Algebraic Topology

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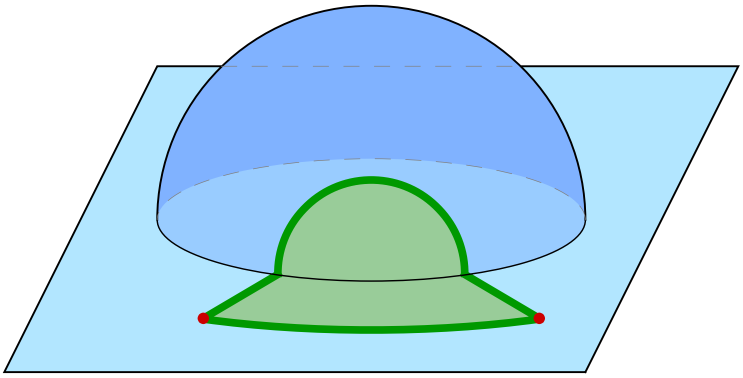
250155, WS 2017, Mo.-Do. $10^{00} - 10^{45}$, SR10,
Oskar-Morgenstern-Platz 1



These lecture notes are inspired to a large extend by the book

R.Stöcker/H.Zieschang: Algebraische Topologie, B.G.Teubner, Stuttgart 1988

which I recommend for many of the topics I could not treat in this lecture course, in particular this concerns the homology of products [**20**, chapter 12], homology with coefficients [**20**, chapter 10], cohomology [**20**, chapter 13–15].

As always, I am very thankful for any feedback in the range from simple typing errors up to mathematical incomprehensibilities.

Vienna, 2000.08.01 Andreas Kriegl

Since Simon Hochgerner pointed out, that I forgot to treat the case $q = n - 1 - r$ for $r < n - 1$ in theorem 8.47, I adopted the proof appropriately.

Vienna, 2000.09.25 Andreas Kriegl

I translated chapter 1 from German to English, converted the whole source from amstex to latex and made some stylistic changes for my lecture course in this summer semester.

Vienna, 2006.02.17 Andreas Kriegl

I am thankfull for the lists of corrections which has been provided by Martin Heuschober and by Stefan Fürdös.

Vienna, 2008.01.30 Andreas Kriegl

I added a chapter on cohomology and on homology with coeffients.

Vienna, 2015.01.27 Andreas Kriegl

Inhaltsverzeichnis

1. Building Blocks and Homeomorphy

In this first chapter we introduce the 'homeomorphy problem'. We will see that even for the best known topological building blocks like ball and spheres this is not easily decided and will be attacked with algebraic methods later on. We will also recall various quotient space constructions and important classes of topological spaces (like manifolds, orbit spaces) constructed from the building blocks.

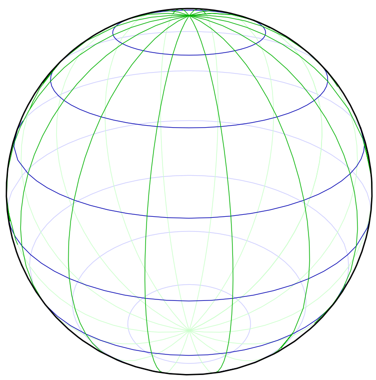
In this chapter I mainly listed the contents in form of short statements. For details please refer to the book.

Ball, sphere and cell

Problem of homeomorphy.
When is $X \cong Y$? Either we find a homeomorphism $f : X \to Y$, or a topological property, which hold for only one of X and Y, or we cannot decide this question.

1.1 Definition of basic building blocks. [20, 1.1.2]

1. $\mathbb{R}$ with the metric given by $d(x,y) := |x-y|$.
2. $I := [0,1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$, the unit interval.
3. $\mathbb{R}^n := \prod_n \mathbb{R} = \prod_{i\in n} \mathbb{R} = \prod_{i=0}^{n-1} \mathbb{R} = \{(x_i)_{i=0,\dots,n-1} : x_i \in \mathbb{R}\}$, with the product topology or, equivalently, with any of the equivalent metrics given by a norm on this vector space.
4. $I^n := \prod_n I = \{(x_i)_{i=0}^{n-1} : 0 \le x_i \le 1 \forall i\} = \{x \in \mathbb{R}^n : \|x - (\frac12,\dots,\frac12)\|_\infty \le \frac12\}$, the n-dimensional unit cube, where $\|x\|_\infty := \max\{|x_i| : i\}$.
5. For subsets $A \subseteq \mathbb{R}^n$ we denote with $\dot A = \partial_{\mathbb{R}^n} A$ the boundary of A in $\mathbb{R}^n$. In particular, $\dot I^n := \partial_{\mathbb{R}^n} I^n = \{(x_i)_i \in I^n : \exists i : x_i \in \{0,1\}\}$, the boundary of I^n in $\mathbb{R}^n$.
6. $D^n := \{x \in \mathbb{R}^n : \|x\|_2 := \sqrt{\sum_{i\in n}(x_i)^2} \le 1\}$, the n-dimensional closed unit ball (with respect to the Euclidean norm).
   A topological space X is called n-BALL iff $X \cong D^n$.
7. $\dot D^n := \partial_{\mathbb{R}^n} D^n = S^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$, the $n-1$-dimensional unit sphere.

   A topological space X is called n-SPHERE iff $X \cong S^n$.
8. $\mathring{D}^n := \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$, the interior of the n-dimensional unit ball.
   A topological space X is called n-CELL iff $X \cong \mathring{D}^n$.

1.2 Definition. [20, 1.1.3] An AFFINE HOMEOMORPHISMS is a mapping of the form $x \mapsto A \cdot x + b$ with an invertible linear A and a fixed vector b.

Hence the ball in $\mathbb{R}^n$ with center b and radius r is homeomorphic to D^n and thus is an n-ball.

1.3 Example. [20, 1.1.4] $\mathring{D}^1 \cong \mathbb{R}$: Use the odd functions $t \mapsto \tan(\frac{\pi}{2}t)$, or $t \mapsto \frac{t}{1-t^2}$ with derivative $t \mapsto \frac{t^2+1}{(t^2-1)^2} > 0$, or $t \mapsto \frac{t}{1-|t|}$ with derivative $t \mapsto \frac{1}{1-|t|} > 0$ and inverse mapping $s \mapsto \frac{t}{1+|t|}$. Note, that any bijective function $f : [0,1) \to [0,+\infty)$ with $f(0) = 0$ extends to an odd function $\tilde{f} : (-1,1) \to \mathbb{R}$ by setting $\tilde{f}(t) := -f(-t)$ for $t < 0$. For $f(t) := \frac{t}{1-t}$ we have $\tilde{f}(t) = -\frac{-t}{1-(-t)} = \frac{t}{1-|t|}$ and for $f(t) := \frac{t}{1-t^2}$ we have $\tilde{f}(t) = -\frac{-t}{1-(-t)^2} = \frac{t}{1-t^2}$. Note that in both cases $f'(0) = \lim_{t\to 0+} f'(t)$ exists and hence $\tilde{f}$ is a C^1 diffeomorphism. However, in the first case $\lim_{t\to 0+} f''(t) = 2$ and hence the odd function $\tilde{f}$ is not C^2.

1.4 Example. [20, 1.1.5] $\mathring{D}^n \cong \mathbb{R}^n$: Use for example $f : x \mapsto \frac{x}{1-\|x\|} = \frac{x}{\|x\|} \cdot f_1(\|x\|)$ with $f_1(t) := \frac{t}{1-t}$ and directional derivative $f'(x)(v) = \frac{1}{1-\|x\|}\, v + \frac{\langle x|v\rangle}{(1-\|x\|)^2\|x\|}\, x \to v$ for $x \to 0$.

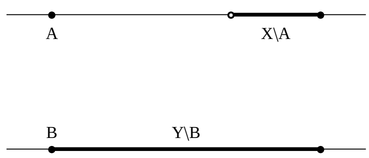
1.5 Corollary. [20, 1.1.6] *$\mathbb{R}^n$ is a cell; products of cells are cells, since $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ by “associativity” of the product.*

1.6 Definition. A PAIR (X, A) of spaces is a topological space X together with a subspace $A \subseteq X$.
A MAPPING $f : (X, A) \to (Y, B)$ of pairs is a continuous mapping $f : X \to Y$ with $f(A) \subseteq B$.
A HOMEOMORPHISM $f : (X, A) \to (Y, B)$ of pairs is a mapping of pairs which is a homeomorphism $f : X \to Y$ and satisfies $f(A) = B$ (and hence induces a homeomorphism $f|_A : A \to B$).

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, hook] \arrow[r, "f|_A"] & B \arrow[u, hook] \end{tikzcd}$$

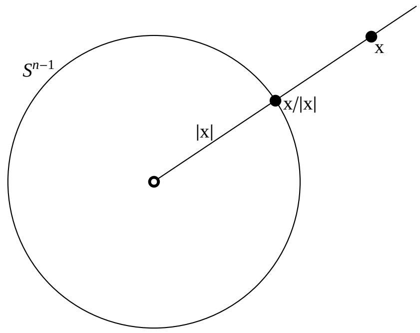
1.7 Definition. [20, 1.3.2] A mapping $f : (X, A) \to (Y, B)$ of pairs is called RELATIVE HOMEOMORPHISM, iff $f : X\setminus A \to Y\setminus B$ is a well-defined homeomorphism.

A homeomorphism of pairs is obviously a relative homeomorphism, but not conversely even if $f|_A : A \to B$ is a homeomorphism: Consider for example $A := \{-1\}$, $X := A \cup (1, 2]$, and $f : t \mapsto t^2 - 2$.



However, for compact X and Y any homeomorphism $f : X\setminus\{x_0\} \to Y\setminus\{y_0\}$ extends to a homeomorphism $\tilde{f} : (X, \{x_0\}) \to (Y, \{y_0\})$ of pairs, since $X \cong (X \setminus \{x_0\})_\infty$, cf. 1.35. Note that Z_∞ denotes the 1-point compactification of the locally compact space Z, see [6, 2.2.5].

1.8 Example. [20, 1.1.15]

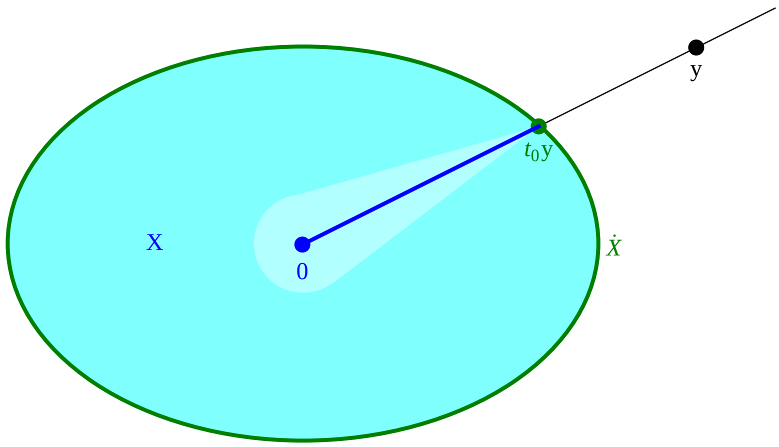


1. $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times (0, +\infty) \cong S^{n-1} \times \mathbb{R}$, via $x \mapsto (\frac{1}{\|x\|}x, \ln(\|x\|))$, $e^t\, y \leftarrow\!\shortmid (y,t)$.
2. $D^n \setminus \{0\} \cong S^{n-1} \times (0,1] \cong S^{n-1} \times (\varepsilon, 1]$, via $(0,1] \cong (\varepsilon, 1]$ and (1).

1.9 Definition. A subset $A \subseteq \mathbb{R}^n$ ist called convex, iff $x + t(y - x) \in A$ for $\forall x, y \in A$, $t \in [0,1]$.

1.10 Theorem. [20, 1.1.8] *$X \subseteq \mathbb{R}^n$ compact, convex, $\mathring{X} \neq \emptyset \Rightarrow (X, \dot{X}) \cong (D^n, S^{n-1})$.*
In particular, X is a ball, $\dot{X}$ is a sphere and $\mathring{X}$ is a cell.
If $X \subseteq \mathbb{R}^n$ is (bounded,) open and convex and not empty, then X is a cell.

Proof. W.l.o.g. let $0 \in \mathring{X}$ (translate X by $-x_0$ with $x_0 \in \mathring{X}$). The mapping $f : \dot{X} \ni x \mapsto \frac{1}{\|x\|}x \in S^{n-1}$ is bijective, since it keeps rays from 0 invariant and since for $y \neq 0$ let $t_0 := \max\{t > 0 : t\,y \in X\}$, then $t\,y \notin X$ for all $t > t_0$ and $t\,y \in \mathring{X}$ for all $0 \leq t < t_0$ (consider the cone with an open 0-neighborhood in X as basis and $t_0\,y$ as apex), hence t_0 is the unique $t > 0$ with $t\,y \in \dot{X}$.



Since $\dot{X}$ is compact f is a homeomorphism. By radial extension we obtain (using 1.8.2) a continuous bijection

$$D^n \setminus \{0\} \cong S^{n-1} \times (0,1] \overset{f \times \mathrm{id}}{\cong} \dot{X} \times (0,1] \to X \setminus \{0\},$$
$$x \mapsto \left(\frac{x}{\|x\|}, \|x\|\right) \mapsto \left(f^{-1}\left(\frac{x}{\|x\|}\right), \|x\|\right) \mapsto \|x\|\, f^{-1}\left(\frac{x}{\|x\|}\right)$$

which extends via $0 \mapsto 0$ to a continuous bijection of the 1-point compactifications and hence a homeomorphism of pairs $(D^n, S^{n-1}) \to (X, \dot{X})$.

The second part follows by applying the first part to $\bar{X}$, a compact convex set with interior X: In order to see this take a point x in the interior of $\bar{X}$. So there exists a open neighborhood of x in $\bar{X}$ and we may assume that this is of the form of an n-simplex (see 3.2) (i.e. a hypertetraeder). Since its vertices are in $\bar{X}$ we can approximate them by points in X and hence x lies inside this approximating simplex contained in X.

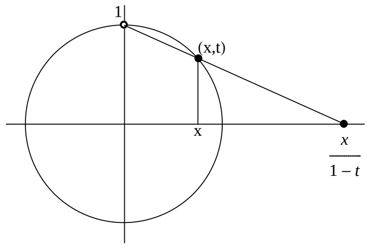
That the boundedness condition can be dropped can be found for a much more general situation in [11, 16.21]. $\square$

1.11 Corollary. [20, 1.1.9] *I^n is an n-ball and $\dot{I}^n$ is an $n-1$-sphere.* $\square$

1.12 Example. [20, 1.1.10] [20, 1.1.11] $D^p \times D^q$ is a ball, hence products of balls are balls, and $\partial(D^p \times D^q) = S^{p-1} \times D^q \cup D^p \times S^{q-1}$ is a sphere:
$D^p \times D^q$ is compact convex, and by exercise (1.1.1A) $\partial(A \times B) = \partial A \times B \cup A \times \partial B$. So by $\boxed{1.10}$ the result follows.

1.13 Remark. [20, 1.1.12] $\boxed{1.10}$ is wrong without convexity or compactness assumption: For compactness this is obvious since D^n is compact. That, for example, a compact annulus is not a ball will follow from $\boxed{2.17}$.

1.14 Example. [20, 1.1.13] $S^n = D^n_+ \cup D^n_-$, $D^n_+ \cap D^n_- = S^{n-1} \times \{0\} \cong S^{n-1}$, where $D^n_\pm := \{(x;t) \in S^n \subseteq \mathbb{R}^n \times \mathbb{R} : \pm t \geq 0\} \cong D^n$ are the southern and northern hemispheres. The stereographic projection $S^n \setminus \{(0,\dots,0;1)\} \cong \mathbb{R}^n$ is given by $(x;t) \mapsto \frac{1}{1-t}x$.



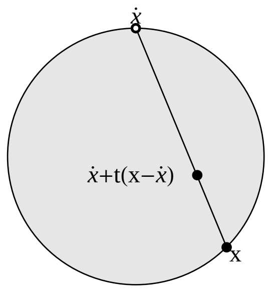
1.15 Corollary. [20, 1.1.14] *$S^n \setminus \{*\}$ is a cell.* $\square$

1.16 Example. [20, 1.1.15.3]
For all $\dot{x} \in S^{n-1}$:

$$D^n \setminus \{\dot{x}\} \cong \mathbb{R}^{n-1} \times [0,+\infty),$$

via

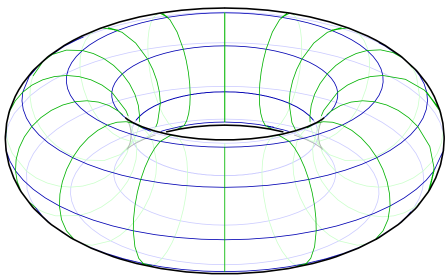
$$\mathbb{R}^{n-1} \times [0,+\infty) \cong (S^{n-1} \setminus \{\dot{x}\}) \times (0,1] \cong D^n \setminus \{\dot{x}\},$$
$$(x,t) \mapsto \dot{x} + t(x - \dot{x}).$$



1.17 Example. [20, 1.1.20] $S^n \not\cong \mathbb{R}^n$ and $D^n \not\cong \mathbb{R}^n$, since $\mathbb{R}^n$ is not compact.

None-homeomorphy of $X = S^1$ with I follows by counting components of $X \setminus \{*\}$.

1.18 Example. [20, 1.1.21] $S^1 \times S^1$ is called torus. It is embeddable into $\mathbb{R}^3$ by $(x,y) = (x_1,x_2;y_1,y_2) \mapsto ((R + r\,y_1)x, r\,y_2)$ with $0 < r < R$. This image is described by the equation $\{(x,y,z) : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\}$. Furthermore, $S^1 \times S^1 \not\cong S^2$ by Jordan's curve theorem, since $(S^1 \times S^1) \setminus (S^1 \times \{1\})$ is connected.



1.19 Theorem (Invariance of a domain). [20, 1.1.16] $\mathbb{R}^n \supseteq X \cong Y \subseteq \mathbb{R}^n$, *$X$ open in $\mathbb{R}^n$ $\Rightarrow Y$ open in $\mathbb{R}^n$.*

We will prove this hard theorem after 8.49.

1.20 Theorem (Invariance of dimension). [20, 1.1.17] $m \neq n \Rightarrow \mathbb{R}^m \not\cong \mathbb{R}^n$, $S^m \not\cong S^n$, $D^m \not\cong D^n$.

Proof. Let $m < n$.

Suppose $\mathbb{R}^n \cong \mathbb{R}^m$, then $\mathbb{R}^n \subseteq \mathbb{R}^n$ is open, but the image $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ is not, a contradiction to 1.19.

$S^m \cong S^n \Rightarrow \mathbb{R}^m \cong S^m \setminus \{x\} \cong S^n \setminus \{y\} \cong \mathbb{R}^n \Rightarrow m = n$.

$f : D^m \cong D^n \Rightarrow \mathring{D}^n \cong f^{-1}(\mathring{D}^n) \subseteq D^m \subseteq \mathbb{R}^m \subset \mathbb{R}^n$ and $f^{-1}(\mathring{D}^n)$ is not open, a contradiction to 1.19. □

1.21 Theorem (Invariance of the boundary). [20, 1.1.18] *$f : D^n \to D^n$ homeomorphism $\Rightarrow f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ homeomorphism of pairs.*

Proof. Let $\dot{x} \in \dot{D}^n$ with $y = f(\dot{x}) \notin \dot{D}^n$. Then $y \in \mathring{D}^n =: U$ and $f^{-1}(U)$ is homeomorphic to U but not open in $\mathbb{R}^n$, since $x \in f^{-1}(U) \cap \dot{D}^n$, a contradiction to 1.19. □

1.22 Definition. [20, 1.1.19] Let X be an n-ball and $f : D^n \to X$ a homeomorphism. The BOUNDARY $\dot{X}$ of X ist defined as the image $f(\dot{D}^n)$. This definition makes sense by 1.21.

Quotient spaces

1.23 Definition. Quotient space. [20, 1.2.1] Cf. [6, 1.2.12]. Let $\sim$ be an equivalence relation on a topological space X. We denote the set of EQUIVALENCE CLASSES $[x]_\sim := \{y \in X : y \sim x\}$ by $X/\sim$. The QUOTIENT TOPOLOGY on $X/\sim$ is the final topology with respect to the mapping $\pi : X \to X/\sim$, $x \mapsto [x]_\sim$ (i.e. the finest topology for which this mapping is continuous, see [6, 1.2.11]).

1.24 Proposition. [20, 1.2.2] *A subset $B \subseteq X/\sim$ is open/closed iff $\pi^{-1}(B)$ is open/closed. The quotient mapping π is continuous and surjective. It is open/closed iff for every open/closed $A \subseteq X$ the saturated hull $\pi^{-1}(\pi(A)) = \{x \in X : \exists a \in A : x \sim a\}$ is open/closed.*

For a proof see [6, 1.2.12].

The image of the closed subset $\{(x, y) : x \cdot y = 1, x, y > 0\} \subseteq \mathbb{R}^2$ under the first projection $\mathrm{pr}_1 : \mathbb{R}^2 \to \mathbb{R}$ is not closed!

1.25 Definition. [20, 1.2.9] A mapping $f : X \to Y$ is called QUOTIENT MAPPING (or final), iff f is surjective and satisfies one of the following equivalent conditions:

1. The induced mapping $X/\sim \to Y$ is a homeomorphism,
   where $x_1 \sim x_2 :\Leftrightarrow f(x_1) = f(x_2)$.
2. A subset $B \subseteq Y$ is open/closed iff $f^{-1}(B)$ is it.
3. A mapping $g : Y \to Z$ is continuous iff $g \circ f$ is it.

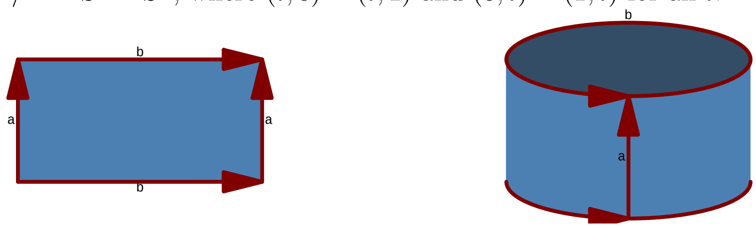
Note that f induces a bijection $\tilde f : X/\sim \to Y$.
(1⇒2) since $\pi : X \to X/\sim$ has this property.
(2⇒3) $g^{-1}(W)$ open $\Leftrightarrow (g \circ f)^{-1}(W) = f^{-1}(g^{-1}W)$ is open.
(3⇒1) $f : X \to Y$ is continuous by (3) for $g := \mathrm{id}_Y$ and $\tilde f : X/\sim \to Y$ is continuous by (1⇒3) for $Y := X/\sim$ and $g := f$.
Conversely, $\tilde f^{-1} : Y \to X/\sim$ is continuous by (3) for $Z := X/\sim$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \arrow[d, "g"] \\ X/\sim \arrow[ur, "\tilde f"] & Z \end{tikzcd}$$

1.26 Examples. [20, 1.2.3]

1. $I/\sim \;\cong S^1$, where $0 \sim 1$: The mapping $t \mapsto e^{2\pi i t}$, $I \to S^1$ factors to a homeomorphism $I/\sim \;\to S^1$, cf. 1.35.
2. $I^2/\sim \;\cong S^1 \times I$, where $(0,t) \sim (1,t)$ for all t.

3. $I^2/\sim \;\cong S^1 \times S^1$, where $(t,0) \sim (t,1)$ and $(0,t) \sim (1,t)$ for all t.



