Exercises for Algebraic Topology

WS 2017

Andreas Kriegl

1.1.
Prove the following statements:

(a) Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. Then $\overline{A} \times \dot{B} \cup \dot{A} \times \overline{B}$ is the boundary of $A \times B$ in $X \times Y$.

(b) Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be convex. Then $A \times B \subseteq \mathbb{R}^{n+m}$ is convex.

1.2.
The convex hull $\langle A \rangle_{\text{cv}}$ of $A \subseteq \mathbb{R}^n$ is defined to be the smallest convex subset of $\mathbb{R}^n$ which contains A. This is the intersection of all convex subsets of $\mathbb{R}^n$ containing A. Show that

$$\langle A \rangle_{\text{cv}} = \Big\{ \sum_{i=0}^{q} \lambda_i \, x_i : q \in \mathbb{N}, \lambda_i \geq 0, x_i \in A, \sum_{i=0}^{q} \lambda_i = 1 \Big\}.$$

1.3.
Give an example of a mapping of pairs $f : (X, A) \to (Y, B)$ which is a relative homeomorphism and for which $f|_A : A \to B$ is a homeomorphism, but which is not a homeomorphism of pairs.

1.4.
For locally compact (T_2) but not compact spaces X the Alexandroff-compactification X_∞ is defined as the disjoint union $X \sqcup \{\infty\}$ with the neighborhoods in X as neighborhoodbasis for the points $x \in X$ and the complements of the compact subsets $K \subseteq X$ in X_∞ as neighborhoodbasis at ∞.
Show that this compactification is up to homeomorphy characterized by the properties that X_∞ is a compact space, X is a topological subspace of X_∞, and $X_\infty \setminus X$ is a single point.
Conclude that for compact spaces X and $x_0 \in X$ we have $X \cong (X \setminus \{x_0\})_\infty$.

1.5.
Show that for any $x, y \in \mathring{D}^n$ there is a homeomorphism of pairs $(D^n, \{x\}) \cong (D^n, \{y\})$.

1.6.
For $R > r > 0$ let the filled torus be the subset of V of $\mathbb{R}^3$ obtained by rotating a closed disk in the x-z-plane with center $(R, 0, 0)$ and radius r around the z-axes. It can be described by $V = \{(x, y, z) : (\sqrt{x^2 + y^2} - R)^2 + z^2 \leq r^2\}$. Show that formula of the embedding described in example [1, 1.18] gives also a homeomorphism $S^1 \times D^2 \cong V$.

1.7.
Show that the mapping $(i_1, \dots, i_n) : X_1 \vee \dots \vee X_n \to X_1 \times \ldots \times X_n$ defined in [1, 1.41] is an embedding.

1.8.
Show: $(S^1 \times S^1)/(S^1 \vee S^1) \cong S^2$.

1.9.
Show that $\mathbb{R}^n / D^n \cong \mathbb{R}^n$ and that $\mathbb{R}^n / \mathring{D}^n$ is not Hausdorff.

1.10.
Show that any continuous $f : X \to Y$ induces a continuous mapping $C(f) : C(X) \to C(Y)$ between the cones, via $f \times I : X \times I \to Y \times I$.

1.11.
The suspension (dt. Einhängung) of a topological space X is $E(X) := C(X)/X$, where X is embedded into $C(X)$ via $x \mapsto (x, 1)$. Show that $f : X \to Y$ induces a mapping $E(f) : E(X) \to E(Y)$. Show furthermore, that $E(D^n) \cong D^{n+1}$ and $E(S^n) \cong S^{n+1}$.

1.12.
Show that the lens space $L(\frac{1}{2})$ is homeomorphic to $\mathbb{P}^3_{\mathbb{R}}$.

1.13.
Describe a mapping $f : S^2 \to S^2 \vee S^1$ such that $(S^2 \vee S^1) \cup_f D^3 \cong S^2 \times S^1$. **Hint:** [1, 1.12]. Note, that the product $p \times q : X \times Y \to X/A \times Y/B$ of the two quotient mappings does not induce a bijection $(X \times Y)/(A \times B) \to X/A \times Y/B$, but only a well-defined surjective continuous mapping. By [1, 1.34] we have the quotient maps induce homeomorphisms $X \setminus A \cong X/A \setminus \{A\}$ and $Y \setminus B \cong Y/B \setminus \{B\}$ and hence a homeomorphism

$$(X \times Y) \setminus (X \times B \cup A \times Y) = (X \setminus A) \times (Y \setminus B) \cong (X/A \setminus \{A\}) \times (Y/B \setminus \{B\}) = (X/A \times Y/B) \setminus (X/A \vee ; Y/B)$$

