurs more often and hence at least $2^{N-i}/2$ times. Then those $2^{N-(i+1)}$ many coordinates of $x_{i+1} := x_i + \sigma_i\, h_i$ are $\frac{i+1}{N}$.

In particular, $\|x_i\| = \frac{i}{N}$ for $i \le N$, since at least one coordinate has this value. Furthermore we have by 15.6

$$\begin{aligned}
1 \le \big|f(x_N) - f(x_0)\big| &\le \sum_{k=0}^{N-1}\Big|f(x_{k+1}) - f(x_k) - f'(x_k)(h_k)\Big| \\
&\le \sum_{k=0}^{N-1}\frac{M}{2}\,\|h_k\|^2 \le N\,\frac{M}{2}\,\frac{1}{N^2},
\end{aligned}$$

hence $M \ge 2N$. □

15.8. Corollary. c_0 is not $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-regular. [132]. *There is no differentiable bump function on c_0 with uniformly continuous derivative.*

Proof. Suppose there exists an $f \in \mathcal{L}\mathrm{ip}^1_{\mathrm{global}}$ with $f(0) = 1$ and $f(x) = 0$ for all $\|x\| \geq 1$. Then the previous lemma applied to $1 - f$ restricted to N-dimensional subspaces shows that the Lipschitz constant M of the derivative has to be greater or equal to N for all N, a contradiction.

This shows even that there exists no differentiable bump function on $c_0(A)$ with uniformly continuous derivative: Otherwise there would exist an $N \in \mathbb{N}$ such that

$$|f(x+h) - f(x) - f'(x)h| \leq \int_0^1 \|f'(x+t\,h) - f'(x)\|\,\|h\|\,dt \leq \tfrac{1}{2}\|h\|,$$

for all $\|h\| \leq \frac{1}{N}$. Hence, the estimation in the proof of 15.7 would give $1 \leq N\frac{1}{2}\frac{1}{N} = \frac{1}{2}$, a contradiction. $\square$

15.9. Positive results on $\mathcal{L}\mathrm{ip}^1_{\mathbf{global}}$-functions. [132].

(1) *Every closed subset of a Hilbert space is the zero-set of a $\mathcal{L}\mathrm{ip}^1_{global}$-function.*
(2) *For every two closed subsets of a Hilbert space which have distance $d > 0$ there exists a $\mathcal{L}\mathrm{ip}^1_{4/d^2}$-function which has value 0 on one set and 1 on the other.*
(3) *Whitney's extension theorem is true for $\mathcal{L}\mathrm{ip}^1_{global}$-functions on closed subsets of Hilbert spaces.*

16. Smooth Partitions of Unity and Smooth Normality

16.1. Definitions

We say that a Hausdorff space X is *smoothly normal* with respect to a subalgebra $\mathcal{S} \subseteq C(X, \mathbb{R})$ or $\mathcal{S}$-*normal*, if for two disjoint closed subsets A_0 and A_1 of X there exists a function $f : X \to \mathbb{R}$ in $\mathcal{S}$ with $f|A_i = i$ for $i = 0$ and $i = 1$. If an algebra $\mathcal{S}$ is specified, then by a smooth function we will mean an element of $\mathcal{S}$. Otherwise it is a C^∞-function.

A $\mathcal{S}$-*partition of unity* on a space X is a set $\mathcal{F}$ of smooth functions $f : X \to \mathbb{R}$ which satisfy the following conditions:

(1) For all $f \in \mathcal{F}$ and $x \in X$ one has $f(x) \geq 0$.
(2) The set $\{\operatorname{carr}(f) : f \in \mathcal{F}\}$ of all carriers is a locally finite covering of X.
(3) The sum $\sum_{f \in \mathcal{F}} f(x)$ equals 1 for all $x \in X$.

Since a family of open sets is locally finite if and only if the family of the closures is locally finite, the foregoing condition (2) is equivalent to:

(2') The set $\{\operatorname{supp}(f) : f \in \mathcal{F}\}$ of all supports is a locally finite covering of X.

The partition of unity is called *subordinated* to an open covering $\mathcal{U}$ of X, if for every $f \in \mathcal{F}$ there exists an $U \in \mathcal{U}$ with $\operatorname{carr}(f) \subseteq U$.

We say that X is *smoothly paracompact* with respect to $\mathcal{S}$ or $\mathcal{S}$-*paracompact* if every open cover $\mathcal{U}$ admits a $\mathcal{S}$-partition $\mathcal{F}$ of unity subordinated to it.

For smoothly paracompact spaces the partition of unity can then even be chosen in such a way that for every $f \in \mathcal{F}$ there exists a $U \in \mathcal{U}$ with $\operatorname{supp}(f) \subseteq U$. This is seen as follows. Since the family of carriers is a locally finite open refinement of $\mathcal{U}$, the topology of X is paracompact. So we may find a finer open cover $\{\tilde{U} : U \in \mathcal{U}\}$ such that the closure of $\tilde{U}$ is contained in U for all $U \in \mathcal{U}$, see [**20**, IX.4.3]. The partition of unity subordinated to this finer cover has the support property for the original one.

Lemma. *Let $\mathcal{S}$ be an algebra which is closed under sums of locally finite families of functions. If $\mathcal{F}$ is an $\mathcal{S}$-partition of unity subordinated to an open covering $\mathcal{U}$, then we may find an $\mathcal{S}$-partition of unity $(f_U)_{U \in \mathcal{U}}$ with* $\operatorname{carr}(f_U) \subseteq U$.

Proof. For every $f \in \mathcal{F}$ we choose a $U_f \in \mathcal{U}$ with $\operatorname{carr}(f) \in U_f$. For $U \in \mathcal{U}$ put $\mathcal{F}_U := \{f : U_f = U\}$ and let $f_U := \sum_{f \in \mathcal{F}_U} f \in \mathcal{S}$. □

16.2. Proposition. Characterization of smooth normality. *Let X be a Hausdorff space with $\mathcal{S} \subseteq C(X, \mathbb{R})$ as in 14.1 Consider the following statements:*

(1) *X is $\mathcal{S}$-normal;*
(2) *For any two closed disjoint subsets $A_i \subseteq X$ there is a function $f \in \mathcal{S}$ with $f|A_0 = 0$ and $0 \notin f(A_1)$;*
(3) *Every locally finite open covering admits $\mathcal{S}$-partitions of unity subordinated to it.*
(4) *For any two disjoint zero-sets A_0 and A_1 of continuous functions there exists a function $g \in \mathcal{S}$ with $g|_{A_j} = j$ for $j \in \{0, 1\}$ and $g(X) \subseteq [0, 1]$;*

(5) *For any continuous function* $f : X \to \mathbb{R}$ *there exists a function* $g \in \mathcal{S}$ *with* $f^{-1}(0) \subseteq g^{-1}(0) \subseteq f^{-1}(\mathbb{R} \setminus \{1\})$.

(6) *The set* $\mathcal{S}$ *is dense in the algebra of continuous functions with respect to the topology of uniform convergence;*

(7) *The set of all bounded functions in* $\mathcal{S}$ *is dense in the algebra of continuous bounded functions on* X *with respect to the supremum norm;*

(8) *The bounded functions in* $\mathcal{S}$ *separate points in the Stone-Čech-compactification* βX *of* X.

The statements (1)-(3) are equivalent, and the weaker statements (4)-(8) are equivalent as well.
If X *is metrizable all statements are equivalent.*
If every open set is the carrier set of a smooth function then X *is* $\mathcal{S}$*-normal.*
If X *is* $\mathcal{S}$*-normal, then it is* $\mathcal{S}$*-regular.*
The space X *is* $\mathcal{S}$*-paracompact if and only if it is paracompact and* $\mathcal{S}$*-normal.*

Proof.
(1) $\Rightarrow$ (2) is obvious.

(2)$\Rightarrow$(1). By assumption, there is a smooth function f_0 with $f_0|A_1 = 0$ and $0 \notin f_0(A_0)$, and again by assumption, there is a smooth function f_1 with $f_1|A_0 = 0$ and $0 \notin f_1(\{x : f_0(x) = 0\})$. The function $f = \frac{f_1}{f_0+f_1}$ has the required properties.

(3) $\Rightarrow$ (1) Let A_0 and A_1 be two disjoint closed subset. Then $\mathcal{U} := \{X \setminus A_1, X \setminus A_0\}$ admits an $\mathcal{S}$-partition of unity $\mathcal{F}$ subordinated to it, and

$$\sum\{f \in \mathcal{F} : \operatorname{carr} f \subseteq X \setminus A_0\}$$

is the required bump function.

(1) $\Rightarrow$ (3) Let $\mathcal{U}$ be a locally finite covering of X. Since X is $\mathcal{S}$-normal, its topology is also normal and therefore for every $U \in \mathcal{U}$ there exists an open set V_U such that $\overline{V_U} \subseteq U$ and $\{V_U : U \in \mathcal{U}\}$ is still an open cover. By assumption, there exist smooth functions $g_U \in \mathcal{S}$ such that $V_U \subseteq \operatorname{carr}(g_U) \subseteq U$, cf. 16.1. The function $g := \sum_U g_U$ is well defined, positive, and smooth since $\mathcal{U}$ is locally finite, and $\{f_U := g_U/g : U \in \mathcal{U}\}$ is the required partition of unity.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (4) Let $A_j := f_j^{-1}(a_j)$ for $j \in \{0,1\}$ with continuous functions f_j. By replacing f_j by $(f_j - a_j)^2$ we may assume that $f_j \geq 0$ and $A_j = f_j^{-1}(0)$. Then $(f_1 + f_2)(x) > 0$ for all $x \in X$, since $A_1 \cap A_2 = \emptyset$. Thus, $f := \frac{f_0}{f_0+f_1}$ is a function in $C(X, [0,1])$ with $f|_{A_j} = j$ for $j \in \{0,1\}$. Now we reason as in (2$\Rightarrow$1): By (5) there exists a $g_0 \in \mathcal{S}$ with $A_0 \subseteq f^{-1}(0) \subseteq g_0^{-1}(0) \subseteq f^{-1}(\mathbb{R} \setminus \{1\}) = X \setminus f^{-1}(1) \subseteq X \setminus A_1$. By replacing g_0 by g_0^2 we may assume that $g_0 \geq 0$.
Applying the same argument to the zero-sets A_1 and $g_0^{-1}(0)$ we obtain a $g_1 \in \mathcal{S}$ with $A_1 \subseteq g_1^{-1}(0) \subseteq X \setminus g_0^{-1}(0)$. Thus, $(g_0 + g_1)(x) > 0$, and hence $g := \frac{g_0}{g_0+g_1} \in \mathcal{S}$ satisfies $g|_{A_j} = j$ for $j \in \{0,1\}$ and $g(X) \subseteq [0,1]$.

(4) $\Rightarrow$ (6) Let f be continuous. Without loss of generality we may assume $f \geq 0$ (decompose $f = f_+ - f_-$). Let $\varepsilon > 0$. Then choose $g_k \in \mathcal{S}$ with image in $[0,1]$, and $g_k(x) = 0$ for all x with $f(x) \leq k\,\varepsilon$, and $g_k(x) = 1$ for all x with $f(x) \geq (k+1)\,\varepsilon$. Let k be the largest integer less or equal to $\frac{f(x)}{\varepsilon}$. Then $g_j(x) = 1$ for all $j < k$, and

$g_j(x) = 0$ for all $j > k$. Hence, the sum $g := \varepsilon \sum_{k\in\mathbb{N}} g_k \in \mathcal{S}$ is locally finite, and $|f(x) - g(x)| < 2\,\varepsilon$.

(6) $\Rightarrow$ (7) This is obvious, since any function approximating a bounded function uniformly is itself bounded.

(7) $\Leftrightarrow$ (8) This follows from the Stone-Weierstraß theorem, since obviously the bounded functions in $\mathcal{S}$ form a subalgebra in $C_b(X) = C(\beta X)$. Hence, it is dense if and only if it separates points in the compact space βX.

(7) $\Rightarrow$ (5) By cutting off f at 0 and at 1, we may assume that f is bounded. By (7) there exists a bounded $g_0 \in \mathcal{S}$ with $\|f - g_0\|_\infty < \frac{1}{2}$. Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $h(t) = 0 \Leftrightarrow t \le \frac{1}{2}$. Then $g := h \circ g_0 \in \mathcal{S}$, and $f(x) = 0 \Rightarrow g_0(x) \le |g_0(x)| \le |f(x)| + \|f - g_0\|_\infty \le \frac{1}{2} \Rightarrow g(x) = h(g_0(x)) = 0$ and also $f(x) = 1 \Rightarrow g_0(x) \ge f(x) - \|f - g_0\|_\infty > 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow g(x) \ne 0$.

If X is metrizable and $A \subseteq X$ is closed, then $\operatorname{dist}(\_, A) : x \mapsto \sup\{\operatorname{dist}(x, a) : a \in A\}$ is a continuous function with $f^{-1}(0) = A$. Thus, (1) and (4) are equivalent.

Let every open subset be the carrier of a smooth mapping, and let A_0 and A_1 be closed disjoint subsets of X. By assumption, there is a smooth function f with $\operatorname{carr}(f) = X \setminus A_0$, hence (2) is valid, i.e. X is $\mathcal{S}$-normal.

Obviously, every $\mathcal{S}$-normal space is $\mathcal{S}$-regular: Take as second closed set in (2) a single point. If we take instead the other closed set as single point, then we have small zero-sets in the sense of [**75**, 19.8].

That a space is $\mathcal{S}$-paracompact if and only if it is paracompact and $\mathcal{S}$-normal follows since (1$\Leftrightarrow$3) and by paracompactness every open covering has a locally finite refinement. □

In [**78**] it is remarked that in an uncountable product of real lines there are open subsets, which are not carrier sets of continuous functions.

Corollary. Denseness of smooth functions. *Let X be $\mathcal{S}$-paracompact, let F be a convenient vector space, and let $U \subseteq X \times F$ be open such that for all $x \in X$ the set $\iota_x^{-1}(U) \subseteq F$ is convex and non-empty, where $\iota_x : F \to X \times F$ is given by $y \mapsto (x, y)$. Then there exists an $f \in \mathcal{S}$ whose graph is contained in U.*

Under the following assumption this result is due to [Bonic, Frampton, 1966]: For $U := \{(x, y) : p(y - g(x)) < \varepsilon(x)\}$, where $g : X \to F$, $\varepsilon : X \to \mathbb{R}^+$ are continuous and p is a continuous seminorm on F.

Proof. For every $x \in X$ let y_x be chosen such that $(x, y_x) \in U$. Next choose open neighborhoods U_x of x such that $U_x \times \{y_x\} \subseteq U$. Since X is $\mathcal{S}$-paracompact there exists a $\mathcal{S}$-partition of unity $\mathcal{F}$ subordinated to the covering $\{U_x : x \in X\}$. In particular, for every $\varphi \in \mathcal{F}$ there exists an $x_\varphi \in X$ with $\operatorname{carr} \varphi \subseteq U_{x_\varphi}$. Now define $f := \sum_{\varphi\in\mathcal{F}} y_{x_\varphi}\, \varphi$. Then $f \in \mathcal{S}$ and for every $x \in X$ we have

$$f(x) = \sum_{\varphi\in\mathcal{F}} y_{x_\varphi}\, \varphi(x) = \sum_{x\in\operatorname{carr}\varphi} y_{x_\varphi}\, \varphi(x) \in \iota_x^{-1}(U),$$

since $\iota_x^{-1}(U)$ is convex, contains y_{x_φ} for $x \in \operatorname{carr}(\varphi) \subseteq U_{x_\varphi}$, and $\varphi(x) \ge 0$ with $1 = \sum_\varphi \varphi(x) = \sum_{x\in\operatorname{carr}\varphi} \varphi(x)$. □

16.3. Lemma. $\mathcal{L}\mathrm{ip}^2$-functions on $\mathbb{R}^n$. [**132**]. *Let $B \in \mathbb{N}$ and $A := \{x \in \mathbb{R}^N : x_i \le 0$ for all i and $\|x\|_2 \le 1\}$. Suppose that $f \in C^3_B(\mathbb{R}^N, \mathbb{R})$ with $f|_A = 0$ and $f(x) \ge 1$ for all x with $\operatorname{dist}(x, A) \ge 1$. Then $N < B^2 + 36\, B^4$.*

Proof. Suppose $N \ge B^2 + 36\, B^4$. Since A is invariant under permutation of coordinates, we may assume that f is symmetric by replacing f with $x \mapsto \frac{1}{N!}\sum_\sigma f(\sigma^* x)$, where σ runs through all permutations, and σ^* just permutes the coordinates. Consider the points $x_j \in \mathbb{R}^N$ for $j = 0, \dots, B^2$ of the form

$$x_j = \Big(\underbrace{\tfrac{1}{B}, \dots, \tfrac{1}{B}}_{j}, \underbrace{-\tfrac{1}{B}, \dots, -\tfrac{1}{B}}_{B^2-j}, \underbrace{0, \dots, 0}_{N-B^2}\Big).$$

Then $\|x_j\|_2 = 1$, $x_0 \in A$ and $d(x_{B^2}, A) \ge 1$. Since f is symmetric and $y_j := \frac{1}{2}(x_j + x_{j+1})$ has equal (vanishing) coordinates with indices $j, B^2 + 1, \dots, N$, we have for the partial derivatives $\partial_j f(y_j) = \partial_k f(y_j)$ for all $k \in \{B^2+1, \dots, N\}$. Thus

$$|\partial_j f(y_j)|^2 = \frac{1}{N - B^2} \sum_{k=B^2+1}^{N} |\partial_k f(y_j)|^2 \le \frac{\|f'(y_j)\|_2^2}{36\, B^4} = \frac{\|f'(y_j)\|^2}{36\, B^4} \le \frac{1}{36\, B^2},$$

since from $f|_A = 0$ we conclude that $f(0) = f'(0) = f''(0) = f'''(0)$ and hence $\|f^{(j)}(y)\| \le B\, \|y\|_2^{3-j}$ for $j \le 3$, see 15.6.

From $\big|f(x+h) - f(x) - f'(x)(h) - \frac{1}{2}f''(x)(h^2)\big| \le B\frac{1}{3!}\, \|h\|^3$ we conclude that

$$\begin{aligned}\big|f(x+h) - f(x-h)\big| &\le \big|f(x+h) - f(x) - f'(x)(h) - \tfrac{1}{2}f''(x)(h^2)\big| \\ &\quad + \big|f(x-h) - f(x) + f'(x)(h) - \tfrac{1}{2}f''(x)(h^2)\big| \\ &\quad + 2|f'(x)(h)| \\ &\le \tfrac{2}{3!}\, B\|h\|_2^3 + 2|f'(x)(h)|.\end{aligned}$$

If we apply this to $x = y_j$ and $h = \frac{1}{B}e_j$, where e_j denotes the j-th unit vector, then we obtain

$$|f(x_{j+1}) - f(x_j)| \le \frac{2}{3!}\, B\, \frac{1}{B^3} + 2|\partial_j f(y_j)|\, \frac{1}{B} \le \frac{2}{3B^2}.$$

Summing up yields $1 \le |f(x_{B^2})| = |f(x_{B^2}) - f(x_0)| \le \frac{2}{3} < 1$, a contradiction. □

16.4. Corollary. ℓ^2 is not $\mathcal{L}\mathrm{ip}^2_{\mathbf{glob}}$-normal . [**132**]. *Let $A_0 := \{x \in \ell^2 : x_j \le 0$ for all j and $\|x\|_2 \le 1\}$ and $A_1 := \{x \in \ell^2 : d(x, A) \ge 1\}$ and $f \in C^3(\ell^2, \mathbb{R})$ with $f|_{A_j} = j$ for $j \in \{0, 1\}$. Then $f^{(3)}$ is not bounded.*

Proof. By the preceding lemma a bound B of $f^{(3)}$ must satisfy for f restricted to $\mathbb{R}^N$, that $N < B^2 + 36B^4$. This is not for all N possible. □

16.5. Corollary. Whitney's extension theorem is false on ℓ^2. [**132**]. *Let $E := \mathbb{R} \times \ell^2 \cong \ell^2$ and $\pi : E \to \mathbb{R}$ be the projection onto the first factor. For subsets $A \subseteq \ell^2$ consider the cone $CA := \{(t, ta) : t \ge 0, a \in A\} \subseteq E$. Let $A := C(A_0 \cup A_1)$ with A_0 and A_1 as in 16.4. Let a jet (f^j) on A be defined by $f^j = 0$ on the cone CA_1 and $f^j(x)(v^1, \dots, v^j) = h^{(j)}(\pi(x))(\pi(v^1), \dots, \pi(v^j))$ for all x in the cone of CA_0, where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ is infinite flat at 0 but with $h(t) \ne 0$ for all $t \ne 0$. This jet has no C^3-prolongation to E.*

Proof. Suppose that such a prolongation f exists. Then $f^{(3)}$ would be bounded locally around 0, hence $f_a(x) := 1 - \frac{1}{h(a)}\, f(a, ax)$ would be a C^3_B function on ℓ^2 for small a, which is 1 on A_1 and vanishes on A_0. This is a contradiction to 16.4.

So it remains to show that the following condition of Whitney [75, 22.2] is satisfied:

$$\Big\| f^j(y) - \sum_{i=0}^{k-j} \frac{1}{i!} f^{j+i}(x)(y-x)^j \Big\| = o(\|x-y\|^{k-j}) \text{ for } A \ni x, y \to a.$$

Let $f_1^j := 0$ and $f_0^j(x) := h^{(j)}(\pi(x)) \circ (\pi \times \ldots \times \pi)$. Then both are smooth on $\mathbb{R} \oplus \ell^2$, and thus Whitney's condition is satisfied on each cone separately. It remains to show this when x is in one cone and y in the other and both tend to 0. Thus, we have to replace f at some places by f_1 and at others by f_0. Since h is infinite flat at 0 we have $\|f_0^j(z)\| = o(\|z\|^n)$ for every n. Furthermore for $x_i \in CA_i$ for $i \in \{0,1\}$ we have that $\|x_1 - x_0\| \geq \sin(\arctan 2 - \arctan 1) \max\{\|x_0\|, \|x_1\|\}$ since $\arctan 2 - \arctan 1$ is the angle between the rays through the x_i and the whole expression is the distance from the larger point to the ray through the smaller one. Thus, we may replace $f_0^j(y)$ by $f_1^j(y)$ and vice versa. So the condition is reduced to the case, where y and z are in the same cone CA_i. □

16.6. Lemma. Smoothly regular strict inductive limits. *Let E be the strict inductive limit of a sequence of C^∞-normal convenient vector spaces E_n such that $E_n \hookrightarrow E_{n+1}$ is closed and has the extension property for smooth functions. Then E is C^∞-regular.*

Proof. Let U be open in E and $0 \in U$. Then $U_n := U \cap E_n$ is open in E_n. We choose inductively a sequence of functions $f_n \in C^\infty(E_n, \mathbb{R})$ such that $\operatorname{supp}(f_n) \subseteq U_n$, $f_n(0) = 1$, and $f_n|_{E_{n-1}} = f_{n-1}$. If f_n is already constructed, we may choose by C^∞-normality a smooth $g : E_{n+1} \to \mathbb{R}$ with $\operatorname{supp}(g) \subseteq U_{n+1}$ and $g|_{\operatorname{supp}(f_n)} = 1$. By assumption, f_n extends to a function $\widetilde{f_n} \in C^\infty(E_{n+1}, \mathbb{R})$. The function $f_{n+1} := g \cdot \widetilde{f_n}$ has the required properties.

Now we define $f : E \to \mathbb{R}$ by $f|E_n := f_n$ for all n. It is smooth since any $c \in C^\infty(\mathbb{R}, E)$ locally factors to a smooth curve into some E_n by 1.8 since a strict inductive limit is regular by [68, 4.8.1], so $f \circ c$ is smooth. Finally, $f(0) = 1$, and if $f(x) \neq 0$ then $x \in E_n$ for some n, and we have $f_n(x) = f(x) \neq 0$, thus $x \in U_n \subseteq U$. □

For counter-examples for the extension property see [75, 21.7] and 21.11. However, for complemented subspaces the extension property obviously holds.

16.7. Proposition. C_c^∞ is C^∞-regular. *The space $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ of smooth functions on $\mathbb{R}^m$ with compact support satisfies the assumptions of 16.6.*

Let $K_n := \{x \in \mathbb{R}^m : \|x\| \leq n\}$. Then $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ is the strict inductive limit of the closed subspaces $C_{K_n}^\infty(\mathbb{R}^m, \mathbb{R}) := \{f : \operatorname{supp}(f) \subseteq K_n\}$, which carry the topology of uniform convergence in all partial derivatives separately. They are nuclear Fréchet spaces and hence separable, see [53, 11.6.2, p231]. Thus they are C^∞-normal by 16.10 below.

In order to show the extension property for smooth functions we prove more generally that for certain sets A the subspace $\{f \in C^\infty(E, \mathbb{R}) : f|_A = 0\}$ is a complemented subspace of $C^\infty(E, \mathbb{R})$. The first result in this direction is:

16.8. Lemma. [114] *The subspace $C_{(-\infty,0]}^\infty(\mathbb{R}, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f(t) = 0 \text{ for } t \leq 0\}$ of the Fréchet space $C^\infty(\mathbb{R}, \mathbb{R})$ is a direct summand.*

Proof. We claim that the following map is a bounded linear mapping being left inverse to the inclusion: $s(g)(t) := g(t) - \sum_{k\in\mathbb{N}} a_k h(-t2^k) g(-t2^k)$ for $t > 0$ and $s(g)(t) = 0$ for $t \le 0$. Where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support satisfying $h(t) = 1$ for $t \in [-1, 1]$ and (a_k) is a solution of the infinite system of linear equations $\sum_{k\in\mathbb{N}} a_k(-2^k)^n = 1$ ($n \in \mathbb{N}$) (the series is assumed to converge absolutely). The existence of such a solution is shown in [**114**] by taking the limit of solutions of the finite subsystems. Let us first show that $s(g)$ is smooth. For $t > 0$ the series is locally around t finite, since $-t2^k$ lies outside the support of h for k sufficiently large. Its derivative $(sg)^{(n)}(t)$ is

$$g^{(n)}(t) - \sum_{k\in\mathbb{N}} a_k(-2^k)^n \sum_{j=0}^{n} \binom{n}{j} h^{(j)}(-t2^k) g^{(n-j)}(-t2^k)$$

and this converges for $t \to 0$ towards $g^{(n)}(0) - \sum_{k\in\mathbb{N}} a_k(-2^k)^n g^{(n)}(0) = 0$. Thus $s(g)$ is infinitely flat at 0 and hence smooth on $\mathbb{R}$. It remains to show that $g \mapsto s(g)$ is a bounded linear mapping. By the uniform boundedness principle 5.26 it is enough to show that $g \mapsto (sg)(t)$ is bounded. For $t \le 0$ this map is 0 and hence bounded. For $t > 0$ it is a finite linear combination of evaluations and thus bounded. □

Now the general result:

16.9. Proposition. *Let E be a convenient vector space, and let p be a smooth seminorm on E. Let $A := \{x : p(x) \ge 1\}$. Then the closed subspace $\{f : f|_A = 0\}$ in $C^\infty(E, \mathbb{R})$ is complemented.*

Proof. Let $g \in C^\infty(E, \mathbb{R})$ be a smooth reparameterization of p with support in $E \setminus A$ equal to 1 near $p^{-1}(0)$. By lemma 16.8, there is a bounded projection $P : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty_{[0,+\infty)}(\mathbb{R}, \mathbb{R})$. The following mappings are smooth in turn by the properties of the cartesian closed smooth calculus, see 3.12:

$$\begin{aligned}
E \times \mathbb{R} \ni (x, t) &\mapsto f(e^t x) \in \mathbb{R}\\
E \ni x &\mapsto f(e^{(\text{-})} x) \in C^\infty(\mathbb{R}, \mathbb{R})\\
E \ni x &\mapsto P\Big(f(e^{(\text{-})} x)\Big) \in C^\infty_{[0,+\infty)}(\mathbb{R}, \mathbb{R})\\
E \times \mathbb{R} \ni (x, r) &\mapsto P\Big(f(e^{(\text{-})} x)\Big)(r) \in \mathbb{R}\\
\operatorname{carr} p \ni x &\mapsto \big(\tfrac{x}{p(x)}, \ln(p(x))\big) \mapsto P\Big(f(e^{(\text{-})} \tfrac{x}{p(x)})\Big)\big(\ln(p(x))\big) \in \mathbb{R}.
\end{aligned}$$

So we get the desired bounded linear projection

$$\bar{P} : C^\infty(E, \mathbb{R}) \to \{f \in C^\infty(E, \mathbb{R}) : f|_A = 0\},$$
$$(\bar{P}(f))(x) := g(x)\, f(x) + (1 - g(x))\, P\Big(f(e^{(\text{-})} \tfrac{x}{p(x)})\Big)\big(\ln(p(x))\big),$$

in fact, $x \in A \Rightarrow g(x) = 0$, $\ln(p(x)) \ge 0 \Rightarrow P\big(f(e^{(\text{-})} \frac{x}{p(x)})\big)\big(\ln(p(x))\big) = 0$ and $f|_A = 0 \Rightarrow \forall t \ge 0 : f(e^t \frac{x}{p(x)}) = 0 \Rightarrow P\big(f(e^{(\text{-})} \frac{x}{p(x)})\big)\big(\ln(p(x))\big) = f\big(e^{\ln(p(x))} \frac{x}{p(x)}\big) = f(x)$. □

16.10. Theorem. Smoothly paracompact Lindelöf. [**132**]. *If X is Lindelöf and $\mathcal{S}$-regular, then X is $\mathcal{S}$-paracompact. In particular, all nuclear Fréchet spaces and strict inductive limits of sequences of such spaces with the extension property for smooth functions are C^∞-paracompact. Furthermore, nuclear Silva spaces, see* [**75**, 52.37], *are C^∞-paracompact.*

The first part was proved by [17] under stronger assumptions. The importance of the proof presented here lies in the fact that we need not assume that $\mathcal{S}$ is local and that $\frac{1}{f} \in \mathcal{S}$ for $f \in \mathcal{S}$. The only things used are that $\mathcal{S}$ is an algebra and for each $g \in \mathcal{S}$ there exists an $h : \mathbb{R} \to [0,1]$ with $h \circ g \in \mathcal{S}$ and $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$. In particular, this applies to $\mathcal{S} = \mathcal{L}\mathrm{ip}^p_{\mathrm{global}}$ and X a separable Banach space.

Proof. Let $\mathcal{U}$ be an open covering of X.

Claim. There exists a sequence of functions $g_n \in \mathcal{S}(X,[0,1])$ such that $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is a locally finite family subordinated to $\mathcal{U}$ and $\{g_n^{-1}(1) : n \in \mathbb{N}\}$ is a covering of X.

For every $x \in X$ there exists a neighborhood $U \in \mathcal{U}$ (since $\mathcal{U}$ is a covering) and hence an $h_x \in \mathcal{S}(X,[0,2])$ with $h_x(x) = 2$ and $\operatorname{carr}(h_x) \subseteq U$ (since X is $\mathcal{S}$-regular). Since X is Lindelöf we find a sequence x_n such that $\{x : h_n(x) > 1 : n \in \mathbb{N}\}$ is a covering of X (we denote $h_n := h_{x_n}$). Now choose an $h \in C^\infty(\mathbb{R},[0,1])$ with $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$. Set

$$g_n(x) := h(n\,(h_n(x)-1)+1) \prod_{j<n} h(n\,(1-h_j(x))+1).$$

Note that

$$h(n\,(h_n(x)-1)+1) = \begin{cases} 0 & \text{for } h_n(x) \leq 1 - \frac{1}{n} \\ 1 & \text{for } h_n(x) \geq 1 \end{cases}$$

$$h(n\,(1-h_j(x))+1) = \begin{cases} 0 & \text{for } h_j(x) \geq 1 + \frac{1}{n} \\ 1 & \text{for } h_j(x) \leq 1 \end{cases}$$

Then $g_n \in \mathcal{S}(X,[0,1])$ and $\operatorname{carr} g_n \subseteq \operatorname{carr} h_n$. Thus, the family $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is subordinated to $\mathcal{U}$.

The family $\{g_n^{-1}(1) : n \in \mathbb{N}\}$ covers X since for each $x \in X$ there exists a minimal n with $h_n(x) \geq 1$, and thus $g_n(x) = 1$.

If we could divide in $\mathcal{S}$, then $f_n := g_n / \sum_j g_j$ would be the required partition of unity (and we do not need the last claim in this strong from).

Instead we proceed as follows. The family $\{\operatorname{carr} g_n : n \in \mathbb{N}\}$ is locally finite: Let m be such that $h_m(x) > 1$, and take $k > m$ so large that $1 + \frac{1}{k} < h_m(x)$, and let $U_x := \{y : h_m(y) > 1 + \frac{1}{k}\}$, which is a neighborhood of x. For $n \geq k$ and $y \in U_x$ we have that $h_m(y) > 1 + \frac{1}{k} \geq 1 + \frac{1}{n}$, hence the $(m+1)$-st factor of g_n vanishes at y, i.e. $\{n : \operatorname{carr} g_n \cap U_x \neq \emptyset\} \subseteq \{1, \dots, k-1\}$.

Now define $f_n := g_n \prod_{j<n}(1-g_j) \in \mathcal{S}$. Then $\operatorname{carr} f_n \subseteq \operatorname{carr} g_n$, hence $\{\operatorname{carr} f_n : n \in \mathbb{N}\}$ is a locally finite family subordinated to $\mathcal{U}$. By induction, one shows that $\sum_{j\leq n} f_j = 1 - \prod_{j\leq n}(1-g_j)$: In fact $\sum_{j\leq n} f_j = f_n + \sum_{j<n} f_j = g_n \prod_{j<n}(1-g_j) + 1 - \prod_{j<n}(1-g_j) = 1 + (g_n - 1)\prod_{j<n}(1-g_j)$. For every $x \in X$ there exists an n with $g_n(x) = 1$, hence $f_k(x) = 0$ for $k > n$ and $\sum_{j=0}^\infty f_j(x) = \sum_{j\leq n} f_j(x) = 1 - \prod_{j\leq n}(1-g_j(x)) = 1$.

Let us consider a nuclear Silva space. Since each Silva space is the strong dual of a Fréchet Schwarz space its c^∞-topology coincides with the locally convex one by 4.11.2. Nuclearity implies that there exists a base of (smooth) Hilbert seminorms. Thus we have C^∞-regularity.

A Silva space is an inductive limit of a sequence of Banach spaces with compact connecting mappings (see [75, 52.37]), and we may assume that the Banach spaces are separable by replacing them by the closures of the images of the connecting mappings, so the topology of the inductive limit is Lindelöf. Therefore, by the first assertion we conclude that the space is C^∞-paracompact.

In order to obtain the statement on nuclear Fréchet spaces we note that these are separable, see [53, 11.6.2, p231], and thus Lindelöf.

A strict inductive limit of a sequence of nuclear Fréchet spaces with the extension property for smooth functions is C^∞-regular by 16.6, and it is also Lindelöf for its c^∞-topology, since this is the inductive topological (not locally convex) limit of the steps. □

Remark. In particular, every separable Hilbert space has $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$-partitions of unity by 15.5, thus there is such a $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$-partition of functions φ subordinated to $\ell^2 \setminus A_0$ and $\ell^2 \setminus A_1$, with A_0 and A_1 mentioned in 16.4. Hence, $f := \sum_{\mathrm{carr}\,\varphi\cap A_0=\emptyset} \varphi \in C^2$ satisfies $f|_{A_j} = j$ for $j = 0, 1$. However, $f \notin \mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$. The reason behind this is that $\mathcal{L}\mathrm{ip}^2_{\mathrm{global}}$ is not a sheaf.

Open problem. *Classically, one proves the existence of continuous partitions of unity from the paracompactness of the space. So the question arises whether theorem 16.10 can be strengthened to: If the initial topology with respect to $\mathcal{S}$ is paracompact, do there exist $\mathcal{S}$-partitions of unity? Or equivalently: Is every paracompact $\mathcal{S}$-regular space $\mathcal{S}$-paracompact?*

16.11. Theorem. Smoothness of separable Banach spaces. *Let E be a separable Banach space. Then the following conditions are equivalent.*

(1) *E has a C^1-norm;*
(2) *E has C^1-bump functions, i.e., E is C^1-regular;*
(3) *The C^1-functions separate closed sets, i.e., E is C^1-normal;*
(4) *E has C^1-partitions of unity, i.e., E is C^1-paracompact;*
(5) *E has no rough norm, i.e. E is Asplund;*
(6) *E' is separable.*

Proof. The implications (1) ⇒ (2) and (4) ⇒ (3) ⇒ (2) are obviously true. The implication (2) ⇒ (4) is 16.10. (2) ⇒ (5) holds by 14.9. (5) ⇒ (6) follows from 14.10 since E is separable. (6) ⇒ (1) is 13.22 together with 13.20. □

A more general result is:

16.12. Result. [55] *Let E be a WCG Banach space. Then the following statements are equivalent:*

(1) *E is C^1-normable;*
(2) *E is C^1-regular;*
(3) *E is C^1-paracompact;*
(4) *E has norm, whose dual norm is LUR;*

(5) *E has a shrinking Markuševič basis, i.e. vectors $x_i \in E$ and $x_i^* \in E'$ with $x_i^*(x_j) = \delta_{i,j}$ and the span of the x_i is dense in E and the span of x_i^* is dense in E'.*

16.13. Results.

(1) [**42**] *([**129**]) Let E' be a WCG Banach space (or even WCD, see [**75**, 53.8]). Then E is C^1-regular.*
(2) [**129**] *Let K be compact with $K^{(\omega_1)} = \emptyset$. Then $C(K)$ is C^1-paracompact. Compare with 13.18.2 and 13.17.5.*
(3) [**43**] *Let E be a subspace of a WCG Banach space. If E is C^k-regular then it is C^k-paracompact. This will be shown in 16.18.*
(4) [**88**] *Let E' be a WCG Banach space. If E is C^k-regular then it is C^k-paracompact.*

16.14. Lemma. Smooth functions on $c_0(\Gamma)$. [**121**]. *The norm-topology of $c_0(\Gamma)$ has a basis which is a countable union of locally finite families of carriers of smooth functions, each of which depends locally only on finitely many coordinates.*

Proof. The open balls $B_r := \{x : \|x\|_\infty < r\}$ are carriers of such functions: In fact, similarly to 13.16 we choose a $h \in C^\infty(\mathbb{R}, \mathbb{R})$ with $h = 1$ locally around 0 and $\operatorname{carr} h = (-1, 1)$, and define $f(x) := \prod_{\gamma\in\Gamma} h(x_\gamma)$.

Let

$$\mathcal{U}_{n,r,q} := \Big\{ B_r + \sum_{i=1}^{n} q_i e_{\gamma_i} : \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma \Big\}$$

where $n \in \mathbb{N}$, $r \in \mathbb{Q}$, $q \in \mathbb{Q}^n$ with $|q_i| > 3r/2$ for $1 \le i \le n$. This is the required countable family.

Claim. The union $\bigcup_{n,r,q} \mathcal{U}_{n,r,q}$ is a basis for the topology.

Let $x \in c_0(\Gamma)$ and $\varepsilon > 0$. Choose $\frac{\varepsilon}{4} \le r < \frac{\varepsilon}{2}$ such that $r \ne |x_\gamma|$ for all γ (note that $|x_\gamma| \ge \varepsilon/4$ only for finitely many γ). Let $\{\gamma_1, \ldots, \gamma_n\} := \{\gamma : |x_\gamma| \ge r\}$. Choose $q_i \in \mathbb{Q}$ with $|q_i - x_{\gamma_i}| < r$ and $|q_i| > 3r/2$. Then

$$x - \sum_{i=1}^{n} q_i\, e_{\gamma_i} \in B_r,$$

and hence

$$x \in B_r + \sum_{i=1}^{n} q_i e_{\gamma_i} \subseteq x + B_{2r} \subseteq \{y : \|y - x\|_\infty < \varepsilon\}.$$

Claim. Each family $\mathcal{U}_{n,r,q}$ is locally finite.

For given $x \in c_0(\Gamma)$, let $\{\gamma_1, \ldots, \gamma_m\} := \{\gamma : |x_\gamma| > \frac{r}{4}\}$ and assume there exists a $y \in (x + B_{\frac{r}{4}}) \cap (B_r + \sum_{i=1}^{n} q_i e_{\beta_i}) \ne \emptyset$. For $y \in x + B_{\frac{r}{4}}$ we have $|y_\gamma| < \frac{r}{2}$ for all $\gamma \notin \{\gamma_1, \ldots, \gamma_m\}$ and for $y \in B_r + \sum_{i=1}^{n} q_i\, e_{\beta_i}$ we have $|y_\gamma| > \frac{r}{2}$ for all $\gamma \in \{\beta_1, \ldots, \beta_n\}$. Hence, $\{\beta_1, \ldots, \beta_n\} \subseteq \{\gamma_1, \ldots, \gamma_m\}$ and $\mathcal{U}_{n,r,q}$ is locally finite. □

16.15. Theorem, Smoothly paracompact metrizable spaces . [**121**]. *Let X be a metrizable smooth space. Then the following are equivalent:*

(1) *X is $\mathcal{S}$-paracompact, i.e. admits $\mathcal{S}$-partitions of unity.*

(2) *X is $\mathcal{S}$-normal.*

(3) *The topology of X has a basis which is a countable union of locally finite families of carriers of smooth functions.*

(4) *There is a homeomorphic embedding $i : X \to c_0(A)$ for some A (with image in the unit ball) such that $ev_a \circ i$ is smooth for all $a \in A$.*

Proof. (1) $\Rightarrow$ (3) Let $\mathcal{U}_n$ be the cover formed by all open balls of radius $1/n$. By (1) there exists a partition of unity subordinated to it. The carriers of these smooth functions form a locally finite refinement $\mathcal{V}_n$. The union of all $\mathcal{V}_n$ is clearly a base of the topology since that of all $\mathcal{U}_n$ is one.

(3) $\Rightarrow$ (2) Let A_0 and A_1 be two disjoint closed subsets of X. Let furthermore $\mathcal{U}_n$ be a locally finite family of carriers of smooth functions such that $\bigcup_n \mathcal{U}_n$ is a basis. Let $W_n^i := \bigcup\{U \in \mathcal{U}_n : U \cap A_i = \emptyset\}$. This is the carrier of the smooth locally finite sum of the carrying functions of the U's. The family $\{W_n^i : i \in \{0,1\}, n \in \mathbb{N}\}$ forms a countable cover of X: $\forall x \in X\ \exists i \in \{0,1\} : x \notin A_i$, hence $\exists n\ \exists U \in \mathcal{U}_n :$ $x \in U \subseteq X \setminus A_i$, i.e. $x \in U \subseteq W_n^i$. By the argument used in the proof of 16.10 we may shrink the W_n^i to obtain a locally finite cover $\{\tilde{W}_n^i : i \in \{0,1\}, n \in \mathbb{N}\}$ of X consisting of carriers. Then $\bigcup_n \tilde{W}_n^0$ is a carrier containing A_1 and avoiding A_0. Now use 16.2.2.

(2) $\Rightarrow$ (1) is lemma 16.2, since metrizable spaces are paracompact.

(3) $\Rightarrow$ (4) Let $\mathcal{U}_n$ be a locally finite family of carriers of smooth functions such that $\mathcal{U} := \bigcup_n \mathcal{U}_n$ is a basis. For every $U \in \mathcal{U}_n$ let $f_U : X \to [0, \frac{1}{n}]$ be a smooth function with carrier U. We define a mapping $\iota : X \to c_0(\mathcal{U})$, by $\iota(x) := (f_U(x))_{U \in \mathcal{U}}$. It is continuous at $x_0 \in X$, since for $n \in \mathbb{N}$ there exists a neighborhood V of x_0 which meets only finitely many sets $U \in \bigcup_{k \le 2n} \mathcal{U}_k$, and so $\|\iota(x) - \iota(x_0)\| \le \frac{1}{n}$ for all $x \in V$ with $|f_U(x) - f_U(x_0)| < \frac{1}{n}$ for those U: For all $x \in V \cap U$ with $U \in \mathcal{U}_k$ we either have $k \le 2n$ and hence $|f_U(x) - f_U(x_0)| < \frac{1}{n}$ or $k > 2n$ and thus $\|f_U\|_\infty \le \frac{1}{k} < \frac{1}{2n}$. The mapping i is even an embedding, since for $x_0 \in U \in \mathcal{U}$ and $x \notin U$ we have $\|\iota(x) - \iota(x_0)\| \ge f_U(x_0) > 0$, thus $\iota^{-1}(\{t : \|t - \iota(x_0)\| < f_U(x_0)\}) \subseteq U$.

(4) $\Rightarrow$ (3) By 16.14 the Banach space $c_0(A)$ has a basis which is a countable union of locally finite families of carriers of smooth functions, all of which depend locally only on finitely many coordinates. The pullbacks of all these functions via ι are smooth on X, and their carriers furnish the required basis. $\square$

16.16. Corollary. Hilbert spaces are C^∞-paracompact. [121]. *Every space $c_0(\Gamma)$ (for arbitrary index set Γ) and every Hilbert space (not necessarily separable) is C^∞-paracompact.*

Proof. The assertion for $c_0(\Gamma)$ is immediate from 16.15.4. For the Hilbert space $\ell^2(\Gamma)$ we use the embedding $\iota : \ell^2(\Gamma) \to c_0(\Gamma \cup \{*\})$ given by

$$\iota(x)^\gamma = \begin{cases} x^\gamma & \text{for } \gamma \in \Gamma \\ \|x\|_2^2 & \text{for } \gamma = * \end{cases}$$

This is an embedding: From $\|x_n - x\|_\infty \to 0$ and boundedness of $\|x_n\|_2$ we conclude by Cauchy-Schwarz inequality that $\langle y, x_n - x\rangle \to 0$ for all $y \in \ell^2$ and hence $\|x_n - x\|_2^2 = \|x_n\|_2^2 + \|x\|_2^2 - 2\langle x, x_n\rangle \to 2\|x\|_2^2 - 2\|x\|_2^2 = 0$. $\square$

16.17. Corollary. *A countable product of $\mathcal{S}$-paracompact metrizable spaces is again $\mathcal{S}$-paracompact.*

Proof. By theorem 16.15 we have appropriate embeddings $\iota_n : X_n \to c_0(A_n)$ with images contained in the unit balls. We consider the embedding $\iota : \prod_n X_n \to c_0(\bigsqcup_n A_n)$ given by $\iota(x)_a = \frac{1}{n}\iota_n(x_n)$ for $a \in A_n$ which has the required properties for theorem 16.15. It is an embedding, since $\iota(x^n) \to \iota(x)$ if and only if $x^n_k \to x_k$ for all k (all but finitely many coordinates are small anyhow). □

53.13. Definition. Projective Resolution of Identity. Let a "long sequence" of continuous projections $P_\alpha \in L(E,E)$ on a Banach space E for all ordinal numbers $\omega \le \alpha \le \operatorname{dens} E$ be given. Recall that $\operatorname{dens}(E)$ is the density of E (a cardinal number, which we identify with the smallest ordinal of same cardinality). Let $E_\alpha := P_\alpha(E)$ and let $R_\alpha := (P_{\alpha+1} - P_\alpha)/(\|P_{\alpha+1} - P_\alpha\|)$ or 0, if $P_{\alpha+1} = P_\alpha$. Then we consider the following properties:

(1) $P_\alpha P_\beta = P_\beta = P_\beta P_\alpha$ for all $\beta \le \alpha$.
(2) $P_{\operatorname{dens} E} = \mathrm{id}_E$.
(3) $\operatorname{dens} P_\alpha E \le \alpha$ for all α.

(4) $\|P_\alpha\| = 1$ for all α.
(5) $\overline{\bigcup_{\beta<\alpha} P_{\beta+1}E} = P_\alpha E$, or equivalently $\overline{\bigcup_{\beta<\alpha} E_\beta} = E_\alpha$ for every limit ordinal $\alpha \le \operatorname{dens} E$.
(6) For every limit ordinal $\alpha \le \operatorname{dens} E$ we have $P_\alpha(x) = \lim_{\beta<\alpha} P_\beta(x)$, i.e. $\alpha \mapsto P_\alpha(x)$ is continuous.

(7) $E_{\alpha+1}/E_\alpha$ is separable for all $\omega \le \alpha < \operatorname{dens} E$.
(8) $(R_\alpha(x))_\alpha \in c_0([\omega, \operatorname{dens} E])$ for all $x \in E$.
(9) $P_\alpha(x) \in \overline{\langle P_\omega(x) \cup \{R_\beta(x) : \omega \le \beta < \alpha\}\rangle}$.

The family $(P_\alpha)_\alpha$ is called *projective resolution of identity* (*PRI*) if it satisfies (1), (2), (3), (4) and (5).

It is called *separable projective resolution of identity* (*SPRI*) if it satisfies (1), (2), (3), (7), (8) and (9). These are the only properties used in 53.20 and they follow for WCD Banach spaces and for duals of Asplund spaces by 53.15. For $C(K)$ with Valdivia compact K this is not clear, see 53.18 and 53.19. However, we still have 53.21 and in 16.18 we don't use (7), but only (8) and (9) which hold also for PRI, see below.