1.27 Proposition. [20, 1.2.10] *Continuous surjective closed/open mappings are obviously quotient-mappings, but not conversely. Continuous surjective mappings from a compact to a T_2-space are quotient-mappings, since the image of each closed subset is compact hence closed.*

1. *f_1, f_2 quotient mappings $\Rightarrow f_1 \circ f_2$ quotient mapping.*
2. *$f_1 \circ f_2$ quotient mapping $\Rightarrow f_1$ quotient mapping.*

Proof. Apply 1.25.3. □

1.28 Proposition. Universal property of $X/\sim$.
[20, 1.2.11] [20, 1.2.6] [20, 1.2.5]
Let $f : X \to Y$ be continuous. Then f is compatible with the equivalence relation (i.e. $x \sim x' \Rightarrow f(x) = f(x')$) iff it factors to a mapping $X/\sim \;\to Y$ over $\pi : X \to X/\sim$. Note that f is compatible with the equivalence relation iff the relation $f \circ \pi^{-1}$ is a mapping. The factorization $X/\sim \;\to Y$ is then given by $f \circ \pi^{-1}$ and is continuous.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, dotted, "!", "f\circ\pi^{-1}"'] & \end{tikzcd}$$

Proof. $(z,y) \in f \circ \pi^{-1} \Leftrightarrow \exists x \in X : f(x) = y, \pi(x) = z$. Thus $f \circ \pi^{-1}$ is a mapping, i.e. y is uniquely determined by z iff $\pi(x) = \pi(x') \Rightarrow f(x) = f(x')$. Continuity of $f \circ \pi^{-1}$ follows from 1.25.3. □

1.29 Proposition. [20, 1.2.4]

Functoriality of formation of quotients. Let $f : X \to Y$ be continuous and compatible with equivalence relations $\sim_X$ on X and $\sim_Y$ on Y, i.e. $x_1 \sim_X x_2 \Rightarrow f(x_1) \sim_Y f(x_2)$. Then there is a unique induced continuous mapping $\tilde f : X/{\sim_X} \to Y/{\sim_Y}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X/{\sim_X} \arrow[r, dotted, "!", "\tilde f"'] & Y/{\sim_Y} \end{tikzcd}$$

If f and f^{-1} are compatible with the equivalence relations and is a homeomorphism, then $\tilde f$ is a homeomorphism.

For a proof see [6, 1.2.11,1.2.12].

1.30 Proposition. [20, 1.2.7] [20, 1.2.12]

The restriction of a quotient-mapping to a closed/open saturated set is a quotient-mapping, i.e. let $f : X \to Y$ be a quotient mapping, $B \subseteq Y$ open (or closed), and $A := f^{-1}(B)$. Then $f|_A : A \to B$ is a quotient mapping.

$$\begin{tikzcd} X \arrow[r, two heads, "f"] & Y \\ f^{-1}(B) \arrow[u, hook] \arrow[r, two heads, "f|_{f^{-1}(B)}"] & B \arrow[u, hook] \end{tikzcd}$$

For example, the restriction of $\pi : I \to I/\dot I$ to the open set $[0, 1)$ is not a quotient mapping.

Proof. Let $U \subseteq B$ with $(f|_A)^{-1}(U)$ open. Then $f^{-1}(U) = (f|_A)^{-1}(U)$ is open and hence $U \subseteq Y$ is open. □

1.31 Corollary. [20, 1.2.8]
*Let $p : X \to Y$ quotient-mapping, $A \subseteq X$ closed/open, $\forall a \in A, x \in X : p(x) = p(a) \Rightarrow x = a$.
Then $p|_A : A \to p(A) \subseteq Y$ is an embedding.*

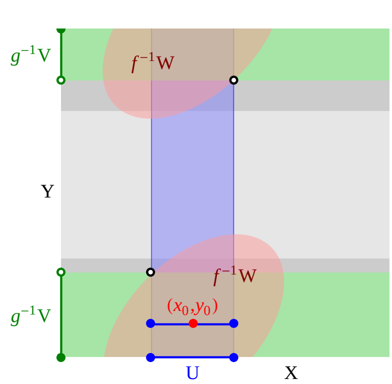
$$\begin{tikzcd} X \arrow[r, two heads, "p"] & Y \\ A \arrow[u, hook] \arrow[r, "\cong", "p|_A"'] & p(A) \arrow[u, hook] \end{tikzcd}$$

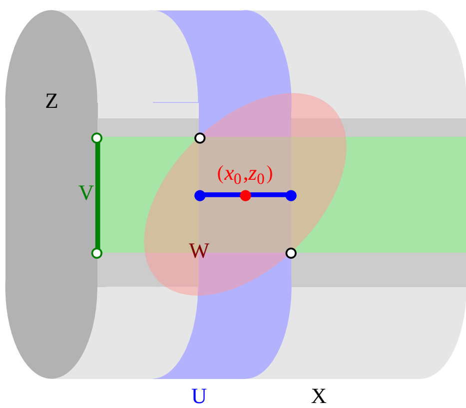
Proof. $\Rightarrow A = p^{-1}(p(A)) \xRightarrow{\boxed{1.30}} p|_A : A \to B := p(A)$ is a quotient mapping and it is injective by assumption, hence a homeomorphism. □

1.32 Proposition. Theorem of Whitehead. [20, 1.2.13] *Let g be a quotient mapping and X be locally compact. Then $X \times g := \mathrm{id}_X \times g$ is a quotient mapping.*

For a counter-example with not locally compact X see [6, 1.2.12]:

Proof.





Let $(x_0, z_0) \in W \subseteq X \times Z$ with open $f^{-1}(W) \subseteq X \times Y$, where $f := X \times g$ for $g : Y \to Z$. We choose $y_0 \in g^{-1}(z_0)$ and a compact neighborhood U of x_0 with

$U \times \{y_0\} \subseteq f^{-1}(W)$. Since $f^{-1}(W)$ is saturated, $U \times g^{-1}(g(y)) \subseteq f^{-1}(W)$ provided $U \times \{y\} \subseteq f^{-1}(W)$. In particular, $U \times g^{-1}(z_0) \subseteq f^{-1}(W)$.
Let $V := \{z \in Z : U \times g^{-1}(z) \subseteq f^{-1}(W)\}$. Then $(x_0, z_0) \in U \times V \subseteq W$ and V is open, since $g^{-1}(V) := \{y \in Y : U \times \{y\} \subseteq f^{-1}(W)\}$ is open (see [6, 2.1.11]). □

1.33 Corollary. [20, 1.2.14] *$f : X \to X'$, $g : Y \to Y'$ quotient mappings, X, Y' locally compact $\Rightarrow f \times g$ quotient mapping.*

Proof.

$$\begin{tikzcd} X \times Y \arrow[r, "f\times Y"] \arrow[d, "X\times g"'] \arrow[dr, "f\times g"] & X' \times Y \arrow[d, "X'\times g"] \\ X \times Y' \arrow[r, "f\times Y'"] & X' \times Y' \end{tikzcd}$$

□

Special cases of quotient mappings

1.34 Proposition. Collapse of a subspace. [20, 1.3.1] [20, 1.3.3]
$A \subseteq X$ closed $\Rightarrow \pi : (X, A) \to (X/A, \{A\})$ is a relative homeomorphism, where $X/A := X/\sim$ with the equivalence relation generated by $\forall a, a' \in A : a \sim a'$.
The functorial property for mappings of pairs is:

$$\begin{tikzcd} (X, A) \arrow[r, "f"] \arrow[d] & (Y, B) \arrow[d] \\ (X/A, A/A) \arrow[r, dotted, "!"] & (Y/B, B/B) \end{tikzcd}$$

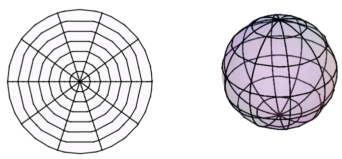
Note that the equivalence class A is a point in X/A, hence $\{A\}$ is a subset of X/A.

Proof. That $\pi : X \setminus A \to X/A \setminus A/A$ is a homeomorphism follows from $\boxed{1.31}$. The functorial property follows from $\boxed{1.28}$. □

1.35 Example. [20, 1.3.4] $X/\emptyset \cong X$ and $X/\{*\} \cong X$. Furthermore, $I/\dot{I} \cong S^1$ (by $\boxed{1.26.1}$) and, more generally, $X/A \cong (X \setminus A)_\infty$, provided X is compact and $A \subseteq X$ is closed: In fact, X/A is compact, $X \setminus A$ is openly embedded into X/A by $\boxed{1.34}$ and $X/A \setminus (X \setminus A)$ is the single point $A \in X/A$. Now use exercise (1.4).

1.36 Example. [20, 1.3.5] $D^n \setminus S^{n-1} = \mathring{D}^n \cong \mathbb{R}^n$ and hence by $\boxed{1.35}$ $D^n/S^{n-1} \cong (D^n \setminus S^{n-1})_\infty \cong (\mathbb{R}^n)_\infty \cong S^n$. Or, explicitly,

$$x \mapsto \Big(t := (1-\|x\|)\pi, \frac{x}{\|x\|}\Big) \mapsto \Big(\sin(t)\frac{x}{\|x\|}, \cos(t)\Big).$$



1.37 Example. [20, 1.3.6] $X \times I$ is called cylinder over X and $CX := (X \times I)/(X \times \{0\})$ is called the cone with base X. $C(S^n) \cong D^{n+1}$, via $(x, t) \mapsto t\,x$.

1.38 Example. [20, 1.3.7] Let (X_j, x_j) be pointed spaces, i.e. X_j a topological space and $x_j \in X_j$ a point in X_j, or, with other words, $(X_j, \{x_j\})$ is a pair of spaces. The 1-point union is

$$\bigvee_{j\in J} X_j = \bigvee_{j\in J} (X_j, x_j) := \Big(\bigsqcup_j X_j\Big)/\{x_j : j\}.$$

By 1.24 the projection $\pi : \bigsqcup_j X_j \to \bigvee_j X_j$ is a closed mapping for T_1-spaces X_j.

1.39 Proposition. [20, 1.3.8] *X_i embeds into $\bigvee_j X_j$ and $\bigvee_j X_j$ is union of the images, which have pairwise as intersection the base point.*

Proof. That the composition $X_i \hookrightarrow \bigsqcup_j X_j \to \bigvee_j X_j$ is continuous and injective is clear. That it is an embedding follows, since by 1.38 the projection π is also a closed mapping. ☐

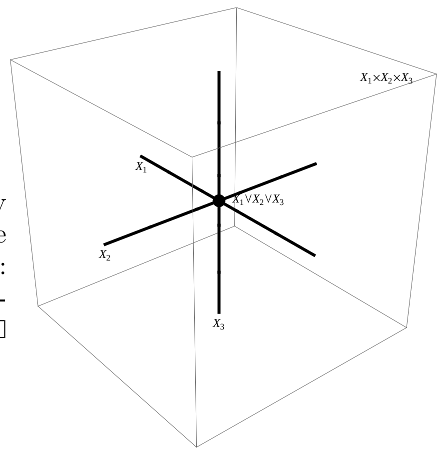
1.40 Proposition. [20, 1.3.9] *Universal and functorial property of the 1-point-union:*

$$\begin{tikzcd}
(X_i, x_i) \arrow[r, "f_i"] \arrow[d] & (Y, y) \\
\bigvee_j X_j \arrow[ur, dotted, "!"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X_i, x_i) \arrow[r, "f_i"] \arrow[d] & (Y_i, y_i) \arrow[d] \\
\bigvee_j X_j \arrow[r, dotted, "!"] & \bigvee_j Y_j
\end{tikzcd}$$

Proof. This follows from 1.28 and 1.29. ☐

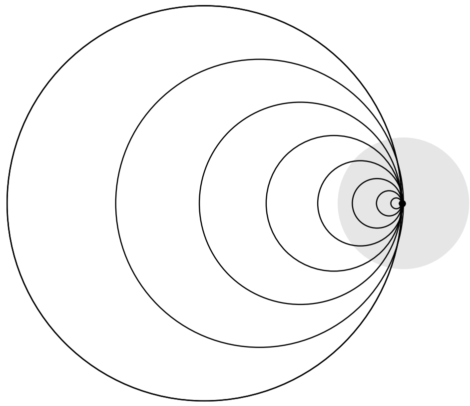
1.41 Proposition. [20, 1.3.10]
Embedding $X_1 \vee \dots \vee X_n \hookrightarrow X_1 \times \ldots \times X_n$.

Proof. Let $i_j : X_j \to \prod_{k=1}^n X_k$ be given by $z \mapsto (x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n)$, where the x_k are the base-points of X_k. Then $\bigsqcup_k i_k : \bigsqcup_k X_k \to \prod_k X_k$ factors to the claimed embedding, see exercise (1.7). ☐

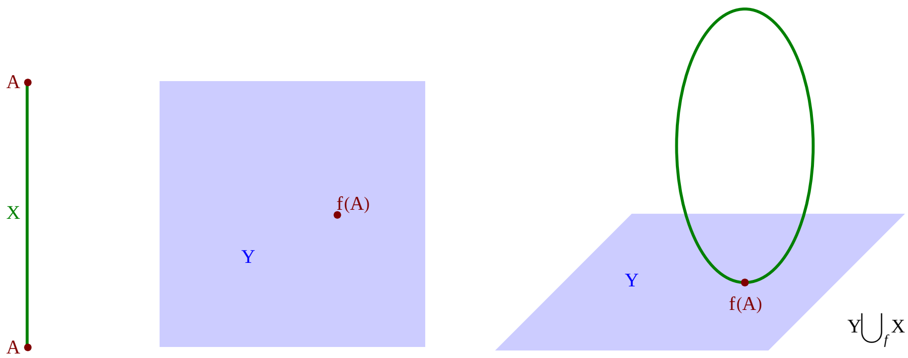


1.42 Example. [20, 1.3.11] 1.41 is wrong for infinite index sets: The open neighborhoods of the base point in $\bigvee_j X_j$ are given by $\bigvee_j U_j$, where U_j is an open neighborhood of the base point in X_j. Hence $\bigvee X_j$ is in general not first countable, whereas the product of countable many metrizable spaces X_j is first countable.

A visualization of the image of $\bigvee_{j\in\mathbb{N}} S^1$ in $\prod_{j\in\mathbb{N}} S^1$ is given by the union of countable many circles in $\mathbb{R}^2$ which intersect only in a single point. This is not their one-point union, since a neighborhood of the single point contains almost all circles completely.



1.43 Definition. Gluing. [20, 1.3.12] $f : X \supseteq A \to Y$ with $A \subseteq X$ closed. $Y \cup_f X := Y \sqcup X/\sim$, where $a \sim f(a)$ for all $a \in A$, is called Y glued with X via f (or X glued to Y along f).



1.44 Proposition. [20, 1.3.13] [20, 1.3.14] *Let $f : X \supseteq A \to Y$ with $A \subseteq X$ closed and $\pi : Y \sqcup X \twoheadrightarrow Y \cup_f X$ be the quotient mapping.*
Then $\pi|_Y : Y \to Y \cup_f X$ is a closed embedding and $\pi|_X : (X, A) \to (Y \cup_f X, \pi(Y))$ is a relative homeomorphism.

$$Y \cup_f X = \underbrace{(\pi(Y) \setminus f(A)) \cup f(A)}_{\cong Y} \cup \underbrace{\pi(X \setminus A)}_{\cong X \setminus A}$$

Proof. That $\pi|_Y : Y \to Y \cup_f X$ is continuous and injective is clear. Now let $B \subseteq Y$ be closed. Then $\pi^{-1}(\pi(B)) = B \sqcup f^{-1}(B)$ is closed and hence also $\pi(B)$.

That $\pi : X \setminus A \to (Y \cup_f X) \setminus \pi(Y)$ is a homeomorphism follows from 1.31. □

1.45 Proposition. [20, 1.3.15] *Universal property of push-outs $Y \cup_f X$:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & Y \cup_f X \arrow[dr, dotted, "!"] & \\
 & & Z
\end{tikzcd}$$

Proof. 1.28. □

1.46 Lemma. *Let $f_i : X_i \supseteq A_i \to Y$ be given, $X := X_1 \sqcup X_2$, $A := A_1 \sqcup A_2 \subseteq X$ and $f := f_1 \sqcup f_2 : X \supseteq A \to Y$. Then $Y \cup_f X \cong (Y \cup_{f_1} X_1) \cup_{f_2} X_2$.*

Proof.

$$\begin{tikzcd}
A_2 \arrow[dr, hook] \arrow[drrrr, "f_2"] \arrow[ddddd, hook] & & & & \\
& A \arrow[rrr, "f"] \arrow[ddd, dotted, "(0)"] & & & Y \arrow[ddl, "i_1"'] \arrow[ddd, hook, "i"] \\
& & A_1 \arrow[ul, hook] \arrow[urr, "f_1"] \arrow[d, hook] & & \\
& & X_1 \arrow[r, "p_1"] \arrow[dl, hook] & Y \cup_{f_1} X_1 \arrow[dr, dotted, "(1)"] \arrow[dd, hook, "i_2"] & \\
& X \arrow[rrr, "p"] \arrow[drr, dotted, "(0)"] & & & Y \cup_f X \arrow[dl, dotted, bend left, "(2)"] \\
X_2 \arrow[ur, hook] \arrow[rrr, "p_2"] & & & (Y \cup_{f_1} X_1) \cup_{f_2} X_2 \arrow[ur, dotted, "(2)"] &
\end{tikzcd}$$

1.47 Example. [[20], 1.3.16]

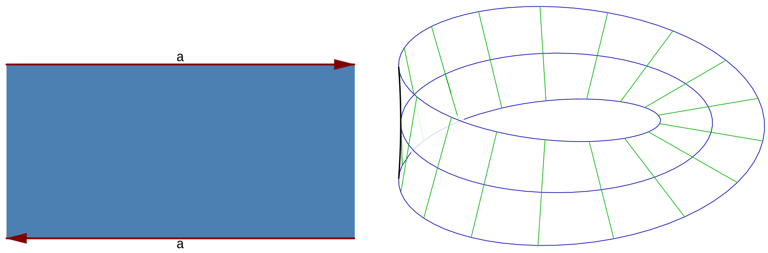
(1) $f : X \supseteq A \to Y := \{*\} \Rightarrow Y \cup_f X \cong X/A$, since X/A satisfies the universal property of the push-out.

(2) $f : X \supseteq \{*\} \to Y \Rightarrow Y \cup_f X \cong X \vee Y$, by definition.

(3) $f : X \supseteq A \to Y$ constant $\Rightarrow Y \cup_f X \cong X/A \vee Y$, since we can compose push-outs:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, hook] & Y_1 \arrow[r, hook, "f_2"] \arrow[d] & Y_2 \arrow[d, hook] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drrr, bend right] & Y_1 \cup_{f_1} X \arrow[r] \arrow[drr, dotted] & Y_2 \cup_{f_2} (Y_1 \cup_{f_1} X) \arrow[dr, dotted] & \\
& & & W
\end{tikzcd}$$

1.48 Example. [[20], 1.3.17] $f : X \supseteq A \to B \subseteq Y$ homeomorphism of closed subsets. $\Rightarrow Y \cup_f X = \pi(X) \cup \pi(Y)$ with $\pi(X) \cong X$, $\pi(Y) \cong Y$ and $\pi(X) \cap \pi(Y) \cong A \cong B$. This follows from 1.44 since $Y \cup_f X \cong X \cup_{f^{-1}} Y$.
Note however, that $Y \cup_f X$ depends not only on $X \supseteq A$ and $Y \supseteq B$ but also on the gluing map $f : A \to B$ as the example $X = I \times I = Y$ and $A = B = I \times \dot{I}$ with $\mathrm{id} \neq f : (x, 1) \mapsto (1 - x, 1)$, $(x, 0) \mapsto (x, 0)$ of a Möbius-strip versus a cylinder shows, see 1.58.



1.49 Proposition. [20, 1.3.18]

$$\begin{tikzcd} X \arrow[d, "\cong"', "F"] & A \arrow[l, hook'] \arrow[r, "f"] \arrow[d, "\cong"', "F|_A"] & Y \arrow[d, "G"', "\cong"] \\ X' & A' \arrow[l, hook'] \arrow[r, "f'"] & Y' \end{tikzcd} \qquad \Longrightarrow \qquad Y \cup_f X \cong Y' \cup_{f'} X'.$$

Proof.
By the push-out property 1.45 we obtain a uniquely determined continuous map $G \cup F : Y \cup_f X \to Y' \cup_{f'} X'$ with $(G \cup F) \circ \pi|_X = \pi'|_{X'} \circ F$ and $(G \cup F) \circ \pi|_Y = \pi'|_{Y'} \circ F$.
Since $G^{-1} \circ f' = G^{-1} \circ f' \circ F \circ F|_A^{-1} = G^{-1} \circ G \circ f \circ F|_A^{-1} = f \circ F^{-1}|_{A'}$ we get similarly $G^{-1} \cup F^{-1} : Y' \cup_{f'} X' \to Y \cup_f X$. On X and Y (resp. X' and Y') they are inverse to each other, hence define a homeomorphism as required. $\square$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "F|_A", "\cong"'] \arrow[dd, hook] & & Y \arrow[dr, "G", "\cong"'] \arrow[dd, hook] & \\ & A' \arrow[rr] \arrow[dd, hook] & & Y' \arrow[dd, hook] \\ X \arrow[rr] \arrow[dr, "F", "\cong"'] & & Y \cup_f X \arrow[dr, dotted, shift left, "G\cup F"] & \\ & X' \arrow[rr] & & Y' \cup_{f'} X' \arrow[ul, dotted, shift left, "G^{-1}\cup F^{-1}"] \end{tikzcd}$$

1.50 Example. [20, 1.3.19]

(1) $Z = X \cup Y$ with X, Y closed and $A := X \cap Y$. $\Rightarrow Z = Y \cup_{\mathrm{id}_A} X$: The canonical mapping $Y \sqcup X \to Z$ induces a continuous bijective mapping $Y \cup_{\mathrm{id}_A} X \to Z$, which is closed and hence a homeomorphism, since $Y \sqcup X \to Z$ is obviously closed.

(2) $Z = X \cup Y$ with X, Y closed, $A := X \cap Y$, and $f : A \to A$ extendable to a homeomorphism of the pair $(X, A) \Rightarrow Z \cong Y \cup_f X$: Apply 1.49 to

$$\begin{tikzcd} X \arrow[d, "\cong"', "\tilde f"] & A \arrow[l, hook'] \arrow[r, "f"] \arrow[d, "\cong"', "f"] & A \arrow[r, hook] \arrow[d, "\cong"', "\mathrm{id}"] & Y \arrow[d, "\cong"', "\mathrm{id}"] \\ X & A \arrow[l, hook'] \arrow[r, "\mathrm{id}"] & A \arrow[r, hook] & Y \end{tikzcd}$$

(3) $D^n \cup_f D^n \cong S^n$ for all homeomorphisms $f : S^{n-1} \to S^{n-1}$: We can extend f radially to a homeomorphism $\tilde f : D^n \to D^n$ by $\tilde f(x) = \|x\|\, f(\frac{x}{\|x\|})$ and can now apply (2).

(4) Gluing two identical cylinders $X \times I$ along any homeomorphism $f : X \times \{0\} \to X \times \{0\}$ yields again the cylinder $X \times I$: Since f extends to a homeomorphism $X \times I \to X \times I$, $(x, t) \mapsto (f(x), t)$ we may apply (2) to obtain $(X \times I) \cup_f (X \times I) \cong (X \times I) \cup_{\mathrm{id}} (X \times I) \cong X \times I$.

Manifolds

1.51 Definition. [20, 1.4.1] [20, 1.5.1] An m-dimensional MANIFOLD (or m-manifold for short) (possibly with boundary) is a topological space X (which we will always require to be Hausdorff and second countable), for which each of its points $x \in X$ has a neighborhood A which is an n-ball, i.e. a homeomorphism $\varphi : A \cong D^m$ (which we call CHART at x) exists. A point $x \in X$ is called BOUNDARY POINT iff for some (and by 1.21 any) chart φ at x the point is mapped to $\varphi(x) \in S^{m-1}$. The set of all boundary points is called the BOUNDARY of X and denoted by ∂X or $\dot{X}$. It is obviously a closed subset of X. A manifold is called CLOSED if it is compact and has empty boundary. Two-dimensional manifolds are called SURFACEs.

1.52 Examples. [20, 1.4.4] [20, 1.4.5]

1. 0-manifolds are discrete countable topological spaces.
2. The connected 1-manifolds are $\mathbb{R}$, S^1, I and $[0, +\infty)$.
3. Quadrics like hyperboloids ($\cong \mathbb{R}^2 \sqcup \mathbb{R}^2$ or $\cong S^1 \times \mathbb{R}$), paraboloids ($\cong \mathbb{R}^2$), and the cylinder $S^1 \times \mathbb{R}$ are surfaces.
4. Let X be a manifold (without boundary) and $A \subseteq X$ be a discrete subset. Then $X \setminus A$ is also a manifold (without boundary).
5. D^m is a manifold with boundary S^{m-1}, so $\mathring{D}^m \cong \mathbb{R}^m$ is a manifold without boundary.
   The halfspace $\mathbb{R}^{m-1} \times [0, +\infty)$ is a manifold with boundary $\mathbb{R}^{m-1} \times \{0\}$.