Whereas $(X \times Y) \setminus (A \times B) \supset (X \times Y) \setminus (X \times B \cup A \times Y)$

1.14.
Consider the subspace $X := S^1 \cup D^1 \subseteq \mathbb{C}$ and a mapping $f : S^1 \to X$ which runs through the top half circle, the diameter D^1, the bottom half circle, and again the diameter. Show that $X \cup_f D^2$ is homeomorphic to the Möbius strip. **Hint:** Use [1, 1.92].

1.15.
Let $\mathbb{Z}$ act on $\mathbb{R}^2$ by $n : (x_1, x_2) \mapsto (x_1 + n, (-1)^n x_2)$. Show that $\mathbb{R}^2/\mathbb{Z}$ is homeomorphic to the open Möbius strip (i.e. the Möbius strip from [1, 1.58] without its boundary S^1).

1.16.
Let G be the subgroup of homeomorphisms on $\mathbb{R}^2$ generated by $(x_1, x_2) \mapsto (x_1 + 1, x_2)$ and $(x_1, x_2) \mapsto (-x_1, x_2 + 1)$. Show that $\mathbb{R}^2/G$ is homeomorphic to Kleins bottle.

1.17.
Let T be the torus into $\mathbb{R}^3$ as in [1, 1.18]. Consider the action of the group $S^0 = \{\pm 1\}$ on T given by

(1) $(x, y, z) \overset{-1}{\longmapsto} (-x, -y, z)$ and show that $T/S^0 \cong S^1 \times S^1$.

(2) $(x, y, z) \overset{-1}{\longmapsto} (x, -y, -z)$ and show that $T/S^0 \cong S^2$.

(3) $(x, y, z) \overset{-1}{\longmapsto} (-x, -y, -z)$ and show that T/S^0 is homeomorphic to Kleins bottle.

1.18.
Let G be a finite group acting on X, $A \subseteq X$ closed with $X = GA$ and $\sim$ die equivalence relation generated by $x \sim gx$ for all x. Show, that the canonical mapping $A/\sim \to X/\sim$ is a homeomorphism. **Hint:** In order to show openness prove that for every open $U \subseteq A$ we have $GU := \bigcup_{g \in G} g(U) = \bigcap_{g \in G} g(V)$, where $V = (X \setminus A) \cup U$ is open in X.

2.1.
Show that $X \times Y$ is contractible provided X and Y are contractible.

2.2.
Show that X is contractible if and only if $\Delta : X \to X \times X$, $x \mapsto (x, x)$ is 0-homotopic.

2.3.
Two homeomorphisms $f_0, f_1 : X \to Y$ are called isotopic, iff there exists a homotopy $t \mapsto f_t$ consisting of homeomorphism $f_t : X \to Y$ only. Let $f : D^n \to D^n$ be a homeomorphism with $f|_{S^{n-1}} = \text{id}$ and $f(0) = 0$. Show that id_{D_n} is isotopic f to via $f_t : x \mapsto t\,\tilde{f}(x/t)$, where $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ is an appropriate extension of f.

2.4.
Show that the pointwise multiplication defines an Abelian group structure on $[X, S^1]$ and, furthermore, that $\deg : [S^1, S^1] \to (\mathbb{Z}, +)$ is a group-homomorphism with respect to this group structure for $X := S^1$.

2.5.
Let $f : D^2 \to \mathbb{R}^2$ be a continuous function with $f|_{S^1}$ odd. Show that there exists a $z \in D^2$ with $f(z) = 0$. Deduce the existence of a solution $(x, y) \in \mathbb{R}^2$ for

$$x \cos(y) = x^2 + y^2 - 1 \text{ and } y \cos(x) = \sin(2\pi(x^2 + y^2))$$

2.6.
Show that S^∞ is contractible.
Hint: Let $p : \mathbb{R}^\infty \setminus \{0\} \to S^\infty$ given by $x \mapsto \frac{x}{\|x\|_2}$, where $\|x\|_2 := \sqrt{\sum_k x_k^2}$. Show that $h_t : (x_0, x_1, x_2, \dots) \mapsto p((1-t)x_0, tx_0 + (1-t)x_1, tx_1 + (1-t)x_2, tx_2 + (1-t)x_3, \dots)$ defines a homotopy between id_{S^∞} and the right shift $S^\infty \to \{x \in S^\infty : x_0 = 0\}$. Now consider the homotopy $(0, x_1, x_2, \dots) \mapsto p(t, (1-t)x_1, (1-t)x_2, \dots)$.