Remark. Note that from (1) we obtain that $P_\alpha^2 = P_\alpha$ and hence $\|P_\alpha\| \ge 1$, and $E_\alpha := P_\alpha(E)$ is the closed subspace $\{x : P_\alpha(x) = x\}$.

Moreover, $P_\alpha P_\beta = P_\beta = P_\beta P_\alpha$ for $\beta \le \alpha$ is equivalent to $P_\alpha^2 = P_\alpha$, $P_\beta(E) \subseteq P_\alpha(E)$ and $\ker P_\beta \supseteq \ker P_\alpha$.
($\Rightarrow$) $P_\beta x = P_\alpha P_\beta x \in P_\alpha(E)$ and $P_\alpha x = 0$ implies that $P_\beta x = P_\beta P_\alpha x$.
($\Leftarrow$) For $x \in E$ there is some $y \in E$ with $P_\beta x = P_\alpha y$, hence $P_\alpha P_\beta x = P_\alpha P_\alpha y = P_\alpha y = P_\beta x$. And $P_\beta(1 - P_\alpha)x = 0$, since $(1 - P_\alpha)x \in \ker P_\alpha \subseteq \ker P_\beta$.

Note that $E_{\alpha+1}/E_\alpha \cong R_\alpha(E)$, since $E_\alpha \hookrightarrow E_{\alpha+1}$ has $P_\alpha|_{E_{\alpha+1}}$ as right inverse, and so $E_{\alpha+1}/E_\alpha \cong \ker(P_\alpha|_{E_{\alpha+1}}) = (1 - P_\alpha)P_{\alpha+1}(E) = (P_{\alpha+1} - P_\alpha)(E)$.

(5) ⇐ (9), since for $x \in E_\alpha$ we have $x = P_\alpha(x)$ and $E_\omega \cup \{R_\beta(x) : \beta < \alpha\} \subseteq E_\alpha$ for all α.

(3) ⇐ (5) & (7) By transfinite induction we get that for successor ordinals $\alpha = \beta + 1$ we have $\operatorname{dens}(E_\alpha) = \operatorname{dens}(E_\beta) + \operatorname{dens}(E_\alpha/E_\beta) = \operatorname{dens}(E_\beta) \le \beta \le \alpha$, since $\operatorname{dens}(E_\alpha/E_\beta) \le \omega$. For limit ordinals it follows from (5), since $\operatorname{dens}(E_\alpha) = \operatorname{dens}(\bigcup_{\beta<\alpha} E_\beta) = \sup\{\operatorname{dens}(E_\beta) : \beta < \alpha\} \le \sup\{\beta : \beta < \alpha\} = \alpha$.

(5) ⇐ (6) is trivial.

(6) ⇐ (4) & (1) & (5) For every limit ordinal $0 < \alpha \le \operatorname{dens} E$ and for all $x \in E$ the net $(P_\beta(x))_{\beta<\alpha}$ converges to $P_\alpha(x)$:
Let first $x \in P_\alpha(E)$ and $\varepsilon > 0$. By (5) there exists a $\gamma < \alpha$ and an $x_\gamma \in P_\gamma(E)$ with $\|x - x_\gamma\| < \varepsilon$. For $\gamma \le \beta < \alpha$ we have by (1) that $P_\beta(x_\gamma) = P_\alpha(x_\gamma)$ and so

$$\begin{aligned}\|P_\alpha(x) - P_\beta(x)\| &= \|P_\alpha(x - x_\gamma) + (P_\alpha(x_\gamma) - P_\beta(x_\gamma)) + P_\beta(x_\gamma - x)\| \\ &\le (\|P_\alpha\| + \|P_\beta\|)\,\|x - x_\gamma\| < 2\,\varepsilon.\end{aligned}$$

If $x \in E$ is arbitrary, then $P_\alpha(x) \in P_\alpha(E)$, hence by (1)

$$P_\beta(x) = P_\beta(P_\alpha(x)) \to P_\alpha(P_\alpha(x)) = P_\alpha(x) \text{ for } \beta \nearrow \alpha.$$

(8) ⇐ (1) & (6) For $\varepsilon > 0$ the set $\{\beta : \beta < \alpha, \|R_\beta(x)\| \ge \varepsilon\}$ is finite: Otherwise there would be a strictly increasing sequence (β_n) such that $\|R_{\beta_n}(x)\| \ge \varepsilon$ and since $\|P_{\alpha+1} - P_\alpha\| \ge 1$ also $\|(P_{\beta_n+1} - P_{\beta_n})(x)\| \ge \varepsilon$. Let $\beta_\infty := \sup_n \beta_n$. Then $\beta_\infty \le \alpha$ is a limit ordinal and $P_{\beta_\infty}(x) = \lim_{\beta<\beta_\infty} P_\beta(x)$ according to (6), a contradiction.

(9) ⇐ (6) We prove by transfinite induction that $P_\alpha(x)$ is in the closure of the linear span of $P_\omega(x) \cup \{R_\beta(x) : \omega \le \beta < \alpha\}$:
For $\alpha = \omega$ this is obviously true. Let now $\alpha = \beta + 1$ and assume $P_\beta(x)$ is in the closure of the linear span of $P_\omega(x) \cup \{R_\gamma(x) : \omega \le \gamma < \beta\}$. Since $P_\alpha(x) = P_\beta(x) + \|P_\alpha - P_\beta\|\, R_\beta(x)$ we get that $P_\alpha(x)$ is in the closure of the linear span of $P_\omega(x) \cup \{R_\gamma(x) : \omega \le \gamma < \beta\} \cup R_\beta(x) = P_\omega(x) \cup \{R_\gamma(x) : \omega \le \gamma < \alpha\}$.
Let now α be a limit ordinal and let $P_\beta(x)$ be in the closure of the linear span of $P_\omega(x) \cup \{R_\gamma(x) : \omega \le \gamma < \alpha\}$ for all $\beta < \alpha$. Then by (6) we get that $P_\alpha(x) = \lim_{\beta<\alpha} P_\beta(x)$ is in this closure as well.

53.19. Remark. *The space $C([0,\alpha])$ has a PRI given by*

$$P_\beta(f)(\mu) := \begin{cases} f(\mu) & \text{for } \mu \le \beta \\ f(\beta) & \text{for } \mu \ge \beta \end{cases}.$$

However, there is no PRI on the hyperplane $E := \{f \in C([0,\omega_1]) : f(\omega_1) = 0\}$ of the space $C[0,\omega_1]$. And, in particular, $C[0,\omega_1]$ is not WCD.

Proof. Obviously, the given P_β satisfy the properties (1), (2), (4), (6), and (7), from 53.13. The remaining property (4) for a PRI in $C([0,\alpha])$, i.e. $\operatorname{dens}(C([0,\beta])) \le \beta$ for all $\omega \le \beta \le \alpha$, follows by combining [Engelking, 1989, Exercise 3.4.H.b(]i.e. $\operatorname{dens}(C(K,\mathbb{R}))$ is at most the weight of K for infinite compact K) and [Engelking, 1989, Theorem 3.1.21(]i.e. the weight of K is at most the cardinality of K).

Assume $\{P_\alpha : \omega \le \alpha \le \omega_1\}$ is some PRI on E. Put $\alpha_0 := \omega_0$. We may find $\beta_0 < \omega_1$ with

$$P_{\alpha_0} E \subseteq E_{\beta_0} := \{f \in E : f(\alpha) = 0 \text{ for } \alpha > \beta_0\},$$

because for each f in dense countable subset $D \subseteq P_{\alpha_0}E$ we find a β_f with $f(\alpha) = 0$ for $\alpha \geq \beta_f$. Since E_{β_0} is separable, there is an $\alpha_0 < \alpha_1 < \omega_1$ such that

$$E_{\beta_0} \subseteq P_{\alpha_1}E,$$

in fact $D \subseteq E_{\beta_0}$ is dense and hence for each $f \in D$ and $n \in \mathbb{N}$ there exists an $\alpha_{f,n} < \omega_1$ and $\tilde{f} \in P_{\alpha_{f,n}}E$ such that $\|f - \tilde{f}\| \leq 1/n$. Then $\alpha_1 := \sup\{\alpha_{f,n} : n \in \mathbb{N}, f \in D\}$ fulfills the requirements.

Now we proceed by induction. Let $\alpha_\infty := \sup_n \alpha_n$ and $\beta_\infty := \sup_n \beta_n$. Then

$$P_{\alpha_\infty}E = \overline{\bigcup_n P_{\alpha_n}E} = F_{\beta_\infty} := \{f \in E : f(\alpha) = 0 \text{ for } \alpha \geq \beta_\infty\}.$$

But F_{β_∞} is not the image of a norm-1 projection: Suppose P were a norm-1 projection on F_{β_∞}. Let $\pi : E \to C(X)$ be the restriction map, where $X := [0, \beta_\infty]$. It is left inverse to the inclusion ι given by $f \mapsto \tilde{f}$ with $\tilde{f}(\gamma) = 0$ for $\gamma \geq \beta_\infty$. Let $\tilde{P} := \pi \circ P \circ \iota \in L(C(X))$. Then $\tilde{P}$ is a norm-1 projection with image $C_{\beta_\infty}(X) := \{f \in C[0, \beta_\infty] : f(\beta_\infty) = 0\}$. Then $C(X) = \ker(\tilde{P}) \oplus C_{\beta_\infty}(X)$. We pick $0 \neq f_0 \in \ker(\tilde{P})$. Since $f_0 \notin \tilde{P}(C(X)) = C_{\beta_\infty}(X) = \ker(\mathrm{ev}_{\beta_\infty})$, we have $f_0(\beta_\infty) \neq 0$, and without loss of generality we may assume that $f_0(\beta_\infty) = 1$. For $f \in C(X)$ we have that $f - \tilde{P}(f) \in \ker \tilde{P}$ and hence there is a $\lambda_f \in \mathbb{R}$ with $f - \tilde{P}(f) = \lambda_f f_0$. In fact evaluating at β_∞ gives $f(\beta_\infty) - 0 = \lambda_f 1$, hence $\tilde{P}(f) = f - f(\beta_\infty) f_0$. Since β_∞ is a limit point, there is for each $\varepsilon > 0$ a $x_\varepsilon < \beta_\infty$ with $f_0(x_\varepsilon) > 1 - \varepsilon$. Now choose $f_\varepsilon \in C(X)$ with $\|f_\varepsilon\| = 1 = -f_\varepsilon(\beta_\infty) = f_\varepsilon(x_\varepsilon)$. Then

$$\begin{aligned}
\|\tilde{P}f_\varepsilon\|_\infty &= \|f_\varepsilon - f_\varepsilon(\beta_\infty) f_0\|_\infty \\
&\geq |f_\varepsilon(x_\varepsilon) - f_\varepsilon(\beta_\infty) f_0(x_\varepsilon)| \\
&\geq 1 + 1(1 - \varepsilon) = 2 - \varepsilon.
\end{aligned}$$

Hence $\|\tilde{P}\| \geq 2$, a contradiction.

Note however that every separable subspace is contained in a 1-complemented separable subspace. $\square$

53.16. Definition. A compact set K is called *Valdivia compact* if there exists some set Γ with $K \subseteq \mathbb{R}^\Gamma$ (i.e. K is *realcompact*) and $\{x \in K : \mathrm{carr}(x) \text{ is countable}\}$ being dense in K.

53.17. Lemma. *For a Valdivia compact set $K \subseteq \mathbb{R}^\Gamma$ we consider the set $E := \{x \in \mathbb{R}^\Gamma : \mathrm{carr}(x)$ is countable$\}$. Let μ be the density number of $K \cap E$. Then there exists an increasing long sequence of subsets $\Gamma_\alpha \subseteq \Gamma$ for $\omega \leq \alpha \leq \mu$ satisfying:*

(i) $|\Gamma_\alpha| \leq \alpha$;
(ii) $\bigcup_{\beta<\alpha} \Gamma_\beta = \Gamma_\alpha$ *for limit ordinals* α;
(iii) $\Gamma_\mu = \bigcup_{x \in K} \mathrm{carr}(x)$;

and such that $K_\alpha := Q_{\Gamma_\alpha}(K) \subseteq K$, where $Q_{\Gamma'} : \mathbb{R}^\Gamma \twoheadrightarrow \mathbb{R}^{\Gamma'} \hookrightarrow \mathbb{R}^\Gamma$, i.e.

$$Q_{\Gamma'}(x)_\gamma := \begin{cases} x_\gamma & \text{for } \gamma \in \Gamma' \\ 0 & \text{for } \gamma \notin \Gamma \setminus \Gamma' \end{cases}.$$

Thus $K_\alpha \subseteq K$ is a retract via Q_{Γ_α}.

Note that for any Valdivia compact set $K \subseteq \mathbb{R}^\Gamma$ we may always replace Γ by $\bigcup_{x \in K} \mathrm{carr}(x) = \bigcup_{x \in K \cap E} \mathrm{carr}(x)$, and then (iii) says $\Gamma_\mu = \Gamma$.

Proof. The proof is based on the following claim: Let $\Delta \subseteq \Gamma$ be a infinite subset. Then there exists some subset $\tilde{\Delta}$ with $\Delta \subseteq \tilde{\Delta} \subseteq \Gamma$ and $|\Delta| = |\tilde{\Delta}|$ and $Q_{\tilde{\Delta}}(K) \subseteq K$.

By induction we construct a sequence $\Delta =: \Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_k \subseteq \dots \subseteq \Gamma$ with $|\Delta_k| = |\Delta_0|$ and $Q_{\Delta_k}(\{x \in K \cap E : \operatorname{carr}(x) \subseteq \Delta_{k+1}\})$ being dense in $Q_{\Delta_k}(K)$:
(k+1) Since $K \cap E$ is dense in K, we have that $Q_{\Delta_k}(K \cap E)$ is dense in $Q_{\Delta_k}(K) \subseteq \mathbb{R}^{\Delta_k} \times \{0\} \subseteq \mathbb{R}^\Gamma$. And since the topology of $\mathbb{R}^{\Delta_k}$ has a basis of cardinality $|\Delta_k|$, there is a subset $D \subseteq K \cap E$ with $|D| \le |\Delta_k|$ and $Q_{\Delta_k}(D)$ dense in $Q_{\Delta_k}(K)$. Let $\Delta_{k+1} := \Delta_k \cup \bigcup_{x \in D} \operatorname{carr}(x)$ then $\Delta_{k+1} \supseteq \Delta_k$ and $|\Delta_{k+1}| = |\Delta_k|$. Furthermore $Q_{\Delta_k}(\{x \in K \cap E : \operatorname{carr}(x) \subseteq \Delta_{k+1}\}) \supseteq Q_{\Delta_k}(D)$ is dense in $Q_{\Delta_k}(K)$.
Now $\tilde{\Delta} := \bigcup_k \Delta_k$ is the required set. In order to show that $Q_{\tilde{\Delta}}(K) \subseteq K$ let $x \in K$ be arbitrary. Since $Q_{\Delta_k}(x)$ is contained in the closure of $Q_{\Delta_k}(\{x_k \in K \cap E : \operatorname{carr}(x_k) \subseteq \Delta_{k+1}\})$ and hence in the closed set $Q_{\Delta_k}(\{x_k \in K : \operatorname{carr}(x_k) \subseteq \tilde{\Gamma}\})$. Thus there is an $x_k \in K$ with $\operatorname{carr}(x_k) \subseteq \tilde{\Gamma}$ and such that x agrees with x_k on Δ_k. Thus $K \ni x_k \to Q_{\tilde{\Delta}}(x)$, since every finite subset of $\tilde{\Delta}$ is contained in some Δ_k and outside $\tilde{\Delta}$ all x_k and $Q_{\tilde{\Delta}}(x)$ are zero. Since K is closed we get $Q_{\tilde{\Delta}}(x) \in K$.

Without loss of generality we may assume that $\mu > \omega$. Let $\{x_\alpha : \omega \le \alpha < \mu\}$ be a dense subset of $K \cap E$. Let $\Gamma_\omega := \operatorname{carr}(x_\omega)$. By transfinite induction we define

$$\Gamma_\alpha := \begin{cases} (\Gamma_\beta \cup \operatorname{carr}(x_\beta))^\sim & \text{for } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} \Gamma_\beta & \text{for limit ordinals } \alpha. \end{cases}$$

Then the Γ_α satisfy all the requirements. □

53.18. Corollary. *Let K be Valdivia compact. Then $C(K)$ has a PRI.*

Proof. We choose Γ_α as in 53.17 and set $K_\alpha := Q_{\Gamma_\alpha}(K)$. Let $Q_\alpha := Q_{\Gamma_\alpha}|_K$. Then Q_α is a continuous retraction.

$$\begin{tikzcd} K_\alpha \arrow[rr, "\mathrm{id}_{K_\alpha}"] \arrow[dr, hook, "\mathrm{incl}_\alpha"'] & & K_\alpha \\ & K \arrow[ur, two heads, "Q_\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} C(K_\alpha) & & C(K_\alpha) \arrow[ll, "\mathrm{id}"'] \arrow[dl, hook, "Q_\alpha^*"] \arrow[d, dotted, two heads] \\ & C(K) \arrow[ul, two heads, "\mathrm{incl}_\alpha^*"] & E_\alpha \arrow[l, dotted, hook] \end{tikzcd}$$

We have $\operatorname{dens}(C(\mathbb{R}^{\Gamma_\alpha})) = |\alpha|$, since we have a base of the topology of this space of that cardinality. Hence $\operatorname{dens}(C(K_\alpha)) \le |\alpha|$. Let $E_\alpha := Q_\alpha^*(C(K_\alpha))$. Then E_α is a closed subspace of $C(K)$ and 53.13.3 holds. Furthermore $P_\alpha := Q_\alpha \circ \mathrm{incl}_\alpha^*$ is a norm-1 projection from $C(K)$ onto E_α. The inclusion $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha \le \beta$ implies 53.13.1. To see 53.13.5 let $\varepsilon > 0$ and choose a finite covering of K_α by sets

$$U_j := \{x \in \mathbb{R}^{\Gamma_\alpha} : |x_\gamma - x^j_\gamma| < \delta_j \text{ for all } \gamma \in \Delta_j\},$$

where $x^j \in \mathbb{R}^{\Gamma_\alpha}$, $\delta_j > 0$ and $\Delta_j \subseteq \Gamma_\alpha$ is finite and such that for $x', x'' \in U_j \cap K$ we have $|f(x') - f(x'')| < \varepsilon$. Now choose $\alpha_0 < \alpha$ such that $\Gamma_{\alpha_0} \supseteq \Delta_j$ for all of the finitely many j. Since the U_j cover K_α, we have $x \in K_\alpha \cap U_j$ for some j and hence $Q_\beta(x) \in K_\alpha \cap U_j$ for all $\alpha_0 \le \beta < \alpha$. Hence $|f(x) - f(Q_\beta(x))| < \varepsilon$ for all $x \in K_\alpha$ and so $\|P_\alpha(f) - P_\beta(f)\| = \|(1 - P_\beta)P_\alpha(f)\| \le \varepsilon$. Thus we have shown that E has a PRI $(P_\alpha)_\alpha$, with all $E_\alpha \cong C(K_\alpha)$ and $\operatorname{dens}(K_\alpha) \le |\Gamma_\alpha| \le \alpha$. □

53.15. Result. *WCD and duals of Asplund spaces have SPRI.*

A Banach space E is called *WCD, weakly countably determined*, if and only if there exists a sequence K_n of weak*-compact subsets of E'' such that for every

$$\forall x \in E\ \forall y \in E'' \setminus E\ \exists n : x \in K_n \text{ and } y \notin K_n.$$

Every WCG Banach space is WCD.

53.20. Theorem. [12, 3.16] *If E is a realcompact (i.e. non-measurable) Banach space admitting a SPRI, then there is a non-measurable set Γ and a injective continuous linear operator $T : E \to c_0(\Gamma)$.*

Proof. We proof by transfinite induction that for every ordinal α with $\alpha \le \mu :=$ dens(E) there is a non-measurable set Γ_α and an injective linear operator $T_\alpha : E_\alpha := P_\alpha(E) \to c_0(\Gamma_\alpha)$ with $\|T_\alpha\| \le 1$.

Note that if E is separable, then there are $x_n^* \in E'$ with $\|x_n^*\| \le 1$, and which are $\sigma(E', E)$ dense in the unit-ball of E'. Then $T : E \to c_0(\mathbb{N})$, defined by $T(x)_n := \frac{1}{n} x_n^*(x)$, satisfies the requirements: It is obviously a continuous linear mapping into c_0, and it remains to show that it is injective. So let $x \ne 0$. By Hahn-Banach there is an $x^* \in E'$ with $x^*(x) = \|x\|$ and $\|x^*\| \le 1$. Hence there is some n with $|(x_n^* - x^*)(x)| < \|x\|$ and hence $x_n^*(x) \ne 0$.
In particular we have $T_\omega : E_\omega \to c_0(\Gamma_\omega)$.

For successor ordinals $\alpha + 1$ we have $E_{\alpha+1} \cong E_\alpha \times (E_{\alpha+1}/E_\alpha) \cong E_\alpha \times R_\alpha(E)$, where $R_\alpha := (P_{\alpha+1} - P_\alpha)/\|P_{\alpha+1} - P_\alpha\|$. Let $T : R_\alpha(E) \to c_0$ be the continuous injection for the, by 53.13.7, separable space $R_\alpha(E)$ with $\|T\| \le 1$. Then we define $\Gamma_{\alpha+1} := \Gamma_\alpha \sqcup \mathbb{N}$ and $T_{\alpha+1} : E_{\alpha+1} \to c_0(\Gamma_{\alpha+1})$ by

$$T_{\alpha+1}(x)_\gamma := \begin{cases} T_\alpha(\frac{P_\alpha(x)}{\|P_\alpha\|})_\gamma & \text{for } \gamma \in \Gamma_\alpha \\ T(R_\alpha(x))_\gamma & \text{for } \gamma \in \mathbb{N} \end{cases}.$$

Now let α be a limit ordinal. We set $\Gamma_\alpha := \Gamma_\omega \sqcup \bigsqcup_{\omega \le \beta < \alpha} \Gamma_{\beta+1}$, and define $T_\alpha : E_\alpha := P_\alpha(E) \to c_0(\Gamma_\alpha)$ by

$$T_\alpha(x)_\gamma := \begin{cases} T_\omega(\frac{P_\omega(x)}{\|P_\omega\|}) & \text{for } \gamma \in \Gamma_\omega \\ T_{\beta+1}(R_\beta(x))_\gamma & \text{for } \gamma \in \Gamma_{\beta+1} \end{cases}$$

We show first that $T_\alpha(x) \in c_0(\Gamma_\alpha)$ for all $x \in E$. So let $\varepsilon > 0$. Then the set $\{\beta : \|R_\beta(x)\| \ge \varepsilon,\ \beta < \alpha\}$ is finite by 53.13.8.
Obviously T_α is linear and $\|T_\alpha\| \le 1$. It is also injective: In fact let $T_\alpha(x) = 0$ for some $x \in E_\alpha$. Then $R_\beta(x) = 0$ for all $\beta < \alpha$ and $P_\omega(x) = 0$, hence by $x = P_\alpha(x) = 0$.
As card(E) is non-measurable, also the smaller cardinal dens(E) is non-measurable. Thus the union Γ_α of non-measurable sets over a non-measurable index set is non-measurable. □

53.21. Corollary. *The WCD Banach spaces and the duals of Asplund spaces continuously and linearly inject into some $c_0(\Gamma)$. The same is true for $C(K)$, where K is Valdivia compact.*

For WCG spaces this is due to [3] and for $C(K)$ with K Valdivia compact it is due to [Argyros, Mercourakis, Negrepontis, 1988.]

Proof. For WCD and duals of Asplund spaces this follows using 53.15. For Valdivia compact spaces K one proceeds by induction on dens(K) and uses the PRI constructed in 53.18. The continuous linear injection $C(K) \to c_0(\Gamma)$ is then given as in 53.20 for $\alpha := \operatorname{dens}(K)$, where T_β exists for $\beta < \alpha$, since $E_\beta \cong C(K_\beta)$ with K_β Valdivia compact and $\operatorname{dens}(K_\beta) \le \beta < \alpha$. □

53.22. Theorem. [9] *Let $T : E \to F$ be a bounded linear surjective mapping between Banach spaces. Then there exists a continuous mapping $S : F \to E$ with $T \circ S = \mathrm{id}$.*

Proof. By the open mapping theorem there is a constant $M_0 > 0$ such that for all $\|y\| \le 1$ there exists an $x \in T^{-1}(y)$ with $\|x\| \le M_0$. In fact there is an M_0 such that $B_{1/M_0} \subseteq T(B_1)$ or equivalently $B_1 \subseteq T(B_{M_0})$. Let $(f_\gamma)_{\gamma\in\Gamma}$ be a continuous partition of unity on $oF := \{y \in F : \|y\| \le 1\}$ with $\operatorname{diam}(\operatorname{supp}(f_\gamma)) \le 1/2$. Choose $x_\gamma \in T^{-1}(\operatorname{carr}(f_\gamma))$ with $\|x_\gamma\| \le M_0$ and for $\|y\| \le 1$ set

$$S_0 y := \sum_{\gamma\in\Gamma} f_\gamma(y) x_\gamma \quad \text{and recursively}$$
$$S_{n+1} y := S_n y + \tfrac{1}{a_n} S_n(a_n(y - T S_n y)),$$

where $a_n := 2^{2^n}$.

By induction we show that the continuous mappings $S_n : \{y : \|y\| \le 1\} \to E$ satisfy $\|y - TS_n y\| \le 1/a_n$ and $\|S_n y\| \le M_n := M_0 \cdot \prod_{k=0}^{n-1}(1 + 1/a_k)$.

($n = 0$) Obviously $\|S_0 y\| \le \sum_\gamma f_\gamma(y)\|x_\gamma\| \le M_0$ and

$$\|y - TS_0 y\| = \Big\|\sum_\gamma f_\gamma(y)(y - Tx_\gamma)\Big\| \le \sum_{\gamma\in\Gamma_y} f_\gamma(y)\,\|y - Tx_\gamma\| \le \frac{1}{2} = \frac{1}{a_0},$$

where $\Gamma_y := \operatorname{carr} f_\gamma$.

($n + 1$) For $\|y\| \le 1$ and $y_n := a_n(y - TS_n y)$ we have $\|y_n\| \le 1$ by induction hypothesis. Then

$$\|S_{n+1} y\| \le \|S_n y\| + \tfrac{1}{a_n}\|S_n y_n\| \le M_n + \tfrac{1}{a_n} M_n = M_{n+1}.$$

Furthermore

$$\begin{aligned}\|y - TS_{n+1} y\| &= \big\|y - TS_n y - \tfrac{1}{a_n} TS_n(a_n(y - TS_n y))\big\|\\ &\le \tfrac{1}{a_n}\|y_n - TS_n y_n\| \le \tfrac{1}{a_n^2} = \tfrac{1}{a_{n+1}}.\end{aligned}$$

Now (S_n) is Cauchy with respect to uniform convergence on $\{y : \|y\| \le 1\}$: In fact

$$\|S_{n+1} y - S_n y\| \le \tfrac{1}{a_n}\|S_n(a_n(y - TS_n y))\| \le \tfrac{M_n}{a_n} \le \tfrac{M_\infty}{a_n},$$

where $M_\infty := \lim_n M_n$. Thus $S := \lim_n S_n$ is continuous and $\|y - TSy\| = \lim_n \|y - TS_n y\| = 0$, i.e. $TSy = y$. Now $S : F \to E$ defined by $S(y) := \|y\| S(\frac{y}{\|y\|})$ and $S(0) := 0$ is the claimed continuous section. □

16.18. Corollary. [43]
Let E be a Banach space with a separable projective resolution of identity, see 53.13. If E is C^k-regular, then it is C^k-paracompact.

Proof. By 53.20 there exists a linear, injective, norm 1 operator $T : E \to c_0(\Gamma_1)$ for some Γ_1 and by 53.13 projections P_α for $\omega \le \alpha \le \operatorname{dens} E$. Let $\Gamma_2 := \{\Delta : \Delta \subseteq [\omega, \operatorname{dens} E),\ \text{finite}\}$. For $\Delta \in \Gamma_2$ choose a dense sequence $(x_n^\Delta)_n$ in the unit sphere of $P_\omega(E) \oplus \bigoplus_{\alpha\in\Delta}(P_{\alpha+1} - P_\alpha)(E)$ and let $y_n^\Delta \in E'$ be such that $\|y_n^\Delta\| = 1$ and $y_n^\Delta(x_n^\Delta) = 1$. For $n \in \mathbb{N}$ let $\pi_n^\Delta : x \mapsto x - y_n^\Delta(x) x_n^\Delta$. Choose a smooth function $h \in C^\infty(E, [0, 1])$ with $h(x) = 0$ for $\|x\| \le 1$ and $h(x) = 1$ for $\|x\| \ge 2$. Let $R_\alpha := (P_{\alpha+1} - P_\alpha)/\|P_{\alpha+1} - P_\alpha\|$.

Now define an embedding as follows: Let $\Gamma := \mathbb{N}^3 \times \Gamma_2 \sqcup \mathbb{N} \times [\omega, \operatorname{dens} E) \sqcup \mathbb{N} \sqcup \Gamma_1$ and let $u : E \to c_0(\Gamma)$ be given by

$$u(x)_\gamma := \begin{cases} \frac{1}{2^{n+m+l}}\, h(m\pi_n^\Delta x) \prod_{\alpha\in\Delta} h(lR_\alpha x) & \text{for } \gamma = (m,n,l,\Delta) \in \mathbb{N}^3 \times \Gamma_2, \\ \frac{1}{2^m}\, h(mR_\alpha x) & \text{for } \gamma = (m,\alpha) \in \mathbb{N} \times [\omega, \operatorname{dens} E), \\ \frac{1}{2}\, h(\frac{x}{m}) & \text{for } \gamma = m \in \mathbb{N}, \\ T(x)_\alpha & \text{for } \gamma = \alpha \in \Gamma_1. \end{cases}$$

Let us first show that u is well-defined and continuous. We do this only for the coordinates in the first row (for the others it is easier, the third has locally even finite support).
Let $x_0 \in E$ and $0 < \varepsilon < 1$. Choose n_0 with $1/2^{n_0} < \varepsilon$. Then $|u(x)_\gamma| < \varepsilon$ for all $x \in X$ and all $\alpha = (m,n,l,\Delta)$ with $m+n+l \geq n_0$.
For the remaining coordinates we proceed as follows: We first choose $\delta < 1/n_0$. By 53.13.8 there is a finite set $\Delta_0 \in \Gamma_2$ such that $\|R_\alpha x_0\| < \delta/2$ for all $\alpha \notin \Delta_0$. For those α and $\|x - x_0\| < \delta/2$ we get

$$\|R_\alpha(x)\| \leq \|R_\alpha(x_0)\| + \|R_\alpha(x-x_0)\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

hence $u(x)_\gamma = 0$ for all $\gamma = (m,n,l,\Delta)$ with $m+n+l < n_0$ and $\Delta \cap ([\omega, \operatorname{dens} E) \setminus \Delta_0) \neq \emptyset$, i.e. $\Delta \not\subseteq \Delta_0$.

For the remaining finitely many coordinates $\gamma = (m,n,l,\Delta)$ with $m+n+l < n_0$ and $\Delta \subseteq \Delta_0$ we may choose a $\delta_1 > 0$ such that $|u(x)_\gamma - u(x_0)_\gamma| < \varepsilon$ for all $\|x - x_0\| < \delta_1$. Thus for $\|x - x_0\| < \min\{\delta/2, \delta_1\}$ we have $|u(x)_\gamma - u(x_0)_\gamma| < 2\varepsilon$ for all $\gamma \in \mathbb{N}^3 \times \Gamma_2$ and $|u(x_0)_\gamma| \geq \varepsilon$ only for $\alpha = (m,n,l,\Delta)$ with $m+n+l < n_0$ and $\Delta \subseteq \Delta_0$.

Since T is injective, so is u. In order to show that u is an embedding let $x_\infty, x_p \in E$ with $u(x_p) \to u(x_\infty)$. Then x_p is bounded, since for $n_0 > \|x_\infty\|$ implies that $h(x_\infty/n_0) = 0$ and from $h(x_p/n_0) \to h(x_\infty/n_0)$ we conclude that $\|x_p/n_0\| \leq 2$ for large p.

Now we show that for any $\varepsilon > 0$ there is a finite ε-net for $\{x_p : p \in \mathbb{N}\}$: For this we choose $m_0 > 2/\varepsilon$. By 53.13.8 there is a finite set Δ_0 and an $n_0 := n \in \mathbb{N}$ such that $\|m_0 \pi_n^{\Delta_0}(x_\infty)\| \leq 1$ and hence $h(m_0\pi_n^{\Delta_0}(x_\infty)) = 0$: In fact by 53.13.9 there is a finite set Δ_0 of α and a linear combination of vectors $R_\alpha(x_\infty)$ with $\alpha \in \Delta_0$, which has distance less than ε from x_∞, let $\delta_0 := \min\{\|R_\alpha(x)\| : \text{ for those } \alpha\} > 0$. Since the $x_n^{\Delta_0}$ are dense in the unit sphere of $P_\omega E \oplus \bigoplus_{\alpha\in\Delta_0} R_\alpha E$ we may choose an n such that $\|x_\infty - \|x_\infty\| x_n^{\Delta_0}\| < \frac{1}{2m_0}$ and hence

$$\begin{aligned} \|\pi_n^{\Delta_0}(x_\infty)\| &= \|x_\infty - y_n^{\Delta_0}(x_\infty) x_n^{\Delta_0}\| \\ &\leq \left\| x_\infty - \|x_\infty\| x_n^{\Delta_0} \right\| + \|x_\infty\| \left\| x_n^{\Delta_0} - y_n^{\Delta_0}(x_n^{\Delta_0}) x_n^{\Delta_0} \right\| \\ &\qquad + \|y_n^{\Delta_0}\| \left\| \|x_\infty\| x_n^{\Delta_0} - x_\infty) \right\| \|x_n^{\Delta_0}\| \\ &\leq \frac{1}{2m_0} + 0 + \frac{1}{2m_0} = \frac{1}{m_0} \end{aligned}$$

Next choose $l_0 := l \in \mathbb{N}$ such that $l_0\delta_0 \geq 2$ and hence $\|l_0 R_\alpha x_\infty\| \geq l_0\delta_0 \geq 2$ for all $\alpha \in \Delta_0$. Then

$$h(m_0\pi_{n_0}^{\Delta_0} x_p) \prod_{\alpha\in\Delta_0} h(l_0\, R_\alpha x_p) \to h(m_0\pi_{n_0}^{\Delta_0} x_\infty) \prod_{\alpha\in\Delta_0} h(l_0\, R_\alpha x_\infty)$$
$$\text{and} \qquad h(l_0 R_\alpha x_p) \to h(l_0 R_\alpha x_\infty) = 1 \text{ for } \alpha \in \Delta_0$$

Hence

$$h(m_0\pi^{\Delta_0}_{n_0}x_p) \to h(m_0\pi^{\Delta_0}_{n_0}x_\infty) = 0,$$

and so $\|\pi^{\Delta_0}_{n_0}x_p\| \le 2/m_0 < \varepsilon$ for all large p. Thus $d(x_p, \mathbb{R}\,x^{\Delta_0}_{n_0}) \le \varepsilon$, hence $\{x_p : p \in \mathbb{N}\}$ has a finite ε-net, since its projection onto the one dimensional subspace $\mathbb{R}x^{\Delta_0}_{n_0}$ is bounded.

Thus $\{x_\infty, x_p : p \in \mathbb{N}\}$ is relatively compact, and hence u restricted to its closure is a homeomorphism onto the image. So $x_p \to x_\infty$.

Now the result follows from 16.15. □

16.19. Corollary. [28]. *Let $c_0(\Gamma) \to E \to F$ be a short exact sequence of Banach spaces and assume F admits C^p-partitions of unity. Then E admits C^p-partitions of unity.*

Proof. Without loss of generality we may assume that the norm of E restricted to $c_0(\Gamma)$ is the supremum norm. Furthermore there is a linear continuous splitting $T : \ell^1(\Gamma) \to E'$ by 13.17.3 and a continuous splitting $S : F \to E$ by 53.22 with $S(0) = 0$. We put $T_\gamma := T(e_\gamma)$ for all $\gamma \in \Gamma$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be a C^p-partition of unity on F with $\operatorname{diam}(\operatorname{carr}(f)) \le 1/n$ for all $f \in \mathcal{F}_n$. Let $\mathcal{F} := \bigsqcup_n \mathcal{F}_n$ and let $\Gamma_2 := \{\Delta \subseteq \Gamma : \Delta \text{ is finite}\}$. For any $f \in \mathcal{F}$ choose $x_f \in S(\operatorname{carr}(f))$ and for any $\Delta \in \Gamma_2$ choose a dense sequence $\{y^\Delta_{f,m} : m \in \mathbb{N}\} \ni 0$ in the linear subspace generated by $\{x_f + e_\gamma : \gamma \in \Delta\}$. Let $\ell^\Delta_{f,m} \in E'$ be such that $\ell^\Delta_{f,m}(y^\Delta_{f,m}) = \|\ell^\Delta_{f,m}\| \cdot \|y^\Delta_{f,m}\| = 1$. Let $\pi^\Delta_{f,m} : E \to E$ be given by $\pi^\Delta_{f,m}(x) := x - \ell^\Delta_{f,m}(x)\, y^\Delta_{f,m}$. Let $h : E \to \mathbb{R}$ be C^p with $h(x) = 0$ for $\|x\| \le 1$ and $h(x) = 1$ for $\|x\| \ge 2$. Let $g : \mathbb{R} \to [-1, 1]$ be C^p with $g(t) = 0$ for $|t| \le 1$ and injective on $\{t : |t| > 1\}$. Now define a mapping $u : E \to c_0(\tilde\Gamma)$, where

$$\tilde\Gamma := (\mathcal{F} \times \Gamma_2 \times \mathbb{N}^2) \sqcup (\mathcal{F} \times \Gamma) \sqcup (\mathcal{F} \times \mathbb{N}) \sqcup \sqcup \mathbb{N} \sqcup \mathbb{N}$$

by

$$u(x)_{\tilde\gamma} := \frac{1}{2^{n+m+j}}\, f(\hat x)\, h(j\, \pi^\Delta_{f,m}(x)) \prod_{\gamma\in\Delta} g(n\, T_\gamma(x - x_f))$$

for $\tilde\gamma = (f, \Delta, j, m) \in \mathcal{F}_n \times \Gamma_2 \times \mathbb{N}^2$, and by

$$u(x)_{\tilde\gamma} := \begin{cases} \frac{1}{2^n}\, f(\hat x)\, g(n\, T_\gamma(x - x_f)) & \text{for } \tilde\gamma = (f, \gamma) \in \mathcal{F}_n \times \Gamma \\ \frac{1}{2^{n+j}}\, f(\hat x)\, h(j\,(x - x_f)) & \text{for } \tilde\gamma = (f, j) \in \mathcal{F}_n \times \mathbb{N} \\ \frac{1}{2^n}\, f(\hat x) & \text{for } \tilde\gamma = f \in \mathcal{F}_n \subseteq \mathcal{F} \\ \frac{1}{2^n}\, h(n\, x) & \text{for } \tilde\gamma = n \in \mathbb{N} \\ \frac{1}{2^n}\, h(x/n) & \text{for } \tilde\gamma = n \in \mathbb{N}. \end{cases}$$

We first claim that u is well-defined and continuous. Every coordinate $x \mapsto u(x)_\gamma$ is continuous, so it remains to show that for every $\varepsilon > 0$ locally in x the set of coordinates γ, where $|u(x)_\gamma| > \varepsilon$ is finite. We do this for the first type of coordinates. For this we may fix n, m and j (since the factors are bounded by 1). Since $\mathcal{F}_n$ is a partition of unity, locally $f(\hat x) \ne 0$ for only finitely many $f \in \mathcal{F}_n$, so we may also fix $f \in \mathcal{F}_n$. For such an f the set $\Delta_0 := \{\gamma : |T_\gamma(x - x_f)| \ge \pi(x - x_f) + \frac{1}{n}\}$ is finite by the proof of 13.17.3. Since $\|\hat x - x_f\| = \|\pi(x - x_f)\| \le 1/n$ be have $g(n\, T_\gamma(x - x_f)) = 0$ for $\gamma \notin \Delta_0$.

Thus only for those Δ contained in the finite set Δ_0, we have that the corresponding coordinate does not vanish.

Next we show that u is injective. Let $x \neq y \in E$.
If $\hat{x} \neq \hat{y}$, then there is some n and a $f \in \mathcal{F}_n$ such that $f(\hat{x}) \neq 0 = f(\hat{y})$. Thus this is detected by the 4th row.
If $\hat{x} = \hat{y}$ then $S\hat{x} = S\hat{y}$ and since $x - S\hat{x}$, $y - S\hat{y} \in c_0(\Gamma)$ there is a $\gamma \in \Gamma$ with

$$T_\gamma(x - S\hat{x}) = (x - S\hat{x})_\gamma \neq (y - S\hat{y})_\gamma = T_\gamma(y - S\hat{y}).$$

We will make use of the following method repeatedly:
For every n there is a $f_n \in \mathcal{F}_n$ with $f_n(\hat{x}) \neq 0$ and hence $\|\hat{x} - \hat{x}_{f_n}\| \leq 1/n$. Since S is continuous we get $x_{f_n} = S(\hat{x}_{f_n}) \to S(\hat{x})$ and thus $\lim_n T_\gamma(x - x_{f_n}) = \lim_n T_\gamma(x - S(\hat{x}_{f_n})) = T_\gamma(x - S(\hat{x}))$.
So we get

$$\lim_n T_\gamma(x - x_{f_n}) = T_\gamma(x - S(\hat{x})) \neq T_\gamma(y - S(\hat{y})) = \lim_n T_\gamma(y - x_{f_n}).$$

If all coordinates for $u(x)$ and $u(y)$ in the second and fourth row would be equal, then

$$g(n\,T_\gamma(x - x_f)) = g(n\,T_\gamma(y - x_f))$$

since $f_n(\hat{x}) \neq 0$ for some n, and hence $\|T_\gamma(x - x_f) - T_\gamma(y - x_f)\| \leq 2/n$, a contradiction.

Now let us show that u is a homeomorphism onto its image. We have to show $x_k \to x$ provided $u(x_k) \to u(x)$.
We consider first the case, where $x = S\hat{x}$. As before we choose $f_n \in \mathcal{F}_n$ with $f_n(\hat{x}) \neq 0$ and get $x_{f_n} = S(\hat{x}_{f_n}) \to S(\hat{x}) = x$. Let $\varepsilon > 0$ and $j > 3/\varepsilon$. Choose an n such that $\|x_{f_n} - x\| < 1/j$. Then $h(j\,(x_{f_n} - x)) = 0$. From the coordinates in the third and fourth row we conclude

$$f(\hat{x}_k)\,h(j\,(x_k - x_{f_n})) \to f(\hat{x})\,h(j\,(x - x_{f_n})) \quad \text{and} \quad f(\hat{x}_k) \to f(\hat{x}) \neq 0.$$

Hence

$$h(j\,(x_k - x_{f_n})) \to h(j\,(x - x_{f_n})) = 0.$$

Thus $\|x_k - x_{f_n}\| < 2/j$ for all large k. But then

$$\|x_k - x\| \leq \|x_k - x_{f_n}\| + \|x_{f_n} - x\| < \frac{3}{j} < \varepsilon,$$

i.e. $x_k \to x$.
Now the case, where $x \neq S\hat{x}$. We show first that $\{x_k : k \in \mathbb{N}\}$ is bounded. Pick $n > \|x\|$. From the coordinates in the last row we get that $\lim_k h(x_k/n) = 0$, i.e. $\|x_k\| \leq 2n$ for all large k.
We claim that for $j \in \mathbb{N}$ there is an $n \in \mathbb{N}$ and an $f \in \mathcal{F}_n$ with $f(\hat{x}) \neq 0$, a finite set $\Delta \subseteq \Gamma$ with $\prod_{\gamma \in \Delta} g(n\,T_\gamma(x - x_f)) \neq 0$ and an $m \in \mathbb{N}$ with $h(j\,\pi^\Delta_{f,m}(x)) = 0$.
From $0 \neq (x - S\hat{x}) \in c_0(\Gamma)$ we deduce that there is a finite set $\Delta \subseteq \Gamma$ with $T_\gamma(x - S\hat{x}) = (x - S\hat{x})_\gamma \neq 0$ for all $\gamma \in \Delta$ and $\operatorname{dist}(x - S\hat{x}, \langle e_\gamma : \gamma \in \Delta\rangle) < 1/(3j)$, i.e. $|(x - S\hat{x})_\gamma| \leq 1/(3j)$ for all $\gamma \notin \Delta$. As before we choose $f_n \in \mathcal{F}_n$ with $f_n(\hat{x}) \neq 0$ and get $x_{f_n} = S(\hat{x}_{f_n}) \to S(\hat{x})$ and

$$\lim_n T_\gamma(x - x_{f_n}) = (x - S\hat{x})_\gamma \neq 0 \text{ for } \gamma \in \Delta.$$

Thus $g(n\,(T_\gamma(x - x_{f_n}))) \neq 0$ for all large n and $\gamma \in \Delta$. Furthermore, $\operatorname{dist}(x, x_{f_n} + \langle e_\gamma : \gamma \in \Delta\rangle) = \operatorname{dist}(x - x_{f_n}, \langle e_\gamma : \gamma \in \Delta\rangle) < 1/(2j)$. Since $\{y^\Delta_{f_n,m} : m \in \mathbb{N}\}$ is dense in $\langle x_{f_n} + e_\gamma : \gamma \in \Delta\rangle$ there is an m such that $\|x - y^\Delta_{f_n,m}\| < 1/(2j)$. Since $\|\pi^\Delta_{f_n,m}\| \leq 2$ we get

$$\begin{aligned}\|\pi^\Delta_{f_n,m}(x)\| &\leq \|x - y^\Delta_{f_n,m}\| + |1 - \ell^\Delta_{f_n,m}(x)|\,\|y^\Delta_{f_n,m}\| \\ &\leq \frac{1}{2j} + \|\ell^\Delta_{f_n,m}\|\,\|x - y^\Delta_{f_n,m}\|\,\|y^\Delta_{f_n,m}\| \leq \frac{1}{2j} + \frac{1}{2j} = \frac{1}{j},\end{aligned}$$

hence $h(j\,\pi^{\Delta}_{f_n,m}(x)) = 0$.

We claim that for every $\varepsilon > 0$ there is a finite ε-net of $\{x_k : k \in \mathbb{N}\}$. Let $\varepsilon > 0$. We choose $j > 4/\varepsilon$ and we pick $n \in \mathbb{N}$, $f \in \mathcal{F}_n$, $\Delta \subseteq \Gamma$ finite, and $m \in \mathbb{N}$ satisfying the previous claim. From $u(x_k) \to u(x)$ we deduce from the coordinates in the first row, that

$$f(\hat{x}_k)\,h(j\,\pi^{\Delta}_{f,m}(x_k)) \prod_{\gamma\in\Delta} g(n\,T_\gamma(x_k - x_f)) \to$$
$$\to f(\hat{x})\,h(j\,\pi^{\Delta}_{f,m}(x)) \prod_{\gamma\in\Delta} g(n\,T_\gamma(x - x_f)) \text{ for } k \to \infty$$

and since by the coordinates in the fourth row $f(\hat{x}_k) \to f(\hat{x}) \neq 0$ we obtain from the coordinates in the second row, that

$$g(n\,T_\gamma(x_k - x_f)) \to g(n\,T_\gamma(x - x_f)) \neq 0 \text{ for } \gamma \in \Delta.$$

Hence

$$h(j\,\pi^{\Delta}_{f,m}(x_k)) \to h(j\,\pi^{\Delta}_{f,m}(x)) = 0.$$

Therefore

$$\|x_k - \ell^{\Delta}_{f,m}(x_k)\,y^{\Delta}_{f,n}\| = \|\pi^{\Delta}_{f,m}(x_k)\| < \frac{1}{j} < \frac{\varepsilon}{4} \text{ for all large } k.$$

Thus there is a finite dimensional subspace in E spanned by $y^{\Delta}_{f,n}$ and finitely many x_k, such that all x_k have distance $\le \varepsilon/4$ from it. Since $\{x_k : k \in \mathbb{N}\}$ are bounded, the compactness of the finite dimensional balls implies that $\{x_k : k \in \mathbb{N}\}$ has an ε-net, hence $\{x_k : k \in \mathbb{N}\}$ is relatively compact, and since u is injective we have $\lim_k x_k = x$.

Now the result follows from 16.15. □

Remark. In general, the existence of C^∞-partitions of unity is not inherited by the middle term of short exact sequences: Take a short exact sequence of Banach spaces with Hilbert ends and non-Hilbertizable E in the middle, as in 13.18.6. If both E and E^* admitted C^2-partitions of unity, then they would admit C^2-bump functions, hence E was isomorphic to a Hilbert space by [94], a contradiction.