1.53 Lemma. *Let $U \subseteq X$ be open in an m-manifold X. Then U is an m-manifold with $\dot{U} = \dot{X} \cap U$.*

Proof. Let $x \in U$ and $\varphi : A \xrightarrow{\cong} D^m =: D$ be a chart at x for X. Then $\varphi(U)$ is an open neighborhood of $\varphi(x)$ in D and hence contains a convex compact neighborhood B which is an m-ball by 1.10. Consequently, $\varphi|_{\varphi^{-1}(B)} : U \supseteq \varphi^{-1}(B) \cong B \subseteq D$ is the required chart at x for U.
We have $x \in \dot{U} :\Leftrightarrow \varphi|_{\varphi^{-1}(B)}(x) \in \dot{B} \Leftrightarrow \varphi(x) \in \dot{D} \Leftrightarrow: x \in \dot{X}$, since $\varphi(x)$ is in the interior $\mathring{B}^D$ of B with respect to the topology of D, $\dot{B} \cap \mathring{B}^D \subseteq \dot{D}$ (since $\mathring{D} \cap \mathring{B}^D \subseteq \mathring{B}$), and $B \cap \dot{D} \subseteq \dot{B}$ (since $\mathring{B} \subseteq \mathring{D} \Rightarrow B \cap \dot{D} = B \cap \mathring{D}^c \subseteq B \cap \mathring{B}^c = \dot{B}$). □

1.54 Proposition. [20, 1.4.2] [20, 1.5.2]
Let $f : X \to Y$ be a homeomorphism between manifolds. Then $f(\dot{X}) = \dot{Y}$.

Proof. Let $x \in X$ and $\varphi : A \cong D^m$ be a chart at x. Then $\varphi \circ f^{-1} : f(A) \to D^m$ is a chart of Y at $f(x)$ and hence $x \in \dot{X} :\Leftrightarrow (\varphi \circ f^{-1})(f(x)) = \varphi(x) \in \dot{D}^m \Leftrightarrow: f(x) \in \dot{Y}$. □

1.55 Proposition. [20, 1.4.3] [20, 1.5.3]
Let X be an m-manifold and $x \in \dot{X}$. Then there exists a neighborhood U of x in X with $(U, U \cap \dot{X}, x) \cong (D^{m-1} \times I, D^{m-1} \times \{0\}, (0, 0))$, a homeomorphism of triples.

Proof. By assumption there exists a neighborhood A of x in X and a homeomorphism $\varphi : A \to D^m$ with $\varphi(x) \in S^{m-1}$. Choose an open neighborhood $W \subseteq A$ of x. Then $\dot{W} = \dot{X} \cap W$ and the manifold W is homeomorphic to $\varphi(W) \subseteq D^m$ by 1.53. Obviously $\varphi(W)$ contains a neighborhood B of $\varphi(x)$ homeomorphic to $D^{m-1} \times I$, where $B \cap S^{m-1}$ corresponds to $D^{m-1} \times \{0\}$, cf. 1.8.2. The set $U := \varphi^{-1}(B)$ is then the required neighborhood:

$$W \cap \dot{X} \overset{1.53}{=\!=} \dot{W} \overset{\varphi}{\cong} \varphi(W)^{\cdot} \overset{1.53}{=\!=} \varphi(W) \cap \dot{D}^m = \varphi(W) \cap S^{m-1}$$

$$U \cap \dot{X} = U \cap W \cap \dot{X} \overset{\varphi}{\cong} B \cap \varphi(W) \cap S^{m-1} \cong D^{m-1} \times \{0\} \quad \square$$

1.56 Corollary. [20, 1.5.4] The boundary $\dot{X}$ of a manifold is a manifold without boundary.

Proof. By 1.55 $\dot{X}$ is locally homeomorphic to $D^{m-1} \times \{0\} \cong D^{m-1}$ and $x \in \dot{X}$ corresponds to 0 thus is not in the boundary of $\dot{X}$. $\square$

1.57 Proposition. [20, 1.5.7] *Let M be an m-dimensional and N an n-dimensional manifold. Then $M \times N$ is an $m+n$-dimensional manifold with boundary $(M \times N)^{\cdot} = \dot{M} \times N \cup_{\mathrm{id}|_{\dot{M}\times\dot{N}}} M \times \dot{N}$. For a manifold X without boundary (like S^1) the cylinder $X \times I$ is a manifold with boundary $X \times \{0, 1\}$.*

This way we get examples of 3-manifolds: $S^2 \times \mathbb{R}$, $S^2 \times I$, and $S^2 \times S^1$.

Proof. 1.12 and 1.50.1. $\square$

1.58 Example. Möbius strip. [20, 1.4.6] The (compact) Möbius-strip X is defined as $I \times I/\sim$, where $(x, 0) \cong (1 - x, 1)$ for all x, cf. 1.48. Its boundary is $(I \times \dot{I})/\sim \cong S^1$ and hence X is not homeomorphic to the cylinder $S^1 \times I$ by 1.54.

An embedding of X into $\mathbb{R}^3$ is given by factoring

$$(\varphi, r) \mapsto \Big(\big(2+(2r-1)\cos\pi\varphi\big)\cos 2\pi\varphi, \big(2+(2r-1)\cos\pi\varphi\big)\sin 2\pi\varphi, (2r-1)\sin\pi\varphi\Big)$$

over the quotient.

The Möbius-strip is not orientable which we will make precise later.

1.59 Proposition. [20, 1.4.7] [20, 1.5.5] *By cutting finitely many disjoint open holes into a manifold one obtains a manifold whose boundary is the union of the boundary of X and the boundaries of the holes. More precisely, let X be an m-manifold and $f_i : D^m \to X$ embeddings with pairwise disjoint images. Let $\mathring{D}_i := \{f_i(x) : |x| < \frac{1}{2}\}$ and $S_i := \{f_i(x) : |x| = \frac{1}{2}\}$. Then $X \setminus \bigcup_{i=1}^n \mathring{D}_i$ is an m-manifold with boundary $\dot{X} \sqcup \bigsqcup_{i=1}^n S_i$.*

The manifold which results by cutting g open holes in the unit-disk D^2 will be denoted D^2_g.

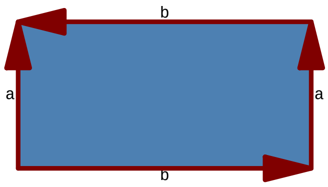
Proof. No point in $\{f_i(x) : |x| < 1\}$ is a boundary point of X, hence the result follows. $\square$

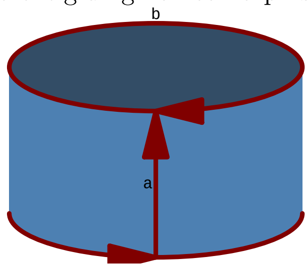
1.60 Proposition. [20, 1.4.8] [20, 1.5.6] *Let X and X' be two manifolds and R and R' components of the corresponding boundaries and $g : R \to R'$ a homeomorphism. Then $X' \cup_g X$ is a manifold in which X and X' are embedded as closed subsets and has boundary $(\dot{X} \setminus R) \cup (\dot{X}' \setminus R')$.*

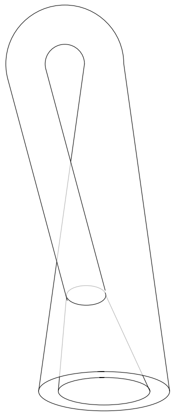
Proof. It is enough to find charts at points x from $R \cup R'$. Let $A \cong D^{m-1} \times I$ and $A' \cong D^{m-1} \times I$ be neighborhoods of $x \in R$ and $g(x) \in R'$ with $\dot{X} \cap A = D^{m-1} \times \{0\}$ and $\dot{X}' \cap A' = D^{m-1} \times \{0\}$ as in 1.55. W.l.o.g. we may assume that $g(\dot{X} \cap A) = \dot{X}' \cap A'$. The image of $A' \sqcup A$ in $X' \cup_g X$ is given by gluing

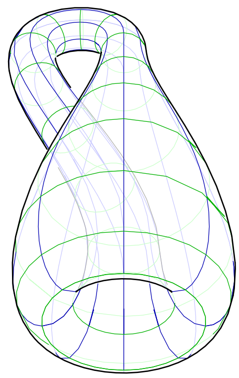
$D^{m-1} \times I \cup D^{m-1} \times I$ along a homeomorphism $D^{m-1} \times \{0\} \to D^{m-1} \times \{0\}$ and hence is by 1.50.3 homeomorphic to $D^{m-1} \times I$ where x corresponds to $(0,0)$. □

1.61 Example. [20, 1.4.9]
$S^1 \times S^1$ can be obtained from two copies of $S^1 \times I$ that way.
The same is true for Klein's bottle but with a different gluing homeomorphism:









1.62 Example. Gluing a handle. [20, 1.4.10] [20, 1.5.8.7] Let X be a surface in which we cut two holes as in 1.59. The surface obtained from X by gluing a handle is $(X \setminus (\mathring{D}^2 \sqcup \mathring{D}^2)) \cup_f (S^1 \times I)$, where $f : S^1 \times I \supseteq S^1 \times \dot{I} \cong S^1 \sqcup S^1 \subseteq D^2 \sqcup D^2$.

More generally, one can glue handles $S^{n-1} \times I$ to n-manifolds.

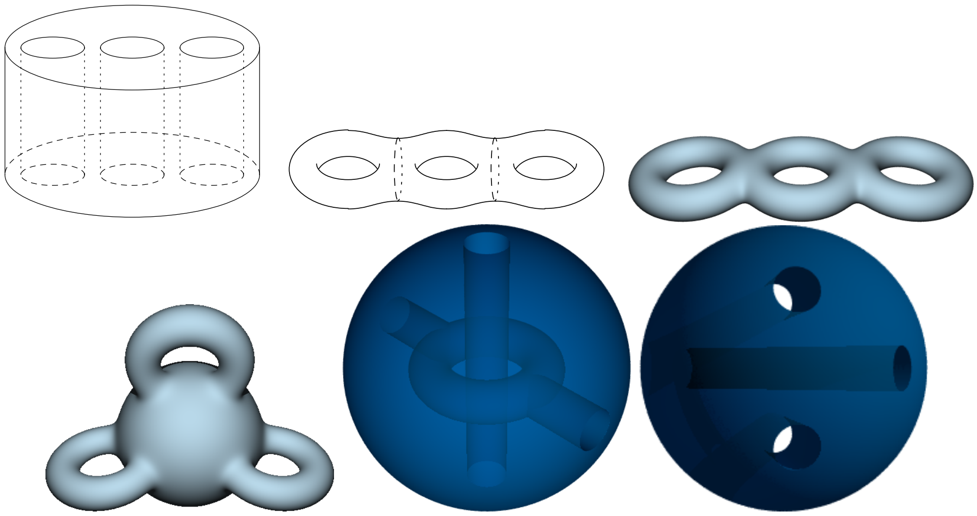
1.63 Example. Connected sum. [20, 1.4.11] [20, 1.5.8.8] The connected sum of two surfaces X_1 and X_2 is given by cutting a whole into each of them and gluing along boundaries of the respective holes. $X_1 \sharp X_2 := (X_1 \setminus \mathring{D}^2) \cup_f (X_2 \setminus \mathring{D}^2)$, where $f : D^2 \supseteq S^1 \cong S^1 \subseteq D^2$.

More generally, one can define analogously the connected sum of n-manifolds. This however depends essentially on the gluing map.

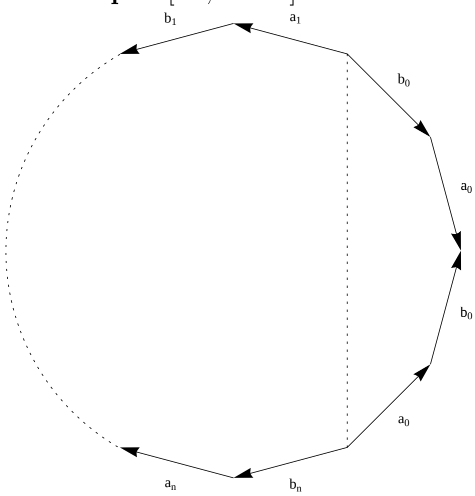
1.64 Example. Doubling of a manifold with boundary. [20, 1.4.12] [20, 1.5.8.9] The doubling of a manifold is given by gluing two copies along their boundaries with the identity: $2X := X \cup_f X$, where $f := \mathrm{id} : \dot{X} \to \dot{X}$.

1.65 Example. [20, 1.4.13] The connected compact oriented surfaces F_g (of genus g) without boundary can be described as:

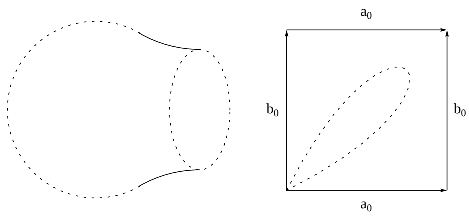
1. boundary $\dot{V}_g$ of a handlebody (pretzel, Brezel) $V_g := D^2_g \times I$ of genus g.
2. doubling $2D^2_g$.
3. connected sum of tori.
4. sphere with g handles.



1.66 Example. [20, 1.4.14]



The compact oriented surface als quotient of a $4g$-polygon. By induction this surface is homeomorphic to those given in 1.65.



1.67 Example. [20, 1.4.15] [20, 1.5.13] The PROJECTIVE PLANE $\mathbb{P}^2$ is defined as $(\mathbb{R}^3 \setminus \{0\})/\!\sim$ with $x \sim \lambda \cdot x$ für $\mathbb{R} \ni \lambda \neq 0$.

More generally, let for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ the PROJECTIVE SPACE be defined by $\mathbb{P}^n_{\mathbb{K}} := (\mathbb{K}^{n+1} \setminus \{0\})/\!\sim$, where $x \sim \lambda x$ for $0 \neq \lambda \in \mathbb{K}$. The quotient mapping $\mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{K}}$ is an open mapping, since the saturated hull of an open subset U is the open double-cone with base U and without its apex.

1.68 Examples. [20, 1.4.17] [20, 1.4.18]

1. $\mathbb{P}^2 \cong D^2/\!\sim$ where $x \sim -x$ for all $x \in S^1$.
2. $\mathbb{P}^n \cong D^n/\!\sim$ where $x \sim -x$ for all $x \in S^{n-1}$:
   Consider a hemisphere $D^n_+ \subseteq S^n$. Then the open quotient mapping $S^n \to \mathbb{P}^n$ restricts to a quotient mapping (by 1.27) on the compact set D^n_+ with associated equivalence relation $x \sim -x$ on $S^{n-1} \subseteq D^n_+$.
3. $\mathbb{P}^2$ can be obtained by gluing a disk to a Möbius strip.
   Consider the closed subsets $A := \{x \in S^2 : x_2 \leq 0, |x_3| \leq 1/2\}$ and $B = \{x \in S^2 : x_3 \geq 1/2\}$. The open quotient mapping induces an homeomorphism on the saturated subset $B \subseteq D^n_+$, i.e. $\pi(B)$ is a 2-ball. The set A is mapped to a

Möbius-strip by 1.29 and 1.58. Since $\pi(B)\cup\pi(A)=\mathbb{P}^2$ and $\pi(B)\cap\pi(A)\cong S^1$ we are done.

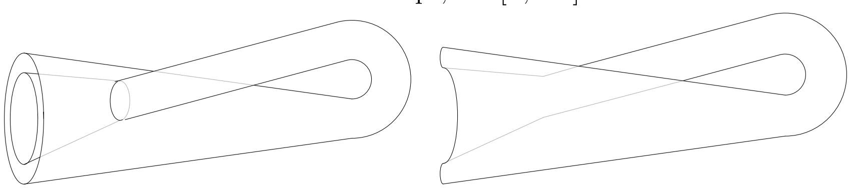
1.69 Proposition. [20, 1.4.16] [20, 1.5.14] [20, 1.6.6] *$\mathbb{P}^n_{\mathbb{K}}$ is a dn-dimensional connected closed manifold, where $d:=\dim_{\mathbb{R}}\mathbb{K}$. The mapping $p:S^{dn-1}\to\mathbb{P}^{n-1}_{\mathbb{K}}$, $x\mapsto[x]$ is a quotient mapping. In particular, $\mathbb{P}^1_{\mathbb{K}}\cong S^d$.*

Proof. Charts are $\varphi_i:\mathbb{K}^n\to\mathbb{P}^n_{\mathbb{K}}$, $(x^1,\dots,x^n)\mapsto[(x^1,\dots,x^i,1,x^{i+1},\dots,x^n)]$ for $i\in\{0,\dots,n\}$ with inverse $[(y^0,\dots,y^n)]\mapsto(\frac{y^0}{y^i},\dots,\frac{y^{i-1}}{y^i},\frac{y^{i+1}}{y^i},\dots,\frac{y^n}{y^i})$.
The restriction $\mathbb{K}^{n+1}\supseteq S^{d(n+1)-1}\to\mathbb{P}^n_{\mathbb{K}}$ is a quotient mapping since $\mathbb{K}^{n+1}\setminus\{0\}\to\mathbb{P}^n_{\mathbb{K}}$ is an open mapping, cf. 1.67, hence $\mathbb{P}^n_{\mathbb{K}}$ is compact. For $\mathbb{K}=\mathbb{R}$ this quotient mapping induces the equivalence relation $x\sim -x$.
For $n=1$ we have $\mathbb{P}^1_{\mathbb{K}}\setminus\varphi_0(\mathbb{K})=\{[(0,1)]\}$, therefore $\mathbb{P}^1_{\mathbb{K}}\cong\mathbb{K}_\infty\cong S^d$. □

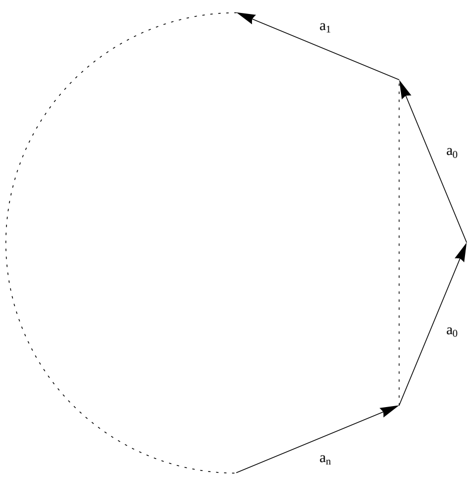
1.70 Example. [20, 1.4.19] The none-oriented connected closed surface N_g of genus g without boundary is

1. connected sum of g projective planes,
2. or equivalently by 1.68.3, a sphere with g Möbius strips glued to it.

Klein's bottle as sum of two Möbius strips, see [8, 9.3]:

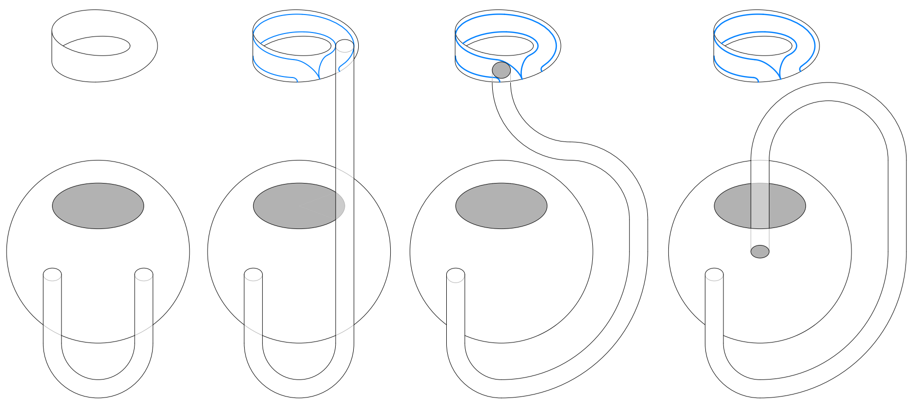


1.71 Proposition. [20, 1.4.20] *The none-orientable connected compact surfaces without boundary as quotient of a $2g$-polygon.*

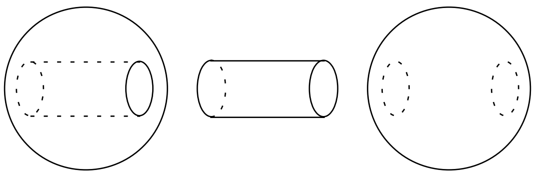


1.72 Theorem. [20, 1.9.1] *Each connected closed surface is homeomorphic to one of the surfaces $S^2=F_0, S^1\times S^1=F_1,\dots$ or $\mathbb{P}^2=N_1,N_2,\dots$.*

For a sketch of proof, see [8, 9.4]



1.73 Example. [20, 1.5.9] Union of filled tori $(D^2\times S^1)\cup_{\mathrm{id}}(S^1\times D^2)=(D^2\times D^2)^\cdot\cong(D^4)^\cdot\cong S^3$ by $\boxed{1.57}$. Other point of view: $S^3=D^3_+\cup_{\mathrm{id}}D^3_-$ and remove a filled cylinder from D_- and glue that to D_+ to obtain two tori. With respect to the stereographic projection the torus $\{(z_1,z_2)\in S^3\subseteq\mathbb{C}^2:|z_1|=r_1,\,|z_2|=r_2\}$ with $r_1^2+r_2^2=1$ corresponds to the torus with the z-axes as its axes and big radius $A:=1/r_1\geq 1$ and small radius $a:=\sqrt{A^2-1}=\frac{r_2}{r_1}$, see [8, 11.6,11.7].



1.74 Example. [20, 1.5.10] More generally, let the homeomorphism $f:S^1\times S^1\to S^1\times S^1$ be given by $f:(z,w)\mapsto(z^aw^b,z^cw^d)$, where $a,b,c,d\in\mathbb{Z}$ with $ad-bc=\pm1$.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\begin{pmatrix} a & b\\ c & d\end{pmatrix}"] \arrow[d] & \mathbb{R}^2 \arrow[d]\\
S^1\times S^1 \arrow[r, "f"] & S^1\times S^1
\end{tikzcd}$$

A meridian $S^1\times\{w\}\subseteq D^2\times S^1$ on the torus is mapped to a curve $t\mapsto(e^{2\pi it},w)\mapsto(w^b\,e^{2\pi iat},w^d\,e^{2\pi ict})$ which winds a-times around the axes and c-times around the core of $S^1\times S^1\hookrightarrow S^1\times D^2\hookrightarrow\mathbb{R}^3$. Similar for a circle of latitude.

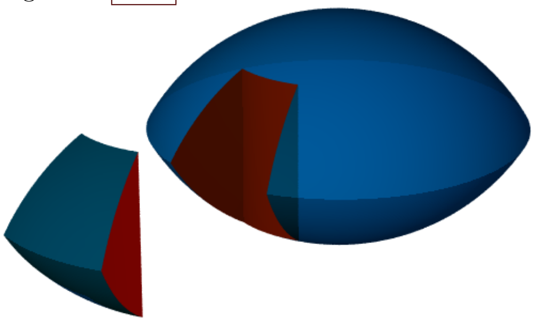
$$M\begin{pmatrix} a & b\\ c & d\end{pmatrix}:=(D^2\times S^1)\cup_f(S^1\times D^2).$$

In $\boxed{1.86}$ together with $\boxed{1.87}$ and $\boxed{1.82}$ we will indicate that M is often not homeomorphic to S^3.

1.75 Example. [20, 1.5.11] Cf. $\boxed{1.60}$. By a Heegard decomposition of a 3-dimensional manifold M one understands a representation of M by gluing two handle bodies V_g (see $\boxed{1.65.1}$) of the same genus g along their boundary.

1.76 Example. [20, 1.5.12] Cf. $\boxed{1.66}$ and $\boxed{1.71}$. For relative prime $1\leq q<p$ let the lens space be $L(\frac{q}{p}):=D^3/\sim$, where $(\varphi,\theta,1)\sim(\varphi-2\pi\frac{q}{p},-\theta,1)$ for $\theta\geq0$ with respect to spherical coordinates, so the northern hemisphere is identified with the southern one rotated by $2\pi\frac{q}{p}$. The interior of D^3 is mapped homeomorphically

to a 3-cell in $L(\frac{q}{p})$ by 1.31. The image of points in the open hemispheres have also such neighborhoods (formed by one half in the one part inside the northern hemisphere and one inside the southern). Each p-points on the equator obtained by recursively turning by $2\pi\frac{q}{p}$ get identified. After squeezing D^3 a little in direction of the axes we may view a neighborhood of a point on the equator as a cylinder over a sector of a circle (a piece of cake) where the flat sides lie on the northern and southern hemisphere. In the quotient p many of these pieces are glued together along their flat sides thus obtaining again a 3-cell as neighborhood. We will come to this description again in 1.87.



Group actions and orbit spaces

1.77 Definition. [20, 1.7.3] Group action of a group G on a topological space X is a group-homomorphism $G \to \mathrm{Homeo}(X)$ into the group of homeomorphisms of X. The ORBIT SPACE is $X/G := X/\sim = \{Gx : x \in X\}$, where $x \sim y :\Leftrightarrow \exists g \in G : y = g \cdot x$. For this we may without loss of generality assume that G is a subgroup of $\mathrm{Homeo}(X)$, since only its image in $\mathrm{Homeo}(X)$ is needed.