2.7.
Let $p, q \in S^1 \times S^1$ be different points. Show that $S^1 \times S^1 \setminus \{p, q\} \sim S^1 \vee S^1 \vee S^1$.

2.8.
Show that $\mathbb{R}^3 \setminus S^1 \sim S^1 \vee S^2$, where $S^1 \subseteq \mathbb{R}^3$ is the unit-circle in $\mathbb{R}^2 \times \{0\}$.

2.9.
Show that $S^3 \setminus S^1 \sim S^1$, where S^1 is the unit-circle in $\mathbb{R}^2 \times \{(0, 0)\}$.

2.10.
Show that the mapping cylinder of $z \mapsto z^2$, $S^1 \to S^1$ is homeomorphic to the Möbius strip.

2.11.
Show that for $f : S^{n-1} \to Y$ one has $M_f / S^{n-1} \sim Y \cup_f D^n$.

2.12.
Show that $O(n) \subseteq GL(n)$ is an SDR. **Hint:** Apply Gram-Schmidt orthonormalization to the columns of $A \in GL(n)$ to obtain $r(A) \in O(n)$. This procedure is given by multiplication with an upper triangular matrix with positive diagonal entries depending smoothly on A. Now deform the matrix to the identity matrix.

3.1.
Let K be a simplicial complex in $\mathbb{R}^n$ and $p \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$. The cone $C(K, p)$ is the set consisting of $\{p\}$, all simplices of K, and all simplices $\langle p, x_0, \dots, x_i \rangle$ for $\langle x_0, \dots, x_i \rangle \in K$. The suspension is $E(K) := C(K, p) \cup C(K, -p)$. Show that $C(K, p)$ and $E(K)$ are simplicial complexes with $|C(K, p)| \cong C(|K|)$ and $|E(K)| = E(|K|)$.

3.2.
The cartesian product of two polyeder is a polyeder. **Hint:** Show that the product of two closed simplices $\bar\sigma$ and $\bar\tau$ can be triangulated using $C((\sigma\times\tau)^\cdot)\cong\bar\sigma\times\bar\tau$.

3.3.
Let K be a simplicial complex and α_i the number of i-simplices of K. The number $\chi(K):=\sum_{i\geq 0}(-1)^i\alpha_i$ is called Euler-characteristic of K. Show that

- For any triangulation K of S^1 we have $\chi(K)=0$.
- $\chi(C(K,p))=1$ for the cone $C(K,p)$ given in exercise (3.1).
- $\chi(E(K))=2-\chi(K)$ for the suspension $E(K)$ given in exercise (3.1).
- $\chi(\dot\sigma)=1+(-1)^n$ where $\dot\sigma:=\{\tau:\tau<\sigma\}$ for any $n+1$-simplex σ.

3.4.
Let $x_0,\dots,x_q$ be vertices of K. Show that $\mathrm{st}_K(x_0)\cap\dots\cap\mathrm{st}_K(x_q)\neq\emptyset\Leftrightarrow\langle x_0,\dots,x_q\rangle\in K$.

3.5.
Show that $S^1\not\simeq S^n$ for $n>1$ and deduce $\mathbb{R}^2\not\cong\mathbb{R}^{n+1}$. **Hint:** [1, 3.32].

4.1.
Find CW-decompositions with as few cells as possible of D^n, $S^1\times I$, the closed Möbiusstrip, and the disk D^2_g with g holes as in [1, 1.65].

4.2.
Show that $S^n\times S^m/S^n\vee S^m$ is a CW-space which is homeomorphic to S^{n+m}.

4.3.
Show that the mapping cylinder of a cellular mapping between CW-spaces is a CW-space.

4.4.
Let X be a CW-space with $\dim(X)<n$. Show that $[X,S^n]=\{0\}$. **Hint:** Use [1, 4.20].

5.1.
Determine the fundamental group of $S^1\times\mathbb{P}^2$, $\mathbb{P}^2\vee\mathbb{P}^2$, $\mathbb{P}^2\times\mathbb{P}^2$, $S^1\times S^m$ for $m\geq 2$, and of $\mathbb{R}^3\setminus S^1$.