16.20. Results on $C(K)$. *Let K be compact. Then for the Banach space $C(K)$ we have:*

(1) [28]. *If $K^{(\omega)} = \emptyset$ then $C(K)$ is C^∞-paracompact.*

(2) [129] *If $K^{(\omega_1)} = \emptyset$ then $C(K)$ is C^1-paracompact.*

(3) [49] *In contrast to (2) there exists a compact space K with $K^{(\omega_1)} = \{*\}$, but such that $C(K)$ has no Gâteaux-differentiable norm. Nevertheless $C(K)$ is C^1-regular by* [50]. *Compare with 13.18.2.*

(4) [101]. *If there exists an ordinal number α with $K^{(\alpha)} = \emptyset$ then the Banach space $C(K)$ is Asplund (and conversely), hence it does not admit a rough norm, by 13.8.*

(5) [22] *There exists a compact K with $K^{(3)} = \emptyset$. Consequently, there is a short exact sequence $c_0(\Gamma_1) \to C(K) \to c_0(\Gamma_2)$, and the space $C(K)$ is Lipschitz homeomorphic to some $c_0(\Gamma)$. However, there is no continuous linear injection of $C(K)$ into some $c_0(\Gamma)$.*

Notes. (1) Applying theorem 16.19 recursively we get the result as in 13.17.5.

16.21. Some radial subsets are diffeomorphic to the whole space

We are now going to show that certain subsets of convenient vector spaces are diffeomorphic to the whole space. So if these subsets form a base of the c^∞-topology of the modeling space of a manifold, then we may choose charts defined on the whole modeling space. The basic idea is to 'blow up' subsets $U \subseteq E$ along all rays starting at a common center. Without loss of generality assume that the center is 0. In order for this technique to work, we need a positive function $\rho : U \to \mathbb{R}$, which should give a diffeomorphism $f : U \to E$, defined by $f(x) := \frac{1}{\rho(x)}x$. For this we need that ρ is smooth, and since the restriction of f to $U \cap \mathbb{R}^+x \to \mathbb{R}^+x$ has to be a diffeomorphism as well, and since the image set is connected, we need that the domain $U \cap \mathbb{R}^+x$ is connected as well, i.e., U has to be radial. Let $U_x := \{t > 0 : tx \in U\}$, and let $f_x : U_x \to \mathbb{R}$ be given by $f(tx) = \frac{t}{\rho(tx)}x =: f_x(t)x$. Since up to diffeomorphisms this is just the restriction of the diffeomorphism f, we need that $0 < f_x'(t) = \frac{\partial}{\partial t}\frac{t}{\rho(tx)} = \frac{\rho(tx) - t\rho'(tx)(x)}{\rho(tx)^2}$ for all $x \in U$ and $0 < t \le 1$. This means that $\rho(y) > \rho'(y)(y)$ for all $y \in U$, which is quite a restrictive condition, and so we want to construct out of an arbitrary smooth function $\rho : U \to \mathbb{R}$, which tends to 0 towards the boundary, a new smooth function ρ satisfying the additional assumption.

Theorem. *Let $U \subseteq E$ be c^∞-open with $0 \in U$ and let $\rho : U \to \mathbb{R}^+$ be smooth, such that for all $x \notin U$ with $tx \in U$ for $0 \le t < 1$ we have $\rho(tx) \to 0$ for $t \nearrow 1$. Then $\operatorname{star} U := \{x \in U : tx \in U \text{ for all } t \in [0,1]\}$ is diffeomorphic to E.*

Proof. By 4.17 $\operatorname{star} U$ is c^∞-open. Note that ρ satisfies on $\operatorname{star} U$ the same boundary condition as on U. So we may assume without loss of generality that U is radial. Furthermore, we may assume that $\rho = 1$ locally around 0 and $0 < \rho \le 1$ everywhere, by composing with some function which is constantly 1 locally around $[\rho(0), +\infty)$.

Now we are going to replace ρ by a new function $\tilde\rho$, and we consider first the case, where $E = \mathbb{R}$. We want that $\tilde\rho$ satisfies $\tilde\rho'(t)t < \tilde\rho(t)$ (which says that the tangent to $\tilde\rho$ at t intersects the $\tilde\rho$-axis in the positive part) and that $\tilde\rho(t) \le \rho(t)$, i.e., $\log \circ \tilde\rho \le \log \circ \rho$, and since we will choose $\tilde\rho(0) = 1 = \rho(0)$ it is sufficient to have $\frac{\tilde\rho'}{\tilde\rho} = (\log \circ \tilde\rho)' \le (\log \circ \rho)' = \frac{\rho'}{\rho}$ or equivalently $\frac{\tilde\rho'(t)t}{\tilde\rho(t)} \le \frac{\rho'(t)t}{\rho(t)}$ for $t > 0$. In order to obtain this we choose a smooth function $h : \mathbb{R} \to \mathbb{R}$ which satisfies $h(t) < 1$, and $h(t) \le t$ for all t, and $h(t) = t$ for t near 0, and we take $\tilde\rho$ as solution of the following ordinary differential equation

$$\tilde\rho'(t) = \frac{\tilde\rho(t)}{t} \cdot h\left(\frac{\rho'(t)t}{\rho(t)}\right) \text{ with } \tilde\rho(0) = 1.$$

Note that for t near 0, we have $\frac{1}{t}h\left(\frac{\rho'(t)t}{\rho(t)}\right) = \frac{\rho'(t)}{\rho(t)}$, and hence locally a unique smooth solution $\tilde\rho$ exists. In fact, we can solve the equation explicitly, since $(\log \circ \tilde\rho)'(t) = \frac{\tilde\rho'(t)}{\tilde\rho(t)} = \frac{1}{t} \cdot h\left(\frac{\rho'(t)t}{\rho(t)}\right)$, and hence $\tilde\rho(s) = \exp(\int_0^s \frac{1}{t} \cdot h(\frac{\rho'(t)t}{\rho(t)})\,dt)$, which is smooth on the same interval as ρ is.

Note that if ρ is replaced by $\rho_s : t \mapsto \rho(ts)$, then the corresponding solution $\widetilde{\rho_s}$ satisfies $\widetilde{\rho_s} = \tilde\rho_s$. In fact,

$$(\log \circ \tilde\rho_s)'(t) = \frac{(\tilde\rho_s)'(t)}{\tilde\rho_s(t)} = \frac{s\tilde\rho'(st)}{\tilde\rho(st)} = \frac{1}{t} \cdot \frac{st\tilde\rho'(st)}{\tilde\rho(st)} = \frac{1}{t}h\Big(\frac{st\rho'(st)}{\rho(st)}\Big) = \frac{1}{t}h\Big(\frac{t(\rho_s)'(t)}{\rho_s(t)}\Big).$$

For arbitrary E and $x \in E$ let $\rho_x : U_x \to \mathbb{R}^+$ be given by $\rho_x(t) := \rho(tx)$, and let $\tilde\rho : U \to \mathbb{R}^+$ be given by $\tilde\rho(x) := \widetilde{\rho_x}(1)$, where $\widetilde{\rho_x}$ is the solution of the differential equation above with ρ_x in place of ρ.

Let us now show that $\tilde\rho$ is smooth. Since U is c^∞-open, it is enough to consider a smooth curve $x : \mathbb{R} \to U$ and show that $t \mapsto \tilde\rho(x(t)) = \tilde\rho_{(x(t))}(1)$ is smooth. This is the case, since $(t,s) \mapsto \frac{1}{s} h\left(\frac{\rho'_{x(t)}(s)s}{\rho_{x(t)}(s)}\right) = \frac{1}{s} h\left(\frac{\rho'(s\,x(t))(s\,x(t))}{\rho(s\,x(t))}\right)$ is smooth, since $\varphi(t,s) := \frac{\rho'(s\,x(t))(s\,x(t))}{\rho(s\,x(t))}$ satisfies $\varphi(t,0) = 0$, and hence $\frac{1}{s}h(\varphi(t,s)) = \frac{\varphi(t,s)}{s} = \frac{\rho'(s\,x(t))(x(t))}{\rho(s\,x(t))}$ locally.

From $\rho_{sx}(t) = \rho(tsx) = \rho_x(ts)$ we conclude that $\widetilde{\rho_{sx}}(t) = \widetilde{\rho_x}(ts)$, and hence $\tilde\rho(sx) = \widetilde{\rho_x}(s)$. Thus, $\tilde\rho'(x)(x) = \frac{\partial}{\partial t}|_{t=1}\tilde\rho(tx) = \frac{\partial}{\partial t}|_{t=1}\tilde\rho_x(t) = \tilde\rho_x'(1) < \tilde\rho_x(1) = \tilde\rho(x)$. This shows that we may assume without loss of generality that $\rho : U \to (0,1]$ satisfies the additional assumption $\rho'(x)(x) < \rho(x)$.

Note that $f_x : t \mapsto \frac{t}{\rho(tx)}$ is bijective from $U_x := \{t > 0 : tx \in U\}$ to $\mathbb{R}^+$, since 0 is mapped to 0, the derivative is positive, and $\frac{t}{\rho(tx)} \to \infty$ if either $\rho(tx) \to 0$ or $t \to \infty$ since $\rho(tx) \le 1$.

It remains to show that the bijection $x \mapsto \frac{1}{\rho(x)}x$ is a diffeomorphism. Obviously, its inverse is of the form $y \mapsto \sigma(y)y$ for some $\sigma : E \to \mathbb{R}^+$. They are inverse to each other so $\frac{1}{\rho(\sigma(y)y)}\sigma(y)y = y$, i.e., $\sigma(y) = \rho(\sigma(y)y)$ for $y \neq 0$. This is an implicit equation for σ. Note that $\sigma(y) = 1$ for y near 0, since ρ has this property. In order to show smoothness, let $t \mapsto y(t)$ be a smooth curve in E. Then it suffices to show that the implicit equation $(\sigma \circ y)(t) = \rho((\sigma \circ y)(t) \cdot y(t))$ satisfies the assumptions of the 2-dimensional implicit function theorem, i.e., $0 \neq \frac{\partial}{\partial\sigma}(\sigma - \rho(\sigma \cdot y(t))) = 1 - \rho'(\sigma \cdot y(t))(y(t))$, which is true, since multiplied with $\sigma > 0$ it equals $\sigma - \rho'(\sigma \cdot y(t))(\sigma \cdot y(t)) > \sigma - \rho(\sigma \cdot y(t)) = 0$. □

Chapter VI
Infinite Dimensional Manifolds

This chapter is devoted to the foundations of infinite dimensional manifolds. We treat here only manifolds described by charts onto c^∞-open subsets of convenient vector spaces.

Note that this limits cartesian closedness of the category of manifolds. For finite dimensional manifolds M, N, P we will show later that $C^\infty(N, P)$ is not locally contractible (not even locally pathwise connected) for the compact-open C^∞-topology if N is not compact, so one has to use a finer structure to make it a manifold $\mathfrak{C}^\infty(N, P)$, see 42.1. But then $C^\infty(M, \mathfrak{C}^\infty(N, P)) \cong C^\infty(M \times N, P)$ if and only if N is compact see 42.14. Unfortunately, $\mathfrak{C}^\infty(N, P)$ cannot be generalized to infinite dimensional N, since this structure becomes discrete. Let us mention, however, that there exists a theory of manifolds and vector bundles, where the structure of charts is replaced by the set of smooth curves supplemented by other requirements, where one gets a cartesian closed category of manifolds and has the compact-open C^∞-topology on $C^\infty(N, P)$ for finite dimensional N, P, see [**115**], [**73**], [**99**].

We start by treating the basic concept of manifolds, existence of smooth bump functions and smooth partitions of unity. Then we investigate tangent vectors seen as derivations or kinematically (via curves): these concepts differ, and we show in 28.4 that even on Hilbert spaces there exist derivations which are not tangent to any smooth curve. In particular, we have different kinds of tangent bundles, the most important ones are the kinematic and the operational one. We treat smooth, real analytic, and holomorphic vector bundles and spaces of sections of vector bundles, we give them structures of convenient vector spaces; they will become important as modeling spaces for manifolds of mappings in chapter IX.

Finally, we discuss Weil functors (certain product preserving functors of manifolds) as generalized tangent bundles. This last section is due to [**77**].

27. Differentiable Manifolds

27.1. Manifolds

A chart (U, u) on a set M is a bijection $u : U \to u(U) \subseteq E_U$ from a subset $U \subseteq M$ onto a c^∞-open subset of a convenient vector space E_U.

For two charts (U_α, u_α) and (U_β, u_β) on M the mapping $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ for $\alpha,\ \beta \in A$ is called the *chart changing*, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. A family $(U_\alpha, u_\alpha)_{\alpha\in A}$ of charts on M is called an *atlas* for M, if the U_α form a cover of M and all chart changings $u_{\alpha\beta}$ are defined on c^∞-open subsets.

An atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$ for M is said to be a C^∞*-atlas*, if all chart changings $u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ are smooth. Two C^∞-atlases are called C^∞*-equivalent*, if their union is again a C^∞-atlas for M. An equivalence class of C^∞-atlases is sometimes called a C^∞*-structure* on M. The union of all atlases in an equivalence class is again an atlas, the *maximal atlas* for this C^∞-structure. A C^∞*-manifold* M is a set together with a C^∞-structure on it. The charts of M are then those of the maximal atlas.

Atlas, structures, and manifolds are called *real analytic* or *holomorphic*, if all chart changings are real analytic or holomorphic, respectively. They are called topological, if the chart changings are only continuous in the c^∞-topology.

A holomorphic manifold is real analytic, and a real analytic one is smooth. By a manifold we will henceforth mean a smooth one.

27.2 Smooth mappings and the topology of manifolds

The *natural topology* on a (smooth) manifold M is the identification topology with respect to some (smooth) atlas $(u_\alpha : M \supseteq U_\alpha \to u_\alpha(U_\alpha) \subseteq E_\alpha)$, where a subset $W \subseteq M$ is open if and only if $u_\alpha(U_\alpha \cap W)$ is c^∞-open in E_α for all α. This topology depends only on the structure and not the specific atlas, since diffeomorphisms are homeomorphisms for the c^∞-topologies. It is also the final topology with respect to all inverses of chart mappings in one atlas. It is also the final topology with respect to all smooth curves defined below.

A mapping $f : M \to N$ between manifolds is called *smooth* if for each chart (U, u) of M and (V, v) of N the domain $u(f^{-1}(V))$ of the composite $v \circ f \circ u^{-1}$ is open and $v \circ f \circ u^{-1}$ is smooth on it, equivalently, for each $x \in M$ and each chart (V, v) on N with $f(x) \in V$ there is a chart (U, u) on M with $x \in U$, $f(U) \subseteq V$, such that $v \circ f \circ u^{-1}$ is smooth. This is the case if and only if $f \circ c$ is smooth for each smooth curve $c : \mathbb{R} \to M$. Obviously, the composite of smooth mappings is smooth.

We will denote by $C^\infty(M, N)$ the space of all C^∞-mappings from M to N.

Likewise, we have the spaces $C^\omega(M, N)$ of real analytic mappings and $\mathcal{H}(M, N)$ of holomorphic mappings between manifolds of the corresponding type. This can be also tested by composing with the relevant types of curves.

A smooth mapping $f : M \to N$ is called a *diffeomorphism* if f is bijective and its inverse is also smooth. Two manifolds are called *diffeomorphic* if there exists a diffeomorphism between them. Likewise, we have *real analytic* and *holomorphic diffeomorphisms*. The latter ones are also called *biholomorphic mappings*.

27.3. Products

Let M and N be smooth manifolds described by smooth atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$ and $(V_\beta, v_\beta)_{\beta\in B}$, respectively. Then the family

$$(U_\alpha \times V_\beta, u_\alpha \times v_\beta : U_\alpha \times V_\beta \to E_\alpha \times F_\beta)_{(\alpha,\beta)\in A\times B}$$

is a smooth atlas for the cartesian product $M \times N$. Beware, however, the manifold topology 27.2 of $M \times N$ may be finer than the product topology, see 4.22. If M and N are metrizable, then it coincides with the product topology, by 4.19. Clearly, the projections

$$M \xleftarrow{\mathrm{pr}_1} M \times N \xrightarrow{\mathrm{pr}_2} N$$

are also smooth. The *product* $(M\times N, \mathrm{pr}_1, \mathrm{pr}_2)$ has the following universal property:

For any smooth manifold P and smooth mappings $f : P \to M$ and $g : P \to N$ the mapping $(f, g) : P \to M \times N$, $(f, g)(x) = (f(x), g(x))$, is the unique smooth mapping with $\mathrm{pr}_1 \circ (f, g) = f$, $\mathrm{pr}_2 \circ (f, g) = g$.

Clearly, we can form products of finitely many manifolds. The reader may now wonder why we do not consider infinite products of manifolds. These have charts which are open for the so called 'box topology'. But then we get 'box products' without the universal property of products. The 'box products', however, have the universal product property for families of mappings such that locally almost all members are constant.

27.4. Separation properties of the manifold topology

For a (smooth) manifold we will additionally require certain (separation-)properties for the natural topology. For finite dimensional manifolds these properties are not inherited from the modelling space $\mathbb{R}^m$, but if one assumes the manifold to be Hausdorff, then the locally compactness of $\mathbb{R}^m$ carries over to the manifold. So its topology is completely regular and even smoothly regular (see 14.1) and paracompactness (or metrizability) of its topology implies then smoothly paracompactness (see 16.1).

For infinite dimensional manifolds the situation is not so simple So let us discuss the relevant notions of Hausdorff:

(1) M is (topologically) Hausdorff, equivalently the diagonal is closed in the product topology on $M \times M$.

(2) The diagonal is closed in the manifold $M \times M$ (Note, that the topology of $M \times M$ may be finer than the product topology).

(3) The smooth functions in $C^\infty(M, \mathbb{R})$ separate points in M. Let us call M *smoothly Hausdorff* if this property holds.

We have the obvious implications (3)⇒(1)⇒(2). We have no counterexamples for the converse implications.

It is not so clear which separation property should be required for a manifold. In order to make some decision, from now on **we require that manifolds are smoothly Hausdorff**. Each convenient vector space has this property. But we will have difficulties with permanence of the property 'smoothly Hausdorff', in particular with quotient manifolds, see for example the discussion [**75**, 27.14] on covering spaces below. For important examples (manifolds of mappings, diffeomorphism groups, etc.) we will prove that they are even smoothly paracompact.

The isomorphism type of the *modeling convenient vector spaces* E_α is constant on the connected components of the manifold M, since the derivatives of the chart changings are linear isomorphisms. A manifold M is called *pure* if this isomorphism type is constant on the whole of M.

Corollary. *If a smooth manifold (which is smoothly Hausdorff) is Lindelöf, and if all modeling vector spaces are smoothly regular, then it is smoothly paracompact.*

If a smooth manifold is metrizable and smoothly normal then it is smoothly paracompact.

Proof. See 16.10 for the first statement and 16.15 for the second one. □

27.5. Lemma. *Let M be a smoothly regular manifold. Then for any manifold N a mapping $f : N \to M$ is smooth if and only if $g \circ f : N \to \mathbb{R}$ is smooth for all $g \in C^\infty(M,\mathbb{R})$. This means that $(M, C^\infty(\mathbb{R}, M), C^\infty(M,\mathbb{R}))$ is a Frölicher space, see* [**75**, 23.1].

Proof. ($\Leftarrow$) Let (V, v) be a chart of M and let $x \in f^{-1}(V)$. We may choose (by smooth regularity) a smooth bump function $g : M \to \mathbb{R}$ with $g = 1$ in a neighborhood $W \subseteq V$ of $f(x)$ and $g = 0$ on a neighborhood of $M \setminus V$. Then $f^{-1}(\operatorname{carr}(g)) = \operatorname{carr}(g \circ f)$ is an open neighborhood of x in N contained in $f^{-1}(V)$. Hence f is continuous. Moreover, $g \cdot (\ell \circ v) : V \to \mathbb{R}$ for $\ell \in E'$ extends by 0 to a smooth mapping g_ℓ on M, hence $g_\ell \circ f$ is smooth and equals $\ell \circ v \circ f$ on $f^{-1}(W)$. Thus $v \circ f|_{f^{-1}(W)}$ is smooth by 2.14.4 since E is convenient, so f is smooth near x. □

27.6. Non-regular manifold

[**89**] Let $0 \neq \lambda \in (\ell^2)^*$, let X be $\{x \in \ell^2 : \lambda(x) \geq 0\}$ with the Moore topology, i.e. for $x \in X$ we take $\{y \in \ell^2 \setminus \ker \lambda : \|y - x\| < \varepsilon\} \cup \{x\}$ for $\varepsilon > 0$ as neighborhood-basis. We set $X^+ := \{x \in \ell^2 : \lambda(x) > 0\} \subseteq \ell^2$.

Then obviously X is Hausdorff (its topology is finer than that of ℓ^2) but not regular: In fact the closed subspace $\ker \lambda \setminus \{0\}$ cannot be separated by open sets from $\{0\}$.

It remains to show that X is a C^∞-manifold. We use the following diffeomorphisms

(1) $S := \{x \in \ell^2 : \|x\| = 1\} \cong_{C^\infty} \ker \lambda$.
(2) $\varphi : \ell^2 \setminus \{0\} \cong_{C^\infty} \ker \lambda \times \mathbb{R}^+$.
(3) $S \cap X^+ \cong_{C^\infty} \ker \lambda$.
(4) $\psi : X^+ \to \ker \lambda \times \mathbb{R}^+$.

(1) This is due to [**10**].

(2) Let $f : S \to \ker \lambda$ be the diffeomorphism of (1) and define the required diffeomorphism to be $\varphi(x) := (f(x/\|x\|), \|x\|)$ with inverse $\varphi^{-1}(y, t) := t\, f^{-1}(y)$.

(3) Take an $a \in (\ker \lambda)^\perp$ with $\lambda(a) = 1$, i.e. $\ker \lambda = a^\perp$. Then $S \cap X^+ \ni y \mapsto \mu_y \cdot y - a \in \ker \lambda = a^\perp$ with $\mu_y \cdot y - a \perp a$, i.e. $\mu_y = \|a\|^2/\langle y, a\rangle$, is the required

diffeomorphism with inverse mapping $a^\perp \ni y \mapsto \nu_y \cdot (a+y) \in X \cap X^+$ with $1 = \|\nu_y(a+y)\|^2$.

($\boxed{4}$) Let $g : S \cap X^+ \to \ker\lambda$ be the diffeomorphism of ($\boxed{3}$) then the desired diffeomorphism is $\psi : x \mapsto (g(x/\|x\|), \|x\|)$.

We now show that there is a norm-preserving homeomorphism of $h : X^+ \cup \{0\} \to \ell^2$, such that $h(0) = 0$ and $h|_{X^+} : X^+ \to \ell^2 \setminus \{0\}$ is a diffeomorphism. We take

$$h(x) := \begin{cases} (\varphi^{-1} \circ \psi)(x) & \text{for } x \in X^+ \\ 0 & \text{for } x = 0 \end{cases}.$$

$$\begin{tikzcd}
(S\cap X^+)\times\mathbb{R}^+ \arrow[dr, "\boxed{3}\times\mathbb{R}^+"] & & S\times\mathbb{R}^+ \arrow[dl, "\boxed{1}\times\mathbb{R}^+"'] \\
X^+ \arrow[u, "\cong"] \arrow[d, hook] & \ker\lambda\times\mathbb{R}^+ \arrow[l, "\psi^{-1}", "\cong"'] & \ell^2\setminus\{0\} \arrow[l, "\varphi", "\cong"'] \arrow[u, "\cong"'] \arrow[d, hook] \\
X^+\cup\{0\} & & \ell^2 \arrow[ll, dotted, "h"'] \\
\{0\} \arrow[u, hook] & & \{0\} \arrow[u, hook] \arrow[ll, hook]
\end{tikzcd}$$

Now we use translates of h as charts $\ell^2 \to X^+ \cup \{x\}$. The chart changes are then diffeomorphisms of $\ell^2 \setminus \{0\}$ and we thus obtained a smooth atlas for $X := \bigcup_{x\in\ker\lambda}(X^+ \cup \{x\})$. The topology described by this atlas is obviously the Moore topology.

If we use instead of X the union $\bigcup_{x\in D}(X^+ \cup \{x\})$, where $D \subseteq \ker\lambda$ is dense and countable. Then the same results are valid, but X is now even second countable.

Note however that a regular space which is locally metrizable is completely regular.

27.7. Proposition. *Let M be a manifold modeled on smoothly regular convenient vector spaces. Then M admits an atlas of charts defined globally on convenient vector spaces.*

Proof. That a convenient vector space is smoothly regular means that the c^∞-topology has a base of carrier sets of smooth functions, see $\boxed{14.1}$. These functions satisfy the assumptions of theorem $\boxed{16.21}$, and hence the stars of these sets with respect to arbitrary points in the sets are diffeomorphic to the whole vector space and still form a base of the c^∞-topology. $\square$

27.11. Submanifolds

A subset N of a manifold M is called a *submanifold*, if for each $x \in N$ there is a chart (U, u) of M such that $u(U \cap N) = u(U) \cap F_U$, where F_U is a c^∞-closed linear subspace of the convenient model space E_U. Then clearly N is itself a manifold with $(U \cap N, u|_{U\cap N})$ as charts, where (U, u) runs through all these *submanifold charts* from above.

A submanifold N of M is called a *splitting submanifold* if there is a cover of N by submanifold charts (U, u) as above such that the $F_U \subset E_U$ are complemented (i.e. splitting) linear subspaces. Then every submanifold chart is splitting.

Note that a closed submanifold of a smoothly paracompact manifold is again smoothly paracompact. Namely, the trace topology is the intrinsic topology on the submanifold since this is true for closed linear subspaces of convenient vector spaces, 4.28.

A mapping $f : N \to M$ between manifolds is called *initial* if it has the following property:

> A mapping $g : P \to N$ from a manifold P ($\mathbb{R}$ suffices) into N is smooth if and only if $f \circ g : P \to M$ is smooth.

Clearly, an initial mapping is smooth and injective. The embedding of a submanifold is always initial. The notion of initial smooth mappings will play an important role in this book whereas that of immersions will be used in finite dimensions only.

In a similar way we shall use the (now obvious) notion of *initial real analytic mappings* between real analytic manifolds and also *initial holomorphic mappings* between complex manifolds.

If $h : \mathbb{R} \to \mathbb{R}$ is a function such that h^p and h^q are smooth for some p, q which are relatively prime in $\mathbb{N}$, then h itself turns out to be smooth, see [Joris, 1982.] So the mapping $f : t \mapsto (t^p, t^q)$, $\mathbb{R} \to \mathbb{R}^2$, is initial, but f is not an immersion at 0.

Smooth mappings $f : N \to M$ which admit local smooth retracts are initial. By this we mean that for each $x \in N$ there are an open neighborhood U of $f(x)$ in M and a smooth mapping $r_x : U \to N$ such that $r \circ f|_{f^{-1}(U)} = \mathrm{Id}_{f^{-1}U}$. We shall meet this class of initial mappings in 43.19.

21.11. Example. *There exists a short exact sequence $\ell^2 \xrightarrow{\iota} E \to \ell^2$, which does not split, see 13.18.6. The square of the norm on the subspace ℓ^2 does not extend to a smooth function on E.*

Proof. Assume indirectly that a smooth extension of the square of the norm exists. Let $2b$ be the second derivative of this extension at 0, then $b(x, y) = \langle x, y\rangle$ for all $x, y \in \ell^2$, and hence the following diagram commutes

$$\begin{tikzcd} \ell^2 \arrow[r, hook, "\iota"] \arrow[d, "\cong"', "\sharp"] & E \arrow[d, "b^\vee"] \\ (\ell^2)^* & E^* \arrow[l, two heads, "\iota^*"'] \end{tikzcd}$$

giving a retraction to ι. □

27.12. Example. We now give an example of an initial smooth mapping f with the following properties:

(1) f is a topological embedding onto a closed subspace and the derivative at each point is an embedding of a closed linear subspace, i.e. f is an immersion.
(2) The image of f is not a submanifold.
(3) The image of f cannot be described locally by a regular smooth equation.

This shows that the notion of an *embedding* is quite subtle in infinite dimensions.

Proof. For this let $\ell^2 \xrightarrow{\iota} E \to \ell^2$ be a short exact sequence, which does not split, see 13.18.6. Choose a $0 \neq \lambda \in E^*$ with $\lambda \circ \iota = 0$ and choose a v with $\lambda(v) = 1$. Now consider $f : \ell^2 \to E$ given by $x \mapsto \iota(x) + \|x\|^2\, v$.

(1) Since f is polynomial it is smooth. We have $(\lambda \circ f)(x) = \|x\|^2$, hence $g \circ f = \iota$, where $g : E \to E$ is given by $g(y) := y - \lambda(y)\, v$. Note however that g is not a diffeomorphism, hence we don't have automatically a submanifold. Thus f and also its differential at every point are topological embeddings. Moreover the image is closed, since $f(x_n) \to y$ implies $\iota(x_n) = g(f(x_n)) \to g(y)$, hence $x_n \to x_\infty$ for some x_∞ and thus $f(x_n) \to f(x_\infty) = y$. Finally f is initial: Namely, let $h : G \to \ell^2$ be such that $f \circ h$ is smooth, then $g \circ f \circ h = \iota \circ h$ is smooth. As a closed linear embedding ι is initial, so h is smooth. Note that λ is an extension of $\|\_\|^2$ along $f : \ell^2 \to E$.

(2) Suppose there were a local diffeomorphism Φ around $f(0) = 0$ and a closed subspace $F < E$ such that locally Φ maps F onto $f(\ell^2)$. Then Φ factors as follows

$$\begin{tikzcd}
\ell^2 \arrow[r, hook, "f"] & E \\
F \arrow[u, dotted, "\varphi"] \arrow[r, hook, "\mathrm{incl}"] & E \arrow[u, "\Phi"', "\cong"]
\end{tikzcd}$$

In fact since $\Phi(F) \subseteq f(\ell^2)$, and f is injective, we have φ as mapping, and since f is initial φ is smooth. By using that $\mathrm{incl} : F \to E$ is initial, we could deduce that φ is a local diffeomorphism. However we only need that $\varphi'(0) : F \to \ell^2$ is a linear isomorphism. Since $f'(0) \circ \varphi'(0) = \Phi'(0)|_F$ is a closed embedding, we have that $\varphi'(0)$ is a closed embedding. In order to see that $\varphi'(0)$ is onto, pick $v \in \ell^2$ and consider the curve $t \mapsto tv$. Then $w : t \mapsto \Phi^{-1}(f(tv)) \in F$ is smooth, and

$$\begin{aligned}
f'(0)\Big(\varphi'(0)\big(w'(0)\big)\Big) &= \tfrac{d}{dt}|_{t=0}(f \circ \varphi)(w(t)) \\
&= \tfrac{d}{dt}|_{t=0}\Phi\big(w(t)\big) = \tfrac{d}{dt}|_{t=0} f(tv) = f'(0)(v)
\end{aligned}$$

and since $f'(0) = \iota$ is injective, we have $\varphi'(0)\big(w'(0)\big) = v$.

$$\begin{tikzcd}
 & \ell^2 \arrow[r, hook, "f"] & E \\
\mathbb{R} \arrow[ur, "t\mapsto tv"] \arrow[dr, dotted, "w"'] & & \\
 & F \arrow[uu, "\varphi"] \arrow[r, hook, "\mathrm{incl}"] & E \arrow[uu, "\Phi"', "\cong"]
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
 & \mathbb{R} & \\
\ell^2 \arrow[ur, "\|_\|^2"] \arrow[rr, hook, "f"] & & E \arrow[ul, "\lambda"'] \\
F \arrow[u, "\varphi"] \arrow[rr, hook, "\mathrm{incl}"] \arrow[d, "\varphi'(0)", "\cong"'] & & E \arrow[u, "\Phi", "\cong"'] \arrow[d, "\Phi'(0)"', "\cong"] \\
\ell^2 \arrow[uu, bend left=60, "k:=\varphi\circ\varphi'(0)^{-1}"] \arrow[rr, hook, "\iota=f'(0)"] & & E \arrow[uu, bend right=60, "\Phi\circ\Phi'(0)^{-1}"']
\end{tikzcd}$$

i.e.

$$\begin{aligned}(\lambda\circ\Phi\circ\Phi'(0)^{-1})\circ\iota\circ\varphi'(0) &= \lambda\circ\Phi\circ\Phi'(0)^{-1}\circ f'(0)\circ\varphi'(0)\\ &= \lambda\circ\Phi\circ\Phi'(0)^{-1}\circ\Phi'(0)\circ\mathrm{incl}\\ &= \lambda\circ\Phi\circ\mathrm{incl} = \lambda\circ f\circ\varphi = \|\_\|^2\circ\varphi.\end{aligned}$$

By composing with $\varphi'(0)^{-1}:\ell^2\to F$ we get an extension $\tilde q$ of $q:=\|\_\|^2\circ k$ to E, where the locally defined mapping $k:=\varphi\circ\varphi'(0)^{-1}:\ell^2\to\ell^2$ is smooth and $k'(0)=\mathrm{id}$. Now $\tilde q''(0):E\times E\to\mathbb{R}$ is an extension of $q''(0):\ell^2\times\ell^2\to\mathbb{R}$, $(v,w)\mapsto 2\langle k'(0)v,k'(0)w\rangle$. Hence the associated $\tilde q''(0)^\vee:E\to E^*$ fits into

$$\begin{tikzcd} \ell^2 \arrow[r, "k'(0)", "\cong"'] \arrow[d, "\sharp", "\cong"'] & \ell^2 \arrow[r, hook, "\iota"] & E \arrow[d, "\tilde q''(0)^\vee"] \\ \ell^2 & \ell^2 \arrow[l, "\cong", "k'(0)^*"'] & E^* \arrow[l, "\iota^*"] \end{tikzcd}$$

and in this way we get a linear retraction for $\iota:\ell^2\to E$. This is a contradiction.

(3) Let us show now the even stronger statement that there is no local regular equation $\rho:E\rightsquigarrow G$ with $f(\ell^2)=\rho^{-1}(0)$ locally and $\ker\rho'(0)=\iota(\ell^2)$. Otherwise we have $\rho'(0)(v)\neq 0$ and hence there is a $\mu\in G'$ with $\mu(\rho'(0)(v))=1$. Thus $\mu\circ\rho:E\rightsquigarrow\mathbb{R}$ is smooth $\mu\circ\rho\circ f=0$ and $(\mu\circ\rho)'(0)(v)=1$. Moreover

$$\begin{aligned}0 &= (\tfrac{d}{dt})^2|_{t=0}(\mu\circ\rho\circ f)(tx)\\ &= \tfrac{d}{dt}|_{t=0}(\mu\circ\rho)'(f(tx))\cdot f'(tx)\cdot x\\ &= (\mu\circ\rho)''(0)(f'(0)x,f'(0)x)+(\mu\circ\rho)'(0)\cdot f''(0)(x,x)\\ &= (\mu\circ\rho)''(0)(\iota(x),\iota(x))+2\|x\|^2\underbrace{(\mu\circ\rho)'(0)}_{=1}\cdot v,\end{aligned}$$

hence $-(\mu\circ\rho)''(0)/2$ is an extension of $\|\_\|^2$ along ι, which is a contradiction to 21.11 . □

27.18. Germs

Let M and N be manifolds, and let $A\subseteq M$ be a closed subset. We consider all smooth mappings $f:U_f\to N$, where U_f is some open neighborhood of A in M, and we put $f\underset{A}{\sim}g$ if there is some open neighborhood V of A with $f\mid V=g\mid V$. This is an equivalence relation on the set of functions considered. The equivalence class of a function f is called the *germ of f along A*, sometimes denoted by $\mathrm{germ}_A f$. As in 8.3 we will denote the space of all these germs by $C^\infty(M\supseteq A,N)$.

If we consider functions on M, i.e. if $N=\mathbb{R}$, we may add and multiply germs, so we get the real commutative algebra of germs of smooth functions. If $A=\{x\}$, this algebra $C^\infty(M\supseteq\{x\},\mathbb{R})$ is sometimes also denoted by $C^\infty_x(M,\mathbb{R})$. We may consider the inductive locally convex vector space topology with respect to the cone

$$C^\infty(M\supseteq\{x\},\mathbb{R})\leftarrow C^\infty(U,\mathbb{R}),$$

where U runs through some neighborhood basis of x consisting of charts, so that each $C^\infty(U,\mathbb{R})$ carries a convenient vector space topology by 2.15 .

This inductive topology is Hausdorff only if x is isolated in M, since the restriction to some one dimensional linear subspace of a modeling space is a projection on a direct summand which is not Hausdorff, by 27.19 below. Nevertheless, multiplication is a bounded bilinear operation on $C^\infty(M \supseteq \{x\}, \mathbb{R})$, so the closure of 0 is an ideal. The quotient by this ideal is thus an algebra with bounded multiplication, denoted by

$$\operatorname{Tay}_x(M, \mathbb{R}) := C^\infty(M \supseteq \{x\}, \mathbb{R})/\overline{\{0\}}.$$

27.19. Lemma. *Let M be a smooth manifold modeled on C_b^∞-regular Banach spaces (see 15.1). Then the closure of 0 in $C^\infty(M \supseteq \{x\}, \mathbb{R})$ is the ideal of all germs which are flat at x of infinite order.*

Proof. This is a local question, so let $x = 0$ in a modeling Banach space E. Let f be a representative in some open neighborhood U of 0 of a flat germ. This means that all iterated derivatives of f at 0 vanish. Let $\rho \in C_b^\infty(E, [0, 1])$ be 0 on a neighborhood of 0 and $\rho(x) = 1$ for $\|x\| > 1$. For $f_n(x) := f(x)\rho(n.x)$ we have $\operatorname{germ}_0(f_n) = 0$, and it remains to show that $n(f - f_n)$ is bounded in $C^\infty(U, \mathbb{R})$. For this we fix a derivative d^k and choose N such that $\|d^{k+1}f(x)\| \le 1$ for $\|x\| \le \frac{1}{N}$. Then for $n \ge N$ we have the following estimate:

$$\begin{aligned}
\|nd^k(f - f_n)(x)\| &\le \sum_{l=0}^{k} \binom{k}{l} n\|d^{k-l}f(x)\| n^l \|d^l(1 - \rho)(nx)\| \\
&\le \sum_{l=0}^{k} \binom{k}{l} n \int_0^1 \frac{(1-t)^{l+1}}{(l+1)!} \|d^{k+1}f(tx)\|\, dt\, \|x\|^{l+1} n^l \|d^l(1-\rho)(nx)\| \\
&\le \begin{cases} 0 & \text{for } \|nx\| > 1 \\ \sum_{l=0}^{k} \binom{k}{l} \frac{1}{l!} \|d^l(1-\rho)\|_\infty & \text{for } \|nx\| \le 1. \end{cases} \quad \square
\end{aligned}$$

27.20. Corollary. *For any C_b^∞-regular Banach space E and $a \in E$ the canonical mapping*

$$\operatorname{Tay}_a(E, \mathbb{R}) \to \prod_{k=0}^{\infty} L_{sym}^k(E, \mathbb{R})$$

is a bornological isomorphism.

Proof. For every open neighborhood U of a in E we have a continuous linear mapping $C^\infty(U, \mathbb{R}) \to \prod_{k=0}^\infty L_{\mathrm{sym}}^k(E, \mathbb{R})$ into the space of formal power series, hence also $C^\infty(E \supseteq \{a\}, \mathbb{R}) \to \prod_{k=0}^\infty L_{\mathrm{sym}}^k(E, \mathbb{R})$, and finally from $\operatorname{Tay}_a(E, \mathbb{R}) \to \prod_{k=0}^\infty L_{\mathrm{sym}}^k(E, \mathbb{R})$. Since E is Banach, the space of formal power series is a Fréchet space and since E is $C_b^\infty(E, \mathbb{R})$-regular the last mapping is injective by 27.19. By E. Borel's theorem 15.4 every bounded subset of the space of formal power series is the image of a bounded subset of $C^\infty(E, \mathbb{R})$. Hence this mapping is a bornological isomorphism and the inductive limit $C^\infty(E \supseteq \{a\}, \mathbb{R})$ is regular. $\square$

27.21. Lemma. *If M is smoothly regular then each germ at a point of a smooth function has a representative which is defined on the whole of M.*

If M is smoothly paracompact then the previous statement is true for germs along closed subsets. $\square$

28. Tangent Vectors

28.1. The tangent spaces of a convenient vector space E

Let $a \in E$. A *kinematic tangent vector* with foot point a is simply a pair (a, X) with $X \in E$. Let $T_aE = E$ be the space of all kinematic tangent vectors with foot point a. It consists of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \to E$ with $c(0) = a$, which explains the choice of the name kinematic.

For each open neighborhood U of a in E (a, X) induces a linear mapping $X_a : C^\infty(U, \mathbb{R}) \to \mathbb{R}$ by $X_a(f) := df(a)(X)$, which is continuous for the convenient vector space topology on $C^\infty(U, \mathbb{R})$ and satisfies $X_a(f \cdot g) = X_a(f) \cdot g(a) + f(a) \cdot X_a(g)$, so it is a *continuous derivation over* ev_a. The value $X_a(f)$ depends only on the germ of f at a.

An *operational tangent vector* of E with foot point a is a bounded derivation $\partial : C^\infty(E \supseteq \{a\}, \mathbb{R}) \to \mathbb{R}$ over ev_a. Let D_aE be the vector space of all these derivations. Any $\partial \in D_aE$ induces a bounded derivation $C^\infty(U, \mathbb{R}) \to \mathbb{R}$ over ev_a for each open neighborhood U of a in E. Moreover any family of bounded derivations $\partial_U : C^\infty(U, \mathbb{R}) \to \mathbb{R}$ over ev_a, which is coherent with respect to the restriction maps, defines an $\partial \in D_aE$. So the vector space D_aE is a closed linear subspace of the convenient vector space $\prod_U L(C^\infty(U, \mathbb{R}), \mathbb{R})$. We equip D_aE with the induced convenient vector space structure. Note that the spaces D_aE are isomorphic for all $a \in E$.

Taylor expansion induces the dashed arrows in the following diagram.

$$\begin{tikzcd}
 & C^\infty(E,\mathbb{R}) \arrow[d] & L^k_{\mathrm{sym}}(E,\mathbb{R}) \\
\overline{\{0\}} \arrow[d, hook] \arrow[dr, hook] & C^\infty(U,\mathbb{R}) \arrow[ur, two heads, "d^k|_0"] \arrow[r, dotted, two heads] \arrow[d] & \prod_{k=1}^d L^k_{\mathrm{sym}}(E,\mathbb{R}) \arrow[u, two heads, "\mathrm{pr}_k"'] \\
\{\infty\text{-flat}\} \arrow[r, hook] \arrow[d, hook] & C^\infty(E \supseteq \{a\},\mathbb{R}) \arrow[ur, dotted, two heads] \arrow[r, dotted] \arrow[dr, two heads] & \prod_{k=1}^\infty L^k_{\mathrm{sym}}(E,\mathbb{R}) \arrow[u, two heads, "\mathrm{pr}"] \\
\{d\text{-flat}\} \arrow[ur, hook] & & C^\infty(E \supseteq \{a\},\mathbb{R})/\overline{\{0\}} \arrow[u, dotted]
\end{tikzcd}$$

Note that all spaces in the right two columns except the top right corner are algebras, the finite product with truncated multiplication. The mappings are algebra-homomorphisms. And the spaces in the left column are the respective kernels. If E is a $C_b^\infty(E, \mathbb{R})$-regular Banach space, then by $\boxed{27.20}$ the vertical dashed arrow is bibounded and since $\mathbb{R}$ is Hausdorff every $\partial \in D_aE$ factors over $\mathrm{Tay}_a(E, \mathbb{R}) := C^\infty(E \supseteq \{a\}, \mathbb{R})/\overline{\{0\}}$. So in this case we can view ∂ as derivation on the algebra of formal power series.

28.2. Degrees of operational tangent vectors

A derivation ∂ is said to have *order* at most d, it vanishes on all d-flat germs, i.e. if it factors over the space $\prod_{k=0}^d L^k_{\mathrm{sym}}(E, \mathbb{R})$ of polynomials of degree at most d. If no such d exists, then it will be called of infinite order.

An operational tangent vector is said to be *homogeneous of order* d if it factors over $L^d_{\mathrm{sym}}(E,\mathbb{R})$, i.e. it corresponds to a bounded linear functional $\ell \in L^d_{\mathrm{sym}}(E,\mathbb{R})'$ via $\partial(f) = \ell(\frac{f^{(d)}(0)}{d!})$. In order that such a functional defines a derivation, we need exactly that

$$\ell\Big(\mathrm{Sym}\Big(\sum_{j=1}^{d-1} L^j_{\mathrm{sym}}(E,\mathbb{R}) \otimes L^{d-j}_{\mathrm{sym}}(E,\mathbb{R})\Big)\Big) = \{0\},$$

i.e. that ℓ vanishes on the subspace

$$\sum_{i=1}^{j-1} L^i_{sym}(E;\mathbb{R}) \vee L^{j-i}_{sym}(E;\mathbb{R})$$

of decomposable elements of $L^j_{sym}(E;\mathbb{R})$, where $L^i_{sym}(E;\mathbb{R}) \vee L^{j-i}_{sym}(E;\mathbb{R})$ denotes the linear subspace generated by all symmetric products $\Phi \vee \Psi$ of the corresponding elements:
In fact, any such ℓ defines an operational tangent vector $\partial^j_\ell|_a \in D_a E$ of order j by

$$\partial^j_\ell|_a(f) := \ell(\tfrac{1}{j!} d^j f(a)).$$

Since ℓ vanishes on decomposable elements we see from the Leibniz rule that ∂^j_ℓ is a derivation, and it is obviously of order j. The inverse bijection is given by $\partial \mapsto (\ell : \Phi \mapsto \partial((\Phi \circ \mathrm{diag})(\quad - a)))$, since the complete polarization of a homogeneous polynomial p of degree j is given by $\frac{1}{j!} d^j p(0)(v_1, \dots, v_j)$, and since the remainder of the Taylor expansion is flat of order $j-1$ at a.

Obviously every derivation of order at most d is a unique sum of homogeneous derivations of order j for $1 \le j \le d$. For $d > 0$ we denote by $D^{[d]}_a E$ the linear subspace of $D_a E$ of operational tangent vectors of homogeneous order d and by $D^{(d)}_a E := \bigoplus_{j=1}^d D^{[j]}$ the subspace of (non homogeneous) operational tangent vectors of order at most d.

28.3. Examples. Queer operational tangent vectors

Let $Y \in E''$ be an element in the bidual of E. Then for each $a \in E$ we have an operational tangent vector $Y_a \in D_a E$, given by $Y_a(f) := Y(df(a))$. So we have a canonical injection $E'' \to D_a E$.

Let $\ell : L^2(E;\mathbb{R}) \to \mathbb{R}$ be a bounded linear functional which vanishes on the subset $E' \otimes E'$. Then for each $a \in E$ we have an operational tangent vector $\partial^{[2]}_\ell|_a \in D_a E$ given by $\partial^{[2]}_\ell|_a(f) := \ell(d^2 f(a))$, since

$$\begin{aligned}\ell(d^2(fg)(a)) &= \ell\Big(d^2 f(a) g(a) + df(a) \otimes dg(a) + dg(a) \otimes df(a) + f(a) d^2 g(a)\Big)\\ &= \ell(d^2 f(a)) g(a) + 0 + f(a) \ell(d^2 g(a)).\end{aligned}$$

Let $E = (\ell^2)^{\mathbb{N}}$ be a countable product of copies of an infinite dimensional Hilbert space. A smooth function on E depends locally only on finitely many Hilbert space variables. Thus, $f \mapsto \sum_n \partial^{[k_n]}_{X_n}(f \circ \mathrm{inj}_n)$ is a well defined operational tangent vector in $D_0 E$ for arbitrary operational tangent vectors X_n of order k_n. If (k_n) is an unbounded sequence and if $X_n \neq 0$ for all n it is not of finite order. But only for $k = 1, 2, 3$ we know that nonzero tangent vectors of order k exist, see 28.4 below.

28.4. Lemma. *If E is an infinite dimensional Hilbert space, there exist nonzero operational tangent vectors of order* $2, 3$.

Proof. We may assume that $E = \ell^2$. For $k = 2$ one knows that the closure of $L(\ell^2, \mathbb{R}) \vee L(\ell^2, \mathbb{R})$ in $L^2_{\text{sym}}(\ell^2, \mathbb{R})$ consists of all symmetric compact operators, and the identity is not compact.

For $k = 3$ we show that for any A in the closure of $L(\ell^2, \mathbb{R}) \vee L^2_{\text{sym}}(\ell^2, \mathbb{R})$ the following condition holds:

(1) $$A(e_i, e_j, e_k) \to 0 \quad \text{for} \quad i, j, k \to \infty.$$

Since this condition is invariant under symmetrization it suffices to consider $A \in \ell^2 \otimes L(\ell^2, \ell^2)$, which we may view as a finite dimensional and thus compact operator $\ell^2 \to L(\ell^2, \ell^2)$. Then $\|A(e_i)\| \to 0$ for $i \to \infty$, since this holds for each continuous linear functional on ℓ^2. The trilinear form $A(x, y, z) := \sum_i x_i y_i z_i$ is in $L^3_{\text{sym}}(\ell^2, \mathbb{R})$ and obviously does not satisfy (1). □

28.5. Proposition. *Let E be a convenient vector space with the following two properties:*

1. *The closure of 0 in $C^\infty(E \supseteq \{0\}, \mathbb{R})$ consists of all flat germs.*
2. *The quotient $C^\infty(E \supseteq \{0\}, \mathbb{R})/\{\infty\text{-flat}\}$ with the bornological quotient topology embeds as topological linear subspace into the space $\prod_k L^k_{sym}(E; \mathbb{R})$ of formal power series.*

Then each operational tangent vector on E is of finite order.