1.78 Examples. [20, 1.7.4]

1. S^1 acts on $\mathbb{C}$ by multiplication $\Rightarrow \mathbb{C}/S^1 \cong [0, +\infty)$.
2. $\mathbb{Z}$ acts on $\mathbb{R}$ by translation $(k, x) \mapsto k + x \Rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$, $\mathbb{R}^2/\mathbb{Z} \cong S^1 \times \mathbb{R}$. ATTENTION: $\mathbb{R}/\mathbb{Z}$ has two meanings.
3. S^0 acts on S^n by reflection (scalar multiplication) $\Rightarrow S^n/S^0 \cong \mathbb{P}^n$.

1.79 Definition. [20, 1.7.5] G is said to ACT FREELY on X, when no $g \neq \mathrm{id}$ has a fixed-point on X, i.e. $gx \neq x$ for all x and $g \neq \mathrm{id}$.

1.80 Theorem. [20, 1.7.6] *Let G act* STRICTLY DISCONTINUOUSLY *on X, i.e. each $x \in X$ has a neighborhood U with $gU \cap U \neq \emptyset \Rightarrow g = \mathrm{id}$. This is in particular the case, when G is finite and acts freely on a T_2 space X.*
Then X/G is a closed m-manifold provided X is one.

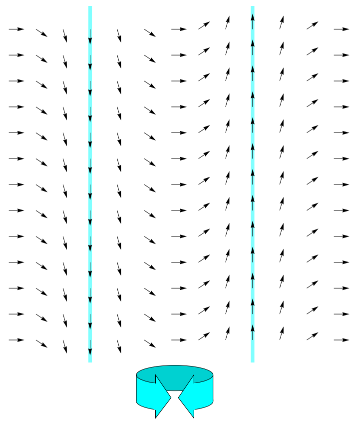
Proof. The quotient mapping $\pi : X \to X/G$ is open since $\pi^{-1}(\pi(W)) = \bigcup_{g\in G} g\,W$ for $W \subseteq X$. Free actions of finite groups on T_2-spaces are strictly discontinuous, since for $x \in X$ and $g \neq \mathrm{id}$ we find disjoint neighborhoods U_g of x and W_g of $g\,x$ and then $U := \bigcap_{g\neq\mathrm{id}} U_g \cap g^{-1}W_g$ is the required neighborhood:
$u \in U \cap gU \Rightarrow g^{-1}u \in U \subseteq g^{-1}W_g \Rightarrow u \in U_g \cap W_g$, a contradiction.

Let now X be an closed m-manifold. Since $U \to \pi(U)$ is a homeomorphism, any chart $A \cong D^m$ with $A \subseteq U$ induces a chart for X/G. In particular, points in X/G are closed (see [8, 19.1.1]), and hence the orbits as inverse images are closed. The orbits have to be discrete, so when X is compact the orbits are finite and hence the

group G is finite.
The quotient manifold is T_2: For $x \not\sim y \in X$ and any $g \in G$ choose disjoint neighborhoods U_g of $g \cdot x$ and W_g of y. Then $U := G \cdot \bigcap_g g^{-1} U_g$ and $W := G \cdot \bigcap_g W_g$ are disjoint saturated neighborhoods of the orbits. In fact, $y \in U \cap W \Rightarrow y' := g_1^{-1} y \in \bigcap_g W_g$ for some $g_1 \in G$ and $y' \in g_1^{-1} G \cdot \bigcap_g g^{-1} U_g$, i.e. $y' = g_2 \cdot g_2^{-1} U_{g_2} = U_{g_2}$ for some $g_2 \in g_1^{-1} G$, a contradiction. □

Example. Orbit spaces need not be Hausdorff.
Consider the ordinary differential equation

$$\frac{dx}{dt} = \cos^2 x, \quad \frac{dy}{dt} = \sin x$$



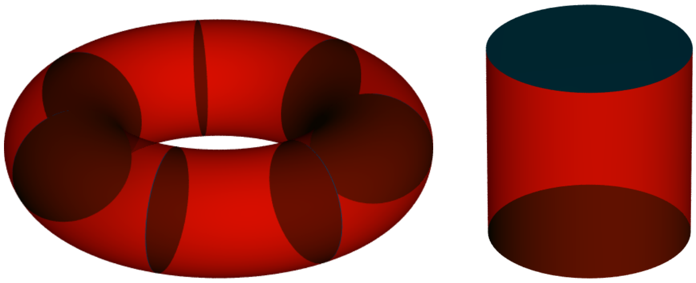
Since this vector field is bounded, the solutions exist globally and we get a smooth function $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ associating to each $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ the solution with value (x, y) at 0 at time t.
If the initial value satisfies $\cos x = 0$ then the solution is $y(t) = y(0) + t \cdot \sin x$. Otherwise we have $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \frac{d}{dx} \frac{1}{\cos x}$, hence it has to be contained in $\{(y, x) : y(x) = \frac{1}{\cos x}\}$. Moreover the time it takes from $x = x_0$ to $x = x_1$ is given by $t(x_1) - t(x_0) = \int_{x_0}^{x_1} \frac{dt}{dx} = \int_{x_0}^{x_1} \frac{1}{\cos^2 x} dx = \tan x|_{x=x_0}^{x_1}$.

Note that the orbit space $\mathbb{R}^2/\mathbb{R}$ is **not Hausdorff** (and $\mathbb{R}^2/\mathbb{Z}$ as well). It consists of a countable union $\bigsqcup_{\mathbb{Z}} \mathbb{R}$ of $\mathbb{R}'s$ together with the points $\pi/2 + \pi \cdot \mathbb{Z}$. A neighborhood basis of $\pi/2 + k\pi$ is given by end-intervals of the two neighboring $\mathbb{R}'s$. However, $\mathbb{Z}$ acts strictly discontinuous on $\mathbb{R}^2$.

We may also form the space $X := ([-\pi/2, \pi/2] \times \mathbb{R})/\sim$, where $(-\pi/2, -t) \sim (\pi/2, t)$. Since the action of $\mathbb{R}$ is compatible with this equivalence relation $\mathbb{R}$ acts fixed-point free on this borderless Möbius strip X as well. The **orbits** of the discrete subgroup $\mathbb{Z} \subseteq \mathbb{R}$ **are** obviously **closed** subsets. However, the **action is not strictly discontinuous**, since for any neighborhood of $[(\pi/2, 0)]_\sim$ some translate by $t \in \mathbb{Z}$ meets it again.

1.81 Example. [20, 1.7.7] Let $1 < p \in \mathbb{N}$ be relative prime to $q_1, \dots, q_k \in \mathbb{Z}$. Then $E_p := \{g \in \mathbb{C} : g^p = 1\} \cong \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ acts freely on $S^{2k-1} \subseteq \mathbb{C}^k$ by $g \cdot (z_1, \dots, z_k) \mapsto (g^{q_1} z_1, \dots, g^{q_k} z_k)$. The GENERAL LENS SPACE $L_{2k-1}(p; q_1, \dots, q_k) := S^{2k-1}/E_p$ of type $(p; q_1, \dots, q_k)$ is a closed manifold of dimension $2k - 1$. Note that this space depends only on $q_j \mod p$ and not on q_j itself, so we may assume $0 < q_j < p$.

In particular, $L_3(p; q, 1) \cong L(\frac{q}{p})$: We may parametrize $S^3 \subseteq \mathbb{C}^2$ by the quotient mapping $f : D^2 \times S^1 \twoheadrightarrow S^3$, $(z_1, z_2) \mapsto (z_1, \sqrt{1 - |z_1|^2}\, z_2)$ and the action of $E_p = \langle g \rangle$ on S^3, where $g := e^{2\pi i/p}$, lifts to the action given by $g \cdot (z_1, z_2) = (g^q z_1, g\, z_2)$. Only the points in $\{z_1\} \times S^1$ for $z_1 \in S^1$ get identified by f. A representative subset of S^3 for the action is given by $\{(z_1, z_2) \in S^3 : |\arg(z_2)| \leq \frac{\pi}{p}\}$, its preimage in $D^2 \times S^1$ is homeomorphic to $D^2 \times I$, and only points $(z_1, 0)$ and $(g^q z_1, 1)$ are in the same orbit. Thus the top $D^2 \times \{1\}$ and the bottom $D^2 \times \{0\}$ rotated by $g^q = e^{2\pi i \frac{q}{p}}$ have to be identified in the orbit space and also the generators $\{z_1\} \times I$ for $z_1 \in S^1$. This gives the description of $L(\frac{q}{p})$ in 1.76.



One has:

- $L_3(p; q_1, q_2) \cong L_3(p; q_2, q_1)$ via the reflection $\mathbb{C} \times \mathbb{C} \supseteq S^3 \to S^3 \subseteq \mathbb{C} \times \mathbb{C}$, $(z_1, z_2) \mapsto (z_2, z_1)$.
- $L_3(p; q\, q_1, q\, q_2) = L_3(p; q_2, q_1)$ for q relative prime to p via the group isomorphism $g \mapsto g^q$.
- $L_3(p; -q_1, q_2) \cong L_3(p; q_1, q_2)$ via $(z_1, z_2) \mapsto (z_1, \overline{z_2})$ and the group isomorphism $g \mapsto g^{-1} = \overline{g}$:

$$\begin{tikzcd}
(z_1, z_2) \arrow[rr] \arrow[d, "g"] & & (z_1, \bar{z}_2) \arrow[d, "\bar{g}"] \\
(g^{q_1} z_1, g^{q_2} z_2) \arrow[r] & (g^{q_1} z_1, \bar{g}^{q_2} \bar{z}_2) \arrow[r, equal] & (\bar{g}^{-q_1} z_1, \bar{g}^{q_2} \bar{z}_2)
\end{tikzcd}$$

1.82 Theorem. [20, 1.9.5] *$L(\frac{q}{p}) \cong L(\frac{q'}{p'}) \Leftrightarrow p = p'$ and ($q \equiv \pm q' \mod p$ or $qq' \equiv \pm 1 \mod p$).*

Proof. ($\Leftarrow$) By $\boxed{1.81}$

- $L_3(p; q, 1) \cong L_3(p; q', 1)$ for $q' \equiv \pm q \mod p$.
- $L_3(p; q, 1) \cong L_3(p; q', 1)$ for $qq' \equiv \pm 1 \mod p$, since $L_3(p; q, 1) \cong L_3(p; q'\, q, q') = L_3(p; \pm 1, q') \cong L_3(p; 1, q') \cong L_3(p; q', 1)$

($\Rightarrow$) is beyond the algebraic methods of this lecture course, see [5] for an elaboration. $\square$

1.83 Definition. [20, 1.7.1] A topological group is a topological space together with a group structure, s.t. $\mu : G \times G \to G$ and $\mathrm{inv} : G \to G$ are continuous.

1.84 Examples of topological groups. [20, 1.7.2]

1. $\mathbb{R}^n$ with addition.
2. $S^0 \subseteq \mathbb{R}$, $S^1 \subseteq \mathbb{C}$ and $S^3 \subseteq \mathbb{H}$ with multiplication, see [8, 14.16].
3. $G \times H$ for topological groups G and H.
4. The general linear group $GL(n) := GL(n, \mathbb{R}) := \{A \in L(\mathbb{R}^n, \mathbb{R}^n) : \det(A) \neq 0\}$ with composition, see [8, 14.1].
5. The special linear group $SL(n) := \{A \in GL(n) : \det(A) = 1\}$, see [8, 14.5].
6. The orthogonal group $O(n) := \{A \in GL(n) : A^t \cdot A = \mathrm{id}\}$ and the (path-)connected component $SO(n) := \{T \in O(n) : \det(T) = 1\}$ of the identity in $O(n)$. As topological space $O(n) \cong SO(n) \times S^0$. For all this see [8, 14.6].
7. $GL(n, \mathbb{C}) := \{A \in L_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) : \det_{\mathbb{C}}(A) \neq 0\}$, see [8, 14.14].
8. The unitary group $U(n) := \{A \in GL(n, \mathbb{C}) : A^* \cdot A = \mathrm{id}\}$ with closed subgroup $SU(n) := \{A \in U(n) : \det_{\mathbb{C}}(A) = 1\}$, see [8, 14.14]. As topological space $U(n) \cong SU(n) \times S^1$, see [10, 1.27]

9. In particular $SO(1) = SU(1) = \{1\}$, $SO(2) \cong U(1) \cong S^1$, $SU(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\} = \{\left(\begin{smallmatrix} a & -\bar{c} \\ c & \bar{a} \end{smallmatrix}\right) : |a|^2 + |c|^2 = 1\} \cong S^3$, $SO(3) \cong \mathbb{P}^3$.

For the last isomorphism consider the surjective mapping $f : [0,\pi] \times S^2 \twoheadrightarrow SO(3)$ given, by associating to an angle $\varphi \in [0,\pi]$ and an unit-vector $x \in S^2$ the rotation $f(\varphi, x)$ by the angle φ around the axes x.

$$\begin{tikzcd} {[0,\pi] \times S^2} \arrow[r, two heads, "f"] \arrow[d, "\mu"] & SO(3) \\ D^3 \arrow[ur, dotted, two heads, "\tilde{f}"'] & \end{tikzcd}$$

This mapping is injective except for $f(0,x) = f(0,x')$ and $f(\pi,x) = f(\pi,-x)$ for all $x, x' \in S^2$. Hence it factors to a surjective mapping $\tilde{f} : D^3 \twoheadrightarrow SO(3)$ over the surjective multiplication $\mu : [0,\pi] \times S^2 \twoheadrightarrow D^3$, $(\varphi, x) \mapsto \frac{\varphi}{\pi} \cdot x$, which is injective except for $\mu(0,x) = \mu(0,x')$ for all $x, x' \in S^2$. Thus $\tilde{f}$ is injective except for $\tilde{f}(y) = \tilde{f}(-y)$ for all $y \in S^2$. This is exactly the equivalence relation defining $\mathbb{P}^3 = D^3/\sim$.

The problem of homeomorphy

Remark. For 3-manifolds one is far from a solution to the classification problem. For $n > 3$ there can be no algorithm.

1.85 Theorem. [20, 1.9.2] *Each closed orientable 3-manifold admits a Heegard-decomposition.*

Hence in order to solve the classification problem it suffices to investigate the homeomorphisms of closed oriented surfaces and determine which gluings give homeomorphic manifolds.

In the following example we study this for the homeomorphisms of the torus considered in 1.74.

1.86 Example. [20, 1.9.3] Let $M := M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M' := M\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $SL(2,\mathbb{Z})$, see 1.74. For $\alpha, \beta, \gamma, \delta \in S^0$ and $m, n \in \mathbb{Z}$ consider the homeomorphisms

$$\begin{aligned} F &: D^2 \times S^1 \to D^2 \times S^1, && (z,w) \mapsto (z^\alpha w^m, w^\beta) \\ G &: S^1 \times D^2 \to S^1 \times D^2, && (z,w) \mapsto (z^\gamma, z^n w^\delta) \end{aligned}$$

If

$$\begin{pmatrix} \gamma & 0 \\ n & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix},$$

i.e.

$$\gamma a = a'\alpha, \quad \gamma b = a'm + b'\beta, \quad na + \delta c = c'\alpha, \quad nb + \delta d = c'm + d'\beta$$

then $(G|_{S^1 \times S^1}) \circ f = f' \circ (F|_{S^1 \times S^1})$ and thus $M \cong M'$ by 1.49.

Reduction:

$(a \le 0)$: $\gamma := -1$, $\alpha := \beta := \delta := 1$, $m := n := 0$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} -a & -b \\ c & d \end{pmatrix}$, i.e. w.l.o.g. $a \ge 0$.

$(ad-bc=-1)$: $\alpha:=\beta:=\gamma:=1$, $\delta:=-1$, $m:=n:=0$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cong M\begin{pmatrix} a & b \\ -c & -d\end{pmatrix}$, i.e. w.l.o.g. $ad-bc=1$.

$(a=0)$: $\Rightarrow bc=-1$. $\alpha:=c$, $\beta:=b$, $\gamma:=1$, $\delta:=1$, $n:=0$, $m:=d$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cong M\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \cong (D^2\cup_{\mathrm{id}} D^2)\times S^1 \cong S^2\times S^1$.

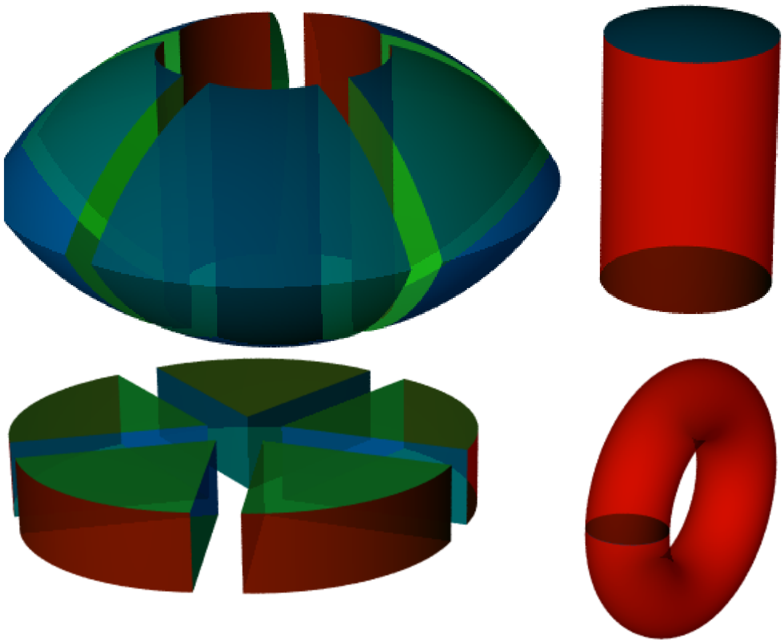
$(a=1)$: $\alpha:=\delta:=a$, $\beta:=ad-bc$, $\gamma:=1$, $m:=b$, $n:=-c$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cong M\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = (D^2\times S^1)\cup_{\mathrm{id}}(S^1\times D^2)\cong S^3$, by $\boxed{1.73}$.

$(ad'-b'c=1)$: $\Rightarrow a(d-d')=c(b-b')$ since $a\,d-b\,c=1$ and $\exists m$: $b-b'=m\,a$, $d-d'=m\,c$ since $gcd(a,c)=1$.
$\alpha:=\beta:=\gamma:=\delta:=1$, $n:=0\Rightarrow M\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cong M\begin{pmatrix} a & b' \\ c & d'\end{pmatrix} =: M(a,c)$.

$(c':=c-na)$: $\alpha:=\beta:=\gamma:=\delta:=-1$, $m:=0\Rightarrow M(a,c)\cong M(a,c')$, i.e. w.l.o.g. $0\le c<a$ (If $c=0\Rightarrow a=1\Rightarrow M(a,c)\cong S^3$).

Thus it suffices to investigate the spaces $M(a.c)$ with $0<c<a$ and $ggT(a,c)=1$ ($\Leftrightarrow \exists b,d: ad-bc=1$).

1.87 Theorem. Heegard-decomposition of lens spaces. [20, 1.9.4] *For relative prime $0<c<a$ we have $L(\frac{c}{a})\cong M(a,c)$.*



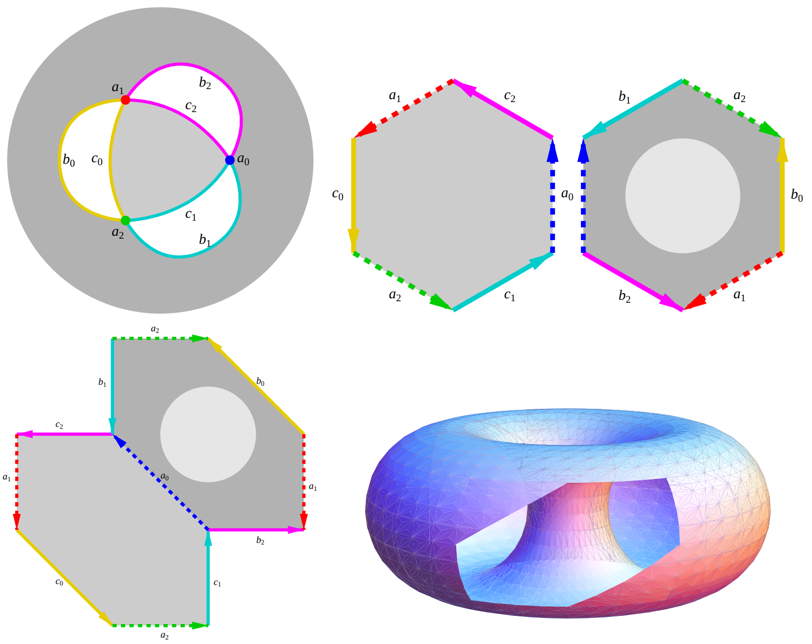
Proof. We start with $L(\frac{c}{a})=D^3/\sim$ (see $\boxed{1.76}$) and drill a cylindrical hole into D^3 and glue its top and bottom via $\sim$ to obtain a filled torus, where collections of a many generators of the cylinder (e.g. the red/green edges) are glued to from a closed curve which winds c-times around the core of the torus (i.e. the axes of the cylinder) and a-times around the axes of the torus. The remaining D^3 with hole is cut into a sectors, each homeomorphic to a piece of a cake, which yield $D^2\times I$ after gluing the blue sides (which correspond to points on S^2) and groups of a many generators of the cylindrical hole are glued to a circle $S^1\times\{t\}$. After gluing the green top and the correspondingly rotated bottom disc we obtain a second filled torus, where the groups of a many generators of the cylinder (e.g. the red/green edges) form a meridian. This is exactly the gluing procedure described in $\boxed{1.74}$ for $M(a,c)$. □

1.88 Definition. [20, 1.9.7] A KNOT is an embedding $S^1 \to \mathbb{R}^3 \subseteq S^3$.

1.89 Definition. [20, 1.9.6] Two embeddings $f, g : X \to Y$ are called TOPOLOGICAL EQUIVALENT, if there exists a homeomorphism $h : Y \to Y$ with $g = h \circ f$. Each two embeddings $S^1 \to \mathbb{R}^2$ are by Schönflies's theorem (which is a strong version of Jordan's theorem) equivalent.

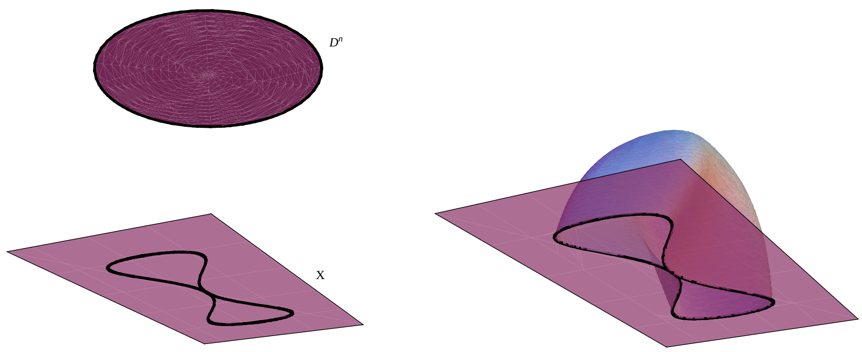
Remark. To each knot we may associated the complement of a tubular neighborhood in S^3. This is a compact connected 3-manifold with a torus as boundary. By a result of [4] a knot is up to equivalence uniquely determined by the homotopy class (see $\boxed{2.34}$) of this manifold.

As another invariant we may consider closed (orientable) surfaces in $\mathbb{R}^3$ of minimal genus which have the knot as boundary.



Gluing cells

1.90 Notation. [20, 1.6.1] $f : D^n \supseteq S^{n-1} \to X$. Consider $X \cup_f D^n$, $p : X \sqcup D^n \to X \cup_f D^n$, $e^n := p(\mathring{D}^n)$, $i := p|_X : X \hookrightarrow X \cup_f D^n =: X \cup e^n$.



By 1.44 $i : X \to X \cup e^n$ is a closed embedding, $X \cap \overset{\circ}{e^n} = \emptyset$ and $p : (D^n, S^{n-1}) \to (X \cup e^n, X)$ is a relative homeomorphism, i.e. $p : \overset{\circ}{D^n} \cong e^n$ is a homeomorphism.

For X T_2 also $X \cup e^n$ is T_2:
Points in X can be separated in X by U_i and the sets $U_i \cup \{tx : 0 < t < 1, f(x) \in U_i\}$ separate them in $X \cup e^n$. When both points are in the open subset e_n, this is obvious. Otherwise one lies in e_n and the other in X, so a sphere in D^n separates them.

Conversely we have:

1.91 Proposition. [20, 1.6.2] *Let Z be T_2, $X \subseteq Z$ closed and $F : (D^n, S^{n-1}) \to (Z, X)$ a relative homeomorphism.*
Then $X \cup_f D^n \cong Z$, where $f := F|_{S^{n-1}}$, via $g := (j \sqcup F) \circ p^{-1}$.