The following exercises (5.2)–(5.5) show, that the isomorphy problem is algorithmically unsolvable for m-manifolds with $m\geq 4$. For this it is enough to show that every finitely presented group appears as fundamental group of such a manifold.

5.2.
Let M be a connected manifold of dimension $m\geq 3$. Show that $\pi_1(M\setminus\mathring{D}_1)\cong\pi_1(M)$ for $M\setminus\mathring{D}_1$ as in [1, 1.59]. Deduce that for the connected sum $M\sharp N$ of [1, 1.63] of two such manifolds we get $\pi_1(M\sharp N)\cong\pi_1(M)\coprod\pi_1(N)$.
Hint: Theorem of Seifert and van Kampen.

5.3.
Show that for $m\geq 4$ the fundamental group of the connected sum M of k copies of $S^1\times S^{m-1}$ is the free group $\langle\{s_1,\dots,s_k\}:\emptyset\rangle$ with k generators.

5.4.
Show that

(a) $\pi_1(M) \cong \pi_1(M \setminus f(S^1 \times \mathring{D}^{m-1}))$ when $f : S^1 \times 2D^{m-1} \to M$ is an embedding into an m-manifold M with $m \geq 3$.

(b) $\pi_1(M \cup_f (D^2 \times S^{m-2})) \cong \pi_1(M)/\langle\{f|_{S^1\times\{*\}}\}\rangle$ when $f : S^1 \times S^{m-2} \to M$ is continuous and $m \geq 4$.

5.5.
Let $G = \langle\{s_1, \dots, s_k\} : \{r_1, \dots, r_l\}\rangle$ be a finitely represented group. Now construct a compact connected manifold without boundary recursively by starting with M from exercise (5.3) and cutting for every $r_i \in \pi_1(M)$ a neighborhood homeomorphic to $S^1 \times D^{m-1}$ of an appropriately choosen representant of r_i and pasting a cylinder $D^2 \times S^{m-2}$ as in exercise (5.4).

6.1.
Consider a torus $T \subseteq \mathbb{R}^3$ with the z-axes as rotation axis. Now glue $g \geq 2$ many handles to T such that the resulting surface F_{g+1} is invariant under rotation R around the z-axes by the angle $2\pi/g$. Let G be the cyclic group generated by R. Show that $F_{g+1}/G \cong F_2$ and hence $F_{g+1} \to F_2$ is a covering map.

6.2.
Consider the covering $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Let $Y := S^1 \vee S^1 \subseteq S^1 \times S^1$ and $X := (p \times p)^{-1}(Y) = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ oder } y \in \mathbb{Z}\}$. Show that:

1. $(p \times p)|_Y : X \to Y$ is an infinite covering.
2. $\pi_1(X)$ is a free group with infinite many generators (Hint: [1, 5.46])
3. Show that the image of $\pi_1(X)$ in $\pi_1(Y)$ is the commutator subgroup of $\pi_1(Y) = \mathbb{Z} \coprod \mathbb{Z}$.
4. Note that this subgroup of the free group with 2 generators is a free group with infinite many generators.

6.3.
Let $Y \to X$ be a n-fold covering map with Y path-connected and X locally path-connected. Show that there is an automorphism which acts cyclically on the standard fibre if and only if the isotropy groups G_y are normal and G/G_y is a cyclic group.

6.4.
Determine the group $\mathrm{Aut}(p)$ of Deck-transformations and the conjugacy class of the covering map p from [1, 6.26].
Hint: Use a maximal tree in the total space Y.

6.5.
Show that the universal coverings of sufficiently connected homotopy equivalent spaces are homotopy equivalent.

6.6.
Show that the universal covering of $S^1 \cup_{g_n} D^2$ from [1, 1.93] is given by $D^2 \times \{0, \dots, n-1\}/\sim$, where $(x, i) \sim (x, j)$ for all $x \in S^1$ and all i, j.

6.7.
Determine the fundamental groups of the torus knots $t \mapsto (e^{2\pi i p t}, e^{2\pi i q t})$, $[0, 1] \to S^1 \times S^1$, with $p, q \geq 2$ relative prime integers.
Hint: Apply the theorem of Seifert and van Kampen to the complement of a neighborhood $S^1 \times \mathring{D}^2$ of the knot in S^3 using the decomposition of S^3 into two filled tori as in [1, 1.73].

Literatur

[1] A. Kriegl. *Algebraic Topology*. Vorlesung, Univ. Wien, 2017. 1, 2, 4, 5