Any C^∞_b-regular Banach space, in particular any Hilbert space has these properties.

Proof. Let $\partial \in D_0 E$ be an operational tangent vector, so it factors over $\operatorname{Tay}_0(E, \mathbb{R}) = C^\infty(E \supseteq \{0\}, \mathbb{R})/\overline{\{0\}}$. By property (1) this space is $C^\infty(E \supseteq \{0\}, \mathbb{R})/\{\infty\text{-flat}\}$. Since ∂ is continuous in the bornological topology, by property (2) and the theorem of Hahn-Banach it extends to a continuous linear functional on the space of all formal power series and thus depends only on finitely many factors.

A C^∞_b-regular Banach space E has property (1) by 27.19, and it has property (2) by E. Borel's theorem 15.4. Hilbert spaces are C^∞_b-regular by 15.5. □

6.6. Definition. Another important additional property for convenient vector spaces E is the approximation property, i.e. the denseness of $E' \otimes E$ in $L(E, E)$. There are at least 3 successively stronger requirements, which have been studied in [1]:
A convenient vector space E is said to have the *bornological approximation property* if $E' \otimes E$ is dense in $L(E, E)$ with respect to the bornological topology. It is said to have the *c^∞-approximation property* if this is true with respect to the c^∞-topology of $L(E, E)$. Finally the *Mackey approximation property* is the requirement, that there is some sequence in $E' \otimes E$ Mackey converging towards id_E.

Note that although the bornological approximation property is the weakest among these 3 conditions, it is difficult to check directly, since the bornologification of $L(E, E)$ is hard to describe explicitly.

6.7. Result. [1, 2.2.9] *The natural topology on*

$$L(C^\infty(\mathbb{R}, \mathbb{R}), C^\infty(\mathbb{R}, \mathbb{R}))$$

of uniform convergence on bounded sets is not bornological.

6.8. Result. [1, 2.5.5] *For any set Γ of non-measurable cardinality the space E of points in $\mathbb{R}^\Gamma$ with countable carrier has the bornological approximation property.*

Note. One first shows that for this space E the topology of uniform convergence on bounded sets is bornological, and the classical approximation property holds for this topology by [53, 21.2.2], since E is nuclear.

6.10. Lemma. [1, 2.1.21] *Let E be a reflexive convenient vector space. Then E has the bornological (resp. c^∞-, resp. Mackey) approximation property if and only if E' has it.*

Proof. For reflexive convenient vector spaces we have:

$$L(E', E') \cong L^2(E', E; \mathbb{R}) \cong L(E, E'') \cong L(E, E),$$

and $E'' \otimes E$ corresponds to $E' \otimes E$ via this isomorphism. So the result follows. $\square$

6.11. Lemma. [1, 2.4.3] *Let E be the product $\prod_{k\in\mathbb{N}} E_k$ of a sequence of convenient vector spaces E_k. Then E has the Mackey (resp. c^∞-) approximation property if and only if all E_k have it.*

Proof. ($\Rightarrow$) follows since one easily checks that these approximation properties are inherited by direct summands.

($\Leftarrow$) Let $(T_n^k)_n$ be Mackey convergent to T^k in $L(E_k, E_k)$. Then one easily checks the Mackey convergence of $(T_n^k)_k \to (T^k)_n$ in $\prod_k L(E_k, E_k) \subseteq L(E, E)$. So the result follows for the Mackey approximation property.

To obtain it also for the c^∞-topology, one first notes that by the argument given in [75, 6.9] it is enough to approximate the identity. Since the c^∞-closure can be obtained as iterated Mackey-adherence by 4.32 this follows now by transfinite induction. $\square$

6.12. Köthe sequence spaces

Recall that a set $\mathcal{P} \subseteq \mathbb{R}_+^\mathbb{N}$ of sequences is called a Köthe set if it is directed upwards with respect to the componentwise partial ordering, see [75, 52.35]. To $\mathcal{P}$ we may associate the set

$$\Lambda(\mathcal{P}) := \{x = (x_n)_n \in \mathbb{R}^\mathbb{N} : (p_n x_n)_n \in \ell^1 \text{ for all } p \in \mathcal{P}\}.$$

A space $\Lambda(\mathcal{P})$ is said to be a Köthe sequence space whenever $\mathcal{P}$ is a Köthe set.

Lemma. *Let $\mathcal{P}$ be a Köthe set for which there exists a sequence μ converging monotonely to $+\infty$ and such that $(\mu_n p_n)_{n\in\mathbb{N}} \in \mathcal{P}$ for each $p \in \mathcal{P}$. Then the Köthe sequence space $\Lambda(\mathcal{P})$ has the Mackey approximation property.*

Proof. The sequence $\left(\sum_{j=1}^n e_j' \otimes e_j\right)_{n\in\mathbb{N}}$ is Mackey convergent in $L(\Lambda(\mathcal{P}), \Lambda(\mathcal{P}))$ to $\mathrm{id}_{\Lambda(\mathcal{P})}$, where e_j and e_j' denote the j-th unit vector in $\Lambda(\mathcal{P})$ and $\Lambda(\mathcal{P})'$ respectively: Indeed, a subset $B \subseteq \Lambda(\mathcal{P})$ is bounded if and only if for each $p \in \mathcal{P}$ there exists $N(p) \in \mathbb{R}$ such that

$$\sum_{k\in\mathbb{N}} p_k |x_k| \le N(p)$$

for all $x = (x_k)_{k\in\mathbb{N}} \in B$. But this implies that

$$\Big\{\mu_{n+1}\Big(\mathrm{id}_{\Lambda(\mathcal{P})} - \sum_{j=1}^{n} e'_j \otimes e_j\Big) : n \in \mathbb{N}\Big\} \subseteq L(\Lambda(\mathcal{P}), \Lambda(\mathcal{P}))$$

is bounded. In fact

$$\Big(\Big(\mathrm{id} - \sum_{j=1}^{n} e'_j \otimes e_j\Big)(x)\Big)_k = \begin{cases} 0 & \text{for } k \le n, \\ x_k & \text{for } k > n. \end{cases}$$

and hence

$$\sum_k p_k \Big|\mu_{n+1}\Big((\mathrm{id} - \sum_{j=1}^{n} e'_j \otimes e_j)(x)\Big)_k\Big| \le \sum_{k>n} p_k |\mu_{n+1} x_k| \le \sum_k p_k \mu_k |x_k| \le N(\mu p) \qquad \square$$

Let α be an unbounded increasing sequence of positive real numbers and $\mathcal{P}_\infty := \{(e^{k\alpha_n})_{n\in\mathbb{N}} : k \in \mathbb{N}\}$. Then the associated Köthe sequence space $\Lambda(\mathcal{P}_\infty)$ is called a POWER SERIES SPACE OF INFINITE TYPE (a Fréchet space by [**53**, 3.6.2]).

6.13. Corollary. *Each power series space of infinite type has the Mackey approximation property.* $\square$

6.14. Theorem. *The following convenient vector spaces have the Mackey approximation property:*

1. *The space $C^\infty(M \leftarrow F)$ of smooth sections of any smooth finite dimensional vector bundle $F \xrightarrow{p} M$ with separable base M, see 6.1 and 30.1.*
2. *The space $C^\infty_c(M \leftarrow F)$ of smooth sections with compact support any smooth finite dimensional vector bundle $F \xrightarrow{p} M$ with separable base M, see 6.2 and 30.4.*
3. *The Fréchet space of holomorphic functions $\mathcal{H}(\mathbb{C}, \mathbb{C})$, see 8.2.*

Proof. The space s of rapidly decreasing sequences coincides with the power series space of infinite type associated to the sequence $(\log(n))_{n\in\mathbb{N}}$. So by 6.13, 6.11 and 6.10 the spaces s, $s^{\mathbb{N}}$ and $s^{(\mathbb{N})} = \big((s')^{\mathbb{N}}\big)'$ have the Mackey approximation property. Now assertions (1) and (2) follow from the isomorphisms $C^\infty_c(M \leftarrow F) = C^\infty(M \leftarrow F) \cong s$ for compact M and $C^\infty(M \leftarrow F) \cong s^{\mathbb{N}}$ for non-compact M (see [**127**] or [**1**, 1.5.16]) and the isomorphism $C^\infty_c(M \leftarrow F) \cong s^{(\mathbb{N})}$ for non-compact M (see [**127**] or [**1**, 1.5.16]).

(3) follows since by [**53**, 2.10.11] the space $\mathcal{H}(\mathbb{C}, \mathbb{C})$ is isomorphic to the (complex) power series space of infinite type associated to the sequence $(n)_{n\in\mathbb{N}}$. $\square$

28.7. Theorem. *Let E be a convenient vector space which has the bornological approximation property. Then we have $D_a E \cong E''$. So if E is in addition reflexive, each operational tangent vector is a kinematic one.*

Proof. We may suppose that $a = 0$. Let $\partial : C^\infty(E \supseteq \{0\}, \mathbb{R}) \to \mathbb{R}$ be a derivation at 0, so it is bounded linear and satisfies $\partial(f \cdot g) = \partial(f) \cdot g(0) + f(0) \cdot \partial(g)$. Then we have $\partial(1) = \partial(1 \cdot 1) = 2\partial(1)$, so ∂ is zero on constant functions.

Since $E' = L(E, \mathbb{R})$ is continuously embedded into $C^\infty(E, \mathbb{R})$, $\partial|_{E'}$ is an element of the bidual E''. Obviously, $\partial - (\partial|_{E'})_0$ is a derivation which vanishes on affine functions. We have to show that it is zero. We call this difference again ∂. For $f \in C^\infty(U,\mathbb{R})$ where U is some radial open neighborhood of 0 we have

$$f(x) = f(0) + \int_0^1 df(tx)(x)dt,$$

thus $\partial(f) = \partial(g)$, where $g(x) := \int_0^1 df(tx)(x)dt$. By assumption, there is a net $\ell_\alpha \in E' \otimes E \subset L(E,E)$ of bounded linear operators with finite dimensional image, which converges to Id_E in the bornological topology of $L(E,E)$. We consider $g_\alpha \in C^\infty(U,\mathbb{R})$, given by $g_\alpha(x) := \int_0^1 df(tx)(\ell_\alpha x)dt$.

Claim. $g_\alpha \to g$ in $C^\infty(U,\mathbb{R})$.

We have $g(x) = h(x,x)$ where $h \in C^\infty(U\times E,\mathbb{R})$ is just $h(x,y) = \int_0^1 df(tx)(y)dt$. By cartesian closedness, the associated mapping $h^\vee : U \to E' \subset C^\infty(E,\mathbb{R}))$ is smooth. Since $' : L(E,E) \to L(E',E')$ is bounded linear, the net ℓ'_α converges to $\mathrm{Id}_{E'}$ in $L(E',E')$. The mapping $(h^\vee)^* : L(E',E') \subset C^\infty(E',E') \to C^\infty(U,E')$ is bounded linear, thus $(h^\vee)^*(\ell'_\alpha)$ converges to $h^\vee$ in $C^\infty(U,E')$. By cartesian closedness, the net $((h^\vee)^*(\ell'_\alpha))^\wedge$ converges to h in $C^\infty(U\times E,\mathbb{R})$. Since the diagonal mapping $\delta : U \to U\times E$ is smooth, the mapping $\delta^* : C^\infty(U\times E,\mathbb{R}) \to C^\infty(U,\mathbb{R})$ is continuous and linear, so finally $g_\alpha = \delta^*(((h^\vee)^*(\ell'_\alpha))^\wedge)$ converges to $\delta^*(h) = g$.

Claim. $\partial(g_\alpha) = 0$ for all α.

Let $\ell_\alpha = \sum_{i=1}^n \varphi_i \otimes x_i \in E' \otimes E \subset L(E,E)$. We have

$$\begin{aligned} g_\alpha(x) &= \int_0^1 df(tx)\Big(\sum_i \varphi_i(x)x_i\Big)dt \\ &= \sum_i \varphi_i(x)\int_0^1 df(tx)(x_i)dt =: \sum_i \varphi_i(x)h_i(x), \\ \partial(g_\alpha) = \partial\Big(\sum_i \varphi_i\cdot h_i\Big) &= \sum_i\Big(\partial(\varphi_i)h_i(0) + \varphi_i(0)\partial(h_i)\Big) = 0. \quad \square \end{aligned}$$

28.8. Remark

There are no nonzero operational tangent vectors of order 2 on E if and only if $E' \vee E' \subset L^2_{sym}(E;\mathbb{R})$ is dense in the bornological topology. This seems to be rather near the bornological approximation property, and one may suspect that theorem 28.7 remains true under this weaker assumption.

28.9. Trivial operational tangent bundle

Let $U \subseteq E$ be an open subset of a convenient vector space E. The operational tangent bundle DU of U is simply the disjoint union $\bigsqcup_{a\in U} D_aE$. Then DU is in bijection to the open subset $U\times D_0E$ of $E\times D_0(E)$ via $\partial_a \mapsto (a, \partial\circ(\quad -a)^*)$. We use this bijection to put a smooth structure on DU. Let now $g : E \supset U \to V \subset F$ be a smooth mapping, then $g^* : C^\infty(W,\mathbb{R}) \to C^\infty(g^{-1}(W),\mathbb{R})$ is bounded and linear for all open $W \subset V$. The adjoints of these mappings uniquely define a mapping $Dg : DU \to DV$ by $(Dg.\partial)(f) := \partial(f\circ g)$.

Lemma. *$Dg : DU \to DV$ is smooth.*

Proof. Via the canonical bijections $DU \cong U \times D_0E$ and $DV \cong V \times D_0F$ the mapping Dg corresponds to

$$\begin{aligned} &U \times D_0E \to V \times D_0F \\ (a, \partial) \mapsto &\big(g(a), \partial \circ (\quad +a)^* \circ g^* \circ (\quad -g(a))^*\big) \\ &= \big(g(a), \partial \circ (g(\quad +a) - g(a))^*\big). \end{aligned}$$

In order to show that this is smooth, its enough to consider the second component and we compose it with the embedding $D_0F \hookrightarrow \prod_{W \ni 0} C^\infty(W, \mathbb{R})'$. The associated mapping $U \times D_0E \times C^\infty(W, \mathbb{R}) \to \mathbb{R}$ is given by

$$(a, \partial, f) \mapsto \partial\big(f \circ (g(\_ + a) - g(a))\big),$$

where $f \circ (g(\_ + a) - g(a))$ is smooth on the open 0-neighborhood $W_a := \{y \in E : g(y + a) - g(a) \in W\} = g^{-1}(g(a) + W) - a$ in E. Now let $a : \mathbb{R} \to U$ be a smooth curve and I a bounded interval in $\mathbb{R}$. Then there exists an open neighborhood $U_{I,W}$ of 0 in E such that $U_{I,W} \subseteq W_{a(t)}$ for all $t \in I$. Then the mapping (1), composed with $a : I \to U$, factors as

$$I \times D_0E \times C^\infty(W, \mathbb{R}) \to C^\infty(U_{I,W}, \mathbb{R})' \times C^\infty(U_{I,W}, \mathbb{R}) \to \mathbb{R},$$

given by

$$(t, \partial, f) \mapsto \Big(\partial_{U_{I,W}}, f \circ \Big(g(\_ + a(t)) - g(a(t))\Big)\Big) \mapsto \partial_{U_{I,W}}(f \circ (g(\_ + a(t)) - g(a(t))))),$$

which is smooth by cartesian closedness. □

28.10. Trivial operational tangent bundle of order at most q

Let E be a convenient vector space. Recall from 28.2 that $D_a^{(k)}E$ is the space of all operational tangent vectors of order $\leq k$. For an open subset U in a convenient vector space E and $k > 0$ we consider the disjoint union

$$D^{(k)}U := \bigsqcup_{a \in U} D_a^{(k)}E \cong U \times D_0^{(k)}E \subseteq E \times D_0^{(k)}E.$$

Lemma. *For a smooth mapping $f : E \supset U \to V \subset F$ the smooth mapping $Df : DU \to DV$ from 28.9 induces smooth mappings $D^{(k)}f : D^{(k)}U \to D^{(k)}V$.*

Proof. We only have to show that D_af maps $D_a^{(k)}E$ into $D_{f(a)}^{(k)}F$, because smoothness follows then by restriction.

The pullback $f^* : C^\infty(V, \mathbb{R}) \to C^\infty(U, \mathbb{R})$ maps functions which are flat of order k at $f(a)$ to functions which are flat of the same order at a. Thus, D_af maps the corresponding annihilator $D_a^{(k)}U$ into the annihilator $D_{f(a)}^{(k)}V$. □

28.11. Lemma.

(1) *The chain rule holds in general: $D(f \circ g) = Df \circ Dg$ and $D^{(k)}(f \circ g) = D^{(k)}f \circ D^{(k)}g$.*

(2) *If $g : E \to F$ is a bounded affine mapping then D_xg commutes with the restriction and the projection to the subspaces of derivations which are homogeneous of degree $k > 1$.*

(3) *If $g : E \to F$ is a bounded affine mapping with linear part $\ell = g - g(0) : E \to F$ then $D_x g : D_x^{[k]} E \to D_{g(x)}^{[k]} F$ is induced by the linear mappings $(L^k_{sym}(\ell;\mathbb{R}))^* : L^k_{sym}(E,\mathbb{R})^* \to L^k_{sym}(F,\mathbb{R})^*$.*

(4) *If $g : E \to \mathbb{R}$ is bounded linear we have $Dg.X_x = D^{(1)}g.X_x^{[1]}$.*

Remark that if g is not affine then in general Dg does not respect the subspaces of derivations which are homogeneous of degree $k > 1$:
In fact let $g : E \to \mathbb{R}$ be a homogeneous polynomial of degree k on which $\partial \in D_0^{[k]} E$ does not vanish. Then by (4) we have that $0 \neq \partial(g) = Dg(\partial) \in \mathbb{R} \cong D_0^{[1]}\mathbb{R} = D_0\mathbb{R}$.

Proof. (1) is obvious.

For (2) let $X_x \in D_x E$ and $f \in C^\infty(F,\mathbb{R})$. Then we have

$$\begin{aligned}
(Dg.X_x)^{[k]}(f) &= (Dg.X_x)(\tfrac{1}{k!} d^k f(g(x))(\quad - g(x))^k) \\
&= \tfrac{1}{k!} X_x(d^k f(g(x))(g(\quad) - g(x))^k) \\
(Dg.X_x^{[k]})(f) &= X_x^{[k]}(f \circ g) \\
&= X_x(\tfrac{1}{k!} d^k (f\circ g)(x)(\quad - x)^k) \\
&= \tfrac{1}{k!} X_x(d^k f(g(x))(\ell(\quad - x))^k).
\end{aligned}$$

These expressions are equal.

$$\begin{tikzcd}
D_x E \arrow[r, two heads] \arrow[d, "Dg"] & D_x^{[k]} E \arrow[r, hook] \arrow[d, "Dg"] & D_x E \arrow[d, "Dg"] \\
D_{g(x)} F \arrow[r, two heads] & D_{g(x)}^{[k]} F \arrow[r, hook] & D_x F
\end{tikzcd}$$

For (3) we take $\varphi \in L^k_{\mathrm{sym}}(E;\mathbb{R})'$ which vanishes on all decomposable forms, and let $X_x = \partial^k_\varphi|_x \in D_x^{[k]} E$ be the corresponding homogeneous derivation. Then

$$\begin{aligned}
(Dg.\partial^k_\varphi|_x)(f) &= \partial^k_\varphi|_x(f \circ g) \\
&= \varphi(\tfrac{1}{k!} d^k(f \circ g)(x)) \\
&= \varphi(\tfrac{1}{k!} d^k f(g(x)) \circ \ell^k) \\
&= (L^k_{\mathrm{sym}}(\ell;\mathbb{R})^* \varphi)(\tfrac{1}{k!} d^k f(g(x))) \\
&= \partial^{[k]}_{L^k_{\mathrm{sym}}(\ell;\mathbb{R})^*\varphi}|_{g(x)}(f).
\end{aligned}$$

$$\begin{tikzcd}
D_x^{[k]} E \arrow[r, hook] \arrow[d, "D^{[k]}g"'] & L^k_{\mathrm{Sym}}(E,\mathbb{R})' \arrow[d, "L^k_{\mathrm{Sym}}(\ell,\mathbb{R})^*"] \\
D_{g(x)}^{[k]} F \arrow[r, hook] & L^k_{\mathrm{Sym}}(F,\mathbb{R})'
\end{tikzcd}$$

(4) is a special case of (2). □

28.12. The operational and the kinematic tangent bundles

Let M be a manifold with a smooth atlas $(M \supset U_\alpha \xrightarrow{u_\alpha} E_\alpha)_{\alpha\in A}$. We consider the following equivalence relation on the disjoint union

$$\bigsqcup_{\alpha\in A} D(u_\alpha(U_\alpha)) := \bigcup_{\alpha\in A} D(u_\alpha(U_\alpha)) \times \{\alpha\},$$
$$(\partial,\alpha) \sim (\partial',\beta) \iff D(u_{\alpha\beta})\partial' = \partial.$$

We denote the quotient set by DM and call it the *operational tangent bundle* of M. Let $\pi_M : DM \to M$ be the obvious foot point projection, let $DU_\alpha = \pi_M^{-1}(U_\alpha) \subset DM$, and let $Du_\alpha : DU_\alpha \to D(u_\alpha(U_\alpha))$ be given by $Du_\alpha([\partial,\alpha]) = \partial$. So $Du_\alpha([\partial',\beta]) = D(u_{\alpha\beta})\partial'$.

The charts (DU_α, Du_α) form a smooth atlas for DM, since the chart changings are given by

$$Du_\alpha \circ (Du_\beta)^{-1} = D(u_{\alpha\beta}) : D(u_\beta(U_{\alpha\beta})) \to D(u_\alpha(U_{\alpha\beta})).$$

This chart changing formula also implies that the smooth structure on DM depends only on the equivalence class of the smooth atlas for M.

The mapping $\pi_M : DM \to M$ is obviously smooth. The natural topology is automatically Hausdorff: X, $Y \in DM$ can be separated by open sets of the form $\pi_M^{-1}(V)$ for $V \subset M$, if $\pi_M(X) \neq \pi_M(Y)$, since M is Hausdorff, and by open subsets of the form $(Tu_\alpha)^{-1}(E_\alpha \times W)$ for W open in E_α, if $\pi_M(X) = \pi_M(Y) \in U_\alpha$.

For $x \in M$ the set $D_xM := \pi_M^{-1}(x)$ is called the *operational tangent space* at x or the *fiber* over x of the operational tangent bundle. It carries a canonical convenient vector space structure induced by $D_x(u_\alpha) := Du_\alpha|_{D_xM} : D_{u_\alpha(x)}E_\alpha \cong D_0(E_\alpha)$ for some (equivalently any) α with $x \in U_\alpha$.

Let us construct now the *kinematic tangent bundle*. We consider the following equivalence relation on the disjoint union

$$\bigcup_{\alpha\in A} U_\alpha \times E_\alpha \times \{\alpha\},$$
$$(x,v,\alpha) \sim (y,w,\beta) \iff x = y \text{ and } d(u_{\alpha\beta})(u_\beta(x))w = v$$

and denote the quotient set by TM, the kinematic tangent bundle of M. Let $\pi_M : TM \to M$ be given by $\pi_M([x,v,\alpha]) = x$, let $TU_\alpha = \pi_M^{-1}(U_\alpha) \subset TM$, and let $Tu_\alpha : TU_\alpha \to u_\alpha(U_\alpha) \times E_\alpha$ be given by $Tu_\alpha([x,v,\alpha]) = (u_\alpha(x), v)$. So $Tu_\alpha([x,w,\beta]) = (u_\alpha(x), d(u_{\alpha\beta})(u_\beta(x))w)$.

The charts (TU_α, Tu_α) form a smooth atlas for TM, since the chart changings are given by

$$Tu_\alpha \circ (Tu_\beta)^{-1} : u_\beta(U_{\alpha\beta}) \times E_\beta \to u_\alpha(U_{\alpha\beta}) \times E_\alpha,$$
$$(x,v) \mapsto (u_{\alpha\beta}(x), d(u_{\alpha\beta})(x)v).$$

This chart changing formula also implies that the smooth structure on TM depends only on the equivalence class of the smooth atlas for M.

The mapping $\pi_M : TM \to M$ is obviously smooth. It is called the *(foot point) projection* of M. The natural topology is automatically Hausdorff; this follows from the bundle property and the proof is the same as for DM above.

For $x \in M$ the set $T_xM := \pi_M^{-1}(x)$ is called the *kinematic tangent space* at x or the *fiber* over x of the tangent bundle. It carries a canonical convenient vector space structure induced by $T_x(u_\alpha) := Tu_\alpha|_{T_xM} : T_xM \to \{x\} \times E_\alpha \cong E_\alpha$ for some (equivalently any) α with $x \in U_\alpha$.

Note that the kinematic tangent bundle TM embeds as a subbundle into DM; also for each $k \in \mathbb{N}$ the same construction as above gives us tangent bundles $D^{(k)}M$ which are subbundles of DM.

28.13. Kinematic tangent vectors as velocity vectors

Let us now give an obvious description of TM as the space of all velocity vectors of curves, which explains the name 'kinematic tangent bundle': We put on $C^\infty(\mathbb{R}, M)$ the equivalence relation : $c \sim e$ if and only if $c(0) = e(0)$ and in one (equivalently each) chart (U, u) with $c(0) = e(0) \in U$ we have $\frac{d}{dt}|_0(u \circ c)(t) = \frac{d}{dt}|_0(u \circ e)(t)$. We have the following diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}, M)/\sim \arrow[d, "\cong"', "\delta"] & & C^\infty(\mathbb{R}, M) \arrow[ll] \arrow[dll, "\delta"'] \arrow[d, "\mathrm{ev}_0"'] \\
TM \arrow[rr, "\pi_M"'] & & M
\end{tikzcd}$$

where to $c \in C^\infty(\mathbb{R}, M)$ we associate the tangent vector $\delta(c) := [c(0), \frac{\partial}{\partial t}\big|_0 (u_\alpha \circ c)(t), \alpha]$. It factors to a bijection $C^\infty(\mathbb{R}, M)/\sim \to TM$, whose inverse associates to $[x, v, \alpha]$ the equivalence class of $t \mapsto u_\alpha^{-1}(u_\alpha(x) + h(t)v)$ for h a small function with $h(t) = t$ near 0.

Since the c^∞-topology on $\mathbb{R} \times E_\alpha$ is the product topology by corollary 4.15, we can choose h uniformly for (x, v) in a piece of a smooth curve. Thus, a mapping $g : TM \to N$ into another manifold is smooth if and only if $g \circ \delta : C^\infty(\mathbb{R}, M) \to N$ maps 'smooth curves' to smooth curves, by which we mean $C^\infty(\mathbb{R}^2, M)$ to $C^\infty(\mathbb{R}, N)$.

28.14. Lemma. *If a smooth manifold M and the squares of its model spaces are smoothly paracompact, then also the kinematic tangent bundle TM is smoothly paracompact.*

If a smooth manifold M and $V \times D_0 V$ for any of its model spaces V are smoothly paracompact, then also the operational tangent bundle DM is smoothly paracompact.

Proof. This is a particular case of [**75**, 29.7] below. □

28.15. Tangent mappings

Let $f : M \to N$ be a smooth mapping between manifolds. Then f induces a linear mapping $D_x f : D_x M \to D_{f(x)} N$ for each $x \in M$ by $(D_x f.\partial_x)(h) = \partial_x(h \circ f)$ for $h \in C^\infty(N \supseteq \{f(x)\}, \mathbb{R})$. These give a mapping $Df : DM \to DN$. If (U, u) is a chart around x and (V, v) is one around $f(x)$, then $Dv \circ Df \circ (Du)^{-1} = D(v \circ f \circ u^{-1})$ is smooth by lemma 28.9. So $Df : DM \to DN$ is smooth.

By lemma 28.10, Df restricts to smooth mappings $D^{(k)}f : D^{(k)}M \to D^{(k)}N$ and to $Tf : TM \to TN$. We check the last statement for open subsets M and N of convenient vector spaces. $(Df.X_a)(g) = X_a(g \circ f) = d(g \circ f)(a)(X) = dg(f(a))df(a)X = (df(a)X)_{f(a)}(g)$.

If $f \in C^\infty(M, E)$ for a convenient vector space E, then $Df : DM \to DE = E \times D_0 E$. We then define the *differential* of f by $df := pr_2 \circ Df : DM \to D_0 E$. It restricts to smooth fiberwise linear mappings $D^{(k)} M \to D_0^{(k)} E$ and $df : TM \to E$.

If $f \in C^\infty(M, \mathbb{R})$, then $df : DM \to \mathbb{R}$. Let Id denote the identity function on $\mathbb{R}$, then $(Tf.\partial_x)(\mathrm{Id}) = \partial_x(\mathrm{Id} \circ f) = \partial_x(f)$, so we have $df(\partial_x) = \partial_x(f)$.

The mapping $f \mapsto df$ is bounded linear $C^\infty(M, \mathbb{R}) \to C^\infty(DM, \mathbb{R})$. That it is linear and has values in this space is obvious. So by the smooth uniform boundedness principle 5.26 it is enough to show that $f \mapsto df.X_x = X_x(f)$ is bounded for all $X_x \in DM$, which is true by definition of DM.

28.16. Remark. Operational tangent vectors on a product

From the construction of the tangent bundle in 28.12 it is immediately clear that

$$TM \xrightarrow{T(\mathrm{pr}_1)} T(M \times N) \xrightarrow{T(\mathrm{pr}_2)} TN$$

is also a product, so that $T(M \times N) = TM \times TN$ in a canonical way.

We investigate $D_0(E \times F)$ for convenient vector spaces. Since D_0 is a functor for 0 preserving maps, we obtain linear sections $D_0(\mathrm{inj}_k) : D_0(E_k) \to D_0(E_1 \times E_2)$ and hence a section $D_0(\mathrm{inj}_1) + D_0(\mathrm{inj}_2) : D_0(E_1) \oplus D_0(E_2) \to D_0(E_1 \oplus E_2)$. The complement of the image is given by the kernel of the linear mapping $(D_0(\mathrm{pr}_1), D_0(\mathrm{pr}_2)) : D_0(E_1 \oplus E_2) \to D_0(E_1) \oplus D_0(E_2)$.

$$\begin{tikzcd}
 & D_0(E_1) \arrow[rr, "\mathrm{id}"] \arrow[dl, hook, "\mathrm{inj}_{D_0(E_1)}"'] \arrow[dr, "D_0(\mathrm{inj}_{E_1})"] & & D_0(E_1) & \\
D_0(E_1) \oplus D_0(E_2) \arrow[rr, tail] & & D_0(E_1 \oplus E_2) \arrow[rr, tail] \arrow[ur, "D_0(\mathrm{pr}_{E_1})"'] \arrow[dr, "D_0(\mathrm{pr}_{E_2})"] & & D_0(E_1) \oplus D_0(E_2) \arrow[ul, "\mathrm{pr}_{D_0(E_1)}"'] \arrow[dl, "\mathrm{pr}_{D_0(E_2)}"] \\
 & D_0(E_2) \arrow[rr] \arrow[ul, hook, "\mathrm{inj}_{D_0(E_2)}"] \arrow[ur, "D_0(\mathrm{inj}_{E_2})"'] & & D_0(E_2) &
\end{tikzcd}$$

Lemma. *In the case $E_1 = \ell^2 = E_2$ this mapping is not injective.*

Proof. The space $L^2(E_1 \times E_2, E_1 \times E_2; \mathbb{R})$ can be viewed as $L^2(E_1, E_1; \mathbb{R}) \times L^2(E_1, E_2; \mathbb{R}) \times L^2(E_2, E_1; \mathbb{R}) \times L^2(E_2, E_2; \mathbb{R})$ and the subspace formed by those forms whose (2,1) and (1,2) components with respect to this decomposition are compact considered as operators in $L(\ell^2, \ell^2) \cong L^2(\ell^2, \ell^2; \mathbb{R})$ is a closed subspace. So, by Hahn-Banach, there is a non-trivial continuous linear functional $\ell : L^2(\ell^2 \times \ell^2, \ell^2 \times \ell^2; \mathbb{R}) \to \mathbb{R}$ vanishing on this subspace. We claim that the linear mapping $\partial : C^\infty(\ell^2 \times \ell^2, \mathbb{R}) \ni f \mapsto \ell(f''(0,0)) \in \mathbb{R}$ is an operational tangent vector of $\ell^2 \times \ell^2$ but not a direct sum of two operational tangent vectors on ℓ^2. In fact, the second derivative of a product h of two functions f and g is given by

$$\begin{aligned}
d^2 h(0,0)(w_1, w_2) = {} & d^2 f(0,0)(w_1, w_2)\, g(0,0) \\
& + df(0,0)(w_1)\, dg(0,0)(w_2) \\
& + df(0,0)(w_2)\, dg(0,0)(w_1) \\
& + f(0,0)\, d^2 g(0,0)(w_1, w_2).
\end{aligned}$$

Thus ∂ is a derivation since the middle terms give finite dimensional operators in $L^2(\ell^2, \ell^2; \mathbb{R})$. It is not a direct sum of two operational tangent vectors on ℓ^2 since functions f depending only on the j-th factor have as second derivative forms with nonzero (j,j) entry only. Hence $D_0(\mathrm{pr}_j)(\partial)(f) = \partial(f \circ \mathrm{pr}_j) = \ell((f \circ \mathrm{pr}_j)''(0)) = 0$, but $\partial \neq 0$. $\square$

29. Vector Bundles

29.1. Vector bundles

Let $p : E \to M$ be a smooth mapping between manifolds. By a *vector bundle chart* on (E, p, M) we mean a pair (U, ψ), where U is an open subset in M, and where ψ is a fiber respecting diffeomorphism as in the following diagram:

$$\begin{tikzcd}
E \mid U := p^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times V \arrow[dl, "\mathrm{pr}_1"] \\
& U. &
\end{tikzcd}$$

Here V is a fixed convenient vector space, called the *standard fiber* or the *typical fiber*, real for the moment.

Two vector bundle charts (U_1, ψ_1) and (U_2, ψ_2) are called *compatible*, if $\psi_1 \circ \psi_2^{-1}$ is a fiber linear isomorphism, i.e., $(\psi_1 \circ \psi_2^{-1})(x, v) = (x, \psi_{1,2}(x)v)$ for some mapping $\psi_{1,2} : U_{1,2} := U_1 \cap U_2 \to GL(V)$. The mapping $\psi_{1,2}$ is then unique and smooth into $L(V, V)$, and it is called the *transition function* between the two vector bundle charts.

A *vector bundle atlas* $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ for $p : E \to M$ is a set of pairwise compatible vector bundle charts (U_α, ψ_α) such that $(U_\alpha)_{\alpha \in A}$ is an open cover of M. Two vector bundle atlases are called *equivalent*, if their union is again a vector bundle atlas.

A (smooth) *vector bundle* $p : E \to M$ consists of manifolds E (the *total space*), M (the *base*), and a smooth mapping $p : E \to M$ (the *projection*) together with an equivalence class of vector bundle atlas: We must know at least one vector bundle atlas. The projection p turns out to be a surjective smooth mapping which has the 0-section as global smooth right inverse. Hence it is a final smooth mapping, see [**75**, 27.15].

If all mappings mentioned above are real analytic we call $p : E \to M$ a *real analytic vector bundle*. If all mappings are holomorphic and V is a complex vector space we speak of a *holomorphic vector bundle*.

30. Spaces of Sections of Vector Bundles

30.1

Let us fix a vector bundle $p : E \to M$ for the moment. On each fiber $E_x := p^{-1}(x)$ (for $x \in M$) there is a unique structure of a convenient vector space, induced by any vector bundle chart (U_α, ψ_α) with $x \in U_\alpha$. So $0_x \in E_x$ is a special element, and $0 : M \to E$, $0(x) = 0_x$, is a smooth mapping, the *zero section*.

A *section* u of $p : E \to M$ is a smooth mapping $u : M \to E$ with $p \circ u = \mathrm{Id}_M$. The *support* of the section u is the closure of the set $\{x \in M : u(x) \neq 0_x\}$ in M. The space of all smooth sections of the bundle $p : E \to M$ will be denoted by either $C^\infty(M \leftarrow E) = C^\infty(E, p, M) = C^\infty(E)$. Also the notation $\Gamma(E \to M) = \Gamma(p) = \Gamma(E)$ is used in the literature. Clearly, it is a vector space with fiber wise addition and scalar multiplication.

If $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is a vector bundle atlas for $p : E \to M$, then any smooth mapping $f_\alpha : U_\alpha \to V$ (the standard fiber) defines a local section $x \mapsto \psi_\alpha^{-1}(x, f_\alpha(x))$ on U_α. If $(g_\alpha)_{\alpha \in A}$ is a partition of unity subordinated to (U_α), then a global section can be formed by $x \mapsto \sum_\alpha g_\alpha(x) \cdot \psi_\alpha^{-1}(x, f_\alpha(x))$. So a smooth vector bundle has "many" smooth sections if M admits enough smooth partitions of unity.

We equip the space $C^\infty(M \leftarrow E)$ with the structure of a convenient vector space given by the closed embedding

$$C^\infty(M \leftarrow E) \to \prod_\alpha C^\infty(U_\alpha, V)$$
$$s \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ (s \mid U_\alpha),$$

where $C^\infty(U_\alpha, V)$ carries the natural structure described in [**75**, 27.17], see also 3.11. This structure is independent of the choice of the vector bundle atlas, because $C^\infty(U_\alpha, V) \to \prod_\beta C^\infty(U_{\alpha\beta}, V)$ is a closed linear embedding for any other atlas $(U_\beta)_\beta$.

Proposition. *The space $C^\infty(M \leftarrow E)$ of sections of the vector bundle (E, p, M) with this structure satisfies the uniform boundedness principle with respect to the point evaluations $\mathrm{ev}_x : C^\infty(M \leftarrow E) \to E_x$ for all $x \in M$.*

If M is a separable manifold modeled on duals of nuclear Fréchet spaces, and if each fiber E_x is a nuclear Fréchet space then $C^\infty(M \leftarrow E)$ is a nuclear Fréchet space and thus smoothly paracompact.

Proof. By definition of the structure on $C^\infty(M \leftarrow E)$ the uniform boundedness principle follows from 5.26 via 5.25.

For the statement about nuclearity note that by 6.1 the spaces $C^\infty(U_\alpha, V)$ are nuclear since we may assume that the U_α form a countable cover of M by charts which are diffeomorphic to c^∞-open subsets of duals of nuclear Fréchet spaces, and closed subspaces of countable products of nuclear Fréchet spaces are again nuclear Fréchet. By 16.10 nuclear Fréchet spaces are smoothly paracompact. □

30.4. Spaces of smooth sections with compact supports

For a smooth vector bundle $p : E \to M$ with finite dimensional SECOND COUNTABLE base M and standard fiber V we denote by $C_c^\infty(M \leftarrow E)$ the vector space of all smooth sections with compact supports in M.

Lemma. *The following structures of a convenient vector space on $C_c^\infty(M \leftarrow E)$ are all equivalent:*

(1) *Let $C_K^\infty(M \leftarrow E)$ be the space of all smooth sections of $E \to M$ with supports contained in the fixed compact subset $K \subset M$, a closed linear subspace of $C^\infty(M \leftarrow E)$. Consider the final convenient vector space structure*

on $C^\infty_c(M\leftarrow E)$ induced by the cone

$$C^\infty_K(M\leftarrow E) \to C^\infty_c(M\leftarrow E)$$

where K runs through a basis for the compact subsets of M. Then $C^\infty_c(M\leftarrow E)$ is even the strict and regular inductive limit of spaces $C^\infty_K(M\leftarrow E)$ where K runs through a countable base of compact sets.

(2) *Choose a second smooth vector bundle $q : E' \to M$ such that the Whitney sum is trivial* [**75**, 29.8]*: $E \oplus E' \cong M \times F$. Then $C^\infty_c(M\leftarrow E)$ can be considered as a closed direct summand of $C^\infty_c(M, F)$.*

The space $C^\infty_c(M\leftarrow E)$ satisfies the uniform boundedness principle with respect to the point evaluations. Moreover, if the standard fiber V is a nuclear Fréchet space and the base M is in addition separable then $C^\infty_c(M\leftarrow E)$ is smoothly paracompact.

Proof. Since $C^\infty_K(M\leftarrow E)$ is closed in $C^\infty(M\leftarrow E)$ the inductive limit $C^\infty_K(M\leftarrow E) \to C^\infty_c(M\leftarrow E)$ is strict. So the limit is regular [**68**, 4.8.1] and hence $C^\infty_c(M\leftarrow E)$ is convenient with the structure in (1). The direct sum property $C^\infty_K(M\leftarrow E) \subset C^\infty_K(M, F)$ from [**75**, 30.3.1] passes through the direct limits, so the equivalence of statements (1) and (2) follows.

We now show that $C^\infty_c(M\leftarrow E)$ satisfies the uniform boundedness principle for the point evaluations. Using description (2) and 5.25 for a direct sum we may assume that the bundle is trivial, hence we only have to consider $C^\infty_c(M, V)$ for a convenient vector space V. Now let F be a Banach space, and let $f : F \to C^\infty_c(M, V)$ be a linear mapping, such that $\mathrm{ev}_x \circ f : F \to V$ is bounded for each $x \in M$. Then by the uniform boundedness principle [**75**, 27.17] it is bounded into $C^\infty(M, V)$. We claim that f has values even in $C^\infty_K(M, V)$ for some K, so it is bounded therein, and hence in $C^\infty_c(M, V)$, as required.

If not we can recursively construct the following data: a discrete sequence (x_n) in M, a bounded sequence (y_n) in the Banach space F, and linear functionals $\ell_n \in V'$ such that

$$|\ell_k(f(y_n)(x_k))| \begin{cases} = 0 & \text{if } n < k, \\ = 1 & \text{if } n = k, \\ < 1 & \text{if } n > k. \end{cases}$$

Namely, we choose $y_n \in F$ and $x_n \in M$ such that $f(y_n)(x_n) \neq 0$ in V, and x_n has distance 1 to $\bigcup_{m<n} \operatorname{supp}(f(y_m))$ (in a complete Riemannian metric, where closed bounded subsets are compact). By shrinking y_n we may get $|\ell_m(f(y_n)(x_m))| < 1$ for $m < n$. Then we choose $\ell_n \in V'$ such that $\ell_n(f(y_n)(x_n)) = 1$.

Then $y := \sum_n \frac{1}{2^n} y_n \in F$, and $f(y)(x_k) \neq 0$ for all k since $|\ell_k(f(y)(x_k))| > 0$. So $f(y) \notin C^\infty_c(M, V)$.

For the last assertion, if the standard fiber V is a nuclear Fréchet space and the base M is separable then $C^\infty(M\leftarrow E)$ is a nuclear Fréchet space by the proposition in 30.1, so each closed linear subspace $C^\infty_K(M\leftarrow E)$ is a nuclear Fréchet space, and by 16.10 the countable strict inductive limit $C^\infty_c(M\leftarrow E)$ is smoothly paracompact. □

Chapter VII
Calculus on Infinite Dimensional Manifolds

In chapter VI we have found that some of the classically equivalent definitions of tangent vectors differ in infinite dimensions, and accordingly we have different kinds of tangent bundles and vector fields. Since this is the central topic of any treatment of calculus on manifolds we investigate in detail Lie brackets for all these notions of vector fields. Only kinematic vector fields can have local flows, and we show that the latter are unique if they exist [**75**, 32.16]. Note also theorem [**75**, 32.18] that any bracket expression of length k of kinematic vector fields is given as the k-th derivative of the corresponding commutator expression of the flows, which is not well known even in finite dimensions.

We also have different kinds of differential forms, which we treat in a systematic way, and we investigate how far the usual natural operations of differential forms generalize. In the end 33.21 the most common type of kinematic differential forms turns out to be the right ones for calculus on manifolds; for them the theorem of De Rham is proved.

We also include a version of the Frölicher-Nijenhuis bracket in infinite dimensions. The Frölicher-Nijenhuis bracket is a natural extension of the Lie bracket for vector fields to a natural graded Lie bracket for tangent bundle valued differential forms (later called vector valued). Every treatment of curvature later in [**75**, 37.3] and [**75**, 37.20] is initially based on the Frölicher-Nijenhuis bracket.

32. Vector Fields

32.1. Vector fields

Let M be a smooth manifold. We may define vector fields to be the sections of one of the tangent bundles $TM \hookrightarrow T''M \hookrightarrow D^{(1)}M \hookrightarrow D^{[1,\infty)}M \hookrightarrow DM$ defined in 28.12 (see 33.1 for $T''M$).

In particular, a *kinematic vector field* X on M is just a smooth section of the kinematic tangent bundle $TM \to M$. The space of all kinematic vector fields will be denoted by $\mathfrak{X}(M) = C^\infty(M\leftarrow TM)$.

By an *operational vector field* X on M we mean a bounded derivation of the sheaf $C^\infty(\_,\mathbb{R})$, i.e. for the open $U\subseteq M$ we are given bounded derivations $X_U : C^\infty(U,\mathbb{R})\to C^\infty(U,\mathbb{R})$ commuting with the restriction mappings.

We shall denote by $\operatorname{Der}(C^\infty(M,\mathbb{R}))$ the SPACE OF ALL OPERATIONAL VECTOR FIELDS on M. We shall equip $\operatorname{Der}(C^\infty(M,\mathbb{R}))$ with the convenient vector space structure induced by the closed linear embedding

$$\operatorname{Der}(C^\infty(M,\mathbb{R}))\hookrightarrow \prod_U L(C^\infty(U,\mathbb{R}),C^\infty(U,\mathbb{R})).$$

Convention

In 32.4 below we will show that for a smoothly regular manifold the space of derivations on the algebra $C^\infty(M,\mathbb{R})$ of globally defined smooth functions coincides with the derivations of the sheaf. Thus we shall follow the convention, that either the manifolds in question are smoothly regular, or that (as defined above) Der means the space of derivations of the corresponding sheaf also denoted by $C^\infty(M,\mathbb{R})$.

32.2. Lemma. *On any manifold M the operational vector fields correspond exactly to the smooth sections of the operational tangent bundle. Moreover we have an isomorphism of convenient vector spaces* $\operatorname{Der}(C^\infty(M,\mathbb{R}))\cong C^\infty(M\leftarrow DM)$.

Proof. Every smooth section $X\in C^\infty(M\leftarrow DM)$ defines an operational vector field by $\partial_U(f)(x):=X(x)(\operatorname{germ}_x f)=pr_2(Df(X(x)))$ for $f\in C^\infty(U,\mathbb{R})$ and $x\in U$. We have that $\partial_U(f)=\operatorname{pr}_2\circ Df\circ X=df\circ X\in C^\infty(U,\mathbb{R})$ by 28.15. Then ∂_U is obviously a derivation, since $df(X_x)=X_x(f)$ by 28.15. The linear mapping $\partial_U : C^\infty(U,\mathbb{R})\to C^\infty(U,\mathbb{R})$ is bounded if and only if $\operatorname{ev}_x\circ\partial_U : C^\infty(U,\mathbb{R})\to\mathbb{R}$ is bounded, by the smooth uniform boundedness principle 5.26, and this is true by 28.15, since $(\operatorname{ev}_x\circ X)(f)=df(X_x)$.

Moreover, the mapping

$$C^\infty(M\leftarrow DM)\to\operatorname{Der}\big(C^\infty(M,\mathbb{R})\big)\hookrightarrow\prod_U L\Big(C^\infty(U,\mathbb{R}),C^\infty(U,\mathbb{R})\Big)$$

given by $X\mapsto(\partial_U)_U$ is linear and bounded, since by the uniform boundedness principle 5.26 this is equivalent to the boundedness of $X\mapsto\partial_U(f)(x)=df(X_x)$ for all open $U\subseteq M$, $f\in C^\infty(U,\mathbb{R})$ and $x\in X$.

Now let conversely ∂ be an operational vector field on M. Then the family $\operatorname{ev}_x\circ\partial_U : C^\infty(U,\mathbb{R})\to\mathbb{R}$, where U runs through all open neighborhoods of x, defines a unique bounded derivation $X_x : C^\infty(M\supseteq\{x\},\mathbb{R})\to\mathbb{R}$, i.e. an element of D_xM. We have to show that $x\mapsto X_x$ is smooth, which is a local question, so we assume that M is a c^∞-open subset of a convenient vector space E. The mapping

$$M\xrightarrow{X} DM\cong M\times D_0E\subseteq M\times\prod_U L\big(C^\infty(U,\mathbb{R}),\mathbb{R}\big)$$

is smooth if and only if for every neighborhood U of 0 in E the component $M\to L(C^\infty(U,\mathbb{R}),\mathbb{R})$, given by $\partial\mapsto X_x(f(\_-x))=\partial_{U_x}(f(\_-x))(x)$ is smooth, where $U_x:=U+x$. By the smooth uniform boundedness principle 5.18 this is the case if and only if its composition with ev_f is smooth for all $f\in C^\infty(U,\mathbb{R})$. If $t\mapsto x(t)$ is a smooth curve in $M\subseteq E$, then there is a $\delta>0$ and an open neighborhood W of $x(0)$ in M such that $W\subseteq U+x(t)$ for all $|t|<\delta$ and hence $X_{x(t)}(f(\_-x(t)))=\partial_W(f(\_-x(t)))(x(t))$, which is by the exponential law smooth in t.