Proof. We consider

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "f=F|_{S^{n-1}}"] \arrow[dd, hook] & & X \arrow[dl] \arrow[dd, hook, "j"] \\
& X \cup_f D^n \arrow[dr, dotted, "g"] & \\
D^n \arrow[ur, "p|_{D^n}"] \arrow[rr, "F"] & & Z
\end{tikzcd}$$

$j : X \hookrightarrow Z$ is closed by assumption and also F, since D^n is compact and Z is T_2. Thus g is closed and obviously bijective and continuous, hence a homeomorphism. □

1.92 Theorem. [20, 1.6.3] *Let $f : S^{n-1} \to X$ be continuous and surjective and X be T_2. Then $p|_{D^n} : D^n \to X \cup_f D^n$ is a quotient mapping.*

Proof. The restriction $p|_{D^n}$ is surjective, since f is. Since D^n is compact and $X \cup_f D^n$ is T_2 by 1.90, p is a quotient mapping by 1.27. □

1.93 Examples. [20, 1.6.4]

(1) $f : S^{n-1} \to \{*\} =: X \Rightarrow X \cup_f D^n \overset{1.47.1}{\cong} D^n/S^{n-1} \overset{1.36}{\cong} S^n$.

(2) $f : S^{n-1} \to X$ constant $\Rightarrow X \cup_f D^n \overset{1.47.3}{\cong} X \vee (D^n/S^{n-1}) \overset{1.36}{\cong} X \vee S^n$.

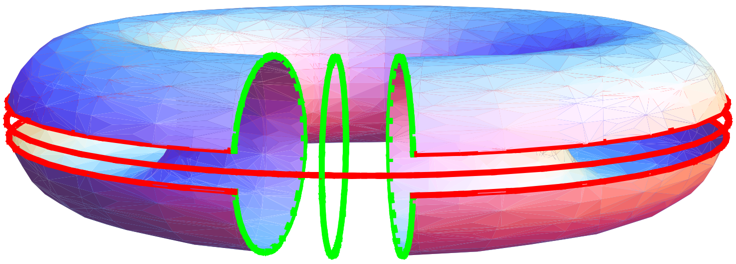
(3) $f = \mathrm{id} : S^{n-1} \to S^{n-1} =: X \Rightarrow X \cup_f D^n \cong D^n$ by 1.92.

(4) $f = \mathrm{incl} : S^{n-1} \hookrightarrow D^n =: X \Rightarrow X \cup_f D^n \cong S^n$ by 1.50.3.

(5) [20, 1.6.10] Let $g_n : S^1 \to S^1$, $z \mapsto z^n$. Then $S^1 \cup_{g_0} D^2 \cong S^1 \vee S^2$ by 2, $S^1 \cup_{g_1} D^2 \cong D^2$ by 3, $S^1 \cup_{g_2} D^2 \cong \mathbb{P}^2$ by 1.68.1, $S^1 \cup_{g_n} D^2 \cong S^1 \cup_{g_{-n}} D^2$ by conjugation $z \mapsto \bar{z}$.

1.94 Theorem. [20, 1.6.9] [20, 1.6.11] *Let $i_j^n : S^1 \hookrightarrow \bigvee_{k=1}^r S^1$ be $z \mapsto z^n$ into the j^{th} summand S^1, furthermore, $B_k := \{\exp(\frac{2\pi i t}{m}) : k-1 \le t \le k\}$ an arc of length $\frac{2\pi}{m}$ and $f_k : B_k \to S^1$, $\exp(\frac{2\pi i t}{m}) \mapsto \exp(2\pi i(t-k+1))$. Finally, for $j_1, \dots, j_m \in \{1, \dots, r\}$ let $i_{j_1}^{n_1} \cdot \dots \cdot i_{j_m}^{n_m} : S^1 \to \bigvee^r S^1$ the mapping which coincides on B_k with $i_{j_k}^{n_k} \circ f_k$, i.e. one runs first n_1-times along the j_1-th summand S^1, etc.*

For $g \ge 1$ and $f := i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \dots \cdot i_{2g-1} \cdot i_{2g} \cdot i_{2g-1}^{-1} \cdot i_{2g}^{-1}$ resp. $f := i_1^2 \cdot i_2^2 \cdot \dots \cdot i_g^2$ we have $\bigvee^{2g} S^1 \cup_f D^2 \cong F_g$ resp. $\bigvee^g S^1 \cup_f D^2 \cong N_g$.



Proof. 1.92 $\Rightarrow X_g := \bigvee S^1 \cup_f D^2 \cong D^2/\sim$ where $x \sim y$ for $x, y \in S^1 \Leftrightarrow f(x) = f(y)$. This is precisely the relation from 1.66, resp. 1.71. □

1.95 Proposition. [20, 1.6.5] [20, 1.6.7] [20, 1.6.8] *We have a closed embedding $\mathbb{P}^{n-1}_{\mathbb{K}} \hookrightarrow \mathbb{P}^n_{\mathbb{K}}$ via $\mathbb{K}^n \cong \mathbb{K}^n \times \{0\} \subseteq \mathbb{K}^{n+1}$. The mapping*

$$F : \mathbb{K}^n \supseteq D^{dn} \to \mathbb{P}^n_{\mathbb{K}}, \quad (x^1, \dots, x^n) \mapsto [(x^1, \dots, x^n, 1 - \|x\|)]$$

defines a relative homeomorphism $F : (D^{dn}, S^{dn-1}) \to (\mathbb{P}^n_{\mathbb{K}}, \mathbb{P}^{n-1}_{\mathbb{K}})$. Thus, by 1.91, $\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}^{n-1}_{\mathbb{K}} \cup_{F|_{S^{dn-1}}} D^{dn}$. Hence we have decompositions into disjoint cells:

$$\mathbb{P}^n_{\mathbb{R}} \cong e^0 \cup e^1 \cup \dots \cup e^n, \quad \mathbb{P}^n_{\mathbb{C}} \cong e^0 \cup e^2 \cup \dots \cup e^{2n}, \quad \text{and } \mathbb{P}^n_{\mathbb{H}} \cong e^0 \cup e^4 \cup \dots \cup e^{4n}$$

Proof. The induced mapping $\mathbb{P}^{n-1}_{\mathbb{K}} \to \mathbb{P}^n_{\mathbb{K}}$ is injective, hence a closed embedding. The charts $\mathbb{K}^n \cong U_{n+1} = \mathbb{P}^n_{\mathbb{K}} \setminus \mathbb{P}^{n-1}_{\mathbb{K}}$, $(x^1, \dots, x^n) \mapsto [(x^1, \dots, x^n, 1)]$ were constructed in the proof of 1.69.

The mapping $D^{dn} \setminus S^{dn-1} \to \mathbb{K}^n$, given by $x \mapsto \frac{x}{1-\|x\|}$, is a homeomorphism as in 1.4, and thus the composite $F|_{D^{dn} \setminus S^{dn-1}}$ is a homeomorphism as well. □

$$\begin{tikzcd}
D^{dn} \arrow[rr, "F"] & & \mathbb{P}^n_{\mathbb{K}} \\
D^{dn} \setminus S^{dn-1} \arrow[u, hook] \arrow[r, "\cong", "1.4"'] & \mathbb{K}^n \arrow[r, "\cong", "1.69"'] & \mathbb{P}^n_{\mathbb{K}} \setminus \mathbb{P}^{n-1}_{\mathbb{K}} \arrow[u, hook]
\end{tikzcd}$$

1.96 Definition. Gluing several cells. [20, 1.6.12] For continuous mappings $f_j : D^n \supseteq S^{n-1} \to X$ for $j \in J$ let

$$X \cup_{(f_j)_j} \bigcup_{j \in J} D^n := X \cup_{\bigsqcup_{j \in J} f_j} \bigsqcup_{j \in J} D^n.$$

1.97 Example. [20, 1.6.13]

(1) $X \cup_{(f_1, f_2)} (D^n \sqcup D^n) \cong (X \cup_{f_1} D^n) \cup_{f_2} D^n$, by 1.46.

(2) Let $f_j := \mathrm{id} : S^{n-1} \to S^{n-1}$ for $j \in \{1,2\}$.

Then $S^{n-1} \cup_{(f_1,f_2)} (D^n \sqcup D^n) \overset{1}{\cong} (S^{n-1} \cup e^n) \cup e^n \overset{1.93.3}{\cong} D^n \cup e^n \overset{1.93.4}{\cong} S^n$.

(3) If $f_j : S^{n-1} \to \{*\} =: X$ for each j, then $X \cup_{(f_j)_j} \bigcup_{j\in J} D^n \cong \bigvee_J S^n$:
By 1.36 $D^n \setminus S^{n-1} \cong \mathbb{R}^n \cong S^n \setminus \{*\}$ extends to a relative homeomorphism $\lambda : (D^n, S^{n-1}) \to (S^n, \{*\})$. Thus also $\bigsqcup_J \lambda = J \times \lambda : (J \times D^n, J \times S^{n-1}) \to (J \times S^n, J \times \{*\})$ is a relative homeomorphism and $J \times \lambda : J \times D^n \to J \times S^n$ is a quotient mapping by 1.32 since J is locally compact as discrete space. Hence also the induced mapping $(J\times D^n)/(J\times S^{n-1}) \to (J\times S^n)/(J\times\{*\}) = \bigvee_j S^n$ is a quotient mapping by 1.27 and is obviously bijective, hence a homeomorphism.

$$\begin{tikzcd}
J \times S^{n-1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "1.36"] & J \times D^n \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "1.32"] & (J \times D^n)/(J \times S^{n-1}) \arrow[r, equal] \arrow[d, tail, two heads, "1.27"] & \bigsqcup_J D^n / \bigsqcup_J S^{n-1} \\
J \times \{*\} \arrow[r, hook] & J \times S^n \arrow[r, two heads] & (J \times S^n)/(J \times \{*\}) \arrow[r, equal] & \bigvee_J S^n
\end{tikzcd}$$

Inductive limits

1.98 Definition. [20, 1.8.1] Let X be a set and $A_j \subseteq X$ topological spaces with $X = \bigcup_{j\in J} A_j$ and such that the trace topology on $A_j \cap A_k$ induced from A_j and from A_k is identical and the intersection closed. We consider the final topology induced on X by all the inclusions $\mathrm{inj}_j : A_j \hookrightarrow X$.
This topology induces on A_j the given topology, moreover $A_j \hookrightarrow X$ is a closed embedding: Let B be closed in A_j, then $B \cap A_k = B \cap (A_j \cap A_k)$ is closed in the topology of A_j and hence also in that of A_k, so B is closed in the final topology on X. Conversely, let $B \subseteq A_j$ be closed in the final topology of X, then $B = B \cap A_j = \mathrm{inj}_j^{-1}(B)$ is closed in A_j.

The canonical mapping $p := \bigsqcup_j \mathrm{inj}_j : \bigsqcup_j A_j \to X$ is a quotient mapping by definition of the final topology (it is clearly onto and $B \subseteq X$ is closed iff $\mathrm{inj}_j^{-1}(B) = B \cap A_j$ is closed in A_j) and thus we have the corresponding universal property:
A mapping $f : X \to Y$ is continuous, iff $f|_{A_j} : A_j \to Y$ is continuous for all j.

1.99 Proposition. [20, 1.8.3] [20, 1.8.4] *Let $\mathcal{A}$ be a closed (locally) finite covering of X. Then X carries the final topology with respect to $\mathcal{A}$.*

Proof. See [6, 1.2.14.3]: Let $B \subseteq X$ be such that $B \cap A \subseteq A$ is closed for all $A \in \mathcal{A}$. In order to show that $B \subseteq X$ is closed it suffices to prove that $\overline{\bigcup_{C\in\mathcal{C}} C} = \bigcup_{C\in\mathcal{C}} \overline{C}$ for locally finite families $\mathcal{C}(:= \{B\cap A : A \in \mathcal{A}\})$. ($\supseteq$) is obvious. ($\subseteq$) Let $x \in \overline{\bigcup_{C\in\mathcal{C}} C}$ and U an open neighborhood of x with $\mathcal{C}_0 := \{C \in \mathcal{C} : C \cap U \neq \emptyset\}$ being finite. Then $x \notin \overline{\bigcup_{C\in\mathcal{C}\setminus\mathcal{C}_0} C}$ and since $x \in \overline{\bigcup_{C\in\mathcal{C}} C} = \overline{\bigcup_{C\in\mathcal{C}_0} C} \cup \overline{\bigcup_{C\in\mathcal{C}\setminus\mathcal{C}_0} C}$ we have $x \in \overline{\bigcup_{C\in\mathcal{C}_0} C} = \bigcup_{C\in\mathcal{C}_0} \overline{C} \subseteq \bigcup_{C\in\mathcal{C}} \overline{C}$. □

1.100 Definition. [20, 1.8.5] Let A_n be an increasing sequence of topological spaces, where each A_n is a closed subspace in A_{n+1}. Then $\bigcup_{n\in\mathbb{N}} A_n$ with the final topology is called (INDUCTIVE) LIMIT $\varinjlim_n A_n$ of the sequence $(A_n)_n$.

1.101 Examples. [20, 1.8.6] [20, 1.8.7]

1. $\mathbb{R}^\infty := \varinjlim_n \mathbb{R}^n$, the space of finite sequences. Let $x \in \mathbb{R}^\infty$ and $\varepsilon_n > 0$. Then $\{y \in \mathbb{R}^\infty : |y_n - x_n| < \varepsilon_n \forall n\}$ is an open neighborhood of x in $\mathbb{R}^\infty$. Conversely, let $U \subseteq \mathbb{R}^\infty$ be an open set containing x. Then there exists an $\varepsilon_1 > 0$ with $K_1 := \{y_1 : |y_1 - x_1| \le \varepsilon_1\} \subseteq U \cap \mathbb{R}^1$. Since $K_1 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2$ is compact, there exists by [6, 2.1.11] an $\varepsilon_2 > 0$ with $K_2 := \{(y_1, y_2) : y_1 \in K_1, |y_2 - x_2| \le \varepsilon_2\} \subseteq U \cap \mathbb{R}^2$. Inductively we obtain ε_n with $\{y \in \mathbb{R}^\infty : |y_k - x_k| \le \varepsilon_k \forall k\} = \bigcup_n K_n \subseteq U$. Thus the sets from above form a basis of the topology.
   In contrast, the sets $\bigcup_n \{y \in \mathbb{R}^n : \|y - x\| < \delta_n\}$ do not from a basis for this topology, since for $\delta_n \searrow 0$ they contain none of the neighborhoods from above, since $x + (\frac{\varepsilon_1}{2}, \dots, \frac{\varepsilon_n}{2}, 0, \dots)$ is not contained therein for n with $\delta_n \le \frac{\varepsilon_1}{2}$.
2. $S^\infty := \varinjlim_n S^n$ is the set of unit vectors in $\mathbb{R}^\infty$.
3. $\mathbb{P}^\infty := \varinjlim_n \mathbb{P}^n$ is the space of lines through 0 in $\mathbb{R}^\infty$.
4. $O(\infty) := \varinjlim_n O(n)$, where $GL(n) \hookrightarrow GL(n+1)$ via $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
5. $SO(\infty) := \varinjlim_n SO(n)$
6. $U(\infty) := \varinjlim_n U(n)$
7. $SU(\infty) := \varinjlim_n SU(n)$

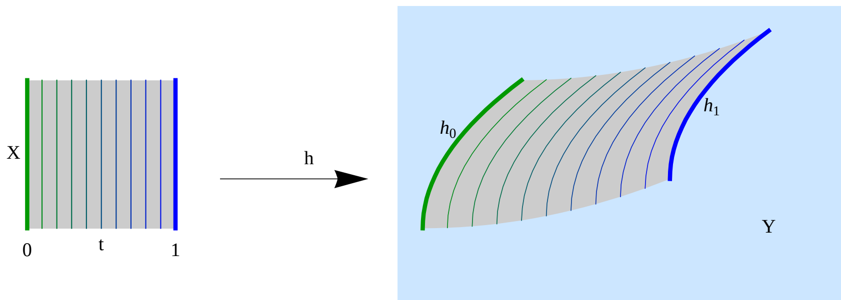
2. Homotopy

In this chapter we introduce the concept of homotopy. This leads to a weakening of the concept of homeomorphy to that of homotopy-equivalence and the special cases of (strict or neighborhood) deformation retracts.

2.1 Definition. [20, 2.1.1] A HOMOTOPY is a mapping $h : I \to C(X, Y)$, which is continuous as mapping $\hat{h} : I \times X \to Y$, where $\hat{h}(t, x) := h(t)(x)$. Note that this implies, that $h : I \to C(X, Y)$ is continuous for the compact-open topology (This is a version of the topology of uniform convergence for general topological spaces instead of uniform spaces Y. A subbasis for it is given by the sets $N_{K,U} := \{f \in C(X, Y) : f(K) \subseteq U\}$ with arbitrary compact $K \subseteq X$ and open $U \subseteq Y$) but not conversely.

Two mappings $h_j : X \to Y$ for $j \in \{0, 1\}$ are called HOMOTOPIC (we write $h_0 \sim h_1$) if there exists a homotopy $h : I \to C(X, Y)$ with $h(j) = h_j$ for $j \in \{0, 1\}$, i.e. a continuous mapping $H : I \times X \to Y$ with and $H(j, x) = h_j(x)$ for all $x \in X$ and $j \in \{0, 1\}$.

$$\begin{tikzcd} \{0,1\} \times X \arrow[r, "h_0 \cup h_1"] \arrow[d, hook] & Y \\ I \times X \arrow[ur, dotted, "H"'] & \end{tikzcd}$$



2.2 Lemma. [20, 2.1.2] *To be homotopic is an equivalence relation on $C(X, Y)$.*

2.3 Definition. [20, 2.1.5] The HOMOTOPY CLASS $[f]$ of a mapping $g \in C(X, Y)$ is $[f] := \{g \in C(X, Y) : g \text{ is homotopic to } f\}$. Let $[X, Y] := \{[f] : f \in C(X, Y)\}$.

2.4 Lemma. [20, 2.1.3] *Homotopy is compatible with the composition.*

For $f : X' \to X$ and $g : Y \to Y'$ let $f^* : C(X, Y) \to C(X', Y)$ be defined by $f^*(k) := k \circ f$ and $g_* : C(X, Y) \to C(X, Y')$ be defined by $g_*(k) := g \circ k$. Finally, let $C(f, g) := f^* \circ g_* = g_* \circ f^* : C(X, Y) \to C(X', Y')$, $k \mapsto g \circ k \circ f$.

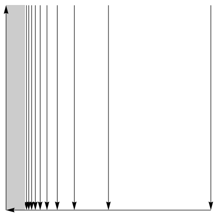
$$\begin{tikzcd} C(X,Y) \arrow[r, "f^*"] \arrow[d, "g_*"'] \arrow[dr, dotted, "C(f,g)"] & C(X',Y) \arrow[d, "g_*"] \\ C(X,Y') \arrow[r, "f^*"'] & C(X',Y') \end{tikzcd}$$

Proof. Let $h : I \to C(X, Y)$ be a homotopy and $f : X' \to X$, $g : Y \to Y'$ be continuous. Then $C(f, g) \circ h := f^* \circ g_* \circ h : I \to C(X', Y')$ is a homotopy $g \circ h_0 \circ f \sim g \circ h_1 \circ f$, since $(C(f, g) \circ h)\hat{} = g \circ \hat{h} \circ (I \times f)$ is continuous. $\square$

2.5 Definition. [20, 2.1.4] A mapping $f : X \to Y$ is called 0-HOMOTOPIC iff it is homotopic to a constant mapping.
A space X is called CONTRACTIBLE, iff id_X is 0-homotopic.

2.6 Remarks. [20, 2.1.6]

(1) *Any two constant mappings into Y are homotopic iff Y is path-connected*: In fact, a path $y : I \to Y$ induces a homotopy $t \mapsto \mathrm{const}_{y(t)}$.
(2) $[\{*\}, Y]$ *is in bijection with the path-components of* Y: homotopy = path.
(3) *Star-shaped subsets $A \subseteq \mathbb{R}^n$ are contractible by scalar-multiplication. In particular, this is true for $A = \mathbb{R}^n$ and for convex subsets $A \subseteq \mathbb{R}^n$.*
(4) For a contractible space X there need not exist an appropriate homotopy h which keeps the point fixed, e.g. the infinite comb (see 2.36.10).



Contractible spaces are path-connected.
(5) *Any composition of a 0-homotopic mapping with any mapping is 0-homotopic*: 2.4.
(6) *If Y is contractible then any two mappings $f_j : X \to Y$ are homotopic, i.e.* $[X, Y] := \{*\}$: 2.4.
(7) *Any continuous none-surjective mapping $f : X \to S^n$ is 0-homotopic*: $S^n \setminus \{*\} \cong \mathbb{R}^n$ by 1.14, now use 3 and 6.
(8) *If X is contractible and Y is path-connected then any two mappings $f_j : X \to Y$ are homotopic, i.e.* $[X, Y] = \{*\}$: 5 and 1.
(9) *Any mapping $f : \mathbb{R}^n \to Y$ is 0-homotopic*: 3 and the arguments in 8.

2.7 Definition. [20, 2.1.7] [20, 2.1.8] [20, 2.1.10]

(1) A homotopy relative $A \subseteq X$ is a homotopy $h : I \to C(X, Y)$ with $\mathrm{incl}^* \circ h : I \to C(X, Y) \to C(A, Y)$ being constant. Two mappings $h_j : X \to Y$ are called homotopic relative $A \subseteq X$, iff there exists a homotopy $h : I \to C(X, Y)$ relative A with boundary values $h(j) = h_j$ for $j \in \{0, 1\}$.
(2) A homotopy of pairs (X, A) and (Y, B) is a homotopy $h : I \to C(X, Y)$ with $h(I)(A) \subseteq B$. Two mappings $h_j : (X, A) \to (Y, B)$ of pairs are called homotopic, iff there exists a homotopy (of pairs) $h : I \to C(X, Y)$ with $h(I)(A) \subseteq B$ and $h(j) = h_j$ for $j \in \{0, 1\}$. We denote with $[h_0]$ also this homotopy class and with $[(X, A), (Y, B)]$ the set of all these classes.
(3) A homotopy of pairs with $A = \{x_0\}$ and $B = \{y_0\}$ is called base-point preserving homotopy. We have $f \sim g : (X, \{x_0\}) \to (Y, \{y_0\})$ iff $f \sim g$ relative $\{x_0\}$.

2.8 Example. [20, 2.1.9] Since I is contractible we have $[I, I] = \{[t \mapsto 0]\}$ by 2.6.6, but $[(I, \dot{I}), (I, \dot{I})] = \{[\mathrm{id}], [t \mapsto 1 - t], [t \mapsto 0], [t \mapsto 1]\}$.

2.9 Lemma. [20, 2.1.11] *Let $p : X' \to X$ be a quotient mapping and let $h : I \to C(X, Y)$ be a mapping for which $p^* \circ h : I \to C(X', Y)$ is a homotopy. Then h is a homotopy.*

Proof. Note that for quotient-mappings p the induced injective mapping p^* is in general not an embedding (we may not find compact inverse images). However $\widehat{p^* \circ h} = \hat{h} \circ (I \times p)$ and $I \times p$ is a quotient-mapping by 1.32. □

2.10 Corollary. [20, 2.1.12]

(1) *Let $p : X' \to X$ be a quotient mapping, $h : I \to C(X', Y)$ be a homotopy and $h_t \circ p^{-1} : X \to Y$ be a well-defined mapping for all t. Then this defines a homotopy $I \to C(X, Y)$ as well: This is just a reformulation of* 2.9.

(2) *Let $f : X \supseteq A \to Y$ be a gluing map and $h : I \to C(X, Z)$ and $k : I \to C(Y, Z)$ be homotopies with $\mathrm{incl}^* \circ h = f^* \circ k$. Then they induce a homotopy $I \to C(Y \cup_f X, Z)$:*
Apply 1 *and* 1.32 *to $p : Y \sqcup X \to Y \cup_f X$.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr, "k_t"] & \\
X \arrow[r] \arrow[drr, "h_t"'] & Y \cup_f X \arrow[dr, dotted, "h"] & \\
& & Z
\end{tikzcd}$$

(3) *Let $h : I \to C(X, Y)$ be a homotopy compatible with equivalence relations $\sim$ on X and on Y, i.e. $x \sim x' \Rightarrow h(t, x) \sim h(t, x')$. Then h factors to a homotopy $I \to C(X/\sim, Y/\sim)$: Apply* 1 *to $(q_Y)_* \circ h : I \to C(X, Y/\sim)$.*

(4) *Each homotopy $h : I \to C((X, A), (Y, B))$ of pairs induces a homotopy $I \to C(X/A, Y/B)$:* 3.