Moreover, the mapping $\mathrm{Der}(C^\infty(M,\mathbb{R})) \to C^\infty(M \leftarrow DM)$ given by $\partial \mapsto X$ is linear and bounded, since by the uniform boundedness principle in proposition 30.1 this is equivalent to the boundedness of $\partial \mapsto X_x \in D_xM \hookrightarrow \prod_U C^\infty(U,\mathbb{R})'$ for all $x \in M$, i.e. to that of $\partial \mapsto X_x(f) = \partial_U(f)(x)$ for all open neighborhoods U of x and $f \in C^\infty(U,\mathbb{R})$, which is obviously true. □

32.3. Lemma. *There is a natural embedding of convenient vector spaces*

$$\mathfrak{X}(M) = C^\infty(M \leftarrow TM) \hookrightarrow C^\infty(M \leftarrow DM) \cong \mathrm{Der}(C^\infty(M,\mathbb{R})).$$

Proof. Since TM is a closed subbundle of DM this is obviously true. □

32.4. Lemma. *Let M be a smoothly regular manifold.*

Then each bounded derivation $X : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is already an operational vector field. Moreover, we have an isomorphism

$$C^\infty(M \leftarrow DM) \cong \mathrm{Der}\Big(C^\infty(M,\mathbb{R}), C^\infty(M,\mathbb{R})\Big)$$

of convenient vector spaces.

Proof. Let ∂ be a bounded derivation of the algebra $C^\infty(M,\mathbb{R})$. If $f \in C^\infty(M,\mathbb{R})$ vanishes on an open subset $U \subset M$ then also $\partial(f)$: For $x \in U$ we take a bump function $g_{x,U} \in C^\infty(M,\mathbb{R})$ at x, i.e. $g_{x,U} = 1$ near x and $\mathrm{supp}(g_{x,U}) \subset U$. Then $\partial(f) = \partial((1-g_{x,U})f) = \partial(1-g_{x,U})f + (1-g_{x,U})\partial(f)$, and both summands are zero near x. So $\partial(f) \mid U = 0$.

Now let $f \in C^\infty(U,\mathbb{R})$ for a c^∞-open subset U of M. We have to show that we can define $\partial_U(f) \in C^\infty(U,\mathbb{R})$ in a unique manner. For $x \in U$ let $g_{x,U} \in C^\infty(M,\mathbb{R})$ be a bump function as before. Then $g_{x,U}f \in C^\infty(M,\mathbb{R})$, and $\partial(g_{x,U}f)$ makes sense. By the argument above, $\partial(gf)$ near x is independent of the choice of g. So let $\partial_U(f)(x) := \partial(g_{x,U}f)(x)$. It has all the required properties since the topology on $C^\infty(U,\mathbb{R})$ is initial with respect to all mappings $f \mapsto g_{x,U}f$ for $x \in U$.

This mapping $\partial \mapsto \partial_U$ is bounded, since by the uniform boundedness principles 5.18 and 5.26 this is equivalent with the boundedness of $\partial \mapsto \partial_U(f)(x) := \partial(g_{x,U}f)(x)$ for all $f \in C^\infty(U,\mathbb{R})$ and all $x \in U$ □

32.5. The operational Lie bracket

Recall that operational vector fields are the bounded derivations of the sheaf $C^\infty(\_,\mathbb{R})$, see 32.1. This is a convenient vector space by 32.2 and 30.1.

If X, Y are two operational vector fields on M, then the mapping $f \mapsto X(Y(f)) - Y(X(f))$ is also a bounded derivation of the sheaf $C^\infty(\_,\mathbb{R})$, as a simple computation shows. We denote it by $[X,Y] \in \mathrm{Der}(C^\infty(\_,\mathbb{R})) \cong C^\infty(M \leftarrow DM)$.

The $\mathbb{R}$-bilinear mapping

$$[\_,\_] : C^\infty(M \leftarrow DM) \times C^\infty(M \leftarrow DM) \to C^\infty(M \leftarrow DM)$$

is called the *Lie bracket*. Note also that $C^\infty(M \leftarrow DM)$ is a module over the algebra $C^\infty(M,\mathbb{R})$ by pointwise multiplication $(f,X) \mapsto fX$, which is bounded.

Theorem. *The Lie bracket* $[\_,\_] : C^\infty(M\leftarrow DM)\times C^\infty(M\leftarrow DM) \to C^\infty(M\leftarrow DM)$ *has the following properties:*

$$\begin{aligned}
&[X,Y] = -[Y,X],\\
&[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]], \quad \textit{the Jacobi identity,}\\
&[fX,Y] = f[X,Y] - (Yf)X,\\
&[X,fY] = f[X,Y] + (Xf)Y.
\end{aligned}$$

The form of the Jacobi identity we have chosen says that $ad(X) = [X,\_]$ is a derivation for the Lie algebra $(C^\infty(M\leftarrow DM),[\_,\_])$.

Proof. All these properties can be checked easily for the commutator $[X,Y] = X\circ Y - Y\circ X$ in the space of bounded derivations of the algebra $C^\infty(U,\mathbb{R})$. □

32.8. Theorem. *The Lie bracket restricts to the following mappings between splitting subspaces*

$$[\_,\_] : C^\infty(M\leftarrow D^{(k)}M)\times C^\infty(M\leftarrow D^{(\ell)}M) \to C^\infty(M\leftarrow D^{(k+\ell)}M).$$

The spaces $\mathfrak{X}(M) = C^\infty(M\leftarrow TM)$ *and* $C^\infty(M\leftarrow D^{[1,\infty)}M) := \bigcup_{1\le i<\infty} C^\infty(M\leftarrow D^{(i)}M)$ *are sub Lie algebras of* $C^\infty(M\leftarrow DM)$.

If $X\in\mathfrak{X}(M)$ *is a kinematic vector field, then* $[X,\_]$ *maps* $C^\infty(M\leftarrow D^{(\ell)}M)$ *into itself.*

This suggests to introduce the notation $D^{(0)} := T$, but here it does not indicate the order of differentiation present in the tangent vector.

32.12. Integral curves

Let $c : J \to M$ be a smooth curve in a manifold M defined on an interval J. It will be called an *integral curve* or *flow line* of a kinematic vector field $X\in\mathfrak{X}(M)$ if $c'(t) = X(c(t))$ holds for all $t\in J$.

For a given kinematic vector field integral curves need not exist locally, and if they exist they need not be unique for a given initial value. This is due to the fact that the classical results on existence and uniqueness of solutions of equations like the inverse function theorem, the implicit function theorem, and the Picard-Lindelöf theorem on ordinary differential equations can be deduced essentially from one another, and all depend on Banach's fixed point theorem. Beyond Banach spaces these proofs do not work any more, since the reduction does no longer lead to a contraction on a metrizable space. We are now going to give examples, which show that almost everything that might fail indeed fails.

Example 1. Let $E := s$ be the Fréchet space of rapidly decreasing sequences. Note that $s = C^\infty(S^1,\mathbb{R})$ by the theory of Fourier series. Consider the continuous linear operator $T : E\to E$ given by $T(x_0,x_1,x_2,\dots) := (0,1^2x_0,2^2x_1,3^2x_2,\dots)$. The ordinary linear differential equation $x'(t) = T(x(t))$ with constant coefficients and initial value $x(0) := (1,0,0,\dots)$ has no solution, since the coordinates would have to satisfy the recursive relation $x_n'(t) = n^2x_{n-1}(t)$, In particular, $x_1'(t) = 0$, and hence we must have $x_n(t) = n!t^n$. But the so defined curve $t\mapsto x(t)$ has only for $t=0$ values in E. Thus, no local solution exists. By recursion one sees that the solution for an arbitrary initial value $x(0)$ should be given by

$$x_n(t)=\sum_{i=0}^{n}\left(\tfrac{n!}{i!}\right)^2 x_i(0)\frac{t^{n-i}}{(n-i)!}.$$

If the initial value is a finite sequence, say $x_n(0)=0$ for $n>N$ and $x_N(0)\neq 0$, then

$$\begin{aligned}
x_n(t)&=\sum_{i=0}^{N}\left(\tfrac{n!}{i!}\right)^2 x_i(0)\frac{t^{n-i}}{(n-i)!}\\
&=\frac{(n!)^2}{(n-N)!}\,t^{n-N}\sum_{i=0}^{N}\left(\tfrac{1}{i!}\right)^2 x_i(0)\tfrac{(n-N)!}{(n-i)!}\,t^{N-i}\\
|x_n(t)|&\geq\frac{(n!)^2}{(n-N)!}\,|t|^{n-N}\left(|x_N(0)|\left(\tfrac{1}{N!}\right)^2-\sum_{i=0}^{N-1}\left(\tfrac{1}{i!}\right)^2|x_i(0)|\tfrac{(n-N)!}{(n-i)!}|t|^{N-i}\right)\\
&\geq\frac{(n!)^2}{(n-N)!}\,|t|^{n-N}\left(|x_N(0)|\left(\tfrac{1}{N!}\right)^2-\sum_{i=0}^{N-1}\left(\tfrac{1}{i!}\right)^2|x_i(0)||t|^{N-i}\right),
\end{aligned}$$

where the first factor does not lie in the space s of rapidly decreasing sequences, and where the second factor is larger than $\varepsilon>0$ for t small enough. So at least for a dense set of initial values this differential equation has no local solution.

This also shows that the theorem of Frobenius is wrong in the following sense: The vector field $x\mapsto T(x)$ generates a 1-dimensional subbundle E of the tangent bundle on the open subset $s\setminus\{0\}$. It is involutive since it is 1-dimensional. But through points representing finite sequences there exist no local integral submanifolds (M with $TM=E|M$). Namely, if c were a smooth non-constant curve with $c'(t)=f(t).T(c(t))$ for some smooth function f, then $x(t):=c(h(t))$ would satisfy $x'(t)=T(x(t))$, where h is a solution of $h'(t)=1/f(h(t))$.

Example 2. Next consider $E:=\mathbb{R}^{\mathbb{N}}$ and the continuous linear operator $T:E\to E$ given by $T(x_0,x_1,\dots):=(x_1,x_2,\dots)$. The corresponding differential equation has solutions for every initial value $x(0)$, since the coordinates must satisfy the recursive relation $x_{k+1}(t)=x_k'(t)$, and hence any smooth function $x_0:\mathbb{R}\to\mathbb{R}$ gives rise to a solution $x(t):=(x_0^{(k)}(t))_k$ with initial value $x(0)=(x_0^{(k)}(0))_k$. So by Borel's theorem there exist solutions to this equation for all initial values and the difference of any two functions with same initial value is an arbitrary infinite flat function. Thus, the solutions are far from being unique. Note that $\mathbb{R}^{\mathbb{N}}$ is a topological direct summand in $C^\infty(\mathbb{R},\mathbb{R})$ via the projection $f\mapsto(f(n))_n$, and hence the same situation occurs in $C^\infty(\mathbb{R},\mathbb{R})$.

Note that it is not possible to choose the solution depending smoothly on the initial value: suppose that x is a local smooth mapping $\mathbb{R}\times E\supset I\times U\to E$ with $x(0,y)=y$ and $\partial_t x(t,y)=T(x(t,y))$, where I is an open interval containing 0 and U is open in E. Then $x_0:I\times U\to\mathbb{R}$ induces a smooth local mapping $x_0{}^\vee:U\to C^\infty(I,\mathbb{R})$, which is a right inverse to the linear infinite jet mapping $j_0^\infty:C^\infty(I,\mathbb{R})\to\mathbb{R}^{\mathbb{N}}=E$. Then the derivative of $x_0{}^\vee$ at any point in U would be a continuous linear right inverse to j_0^∞, which does not exist (since $\mathbb{R}^{\mathbb{N}}$ does not admit a continuous norm, whereas $C^\infty(I,\mathbb{R})$ does for compact I, see also [**123**, IV.3.9]).

Also in this example the theorem of Frobenius is wrong, now in the following sense: On the complement of $T^{-1}(0) = \mathbb{R} \times 0$ we consider again the 1-dimensional subbundle generated by the vector field T. For every smooth function $f \in C^\infty(\mathbb{R}, \mathbb{R})$ the infinite jet $t \mapsto j_t^\infty(f)$ is an integral curve of T. We show that integral curves through a fixed point sweep out arbitrarily high dimensional submanifolds of $\mathbb{R}^\mathbb{N}$: Let $\varphi : \mathbb{R} \to [0,1]$ be smooth, $\varphi(t) = 0$ near $t = 0$, and $\varphi(t) = 1$ near $t = 1$. For each $(s_2, \dots, s_N)$ we get an integral curve

$$t \mapsto j_t \Big(t + \frac{s_2}{2!}\varphi(t)(t-1)^2 + \frac{s_3}{3!}\varphi(t)(t-1)^3 + \dots + \frac{s_N}{N!}\varphi(t)(t-1)^N \Big)$$

connecting $(0, 1, 0, \dots)$ with $(1, 1, s_2, s_3, \dots, s_N, 0, \dots)$, and for small s this integral curve lies in $\mathbb{R}^\mathbb{N} \setminus 0$.

Problem: Can any two points be joined by an integral curve in $\mathbb{R}^\mathbb{N} \setminus 0$: One has to find a smooth function on $[0, 1]$ with prescribed jets at 0 and 1 which is nowhere flat in between.

Example 3. Let now $E := C^\infty(\mathbb{R}, \mathbb{R})$, and consider the continuous linear operator $T : E \to E$ given by $T(x) := x'$. Let $x : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ be a solution of the equation $x'(t) = T(x(t))$. In terms of $\hat{x} : \mathbb{R}^2 \to \mathbb{R}$ this says $\frac{\partial}{\partial t}\hat{x}(t, s) = \frac{\partial}{\partial s}\hat{x}(t, s)$. Hence, $r \mapsto \hat{x}(t - r, s + r)$ has vanishing derivative everywhere, and so this function is constant, and in particular $x(t)(s) = \hat{x}(t, s) = \hat{x}(0, s + t) = x(0)(s + t)$. Thus, we have a smooth solution x uniquely determined by the initial value $x(0) \in C^\infty(\mathbb{R}, \mathbb{R})$, which even describes a flow for the vector field T in the sense of 32.13 below. In general however, this solution is not real-analytic, since for any $x(0) \in C^\infty(\mathbb{R}, \mathbb{R})$ which is not real-analytic in a neighborhood of a point s the composite $\mathrm{ev}_s \circ x = x(s + \_)$ is not real-analytic around 0.

32.13. The flow of a vector field

Let $X \in \mathfrak{X}(M)$ be a kinematic vector field. A *local flow* Fl^X for X is a smooth mapping $M \times \mathbb{R} \supset U \xrightarrow{\mathrm{Fl}^X} M$ defined on a c^∞-open neighborhood U of $M \times 0$ such that

(1) $U \cap (\{x\} \times \mathbb{R})$ is a connected open interval.

(2) If $\mathrm{Fl}^X_s(x)$ exists then $\mathrm{Fl}^X_{t+s}(x)$ exists if and only if $\mathrm{Fl}^X_t(\mathrm{Fl}^X_s(x))$ exists, and we have equality.

(3) $\mathrm{Fl}^X_0(x) = x$ for all $x \in M$.

(4) $\frac{d}{dt}\mathrm{Fl}^X_t(x) = X(\mathrm{Fl}^X_t(x))$.

In formulas similar to (4) we will often omit the point x for sake of brevity, without signalizing some differentiation in a space of mappings. The latter will be done whenever possible in section 42.

32.14. Lemma. *Let $X \in \mathfrak{X}(M)$ be a kinematic vector field which admits a local flow Fl^X_t. Then for each integral curve c of X we have $c(t) = \mathrm{Fl}^X_t(c(0))$, thus there exists a unique maximal flow. Furthermore, X is Fl^X_t-related to itself, i.e., $T(\mathrm{Fl}^X_t) \circ X = X \circ \mathrm{Fl}^X_t$.*

Proof. We compute

$$\begin{aligned}\tfrac{d}{dt}\operatorname{Fl}^X(-t,c(t)) &= -\tfrac{d}{ds}|_{s=-t}\operatorname{Fl}^X(s,c(t)) + \tfrac{d}{ds}|_{s=t}\operatorname{Fl}^X(-t,c(s))\\ &= -\tfrac{d}{ds}|_{s=0}\operatorname{Fl}^X_{-t}\operatorname{Fl}^X(s,c(t)) + T(\operatorname{Fl}^X_{-t}).c'(t)\\ &= -T(\operatorname{Fl}^X_{-t}).X(c(t)) + T(\operatorname{Fl}^X_{-t}).X(c(t)) = 0.\end{aligned}$$

Thus, $\operatorname{Fl}^X_{-t}(c(t)) = c(0)$ is constant, so $c(t) = \operatorname{Fl}^X_t(c(0))$. For the second assertion we have $X\circ\operatorname{Fl}^X_t = \frac{d}{dt}\operatorname{Fl}^X_t = \frac{d}{ds}|_0\operatorname{Fl}^X_{t+s} = \frac{d}{ds}|_0(\operatorname{Fl}^X_t\circ\operatorname{Fl}^X_s) = T(\operatorname{Fl}^X_t)\circ\frac{d}{ds}|_0\operatorname{Fl}^X_s = T(\operatorname{Fl}^X_t)\circ X$, where we omit the point $x\in M$ for the sake of brevity. □

33. Differential Forms

This section is devoted to the search for the right notion of differential forms which are stable under Lie derivatives $\mathcal{L}_X$, exterior derivative d, and pullback f^*. Here chaos breaks out (as one referee has put it) since the classically equivalent descriptions of differential forms give rise to many different classes; in the table 33.21 we shall have 12 classes. But fortunately it will turn out in 33.22 that there is only one suitable class satisfying all requirements, namely

$$\Omega^k(M) := C^\infty(L^k_{\mathrm{alt}}(TM, M\times\mathbb{R})).$$

33.1. Cotangent bundles

We consider the contravariant smooth functor which associates to each convenient vector space E its dual E' of bounded linear functionals, and we apply it to the kinematic tangent bundle TM described in 28.12 of a smooth manifold M (see [**75**, 29.5]) to get the *kinematic cotangent bundle* $T'M$. A smooth atlas $(U_\alpha, u_\alpha : U_\alpha\to E_\alpha)$ of M gives the cocycle of transition functions

$$U_{\alpha\beta}\ni x\mapsto d(u_\beta\circ u_\alpha^{-1})(u_\alpha(x))^*\in GL(E'_\beta, E'_\alpha).$$

If we apply the same duality functor to the operational tangent bundle DM described in 28.12 we get the *operational cotangent bundle* $D'M$. A smooth atlas $(U_\alpha, u_\alpha : U_\alpha\to E_\alpha)$ of M now gives rise to the following cocycle of transition functions

$$U_{\alpha\beta}\ni x\mapsto D(u_\beta\circ u_\alpha^{-1})(u_\alpha(x))^*\in GL((D_0E_\beta)', (D_0E_\alpha)'),$$

see 28.9 and 28.12.

For each $k\in\mathbb{N}$ we get the *operational cotangent bundle* $(D^{(k)})'M$ *of order* $\le k$, which is described by the same cocycle of transition functions but now restricted to have values in $GL((D^{(k)}_0E_\beta)', (D^{(k)}_0E_\alpha)')$, see 28.10.

33.2. 1-forms

Let M be a smooth manifold. We may define 1-forms to be the sections of one of the cotangent bundles. In particular, a *kinematic 1-form* is just a smooth section of the kinematic cotangent bundle $T'M$. So $C^\infty(M\leftarrow T'M)$ denotes the convenient vector space (with the structure from 30.1) of all kinematic 1-forms on M.

An *operational 1-form* is just a smooth section of the operational cotangent bundle $D'M$. So $C^\infty(M\leftarrow D'M)$ denotes the convenient vector space (with the structure from 30.1) of all operational 1-forms on M.

For each $k \in \mathbb{N}$ we get the convenient vector space $C^\infty(M\leftarrow(D^{(k)})'(M))$ of all *operational 1-forms of order* $\le k$, a closed linear subspace of $C^\infty(M\leftarrow D'M)$.

On the other hand, we may consider 1-forms as $C^\infty(M,\mathbb{R})$-module homomorphisms for the spaces of vector fields defined in 32.1 to $C^\infty(M,\mathbb{R})$. In particular and more precisely, a *modular 1-form* is a bounded linear sheaf homomorphism $\omega : \operatorname{Der}(C^\infty(\_,\mathbb{R})) \to C^\infty(\_,\mathbb{R})$ which satisfies $\omega_U(f\cdot X) = f\cdot\omega_U(X)$ for $X \in \operatorname{Der}(C^\infty(U,\mathbb{R})) = C^\infty(U\leftarrow DU)$ and $f \in C^\infty(U,\mathbb{R})$ for each open $U \subset M$. We denote the space of all modular 1-forms by

$$\operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$$

and we equip it with the initial structure of a convenient vector space induced by the closed linear embedding

$$\operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big) \hookrightarrow \prod_U L\Big(C^\infty(U\leftarrow DU), C^\infty(U,\mathbb{R})\Big).$$

Convention

Similarly as in 32.1, we shall follow the convention that either the manifolds in question are smoothly regular or that Hom means the space of sheaf homomorphisms (as defined above) between the sheafs of sections like $C^\infty(M\leftarrow DM)$ of the respective vector bundles. This is justified by 33.3 below.

33.3. Lemma. *If M is smoothly regular, the bounded $C^\infty(M,\mathbb{R})$-module homomorphisms $\omega : C^\infty(M\leftarrow DM) \to C^\infty(M,\mathbb{R})$ are exactly the modular 1-forms and this identification is an isomorphism of the convenient vector spaces.*

Proof. If $X \in C^\infty(M\leftarrow DM)$ vanishes on an open subset $U \subset M$ then also $\omega(X)$: For $x \in U$ we take a bump function $g \in C^\infty(M,\mathbb{R})$ at x, i.e. $g = 1$ near x and $\operatorname{supp}(g) \subset U$. Then $\omega(X) = \omega((1-g)X) = (1-g)\omega(X)$ which is zero near x. So $\omega(X) \mid U = 0$.

Now let $X \in C^\infty(U\leftarrow DU)$ for a c^∞-open subset U of M. We have to show that we can define $\omega_U(X) \in C^\infty(U,\mathbb{R})$ in a unique manner. For $x \in U$ let $g \in C^\infty(M,\mathbb{R})$ be a bump function at x, i.e. $g = 1$ near x and $\operatorname{supp}(g) \subset U$. Then $gX \in C^\infty(M\leftarrow DM)$, and $\omega(gX)$ makes sense. By the argument above, $\omega(gX)(x)$ is independent of the choice of g. So let $\omega_U(X)(x) := \omega(gX)(x)$. It has all required properties since the topology on $C^\infty(U\leftarrow DU)$ is initial with respect to all mappings $X \mapsto gX$, where g runs through all bump functions as above.

That this identification furnishes an isomorphism of convenient vector spaces can be seen as in 32.4. □

33.4. Lemma. *On any manifold M the space of operational 1-forms is a closed linear subspace of that of modular 1-forms:*

$$C^\infty(M\leftarrow D'M) \hookrightarrow \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$$

The closed vector bundle embedding $TM \to DM$ induces a bounded linear mapping $C^\infty(M\leftarrow D'M) \to C^\infty(M\leftarrow T'M)$.

We do not know whether $C^\infty(M\leftarrow D'M) \to C^\infty(M\leftarrow T'M)$ is surjective or even final.

Proof. A smooth section $\omega \in C^\infty(M\leftarrow D'M)$ defines a modular 1-form which assigns $\omega_U(X)(x) := \omega(x)(X(x))$ to $X \in C^\infty(U\leftarrow DU)$ and $x \in U$, by 32.2, since this gives a bounded sheaf homomorphism which is $C^\infty(\_,\mathbb{R})$-linear.

To show that this gives an embedding onto a c^∞-closed linear subspace we consider the following diagram, where (U_α) runs through an open cover of charts of M. Then the vertical mappings are closed linear embeddings by 30.1, 33.1, and 32.2.

$$\begin{array}{ccc}
C^\infty(M\leftarrow D'M) & \longrightarrow & \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big) \\
\downarrow & & \downarrow \\
\prod_\alpha C^\infty(U_\alpha, (D_0E_\alpha)') & \longrightarrow & \prod_\alpha L\Big(C^\infty(U_\alpha\leftarrow DU_\alpha), C^\infty(U_\alpha,\mathbb{R})\Big) \\
\downarrow & & \downarrow \\
\prod_\alpha C^\infty(U_\alpha\times D_0E_\alpha,\mathbb{R}) & \longrightarrow & \prod_\alpha C^\infty\Big(C^\infty(U_\alpha, D_0E_\alpha)\times U_\alpha,\mathbb{R}\Big)
\end{array}$$

The horizontal bottom arrow is the mapping $f \mapsto ((X,x)\mapsto f(x,X(x)))$, which is an embedding since $(X,x)\mapsto(x,X(x))$ has $(x,Y)\mapsto(\operatorname{const}(Y),x)$ as smooth right inverse. $\square$

33.5. Lemma. *Let M be a smooth manifold such that for all model spaces E the convenient vector space D_0E has the bornological approximation property [**75**, 28.6]. Then*

$$C^\infty(M\leftarrow D'M) \cong \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big).$$

If all model spaces E have the bornological approximation property then $D_0E = E''$, and the space E'' also has the bornological approximation property. So in this case,

$$\operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big) \cong C^\infty(M\leftarrow T'''M).$$

If, moreover, all E are reflexive, we have

$$\operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big) \cong C^\infty(M\leftarrow T'M),$$

as in finite dimensions.

Proof. By lemma 33.4 the space $C^\infty(M\leftarrow D'M)$ is a closed linear subspace of the convenient vector space $\operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$. We have to show that any sheaf homomorphism $\omega \in \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$ lies in $C^\infty(M\leftarrow D'M)$. This is a local question, hence we may assume that M is a c^∞-open subset of E.

We have to show that for each $X \in C^\infty(U, D_0E)$ the value $\omega_U(X)(x)$ depends only on $X(x) \in D_0E$. So let $X(x) = 0$, and we have to show that $\omega_U(X)(x) = 0$.

By assumption, there is a net $\ell_\alpha \in (D_0E)' \otimes D_0E \subset L(D_0E, D_0E)$ of bounded linear operators with finite dimensional images, which converges to Id_{D_0E} in the bornological topology of $L(D_0E, D_0E)$. Then $X_\alpha := \ell_\alpha \circ X$ converges to X in $C^\infty(U, D_0E)$ since $X^* : L(D_0E, D_0E) \to C^\infty(U, D_0E)$ is continuous linear. It remains to show that $\omega_U(X_\alpha)(x) = 0$ for each α.

We have $\ell_\alpha = \sum_{i=1}^n \varphi_i \otimes \partial_i \in (D_0E)' \otimes D_0E$, hence $X_\alpha = \sum(\varphi_i \circ X).\partial_i$ and $\omega_U(X_\alpha)(x) = \sum \varphi_i(X(x)).\omega_U(\partial_i)(x) = 0$ since $X(x) = 0$.

So we get a fiber linear mapping $\omega : DM \to M \times \mathbb{R}$ which is given by $\omega(X_x) = (x, \omega_U(X)(x))$ for any $X \in C^\infty(U \leftarrow DU)$ with $X(x) = X_x$. Obviously, $\omega : DM \to M \times \mathbb{R}$ is smooth and gives rise to a smooth section of $D'M$.

If E has the bornological approximation property, then by 28.7 we have $D_0E = E''$. If ℓ_α is a net of finite dimensional bounded operators which converges to Id_E in $L(E, E)$, then the finite dimensional operators ℓ_α^{**} converge to $\mathrm{Id}_E'' = \mathrm{Id}_{E''}$ in $L(E'', E'')$, in the bornological topology. The rest follows from theorem 28.7 □

33.6. Queer 1-forms

Let E be a convenient vector space without the bornological approximation property, for example an infinite dimensional Hilbert space. Then there exists a bounded linear functional $\alpha \in L(E, E)'$ which vanishes on $E' \otimes E$ such that $\alpha(\mathrm{Id}_E) = 1$. Then $\omega_U : C^\infty(U, E) \to C^\infty(U, \mathbb{R})$, given by $\omega_U(X)(x) := \alpha(dX(x))$, is a bounded sheaf homomorphism which is a module homomorphism, since $\omega_U(f.X)(x) = \alpha(df(x) \otimes X(x) + f(x).dX(x)) = f(x)\omega_U(X)(x)$. Note that $\omega_U(X)(x)$ does not depend only on $X(x)$. So there are many 'kinematic modular 1-forms' which are not kinematic 1-forms.

This process can be iterated to involve higher derivatives like for derivations, see 28.2, but we resist the temptation to pursue this task. It would be more interesting to produce queer modular 1-forms which are not operational 1-forms.

33.7. k-forms

Since the natural mapping $\Lambda^k E^* \to L^k_{\mathrm{alt}}(E, \mathbb{R})$ is usually not an isomorphism for convenient vector spaces E, we have multiple ways to define k-forms for $k \geq 2$. For a smooth manifold M there are at least 10 interesting spaces of k-forms, see the diagram below where $A := C^\infty(M,\mathbb{R})$.

$$\begin{tikzcd}
\Lambda^k_A C^\infty(M\leftarrow D'M) \arrow[rr] \arrow[dr] \arrow[ddd] & & \Lambda^k_A C^\infty(M\leftarrow T'M) \arrow[dr] \arrow[ddd] & \\
& C^\infty\Big(M\leftarrow\Lambda^k(D'M)\Big) \arrow[rr] \arrow[d] & & C^\infty\Big(M\leftarrow\Lambda^k(T'M)\Big) \arrow[d] \\
& C^\infty\Big(M\leftarrow L^k_{\text{alt}}(DM, M\times\mathbb{R})\Big) \arrow[rr, dashed] \arrow[dd] & & C^\infty\Big(M\leftarrow L^k_{\text{alt}}(TM, M\times\mathbb{R})\Big) \arrow[dd] \\
\Lambda^k_A \operatorname{Hom}_A\Big(C^\infty(M\leftarrow DM), A\Big) \arrow[rr] \arrow[dr] & & \Lambda^k_A \operatorname{Hom}_A\Big(C^\infty(M\leftarrow TM), A\Big) \arrow[dr] & \\
& \operatorname{Hom}^{k,\text{alt}}_A\Big(C^\infty(M\leftarrow DM), A\Big) \arrow[rr] & & \operatorname{Hom}^{k,\text{alt}}_A\Big(C^\infty(M\leftarrow TM), A\Big)
\end{tikzcd}$$

Here Λ^k is the bornological exterior product which was treated in 5.9. One could also start from other tensor products. By $\Lambda^k_A = \Lambda^k_{C^\infty(M,\mathbb{R})}$ we mean the convenient module exterior product, the subspace of all skew symmetric elements in the k-fold bornological tensor product over A, see 5.21. By $\operatorname{Hom}^k_{C^\infty(M,\mathbb{R}),\text{alt}} = \operatorname{Hom}^{k,\text{ alt}}_{C^\infty(M,\mathbb{R})}$ we mean the convenient space of all bounded homomorphism between the respective sheaves of convenient modules over the sheaf of smooth functions.

33.8. Wedge product

For differential forms φ of degree k and ψ of degree ℓ and for (local) vector fields X_i (or tangent vectors) we put

$$\begin{aligned}
&(\varphi\wedge\psi)(X_1,\dots,X_{k+\ell}) = \\
&\qquad = \tfrac{1}{k!\,\ell!}\sum_{\sigma\in\mathcal{S}_{k+\ell}} \operatorname{sign}\sigma\cdot\varphi(X_{\sigma 1},\dots,X_{\sigma k}).\psi(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)}).
\end{aligned}$$

This is well defined for differential forms in each of the spaces in 33.7 and others (see 33.12 below) and gives a differential form of the same type of degree $k+\ell$. The wedge product is associative, i.e $(\varphi\wedge\psi)\wedge\tau = \varphi\wedge(\psi\wedge\tau)$, and graded commutative, i. e. $\varphi\wedge\psi = (-1)^{k\ell}\psi\wedge\varphi$. These properties are proved in multilinear algebra. There arise several kinds of algebras of differential forms.

33.9. Pullback of differential forms

Let $f : N \to M$ be a smooth mapping between smooth manifolds, and let φ be a differential form on M of degree k in any of the following spaces:

$$C^\infty\Big(M\leftarrow L^k_{\text{alt}}(D^\alpha M, M\times\mathbb{R})\Big) \text{ for } D^\alpha\in\{D, D^{(k)}, D^{[1,\infty)}, T\}.$$

In this situation the pullback $f^*\varphi$ is defined for tangent vectors $X_i \in D^\alpha_x N$ by

$$(f^*\varphi)_x(X_1,\dots,X_k) := \varphi_{f(x)}(D^\alpha_x f.X_1,\dots,D^\alpha_x f.X_k).$$

Then we have $f^*(\varphi \wedge \psi) = f^*\varphi \wedge f^*\psi$, so the linear mapping f^* is an algebra homomorphism. Moreover, we have $(g\circ f)^* = f^*\circ g^*$ if $g : M \to P$, and $(\mathrm{Id}_M)^* = \mathrm{Id}$, and $(f, \varphi) \mapsto f^*\varphi$ is smooth in all these cases.

If $f : N \to M$ is a local diffeomorphism, then we may define the pullback $f^*\varphi$ also for a modular differential form $\varphi \in \mathrm{Hom}^{k,\text{ alt}}_{C^\infty(M,\mathbb{R})}(C^\infty(M\leftarrow D^\alpha M), C^\infty(M, \mathbb{R}))$, by

$$(f^*\varphi)|_U(X_1, \dots, X_k) := \varphi|_{f(U)}(D^\alpha f \circ X_1 \circ (f|U)^{-1}, \dots, D^\alpha f \circ X_k \circ (f|U)^{-1}) \circ f.$$

These two definitions are intertwined by the canonical mappings between different spaces of differential forms.

33.10. Insertion operator

For a vector field $X \in C^\infty(M\leftarrow D^\alpha M)$ where $D^\alpha \in \{D, D^{(k)}, D^{[1,\infty)}, T\}$ we define the *insertion operator*

$$\begin{aligned} i_X = i(X) : \mathrm{Hom}^{k,\text{ alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^\alpha M), C^\infty(M, \mathbb{R})\Big) \to \\ \to \mathrm{Hom}^{k-1,\text{ alt}}_{C^\infty(M,\mathbb{R})}(C^\infty(M\leftarrow D^\alpha M), C^\infty(M, \mathbb{R})) \\ (i_X\varphi)(Y_1, \dots, Y_{k-1}) := \varphi(X, Y_1, \dots, Y_{k-1}). \end{aligned}$$

It restricts to operators

$$i_X = i(X) : C^\infty\Big(M\leftarrow L^k_{\mathrm{alt}}(D^\alpha M, M \times \mathbb{R})\Big) \to C^\infty\Big(M\leftarrow L^{k-1}_{\mathrm{alt}}(D^\alpha M, M \times \mathbb{R})\Big).$$

33.11. Lemma. *i_X is a graded derivation of degree -1, so we have $i_X(\varphi \wedge \psi) = i_X\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge i_X\psi$.*

Proof. We have

$$\begin{aligned} (i_{X_1}(\varphi \wedge \psi))(X_2, \dots, X_{k+\ell}) &= (\varphi \wedge \psi)(X_1, \dots, X_{k+\ell}) \\ &= \tfrac{1}{k!\,\ell!} \sum_\sigma \mathrm{sign}(\sigma)\, \varphi(X_{\sigma 1}, \dots, X_{\sigma k})\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}). \end{aligned}$$

$$\begin{aligned} (i_{X_1}\varphi \wedge \psi &+ (-1)^k\varphi \wedge i_{X_1}\psi)(X_2, \dots, X_{k+\ell}) \\ &= \tfrac{1}{(k-1)!\,\ell!} \sum_\sigma \mathrm{sign}(\sigma)\, \varphi(X_1, X_{\sigma 2}, \dots, X_{\sigma k})\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}) \\ &\quad + \frac{(-1)^k}{k!\,(\ell-1)!} \sum_\sigma \mathrm{sign}(\sigma)\, \varphi(X_{\sigma 2}, \dots, X_{\sigma(k+1)})\psi(X_1, X_{\sigma(k+2)}, \dots). \end{aligned}$$

Using the skew symmetry of φ and ψ we may distribute X_1 to each position by adding an appropriate sign. These are $k + \ell$ summands. Since $\frac{1}{(k-1)!\,\ell!} + \frac{1}{k!\,(\ell-1)!} = \frac{k+\ell}{k!\,\ell!}$, and since we can generate each permutation in $\mathcal{S}_{k+\ell}$ in this way, the result follows. □

33.12. Exterior derivative

Let $U \subset E$ be c^∞-open in a convenient vector space E, and let $\omega \in C^\infty(U, L^k_{\mathrm{alt}}(E; \mathbb{R}))$ be a kinematic k-form on U. We define the exterior derivative $d\omega \in C^\infty(U, L^{k+1}_{\mathrm{alt}}(E; \mathbb{R}))$

as the skew symmetrization of the derivative $\omega'(x) : E \to L^k_{\mathrm{alt}}(E;\mathbb{R})$ (sorry for the two notions of d, it's only local); i.e.

$$(1)\qquad (d\omega)(x)(X_0,\dots,X_k) = \sum_{i=0}^{k}(-1)^i\omega'(x)(X_i)(X_0,\dots,\widehat{X_i},\dots,X_k)$$
$$= \sum_{i=0}^{k}(-1)^i d(\omega(\_)(X_0,\dots,\widehat{X_i},\dots,X_k))(x)(X_i)$$

where $X_i \in E$. Next we view the X_i as 'constant vector fields' on U and try to replace them by kinematic vector fields. Let us compute first for $X_j \in C^\infty(U,E)$, where we suppress obvious evaluations at $x \in U$:

$$\sum_i(-1)^i X_i(\omega\circ(X_0,\dots,\widehat{X_i},\dots,X_k))(x) =$$
$$= \sum_i(-1)^i(\omega'(x).X_i)(X_0,\dots,\widehat{X_i},\dots,X_k)+$$
$$+\sum_{j<i}(-1)^i\omega\circ(X_0,\dots,dX_j(x).X_i,\dots,\widehat{X_i},\dots,X_k)+$$
$$(2)\qquad +\sum_{i<j}(-1)^i\omega\circ(X_0,\dots,\widehat{X_i},\dots,dX_j(x).X_i,\dots,X_k) =$$
$$= \sum_i(-1)^i(\omega'(x).X_i)(X_0,\dots,\widehat{X_i},\dots,X_k)+$$
$$+\sum_{j<i}(-1)^{i+j}\omega\circ(dX_j(x).X_i - dX_i(x).X_j,X_0,\dots,\widehat{X_j},\dots,\widehat{X_i},\dots,X_k)$$
$$= \sum_i(-1)^i(\omega'(x).X_i)(X_0,\dots,\widehat{X_i},\dots,X_k)+$$
$$+\sum_{j<i}(-1)^{i+j}\omega\circ([X_i,X_j],X_0,\dots,\widehat{X_j},\dots,\widehat{X_i},\dots,X_k).$$

Combining 2 and 1 gives the *global formula for the exterior derivative*

$$(3)\qquad (d\omega)(x)(X_0,\dots,X_k) = \sum_{i=0}^{k}(-1)^i X_i(\omega\circ(X_0,\dots,\widehat{X_i},\dots,X_k))+$$
$$+\sum_{i<j}(-1)^{i+j}\omega\circ([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k).$$

Formula 3 defines the exterior derivative for modular forms on $C^\infty(M\leftarrow D^\alpha M)$ for each $D^\alpha \in \{T,D,D^{[1,\infty)}\}$, since it gives multilinear module homomorphisms by the Lie module properties of the Lie bracket, see 32.5 and 32.8.

The local formula 1 gives the exterior derivative on $C^\infty\big(M\leftarrow L^k_{\mathrm{alt}}(TM, M\times\mathbb{R})\big)$: Local expressions 1 for two different charts describe the same differential form since both can be written in the global form 3, and the canonical mapping $C^\infty\big(M\leftarrow L^k_{\mathrm{alt}}(TM,M\times\mathbb{R})\big) \to \mathrm{Hom}^{k,\,\mathrm{alt}}_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M), C^\infty(M,\mathbb{R}))$ is injective, since we use sheaves on the right hand side.

The first line of the local formula 1 gives an exterior derivative d^{loc} also on the space $C^\infty\big(U\leftarrow L^k_{\mathrm{alt}}(DU,\mathbb{R})\big)$, where U is an open subset in a convenient vector space E, if we replace $\omega'(x)$ by $D_x\omega : D_0E \to D_0(L^k_{\mathrm{alt}}(D_0E,\mathbb{R}))$ composed with the canonical mapping

$$D_0(L^k_{\mathrm{alt}}(D_0E,\mathbb{R})) \xrightarrow{(\_)^{[1]}} D_0(L^k_{\mathrm{alt}}(D_0E,\mathbb{R})) \xrightarrow{(\partial^{[1]})^{-1}} L^k_{\mathrm{alt}}(D_0E,\mathbb{R})'' = \\ = (\Lambda^k(D_0E))''' \xrightarrow{\iota^*} (\Lambda^k(D_0E))' = L^k_{\mathrm{alt}}(D_0E,\mathbb{R}).$$

Here $\iota : \Lambda^k D_0E \to (\Lambda^k D_0E)''$ is the canonical embedding into the bidual. If we replace the derivative by D in the second expression of the local formula $\boxed{1}$ we get the same expression. For $\omega \in C^\infty(U, L^k_{\mathrm{alt}}(D_0E,\mathbb{R}))$ we have

$$\begin{aligned}
(d^{\mathrm{loc}}\omega)(x)(X_0,\dots,X_k) &= \sum_{i=0}^k (-1)^i D_x(\omega(_)(X_0,\dots,\widehat{X_i},\dots,X_k))(X_i) \\
&= \sum_{i=0}^k (-1)^i D_x(\mathrm{ev}_{(X_0,\dots,\widehat{X_i},\dots,X_k)} \circ \omega)(X_i) \\
&= \sum_{i=0}^k (-1)^i D_{\omega(x)}(\mathrm{ev}_{(X_0,\dots,\widehat{X_i},\dots,X_k)}).D_x\omega.X_i \\
&= \sum_{i=0}^k (-1)^i (D^{(1)}_{\omega(x)}\, \mathrm{ev}_{(X_0,\dots,\widehat{X_i},\dots,X_k)}).(D_x\omega.X_i)^{[1]} \qquad \text{by } \boxed{28.11.4} \\
&= \sum_{i=0}^k (-1)^i (\mathrm{ev}_{(X_0\wedge\dots\widehat{X_i}\dots\wedge X_k)})^{**}.(\partial^{[1]})^{-1}.(D_x\omega.X_i)^{[1]} \qquad \text{by } \boxed{28.11.3} \\
&= \sum_{i=0}^k (-1)^i\, \mathrm{ev}_{(X_0\wedge\dots\widehat{X_i}\dots\wedge X_k)}.\iota^*.(\partial^{[1]})^{-1}.(D_x\omega.X_i)^{[1]} \\
&= \sum_{i=0}^k (-1)^i \Big(\iota^* \circ (\partial^{[1]})^{-1} \circ (_)^{[1]} \circ D_x\omega\Big)(X_i)(X_0,\dots,\widehat{X_i},\dots,X_k),
\end{aligned}$$

since the following diagram commutes:

$$\begin{tikzcd}
(\Lambda^k D_0E)' \arrow[rr, "\mathrm{ev}_{(X_0\wedge\dots\widehat{X_i}\dots\wedge X_k)}"] & & \mathbb{R} \\
(\Lambda^k D_0E)''' \arrow[u, "\iota^*"] \arrow[rr, "(\mathrm{ev}_{(X_0\wedge\dots\widehat{X_i}\dots\wedge X_k)})^{**}"] & & \mathbb{R} \arrow[u, equal]
\end{tikzcd}$$

The local formula $\boxed{1}$ describes by a similar procedure the local exterior derivative d^{loc} also on $C^\infty\big(M\leftarrow L^k_{\mathrm{alt}}(D^{[1,\infty)}M,\mathbb{R})\big)$.

For the forms of tensorial type (i.e. involving Λ^k) there is no exterior derivative in general, since the derivative is not tensorial in general.

For a manifold M let us now consider the following diagram of certain spaces of differential forms.

$$\begin{tikzcd}
C^\infty\Big(M\leftarrow L^k_{\text{alt}}(DM, M\times\mathbb{R})\Big) \arrow[r, dashed] \arrow[d] & \operatorname{Hom}^{k,\text{alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big) \arrow[d] \\
C^\infty\Big(M\leftarrow L^k_{\text{alt}}(D^{[1,\infty)}M, M\times\mathbb{R})\Big) \arrow[r, dashed] \arrow[d] & \operatorname{Hom}^{k,\text{alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^{[1,\infty)}M), C^\infty(M,\mathbb{R})\Big) \arrow[d] \\
C^\infty\Big(M\leftarrow L^k_{\text{alt}}(TM, M\times\mathbb{R})\Big) \arrow[r] & \operatorname{Hom}^{k,\text{alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow TM), C^\infty(M,\mathbb{R})\Big)
\end{tikzcd}$$

If M is a c^∞-open subset in a convenient vector space E, on the two upper left spaces there exists only the local (from formula 1) exterior derivative d^{loc}. On all other spaces the global (from formula 3) exterior derivative d makes sense. All canonical mappings in this diagram commute with the exterior derivatives except the dashed ones. In fact, the following example 33.13 shows that

1. The dashed arrows do not commute with the respective exterior derivatives.
2. The (global) exterior derivative does not respect the spaces on the left hand side of the diagram except the bottom one.
3. The dashed arrows are not surjective.

The example 33.14 shows that the local exterior derivative on the two upper left spaces does not commute with pullbacks of smooth mappings, not even of diffeomorphisms, in general. So it does not even exist on manifolds. Furthermore, $d^{\text{loc}}\circ d^{\text{loc}}$ is more interesting than 0, see example 33.16.

33.13. Example. *Let U be c^∞-open in a convenient vector space E. If $\omega\in C^\infty(U,E''')=C^\infty(U,L(D^{(1)}_0E,\mathbb{R}))$ then in general the exterior derivative*

$$d\omega\in\operatorname{Hom}^{2,\,alt}_{C^\infty(U,\mathbb{R})}\Big(C^\infty(U\leftarrow DU), C^\infty(U,\mathbb{R})\Big)$$

is not contained in $C^\infty\Big(U\leftarrow L^2_{alt}(DU, U\times\mathbb{R})\Big)$.

Proof. Let $X,Y\in C^\infty(U,E'')$. The Lie bracket $[X,Y]$ is given in [**75**, 32.7], and ω depends only on the $D^{(1)}$-part of the bracket. Thus, we have

$$\begin{aligned}
d\omega(X,Y)(x) &= X(\omega(Y))(x)-Y(\omega(X))(x)-\omega([X,Y])(x)\\
&=\langle X(x), d\langle\omega,Y\rangle_{E''}(x)\rangle_{E'}-\langle Y(x), d\langle\omega,X\rangle_{E''}(x)\rangle_{E'}\\
&\quad-\langle\omega(x),(dY(x)^t)^*.X(x)-(dX(x)^t)^*.Y(x)\rangle_{E''}\\
&=\langle X(x),\langle d\omega(x),Y(x)\rangle_{E''}\rangle_{E'}+\langle X(x),\langle\omega(x),dY(x)\rangle_{E''}\rangle_{E'}-\\
&\quad-\langle Y(x),\langle d\omega(x),X(x)\rangle_{E''}\rangle_{E'}-\langle Y(x),\langle\omega(x),dX(x)\rangle_{E''}\rangle_{E'}\\
&\quad-\langle\omega(x),(dY(x)^*\circ\iota_{E'})^*.X(x)\rangle_{E''}+\langle\omega(x),(dX(x)^*\circ\iota_{E'})^*.Y(x)\rangle_{E''}.
\end{aligned}$$

Let us treat the terms separately which contain derivatives of X or Y. Choosing X constant (but arbitrary) we have to consider only the following expression:

$$\begin{aligned}\langle X(x), \langle\omega(x), dY(x)\rangle_{E''}\rangle_{E'} &- \langle\omega(x), (dY(x)^* \circ \iota_{E'})^*.X(x)\rangle_{E''} = \\ &= \langle X(x), \omega(x)\circ dY(x)\rangle_{E'} - \langle\omega(x), \iota_{E'}^*.dY(x)^{**}.X(x)\rangle_{E''} \\ &= \langle X(x), dY(x)^*.\omega(x)\rangle_{E'} - \langle\iota_{E'}^{**}.\omega(x), dY(x)^{**}.X(x)\rangle_{E''''} \\ &= \langle\iota_{E'''}.\omega(x), dY(x)^{**}.X(x)\rangle_{E''''} - \langle\iota_{E'}^{**}.\omega(x), dY(x)^{**}.X(x)\rangle_{E''''} \\ &= \langle(\iota_{E'''} - \iota_{E'}^{**}).\omega(x), dY(x)^{**}.X(x)\rangle_{E''''},\end{aligned}$$

which is not 0 in general since $\ker(\iota_{E'''} - \iota_{E'}^{**}) = \iota_{E'}(E')$ at least for Banach spaces, see [**23**, 1.15], applied to $\iota_{E'}$. So we may assume that $(\iota_{E'''} - \iota_{E'}^{**}).\omega(x) \neq 0 \in E''''$. We choose a non-reflexive Banach space which is isomorphic to its bidual ([**52**]) and we choose as $dY(x)$ this isomorphism, then $dY(x)^{**}$ is also an isomorphism, and a suitable $X(x)$ makes the expression nonzero. □

Note that this also shows that for general convenient vector spaces E the exterior derivative $d\omega$ is in $C^\infty(U, L^2_{\text{alt}}(D_0^{(1)}E, \mathbb{R}))$ only if $\omega \in C^\infty(M\leftarrow T'M)$. Note that even for $\omega : U \to E'''$ a constant 1-form of order 1 we need not have $d\omega = 0$.

33.14. Example. *There exist c^∞-open subsets U and V in a Banach space E, a diffeomorphism $f : U \to V$, and a 1-form $\omega \in C^\infty(U, L(E'', \mathbb{R}))$ such that $d^{loc} f^*\omega \neq f^* d^{loc}\omega$.*

Proof. We start in a more general situation. Let $f : U \to V \subset F$ be a smooth mapping, and let $X_x, Y_x \in D_x^{(1)}U = E''$. Then we have

$$\begin{aligned}d^{\text{loc}}(f^*\omega)_x(X_x, Y_x) &= D_x(f^*\omega(\_).Y_x).X_x - D_x(f^*\omega(\_).X_x).Y_x \\ &= D_x(\omega(f(\_)).D_{(\_)}f.Y_x).X_x - \dots \\ &= X_x\langle\omega\circ f, D_{(\_)}f.Y_x\rangle_{F''} - \dots \\ &= d\langle\omega\circ f, df(\_)^{**}.Y_x\rangle_{F''}(x)^{**}.X_x - \dots \quad \text{by [75, 32.6]} \\ &= d\langle\omega(f(\_)), df(x)^{**}.Y_x\rangle_{F''}(x)^{**}.X_x + \\ &\quad + d\langle\omega(f(x)), df(\_)^{**}.Y_x\rangle_{F''}(x)^{**}.X_x - \dots \qquad \text{by [75, 32.6]}\\ f^*(d^{\text{loc}}\omega)_x(X_x, Y_x) &= (d^{\text{loc}}\omega)_{f(x)}(D_xf.X_x, D_xf.Y_x) \\ &= D_{f(x)}(\omega_{(\_)}.D_xf.Y_x).D_xf.X_x - D_{f(x)}(\omega_{(\_)}.D_xf.X_x).D_xf.Y_x \\ &= d\langle\omega(\_), df(x)^{**}.Y_x\rangle_{F''}(f(x))^{**}.df(x)^{**}.X_x - \dots\end{aligned}$$

Recall that for $\ell \in H' = L(H, \mathbb{R})$ the bidual mapping satisfies $L(H'', \mathbb{R}) \ni \ell^{**} = \iota_{H'}(\ell) \in H'''$. Then for the difference we get

$$\begin{aligned}&d^{\text{loc}}(f^*\omega)_x(X_x, Y_x) - f^*(d^{\text{loc}}\omega)_x(X_x, Y_x) \\ &\quad = d\langle\omega(f(x)), df(\_)^{**}.Y_x\rangle_{F''}(x)^{**}.X_x - d\langle\omega(f(x)), df(\_)^{**}.X_x\rangle_{F''}(x)^{**}.Y_x \\ &\quad = \langle i_{F'''}\omega(f(x)), d(df(\_)^{**}.Y_x)(x)^{**}.X_x - d(df(\_)^{**}.X_x)(x)^{**}.Y_x\rangle_{F''''}.\end{aligned}$$

This expression does not vanish in general, e.g., when the following choices are made: We put $\omega(f(x)) = \iota_{F'}.\ell = \ell^{**}$ for $\ell \in F'$, and we have

$$\begin{aligned}d(d(\ell\circ f)(\_)^{**}Y_x)(x)^{**}.X_x &= d(d\langle\ell, f\rangle_F(\_)^{**}Y_x)(x)^{**}.X_x \\ &= d\langle\iota_{F'}\ell, df(\_)^{**}Y_x\rangle_{F''}(x)^{**}.X_x \\ &= \langle\iota_{F'''}\ell^{**}, d(df(\_)^{**}Y_x)(x)^{**}.X_x\rangle_{F''''},\end{aligned}$$

which is not symmetric in general for $\ell\circ f = \mathrm{ev}: G'\times G\to\mathbb{R}$ (for a non reflexive Banach space G) by the argument in [**75**, 32.7]. It remains to show that such a factorization of ev over a diffeomorphism f and $\ell\in(G'\times G)'$ is possible. Choose $(\alpha,x)\in G'\times G$ such that $\langle\alpha,x\rangle=1$, and consider

$$\begin{aligned} G'\times G = G'\times\ker\alpha\times\mathbb{R}.x &\xrightarrow{f} G'\times\ker\alpha\times\mathbb{R}.x \xrightarrow{\ell}\mathbb{R}\\ (\beta,y,tx)&\mapsto(\beta,y,\langle\beta,y+tx\rangle_G.x)\mapsto\langle\beta,y+tx\rangle_G\\ (\beta,y,\tfrac{t-\langle\beta,y\rangle}{\langle\beta,x\rangle}.x)&\longleftarrow(\beta,y,tx). \quad\square \end{aligned}$$

33.15. Proposition. *Let $f:M\to N$ be a smooth mapping between smooth manifolds. Then we have*

$$f^*\circ d = d\circ f^*: C^\infty\Big(N\leftarrow L^k_{\mathrm{alt}}(TN,N\times\mathbb{R})\Big)\to C^\infty\Big(M\leftarrow L^{k+1}_{alt}(TM,M\times\mathbb{R})\Big).$$

Proof. Since by 33.12 the local and global formula for the exterior derivative coincide on spaces $C^\infty(L^k_{\mathrm{alt}}(D^\alpha M,M\times\mathbb{R}))$ we shall prove the result with help of the local formula. So we may assume that $f:U\to V$ is smooth between c^∞-open sets in convenient vector spaces E and F, respectively. Note that we may use the global formula only if f is a local diffeomorphism, see 33.9.

For $\omega\in C^\infty(V,L^k_{\mathrm{alt}}(F,\mathbb{R}))$, $x\in U$, and $X_i\in E$ we have

$$(f^*\omega)(x)(X_1,\dots,X_k)=\omega(f(x))(df(x).X_1,\dots,df(x).X_k),$$

so by 33.12.1 we may compute

$$\begin{aligned}
(df^*\omega)(x)(X_0,\dots,X_k) &= \sum_{i=0}^k(-1)^i d(f^*\omega)(x)(X_i)(X_0,\dots,\widehat{X_i},\dots,X_k)\\
&= \sum_{i=0}^k(-1)^i\,(d\omega(f(x)).df(x).X_i)(df(x).X_0,\dots,\widehat{\ i\ },\dots,df(x).X_k)\\
&\quad+\sum_{i=0}^k(-1)^i\sum_{j<i}\omega(f(x))(df(x).X_0,\dots,d^2f(x).(X_i,X_j),\dots,\widehat{\ i\ },\dots,df(x).X_k)\\
&\quad+\sum_{i=0}^k(-1)^i\sum_{j>i}\omega(f(x))(df(x).X_0,\dots,\widehat{\ i\ },\dots,d^2f(x).(X_i,X_j),\dots,df(x).X_k)\\
&= \sum_{i=0}^k(-1)^i\,d\omega(f(x))(df(x).X_0,\dots,df(x).X_k)\\
&\quad+\sum_{j<i}(-1)^{i+j}\omega(f(x))(d^2f(x).(X_i,X_j)-d^2f(x).(X_j,X_i),\\
&\qquad\qquad df(x).X_0,\dots,\widehat{\ j\ },\dots,\widehat{\ i\ },\dots,df(x).X_k)\\
&= (f^*d\omega)(x)(X_0,\dots,X_k)+0. \quad\square
\end{aligned}$$

33.16. Example. *There exists a smooth function*

$$f\in C^\infty(E,\mathbb{R})=C^\infty(E,L^0_{alt}(D^{(1)}E,\mathbb{R}))$$

such that

$$0\neq d^{loc}d^{loc}f\in C^\infty(E,L^2_{alt}(D^{(1)}E,\mathbb{R})).$$

Proof. Let $f \in C^\infty(E,\mathbb{R})$, $X_x, Y_x \in D^{(1)}_x E = E''$. Then we have

$$\begin{aligned}
(d^{\mathrm{loc}} f)_x(X_x) &= df(x)^{**}.X_x = \langle \iota_{F'}.df(x), X_x\rangle_{E''} \\
&= \langle X_x, df(x)\rangle_{E'} \\
(d^{\mathrm{loc}} d^{\mathrm{loc}} f)_x(X_x, Y_x) &= \\
&= d\langle Y_x, df(_)\rangle_{E'}(x)^{**}.X_x - d\langle X_x, df(_)\rangle_{E'}(x)^{**}.Y_x \\
&= \langle \iota_{E''}.Y_x, d(df)(x)^{**}.X_x\rangle_{E'''} - \langle \iota_{E''}.X_x, d(df)(x)^{**}.Y_x\rangle_{E'''} \\
&= \langle d(df)(x)^{**}.X_x, Y_x\rangle_{E''} - \langle d(df)(x)^{**}.Y_x, X_x\rangle_{E''},
\end{aligned}$$

which does not vanish in general by the argument in [**75**, 32.7]. □

33.17. Lie derivatives

Let D^α denote one of T, D, or $D^{[1,\infty)}$. For a vector field $X \in C^\infty(M\leftarrow D^\alpha M)$ and $\omega \in \mathrm{Hom}^{k,\,\mathrm{alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^\alpha M), C^\infty(M,\mathbb{R})\Big)$ we define the *Lie derivative* $\mathcal{L}_X\omega$ of ω along X by

$$(\mathcal{L}_X\omega)|_U(Y_1,\dots,Y_k) = X(\omega(Y_1,\dots,Y_k)) - \sum_{i=1}^k \omega|_U(Y_1,\dots,[X,Y_i],\dots,Y_k),$$

for $Y_1,\dots,Y_k \in C^\infty(U\leftarrow D^\alpha U)$. From 32.5 it follows that

$$\mathcal{L}_X\omega \in \mathrm{Hom}^{k,\,\mathrm{alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^\alpha M), C^\infty(M,\mathbb{R})\Big).$$

33.18. Theorem. *The following formulas hold for $C^\infty\Big(M\leftarrow L^k_{alt}(TM, M\times\mathbb{R})\Big)$ and for the spaces $\mathrm{Hom}^{k,\,alt}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^\alpha M), C^\infty(M,\mathbb{R})\Big)$ where D^α is any of T, D, or $D^{[1,\infty)}$.*

(1) $i_X(\varphi\wedge\psi) = i_X\varphi\wedge\psi + (-1)^{\deg\varphi}\varphi\wedge i_X\psi$.

(2) $\mathcal{L}_X(\varphi\wedge\psi) = \mathcal{L}_X\varphi\wedge\psi + \varphi\wedge\mathcal{L}_X\psi$.

(3) $d(\varphi\wedge\psi) = d\varphi\wedge\psi + (-1)^{\deg\varphi}\varphi\wedge d\psi$.

(4) $d^2 = d\circ d = \frac{1}{2}[d,d] = 0$.

(5) $[\mathcal{L}_X, d] = \mathcal{L}_X\circ d - d\circ\mathcal{L}_X = 0$.

(6) $[i_X, d] = i_X\circ d + d\circ i_X = \mathcal{L}_X$.

(7) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X\circ\mathcal{L}_Y - \mathcal{L}_Y\circ\mathcal{L}_X = \mathcal{L}_{[X,Y]}$.

(8) $[\mathcal{L}_X, i_Y] = \mathcal{L}_X i_Y - i_Y\mathcal{L}_X = i_{[X,Y]}$.

(9) $[i_X, i_Y] = i_X i_Y + i_Y i_X = 0$.

(10) $\mathcal{L}_{f.X}\varphi = f.\mathcal{L}_X\varphi + df\wedge i_X\varphi$.

33.21. Review of operations on differential forms

Space of differential forms	$\mathcal{L}_X$	d	f^*
$\operatorname{Hom}^{*,\text{ alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$	+	+	diff
$\Lambda^*_{C^\infty(M,\mathbb{R})} \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow DM), C^\infty(M,\mathbb{R})\Big)$	–	–	diff
$\Lambda^*_{C^\infty(M,\mathbb{R})} C^\infty(M\leftarrow D'M)$	–	–	+
$C^\infty\big(M\leftarrow \Lambda^*(D'M)\big)$	–	–	+
$C^\infty\Big(M\leftarrow L^*_{\text{alt}}(DM, M\times\mathbb{R})\Big)$	flow	–	+
$\operatorname{Hom}^{*,\text{ alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^{[1,\infty)}M), C^\infty(M,\mathbb{R})\Big)$	+	+	diff
$C^\infty\Big(M\leftarrow L^*_{\text{alt}}(D^{[1,\infty)}M, M\times\mathbb{R})\Big)$	flow	–	+
$\operatorname{Hom}^{*,\text{ alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow D^{(1)}M), C^\infty(M,\mathbb{R})\Big)$	flow	?	diff
$C^\infty\Big(M\leftarrow L^*_{\text{alt}}(D^{(1)}M, M\times\mathbb{R})\Big)$	flow	–	+
$\operatorname{Hom}^{*,\text{ alt}}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow TM), C^\infty(M,\mathbb{R})\Big)$	+	+	diff
$\Lambda^*_{C^\infty(M,\mathbb{R})} \operatorname{Hom}_{C^\infty(M,\mathbb{R})}\Big(C^\infty(M\leftarrow TM), C^\infty(M,\mathbb{R})\Big)$	–	–	diff
$\Lambda^*_{C^\infty(M,\mathbb{R})} C^\infty(M\leftarrow T'M)$	–	–	+
$C^\infty\big(M\leftarrow \Lambda^*(T'M)\big)$	–	–	+
$C^\infty\Big(M\leftarrow L^*_{\text{alt}}(TM, M\times\mathbb{R})\Big)$	+	+	+

In this table a '–' means that the space is not invariant under the operation on top of the column, a '+' means that it is invariant, 'diff' means that it is invariant under f^* only for diffeomorphisms f, and 'flow' means that it is invariant under $\mathcal{L}_X$ for all kinematic vector fields X which admit local flows. Moreover, Λ^*, $\operatorname{Hom}^{*,\text{alt}}_A$, and L^*_{alt} denote the $\mathbb{N}$-graded spaces with these spaces for $*$ replaced by $k \in \mathbb{N}$ as k-homogeneous parts.

33.22. Remark

From the table 33.21 we see that for many purposes only one space of differential forms is fully suited. We will denote from now on by

$$\Omega^k(M) := C^\infty\Big(M\leftarrow L^k_{\text{alt}}(TM, M\times\mathbb{R})\Big)$$

the *space of differential forms*, for a smooth manifold M. By 30.1 it carries the structure of a convenient vector space induced by the closed embedding

$$\begin{aligned}\Omega^k(M) &\to \prod_\alpha C^\infty\big(U_\alpha, L^k_{\text{alt}}(E,\mathbb{R})\big)\\ s &\mapsto pr_2\circ\psi_\alpha\circ(s \mid U_\alpha),\end{aligned}$$

where $(U_\alpha, u_\alpha : U_\alpha \to E)$ is a smooth atlas for the manifold M, and where $\psi_\alpha := L^k_{\text{alt}}(Tu_\alpha^{-1}, \mathbb{R}))$ is the induced vector bundle chart.

Similarly, we denote by

$$\Omega^k(M,V) := C^\infty\Big(M\leftarrow L^k_{\text{alt}}(TM, M\times V)\Big)$$

the *space of differential forms with values in a convenient vector space* V, and by

$$\Omega^k(M;E) := C^\infty\Big(M \leftarrow L^k_{\text{alt}}(TM,E)\Big)$$

the *space of differential forms with values in a vector bundle* $p: E \to M$.

Lemma. *The space $\Omega^k(M)$ is isomorphic as convenient vector space to the closed linear subspace of $C^\infty(TM \times_M \ldots \times_M TM, \mathbb{R})$ consisting of all fiberwise k-linear alternating smooth functions in the vector bundle structure $TM \oplus \cdots \oplus TM$ from* [**75**, 29.5].

Proof. By [**75**, 27.17], the space $C^\infty(TM \times_M \ldots \times_M TM, \mathbb{R})$ carries the initial structure with respect to the closed linear embedding

$$C^\infty\Big(TM \times_M \ldots \times_M TM, \mathbb{R}\Big) \to \prod_\alpha C^\infty(u_\alpha(U_\alpha) \times E \times \ldots \times E, \mathbb{R}),$$

and $C^\infty\big(u_\alpha(U_\alpha) \times E \times \ldots \times E, \mathbb{R}\big)$ contains an isomorphic copy of $C^\infty(U_\alpha, L^k_{\text{alt}}(E, \mathbb{R}))$ as closed linear subspace by cartesian closedness. □

Corollary. *All the important mappings are smooth:*

$$\begin{aligned} d &: \Omega^k(M) \to \Omega^{k+1}(M) \\ i &: \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M) \\ \mathcal{L} &: \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^k(M) \\ f^* &: \Omega^k(M) \to \Omega^k(N) \end{aligned}$$

where $f: N \to M$ is a smooth mapping. The last mappings is even smooth considered as mapping $(f,\omega) \mapsto f^*\omega$, $C^\infty(N,M) \times \Omega^k(M) \to \Omega^k(N)$.

Recall once more the formulas for $\omega \in \Omega^k(M)$ and $X_i \in \mathfrak{X}(M)$, from 33.12.3, 33.10, 33.17 *:*

$$\begin{aligned} (d\omega)(x)(X_0, \ldots, X_k) &= \sum_{i=0}^k (-1)^i X_i\big(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \\ (i_X\varphi)(X_1, \ldots, X_{k-1}) &= \varphi(X, X_1, \ldots, X_{k-1}), \\ (\mathcal{L}_X\omega)(X_1, \ldots, X_k) &= X(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^k \omega(X_1, \ldots, [X, X_i], \ldots, X_k). \end{aligned}$$

Proof. For d we use the local formula 33.12.1, smoothness of i is obvious, and for the Lie derivative we may use formula 33.18.6. The pullback mapping f^* is induced from $Tf \times \ldots \times Tf$. □

Chapter IX
Manifolds of Mappings

Manifolds of smooth mappings between finite dimensional manifolds are the foremost examples of infinite dimensional manifolds, and in particular diffeomorphism groups can only be treated in a satisfactory manner at the level of generality developed in this book: One knows from [**102**] that a Banach Lie group acting effectively on a finite dimensional compact manifold is necessarily finite dimensional. So there is **no way** to model the diffeomorphism group on Banach spaces as a manifold.

The space of smooth mappings $C^\infty(M,N)$ carries a natural atlas with charts induced by any exponential mapping on N 42.1, which permits us also to consider certain infinite dimensional manifolds N in 42.4. Unfortunately, for noncompact M, the space $C^\infty(M,N)$ is not locally contractible in the compact-open C^∞-topology, and the natural chart domains are quite small: Namely, the natural model spaces turn out to be convenient vector spaces of sections with compact support of vector bundles f^*TN, which have been treated in detail in section 30. Thus, the manifold topology on $C^\infty(M,N)$ is finer than the Whitney C^∞-topology, and we denote by $\mathfrak{C}^\infty(M,N)$ the resulting smooth manifold (otherwise, e.g. $C^\infty(\mathbb{R},\mathbb{R})$ would have two meanings).

With a careful description of the space of smooth curves 42.5 we can later often avoid the explicit use of the atlas, for example when we show that the composition mapping is smooth in 42.13. Since we insist on charts the exponential law for manifolds of mappings holds only for a compact source manifold M, 42.14.

If we insist that the exponential law should hold for manifolds of mappings between all (even only finite dimensional) manifolds, then one is quickly lead to a more general notion of a manifold, where an atlas of charts is replaced by the system of all smooth curves. One is lead to further requirements: tangent spaces should be convenient vector spaces, the tangent bundle should be trivial along smooth curves via a kind of parallel transport, and a local addition as in 42.4 should exist. In this way one obtains a cartesian closed category of smooth manifolds and smooth mappings between them, where those manifolds with Banach tangent spaces are exactly the classical smooth manifolds with charts. Theories along these lines can be found in [**73**], [**99**], and [Kriegl, 1984]. Unfortunately they found no applications, and even the authors were not courageous enough to pursue them further and to include them in this book. But we still think that it is a valuable theory, since for instance the diffeomorphism group $\operatorname{Diff}(M)$ of a non-compact finite dimensional smooth manifold M with the compact-open C^∞-topology is a Lie group in this sense with the space of all vector fields on M as Lie algebra. Also, in section [**75**, 45] results will appear which indicate that ultimately this is a more natural setting.

Let us return (after discussing non-contents) to describing the contents of this chapter. For the tangent space we have a natural diffeomorphism $T\mathfrak{C}^\infty(M,N) \cong \mathfrak{C}^\infty_c(M,TN) \subset \mathfrak{C}^\infty(M,TN)$, see [**75**, 42.17]. In the same manner we also treat manifolds of real analytic mappings from a compact manifold M into N.

In section 43 on diffeomorphism groups we first show that the group $\operatorname{Diff}(M)$ is a regular smooth Lie group 43.1. The proof clearly shows the power of our calculus: It is quite obvious that the inversion is smooth, whereas more traditional treatments as in [**84**], [**96**], and [Michor, 1980c] needed specially tailored inverse function theorems in infinite dimensions. The Lie algebra of the diffeomorphism group is the space $\mathfrak{X}_c(M)$ of all vector fields with compact support on M, with the negative of the usual Lie bracket. The exponential mapping exp is the flow mapping to time 1, but it is not surjective on any neighborhood of the identity 43.2, and $\operatorname{Ad}\circ\exp : \mathfrak{X}_c(M) \to L(\mathfrak{X}_c(M), \mathfrak{X}_c(M))$ is not real analytic, [**75**, 43.3]. Real analytic diffeomorphisms on a real analytic compact manifold form a regular real analytic Lie group [**75**, 43.4]. Also regular Lie groups are the subgroups of volume preserving 43.7, symplectic 43.12, exact symplectic [**75**, 43.13], or contact diffeomorphisms 43.19.

In section [**75**, 44] we treat principal bundles with a diffeomorphism group as structure group. The first example is the space of all embeddings between two manifolds 44.1, a sort of nonlinear Grassmann manifold, in particular if the image space is an infinite dimensional convenient vector space which leads to a smooth manifold which is a classifying space for the diffeomorphism group of a compact manifold 44.24. Another example is the nonlinear frame bundle of a fiber bundle with compact fiber [**75**, 44.5], for which we investigate the action of the gauge group on the space of generalized connections [**75**, 44.14] and show that in the smooth case there never exist slices [**75**, 44.19], [**75**, 44.20].

In section [**75**, 45] we compute explicitly all geodesics for some natural (pseudo) Riemannian metrics on the space of all Riemannian metrics. Section [**75**, 46] is devoted to the Korteweg–De Vrieß equation which is shown to be the geodesic equation of a certain right invariant Riemannian metric on the Virasoro group. Here we also compute the curvature [**75**, 46.13] and the Jacobi equation [**75**, 46.14].

41.5. The compact-open topology on spaces of continuous mappings

Let M and N be Hausdorff topological spaces. The best known topology on the space $C(M,N)$ of all continuous mappings is the *compact-open topology* or *CO-topology.* A subbasis for this topology consists of all sets of the form $\{f \in C(M,N) : f(K) \subseteq U\}$, where K runs through all compact subsets in M and U through all open subsets of N. This is a Hausdorff topology, since it is finer than the topology of pointwise convergence.

It is easy to see that if M has a countable basis of the compact sets and is compactly generated (4.7.(i), i.e., M carries the final topology with respect to the inclusions of its compact subsets), and if N is a complete metric space, then there exists a complete metric on $(C(M,N), CO)$, so it is a Baire space.

41.6. The graph topology

For $f \in C(M,N)$ let $\operatorname{graph}_f : M \to M \times N$ be given by $\operatorname{graph}_f(x) = (x, f(x))$, the graph mapping of f.

The *WO-topology* or *wholly open topology* on $C(M,N)$ is given by the basis $\{f \in C(M,N) : f(M) \subset U\}$, where U runs through all open sets in N. It is not Hausdorff, since mappings with the same image cannot be separated.

The *graph topology* or *WO*^{0}*-topology* on $C(M,N)$ is induced by the mapping

$$\operatorname{graph} : C(M,N) \to (C(M, M \times N),\ \text{WO-topology}).$$

A basis for it is given by all sets of the form $\{f \in C(M,N) : \operatorname{graph}_f(M) \subseteq U\}$, where U runs through all open sets in $M \times N$. This topology is Hausdorff since it is finer than the compact-open topology. Note that a continuous mapping $g : N \to P$ induces a continuous mapping $g_* : C(M,N) \to C(M,P)$ for the WO^0-topology, since $\operatorname{graph}_{g\circ f} = (\operatorname{Id} \times g) \circ \operatorname{graph}_f$.

If M is paracompact and (N,d) is a metric space, then for $f \in C(M,N)$ the sets $\{g \in C(M,N) : d(g(x), f(x)) < \varepsilon(x) \text{ for all } x \in M\}$ form a basis of neighborhoods, where ε runs through all positive continuous functions on M. This is easily seen.

41.7. Lemma. *Let N be metrizable, and let M satisfy one of the following conditions:*

(1) *M is locally compact with a countable basis of open sets.*
(2) $M = \mathbb{R}^{(\mathbb{N})}$.

Then for any sequence (f_n) in $C(M,N)$ the following holds: (f_n) converges to f in the WO^0-topology if and only if there exists a compact set $K \subseteq M$ such that f_n equals f off K for all but finitely many n, and $f_n|K$ converges to $f|K$ uniformly.

Note that in case (2) we get $f_n = f$ for all but finitely many n, since f differs from f_n on a c^∞-open subset.

Proof. Clearly, the condition above implies convergence. Conversely, let (f_n) and f in $C(M,N)$ be such that the condition does not hold. In case (1) let $K_n \subset K_{n+1}^o$ be a basis of the compact sets in M, and in case (2) let $K_n := \{x \in \mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})} : |x^i| \le n \text{ for } i \le n\}$. Then either f_n does not converge to f in the compact-open topology, or there exists $x_n \notin K_n$ with $d(f_n(x_n), f(x_n)) =: \varepsilon_n > 0$. Then (x_n) is without cluster point in M: This is obvious in case (1), and in case (2) this can be seen by the following argument: Assume that there exists a cluster point y. Let N be so large that $\operatorname{supp}(y) \subset \{0, \dots, N\}$ and $|y^i| \le N - 1$ for all i. Then we define $k_n \in \mathbb{N}$ and $\delta_n > 0$ by

$$\begin{cases} k_n := n, \delta_n := 1 & \text{for } n \le N \text{ or } \operatorname{supp}(x_n) \subseteq \{1, \dots, n\} \\ k_n := \min\{i > n : x_n^i \neq 0\}, \delta_n := |x_n^{k_n}| & \text{otherwise} \end{cases}$$

Then $x_n - y \notin U := \{z : |z^{k_i}| < \delta_i \text{ for all } i\}$ for $n > N$, so y cannot be a cluster point.

The set $\{(x, y) \in M \times N : \text{ if } x = x_n \text{ for some } n \text{ then } d(f(x_n), y) < \varepsilon_n\}$ is an open neighborhood of $\operatorname{graph}_f(M)$ not containing any $\operatorname{graph}_{f_n}(M)$. So f_n cannot converge to f in the WO^0-topology. □

41.8. Lemma. *Let E be a convenient vector space, and suppose that M satisfies the following condition:*

(1) *Each neighborhood of each point contains a sequence without cluster point in M.*

Then for $f \in C(M, E)$ we have $tf \to 0$ in the WO^0-topology for $t \to 0$ in $\mathbb{R}$ if and only if $f = 0$.

Moreover, each open subset in an infinite dimensional locally convex space has property (1).

Proof. The mapping $f \mapsto g \circ f$ is continuous in the WO^0-topologies, so by composing with bounded linear functionals on E we may suppose that $E = \mathbb{R}$.

Suppose that $f \neq 0$, say $f(x) = 2$ for some x. Then $f(y) > 1$ for y in some neighborhood U of x, which contains a sequence x_n without cluster point in M. Then $\{(x, y) \in M \times \mathbb{R} : \text{ if } x = x_n \text{ then } y < 1/n\}$ is an open neighborhood of $\operatorname{graph}_0(M)$ not containing any $\operatorname{graph}_{tf}(M)$ for $t \neq 0$. So tf cannot converge to 0 in the WO^0-topology.

For the last assertion we have to show that the unit ball of each seminorm p in an infinite dimensional locally convex vector space M contains a sequence without cluster point. If the seminorm has non-trivial kernel $p^{-1}(0)$ then $(n.x)_n$ for $0 \neq x \in p^{-1}(0)$ has this property. If p has trivial kernel, it is a norm, and the unit ball in the normed space (M, p) contains a sequence without cluster point, since otherwise the unit ball would be compact, and (M, p) would be finite dimensional. This sequence has also no cluster point in M, since M has a finer topology. □

41.9. The CO^k-topology on spaces of smooth mappings

Let M and N be smooth manifolds, possibly infinite dimensional. For $0 \le k \le \infty$ the *compact-open C^k-topology* or *CO^k-topology* on the space $C^\infty(M, N)$ of all smooth mappings $M \to N$ is induced by the k-jet extension 41.3 from the CO-topology

$$j^k : C^\infty(M, N) \to (C(M, J^k(M, N)), \mathrm{CO}).$$

We conclude with some remarks. If M is infinite dimensional it would be more natural to replace the system of compact sets in M by the system of all subsets on which each smooth real valued function is bounded. Since these topologies will play only minor roles in this book we do not develop them here.

41.10. Whitney C^k-topology

Let M and N be smooth manifolds, possibly infinite dimensional. For $0 \le k \le \infty$ the *Whitney C^k-topology* or *WO^k-topology* on the space $C^\infty(M,N)$ of all smooth mappings $M \to N$ is induced by the k-jet extension 41.3 from the WO-topology

$$j^k : C^\infty(M,N) \to (C(M, J^k(M,N)), \mathrm{WO}).$$

A basis for the open sets is given by all sets of the form $\{f \in C^\infty(M,N) : j^k f(M) \subset U\}$, where U runs through all open sets in the smooth manifold $J^k(M,N)$. A smooth mapping $g : N \to P$ induces a smooth mapping $J^k(M,g) : J^k(M,N) \to J^k(M,P)$ by 41.3.4, and thus in turn a continuous mapping $g_* : C^\infty(M,N) \to C^\infty(M,P)$ for the WO^k-topologies for each k.

For a convenient vector space E and for a manifold M modeled on infinite dimensional Fréchet spaces (so that there the c^∞-topology coincides with the locally convex one) we see from 41.8 that for $f \in C^\infty(M,E)$ we have $t.f \to 0$ for $t \to 0$ in the WO^k-topology if and only if $f = 0$. So $(C^\infty(M,E), \mathrm{WO}^k)$ does not contain a non-trivial topological vector space if M is infinite dimensional.

If M is compact, then the WO^k-topology and the CO^k-topology coincide on the space $C^\infty(M,N)$ for all k.

41.11. Lemma. *Let M, N be smooth manifolds, where M is finite dimensional and second countable, and where N is metrizable. Then $J^\infty(M,N)$ is also a metrizable manifold. If, moreover, N is second countable then also $J^\infty(M,N)$ is also second countable.*

Let $K_n \subset K_{n+1}^o \subset K_{n+1}$ be a compact exhaustion of M. Then the following is a basis of open sets for the Whitney C^∞-topology:

$$M(U,m) := \{f \in C^\infty(M,N) : j^{m_n} f(M \setminus K_n^o) \subset U_n\},$$

where (m_n) is any sequence in $\mathbb{N}$ and where $U_n \subset J^{m_n}(M,N)$ is an open subset.

Proof. Looking at 41.3 we see that $J^\infty(M,N)$ is a bundle over $M \times N$ with Fréchet spaces as fibers, so it is metrizable. We can also write

$$M(U,m) := \{f \in C^\infty(M,N) : j^\infty f(M \setminus K_n^o) \subset (\pi^\infty_{m_n})^{-1} U_n\}.$$

By pulling up to higher jet bundles, we may assume that m_n is strictly increasing. If we put $V_n = (\pi^\infty_{m_n})^{-1} U_n$, we may then replace V_n by $V_0 \cap \dots \cap V_n$ without changing $M(U,m)$. But then we may replace $M \setminus K_n^o$ by $K_{n+1} \setminus K_n^o$ without changing the set. Using that $J^\infty(M,N)$ carries the initial topology with respect to all projections $\pi^\infty_l : J^\infty(M,N) \to J^l(M,N)$ by 41.3.6, we get an equivalent basis of open sets given by

$$M(U) := \{f \in C^\infty(M,N) : j^\infty f(K_{n+1} \setminus K_n^o) \subset U_n\},$$

where now $U_n \subset J^\infty(M,N)$ is a sequence of open sets. It is obvious that this basis generates a topology which is finer than the WO^∞-topology. To show the converse let $f \in M(U)$. Let d be a compatible metric on the metrizable manifold $J^\infty(M,N)$, and let $0 < \varepsilon_n$ be smaller than the distance between the compact set $j^\infty f(K_{n+1} \setminus K_n^o)$ and the complement of its open neighborhood U_n. Let ε be a positive continuous function on M such that $0 < \varepsilon(x) < \varepsilon_n$ for $x \in K_{n+1} \setminus K_n^o$, and consider the open set $W := \{\sigma \in J^\infty(M,N) : d(\sigma, j^\infty f(\alpha(\sigma))) < \varepsilon(\alpha(\sigma))\}$ in $J^\infty(M,N)$. Then $f \in \{g \in C^\infty(M,N) : j^\infty g(M) \subset W\} \subseteq M(U)$. □

41.12. Corollary. *Let M, N be smooth manifolds, where M is finite dimensional and second countable, and where N is metrizable. Then the CO^k-topology is metrizable. If N is also second countable then so is the CO^k-topology.*

Proof. Use 41.11 and [20, X, 3.3]. □

41.13. Comparison of topologies on $C^\infty(M, E)$

Let $p : E \to M$ be a smooth finite dimensional vector bundle over a finite dimensional second countable base manifold M. We consider the space $C^\infty_c(M\leftarrow E)$ of all smooth sections of E with compact support, equipped with the bornological locally convex topology from 30.4,

$$C^\infty_c(M\leftarrow E) = \varinjlim_K C^\infty_K(M\leftarrow E),$$

where K runs through all compact sets in M and each of the spaces $C^\infty_K(M\leftarrow f^*TN)$ is equipped with the topology of uniform convergence (on K) in all derivatives separately, as in 30.4, reformulated for the bornological topologies. Consider also the space $C^\infty(M, E)$ of all smooth mappings $M \to E$, equipped with the Whitney C^∞-topology, and the subspace $C^\infty(M\leftarrow E)$ of all smooth sections, with the induced topology.

Lemma. *Then the canonical injection*

$$C^\infty_c(M\leftarrow E) \to C^\infty(M, E)$$

is a topological embedding. The subspace $C^\infty(M\leftarrow E)$ is a vector space, but scalar multiplication is jointly continuous in the induced topology on it if and only if M is compact or the fiber is 0. The maximal topological vector space contained in $C^\infty(M\leftarrow E)$ is just $C^\infty_c(M\leftarrow E)$.

Proof. That the injection is an embedding is clear by contemplating the description of the Whitney C^∞-topology given in lemma 41.11, which obviously is the inductive limit topology $\varinjlim C^\infty_{K_n}(E)$. The rest follows from 41.7 since $t.f \to 0$ for $t \to 0$ in in $C^\infty(M,E)$ for WO^∞ if and only if $t.j^\infty f \to 0$ in $C^\infty(M, J^\infty(E))$ for the WO^0-topology. □

41.14. Tubular neighborhoods

Let M be an (embedded) submanifold of a smooth finite dimensional manifold N. Then the *normal bundle* of M in N is the vector bundle $\mathcal{N}(M) := (TN|M)/TM \xrightarrow{\pi} M$ with fiber T_xN/T_xM over a point $x \in M$. A *tubular neighborhood* of M in N consists of:

(1) A fiberwise radial open neighborhood $\tilde{U} \subset \mathcal{N}(M)$ of the 0-section in the normal bundle
(2) A diffeomorphism $\varphi : \tilde{U} \to U \subset N$ onto an open neighborhood U of M in N, which on the 0-section coincides with the projection of the normal bundle.

It is well known that tubular neighborhoods exist.

42. Manifolds of Mappings

The aim is to turn $C^\infty(M,N)$ for (finite dimensional) smooth manifolds M and N into a smooth manifold which hopefully satisfies the exponential law

$$C^\infty(P, C^\infty(M,N)) \cong C^\infty(P\times M, N)$$

for any other (finite dimensional) smooth manifold P. Since for finite dimensional vector spaces M and N the appropriate topology for the exponential law on the vector space $C^\infty(M,N)$ is that of uniform convergence of each derivative on each compact set, one would expect that this topology is also appropriate in the general case.

a.1 Example. Note however, that $C(\bigsqcup_{\mathbb{N}} S^1, S^1)$ is not locally path connected in the compact open topology, since any neighborhood of $f_0 := \bigcup_{\mathbb{N}} \mathrm{id}_{S^1}$ contains $f_n := \mathrm{id} \sqcup \dots \sqcup \mathrm{id} \sqcup p_2 \sqcup \mathrm{id} \sqcup \dots$ for sufficiently large n, where $p_2 : z \mapsto z^2$ sits on the n-th summand S^1. If f_n could be connected to f_0 by a path $h : [0,1] \to C(\bigsqcup_{\mathbb{N}} S^1, S^1)$, then $\hat{h} : [0,1] \times \bigsqcup_{\mathbb{N}} S^1 \to S^1$ would be a homotopy between f_0 and f_n. But then composition with the embedding $\mathrm{id} \times \mathrm{inj}_n : [0,1] \times S^1 \to [0,1] \times \bigsqcup_{\mathbb{N}} S^1$ would be a homotopy between id_{S^1} and p_2, which is impossible since their winding numbers are 1 and 2.

This counter example may not be completely satisfying, since the domain manifold is not connected. So let us give another.

a.2 Example. Let $M := \mathbb{C} \setminus \mathbb{Z}$ and $N := \mathbb{C} \setminus \{0\}$ and consider maps of the form $x + iy \mapsto h_0(x) + ih_1(x)y$, where $h_0, h_1 \in C^\infty(\mathbb{R}, \mathbb{R})$ which $h_0^{-1}(0) \subseteq \mathbb{Z}$ and $0 \notin h_1(\mathbb{Z})$. They are well-defined $M \to N$, since $h_0(x) + ih_1(x)y = 0 \Rightarrow x \in h_0^{-1}(0) \subseteq \mathbb{Z} \Rightarrow 0 \notin h_1(x) \Rightarrow y = 0$. In particular, let $f_0 : M \to N$ be such a map with $h_0(x) := 2\sin(\pi x)$ and $h_1(x) := 1$ for all x and for $n > 0$ let f_n be such a map with the same h_0 but $h_1(x) := h(x-n)$, where $h \in C^\infty(\mathbb{R}, [-1,1])$ is equal to 1 outside $(-1,1)$ and equal to -1 on $[-1/2, 1/2]$. Then $f_n = f_0$ outside $(n-1, n+1)$ and hence any neighborhood of f_0 contains f_n for sufficiently large n. Any continuous curve in $C^\infty(M,N) \subseteq C(M,N)$ connecting f_0 and f_{2n} would yield a homotopy between f_0 and f_{2n} and hence also a homotopy between $p \circ f_0 \circ i_{2n}$ and $p \circ f_{2n} \circ i_{2n}$, where $p : \mathbb{C} \setminus \{0\} \to S^1$ is given by $p : z \mapsto \frac{z}{|z|}$ and $i_{2n} : S^1 \to \mathbb{C} \setminus \mathbb{Z}$ is given by $x + i\,y \mapsto 2n + \frac{1}{\pi} \arcsin\left(\frac{x}{2}\right) + i\,y$. Then

$$(f_0 \circ i_{2n})(x + i\,y) = h_0\Big(2n + \frac{1}{\pi}\arcsin\left(\frac{x}{2}\right)\Big) + i\,y = x + i\,y$$
$$(f_{2n} \circ i_{2n})(x + i\,y) = h_0\Big(2n + \frac{1}{\pi}\arcsin\left(\frac{x}{2}\right)\Big) - i\,y = x - i\,y.$$

So the winding numbers of $p \circ f_0 \circ i_{2n}$ and $p \circ f_{2n} \circ i_{2n}$ are 1 and -1, a contradiction.

From this one can construct an even more geometric example using $N := S^1$ and for $M \subseteq \mathbb{R}^3$ the connected sum of the tori centered at $(n,0,0)$ with rotation axes $(0,0,1)$ for $n \in \mathbb{Z}$. The embedding i_{2n} factors over $\mathrm{pr}_1 : \mathbb{C} \times \mathbb{R} \supseteq M \to \mathbb{C} \setminus \mathbb{Z}$, thus $p \circ f_0 \circ \mathrm{pr}_1$ and $p \circ f_{2n} \circ \mathrm{pr}_1$ are non-homotopic in $C^\infty(M,N)$.

But is the topology used in the previous two examples really the appropriate topology on $C^\infty(M,N)$? Infinite dimensional manifolds modelled on convenient vector space should be considered with the final topology with respect to (the inverse of) their charts $u : C^\infty(M,N) \supseteq U \to u(U) \subseteq E$, where $u(U)$ is supplied with the c^∞-topology (inherited from E). This final topology is just the final topology with respect to the smooth curves into $C^\infty(M,N)$, which, by exponential law, should correspond to the smooth mappings $\mathbb{R}\times M \to N$. Obviously this topology is locally (C^∞-)path connected. Thus it cannot be the topology used in the two examples above.

a.3 Example. Let M be the 0-dimensional discrete manifold $\mathbb{N}$ and $N := S^1$. We claim the product topology on $(S^1)^{\mathbb{N}} = C^\infty(M,N)$ is the final topology with respect to the smooth curves $c \in C^\infty(\mathbb{R}, C^\infty(M,N)) \cong C^\infty(\mathbb{R}\times\mathbb{N}, N) \cong C^\infty(\mathbb{R}, S^1)^{\mathbb{N}}$, given by the eponential law. Since these curves are obviously continuous for the (metrizable) product topology on the range space, the final topology has to be finer or equal.

Let conversely U be open in the final topology. Suppose there exists a $z_\infty \in U \subseteq (S^1)^{\mathbb{N}}$ for which U is not a neighborhood of z_∞ in the product topology. So there exists a sequence $z_n \in (S^1)^{\mathbb{N}} \setminus U$ which converges to z_∞. We lift the coordinates $z_n^k z_\infty^{-1} \in S^1$ of $\frac{z_n}{z_\infty}$ to $x_n^k \in \mathbb{R}$ along $\exp : \mathbb{R} \to S^1$ such that $x_n^k \to 0$ for $n \to \infty$ and each k. By passing to a common subsequence (given by $j(n) := n + \max\{i : \exists k \leq n : |x_i^k| > \frac{1}{2^n}\}$) we get that $\{2^n x_{j(n)}^k : n \in \mathbb{N}\}$ is bounded for each k. So there exist smooth curves $c^k : \mathbb{R} \to \mathbb{R}$ with $c^k(1/n) = x_{j(n)}^k$ by the special curve lemma. The curve $\tilde{c} := (x_\infty^k \cdot \exp\circ c^k)_{k\in\mathbb{N}} : \mathbb{R} \to (S^1)^{\mathbb{N}}$ is then a smooth curve with $\tilde{c}(\frac{1}{n}) = z_{j(n)}$, a contradiction to the openness of U in the final topology.

Moreover, this topology is (locally) connected but not locally simply connected:
Obviously we may connect $z_0, z_1 \in (S^1)^{\mathbb{N}}$ by parametrizing in the k-th factor one of the two arcs von z_0^k to z_1^k for each $k \in \mathbb{N}$.
Let U be any neighborhood of $z_0 := (1,1,\dots) \in (S^1)^{\mathbb{N}}$. Then for sufficently large n the loop $\mathrm{inj}_n : S^1 \to (S^1)^{\mathbb{N}}$, $z \mapsto (1,\dots,1,z,1,\dots)$ has image in U but cannot be 0-homotopic, otherwise $\mathrm{pr}_n \circ \mathrm{inj}_n = \mathrm{id}_{S^1}$ would be 0-homotopic.

42.1a Trying to find charts for $C^\infty(M,N)$. Let $f \in C^\infty(M,N)$. The basic idea is, that for mappings g sufficiently near to f and any $x \in M$ the point $g(x)$ should be connectable with $f(x)$ by a geodesic starting at $f(x)$ with some initial vector $\sigma(x) \in T_{f(x)}N$, i.e. $g(x) = \exp_{f(x)}(\sigma(x))$, where we have choosen some fixed Riemannian metric on N. We may assume that the exponential mapping exp of the Riemannian metric is defined on an open neighborhood U of the zero section $N \subseteq TN$ such that $(\pi_N, \exp) : TN \supseteq U \xrightarrow{\cong} V \subseteq N \times N$ is a smooth diffeomorphism onto an open neighborhood V of the diagonal $N \subseteq N \times N$. Thus the g correspond to sections σ along f or, equivalently, sections s of the pullback bundle $f^*(TN) \to M$:

$$\begin{tikzcd}
M \times_N TN \arrow[r, equal] \arrow[rr, bend left, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] & f^*TN \arrow[r, "\pi_N^* f"'] \arrow[d, "f^*\pi_N"] & TN \arrow[d, "\pi_N"] \\
 & M \arrow[ul, dotted, bend left, "s=(\mathrm{id},\sigma)"] \arrow[ur, dotted, "\sigma"'] \arrow[r, "f"'] & N.
\end{tikzcd}$$

Now let

$$\begin{aligned}
U_f &:= \Big\{ g \in C^\infty(M,N) : (f(x),\ g(x)) \in V \text{ for all } x \in M \Big\}, \\
u_f &: U_f \to C^\infty(M \leftarrow f^*TN), \\
u_f(g)(x) &:= \Big(x, \exp_{f(x)}^{-1}\big(g(x)\big)\Big) = \Big(x, \big((\pi_N, \exp)^{-1} \circ (f,g)\big)(x)\Big).
\end{aligned}$$

Then u_f is a bijective mapping from U_f onto the set

$$u_f(U_f) = \Big\{ s \in C^\infty(M \leftarrow f^*TN) : s(M) \subseteq f^*U := (\pi_N^* f)^{-1}(U) \Big\},$$

whose inverse is given by

$$u_f^{-1}(s) = \exp \circ \sigma := \exp \circ \pi_N^* f \circ s.$$

We will have to show that $u_f(U_f)$ is c^∞-open is some space of sections.

Now we consider the atlas $(U_f, u_f)_{f \in C^\infty(M,N)}$ for $C^\infty(M,N)$. Its chart change mappings are given for $s \in u_{f_1}(U_{f_2} \cap U_{f_1}) \subseteq C^\infty(M \leftarrow {f_1}^*TN)$ by

$$\begin{aligned}(u_{f_2} \circ u_{f_1}^{-1})(s) &= \Big(\mathrm{id}_M, (\pi_N, \exp)^{-1} \circ (f_2, \exp \circ \pi_N^* f_1 \circ s)\Big) \\ &= (\tau_{f_2}^{-1} \circ \tau_{f_1})_*(s) \in C^\infty(M \leftarrow {f_2}^*TN),\end{aligned}$$

where $\tau_f(x, Y) := (x, \exp_{f(x)}(Y))$ is a smooth diffeomorphism

$$\tau_f : f^*TN \supseteq f^*U \xrightarrow{\cong} (f \times \mathrm{id}_N)^{-1}(V) \subseteq M \times N$$

which is fiber respecting over M:

$$\begin{tikzcd}
f^*U \arrow[r] \arrow[d, hook] \arrow[rrr, dotted, bend left, "\tau_f"] & U \arrow[r, tail, "\cong", "(\pi_N, \exp)"'] \arrow[d, hook] & V \arrow[d, hook] & (f \times N)^{-1}(V) \arrow[l] \arrow[d, hook] \\
f^*TN \arrow[r, "\pi_N^* f = \mathrm{pr}_2"] \arrow[d, "f^*\pi_N = \mathrm{pr}_1"] & TN \arrow[d, "\pi_N"] \arrow[rd, bend left, "\exp"] & N \times N \arrow[d, "\mathrm{pr}_2"] \arrow[ld, "\mathrm{pr}_1"] & M \times N \arrow[l, "f \times N"] \arrow[d, "\mathrm{pr}_1"] \\
M \arrow[r, "f"] \arrow[ru, "\sigma"] \arrow[uu, bend left, "s"] \arrow[rr, bend right, "u_f^{-1}(s)"'] \arrow[rrr, bend right=40, "\mathrm{id}"'] & N & N & M \arrow[ll, bend left, "f"]
\end{tikzcd}$$

We will have to show that the chart change $u_{f_2} \circ u_{f_1}^{-1} = (\tau_{f_2}^{-1} \circ \tau_{f_1})_*$ is defined on a c^∞-open subset and is smooth.