(5) *Homotopies $h^j : I \to C((X_j, x_j^0), (Y_j, y_j^0))$ induce a homotopy $\bigvee_j h^j : I \to C((\bigvee_j X_j, x^0), (\bigvee_j Y_j, y^0))$: Apply* 4 *to the homotopy $h : I \to C((\bigsqcup_j X_j, \{x_j^0 : j\}), (\bigsqcup_j Y_j, \{y_j^0 : j\}))$.*

2.11 Example. [20, 2.1.13]

(1) Let $h_t : (X, I) \to (X, I)$ be given by $h_t(x, s) := (x, ts)$. This induces a contraction of the cone $CX := (X \times I)/(X \times \{0\})$ to its apex by 2.10.3.

(2) The contraction of $D^n = CS^{n-1}$ given by 1 is not compatible with the equivalence relation describing $D^n/S^{n-1} \cong S^n$, hence induces no contraction of S^n. We will see in 2.17 and 8.43, that S^n is not contractible at all.

Homotopy classes for mappings of the circle

2.12 Definition. [20, 2.2.1] We consider the (periodic) quotient mapping (and group homomorphism) $p : \mathbb{R} \twoheadrightarrow S^1$, $t \mapsto e^{2\pi i t}$ as well as its restriction $p|_I : I \twoheadrightarrow S^1$.

A mapping $\varphi : I \to \mathbb{R}$ factors to a well defined mapping $\overline{\varphi} := p \circ \varphi \circ p^{-1} : S^1 \to S^1$ if and only if $n := \varphi(1) - \varphi(0) \in \mathbb{Z}$.

$$\begin{tikzcd}
I \arrow[r, "\varphi"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "p"] \\
S^1 \arrow[r, dotted, "\overline{\varphi}"] & S^1
\end{tikzcd}$$

Conversely:

2.13 Lemma. [20, 2.2.2]
Let $f : S^1 \to S^1$ be continuous, then there exists a unique continuous $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ with $f = f(1) \cdot \bar{\varphi}$.

$$\begin{tikzcd}
(\mathbb{R}, 0) \arrow[r, dotted, "\varphi"] \arrow[d, "p"] & (\mathbb{R}, 0) \arrow[d, "p"] \\
(S^1, 1) \arrow[r, "f(1)^{-1} f"] & (S^1, 1)
\end{tikzcd}$$

Proof. Replace f by $f(1)^{-1} \cdot f$, i.e. w.l.o.g. $f(1) = 1$. Let $h := f \circ p : \mathbb{R} \to S^1$. Then h is periodic, uniformly continuous and $h(0) = 1$. So choose $\delta > 0$ with $|t - t'| \leq \delta \Rightarrow |h(t) - h(t')| < 2$ and hence $\frac{h(t)}{h(t')} \neq -1$. Let $t_j := j\,\delta$. The mapping $t \mapsto e^{it}$ is a homeomorphism $(-\pi,\pi) \to S^1 \setminus \{-1\}$. Let $\arg : S^1 \setminus \{-1\} \to (-\pi,\pi) \subseteq \mathbb{R}$ denote its inverse, i.e. $p(\frac{\arg(z)}{2\pi}) = z$. Then for $t_j \le t \le t_{j+1}$ let

$$\varphi(t) := \frac{1}{2\pi}\left(\arg\frac{h(t_1)}{h(t_0)} + \dots + \arg\frac{h(t)}{h(t_j)}\right),$$

which gives the desired lifting.

This lifting is unique, since the difference of two such liftings has image in the discrete subset $p^{-1}(1) \subseteq \mathbb{R}$, and hence is constant (=0). □

2.14 Definition. [20, 2.2.3] Let $f : S^1 \to S^1$ be continuous and φ as in 2.13, then $\deg f := \varphi(1) \in p^{-1}(1) = \mathbb{Z}$ is called MAPPING DEGREE of f.

2.15 Theorem. [20, 2.2.4] deg *induces an isomorphism* $[S^1, S^1] \cong \mathbb{Z}$ *of semigroups. In more detail:*

(1) *The mapping* $g_n : z \mapsto z^n$ *from* 1.93.5 *has degree* n.
(2) *Two mappings are homotopic iff they have the same degree.*
(3) $\deg(f_1 \circ f_2) = \deg(f_1) \cdot \deg(f_2)$.

Proof. 1 follows since $\varphi(t) = n \cdot t$.

2 Let f be a homotopy $I \to C(S^1, S^1)$. Then, by 2.13, there exists a lifting $\varphi : I \to C(\mathbb{R}, \mathbb{R})$ with $p(\varphi_t(z)) = f_t(1)^{-1} \cdot f_t(p(z))$. This φ is a homotopy, since we can use for each h_t the same δ in the proof of 2.13. In particular $\varphi_t(1) \in p^{-1}(1) = \mathbb{Z}$ and hence is constant. So $\deg(f_0) = \varphi_0(1) = \varphi_1(1) = \deg(f_1)$.

Conversely, we define $\varphi : I \to C(\mathbb{R}, \mathbb{R})$ by $\varphi_t := (1-t)\varphi_0 + t\varphi_1$. Then this induces a homotopy $f : I \to C(S^1, S^1)$ by 2.12, since $\varphi_t(1) = \deg(f_0) = \deg(f_1) \in \mathbb{Z}$.

3 Let $n := \deg(f_1)$ and $m := \deg(f_2)$. Obviously, $g_n \circ g_m = g_{nm}$. By 1 and 2 $f_1 \sim g_n$ and $f_2 \sim g_m$, hence $f_1 \circ f_2 \sim g_n \circ g_m = g_{nm}$ and thus $\deg(f_1 \circ f_2) = nm$. □

2.16 Remarks. [20, 2.2.5]

(1) $\deg(\mathrm{id}) = 1$: $\mathrm{id} = g_1$; f 0-homotopic $\Rightarrow \deg(f) = 0$: $f \sim g_0$; $\deg(g_{-1} : z \mapsto \bar{z}) = -1$ by 2.15.1.
(2) f homeomorphism $\Rightarrow \deg(f) \in \{\pm 1\}$, by 2.15.3 since $\deg(f)$ is invertible in $\mathbb{Z}$.
(3) $\mathrm{incl} : S^1 \hookrightarrow \mathbb{C} \setminus \{0\}$ is not 0-homotopic, since id_{S^1} is not: $\deg(\mathrm{id}) = 1$ and 2.4 applied to $\mathbb{C} \setminus \{0\} \twoheadrightarrow S^1$. We can use $[S^n, X]$ to detect "holes" in X.
(4) The two natural inclusions $\mathrm{inc}_i : S^1 \hookrightarrow S^1 \times S^1$ are not homotopic: $\mathrm{pr}_1 \circ \mathrm{inc}_1 = \mathrm{id}$, $\mathrm{pr}_1 \circ \mathrm{inc}_2 \sim 0$.

2.17 Lemma. [20, 2.2.6] S^1 *is not contractible.*

Proof. $\deg(\mathrm{id}) = 1$. □

2.18 Definition. [20, 2.3.1] A subspace $A \subseteq X$ is called RETRACT iff there exists an $r : X \to A$ with $r|_A = \mathrm{id}_A$, i.e. an extension $r : X \to A$ (called a RETRACTION) of id_A.

Being a retract is a transitive relation. Retracts in Hausdorff spaces are closed ($A = \{x \in X : r(x) = x\}$)

2.19 Lemma. [20, 2.3.2]

(1) *A subspace $A \subseteq X$ is a retract of X iff every function $f : A \to Y$ can be extended to $\tilde{f} : X \to Y$.*

(2) *Let $A \subseteq X$ be closed. Then a function $f : A \to Y$ can be extended to X iff Y is a retract of $Y \cup_f X$.*

Proof. For 1 we prove that id_A can be extended iff any $f : A \to Y$ can be extended:

1
$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, hook] \arrow[ddr, bend right, "f"'] & & A \arrow[ddl, bend left, "f"] \\
& X \arrow[ur, dotted, "r"] \arrow[d, dotted, "\tilde{f}"] & \\
& Y &
\end{tikzcd}$$

2
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \arrow[r] \arrow[drr, dotted, "\tilde{f}"'] & Y \cup_f X \arrow[dr, dotted, "r"] & \\
& & Y
\end{tikzcd}$$

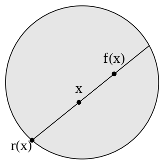
Thus the extensions $\tilde{f}$ of $f : A \to Y$ correspond to retractions $r = \mathrm{id}_Y \cup \tilde{f}$ of $Y \subseteq Y \cup_f X$. □

2.20 Lemma. [20, 2.2.7] *There is no retraction of D^2 to $S^1 \hookrightarrow D^2$.*

Proof. Otherwise, let $r : D^2 \to S^1$ be a retraction to $\iota : S^1 \hookrightarrow D^2$. Then $\mathrm{id} = r \circ \iota \sim r \circ 0 = 0$, a contradiction to 2.16.1. □

2.21 Lemma. Brouwer's fixed point theorem. [20, 2.2.8]
Every continuous mapping $f : D^2 \to D^2$ has a fixed point.

Proof.
Assume $f(x) \neq x$ and let $r(x)$ be the unique intersection point of the ray from $f(x)$ to x with S^1. Then r is a retraction, a contradiction to 2.20. □



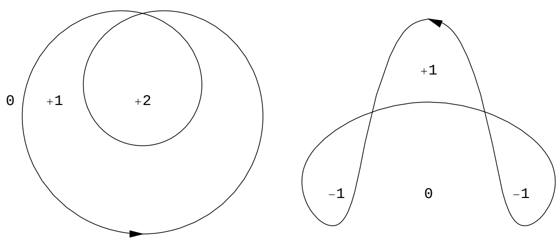
2.22 Lemma. Fundamental theorem of algebra. [20, 2.2.9]
Every nonconstant polynomial has a root.

Proof. Let $p(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$ be a polynomial without root and $n \geq 1$, $s := |a_0| + \cdots + |a_{n-1}| + 1 \geq 1$ and $z \in S^1$. Then

$$\begin{aligned}
|p(sz) - (sz)^n| &\leq |a_0| + s|a_1| + \cdots + s^{n-1}|a_{n-1}| \\
&\leq s^{n-1}(|a_0| + \cdots + |a_{n-1}|) < s^n = |(sz)^n|.
\end{aligned}$$

Hence $0 \notin \overline{p(sz), (sz)^n}$. Thus $z \mapsto s^n\, z^n$, $S^1 \to \mathbb{C}\setminus\{0\}$ is homotopic to $z \mapsto p(sz)$ and consequently 0-homotopic. Hence $0 \sim g_n : z \mapsto z^n$, a contradiction to 2.15. □

2.23 Definition. [20, 2.2.10]
The DEGREE of $f : S^1 \to \mathbb{R}^2$ with respect to $z_0 \notin f(S^1)$ is the degree of $x \mapsto \frac{f(x)-z_0}{|f(x)-z_0|}$ and will be denoted by $U(f, z_0)$ the TURNING (WINDING) NUMBER of f around z_0.



2.24 Lemma. [20, 2.2.11] *If z_0 and z_1 are in the same component of $\mathbb{C} \setminus f(S^1)$ then $U(f, z_0) = U(f, z_1)$.*

Proof. Let $t \mapsto z_t$ be a path in $\mathbb{C} \setminus f(S^1)$. Then $t \mapsto (x \mapsto \frac{f(x)-z_t}{|f(x)-z_t|})$ is a homotopy and hence $U(f, z_0) = U(f, z_1)$ by 2.15. $\square$

2.25 Lemma. [20, 2.2.12] *There is exactly one unbounded component of $\mathbb{C} \setminus f(S^1)$ and for z in this component we have $U(f, z) = 0$.*

Proof. For x' outside a sufficiently large disk containing $f(S^1)$ (this complement is connected and contained in the (unique) unbounded component) the mapping

$$t \mapsto \left(x \mapsto \frac{tf(x) - x'}{|tf(x) - x'|} \right)$$

is a homotopy showing that $x \mapsto \frac{f(x)-x'}{|f(x)-x'|}$ is 0-homotopic and hence $U(f, x') = 0$ and thus $U(f, \_) = 0$ on the unbounded component by 2.24. $\square$

By Jordan's curve theorem there are exactly two components for an embedding $f : S^1 \to \mathbb{C}$ and $U(f, z) \in \{\pm 1\}$ for z in the bounded component.

2.26 Theorem. [20, 2.3.3] *A mapping $f : X \to Y$ is 0-homotopic iff there exists an extension $\tilde{f} : CX \to Y$ with $\tilde{f}|_X = f$.*

Proof. We prove that homotopies $h : X \times I \to Y$ with constant h_0 correspond to extensions $\tilde{h} : CX \to Y$ of h_1. $\square$

$$\begin{tikzcd}
X\times\{1\} \arrow[rr, equal] \arrow[d, hook] & & X \arrow[dl, hook'] \arrow[dd, "h_1=f"] \\
X\times I \arrow[r, two heads] \arrow[drr, dotted, "h"'] & CX \arrow[dr, dotted, "\tilde{h}"] & \\
X\times\{0\} \arrow[u, hook'] \arrow[rr, "h_0=\text{const}"'] & & Y
\end{tikzcd}$$

2.27 Theorem of Borsuk and Ulam. [20, 2.2.13]
For every continuous mapping $f : S^2 \to \mathbb{R}^2$ there is a $z \in S^2$ with $f(z) = f(-z)$.

Proof. Suppose indirectly that $f(x) \neq f(-x)$ for all $x \in S^2$. Consider $f_1 : S^2 \to S^1$, $x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ and $f_2 : CS^1 \cong D^2 \to S^2 \to S^1$, $x \mapsto f_1(x, \sqrt{1 - |x|^2})$. Then $g := f_2|_{S^1} \sim 0$ via f_2 by 2.26. Let $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the lift of $g(1)^{-1}g$ from 2.13 and hence $\varphi(1) =: \deg(g) = 0$. Since f_1 and thus also g is odd, we have $g(\exp(2\pi i(t + \frac{1}{2}))) = g(-\exp(2\pi it)) = -g(\exp(2\pi it))$ for all t. Hence

$$\begin{aligned}
\exp\Big(2\pi i\varphi\big(t + \frac{1}{2}\big)\Big) &= g(1)^{-1}g\Big(\exp\big(2\pi i(t + \frac{1}{2})\big)\Big) = -g(1)^{-1}g\big(\exp(2\pi it)\big) \\
&= -\exp\big(2\pi i\varphi(t)\big) = \exp\Big(2\pi i\big(\varphi(t) + \frac{1}{2}\big)\Big).
\end{aligned}$$

Hence $k := \varphi(t + \frac{1}{2}) - \varphi(t) - \frac{1}{2} \in \mathbb{Z}$ and is independent on t. For $t = 0$ we get $\varphi(\frac{1}{2}) = k + \frac{1}{2}$ and for $t = \frac{1}{2}$ we get $\deg(g) = \varphi(1) = \varphi(\frac{1}{2}) + \frac{1}{2} + k = 2k + 1 \neq 0$, a contradiction. $\square$

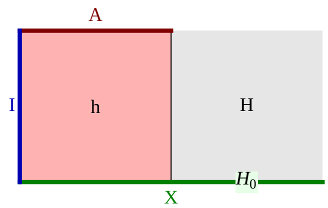
2.28 Ham-Sandwich-Theorem. [20, 2.2.14]
Let A_0, A_1, A_2 be bounded measurable subsets of $\mathbb{R}^3$.
Then there is a plane which cuts A_0, A_1 and A_2 in exactly equal parts.

Proof. We denote the halfspaces with $H_{a,d} := \{x \in \mathbb{R}^3 : \langle x, a\rangle \leq d\}$ and the volume of the trace of A_j on this halfspace with $\mu_j(a, d) := \mu(A_j \cap H_{a,d})$. Then $\mu_j : S^2 \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $\mu_j(-a, -d) + \mu_j(a, d) = \mu(A_j)$ and monotone increasing with respect to d. Let d_a be the midpoint of the closed interval

$I_a := \{d : \mu_0(a,d) = \mu(A_0)/2\}$. For $d \in I_a$ we have $\mu_0(a,d) = \frac{\mu(A_0)}{2} = \mu_0(-a,-d)$ and hence $d_{-a} = -d_a$.
Moreover, $a \mapsto d_a$ is continuous: let $d_- := \min I_{a_0}$ and $d_+ := \max I_{a_0}$. Then $\mu_0(a_0,d) < \mu(A_0)/2$ for all $d < d_-$ and by continuity of μ_0 there exists for $\varepsilon > 0$ a $\delta > 0$ such that $\mu_0(a, d_- - \varepsilon) < \mu(A_0)/2$ for all $|a - a_0| < \delta$ and analogously $\mu_0(a, d_+ + \varepsilon) > \mu(A_0)/2$ for all $|a - a_0| < \delta$, thus $I_a \subseteq [d_- - \varepsilon, d_+ + \varepsilon]$. In case $d_- = d_+$ we get $|d_a - d_{a_0}| \le \varepsilon$. Otherwise $d \mapsto \mu(a_0,d) = \mu(A_0)/2$ is constant on $[d_-, d_+]$ and thus $\mu(A_0 \cap (H_{a_0,d_+} \setminus H_{a_0,d_-})) = \mu(a_0,d_+) - \mu(a_0,d_-) = 0$. Hence we may assume that $\delta > 0$ is so small, that $\mu(a,d) = \mu(A_0)/2$ for all $|a - a_0| < \delta$ and all $d_- + \varepsilon < d < d_+ - \varepsilon$. So again $|d_a - d_{a_0}| \le \varepsilon$.
Now let $f : S^2 \to \mathbb{R}^2$ be given by $f(a) := (\mu_1(a,d_a), \mu_2(a,d_a))$. By $\boxed{2.27}$ there exists a point $b \in S^2$ with $f(b) = f(-b)$. Since $d_{-a} = -d_a$ we have that $f(-b)$ is the volume of A_1 and A_2 on the complement of H_{a,d_a}. $\square$

2.29 Definition. [20, 2.3.4] A pair (X,A) is said to have the general HOMOTOPY EXTENSION PROPERTY (HEP) (equiv. is said to be a COFIBRATION) iff A is closed in X and we have

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\mathrm{ins}_0"'] & X \arrow[d, "\mathrm{ins}_0"'] \arrow[ddr, bend left, "H_0"] & \\ A \times I \arrow[r, hook] \arrow[drr, bend right, "h"'] & X \times I \arrow[dr, dotted, "H"] & \\ & & Y \end{tikzcd}$$



or, equivalently,

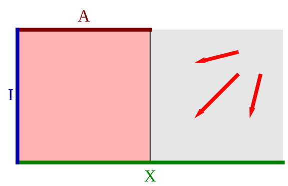
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\check{h}"'] & X \arrow[d, "H_0"] \arrow[dl, dotted, "\check{H}"'] \\ C(I,Y) \arrow[r, "\mathrm{ev}_0"'] & Y \end{tikzcd}$$

This is dual to the notion of FIBRATION (mappings with the homotopy lifting property):

$$\begin{tikzcd} A & X \arrow[l, two heads] \\ Y \times I \arrow[u, "h"] \arrow[ur, dotted, "H"] & Y \arrow[l, hook', "\mathrm{ins}_0"] \arrow[u, "H_0"'] \end{tikzcd}$$

2.30 Theorem. [20, 2.3.5]
(X,A) has HEP $\Leftrightarrow$ $L := X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof. (X,A) has HEP $\Leftrightarrow$
$\Leftrightarrow$ each $f : L \to Y$ extends to $X \times I$
$\overset{\boxed{2.19}}{\Longleftrightarrow}$ $L \subseteq X \times I$ is a retract. $\square$



2.31 Remarks. [20, 2.3.6]

(1) *The pair (D^n, S^{n-1}) has the HEP*: Radial projection from the axis at some point above the cylinder is a retraction.

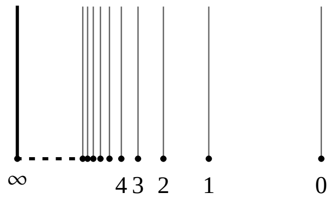
(2) *If (X,A) has HEP then $(Y \cup_f X, Y)$ has HEP for each $f : A \to Y$*:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[r, "h"] \arrow[d, hook] & C(I,Z) \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r] \arrow[urr, dotted, "(1)"] & Y \cup_f X \arrow[r, "H_0"'] \arrow[ur, dotted, "(2)"] & Z \end{tikzcd}$$

(3) *If Z is obtained from Y by gluing cells, then (Z,Y) has HEP*: $\Leftarrow$ $\boxed{1}$, $\boxed{2}$.

(4) *The pair* $(\mathbb{N}_\infty, \{\infty\})$ *does not have HEP.*

Otherwise, for $x \neq \infty$ the map $t \mapsto r(x,t)$, $I \to L$, maps $0 \mapsto (x,0) \Rightarrow r(\{x\} \times I) \subseteq L \cap (\{x\} \times I) = \{(x,0)\}$, but $r(x,1)$ is near $r(\infty,1) = (\infty,1)$ for x near ∞.

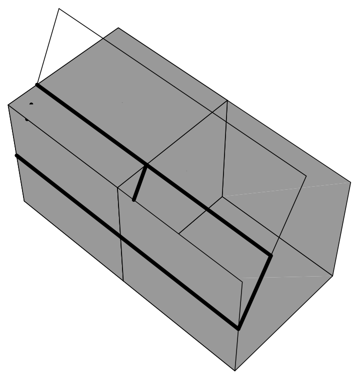


2.32 Remark. [20, 2.3.7] Let (X,A) has HEP.

(1) *If* $f \sim g : A \to Y$ *and* f *extends to* X *then so does* g: By Definition of HEP.
(2) *If* $f : X \to Y$ *is 0-homotopic on* A*, then there exists a mapping* g *homotopic to* f*, which is constant on* A:
Consider f on $X \times \{0\}$ and the given homotopy on $A \times I$.
(3) *If* $A = \{x_0\}$ *and* Y *is path-connected, then every mapping* $X \to Y$ *is homotopic to a base-point preserving one*:
Consider f on $X \times \{0\}$ and a path w on $\{x_0\} \times I$ between $f(x_0)$ and y_0.
(4) *There exists a continuous* $u : X \to I$ *with* $A = u^{-1}(0)$:
Define $u(x) := \sup\{t - \mathrm{pr}_2(r(x,t)) : t \in I\}$. Then $u : X \to I$ is continuous, $u|_A = 0$, and $u(x) = 0 \Rightarrow t \leq \mathrm{pr}_2(r(x,t)) \Rightarrow \mathrm{pr}_2(r(x,t)) \neq 0$ for $t > 0$, thus $r(x,t) \in A \times I$ for $t > 0$ and hence also $(x,0) = r(x,0) \in A \times I$, i.e. $x \in A$.
(5) *For closed subsets* A *of metric spaces* Y *there always exists a function* $u : Y \to I$ *as in* 4: Define $u(y) := d(y,A) = \inf\{d(y,a) : a \in A\}$.

2.33 Theorem. [20, 2.3.8] *If* (X,A) *has HEP, then so has* $(X \times I, X \times \dot{I} \cup A \times I)$.

Proof.
We use 2.30 to show that $X \times I \times I$ has $L := X \times I \times \{0\} \cup (X \times \dot{I} \cup A \times I) \times I$ as retract. For this we consider planes E through the axis $X \times (1/2, 2)$. For planes intersecting the bottom $X \times I \times \{0\}$ we take the retraction r of the intersection $E \cap (X \times I \times I) \cong X \times I$ (via horizontal projection) onto the intersection $E \cap L \cong X \times \{0\} \cup A \times I$. For the other planes meeting the sides we take the retraction r of the intersection $E \cap (X \times I \times I) \cong X \times [0, s/4] \cong X \times [0,s]$ (via vertical projection) onto the intersection $E \cap L \cong X \times \{0\} \cap A \times [0,s]$. For this we have to use that the retraction $r : (x,t) \mapsto (r_1(x,t), r_2(x,t))$ given by 2.30 can be chosen such that $r_2(x,t) \leq t$ by replaceing $r_2(x,t)$ by $\min\{t, r_2(x,t)\}$.



□

Homotopy equivalences

2.34 Definition. [20, 2.4.1] [20, 2.4.2] [20, 2.4.3]

(1) A homotopy equivalence is a mapping having an inverse up to homotopy. It is enough to assume a homotopy left inverse l and a homotopy right inverse r, i.e. $[l] \circ [f] = [\mathrm{id}]$ and $[f] \circ [r] = [\mathrm{id}]$, since then $[f] \circ [l] = [f] \circ [l] \circ [\mathrm{id}] = [f] \circ [l] \circ [f] \circ [r] = [f] \circ [\mathrm{id}] \circ [r] = [f] \circ [r] = [\mathrm{id}]$. Two spaces are

called HOMOTOPY EQUIVALENT (and we write $\sim$) iff there exists a homotopy equivalence between them.

(2) A continuous mapping between pairs is called HOMOTOPY EQUIVALENCE OF PAIRS, iff there is a mapping of pairs in the opposite direction which is inverse up to homotopy of pairs.

(3) A subspace $A \subseteq X$ is called DEFORMATION RETRACT (DR) iff there is a homotopy $h_t : X \to X$ with $h_0 = \mathrm{id}_X$ and $h_1 : X \to A \subseteq X$ being a retraction to $A \hookrightarrow X$.

(4) The subspace $A \subseteq X$ is called STRICT DEFORMATION RETRACT (SDR) iff, in addition to 3, h_t is a homotopy rel. A and there exists a continuous $u : X \to I$ with $A = u^{-1}(0)$. The later condition is not assumed in [20, 2.4.3]

(5) A subspace $A \subseteq X$ is called NEIGHBORHOOD DEFORMATION RETRACT (NDR) iff there exists a continuous $u : X \to I$ with $A = u^{-1}(0)$ and a homotopy $h_t : X \to X$ relative A with $h_0 = \mathrm{id}_X$ and $h_1(x) \in A$ for $u(x) < 1$.
Note that the SDRs are exactly the NDRs for which u can be choosen with $u(x) < 1$ for all $x \in X$ (replace u by $\frac{u}{2}$).
For NDR it suffices to assume that the homotopy h_t is a homotopy on $U := \{x : u(x) < 1\}$, since we can replace it by the new homotopy $\tilde{h}_t(x) := h_{t\,\max(0,\min(1,2-3u(x)))}(x)$ for all $x \in U(A)$, i.e. $u(x) < 1$. Then

$$\tilde{h}_t(x) = \begin{cases} x & \text{for } x \in A \text{ or } u(x) \geq \frac{2}{3} \\ h_1(x) & \text{for } t = 1 \text{ and } u(x) \leq \frac{1}{3} \end{cases}$$

Thus $\tilde{h}_t$ extends by id to a homotopy of X and with $\tilde{u}(x) := \min\{1, 3u(x)\}$ we get the NDR property.

2.35 Theorem. [20, 2.4.4] *For (X, A) with HEP the following is equivalent:*

(1) *$A \hookrightarrow X$ is a homotopy-equivalence;*
(2) *A is a DR of X;*
(3) *A is an SDR of X.*

The implications (3⇒2⇒1) are true without assuming HEP.

Proof. (3⇒2) is obvious.

(2⇒1) Let h_t be a homotopy from id_X to a retraction $h_1 : X \to A \subseteq X$. Then h_1 is a homotopy inverse to $\iota : A \hookrightarrow X$, since $h_1 \circ \iota = \mathrm{id}_A$ and $\iota \circ h_1 = h_1 \sim h_0 = \mathrm{id}_X$.

(1⇒2) Let g be a homotopy inverse to $\iota : A \hookrightarrow X$. Since $g \circ \iota \sim \mathrm{id}_A$ and $g : X \to A$ is an extension of $g \circ \iota$, we conclude from 2.32.1 that $\mathrm{id}_A : A \to A$ has an extension $r : X \to A \subseteq X$, i.e. a retraction. Moreover, $\mathrm{id}_X \sim \iota \circ g = r \circ \iota \circ g \sim r \circ \mathrm{id}_X = r$.

(2⇒3) Let $h_t : X \to X$ be a homotopy from $h_0 = \mathrm{id}_X$ to a retraction $h_1 = r : X \to A \subseteq X$ and let $H_t : W := X \times \dot{I} \cup A \times I \to X$ be given by

$$H_t(x, s) := \begin{cases} h_{st}(r(x)) & \text{für } s = 1 \text{ (the back side)} \\ h_{st}(x) & \text{elsewhere, i.e. for } x \in A \text{ or } s = 0 \text{ (front) or even } t = 1 \text{ (top).} \end{cases}$$

Because of $r(x) = x$ for $x \in A$ the definition coincides on the intersection. Since the expression for H_1 works on $X \times I$ and $(X \times I, W)$ has HEP by 2.33 we can extend H_0 to $X \times I$ by 2.32.1. This is the required deformation $\mathrm{id}_X \sim r$ rel. A.

Since (X, A) has HEP we have $A = u^{-1}(0)$ for a $u : X \to I$ by 2.32.4. □

2.36 Remarks. [20, 2.4.5]

(1) *X is contractible iff it is homotopy-equivalent to a point*:
X is contractible $:\Leftrightarrow \mathrm{id}_X \sim \mathrm{const}_* \Leftrightarrow \{*\} \subseteq X$ is a DR $\Leftrightarrow \{*\} \overset{\sim}{\hookrightarrow} X$.

(2) *Every set being star-shaped with respect to some point, has this point as SDR. Furthermore, $S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$ is SDR*: The radial homotopy from [2.6.3] is the strict deformation.

(3) *Composition of (S)DRs are (S)DRs*:

$$h(t,x) := \begin{cases} h^2(2t,x) & \text{for } t \le \frac{1}{2} \\ h^1(2t-1, h^2(1,x)) & \text{for } t \ge \frac{1}{2} \end{cases} \quad \text{where} \quad \bullet \overset{h^1}{\underset{\sim}{\hookrightarrow}} \bullet \overset{h^2}{\underset{\sim}{\hookrightarrow}} \bullet$$

and $u := \max\{u_2, u_1 \circ h_1^2\}$.

(4) *If $\{*\}$ is an (S)DR of Y, then so is $X \times \{*\}$ of $X \times Y$ and of $X \vee Y \subseteq X \times Y$*: Use $h_t(x,y) := (x, h_t(y))$ and $u(x,y) := u(y)$.

(5) *If (X,A) is an NDR and (Y,B) is an NDR (SDR), then $(X \times Y, X \times B \cup A \times Y)$ is an NDR (SDR)*: Let

$$h_t(x,y) := \begin{cases} \Big(h_{t\frac{u(y)}{u(x)}}(x), h_t(y)\Big) & \text{for } 0 \neq u(x) \ge u(y) \\ \Big(h_t(x), h_{t\frac{u(x)}{u(y)}}(y)\Big) & \text{for } u(x) \le u(y) \neq 0 \\ (x,y) & \text{for } u(x) = 0 = u(y) \end{cases}$$

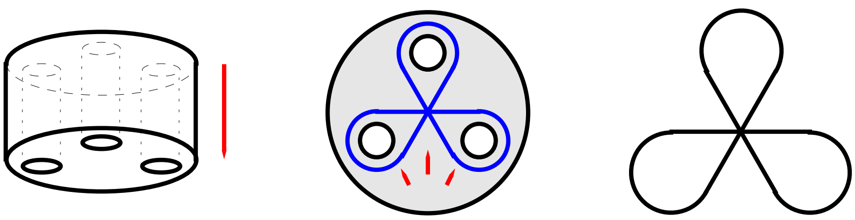
and $u(x,y) = \min\{u(x), u(y)\}$. The continuity of $(t,x,y) \mapsto h_t(x,y)$ follows, since $h_t(x) \to h_t(x_0) = x_0$ for $x \to x_0 \in A$ uniformly in t and similarly for $h_t(y)$.
If $u(x,y) < 1$ and say $u(x) \le u(y)$ then $u(x) < 1$ and hence $h_1(x) \in A$ and thus $h_1(x,y) \in A \times X$.

(6) *The complement of any k-dimensional affine subspace of $\mathbb{R}^n$ has S^{n-k-1} as SDR*: $\mathbb{R}^n \setminus \mathbb{R}^k = \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) \sim \{0\} \times (\mathbb{R}^{n-k} \setminus \{0\}) \cong \mathbb{R}^{n-k} \setminus \{0\} \sim S^{n-k-1}$ by [2], [4], and [3].

(7) *$X \times \{0\}$ is an SDR of $X \times I$ and consequently the apex $X \times \{0\} \in C(X)$ is an SDR of CX*: By [2], [4], and [2.10.4].

(8) *S^1 is a DR of $X \times S^1$ for every contractible X and also of the Möbius strip*: By [1], [4], and using $I \times \{0\} \subseteq I \times [-1,1]$ for the Möbius strip.

(9) *Every handle-body of genus g has $S^1 \vee \dots \vee S^1$ as SDR.*



(10) The infinite comb (see [2.6.4]) has $(+\infty, 1)$ as DR but not as SDR.

2.37 Proposition. [20, 2.4.6] *If A is an NDR (SDR) in X and $f : A \to Y$ is continuous, then Y is an NDR (SDR) in $Y \cup_f X$.*

Proof.

Idea:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[dr, "\mathrm{id}"] & \\ X \arrow[r] \arrow[dr, "h_t"'] & Y \cup_f X \arrow[dr, dotted, "\tilde h_t"] & Y \arrow[d, hook] \\ & X \arrow[r] & Y \cup_f X \end{tikzcd}$$

Details:
$$\begin{tikzcd} I \times A \arrow[r, "I\times f"] \arrow[d, hook] & I \times Y \arrow[d, hook] \arrow[dr, "\mathrm{pr}_2"] & \\ I \times X \arrow[r] \arrow[dr, "h"'] & I \times (Y \cup_f X) \arrow[dr, dotted, "\tilde h"] & Y \arrow[d, hook] \\ & X \arrow[r] & Y \cup_f X \end{tikzcd}$$

Note that the product of a pushout with a locally compact space is again a pushout. Let $u : Y \cup_f X \to I$ be given by $u(y) := 0$ for $y \in Y$ and $u([x]) := u(x)$ for $x \in X$.

2.38 Corollary. [20, 2.4.7] *If Z is built from Y by gluing simultaneously cells, then Y is an SDR in $Z \setminus P$, where P is given by picking in every cell a single point.*

Proof. Use $\boxed{2.36.2}$ and $\boxed{2.37}$. □

2.39 Example. [20, 2.4.8] The pointed compact surfaces have $S^1 \vee \dots \vee S^1$ as SDR.

Proof. By $\boxed{1.94}$ they are $S^1 \vee \dots \vee S^1 \cup_f (D^2 \setminus \{0\})$. Now use $\boxed{2.38}$. □

2.40 Theorem. [20, 2.4.9] *For a pair (X, A) and $L := X \times \{0\} \cup A \times I \subseteq X \times I$ the following statements are equivalent:*

(1) *(X, A) is NDR;*
(2) *$(X \times I, L)$ is SDR;*
(3) *L is a retract of $X \times I$;*
(4) *(X, A) has HEP.*

Proof.

(1⇒2) By $\boxed{2.36.5}$, since (X, A) is NDR and $(I, \{0\})$ is SDR.

(2⇒3) Take $r := h_1$.

(3⇔4) is $\boxed{2.30}$.

(3⇒1) Let $r = (r_1, r_2)$ be a retraction for $L \hookrightarrow X \times I$.
Define $u(x) := \sup\{t - r_2(x, t) : t \in I\}$ and $h_t(x) := r_1(x, t)$. Then $A = u^{-1}(0)$ as in $\boxed{2.32.4}$. Furthermore, $h_0(x) = r_1(x, 0) = x$, $h_t(a) = r_1(a, t) = a$ for all $a \in A$, and $u(x) < 1 \Rightarrow r_2(x, 1) > 0 \Rightarrow h_1(x) = r_1(x, 1) \in A$. □

2.41 Dependencies for closed subspaces $A \hookrightarrow X$.

$$\begin{tikzcd} & & \mathrm{SDR} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\ & \mathrm{DR} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \mathrm{NDR} \arrow[r, Leftrightarrow, "\boxed{2.40}"] & \mathrm{HEP} \arrow[d, Rightarrow, "\boxed{2.32.4}"] \\ \mathrm{Retr.} & & A \sim X & & A = u^{-1}(0) \end{tikzcd}$$

2.42 Counter-Examples.

Prop.	$L \subseteq E$	$\{(\infty,1)\} \subseteq E$	$\{\infty\} \subseteq \mathbb{N}_\infty$	$S^1 \subseteq D^2$	$\{0\} \subseteq \prod_I I$
SDR	−	−	−	−	−
NDR=HEP	−	−	−	+	−
DR	−	+	−	−	+
Retract	+	+	+	−	+
$A \sim X$	+	+	−	−	+
$A = u^{-1}(0)$	+	+	+	+	−

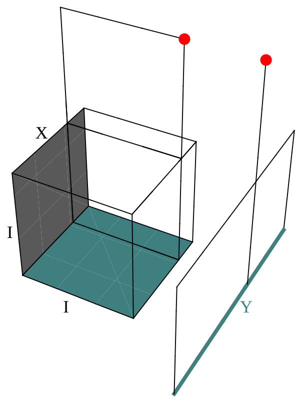
Here

- $(\mathbb{N}_\infty, \infty) \cong (\{\frac{1}{n} : 0 \neq n \in \mathbb{N}\} \cup \{0\}, 0)$,
- $E := \mathbb{N}_\infty \times I \cup [0, +\infty] \times \{0\}$ is the infinite comb,
- and $L := \{\infty\} \times I \cup [0, +\infty] \times \{0\} \subseteq E$.

2.43 Definition. [20, 2.4.10] The MAPPING CYLINDER M_f of a mapping $f : X \to Y$ is given by $Y \cup_f (X \times I)$, where f is considered as mapping $X \times \{1\} \cong X \to Y$.

We have the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i", "\text{HEP}"'] & & Y \arrow[dl, hook, "\text{SDR}", shift left] \\ & M_f \arrow[ur, "r", shift left] & \end{tikzcd}$$



where $f = r \circ i$ and i is a closed embedding with HEP and $Y \to M_f$ a SDR (along the generators $X \times I$) with retraction r (by 2.36.7 and 2.37). To see the HEP, construct a retraction $M_f \times I \to M_f \times \{0\} \cup X \times I$ by projecting radially in the plane $\{x\} \times I \times I$ from $\{x\} \times \{1\} \times \{2\}$ and use 2.30.

2.44 Corollary. [20, 2.4.12] *Two spaces are homotopy equivalent iff there exists a third one which contains both as SDRs.*

Proof. ($\Rightarrow$) Use the mapping cylinder as third space. Since f is a homotopy equivalence, so is $i : X \to M_f$ by 2.43 and by the HEP it is an SDR by 2.35.

($\Leftarrow$) Use that SDRs are by 2.35 always homotopy equivalences. □

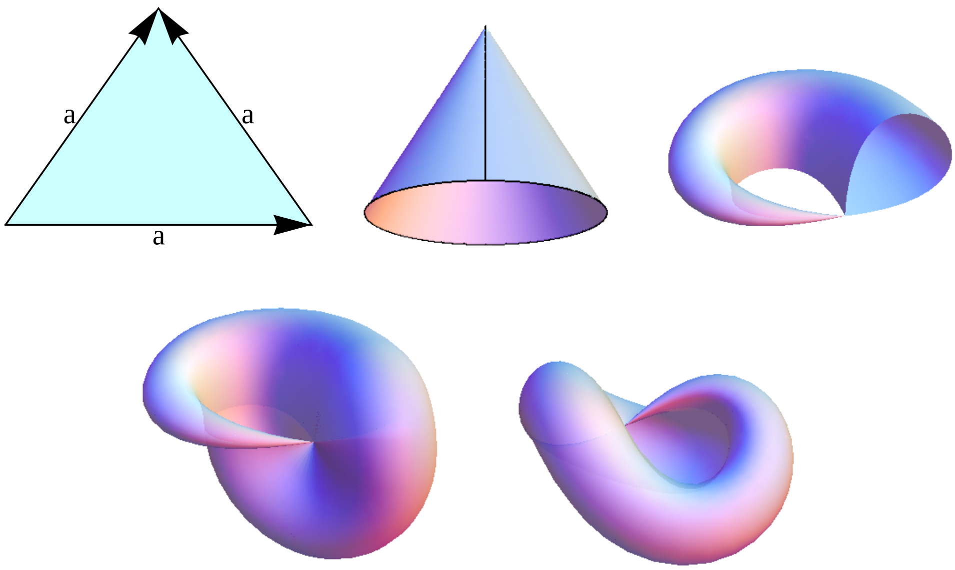
2.45 Proposition. [20, 2.4.13] *Assume (X, A) has HEP and $f_j : X \supseteq A \to Y$ are homotopic. Then $Y \cup_{f_0} X$ and $Y \cup_{f_1} X$ are homotopy equivalent rel. Y.*

Proof.
Consider the homotopy $f : A \times I \to Y$ and the space $Z := Y \cup_f (X \times I)$. We show that $Y \cup_{f_j} X$ are SDRs of Z and hence are homotopy equivalent by 2.44.

Here we use that if the composite of push-out and a commuting square is a push-out then so is the second square, cf. 1.47. $\square$

$$\begin{tikzcd}
A \arrow[rrr, "f_j"] \arrow[dr, hook, "\mathrm{ins}_j"] \arrow[dd, hook] & & & Y \arrow[dd] \\
& A \times I \arrow[urr, "f"'] \arrow[dd, hook] & & \\
X \arrow[rrr] \arrow[dr, hook] & & \arrow[ur, phantom, "\mathrm{PO}"] & Y \cup_{f_j} X \arrow[d, "\mathrm{SDR}"', "\boxed{2.37}"] \\
& L \arrow[urr] \arrow[d, hook, "\mathrm{SDR}", "\boxed{2.40.2}"'] \arrow[ul, phantom, "\mathrm{PO}"] & \arrow[r, phantom, "\mathrm{PO}"] & Z \\
& X \times I \arrow[urr, "f"'] & &
\end{tikzcd}$$

2.46 Example. [20, 2.4.14] The dunce hat D, i.e. a triangle with sides a, a, a^{-1} identified, is contractible:
By 1.92, 2.31.1, 2.45, and 1.93.3 we have $D \cong S^1 \cup_f D^2 \sim S^1 \cup_{\mathrm{id}} D^2 \cong D^2$.



2.47 Proposition. [20, 2.4.15] *Let A be contractible and let (X, A) have the HEP. Then the projection $X \twoheadrightarrow X/A$ is a homotopy equivalence.*

Proof. Consider

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f_t"'] & X \arrow[r, two heads] \arrow[d, dotted, "F_t"'] & X/A \arrow[d, dotted, "\tilde F_t"] \arrow[dl, dotted, "\tilde R"'] \\
A \arrow[r, hook] & X \arrow[r, two heads] & X/A
\end{tikzcd}$$

Then $\tilde R$, being given by factoring F_1 (since $F_1(A) = \{*\}$), is the desired homotopy inverse to $X \to X/A$ (since $F_0 = \mathrm{id}_A$ and $\tilde F_0 = \mathrm{id}_{X/A}$). $\square$

3. Simplicial Complexes

In this chapter we consider topological spaces (the so-called polyhedra) which can be treated by combinatorial methods (so called simplicial complexes) and we will prove homotopy properties for them.

Basic concepts

3.1 Remark (Points in general position). [20, 3.1.1] A finite set of points $x_0, \dots, x_q$ in $\mathbb{R}^n$ is said to be IN GENERAL POSITION if one of the following equivalent conditions is satisfied:

1. The affine subspace $\{\sum_{i=0}^q \lambda_i x_i : \sum_{i=0}^q \lambda_i = 1\}$ generated by the x_i has dimension q;
2. No strict subset of $\{x_0, \dots, x_q\}$ generates the same affine subspace;
3. The vectors $x_i - x_0$ for $i > 0$ are linear independent;
4. The representation $\sum_{i=0}^q \lambda_i x_i$ with $\sum_{i=0}^q \lambda_i = 1$ is unique.

These statements are equivalent, since

$$\sum_i \lambda_i x_i = (1 - \sum_{i\neq 0} \lambda_i) x_0 + \sum_{i\neq 0} \lambda_i x_i = x_0 + \sum_{i\neq 0} \lambda_i (x_i - x_0)$$

So

$$\Big\{\sum_i \lambda_i x_i : \sum_i \lambda_i = 1\Big\} = x_0 + \Big\{\sum_{i\neq 0} \lambda_i (x_i - x_0)\Big\}$$

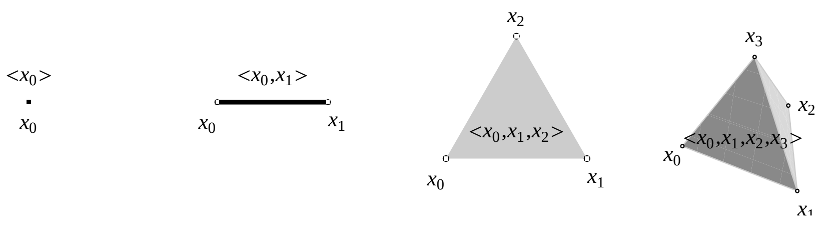
3.2 Definition (Simplex). [20, 3.1.2] The SIMPLEX of dimension q (or short: q-simplex) generated by points $x_0, \dots, x_q$ in general position is the set

$$\sigma := \langle x_0, \dots, x_q \rangle := \Big\{\sum_i \lambda_i x_i : \sum_i \lambda_i = 1, \forall i : \lambda_i > 0\Big\}$$

Its closure in $\mathbb{R}^n$ is the convex hull

$$\bar\sigma := \Big\{\sum_i \lambda_i x_i : \sum_i \lambda_i = 1, \forall i : \lambda_i \geq 0\Big\}.$$

The points x_i are called the VERTICES of σ. Note that as extremal points of $\bar\sigma$ they are uniquely determined. The set $\dot\sigma := \bar\sigma \setminus \sigma$ is called boundary of σ.



3.3 Lemma. [20, 3.1.3] *Let σ be a q-simplex. Then $(\bar\sigma, \dot\sigma) \cong (D^q, S^{q-1})$.*

Proof. Use 1.10 for the affine subspace generated by σ. □

3.4 Definition (Faces). [20, 3.1.4] Let σ and τ be simplices in $\mathbb{R}^n$. Then τ is called FACE of σ (and we write $\tau \leq \sigma$) iff the vertices of τ form a subset of those of σ.

3.5 Remark. [20, 3.1.5]

(1) Every q-simplex has $2^{q+1}-1$ many faces and it has $\binom{q+1}{p+1}$ many faces of dimension p: In fact this is the number of none-void subsets (of cardinality $p+1$) of $\{x_0,\dots,x_q\}$.

(2) *The relation of being a face is transitive.*

(3) *The closure of a simplex σ is the disjoint union of all its faces $\overline{\sigma} = \bigcup_{\tau\le\sigma}\tau$:* Remove all summands $\lambda_i x_i$ in $x=\sum_i \lambda_i x_i$ for which $\lambda_i = 0$ to get the face containing x.

3.6 Definition (Simplicial Complex). [20, 3.1.6] A SIMPLICIAL COMPLEX K is a finite set of simplices in some $\mathbb{R}^n$ with the following properties:

1. $\sigma\in K,\ \tau\le\sigma \Rightarrow \tau\in K$.

2. $\sigma,\tau\in K,\ \sigma\ne\tau \Rightarrow \sigma\cap\tau=\emptyset$.

The 0-simplices $\{x_0\}$ (or their elements x_0) are called VERTICES and the 1-simplices are called EDGES of K.
The number $\max\{\dim\sigma : \sigma\in K\}$ is called DIMENSION of K.

3.7 Definition (Triangulation). [20, 3.1.7] For a simplicial complex K the subspace $|K| := \bigcup_{\sigma\in K}\sigma$ is called the UNDERLYING TOPOLOGICAL SPACE. Every space which is homeomorphic to the underlying space of a simplicial complex is called POLYHEDRA. A corresponding simplicial complex is called a TRIANGULATION of the space.

3.8 Remark. [20, 3.1.8] *We have $|K| = \bigcup_{\sigma\in K}\bar\sigma$ and hence every polyhedra is compact and metrizable:*

$$\bigcup_{\sigma\in K}\bar\sigma \overset{\boxed{3.5.3}}{=\!=} \bigcup_{\sigma\in K}\bigcup_{\tau\le\sigma}\tau \overset{\boxed{3.6.1}}{=\!=} \bigcup_{\tau\in K}\tau = |K|.$$

Moreover, *$\bar\sigma\cap\bar\tau$ is a either empty or the closure of a common face:*

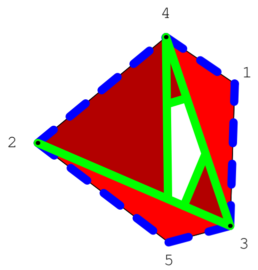
$$x\in\bar\sigma\cap\bar\tau \overset{\boxed{3.5.3}}{\Longrightarrow} x\in\sigma_1\le\sigma, x\in\tau_1\le\tau \overset{\boxed{3.6.2}}{\Longrightarrow} \sigma_1=\tau_1\subseteq\bar\sigma\cap\bar\tau$$

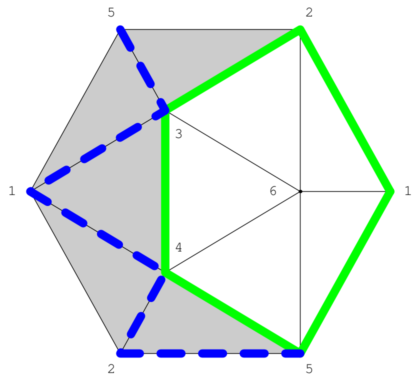
and each closed convex subset of $\bar\sigma$ (like $\bar\sigma\cap\bar\tau$) which consists only of whole faces has to be the closure of some face, namely the simplex generated by all vertices of these faces.