30.8. Lemma. Curves in spaces of sections.

(1) *For a smooth vector bundle $p : E \to M$ a curve $c : \mathbb{R} \to C^\infty(M \leftarrow E)$ is smooth if and only if $c^\wedge : \mathbb{R} \times M \to E$ is smooth.*

(2) *For a holomorphic vector bundle $p : E \to M$ a curve $c : \mathbb{D} \to \mathcal{H}(M \leftarrow E)$ is holomorphic if and only if $c^\wedge : \mathbb{D} \times M \to E$ is holomorphic.*

(3) *For a real analytic vector bundle $p : E \to M$ a curve $c : \mathbb{R} \to C^\omega(M \leftarrow E)$ is real analytic if and only if the associated mapping $c^\wedge : \mathbb{R} \times M \to E$ is real analytic.*

(4) *For a real analytic vector bundle $p : E \to M$ a curve $c : \mathbb{R} \to C^\omega(M \leftarrow E)$ is smooth if and only if $c^\wedge : \mathbb{R} \times M \to E$ is $C^{\infty,\omega}$, see [**75**, 30.7]. A curve $c : \mathbb{R} \to C^\infty(M \leftarrow E)$ is real analytic if and only if $c^\wedge : \mathbb{R} \times M \to E$ is $C^{\omega,\infty}$, see 11.20.*

Proof. By the descriptions of the structures (30.1 for the smooth case, [**75**, 30.5] for the holomorphic case, and [**75**, 30.6] for the real analytic case) we may assume that M is open in a convenient vector space F, and we may consider functions with values in the standard fiber instead of sections. The statements then follow from the respective exponential laws (3.12 for the smooth case, 7.22 for the holomorphic case, 11.18 for the real analytic case, and the definition in 11.20 for the $C^{\infty,\omega}$ and $C^{\omega,\infty}$ cases). □

30.9. Lemma. Curves in spaces of sections with compact support.

(1) *For a smooth vector bundle $p : E \to M$ with finite dimensional base manifold M a curve $c : \mathbb{R} \to C_c^\infty(M \leftarrow E)$ is smooth if and only if $c^\wedge : \mathbb{R} \times M \to E$ is smooth and satisfies the following condition:*

> *For each compact interval $[a,b] \subset \mathbb{R}$ there is a compact subset $K \subset M$ such that $c^\wedge(t,x)$ is constant in $t \in [a,b]$ for each $x \in M \setminus K$.*

(2) *For a real analytic finite dimensional vector bundle $p : E \to M$ a curve $c : \mathbb{R} \to C_c^\infty(M \leftarrow E)$ is real analytic if and only if $c^\wedge$ satisfies the condition of (1) above and $c^\wedge : \mathbb{R} \times M \to E$ is $C^{\omega,\infty}$, see [75, 30.7].*

Compare this with 42.5 and 42.12.

Proof. By lemma 30.4.1 a curve $c : \mathbb{R} \to C_c^\infty(M \leftarrow E)$ is smooth if it factors locally as a smooth curve into some step $C_K^\infty(M \leftarrow E)$ for some compact K in M, and this is by 30.8.1 equivalent to smoothness of $c^\wedge$ and to condition (1). An analogous proof applies to the real analytic case. □

30.10. Corollary. *Let $p : E \to M$ and $p' : E' \to M$ be smooth vector bundles with finite dimensional base manifold. Let $W \subseteq E$ be an open subset, and let $f : W \to E'$ be a fiber respecting smooth (nonlinear) mapping. Then*

$$C_c^\infty(M \leftarrow W) := \{s \in C_c^\infty(M \leftarrow E) : s(M) \subseteq W\}$$

is c^∞-open in the convenient vector space $C_c^\infty(M \leftarrow E)$. The mapping

$$f_* : C_c^\infty(M \leftarrow W) \to C_c^\infty(M \leftarrow E')$$

is smooth with derivative

$$(d_v f)_* : C_c^\infty(M \leftarrow W) \times C_c^\infty(M \leftarrow E) \to C_c^\infty(M \leftarrow E'),$$

where the vertical derivative $d_v f : W \times_M E \to E'$ is given by

$$d_v f(u,w) := \tfrac{d}{dt}|_0 f(u + tw).$$

If the vector bundles and f are real analytic then $f_* : C_c^\infty(M \leftarrow W) \to C_c^\infty(M \leftarrow E')$ is real analytic with derivative $(d_v f)_*$.

If M is compact and the vector bundles and f are real analytic then $C^\omega(M \leftarrow W) := \{s \in C^\omega(M \leftarrow E) : s(M) \subseteq W\}$ is open in the convenient vector space $C^\omega(M \leftarrow E)$, and the mapping $f_* : C^\omega(M \leftarrow W) \to C^\omega(M \leftarrow E')$ is real analytic with derivative $(d_v f)_*$.

Proof. The set $C_c^\infty(M \leftarrow W)$ is open in $C_c^\infty(M \leftarrow E)$ since its intersection with each $C_K^\infty(M \leftarrow E)$ is open therein, see corollary 4.16, and the colimit is strict and regular by 30.4. Then f_* has all the stated properties, since it preserves (by [75, 30.7] for $C^{\infty,\omega}$) the respective classes of curves which are described in 30.8 and 30.9. The derivative can be computed pointwise on M. □

42.1. Theorem. Manifold structure of $\mathfrak{C}^\infty(M,N)$. *Let M and N be smooth finite dimensional manifolds. Then the space $\mathfrak{C}^\infty(M,N)$ of all smooth mappings* *from M to N is a smooth manifold, modeled on spaces $C_c^\infty(M\leftarrow f^*TN)$ of smooth sections with compact support of pullback bundles along $f : M \to N$ over M.*

Proof. As indicated in 42.1a we choose a smooth Riemannian metric on N and open neighborhoods U of the zero section $N \hookrightarrow TN$ and V of the diagonal $N \hookrightarrow N \times N$ such that $(\pi_N, \exp) : U \to V$ is a smooth diffeomorphism. Since we will use spaces $C_c^\infty(M\leftarrow f^*TN)$ of sections with compact support as modelling vector spaces, we need to consider the equivalence relation on $C^\infty(M, N)$ given by $f \sim g$ if f and g agree off some compact subset in M. Then we define charts

$$\begin{aligned}
U_f &:= \Big\{g \in C^\infty(M, N) : (f(x),\ g(x)) \in V \text{ for all } x \in M, g \sim f\Big\},\\
&u_f : U_f \to C_c^\infty(M\leftarrow f^*TN),\\
u_f(g)(x) &:= \Big(x, \exp^{-1}_{f(x)}(g(x))\Big) = \Big(x, \big((\pi_N, \exp)^{-1} \circ (f, g)\big)(x)\Big).
\end{aligned}$$

These are bijections onto

$$u_f(U_f) = \Big\{s \in C_c^\infty(M\leftarrow f^*TN) : s(M) \subseteq f^*U = (\pi_N^* f)^{-1}(U)\Big\},$$

with inverse given by $u_f^{-1}(s) = \exp \circ \pi_N^* f \circ s$, where we view $U \to N$ as fiber bundle. The image $u_f(U_f)$ is c^∞-open in $C_c^\infty(M\leftarrow f^*TN)$ by 30.10.

The chart change mappings for the atlas $(U_f, u_f)_{f\in\mathfrak{C}^\infty(M,N)}$ for $\mathfrak{C}^\infty(M, N)$ are given for $s \in u_{f_1}(U_{f_2} \cap U_{f_1}) \subseteq C^\infty(M\leftarrow {f_1}^*TN)$ by

$$\begin{aligned}
(u_{f_2} \circ u_{f_1}^{-1})(s) &= \Big(\mathrm{id}_M, (\pi_N, \exp)^{-1} \circ (f_2, \exp \circ \pi_N^* f_1 \circ s)\Big)\\
&= (\tau_{f_2}^{-1} \circ \tau_{f_1})_*(s) \in C^\infty(M\leftarrow {f_2}^*TN),
\end{aligned}$$

where $\tau_f(x, Y) := (x, \exp_{f(x)}(Y))$ is a smooth diffeomorphism

$$\tau_f : f^*TN \supseteq f^*U \xrightarrow{\cong} (f \times \mathrm{id}_N)^{-1}(V) \subseteq M \times N$$

which is fiber respecting over M. Thus $\tau_{f_2}^{-1}\circ\tau_{f_1}$ is a fibre respecting diffeomorphism from an open set in ${f_1}^*TN$ onto one in ${f_2}^*TN$, hence $(\tau_{f_2}^{-1} \circ \tau_{f_1})_*$ is smooth by 30.10.

Finally, following 27.1, the natural topology on $\mathfrak{C}^\infty(M, N)$ is the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is obviously finer than the topology of pointwise convergence and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the smooth structure does not depend on the choice of the smooth Riemannian metric on N. □

42.3. Proposition. *For finite dimensional second countable manifolds M, N the smooth manifold $\mathfrak{C}^\infty(M, N)$ has separable connected components and is smoothly paracompact and Lindelöf. If M is compact, it is metrizable.*

Proof. Each connected component of a mapping f is contained in the open equivalence class $\{g : g \sim f\}$ of f consisting of those smooth mappings which differ from f only on compact subsets. This equivalence class is the countable inductive limit in the category of topological spaces of the sets $\{g : g = f \text{ off } K\}$ of all mappings which differ from f only on members K_n of a countable exhaustion of M with compact sets, see 30.9, since a smooth curve locally has values in these steps $\{g : g = f \text{ off } K_n\}$. By 41.12 the steps are metrizable and second countable. Thus, $\{g : g \sim f\}$ is Lindelöf and separable. Since its model spaces $C_c^\infty(M\leftarrow h^*TN)$

are smoothly paracompact by 30.4, by 16.10 the space $\{g : g \sim f\}$ is smoothly paracompact, and since $\mathfrak{C}^\infty(M,N)$ is the disjoint union of such open sets, it is smoothly paracompact, too. □

42.4. Manifolds of mappings with an infinite dimensional range space

The method of proof of theorem 42.1 carries over to spaces $C^\infty(M,\mathcal{N})$, where M is a finite dimensional smooth manifold, and where $\mathcal{N}$ is a possibly infinite dimensional manifold which is required to admit an analogue of the exponential mapping used above, i.e., a smooth mapping $\alpha : T\mathcal{N} \supset U \to \mathcal{N}$, defined on an open neighborhood of the zero section in $T\mathcal{N}$, which satisfies

(1) $(\pi_\mathcal{N}, \alpha) : T\mathcal{N} \supset U \to \mathcal{N} \times \mathcal{N}$ is a diffeomorphism onto a c^∞-open neighborhood of the diagonal.
(2) $\alpha(0_x) = x$ for all $x \in \mathcal{N}$.

A smooth mapping α with these properties is called a *local addition* on $\mathcal{N}$.

42.5. Lemma. Smooth curves in $\mathfrak{C}^\infty(M,\mathcal{N})$. *Let M and $\mathcal{N}$ be smooth manifolds with M finite dimensional and $\mathcal{N}$ admitting a smooth local addition. Then the smooth curves c in $\mathfrak{C}^\infty(M,\mathcal{N})$ correspond exactly to the smooth mappings $c^\wedge \in C^\infty(\mathbb{R} \times M,\mathcal{N})$ which satisfy the following property:*

(1) *For each compact interval $[a,b] \subset \mathbb{R}$ there is a compact subset $K \subset M$ such that $c^\wedge(t,x)$ is constant in $t \in [a,b]$ for all $x \in M \setminus K$.*

*In particular, the identity induces a smooth mapping $\mathfrak{C}^\infty(M,\mathcal{N}) \to C^\infty(M,\mathcal{N})$ into the Frölicher space $C^\infty(M,\mathcal{N})$ discussed in [**75**, 23.2.3], which is a diffeomorphism if and only if M is compact or $\mathcal{N}$ is discrete.*

Proof. Since $\mathbb{R}$ is locally compact, property (1) is equivalent to

(2) For each $t \in \mathbb{R}$ there is an open neighborhood U of t in $\mathbb{R}$ and a compact $K \subset M$ such that the restriction has the property that $c^\wedge(t,x)$ is constant in $t \in U$ for all $x \in M \setminus K$.

Since this is a local condition on $\mathbb{R}$, and since smooth curves in $\mathfrak{C}^\infty(M,\mathcal{N})$ locally take values in charts as in the proof of theorem 42.1, it suffices to describe the smooth curves in the space $C^\infty_c(M\leftarrow E)$ of sections with compact support of a vector bundle $(p : E \to M, V)$ with finite dimensional base manifold M, with the structure described in 30.4. This was done in 30.9. □

42.6. Theorem. C^ω-manifold structure of $C^\omega(M,\mathcal{N})$. *Let M and $\mathcal{N}$ be real analytic manifolds, let M be compact, and let $\mathcal{N}$ be either finite dimensional, or let us assume that $\mathcal{N}$ admits a real analytic local addition in the sense of 42.4.*

Then the space $C^\omega(M,\mathcal{N})$ of all real analytic mappings from M to $\mathcal{N}$ is a real analytic manifold, modeled on spaces $C^\omega(M\leftarrow f^*T\mathcal{N})$ of real analytic sections of pullback bundles along $f : M \to \mathcal{N}$ over M.

Proof. The proof is a variant of the proof of 42.4, using a real analytic Riemannian metric if $\mathcal{N}$ is finite dimensional, and the given real analytic local addition otherwise. For finite dimensional $\mathcal{N}$ a detailed proof can be found in [**76**]. □

42.7. Lemma. *Let M, N be real analytic finite dimensional manifolds. Then the space $C^\omega(M,N)$ of all real analytic mappings is dense in $C^\infty(M,N)$, in the Whitney C^∞-topology.*

This is not true in the manifold topology of $\mathfrak{C}^\infty(M,N)$ used in 42.1, if M is not compact, because of the compact support condition used there.

Proof. By [**45**, theorem 3], there is a real analytic embedding $i : N \to \mathbb{R}^k$ on a closed submanifold, for some k. We use the constant standard inner product on $\mathbb{R}^k$ to obtain a real analytic tubular neighborhood U of $i(N)$ with projection $p : U \to i(N)$. By [**45**, proposition 8] applied to each coordinate of $\mathbb{R}^k$, the space $C^\omega(M,\mathbb{R}^k)$ of real analytic $\mathbb{R}^k$-valued functions is dense in the space $C^\infty(M,\mathbb{R}^k)$ of smooth functions, in the Whitney C^∞-topology. If $f : M \to N$ is smooth we may approximate $i \circ f$ by real analytic mappings g in $C^\omega(M,U)$, then $p \circ g$ is real analytic $M \to i(N)$ and approximates $i \circ f$. □

42.8. Theorem. C^ω-manifold structure on $\mathfrak{C}^\infty(M,N)$. *Let M and N be real analytic finite dimensional manifolds, with M compact. Then the smooth manifold $\mathfrak{C}^\infty(M,N)$ with the structure from 42.1 is even a real analytic manifold.*

Proof. For a fixed real analytic exponential mapping on N the charts (U_f,u_f) from 42.1 for $f \in C^\omega(M,N)$ form a smooth atlas for $\mathfrak{C}^\infty(M,N)$, since $C^\omega(M,N)$ is dense in $\mathfrak{C}^\infty(M,N)$ by 42.7.

The chart changings $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ are real analytic by 30.10. □

42.12. Lemma. Real analytic curves in spaces of mappings. *Let M and N be finite dimensional real analytic manifolds with M compact.*

(1) *A curve $c : \mathbb{R} \to C^\omega(M,N)$ is real analytic if and only if the associated mapping $c^\wedge : \mathbb{R} \times M \to N$ is real analytic.*
The curve $c : \mathbb{R} \to C^\omega(M,N)$ is smooth if and only if $c^\wedge : \mathbb{R} \times M \to N$ satisfies the following condition:
> *For each n there is an open neighborhood U_n of $\mathbb{R} \times M$ in $\mathbb{R} \times M_\mathbb{C}$ and a (unique) C^n-extension $\tilde{c} : U_n \to N_\mathbb{C}$ such that $\tilde{c}(t, \_)$ is holomorphic for all $t \in \mathbb{R}$.*

(2) *The curve $c : \mathbb{R} \to \mathfrak{C}^\infty(M,N)$ is real analytic if and only if $c^\wedge$ satisfies the following condition:*
> *For each n there is an open neighborhood U_n of $\mathbb{R} \times M$ in $\mathbb{C} \times M$ and a (unique) C^n-extension $\tilde{c} : U_n \to N_\mathbb{C}$ such that $\tilde{c}(\_, x)$ is holomorphic for all $x \in M$.*

Note that the two conditions are in fact local in $\mathbb{R}$. We need N finite dimensional since we need an extension $N_\mathbb{C}$ of N to a complex manifold.

Proof. This follows from the corresponding statement 30.8 for spaces of sections of vector bundles, and from the chart structure on $C^\omega(M,N)$ and $\mathfrak{C}^\infty(M,N)$. □

42.13. Theorem. Smoothness of composition. *If M, $\mathcal{N}$ are smooth manifolds with M finite dimensional and $\mathcal{N}$ admitting a smooth local addition, then the evaluation mapping $\mathrm{ev} : \mathfrak{C}^\infty(M,\mathcal{N}) \times M \to \mathcal{N}$ is smooth.*

If P is another smooth finite dimensional manifold, then the composition mapping

$$\operatorname{comp} : \mathfrak{C}^\infty(M, N) \times \mathfrak{C}^\infty_{prop}(P, M) \to \mathfrak{C}^\infty(P, N)$$

is smooth, where $\mathfrak{C}^\infty_{prop}(P, M)$ denotes the space of all proper smooth mappings $P \to M$ (i.e. compact sets have compact inverse images). This space is open in $\mathfrak{C}^\infty(P, M)$.

In particular, $f_* : \mathfrak{C}^\infty(M, \mathcal{N}) \to \mathfrak{C}^\infty(M, \mathcal{N}')$ and $g^* : \mathfrak{C}^\infty(M, \mathcal{N}) \to \mathfrak{C}^\infty(P, \mathcal{N})$ are smooth for $f \in C^\infty(\mathcal{N}, \mathcal{N}')$ and $g \in \mathfrak{C}^\infty_{prop}(P, M)$.

The corresponding statement for real analytic mappings can be shown along similar lines, using 42.12. But we will give another proof in 42.15 below.

Proof. Using the description of smooth curves in $\mathfrak{C}^\infty(M, \mathcal{N})$ given in 42.5, we immediately see that $(\operatorname{ev} \circ (c_1, c_2))(t) = c_1^\wedge(t, c_2(t))$ is smooth for each smooth $(c_1, c_2) : \mathbb{R} \to \mathfrak{C}^\infty(M, \mathcal{N}) \times M$, so ev is smooth as claimed.

The space of proper mappings $\mathfrak{C}^\infty_{\text{prop}}(P, M)$ is open in the manifold $\mathfrak{C}^\infty(P, M)$ since property 42.5.1 shows that smooth curves stay locally in $\mathfrak{C}^\infty_{\text{prop}}(P, M)$. Let $(c_1, c_2) : \mathbb{R} \to \mathfrak{C}^\infty(M, \mathcal{N}) \times \mathfrak{C}^\infty_{\text{prop}}(P, M)$ be a smooth curve. Then we have $(\operatorname{comp} \circ (c_1, c_2))(t)(p) = c_1^\wedge(t, c_2^\wedge(t, p))$, and one may check that this has again property 42.5.1, so it is a smooth curve in $\mathfrak{C}^\infty(P, \mathcal{N})$. Thus, comp is smooth. $\square$

42.14. Theorem. Exponential law. *Let $\mathcal{M}$, M, and $\mathcal{N}$ be smooth manifolds with M finite dimensional and $\mathcal{N}$ admitting a smooth local addition.*

Then we have a canonical injection

$$C^\infty(\mathcal{M}, \mathfrak{C}^\infty(M, \mathcal{N})) \subseteq C^\infty(\mathcal{M} \times M, \mathcal{N}),$$

where the image in the right hand side consists of all smooth mappings $f : \mathcal{M} \times M \to \mathcal{N}$ which satisfy the following property

1. *If $\mathcal{M}$ is locally metrizable then for each point $x \in \mathcal{M}$ there is an open neighborhood $\mathcal{U}$ and a compact set $K \subset M$ such that $f(x, y)$ is constant in $x \in \mathcal{U}$ for all $y \in M \setminus K$.*
2. *For general $\mathcal{M}$: For each $c \in C^\infty(\mathbb{R}, \mathcal{M})$ and each $t \in \mathbb{R}$ there exists a neighborhood U of t and a compact set $K \subset M$ such that $f(c(s), y)$ is constant in $s \in U$ for each $y \in M \setminus K$.*

Under the assumption that $\mathcal{N}$ admits smooth functions which separate points, we have equality if and only if M is compact, or $\mathcal{N}$ is discrete, or each $f \in C^\infty(\mathcal{M}, \mathbb{R})$ is constant along all smooth curves into $\mathcal{M}$.

If M and $\mathcal{N}$ are real analytic manifolds with M compact we have

$$C^\omega(\mathcal{M}, C^\omega(M, \mathcal{N})) = C^\omega(\mathcal{M} \times M, \mathcal{N})$$

for each real analytic (possibly infinite dimensional) manifold $\mathcal{M}$.

Proof. The smooth case is simple: The description (1) of the image follows directly from the characterization of all smooth curves in $\mathfrak{C}^\infty(M, \mathcal{N})$ given in the proof of 42.5.

It remains to show that for locally metrizable $\mathcal{M}$ a smooth mapping $f : \mathcal{M} \to \mathfrak{C}^\infty(M, \mathcal{N})$ satisfies condition (1). Since f is smooth, locally it has values in a chart, so we may assume that $\mathcal{M}$ is open in a Fréchet space by 4.19, and that f has values in $C_c^\infty(M\leftarrow E)$ for some vector bundle $p : E \to M$.

We claim that f locally factors into some $C^\infty_{K_n}(E)$ where (K_n) is an exhaustion of M by compact subsets such that K_n is contained in the interior of K_{n+1}. If not there exist a (fast) converging sequence (y_n) in $\mathcal{M}$ and $x_n \notin K_n$ such that $f(y_n)(x_n) \neq 0$. One may find a proper smooth curve $e : \mathbb{R} \to M$ with $e(n) = x_n$ and a smooth curve $g : \mathbb{R} \to \mathcal{M}$ with $g(1/n) = y_n$. Then by 30.4, $\mathrm{Pt}(e, \_)^* \circ f \circ g$ is a smooth curve into $C_c^\infty(\mathbb{R}, E_{e(0)})$. Since the latter space is a strict inductive limit of spaces $C_I^\infty(\mathbb{R}, E_{e(0)})$ for compact intervals I, the curve $\mathrm{Pt}(e, \_)^* \circ f \circ g$ locally factors into some $C_I^\infty(\mathbb{R}, E_{e(0)})$, but $(e^* \circ f \circ g)(1/n)(n) = f(y_n)(x_n) \neq 0$, a contradiction.

We check now the statement on equality: if M is compact, or if $\mathcal{N}$ is discrete then (2) is automatically satisfied. If each $f \in C^\infty(\mathcal{M}, \mathbb{R})$ is constant along all smooth curves into $\mathcal{M}$, we may check global constancy in (2) by composing with smooth functions on $\mathcal{N}$ which separate points there.

For the converse, we may assume that there are a function $f \in C^\infty(\mathcal{M}, \mathbb{R})$, a curve $c \in C^\infty(\mathbb{R}, \mathcal{M})$ such that $f\circ c$ is not constant, and an injective smooth curve $e : \mathbb{R} \to \mathcal{N}$. Then $\mathcal{M} \times M \ni (x, y) \mapsto e(f(x))$ is in $C^\infty(\mathcal{M} \times M, \mathcal{N}) \setminus C^\infty(\mathcal{M}, \mathfrak{C}^\infty(M, \mathcal{N}))$ since condition (2) is violated for the curve c.

Now we treat the real analytic case. Let $f^\wedge \in C^\omega(\mathcal{M}\times M, \mathcal{N}) \subset C^\infty(\mathcal{M}\times M, \mathcal{N}) = C^\infty(M, \mathfrak{C}^\infty(M, \mathcal{N}))$. So we may restrict f to a neighborhood U in $\mathcal{M}$, where it takes values in a chart U_g of $C^\infty(M, \mathcal{N})$ for $g \in C^\omega(M, \mathcal{N})$. Then $f(U) \subset U_g \cap C^\omega(M, \mathcal{N})$, one of the canonical charts of $C^\omega(M, \mathcal{N})$. So we may assume that $f : U \to C^\omega(M\leftarrow g^*T\mathcal{N})$. For a real analytic vector bundle atlas (U_α, ψ_α) of $g^*T\mathcal{N}$ the composites $U \to C^\omega(M\leftarrow g^*T\mathcal{N}) \to C^\omega(U_\alpha, \mathbb{R}^n)$ are real analytic by applying cartesian closedness 11.18 to the mapping $(x, y) \mapsto \psi_\alpha(\pi_{\mathcal{N}}, \exp)^{-1}(g(y), f^\wedge(x, y))$. By the description [**75**, 30.6] of the structure on $C^\omega(M\leftarrow g^*T\mathcal{N})$, the chart representation of f is real analytic, so f is it also.

Let conversely $f : \mathcal{M} \to C^\omega(M, \mathcal{N})$ be real analytic. Then its chart representation is real analytic and we may use cartesian closedness in the other direction to conclude that $f^\wedge$ is real analytic. □

42.15. Corollary. *If M and $\mathcal{N}$ are real analytic manifolds with M compact and $\mathcal{N}$ admitting a real analytic local addition, then the evaluation mapping $\mathrm{ev} : C^\omega(M, \mathcal{N}) \times M \to \mathcal{N}$ is real analytic.*

If P is another compact real analytic manifold, then the composition mapping $\mathrm{comp} : C^\omega(M, \mathcal{N}) \times C^\omega(P, M) \to C^\omega(P, \mathcal{N})$ is real analytic.

In particular, $f_* : C^\omega(M, \mathcal{N}) \to C^\omega(M, \mathcal{N}')$ and $g^* : C^\omega(M, \mathcal{N}) \to C^\omega(P, \mathcal{N})$ are real analytic for real analytic $f : \mathcal{N} \to \mathcal{N}'$ and $g \in C^\omega(P, M)$.

Proof. The mapping $\mathrm{ev}^\vee = \mathrm{Id}_{C^\omega(M,\mathcal{N})}$ is real analytic, so ev too, by 42.14. The mapping $\mathrm{comp}^\wedge = \mathrm{ev} \circ(\mathrm{Id}_{C^\omega(M,\mathcal{N})} \times \mathrm{ev}) : C^\omega(M, \mathcal{N}) \times C^\omega(P, M) \times P \to C^\omega(M, \mathcal{N}) \times M \to \mathcal{N}$ is real analytic, thus comp too. □

42.10. Lemma. *Let M and N be real analytic finite dimensional manifolds with M compact. Let (U_α, u_α) be a real analytic atlas for M, and let $i : N \to \mathbb{R}^n$ be a closed real analytic embedding into some $\mathbb{R}^n$. Let $\mathcal{M}$ be a possibly infinite dimensional real analytic manifold.*

Then $f:\mathcal{M}\to C^\omega(M,N)$ *is real analytic or smooth if and only if* $C^\omega(u_\alpha^{-1},i)\circ f:\mathcal{M}\to C^\omega(u_\alpha(U_\alpha),\mathbb{R}^n)$ *is real analytic or smooth, respectively.*

Furthermore, $f:\mathcal{M}\to\mathfrak{C}^\infty(M,N)$ *is real analytic or smooth if and only if the mapping* $C^\infty(u_\alpha^{-1},i)\circ f:\mathcal{M}\to C^\infty(u_\alpha(U_\alpha),\mathbb{R}^n)$ *is real analytic or smooth, respectively.*

Proof. The statement that i_* is initial is obvious. So we just have to treat $C^\infty(u_\alpha^{-1},N)$. The corresponding statement for spaces of sections of vector bundles are [**75**, 30.6] for the real analytic case and 30.1 for the smooth case. So if f takes values in a chart domain U_g of $C^\infty(M,N)$ for a real analytic $g:M\to N$, the result follows. Recall from the proof of 42.1 that $U_g=\{h\in C^\beta(M,N):(g(x),h(x))\in V\}$ where V is a fixed open neighborhood of the diagonal in $N\times N$, and where $\beta=\infty$ or ω. Let $f(z_0)\in U_g$ for $z_0\in\mathcal{M}$. Since M is covered by finitely many of its charts U_α, and since by assumption $f(z)|U_\alpha$ is near $f(z_0)|U_\alpha$ for z near z_0, so $f(z)\in U_g$ for z near z_0 in $\mathcal{M}$. So f takes values locally in charts, and the result follows. □

42.11. Corollary. *Let* M *and* N *be finite dimensional real analytic manifolds with* M *compact. Then the inclusion* $C^\omega(M,N)\to\mathfrak{C}^\infty(M,N)$ *is real analytic.*

Proof. Use the chart description and lemma 11.3. □

42.16. Lemma. *Let* M_i *and* N_i *be finite dimensional real analytic manifolds with* M_i *compact. Then for* $f\in C^\infty(N_1,N_2)$ *the push forward* $f_*:\mathfrak{C}^\infty(M,N_1)\to\mathfrak{C}^\infty(M,N_2)$ *is real analytic if and only if* f *is real analytic. For* $f\in C^\infty(M_2,M_1)$ *the pullback* $f^*:\mathfrak{C}^\infty(M_1,N)\to\mathfrak{C}^\infty(M_2,N)$ *is, however, always real analytic.*

Proof. If f is real analytic and if $g\in C^\omega(M,N_1)$, then the mapping

$$u_{f\circ g}\circ f_*\circ u_g^{-1}:C^\infty(M\leftarrow g^*TN_1)\to C^\infty(M\leftarrow(f\circ g)^*TN_2)$$

is a push forward by the real analytic mapping

$$\Big(\mathrm{pr}_1,(\pi,\exp^{N_2})^{-1}\circ(f\circ g\circ\mathrm{pr}_1,f\circ\exp^{N_1}\circ\mathrm{pr}_2)\Big):g^*TN_1\to(f\circ g)^*TN_2,$$

which is real analytic by 30.10.

The canonical mapping $\mathrm{ev}_x:\mathfrak{C}^\infty(M,N_2)\to N_2$ is real analytic since $\mathrm{ev}_x|U_g=\exp^{N_2}\circ\mathrm{ev}_x\circ u_g:U_g\to C^\infty(M\leftarrow g^*TN_2)\to T_{g(x)}N_2\to N_2$, where the second ev_x is linear and bounded. Furthermore, $\mathrm{const}:N_1\to\mathfrak{C}^\infty(M,N_1)$ is real analytic since the mapping $u_g\circ\mathrm{const}:y\mapsto(x\mapsto(\pi_{N_1},\exp^{N_1})^{-1}(g(x),y))$ is locally real analytic into $C^\omega(M\leftarrow g^*TN_1)$ and hence into $C^\infty(M\leftarrow g^*TN_1)$.

If f_* is real analytic, also $f=\mathrm{ev}_x\circ f_*\circ\mathrm{const}$ is.

For the second statement choose real analytic atlas (U_α^i,u_α^i) of M_i such that $f(U_\alpha^2)\subseteq U_\alpha^1$ and a closed real analytic embedding $j:N\to\mathbb{R}^n$. Then the diagram

$$\begin{tikzcd}
\mathfrak{C}^\infty(M_1,N) \arrow[r, "f^*"] \arrow[d, "\mathfrak{C}^\infty((u_\alpha^1)^{-1},j)"'] & \mathfrak{C}^\infty(M_2,N) \arrow[d, "\mathfrak{C}^\infty((u_\alpha^2)^{-1},j)"] \\
\mathfrak{C}^\infty(u_\alpha^1(U_\alpha^1),\mathbb{R}^n) \arrow[r, "(u_\alpha^2\circ f\circ(u_\alpha^1)^{-1})^*"] & \mathfrak{C}^\infty(u_\alpha^2(U_\alpha^2),\mathbb{R}^n)
\end{tikzcd}$$

commutes, the bottom arrow is a continuous and linear mapping, so it is real analytic. Thus, by 42.10, the mapping f^* is real analytic. $\square$

43. Diffeomorphism Groups

43.1. Theorem. Diffeomorphism group. *For a smooth manifold M the group $\operatorname{Diff}(M)$ of all smooth diffeomorphisms of M is an open submanifold of $\mathfrak{C}^\infty(M, M)$, composition and inversion are smooth. It is a regular Lie group in the sense of 38.4.*

The Lie algebra of the smooth infinite dimensional Lie group $\operatorname{Diff}(M)$ is the convenient vector space $C_c^\infty(M \leftarrow TM)$ of all smooth vector fields on M with compact support, equipped with the negative of the usual Lie bracket. The exponential mapping $\exp : C_c^\infty(M \leftarrow TM) \to \operatorname{Diff}^\infty(M)$ is the flow mapping to time 1, and it is smooth.

Proof. We first show that $\operatorname{Diff}(M)$ is open in $\mathfrak{C}^\infty(M, M)$. Let $c : \mathbb{R} \to \mathfrak{C}^\infty(M, M)$ be a smooth curve such that $c(0)$ is a diffeomorphism. We have to show that then $c(t)$ also is a diffeomorphism for all small t. Since composition from the right with $c(0)^{-1}$ is smooth by 42.13 we may assume that $c(0) = \mathrm{id}$. Using 42.5 we choose a compact set $K_1 \subseteq M$ such that $c(t)(x) = x$ for all $t \in [-1, 1]$ and $x \notin K_1$.

For $x \in M$ choose a chart $u : M \supseteq U \to u(U) \subseteq \mathbb{R}^m$ centered at x. Let $K \subseteq U$ be a compact neighborhood of x. Then $\{0\} \times K \subseteq (c^\wedge)^{-1}(U)$, hence we find a $\delta > 0$ such that $c(t)(K) \subseteq U$ for all $|t| < \delta$ and we may consider the chart representation $\bar{c}(t) : u(K) \to u(U) \subseteq \mathbb{R}^m$. Since $\bar{c}(0) = \mathrm{id}$ we have $\bar{c}(0)'(0) = \mathrm{id}$ and hence $\bar{c}(t)'(y) \in GL(m)$ for all small t and all y near 0. We may cover K_1 by the corresponding neighborhoods of x in M for finitely many $x \in M$ and take intersection of the 0-neighborhoods for the t. Thus for all $t \in [-1, 1]$ in this intersection, $c(t)$ is a local diffeomorphisms near any $x \in M$.

The mapping $c(t)$ stays injective for t near 0: Let $K_2 := c^\wedge([-1, 1] \times K_1)$. If $c(t)$ does not stay injective for t near 0 then there are $t_n \to 0$ and $x_n \neq y_n$ in M with $c(t_n)(x_n) = c(t_n)(y_n)$. We claim that $x_n, y_n \in K_2$: If $x_n \notin K_2 \supseteq c(0)(K_1) = K_1$ then $c(t_n)(y_n) = c(t_n)(x_n) = x_n \neq y_n$, so $y_n \in K_1$ and hence $x_n = c(t_n)(x_n) = c(t_n)(y_n) \in c^\wedge([-1, 1] \times K_1) = K_2$, a contradiction.
Passing to subsequences we may assume that $x_n \to x$ and $y_n \to y$ in K_2. By continuity of $c^\wedge$, we get $c(0)(x) = c(0)(y)$, so $x = y$. The mapping $(t, z) \mapsto (t, c(t)(z))$ is a diffeomorphism near $(0, x)$, since it is an immersion. But then $c(t_n)(x_n) \neq c(t_n)(y_n)$ for large n.

The mapping $c(t)$ stays surjective for t near 0: In the situation of the last paragraph $c(t)(M) = c(t)(K_2) \cup (M \setminus K_1^{\text{interior}})$ is closed in M for $|t| \leq 1$ and also open for t near 0, since $c(t)$ is a local diffeomorphism. It meets each connected component of M since $c(t)$ is homotopic to $c(0)$. Thus, $c(t)(M) = M$.

Therefore, $\operatorname{Diff}(M)$ is an open submanifold of $\mathfrak{C}^\infty_{\text{prop}}(M, M)$, so composition is smooth by 42.13. To show that the inversion inv is smooth, we consider a smooth curve $c : \mathbb{R} \to \operatorname{Diff}(M) \subset \mathfrak{C}^\infty(M, M)$. Then the mapping $c^\wedge : \mathbb{R} \times M \to M$ satisfies 42.5.1, and $(\operatorname{inv} \circ c)^\wedge$ fulfills the finite dimensional implicit equation $c^\wedge(t, (\operatorname{inv} \circ c)^\wedge(t, m)) = m$ for all $t \in \mathbb{R}$ and $m \in M$. By the finite dimensional implicit function theorem, $(\operatorname{inv}\circ c)^\wedge$ is smooth in (t,m). Property 42.5.1 is obvious. Hence, inv maps smooth curves to smooth curves and is thus smooth. (This proof is by far simpler than the original one, see [**95**], and shows the power of the Frölicher-Kriegl calculus.)

By the chart structure from 42.1, or directly from theorem [**75**, 42.17], we see that the tangent space $T_e\operatorname{Diff}(M)$ equals the space $C^\infty_c(M\leftarrow TM)$ of all vector fields with compact support. Likewise $T_f\operatorname{Diff}(M) = C^\infty_c(M\leftarrow f^*TM)$, which we identify with the space of all vector fields with compact support along the diffeomorphism f. Right translation μ^f is given by $\mu^f(g) = f^*(g) = g\circ f$, thus $T(\mu^f).X = X\circ f$, and for the flow Fl^X_t of the vector field with compact support X we have $\frac{d}{dt}\operatorname{Fl}^X_t = X\circ\operatorname{Fl}^X_t = T(\mu^{\operatorname{Fl}^X_t}).X$. So the one parameter group $t\mapsto\operatorname{Fl}^X_t\in\operatorname{Diff}(M)$ is the integral curve of the right invariant vector field $R_X: f\mapsto T(\mu^f).X = X\circ f$ on $\operatorname{Diff}(M)$. Thus, the exponential mapping of the diffeomorphism group is given by $\exp = \operatorname{Fl}_1 : C^\infty_c(M\leftarrow TM)\to\operatorname{Diff}(M)$. To show that is smooth we consider a smooth curve in $C^\infty_c(M\leftarrow TM)$, i.e., a time dependent vector field with compact support X_t. We may view it as a complete vector field $(0_t, X_t)$ on $\mathbb{R}\times M$ whose smooth flow respects the level surfaces $\{t\}\times M$ and is smooth. Thus, $\exp\circ X = (\operatorname{pr}_2\circ\operatorname{Fl}^{(0,X)}_1)^\vee$ maps smooth curves to smooth curves and is smooth itself. Again one may compare this simple proof with the original one [**98**, section 4].

To see that $\operatorname{Diff}(M)$ is a regular Lie group note that the evolution is given by integrating time dependent vector fields with compact support,

$$\begin{aligned}\operatorname{evol}(t\mapsto X_t) &= \varphi(1,\_)\\ \tfrac{\partial}{\partial t}\varphi(t,x) = X(t,\varphi(t,x)),&\quad \varphi(0,x)=x.\end{aligned}$$

Let us finally compute the Lie bracket on $C^\infty_c(M\leftarrow TM)$ viewed as the Lie algebra of $\operatorname{Diff}(M)$. For $X\in C^\infty_c(M\leftarrow TM)$ let L_X denote the left invariant vector field on $\operatorname{Diff}(M)$. Its flow is given by $\operatorname{Fl}^{L_X}_t(f) = f\circ\exp(tX) = f\circ\operatorname{Fl}^X_t = (\operatorname{Fl}^X_t)^*(f)$. From [**75**, 32.15] we get $[L_X, L_Y] = \frac{d}{dt}|_0(\operatorname{Fl}^{L_X}_t)^*L_Y$, so for $e=\operatorname{Id}_M$ we have

$$\begin{aligned}[L_X,L_Y](e) &= (\tfrac{d}{dt}|_0(\operatorname{Fl}^{L_X}_t)^*L_Y)(e)\\ &= \tfrac{d}{dt}|_0(T(\operatorname{Fl}^{L_X}_{-t})\circ L_Y\circ\operatorname{Fl}^{L_X}_t)(e)\\ &= \tfrac{d}{dt}|_0 T(\operatorname{Fl}^{L_X}_{-t})(L_Y(e\circ\operatorname{Fl}^X_t))\\ &= \tfrac{d}{dt}|_0 T((\operatorname{Fl}^X_{-t})^*)(T(\operatorname{Fl}^X_t)\circ Y)\\ &= \tfrac{d}{dt}|_0(T(\operatorname{Fl}^X_t)\circ Y\circ\operatorname{Fl}^X_{-t}),\quad\text{by [75, 42.18]}\\ &= \tfrac{d}{dt}|_0(\operatorname{Fl}^X_{-t})^*Y = -[X,Y].\end{aligned}$$

Another proof using [**75**, 36.10] is as follows:

$$\begin{aligned}\operatorname{Ad}(\exp(sX))Y &= \tfrac{\partial}{\partial t}\big|_0\exp(sX)\circ\exp(tY)\circ\exp(-sX)\\ &= T(\operatorname{Fl}^X_s)\circ Y\circ\operatorname{Fl}^X_{-s} = (\operatorname{Fl}^X_{-s})^*Y,\end{aligned}$$

thus

$$\tfrac{\partial}{\partial t}\big|_0\operatorname{Ad}(\exp(tX))Y = \tfrac{\partial}{\partial t}\big|_0(\operatorname{Fl}^X_{-t})^*Y = -[X,Y]$$

is the negative of the usual Lie bracket on $C^\infty_c(M\leftarrow TM)$. □

It is well known that the space $\operatorname{Diff}(M)$ of all diffeomorphisms of M is open in $C^\infty(M,M)$ even for the Whitney C^∞-topology, see 41.10; proofs can be found in [**51**, p. 38] or [**95**, section 5].

38.4. Definition. Regular Lie groups

If for each $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists $g \in C^\infty(\mathbb{R}, G)$ satisfying

$$\begin{cases} g(0) = e \\ \frac{\partial}{\partial t} g(t) = T_e(\mu^{g(t)})X(t) = R_{X(t)}(g(t)), \\ \quad \text{or } \kappa^r(\frac{\partial}{\partial t} g(t)) = \delta^r g(\partial_t) = X(t) \end{cases}$$

then we write

$$\operatorname{evol}_G^r(X) = \operatorname{evol}_G(X) := g(1),$$
$$\operatorname{Evol}_G^r(X)(t) := \operatorname{evol}_G(s \mapsto tX(ts)) = g(t),$$

and call them the *right evolution* of the curve X in G. By lemma [75, 38.3], the solution of the differential equation 1 is unique, and for global existence it is sufficient that it has a local solution. Then

$$\operatorname{Evol}_G^r : C^\infty(\mathbb{R}, \mathfrak{g}) \to \{g \in C^\infty(\mathbb{R}, G) : g(0) = e\}$$

is bijective with inverse δ^r. The Lie group G is called a *regular Lie group* if $\operatorname{evol}^r : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ exists and is smooth.

43.2. Example

The exponential mapping $\exp : C_c^\infty(M \leftarrow TM) \to \operatorname{Diff}(M)$ satisfies $T_0 \exp = \operatorname{Id}$, but it is not locally surjective near Id_M: This is due to [38] and [62]. The strongest result in this direction is [44], where it is shown, that $\operatorname{Diff}(M)$ contains a smooth curve through Id_M whose points (sauf Id_M) are free generators of an arcwise connected free subgroup which meets the image of exp only at the identity.

We shall prove only a weak version of this for $M = S^1$. For large $n \in \mathbb{N}$ we consider the diffeomorphism

$$f_n(\theta) = \theta + \tfrac{2\pi}{n} + \tfrac{1}{2^n} \sin^2(\tfrac{n\theta}{2}) \mod 2\pi;$$

(the subgroup generated by) f_n has just one periodic orbit and this is of period n, namely $\{\frac{2\pi k}{n} : k = 0, \dots, n-1\}$. For even n the diffeomorphism f_n cannot be written as $g \circ g$ for a diffeomorphism g (so f_n is not contained in a flow), by the following argument: If $g \circ g$ has exactly one periodic orbit (say through x) and this is of even period, then the orbit through x is a periodic orbit of g. If it had odd order for g, then the orbit of $g \circ g$ through x would have the same order. Thus it has even order for g and so x and gx would yield two different orbits for $g \circ g$, a contradiction.

43.6. Examples

Example 1. Let $\mathfrak{g} \subset \mathfrak{X}_c(\mathbb{R}^2)$ be the closed Lie subalgebra of all vector fields with compact support on $\mathbb{R}^2$ of the form $X(x, y) = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$ where g vanishes on the strip $0 \le x \le 1$.
Claim. There is no Lie subgroup G of $\operatorname{Diff}(\mathbb{R}^2)$ corresponding to $\mathfrak{g}$.
If G exists then there is a smooth curve $t \mapsto f_t \in G \subset \operatorname{Diff}_c(\mathbb{R}^2)$. Then $X_t := (\frac{\partial}{\partial t} f_t) \circ f_t^{-1}$ is a smooth curve in $\mathfrak{g}$, and we may assume that $X_0 = f\frac{\partial}{\partial x}$ where $f = 1$ on a large ball. But then $\operatorname{Ad}^G(f_t) = f_t^* : \mathfrak{g} \not\to \mathfrak{g}$, a contradiction.

So we see that on any manifold of dimension greater than 2 there are closed Lie subalgebras of the Lie algebra of vector fields with compact support which do not admit Lie subgroups.

Example 2. The space $\mathfrak{X}_K(M)$ of all vector fields with support in some open set U is an ideal in $\mathfrak{X}_c(M)$, the corresponding Lie group is the connected component $\mathrm{Diff}_U(M)_0$ of the group of all diffeomorphisms which equal Id off some compact in U, but this is not a normal subgroup in the connected component $\mathrm{Diff}_c(M)_0$, since we may conjugate the support out of U.

Note that this examples do not work for the Lie group of real analytic diffeomorphisms on a compact manifold.

43.7. Theorem. [**30**] *Let M be a compact orientable manifold, let μ_0 be a positive volume form on M with total mass 1. Then the regular Lie group $\mathrm{Diff}_+(M)$ of all orientation preserving diffeomorphisms splits smoothly as $\mathrm{Diff}_+(M) = \mathrm{Diff}(M,\mu_0)\times \mathrm{Vol}(M)$, where $\mathrm{Diff}(M,\mu_0)$ is the regular Lie group of all μ_0-preserving diffeomorphisms, and $\mathrm{Vol}(M)$ is the space of all volume forms of total mass 1.*

If (M,μ_0) is real analytic, then $\mathrm{Diff}^\omega_+(M)$ splits real analytically as $\mathrm{Diff}^\omega_+(M) = \mathrm{Diff}^\omega(M,\mu_0)\times \mathrm{Vol}^\omega(M)$, where $\mathrm{Diff}^\omega(M,\mu_0)$ is the Lie group of all μ_0-preserving real analytic diffeomorphisms, and $\mathrm{Vol}^\omega(M)$ is the space of all real analytic volume forms of total mass 1.

Proof. Note that $\mathrm{Diff}_+(M)$ is open in $\mathrm{Diff}(M)$ and $\mathrm{Vol}(M)$ is open in the closed hyperplane $\{\omega\in\Omega^m(M) : \int_M \omega = 1\}$.

We show first that there exists a smooth mapping $\tau : \mathrm{Vol}(M)\to \mathrm{Diff}_+(M)$ such that $\tau(\mu)^*\mu_0 = \mu$.

We put $\mu_t = \mu_0 + t(\mu-\mu_0)$. We want a smooth curve $t\mapsto f_t\in \mathrm{Diff}_+(M)$ with $f_t^*\mu_t = \mu_0$. We have $\frac{\partial}{\partial t} f_t = X_t\circ f_t$ for a time dependent vector field X_t on M. Then $0 = \frac{\partial}{\partial t} f_t^*\mu_t = f_t^*\mathcal{L}_{X_t}\mu_t + f_t^*\frac{\partial}{\partial t}\mu_t = f_t^*(\mathcal{L}_{X_t}\mu_t + (\mu-\mu_0))$, so $\mathcal{L}_{X_t}\mu_t = \mu_0-\mu$ and $\mathcal{L}_{X_t}\mu_t = di_{X_t}\mu_t + i_{X_t}0 = d\omega$ for some $\omega\in\Omega^{\dim M-1}(M)$. Now we choose ω such that $d\omega = \mu_0-\mu$, and we choose it smoothly and in the real analytic case even real analytically depending on μ by the theorem of Hodge, as follows: For any $\alpha\in\Omega(M)$ we have $\alpha = \mathcal{H}\alpha + d\delta G\alpha + \delta G d\alpha$, where $\mathcal{H}$ is the projection on the space of harmonic forms, $\delta = *d*$ is the codifferential, $*$ is the Hodge-star operator, and G is the Green operator, see [**130**]. All these are bounded linear operators, G is even compact. So we may choose $\omega = \delta G(\mu_0-\mu)$. Then the time dependent vector field X_t is uniquely determined by $i_{X_t}\mu_t = \omega$ since μ_t is nowhere 0. Let f_t be the evolution operator of X_t, and put $\tau(\mu) = f_1^{-1}$.

Now we may prove the theorem itself. We define a mapping $\Psi : \mathrm{Diff}_+(M)\to \mathrm{Diff}(M,\mu_0)\times\mathrm{Vol}(M)$ by $\Psi(f) := (f\circ\tau(f^*\mu_0)^{-1}, f^*\mu_0)$, which is smooth or real analytic by 42.15 and [**75**, 43.4]. An easy computation shows that the inverse is given by the restriction of the smooth (or real analytic) mapping $\Psi : \mathrm{Diff}_+(M)\times\mathrm{Vol}(M)\to\mathrm{Diff}_+(M)$, $\Psi^{-1}(g,\mu) = g\circ\tau(\mu)$.

That $\mathrm{Diff}(M,\mu_0)$ is regular follows from [**75**, 38.7], where we use the mapping $p : \mathrm{Diff}_+(M)\to\Omega^{\max}(M)$, given by $p(f) := f^*\mu_0-\mu_0$. $\square$

We next treat the Lie group of symplectic diffeomorphisms.

43.8. Symplectic manifolds

Let M be a smooth manifold of dimension $2n \geq 2$. A *symplectic form* on M is a closed 2-form σ such that $\sigma^n = \sigma \wedge \dots \wedge \sigma \in \Omega^{2n}(M)$ is nowhere 0. The pair (M, σ) is called a *symplectic manifold*. See section [**75**, 48] for a treatment of infinite dimensional symplectic manifolds.

43.11. Lemma. *Let M be a smooth finite dimensional manifold, let $N \subset M$ be a closed submanifold, and let σ_0 and σ_1 be symplectic forms on M which are equal along N.*

Then there exist: A diffeomorphism $f : U \to V$ between two open neighborhoods U and V of N in M which satisfies $f|N = \mathrm{Id}_N$, $Tf|(TM|N) = \mathrm{Id}_{TM|N}$, and $f^*\sigma_1 = \sigma_0$.

If all data are real analytic then the diffeomorphism can be chosen real analytic, too.

Proof. Let $\sigma_t = \sigma_0 + t(\sigma_1 - \sigma_0)$ for $t \in [0, 1]$. Since the restrictions of σ_0 and σ_1 to $\Lambda^2 TM|N$ are equal, there is an open neighborhood U_1 of N in M such that σ_t is a symplectic form on U_1, for all $t \in [0, 1]$. If $i : N \to M$ is the inclusion, we also have $i^*(\sigma_1 - \sigma_0) = 0$, so by lemma [**75**, 43.10] there is a smaller open neighborhood U_2 of N such that $\sigma_1|U_2 - \sigma_0|U_2 = d\varphi$ for some $\varphi \in \Omega^1(U_2)$ with $\varphi_x = 0$ for $x \in N$, such that also all first derivatives of φ vanish along N.

Let us now consider the time dependent vector field $X_t := -(\sigma_t{}^\vee)^{-1} \circ \varphi$, which vanishes together with all first derivatives along N. Let f_t be the curve of local diffeomorphisms with $\frac{\partial}{\partial t} f_t = X_t \circ f_t$, then $f_t|N = \mathrm{Id}_N$ and $Tf_t|(TM|N) = \mathrm{Id}$. There is a smaller open neighborhood U of N such that f_t is defined on U for all $t \in [0, 1]$. Then we have

$$\begin{aligned}\tfrac{\partial}{\partial t}(f_t^*\sigma_t) &= f_t^*\mathcal{L}_{X_t}\sigma_t + f_t^*\tfrac{\partial}{\partial t}\sigma_t = f_t^*(di_{X_t}\sigma_t + \sigma_1 - \sigma_0)\\ &= f_t^*(-d\varphi + \sigma_1 - \sigma_0) = 0,\end{aligned}$$

so $f_t^*\sigma_t$ is constant in t, equals $f_0^*\sigma_0 = \sigma_0$, and finally $f_1^*\sigma_1 = \sigma_0$ as required. □

43.12. Theorem. *Let (M, σ) be a finite dimensional symplectic manifold. Then the group $\mathrm{Diff}(M, \sigma)$ of symplectic diffeomorphisms is a smooth regular Lie group and a closed submanifold of $\mathrm{Diff}(M)$. The Lie algebra of $\mathrm{Diff}(M, \sigma)$ agrees with $\mathfrak{X}_c(M, \sigma)$.*

If moreover (M, σ) is a compact real analytic symplectic manifold, then the group $\mathrm{Diff}^\omega(M, \sigma)$ of real analytic symplectic diffeomorphisms is a real analytic regular Lie group and a closed submanifold of $\mathrm{Diff}^\omega(M)$.

Proof. The smooth and the real analytic cases will be proved simultaneously; only once we will need an extra argument for the latter.

Consider a local addition $\alpha : TM \to M$ in the sense of 42.4, so that $(\pi_M, \alpha) : TM \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal, and $\alpha(0_x) = x$. Let us compose α from the right with a fiber respecting diffeomorphism $TM^* \to TM$ (coming from the symplectic structure or from a Riemannian metric) and call the result again $\alpha : T^*M \to M$. Then $(\pi_M, \alpha) : T^*M \to M \times M$ also is a diffeomorphism onto an open neighborhood of the diagonal, and $\alpha(0_x) = x$.

We consider now two symplectic structures on T^*M, namely the canonical symplectic structure $\sigma_0 = \sigma_M$, and $\sigma_1 := (\pi_M, \alpha)^*(\operatorname{pr}_1^*\sigma - \operatorname{pr}_2^*\sigma)$. Both have vanishing pullbacks on the zero section $0_M \subset T^*M$.

Claim. In this situation, there exists a diffeomorphism $\varphi : V_0 \to V_1$ between two open neighborhoods V_0 and V_1 of the zero section in T^*M which is the identity on the zero section and satisfies $\varphi^*\sigma_1 = \sigma_0$.

First we solve the problem along the zero section, i.e., in $T(T^*M)|0_M$. There is a vector bundle isomorphism $\gamma : T(T^*M)|0_M \to T(T^*M)|0_M$ over the identity on 0_M, which is the identity on $T(0_M)$ and maps the symplectic structure σ_0 on each fiber to σ_1. In the smooth case, by using a partition of unity it suffices to construct γ locally. But locally σ_i can be described by choosing a Lagrange subbundle $L_i \subset T(T^*M)|0_M$ which is a complement to $T0_M$. Then σ_i is completely determined by the duality between $T0_M$ and W_i induced by it, and a smooth γ is then given by the resulting isomorphism $W_0 \to W_1$.

In the real analytic case, in order to get a real analytic γ, we consider the principal fiber bundle $P \to 0_M$ consisting of all $\gamma_x \in GL(T_{0_x}(T^*M))$ with $\gamma_x|T_{0_x}(0_M) = \operatorname{Id}$ and $\gamma_x^*\sigma_1 = (\sigma_0)_{0_x}$. The proof above shows that we may find a smooth section of P. By lemma [**75**, 30.12], there also exist real analytic sections.

Next we choose a diffeomorphism $h : V_0 \to V_1$ between open neighborhoods of 0_M in T^*M such that $Th|0_M = \gamma$, which can be constructed as follows: Let $u : \mathcal{N}(0_M) \to V_0$ be a tubular neighborhood of the zero section, where $\mathcal{N}(0_M) = (T(T^*M)|0_M)/T(0_M)$ is the normal bundle of the zero section. Clearly, γ induces a vector bundle automorphism of this normal bundle, and $h = u \circ \gamma \circ u^{-1}$ satisfies all requirements.

Now σ_0 and $h^*\sigma_1$ agree along the zero section 0_M, so we may apply lemma 43.11, which implies the claim with possibly smaller V_i.

We consider the diffeomorphism $\rho := (\pi_M, \alpha) \circ \varphi : T^*M \supset V_0 \to V_2 \subset M \times M$ from an open neighborhood of the zero section to an open neighborhood of the diagonal, and we let $U \subseteq \operatorname{Diff}(M)$ be the open neighborhood of Id_M consisting of all $f \in \operatorname{Diff}(M)$ with compact support such that $(\operatorname{Id}_M, f)(M) \subset V_2$, i.e. the graph $\{(x, f(x)) : x \in M\}$ of f is contained in V_2, and $\pi_M : \rho^{-1}(\{(x, f(x)) : x \in M\}) \to M$ is still a diffeomorphism.

For $f \in U$ the mapping $(\operatorname{Id}_M, f) : M \to \operatorname{graph}(f) \subset M \times M$ is the natural diffeomorphism onto the graph of f, and the latter is a Lagrangian submanifold if and only if

$$0 = (\operatorname{Id}_M, f)^*(\operatorname{pr}_1^*\sigma - \operatorname{pr}_2^*\sigma) = \operatorname{Id}_M^*\sigma - f^*\sigma.$$

Therefore, $f \in \operatorname{Diff}(M, \sigma)$ if and only if the graph of f is a Lagrangian submanifold of $(M \times M, \operatorname{pr}_1^*\sigma - \operatorname{pr}_2^*\sigma)$. Since $\rho^*(\operatorname{pr}_1^*\sigma - \operatorname{pr}_2^*\sigma) = \sigma_0$ this is the case if and only if $\{\rho^{-1}(x, f(x)) : x \in M\}$ is a Lagrange submanifold of (T^*M, σ_0).

We consider now the following smooth chart of $\operatorname{Diff}(M)$ which is centered at the identity:

$$\operatorname{Diff}(M) \supset U \xrightarrow{u} u(U) \subset C_c^\infty(M \leftarrow T^*M) = \Omega_c^1(M),$$
$$u(f) := \rho^{-1} \circ (\operatorname{Id}_M, f) \circ (\pi_M \circ \rho^{-1} \circ (\operatorname{Id}_M, f))^{-1} : M \to T^*M.$$

Then $f \in U \cap \operatorname{Diff}(M, \sigma)$ if and only if $u(f)$ is a closed form, since $u(f)(M) = \{\rho^{-1}(x, f(x)) : x \in M\}$ is a Lagrange submanifold if and only if f is symplectic. Thus, (U, u) is a smooth chart of $\operatorname{Diff}(M)$ which is a submanifold chart for

$\mathrm{Diff}(M,\sigma)$. For arbitrary $g \in \mathrm{Diff}(M,\sigma)$ we consider the smooth submanifold chart

$$\mathrm{Diff}(M) \supset U_g := \{f : f \circ g^{-1} \in U\} \xrightarrow{u_g} u_g(U_g) \subset C^\infty_c(M \leftarrow T^*M) = \Omega^1_c(M),$$
$$u_g(f) := u(f \circ g^{-1}).$$

Hence, $\mathrm{Diff}(M,\sigma)$ is a closed smooth submanifold of $\mathrm{Diff}(M)$ and a smooth Lie group, since composition and inversion are smooth by restriction. If M is compact then the space of closed 1-forms is a direct summand in $\Omega^1(M)$ by Hodge theory, as in the proof of 43.7, so in this case $\mathrm{Diff}(M,\sigma)$ is even a splitting submanifold of $\mathrm{Diff}(M)$. The embedding $\mathrm{Diff}(M,\sigma) \to \mathrm{Diff}(M)$ is smooth, thus it induces a bounded injective homomorphism of Lie algebras which is an embedding onto a closed Lie subalgebra, which we shall soon identify with $\mathfrak{X}_c(M,\sigma)$.

Suppose that $X : \mathbb{R} \to \mathfrak{X}_c(M,\sigma)$ is a smooth curve, and consider the evolution curve $f(t) = \mathrm{Evol}^r_{\mathrm{Diff}(M)}(X)(t)$, which is the solution of the differential equation $\frac{\partial}{\partial t} f(t) = X(t) \circ f(t)$ on M. Then $f : \mathbb{R} \to \mathrm{Diff}(M)$ actually has values in $\mathrm{Diff}(M,\sigma)$, since $\frac{\partial}{\partial t} f_t^* \sigma = f_t^* \mathcal{L}_{X_t} \sigma = 0$. So the restriction of $\mathrm{evol}^r_{\mathrm{Diff}(M)}$ to $\mathfrak{X}_c(M,\sigma)$ is smooth into $\mathrm{Diff}(M,\sigma)$ and thus gives $\mathrm{evol}^r_{\mathrm{Diff}(M,\sigma)}$. We take now the right logarithmic derivative of $f(t)$ in $\mathrm{Diff}(M,\sigma)$ and get a smooth curve in the Lie algebra of $\mathrm{Diff}(M,\sigma)$ which maps to $X(t)$. Thus, the Lie algebra of $\mathrm{Diff}(M,\sigma)$ is canonically identified with $\mathfrak{X}_c(M,\sigma)$.

Note that this proof of regularity is an application of the method from [**75**, 38.7], where $p(f) := f^*\sigma - \sigma$, $p : \mathrm{Diff}(M) \to \Omega^2(M)$. □

43.15. Contact manifolds

Let M be a smooth manifold of dimension $2n+1 \geq 3$. A *contact form* on M is a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n \in \Omega^{2n+1}(M)$ is nowhere zero. This is sometimes called an *exact contact structure*. The pair (M,α) is called a *contact manifold*.

A contact form can be put into the following normal form: For each $x \in M$ there is a chart $M \supset U \xrightarrow{u} u(U) \subset \mathbb{R}^{2n+1}$ centered at x such that $\alpha|U = u^1\, du^{n+1} + u^2\, du^{n+2} + \dots + u^n\, du^{2n} + du^{2n+1}$. This follows from proposition [**75**, 43.18] below, for a simple direct proof see [**85**].

The vector subbundle $\ker(\alpha) \subset TM$ is called the *contact distribution*. It is as non-involutive as possible, since $d\alpha$ is even non-degenerate on each fiber $\ker(\alpha)_x = \ker(\alpha_x) \subset T_xM$. The *characteristic vector field* $X_\alpha \in \mathfrak{X}(M)$ is the unique vector field satisfying $i_{X_\alpha}\alpha = 1$ and $i_{X_\alpha} d\alpha = 0$.

Note that $X \mapsto (i_X d\alpha, i_X\alpha)$ is isomorphic $TM \to \{\varphi \in T^*M : i_{X_\alpha}\varphi = 0\} \times \mathbb{R}$, but we shall use the isomorphism of vector bundles

$$TM \to T^*M, \quad X \mapsto i_X d\alpha + \alpha(X).\alpha,$$

A diffeomorphism $f \in \mathrm{Diff}(M)$ with $f^*\alpha = \lambda_f.\alpha$ for a nowhere vanishing function $\lambda_f \in C^\infty(M, \mathbb{R} \setminus 0)$ is called a *contact diffeomorphism*. The group of all contact diffeomorphisms will be denoted by $\mathrm{Diff}(M,\alpha)$.

A vector field $X \in \mathfrak{X}(M)$ is called a contact vector field if $\mathcal{L}_X\alpha = \mu_X.\alpha$ for a smooth function $\mu_X \in C^\infty(M,\mathbb{R})$. The linear space of all contact vector fields will be denoted by $\mathfrak{X}(M,\alpha)$; it is clearly a Lie algebra. Contraction with α is a linear mapping also denoted by $\alpha : \mathfrak{X}(M,\alpha) \to C^\infty(M,\mathbb{R})$. It is bijective since we may apply i_{X_α} to $\mu_X.\alpha = \mathcal{L}_X\alpha = i_X\,d\alpha + d(\alpha(X))$ to get $\mu_X = 0 + X_\alpha(\alpha(X))$, and since by using (1) we may reconstruct X from $\alpha(X)$ as

$$\begin{aligned} i_X\,d\alpha + \alpha(X).\alpha &= \mu_X.\alpha - d(\alpha(X)) + \alpha(X).\alpha \\ &= X_\alpha(\alpha(X)).\alpha - d(\alpha(X)) + \alpha(X).\alpha. \end{aligned}$$

Note that the inverse $f \mapsto \operatorname{grad}^\alpha(f)$ of $\alpha : \mathfrak{X}(M,\alpha) \to C^\infty(M,\mathbb{R})$ is a linear differential operator of order 1.

A smooth mapping $f : L \to M$ is called a *Legendre mapping* if $f^*\alpha = 0$. If f is also an embedding and $\dim M = 2\dim L + 1$, then the image $f(L)$ is called a *Legendre submanifold* of M.

43.16. Lemma. *Let X_t be a time dependent vector field on M, and let f_t be the local curve of local diffeomorphisms with $\frac{\partial}{\partial t} f_t \circ f_t^{-1} = X_t$ and $f_0 = \mathrm{Id}$. Then $\mathcal{L}_{X_t}\alpha = \mu_t\alpha$ if and only if $f_t^*\alpha = \lambda_t.\alpha$, where λ_t and μ_t are related by $\frac{\partial_t\lambda_t}{\lambda_t} = f_t^*\mu_t$.*

Proof. The two following equations are equivalent:

$$\begin{aligned} \alpha &= \frac{1}{\lambda_t} f_t^*\alpha, \\ 0 &= \tfrac{\partial}{\partial t}\left(\frac{1}{\lambda_t} f_t^*\alpha\right) = -\frac{\frac{\partial}{\partial t}\lambda_t}{\lambda_t^2} f_t^*\alpha + \frac{1}{\lambda_t} f_t^*\mathcal{L}_{X_t}\alpha = \frac{1}{\lambda_t} f_t^*(-\mu_t.\alpha + \mathcal{L}_{X_t}\alpha). \quad \square \end{aligned}$$

43.19. Theorem. *Let (M,α) be a finite dimensional contact manifold. Then the group $\mathrm{Diff}(M,\alpha)$ of contact diffeomorphisms is a smooth regular Lie group. The injection $i : \mathrm{Diff}(M,\alpha) \to \mathrm{Diff}(M)$ is smooth, $T_{\mathrm{Id}}i$ maps the Lie algebra of $\mathrm{Diff}(M,\alpha)$ isomorphically onto $\mathfrak{X}_c(M,\alpha)$ with the negative of the usual Lie bracket, and locally there exist smooth retractions to i, so i is an initial mapping, see 27.11.*

If (M,α) is in addition a real analytic and compact contact manifold then all assertions hold in the real analytic sense.

Proof. For a contact manifold (M,α) let $\widehat{M} = M \times M \times (\mathbb{R}\setminus 0)$, with the contact structure $\hat\alpha = t.\mathrm{pr}_1^*\alpha - \mathrm{pr}_2^*\alpha$, where $t = \mathrm{pr}_3 : M \times M \times (\mathbb{R}\setminus 0) \to \mathbb{R}$. Let $f \in \mathrm{Diff}(M,\alpha)$ be a contact diffeomorphism with $f^*\alpha = \lambda_f.\alpha$. Inserting the characteristic vector field X_α into this last equation we get

$$\lambda_f = i_{X_\alpha}\lambda_f\alpha = i_{X_\alpha}(f^*\alpha) = f^*(i_{f_*X_\alpha}\alpha).$$

Thus, f determines λ_f, and for an arbitrary diffeomorphism $f \in \mathrm{Diff}(M)$ we may define a smooth function λ_f by 1. Then $\lambda_f \in C^\infty(M,\mathbb{R}\setminus 0)$ if f is near a contact diffeomorphism in the Whitney C^0-topology. We consider its *contact graph* $\Gamma_f : M \to \widehat{M}$, given by $\Gamma_f(x) := (x, f(x), \lambda_f(x))$, a section of the surjective submersion $\mathrm{pr}_1 : \widehat{M} \to M$. Note that Γ_f is a Legendre mapping if and only if f is a contact diffeomorphism, $f \in \mathrm{Diff}(M,\alpha)$, since $\Gamma_f^*\hat\alpha = \lambda_f.\alpha - f^*\alpha$.

Let us now fix a contact diffeomorphism $f \in \mathrm{Diff}(M,\alpha)$ with $f^*\alpha = \lambda_f.\alpha$. By proposition [**75**, 43.18], and also using the diffeomorphism $\Gamma_f : M \to \Gamma_f(M)$ there are: an open neighborhood U_f' of $\Gamma_f(M) \subset \widehat{M}$, an open neighborhood V_f' of the zero section 0_M in $T^*M \times \mathbb{R}$, and a diffeomorphism $\widehat{M} \supset U_f' \xrightarrow{\varphi_f} V_f' \subset T^*M \times \mathbb{R}$, such that the restriction $\varphi_f|\Gamma_f(M)$ equals the inverse of $\Gamma_f : 0_M \cong M \to \Gamma_f(M)$, and $\varphi_f^*(\theta_M - dt) = \hat\alpha$.

Now let $\tilde{U}_f$ be the open set of all diffeomorphisms $g \in \operatorname{Diff}(M)$ such that g equals f off some compact subset of M, $\Gamma_g(M) \subset U'_f \subset \widehat{M}$, and $\pi \circ \varphi_f \circ \Gamma_g : M \to M$ is a diffeomorphism, where $\pi : T^*M \times \mathbb{R} \to M$ is the vector bundle projection. For $g \in \tilde{U}_f$ and

$$\begin{aligned} s_f(g) &:= (\varphi_f \circ \Gamma_g) \circ (\pi \circ \varphi_f \circ \Gamma_g)^{-1} \in C_c^\infty(M \leftarrow T^*M \times \mathbb{R}) \\ &=: (\sigma_f(g), u_f(g)) \in \Omega_c^1(M) \times C_c^\infty(M, \mathbb{R}) \end{aligned}$$

the following conditions are equivalent:

(2) g is a contact diffeomorphism.

(3) $\Gamma_g(M)$ is a Legendre submanifold of $(\widehat{M}, \hat{\alpha})$.

(4) $\varphi_f(\Gamma_g(M))$ is a Legendre submanifold of $(T^*M \times \mathbb{R}, \theta_M - dt)$.

(5) The section $s_f(g)$ satisfies $s_f(g)^*(\theta_M - dt) = 0$, equivalently (by [**75**, 43.17]) $\sigma_f(g) = d(u_f(g))$.

Let us now consider the following diagram:

$$\begin{tikzcd}
\operatorname{Diff}(M) & \tilde{U}_f \arrow[l, hook'] \arrow[r, "s_f"] & \tilde{V}_f \arrow[r, hook] & C_c^\infty(M \leftarrow T^*M \times \mathbb{R}) \\
\operatorname{Diff}(M, \alpha) \arrow[u, hook] & U_f \arrow[l, hook'] \arrow[u, hook] \arrow[r, "u_f", "\cong"'] & V_f \arrow[u, hook, "j"] \arrow[r, hook] & C_c^\infty(M, \mathbb{R}). \arrow[u, hook, "j"', "\text{linear, splitting}"]
\end{tikzcd}$$

In this diagram we put $j(h) := (dh, h)$, a bounded linear splitting embedding. We let $\tilde{V}_f \subset C_c^\infty(M \leftarrow T^*M \times \mathbb{R})$ be the open set of all $(\omega, h) \in \Omega_c^1(M) \times C_c^\infty(M, \mathbb{R})$ with $(\omega, h)(M) \subset V'_f$ and such that $\operatorname{pr}_1 \circ \varphi_f^{-1} \circ (\omega, h) : M \to M$ is a diffeomorphism. We also consider the smooth mapping

$$\begin{aligned} &w_f : \tilde{V}_f \to \operatorname{Diff}(M) \\ &w_f(\omega, h) := \operatorname{pr}_2 \circ \varphi_f^{-1} \circ (\omega, h) \circ (\operatorname{pr}_1 \circ \varphi_f^{-1} \circ (\omega, h))^{-1} : M \to M, \end{aligned}$$

and let $V_f = (w_f \circ j)^{-1} \tilde{U}_f$. Then $w_f \circ s_f = \operatorname{Id}$, and so we may use as chart mappings for $\operatorname{Diff}(M, \alpha)$:

$$\begin{aligned} u_f : U_f := \tilde{U}_f \cap \operatorname{Diff}(M, \alpha) &\to V_f := (w_f \circ j)^{-1}(\tilde{U}_f) \subset C_c^\infty(M, \mathbb{R}), \\ u_f(g) &:= \operatorname{pr}_2 \circ (\varphi_f \circ \Gamma_g) \circ (\pi \circ \varphi_f \circ \Gamma_g)^{-1} \in C^\infty(M, \mathbb{R}), \\ u_f^{-1}(h) &= (w_f \circ j)(h) = w_f(dh, h). \end{aligned}$$

The chart change mapping $u_k \circ u_f^{-1}$ is defined on an open subset and is smooth, because $u_k \circ u_f^{-1} = \operatorname{pr}_2 \circ s_k \circ w_f \circ j$, and s_k and w_f are smooth by 42.13, 43.1, and by [**75**, 42.20]. Thus, the resulting atlas $(U_f, u_f)_{f \in \operatorname{Diff}(M,\alpha)}$ is smooth, and $\operatorname{Diff}(M, \alpha)$ is a smooth manifold in such a way that the injection $i : \operatorname{Diff}(M, \alpha) \to \operatorname{Diff}(M)$ is smooth.

Note that $s_f \circ w_f \neq \operatorname{Id}$, so we cannot construct (splitting) submanifold charts in this way.

But there exist local smooth retracts $u_f^{-1} \circ \operatorname{pr}_2 \circ s_f : (\operatorname{pr}_2 \circ s_f)^{-1}(V_f) \to U_f$. Therefore, the injection i has the property that a mapping into $\operatorname{Diff}(M, \alpha)$ is smooth if and only if its prolongation via i into $\operatorname{Diff}(M)$ is smooth. Thus, $\operatorname{Diff}(M, \alpha)$ is a Lie group, and from [**75**, 38.7] we may conclude that it is a regular Lie group.

A direct proof of regularity goes as follows: From lemma 43.16 and [**75**, 36.6] we see that $T_{\mathrm{Id}} i$ maps the Lie algebra of $\operatorname{Diff}(M, \alpha)$ isomorphically onto the Lie algebra $\mathfrak{X}_c(M,\alpha)$ of all contact vector fields with compact support. It also follows from lemma 43.16 that we have for the evolution operator

$$\operatorname{Evol}^r_{\operatorname{Diff}(M)}|C^\infty(\mathbb{R},\mathfrak{X}_c(M,\alpha)) = \operatorname{Evol}^r_{\operatorname{Diff}(M,\alpha)}$$

so that $\operatorname{Diff}(M,\alpha)$ is a regular Lie group. □

44.1. Theorem. Principal bundle of embeddings. *Let M and N be smooth finite dimensional manifolds, connected and second countable such that $\dim M \le \dim N$.*

Then the set $\operatorname{Emb}(M,N)$ of all smooth embeddings $M \to N$ is an open submanifold of $\mathfrak{C}^\infty(M,N)$. It is the total space of a smooth principal fiber bundle with structure group $\operatorname{Diff}(M)$, whose smooth base manifold is the space $B(M,N)$ of all submanifolds of N of type M.

The open subset $\operatorname{Emb}_{prop}(M,N)$ of proper (equivalently closed) embeddings is saturated under the $\operatorname{Diff}(M)$-action, and is thus the total space of the restriction of the principal bundle to the open submanifold $B_{closed}(M,N)$ of $B(M,N)$ consisting of all closed submanifolds of N of type M.

This result is based on an idea implicitly contained in [**131**], it was fully proved by [**11**] for compact M and for general M by [**97**]. The clearest presentation was in [**95**, section 13].

44.2. Result. [**21**]. *Let M and N be smooth manifolds. Then the diffeomorphism group $\operatorname{Diff}(M)$ acts smoothly from the right on the manifold $\operatorname{Imm}_{prop}(M,N)$ of all smooth proper immersions $M \to N$, which is an open subset of $\mathfrak{C}^\infty(M,N)$.*

Then the space of orbits $\operatorname{Imm}_{prop}(M,N)/\operatorname{Diff}(M)$ is Hausdorff in the quotient topology.

Let $\operatorname{Imm}_{free,\ prop}(M,N)$ be set of all proper immersions, on which $\operatorname{Diff}(M)$ acts freely. Then this is open in $\mathfrak{C}^\infty(M,N)$ and it is the total space of a smooth principal fiber bundle

$$\operatorname{Imm}_{free,prop}(M,N) \to \operatorname{Imm}_{free,prop}(M,N)/\operatorname{Diff}(M).$$

44.21. A classifying space for the diffeomorphism group

Let ℓ^2 be the Hilbert space of square summable sequences, and let S be a compact manifold. By a slight generalization of theorem 44.1 (we use a Hilbert space instead of a Riemannian manifold N), the space $\operatorname{Emb}(S,\ell^2)$ of all smooth embeddings is an open submanifold of $C^\infty(S,\ell^2)$, and it is also the total space of a smooth principal bundle with structure group $\operatorname{Diff}(S)$ acting from the right by composition. The base space $B(S,\ell^2) := \operatorname{Emb}(S,\ell^2)/\operatorname{Diff}(S)$ is a smooth manifold modeled on Fréchet spaces which are projective limits of Hilbert spaces. $B(S,\ell^2)$ is a Lindelöf space in the quotient topology, and the model spaces admit bump functions, thus $B(S,\ell^2)$ admits smooth partitions of unity, by 16.10. We may view $B(S,\ell^2)$ as the space of all submanifolds of ℓ^2 which are diffeomorphic to S, a nonlinear analog of the infinite dimensional Grassmannian.

44.22. Lemma. *The total space $\operatorname{Emb}(S,\ell^2)$ is contractible.*

Therefore, by the general theory of classifying spaces the base space $B(S,\ell^2)$ is a classifying space of $\mathrm{Diff}(S)$. We will give a detailed description of the classifying process in 44.24.

Proof. We consider the continuous homotopy $A : \ell^2\times[0,1] \to \ell^2$ through isometries which is given by $A_0 = \mathrm{Id}$ and by

$$A_t(a_0, a_1, a_2, \dots) = (a_0, \dots, a_{n-2}, a_{n-1}\cos\theta_n(t), a_{n-1}\sin\theta_n(t),$$
$$a_n\cos\theta_n(t), a_n\sin\theta_n(t), a_{n+1}\cos\theta_n(t), a_{n+1}\sin\theta_n(t), \dots)$$

for $\frac{1}{n+1} \le t \le \frac{1}{n}$, where $\theta_n(t) = \varphi(n((n+1)t-1))\frac{\pi}{2}$ for a fixed smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ which is 0 on $(-\infty, 0]$, grows monotonely to 1 in $[0,1]$, and equals 1 on $[1,\infty)$.

Then $A_{1/2}(a_0, a_1, a_2, \dots) = (a_0, 0, a_1, 0, a_2, 0, \dots)$ is in ℓ^2_{even} and on the other hand $A_1(a_0, a_1, a_2, \dots) = (0, a_0, 0, a_1, 0, a_2, 0, \dots)$ is in ℓ^2_{odd}. The same homotopy makes sense as a mapping $A : \mathbb{R}^\infty \times \mathbb{R} \to \mathbb{R}^{(\mathbb{N})}$, and here it is easily seen to be smooth: a smooth curve in $\mathbb{R}^{(\mathbb{N})}$ is locally bounded and thus locally takes values in a finite dimensional subspace $\mathbb{R}^N \subset \mathbb{R}^{(\mathbb{N})}$. The image under A then has values in $\mathbb{R}^{2N} \subset \mathbb{R}^{(\mathbb{N})}$, and the expression is clearly smooth as a mapping into $\mathbb{R}^{2N}$. This is a variant of a homotopy constructed by [**107**].

Given two embeddings e_1 and $e_2 \in \mathrm{Emb}(S,\ell^2)$ we first deform e_1 through embeddings to $e_1' \in \mathrm{Emb}(S,\ell^2_{\text{even}})$, and e_2 to $e_2' \in \mathrm{Emb}(S,\ell^2_{\text{odd}})$. Then we connect them by $te_1' + (1-t)e_2'$ which is a smooth embedding for all t since the values are always orthogonal. □

44.23

We consider the smooth action $\mathrm{ev} : \mathrm{Diff}(S)\times S \to S$ and the associated bundle $\mathrm{Emb}(S,\ell^2)[S,\mathrm{ev}] = \mathrm{Emb}(S,\ell^2)\times_{\mathrm{Diff}(S)} S$ which we call $E(S,\ell^2)$, a smooth fiber bundle over $B(S,\ell^2)$ with standard fiber S. In view of the interpretation of $B(S,\ell^2)$ as the nonlinear Grassmannian, we may visualize $E(S,\ell^2)$ as the "universal S-bundle" as follows: $E(S,\ell^2) = \{(N,x) \in B(S,\ell^2)\times\ell^2 : x \in N\}$ with the differentiable structure from the embedding into $B(S,\ell^2)\times\ell^2$.

The tangent bundle $TE(S,\ell^2)$ is then the space of all (N,x,ξ,v) where $N \in B(S,\ell^2)$, $x \in N$, ξ is a vector field along and normal to N in ℓ^2, and $v \in T_x\ell^2$ such that the part of v normal to T_xN equals $\xi(x)$. This follows from the description of the principal fiber bundle $\mathrm{Emb}(S,\ell^2) \to B(S,\ell^2)$ given in 44.1 combined with [**75**, 42.17]. Obviously, the vertical bundle $VE(S,\ell^2)$ consists of all (N,x,v) with $x \in N$ and $v \in T_xN$. The orthonormal projection $p_{(N,x)} : \ell^2 \to T_xN$ defines a connection $\Phi^{\text{class}} : TE(S,\ell^2) \to VE(S,\ell^2)$ which is given by $\Phi^{\text{class}}(N,x,\xi,v) = (N,x,p_{(N,x)}v)$. It will be called the *classifying connection* for reasons to be explained in the next theorem.

44.24. Theorem. Classifying space for $\mathrm{Diff}(S)$**.**
The fiber bundle $(E(S,\ell^2) \to B(S,\ell^2), S)$ is classifying for S-bundles and Φ^{class} is a classifying connection:

For each finite dimensional bundle $(p : E \to M, S)$ and each connection Φ on E there is a smooth (classifying) mapping $f : M \to B(S,\ell^2)$ such that (E,Φ) is isomorphic to $(f^*E(S,\ell^2), f^*\Phi^{class})$. Homotopic maps pull back isomorphic S-bundles *and conversely (the homotopy can be chosen smooth). The pulled back connection is invariant under a homotopy H if and only if $i(C^{class}T_{(x,t)}H.(0_x, \frac{d}{dt}))\mathcal{R}^{class} = 0$ where C^{class} is the horizontal lift of Φ^{class}, and $\mathcal{R}^{class}$ is its curvature .*

Since S is compact the classifying connection Φ^{class} is complete, and its parallel transport Pt^{class} has the following classifying property:

$$\tilde{f} \circ \mathrm{Pt}^{f^*\Phi^{class}}(c,t) = \mathrm{Pt}^{class}(f \circ c, t) \circ \tilde{f},$$

where $\tilde{f} : E \cong f^*E(S,\ell^2) \to E(S,\ell^2)$ is the fiberwise diffeomorphic which covers the classifying mapping $f : M \to B(S,\ell^2)$.

47.2. Example: The sphere S^∞

This is the set $\{x \in R^{(\mathbb{N})} : \langle x,x\rangle = 1\}$, the usual infinite dimensional sphere used in algebraic topology, the topological inductive limit of $S^n \subset S^{n+1} \subset \dots$. The inductive limit topology coincides with the subspace topology since clearly $\varinjlim S^n \to S^\infty \subset \mathbb{R}^{(\mathbb{N})}$ is continuous, S^∞ as closed subset of $\mathbb{R}^{(\mathbb{N})}$ with the c^∞-topology is compactly generated, and since each compact set is contained in a step of the inductive limit.

We show that S^∞ is a smooth manifold by describing an explicit smooth atlas, the *stereographic atlas*. Choose $a \in S^\infty$ ("south pole"). Let

$$\begin{aligned} U_+ &:= S^\infty \setminus \{a\}, & u_+ &: U_+ \to \{a\}^\perp, & u_+(x) &= \tfrac{x-\langle x,a\rangle a}{1-\langle x,a\rangle}, \\ U_- &:= S^\infty \setminus \{-a\}, & u_- &: U_- \to \{a\}^\perp, & u_-(x) &= \tfrac{x-\langle x,a\rangle a}{1+\langle x,a\rangle}. \end{aligned}$$

From an obvious drawing in the 2-plane through 0, x, and a it is easily seen that u_+ is the usual stereographic projection. We also get

$$u_+^{-1}(y) = \tfrac{|y|^2-1}{|y|^2+1}a + \tfrac{2}{|y|^2+1}y \qquad \text{for } y \in \{a\}^\perp \setminus \{0\}$$

and $(u_- \circ u_+^{-1})(y) = \frac{y}{|y|^2}$. The latter equation can directly be seen from the drawing using the intersection theorem.

The two stereographic charts above can be extended to charts on open sets in $\mathbb{R}^{(\mathbb{N})}$ in such a way that S^∞ becomes a splitting submanifold of $\mathbb{R}^{(\mathbb{N})}$:

$$\begin{aligned} \tilde{u}_+ &: \mathbb{R}^{(\mathbb{N})} \setminus [0,+\infty)a \to a^\perp + (-1,+\infty)a \\ \tilde{u}_+(z) &:= u_+(\tfrac{z}{|z|}) + (|z|-1)a \\ &= (1+\langle z,a\rangle)u_+^{-1}(z - \langle z,a\rangle a) \end{aligned}$$

Since the model space $\mathbb{R}^{(\mathbb{N})}$ of S^∞ has the bornological approximation property by [**75**, 28.6], and is reflexive, by 28.7 the operational tangent bundle of S^∞ equals the kinematic one: $DS^\infty = TS^\infty$.

We claim that TS^∞ is diffeomorphic to $\{(x,v) \in S^\infty \times \mathbb{R}^{(\mathbb{N})} : \langle x,v\rangle = 0\}$.
The $X_x \in T_xS^\infty$ are exactly of the form $c'(0)$ for a smooth curve $c : \mathbb{R} \to S^\infty$ with $c(0) = x$ by 28.13. Then $0 = \frac{d}{dt}|_0\langle c(t), c(t)\rangle = 2\langle x, X_x\rangle$. For $v \in x^\perp$ we use $c(t) = \cos(|v|t)x + \sin(|v|t)\frac{v}{|v|}$.

The construction of S^∞ works for any positive definite bounded bilinear form on any convenient vector space.

The sphere is smoothly contractible, by the following argument: We consider the homotopy $A : \mathbb{R}^{(\mathbb{N})} \times [0,1] \to \mathbb{R}^{(\mathbb{N})}$ through isometries which is given by $A_0 = \mathrm{Id}$ and by 44.22

$$A_t(a_0, a_1, a_2, \dots) = (a_0, \dots, a_{n-2}, a_{n-1}\cos\theta_n(t), a_{n-1}\sin\theta_n(t), \\ a_n\cos\theta_n(t), a_n\sin\theta_n(t), a_{n+1}\cos\theta_n(t), a_{n+1}\sin\theta_n(t), \dots)$$

for $\frac{1}{n+1} \le t \le \frac{1}{n}$, where $\theta_n(t) = \varphi(n((n+1)t-1))\frac{\pi}{2}$ for a fixed smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ which is 0 on $(-\infty, 0]$, grows monotonely to 1 in $[0,1]$, and equals 1 on $[1,\infty)$. The mapping A is smooth since it maps smooth curves (which locally map into some $\mathbb{R}^N$) to smooth curves (which then locally have values in $\mathbb{R}^{2N}$). Then $A_{1/2}(a_0, a_1, a_2, \dots) = (a_0, 0, a_1, 0, a_2, 0, \dots)$ is in $\mathbb{R}^{(\mathbb{N})}_{\text{even}}$, and on the other hand $A_1(a_0, a_1, a_2, \dots) = (0, a_0, 0, a_1, 0, a_2, 0, \dots)$ is in $\mathbb{R}^{(\mathbb{N})}_{\text{odd}}$. This is a variant of a homotopy constructed by [107]. Now $A_t|S^\infty$ for $0 \le t \le 1/2$ is a smooth isotopy on S^∞ between the identity and $A_{1/2}(S^\infty) \subset \mathbb{R}^{(\mathbb{N})}_{\text{even}}$. The latter set is contractible in a chart.

One may prove in a simpler way that S^∞ is contractible with a real analytic homotopy with one corner: roll all coordinates one step to the right and then contract in the stereographic chart opposite to $(1, 0, \dots)$.

47.3. Example. The Grassmannians and the Stiefel manifolds

The *Grassmann manifold* $G(k,\infty;\mathbb{R}) = G(k,\infty)$ is the set of all k-dimensional linear subspaces of the space of all finite sequences $\mathbb{R}^{(\mathbb{N})}$. The *Stiefel manifold of orthonormal k-frames* $O(k,\infty;\mathbb{R}) = O(k,\infty)$ is the set of all linear isometries $\mathbb{R}^k \to \mathbb{R}^{(\mathbb{N})}$, where the latter space is again equipped with the standard weak inner product described at the beginning of 47.2. The *Stiefel manifold of all k-frames* $GL(k,\infty;\mathbb{R}) = GL(k,\infty;\mathbb{R})$ is the set of all injective linear mappings $\mathbb{R}^k \to \mathbb{R}^{(\mathbb{N})}$.

There is a canonical *transposition mapping* $(\_)^t : L(\mathbb{R}^k, \mathbb{R}^{(\mathbb{N})}) \to L(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^k)$ which is given by

$$A^t : \mathbb{R}^{(\mathbb{N})} \xrightarrow{\text{incl}} \mathbb{R}^{\mathbb{N}} = \left(\mathbb{R}^{(\mathbb{N})}\right)' \xrightarrow{A'} (\mathbb{R}^k)' = \mathbb{R}^k$$

and satisfies $\langle A^t(x), y\rangle = \langle x, A(y)\rangle$. The transposition mapping is bounded and linear, so it is real analytic. Then we have

$$GL(k,\infty) = \{A \in L(\mathbb{R}^k, \mathbb{R}^{(\mathbb{N})}) : A^t \circ A \in GL(k)\},$$

since $A^t \circ A \in GL(k)$ if and only if $\langle Ax, Ay\rangle = \langle A^tAx, y\rangle = 0$ for all y implies $x = 0$, which is equivalent to A injective. So in particular $GL(k,\infty)$ is open in $L(\mathbb{R}^k, \mathbb{R}^{(\mathbb{N})})$. The Lie group $GL(k)$ acts freely from the right on the space $GL(k,\infty)$. Two elements of $GL(k,\infty)$ lie in the same orbit if and only if they have the same image in $\mathbb{R}^{(\mathbb{N})}$. We have a surjective mapping $\pi : GL(k,\infty) \to G(k,\infty)$, given by $\pi(A) = A(\mathbb{R}^k)$, where the inverse images of points are exactly the $GL(k)$-orbits. Similarly, we have

$$O(k,\infty) = \{A \in L(\mathbb{R}^k, \mathbb{R}^{(\mathbb{N})}) : A^t \circ A = \mathrm{Id}_k\}.$$

The Lie group $O(k)$ of all isometries of $\mathbb{R}^k$ acts freely from the right on the space $O(k,\infty)$. Two elements of $O(k,\infty)$ lie in the same orbit if and only if they have the same image in $\mathbb{R}^{(\mathbb{N})}$. The projection $\pi : GL(k,\infty) \to G(k,\infty)$ restricts to a surjective mapping $\pi : O(k,\infty) \to G(k,\infty)$, and the inverse images of points are now exactly the $O(k)$-orbits.

47.6. Theorem. *The principal bundle $(O(k,\infty),\pi,G(k,\infty))$ is classifying for finite dimensional principal $O(k)$-bundles and carries a universal real analytic $O(k)$-connection $\omega\in\Omega^1(O(k,\infty),\mathfrak{o}(k))$.*

This means: For each finite dimensional smooth or real analytic principal $O(k)$-bundle $P\to M$ with principal connection ω_P there is a smooth or real analytic mapping $f:M\to G(k,\infty)$ such that the pullback $O(k)$-bundle $f^*O(k,\infty)$ is isomorphic to P and the pullback connection $f^*\omega$ equals ω_P via this isomorphism.

For ∞ replaced by a large N and bundles where the dimension of the base is bounded this is due to [**110**].

47.9. Theorem. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(\infty)$. Then there is a smoothly arcwise connected splitting regular Lie subgroup G of $GL(\infty)$ whose Lie algebra is $\mathfrak{g}$. The exponential mapping of $GL(\infty)$ restricts to that of G, which is a local real analytic diffeomorphism near zero. The Campbell-Baker-Hausdorff formula gives a real analytic mapping near 0 and has the usual properties, also on G.*

Proof. Let $\mathfrak{g}_n:=\mathfrak{g}\cap\mathfrak{gl}(n)$, a finite dimensional Lie subalgebra of $\mathfrak{g}$. Then $\bigcup\mathfrak{g}_n=\mathfrak{g}$. The convenient structure $\mathfrak{g}=\varinjlim_n\mathfrak{g}_n$ coincides with the structure inherited as a complemented subspace, since $\mathfrak{gl}(\infty)$ carries the finest locally convex structure.

So for each n there is a connected Lie subgroup $G_n\subset GL(n)$ with Lie algebra $\mathfrak{g}_n$. Since $\mathfrak{g}_n\subset\mathfrak{g}_{n+1}$ we have $G_n\subset G_{n+1}$, and we may consider $G:=\bigcup_n G_n\subset GL(\infty)$. Each $g\in G$ lies in some G_n and may be connected to Id via a smooth curve there, which is also smooth curve in G, so G is smoothly arcwise connected.

All mappings $\exp|\mathfrak{g}_n:\mathfrak{g}_n\to G_n$ are local real analytic diffeomorphisms near 0, so $\exp:\mathfrak{g}\to G$ is also a local real analytic diffeomorphism near zero onto an open neighborhood of the identity in G. A similar argument applies to evol so that G is regular. The rest is clear. □

47.10. Examples

In the following we list some of the well known examples of simple infinite dimensional Lie groups which fit into the picture treated in this section. The reader can easily continue this list, especially by complex versions.

The Lie group

$SL(\infty)$ is the inductive limit

$$\begin{aligned}SL(\infty)&=\{A\in GL(\infty):\det(A)=1\}\\&=\varinjlim_{n\to\infty}SL(n)\subset GL(\infty),\end{aligned}$$

the connected Lie subgroup with Lie algebra $\mathfrak{sl}(\infty)=\{X\in\mathfrak{gl}(\infty):\operatorname{tr}(X)=0\}$.

The Lie group $SO(\infty,\mathbb{R})$

is the inductive limit

$$\begin{aligned} SO(\infty) &= \{A \in GL(\infty) : \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^{(\mathbb{N})} \text{ and } \det(A) = 1\} \\ &= \varinjlim_{n\to\infty} SO(n) \subset GL(\infty), \end{aligned}$$

the connected Lie subgroup of $GL(\infty)$ with the Lie algebra $\mathfrak{o}(\infty) = \{X \in \mathfrak{gl}(\infty) : X^t = -X\}$ of skew elements.

The Lie group

$O(\infty)$ is the inductive limit

$$\begin{aligned} O(\infty) &= \{A \in GL(\infty) : \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^{(\mathbb{N})}\} \\ &= \varinjlim_{n\to\infty} O(n) \subset GL(\infty). \end{aligned}$$

It has two connected components, that of the identity is $SO(\infty)$.

The Lie group

$Sp(\infty,\mathbb{R})$ is the inductive limit

$$\begin{aligned} Sp(\infty,\mathbb{R}) &= \{A \in GL(\infty) : A^t J A = J\} \\ &= \varinjlim_{n\to\infty} Sp(2n,\mathbb{R}) \subset GL(\infty), \text{ where} \\ J &= \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \in L(\mathbb{R}^{(\mathbb{N})}, \mathbb{R}^{(\mathbb{N})}). \end{aligned}$$

It is the connected Lie subgroup of $GL(\infty)$ with the Lie algebra $\mathfrak{sp}(\infty,\mathbb{R}) = \{X \in \mathfrak{gl}(\infty) : X^t J + JX = 0\}$ of symplectically skew elements.

47.11. Theorem. *The following manifolds are real analytically diffeomorphic to the homogeneous spaces indicated:*

$$\begin{gathered} GL(k,\infty) \cong GL(\infty) \Big/ \begin{pmatrix} \mathrm{Id}_k & L(\mathbb{R}^k, \mathbb{R}^{\infty-k}) \\ 0 & GL(\infty-k) \end{pmatrix}, \\ O(k,\infty) \cong O(\infty)/(\mathrm{Id}_k \times O(\infty-k)), \\ G(k,\infty) \cong O(\infty)/(O(k) \times O(\infty-k)). \end{gathered}$$

The universal vector bundle $(E(k,\infty),\pi,G(k,\infty),\mathbb{R}^k)$ *is defined as the associated bundle*

$$\begin{aligned} E(k,\infty) &= O(k,\infty)[\mathbb{R}^k] \\ &= \{(Q,x) : x \in Q\} \subset G(k,\infty)\times\mathbb{R}^{(\mathbb{N})}. \end{aligned}$$

The tangent bundle of the Grassmannian is then given by

$$TG(k,\infty) = L(E(k,\infty), E(k,\infty)^\perp).$$

Proof. This is a direct consequence of the chart construction of $G(k,\infty)$. $\square$

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