3.9 Remarks. [20, 3.1.9]

1. Regular polyhedra are triangulations of a 2-sphere.

2. *There is a triangulation of the Möbius strip by 5 triangles.*





3. *There is a (minimal) triangulation of the projective plane by 10 triangles*: Add the cone with the boundary of the Möbiusstrip as base and an apex not in $\mathbb{R}^3$.
4. One can show, that every compact surface, every compact 3-dimensional manifold and every compact differentiable manifold has a triangulation.
5. It is not known whether every compact manifold has a triangulation.
6. Every ball (and every sphere) has the n-simplex together with all its faces as a triangulation.
7. A countable union of circles tangent at some point is not a polyhedra, since it needs infinite many 1-simplices for a decomposition.

3.10 Definition (Carrier Simplex). [20, 3.1.10] For every $x \in |K|$ exists a unique simplex $\sigma \in K$ with $x \in \sigma$ by 3.6.2. It is called the CARRIER SIMPLEX of x and denoted $\operatorname{carr}_K(x)$.

3.11 Lemma. [20, 3.1.11] *Every point $x \in |K|$ has a unique representation $x = \sum_i \lambda_i x_i$, with $\sum_i \lambda_i = 1$ and $\lambda_i > 0$ and vertices $\{x_i\}$ of K. The x_i are the vertices of the carrier simplex $\operatorname{carr}_K(x)$ of x.*

Conversely, any point $x = \sum_i \lambda_i x_i$, with $\sum_i \lambda_i = 1$ and $\lambda_i > 0$ and such that the set of those x_i generate a simplex $\sigma \in K$, belongs to $|K|$. □

3.12 Definition. [20, 3.1.12] A SUBCOMPLEX is a subset $L \subseteq K$, that is itself a simplicial complex. This is exactly the case if $\tau \leq \sigma \in L \Rightarrow \tau \in L$ (condition 3.6.2 is obvious).

3.13 Lemma. [20, 3.1.13] *A subset $L \subseteq K$ is a subcomplex iff $|L|$ is closed in $|K|$.*

Proof. ($\Rightarrow$) since $|L|$ is compact by 3.8.
($\Leftarrow$) $\tau \leq \sigma \in L \Rightarrow \tau \subseteq \bar\sigma \subseteq |L| = \bigcup_{\rho \in L} \rho \Rightarrow \exists \rho : \tau = \rho \in L$, by 3.5.3 and then 3.6.2. □

3.14 Definition (Components of a Complex). [20, 3.1.14] Two simplices σ and τ are called CONNECTIBLE in K iff there are simplices $\sigma_0 = \sigma, \dots, \sigma_r = \tau$ with $\bar\sigma_j \cap \bar\sigma_{j+1} \neq \emptyset$. The equivalence classes with respect to being connectible are called the COMPONENTS of K. If there is only one component then K is called CONNECTED.

3.15 Lemma. [20, 3.1.15] *The components of K are subcomplexes and their underlying spaces are the path-components (connected components) of $|K|$.*

Proof. Since $\overline{\sigma}$ is a closed convex subset of some $\mathbb{R}^n$, it is path-connected and hence the underlying subspace of a component is (path-)connected. Conversely, if two simplices σ and τ belong to the same path-component of the underlying space, then there is a curve c connecting σ with τ. This curve meets finitely many simplices $\sigma_0 = \sigma, \dots, \sigma_N = \tau$ and we may assume that it meets σ_i before σ_j for $i < j$. By induction we show that all $\overline{\sigma_i}$ belong to the same component of K. In fact if $\sigma_0, \dots, \sigma_{i-1}$ does so, then let $t_0 := \min\{t \in [0,1] : c(t) \in \overline{\sigma_i}\}$. Then $c(t) \in \bigcup_{j<i} \sigma_j$ for $t < t_0$ and hence $c(t_0) \in \bigcup_{j<i} \overline{\sigma_j} \cap \overline{\sigma_i}$. Thus $\overline{\sigma_i} \cap \overline{\sigma_j} \neq \emptyset$ for some $j < i$. □

3.16 Definition (Simplicial Mapping). [20, 3.1.16] A mapping $\varphi : K \to L$ between simplicial complexes is called SIMPLICIAL MAPPING iff

1. It maps vertices to vertices (and we write $\varphi(\{x\}) =: \{\varphi(x)\}$); And
2. If σ is generated by vertices $x_0, \dots, x_q$ then $\varphi(\sigma)$ is generated by the vertices $\varphi(x_0), \dots, \varphi(x_q)$, i.e. $\varphi(\langle x_0, \dots, x_q\rangle) = \langle\{\varphi(x_i) : 0 \le i \le q\}\rangle$.
   Attention: It is not assumed, that the $\varphi(x_i)$ are pairwise distinct, so we need to consider simplices generated by a finite **set** of vertices.

3.17 Lemma. [20, 3.1.17]

1. *A simplicial mapping is uniquely determined by its action on the vertices.*
2. *If $\sigma \le \tau \in K$ then $\varphi(\sigma) \le \varphi(\tau) \in L$.*
3. $\dim(\varphi(\sigma)) \le \dim \sigma$.

Proof. This follows immediately, since $\varphi(\langle x_0, \dots, x_q\rangle) = \langle\{\varphi(x_i) : 0 \le i \le q\}\rangle$. □

3.18 Definition (Underlying continuous Mapping). [20, 3.1.18] Let $\varphi : K \to L$ be a simplicial mapping. Then, by 3.11,

$$|\varphi|\Big(\sum_i \lambda_i x_i\Big) := \sum_i \lambda_i \varphi(x_i) \text{ for } x_i \in K,\ \sum_i \lambda_i = 1 \text{ and } \lambda_i \ge 0$$

describes a welldefined continuous mapping $|\varphi| : |K| \to |L|$ (which is affine on every closed simplex $\bar{\sigma}$).

3.19 Remark. [20, 3.1.19] There are only finitely many simplicial mappings from K to L. For every simplicial mapping φ the mapping $|\varphi|$ is not dimension increasing.

3.20 Lemma. [20, 3.1.21]

1. *A mapping $\varphi : K \to L$ is a simplicial isomorphism (i.e. has an inverse, which is simplicial) iff it is simplicial and bijective.*
2. *For every simplicial isomorphism φ the mapping $|\varphi|$ is a homeomorphism.*

Proof. (1, $\Leftarrow$) We have to show that the inverse of a bijective simplicial mapping is simplicial.

Let $\xi = \{x\}$ be a vertex of L and $\varphi(\sigma) = \xi$. We have to show that σ is a 0-simplex. Let $x_0, \dots, x_q$ be the vertices of σ. By 3.16.2 the $\varphi(x_0), \dots, \varphi(x_q)$ generate the simplex $\xi = \varphi(\sigma)$ and hence have to be equal to the single vertex x of ξ. Since φ is injective $q = 0$ and $\sigma = \{x_0\}$. Hence φ is bijective on the vertices.

Now let $\tau = \varphi(\sigma)$ be a simplex in L with vertices $y_0, \dots, y_q$. Let $x_0, \dots, x_p$ be the vertices of σ. Since φ is injective and simplicial the images $\varphi(x_0), \dots, \varphi(x_p)$ are distinct and generate the simplex $\varphi(\sigma)$ by 3.16.2, hence are exactly the vertices $y_0, \dots, y_q$ of τ. Thus $p = q$ and w.l.o.g. $\varphi(x_j) = y_j$ for all j. So σ is generated by the $\varphi^{-1}(y_j) = x_j$. $\square$

Simplicial approximation

3.21 Definition (Simplicial Approximation). [20, 3.2.4] Let K and L be two simplicial complexes, $f : |K| \to |L|$ be continuous. Then a simplicial mapping $\varphi : K \to L$ is called SIMPLICIAL APPROXIMATION for f iff for all $x \in |K|$ we have $|\varphi|(x) \in \overline{\operatorname{carr}_L(f(x))}$, i.e. $f(x) \in \sigma \in L \Rightarrow |\varphi|(x) \in \bar{\sigma}$. This can be expressed shortly by $\forall \sigma \in L : |\varphi|(f^{-1}(\sigma)) \subseteq \bar{\sigma}$. In particular, for every $x \in |K|$ there is then a simplex $\sigma \in L$ (namely $\sigma := \operatorname{carr}_L(f(x))$) with $f(x), |\varphi|(x) \in \bar{\sigma}$. Note that $|\varphi|(\bar{\sigma}) = \overline{\varphi(\sigma)}$.

3.22 Lemma. [20, 3.2.5] *Let φ be a simplicial approximation of f, then $|\varphi| \sim f$.*

Proof. Connect $|\varphi|(x)$ to $f(x)$ by the segment in $\overline{\operatorname{carr}_L f(x)}$. $\square$

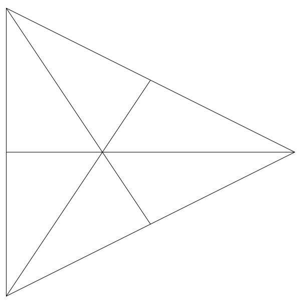
3.23 Example. [20, 3.2.6]

1. Let $K := \dot{\sigma}^2$. Then $X := |K| \cong S^1$. If $\varphi : K \to K$ is simplicial, then either φ is bijective or not surjective, so $|\varphi|$ has degree in $\{\pm 1, 0\}$ by 2.16.2 and 2.6.7. Thus every continuous map $f : X \to X$ with $|\deg(f)| > 1$ has no simplicial approximation.
2. For $f : t \mapsto 4t(1-t)$ from $[0,1] \to [0,1]$ there is no simplicial approximation $\varphi : K \to K := \{\langle 0 \rangle, \langle 1 \rangle, \langle 0,1 \rangle\}$: In fact, $\operatorname{carr}(f(j)) = \{j\}$ for $j \in \{0,1\}$ and $\operatorname{carr}(f(\frac{1}{2})) = \{1\}$, hence any such φ must satisfy $\varphi(0) = \varphi(1) = 0$ and thus $|\varphi|(\frac{1}{2}) = 0 \notin \overline{\{1\}}$.

In order to get simplicial approximations we have to refine the triangulation of $|K|$. This can be done with the following barycentric refinement.

3.24 Definition (Barycentric Refinement). [20, 3.2.1] The BARYCENTER $\hat{\sigma}$ of a q-simplex σ with vertices x_i is given by

$$\hat{\sigma} = \frac{1}{q+1} \sum_{i=0}^{q} x_i.$$



For every simplicial complex K the BARYCENTRIC REFINEMENT K' is given by all simplices having as vertices the barycenter of strictly increasing sequences of faces of a simplex in K, i.e.

$$K' := \{\langle \hat{\sigma}_0, \dots, \hat{\sigma}_q \rangle : \sigma_0 < \dots < \sigma_q \in K\}.$$

3.25 Theorem. [20, 3.2.2] *For every simplicial complex K the barycentric refinement K' is a simplicial complex of the same dimension* $\dim K$ *and the same underlying space but with* $\max\{d(\sigma') : \sigma' \in K'\} \le \frac{\dim K}{1+\dim K} \max\{d(\sigma) : \sigma \in K\}$. *Here $d(\sigma) := \sup\{|x-y| : x, y \in \sigma\}$ denotes the diameter of σ.*

Proof. If $\sigma_0 < \cdots < \sigma_q$, then their barycenters $\hat\sigma_0, \dots, \hat\sigma_q$ all lie in $\bar\sigma_q$ and are in general position: In fact, let $\sigma_i = \langle x_0, \dots, x_{n_i}\rangle$ with $i \mapsto n_i$ strictly increasing and

$$x = \sum_{i=0}^{q} \lambda_i \hat\sigma_i = \sum_i \lambda_i \frac{1}{n_i+1} \sum_{j=0}^{n_i} x_j = \sum_j x_j \underbrace{\sum_{\substack{i\\ n_i \ge j}} \lambda_i \frac{1}{n_i+1}}_{=:\mu_j} \quad \text{with} \quad \sum_i \lambda_i = 1.$$

Then

$$\sum_j \mu_j = \sum_j \sum_{\substack{i\\ n_i\ge j}} \lambda_i \frac{1}{n_i+1} = \sum_i \sum_{\substack{j\\ n_i \ge j}} \lambda_i \frac{1}{n_i+1} = \sum_i \lambda_i = 1.$$

Since the x_i are in general position the μ_j are uniquely determined and thus also the $\lambda_i = (n_i+1)\Big(\mu_{n_i} - \sum_{i'>i} \lambda_{i'} \frac{1}{n_{i'}+1}\Big)$.

We show now by induction on $q := \dim(\sigma)$ that for $\sigma \in K$ the set $\{\sigma' \in K' : \sigma' \subseteq \sigma\}$ is a disjoint partition of σ: For ($q = 0$) this is obvious. For ($q > 0$) and $x \in \sigma \setminus \{\hat\sigma\}$ the half-line from $\hat\sigma$ through x meets $\dot\sigma$ in some point y_x. By induction hypothesis $\exists! \tau' \in K' : y_x \in \tau'$. Thus y_x is a positive convex combination of $\hat\tau_0, \dots, \hat\tau_j$ with $\tau_0 < \cdots < \tau_j$. Hence x is a positive convex combination of $\hat\tau_0, \dots, \hat\tau_j, \hat\sigma$.

Finally, let $x' \ne y'$ be two vertices of some $\sigma' \in K'$, i.e. $x' = \frac{1}{r+1}(x_0 + \cdots + x_r)$ and $y' = \frac{1}{s+1}(x_0 + \cdots + x_s)$ with $r < s \le q \le \dim K$ for some simplex $\sigma = \langle x_0, \dots, x_q\rangle \in K$. Then

$$|x' - y'| \le \frac{1}{r+1} \sum_i |x_i - y'| \le \max\{|x_i - y'| : i\}$$
$$|x_i - y'| \le \frac{1}{s+1} \sum_{j\ne i} |x_i - x_j| \le \frac{s}{1+s} d(\sigma) \le \frac{\dim K}{1+\dim K} d(\sigma). \quad \square$$

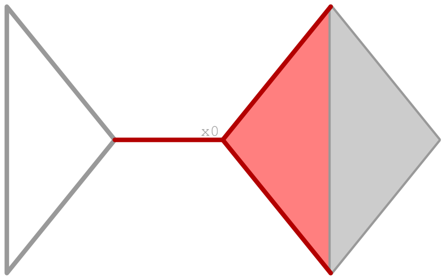
3.26 Corollary. [20, 3.2.3] *For every simplicial complex K and every $\varepsilon > 0$ there is an iterated barycentric refinement $K^{(q)}$ (for some $q \in \mathbb{N}$) with $d(\sigma) < \varepsilon$ for all $\sigma \in K^{(q)}$.*

Proof. $\left(\frac{\dim K}{1+\dim K}\right)^q \to 0$ for $q \to \infty$. $\square$

3.27 Definition. Star of a Vertex. [20, 3.2.8] Let $\xi = \{x\}$ be a vertex of K. Then the STAR of ξ in K is defined as

$$\operatorname{st}_K(\xi) := \bigcup_{\xi \le \sigma \in K} \sigma = \Big\{ y \in |K| : x \in \overline{\operatorname{carr}_K(y)} \Big\},$$

i.e. $y \in \operatorname{st}_K(\xi) \Leftrightarrow \exists(!)\sigma : y \in \sigma$ and $\xi \le \sigma \Leftrightarrow \xi = \{x\} \le \operatorname{carr}_K(y) \Leftrightarrow x \in \overline{\operatorname{carr}_K(y)}$.



3.28 Lemma. [20, 3.2.9] *The family of stars of vertices of K forms an open covering of $|K|$. For every open covering $\mathcal{U}$ of $|K|$ there is a refinement by the stars of some iterated barycentric refinement $K^{(q)}$ of K.*

Proof. For vertices $\xi = \{x\}$ of K let $K_x := \{\sigma \in K : x \text{ is not vertex of } \sigma\}$. Then K_x is a subcomplex of K and hence $\mathrm{st}_K(\xi) = |K| \setminus |K_x|$ is open in $|K|$.

If $\sigma \in K$ and x is any vertex of σ then obviously $\sigma \subseteq \mathrm{st}_K(\{x\})$ and hence the stars form a covering.

By the Lebesgue-covering lemma (see [6, 3.3.3] or [7, 5.1.5]) applied to the compact metric space $|K|$, there is a $\delta > 0$ such that each set of diameter less then δ is contained in some $U \in \mathcal{U}$. Choose by 3.25 a barycentric refinement $K^{(q)}$, such that $d(\sigma) < \frac{\delta}{2}$ for all $\sigma \in K^{(q)}$. For every $y \in \mathrm{st}_{K^{(q)}}(\{x\})$ we have $d(y,x) \leq \max\{d(\sigma) : \sigma\}$ hence $d(\mathrm{st}_{K^{(q)}}(\{x\})) \leq 2 \max\{d(\sigma) : \sigma\} < \delta$, and thus the stars form a refinement of $\mathcal{U}$. □

3.29 Corollary. Simplicial Approximation. [20, 3.2.7] *For every continuous map $f : |K| \to |L|$ there is a simplicial approximation $\varphi : K^{(q)} \to L$ of f for some iterated barycentric refinement $K^{(q)}$.*

Proof. Let q be chosen so large, that by 3.28 the stars of $K^{(q)}$ form a refinement of the open covering $\{f^{-1}(\mathrm{st}_L(\{y\})) : \{y\} \in L\}$. For sake of simplicity we write K instead of $K^{(q)}$. Thus for every vertex $\xi \in K$ we may choose a vertex denoted $\varphi(\xi) \in L$ with $f(\mathrm{st}_K(\xi)) \subseteq \mathrm{st}_L(\varphi(\xi))$. For $\sigma \in K$ with vertices $x_0, \dots, x_p$ define $\varphi(\sigma)$ to be the simplex generated by the $\varphi(\{x_i\})$. In order to see that φ is a simplicial mapping, we have to show that this simplex belongs to L. Let $x \in \sigma$ be any point in σ. Since $\sigma \subseteq \bigcap_i \mathrm{st}_K(\{x_i\})$ we get $f(x) \in f(\sigma) \subseteq f(\bigcap_i \mathrm{st}_K(\{x_i\})) \subseteq \bigcap_i f(\mathrm{st}_K(\{x_i\})) \subseteq \bigcap_i \mathrm{st}_L(\varphi(\{x_i\}))$. Thus $f(x) \in \mathrm{st}_L(\varphi(\{x_i\}))$, i.e. $\varphi(\{x_i\}) \leq \mathrm{carr}_L(f(x)) =: \tau \in L$, for all i. Hence $|\varphi|(x) \in \varphi(\sigma) := \langle \varphi(\{x_0\}), \dots, \varphi(\{x_p\}) \rangle \leq \tau \in L$ and φ is a simplicial approximation of f. □

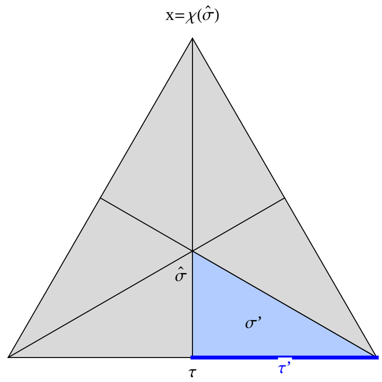
3.30 Corollary. [20, 3.2.10] *Let X and Y be polyhedra. Then $[X,Y]$ is countable.* □

3.31 Remark. [20, 3.2.11]

We obtain a simplicial approximation $\chi : K' \to K$ of $\mathrm{id} : |K'| \to |K|$ by choosing for every vertex $\hat\sigma \in K'$ a vertex $\chi(\hat\sigma)$ of σ:
Let $\hat\sigma_0, \dots, \hat\sigma_p$ be the vertices of some simplex $\sigma' \in K'$ with $\sigma_0 < \dots < \sigma_p$ and hence $\sigma' \subseteq \sigma_p$. Then the $\chi(\hat\sigma_j)$ are vertices of $\sigma_j \leq \sigma_p$ and hence they generate a face of $\sigma_p \in K$. Thus χ extends to a simplicial mapping.
Let $x \in \sigma'$. Then $|\chi|(x) \in \chi(\sigma') \subseteq \overline{\sigma_p} = \overline{\mathrm{carr}_K(x)}$, hence χ is a simplicial approximation of id.

Let σ be any q-simplex of K. Then there exists a unique simplex $\sigma' \subseteq \sigma$ which is mapped by χ to σ and all other $\sigma' \subseteq \sigma$ are mapped to true faces of σ.

Proof. We use induction on q. For $q = 0$ this is obvious, since χ is the identity on $\hat\sigma = \sigma$. If $q > 0$ and $x := \chi(\hat\sigma)$ let τ be the face of σ opposite to x. By induction hypothesis there is a unique $\tau' \subseteq \tau$ of K' which is mapped to τ. But then the simplex σ' generated by τ' and $\hat\sigma$ is the unique simplex mapped to σ: In fact, any simplex $\langle\hat\sigma_0, \dots, \hat\sigma_r\rangle \subseteq \sigma$ that is mapped via χ to σ has to satisfy $r \geq \dim\sigma$ and $\sigma_0 < \dots < \sigma_r = \sigma$, hence $r = \dim(\sigma)$. Since $\chi(\hat\sigma) = x$ we have that $\chi(\hat\sigma_0), \dots, \chi(\hat\sigma_{r-1})$ generate τ and thus τ' is the simplex with vertices $\hat\sigma_0, \dots, \hat\sigma_{r-1}$ by induction hypothesis. □



Freeing by deformations

3.32 Proposition. [20, 3.3.2] *Let K be a simplicial complex and $\dim K < n$. Then every continuous $f : |K| \to S^n$ is 0-homotopic. In particular, this is true for $K := \dot\sigma^{k+1}$ with $\dim K = k < n$.*

Proof. By 3.29 there exists a simplicial approximation φ of $f : |K| \to S^n = |\dot\sigma^{n+1}|$ for some iterated barycentric refinement of K. Then $|\varphi| : |K| \to S^n$ cannot be surjective (since $\dim K < n$) and hence $f \sim |\varphi|$ is 0-homotopic since $S^n \setminus \{*\}$ is contractible. □

3.33 Theorem. Freeing of a point. [20, 3.3.3] *Let (K, L) be a simplicial pair and e^n be an n-cell with $\dim K < n$. Then every $f_0 : (|K|, |L|) \to (e^n, e^n \setminus \{0\})$ is homotopic relative $|L|$ to a mapping $f_1 : |K| \to e^n \setminus \{0\}$.*

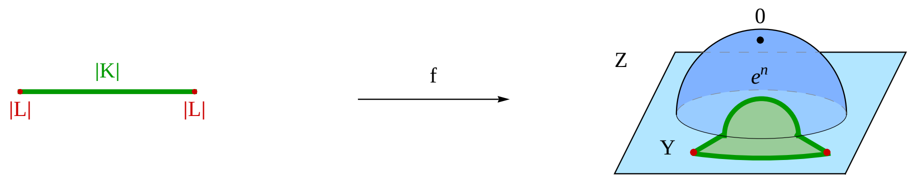
Proof. We first show this result for $(|K|, |L|) = (D^k, S^{k-1})$. By 2.36.6 we have $e^n \setminus \{0\} \sim S^{n-1}$. Hence $f_0|_{S^{k-1}} : S^{k-1} \to e^n \setminus \{0\}$ is 0-homotopic by 3.32. By 2.26 this homotopy gives an extension $f_1 : D^k = C(S^{k-1}) \to e^n \setminus \{0\}$. Consider a mapping $h : (D^k \times I)^{\cdot} \to e^n$ which is f_1 on the top, and is f_0 on the bottom and on $S^{k-1} \times \{t\}$ for all $t \in I$. Since e^n is contractible this mapping h is 0-homotopic by 2.6.6 and hence extends to $C((D^k \times I)^{\cdot}) \cong D^k \times I$ again by 2.26. This extension is the desired homotopy.

For the general case we proceed by induction on the number of cells in $K \setminus L$. For $K = L$ the homotopy is constant f_0. So let $K \supset L$ and take $\sigma \in K \setminus L$ of maximal dimension. Then $K_\sigma := K \setminus \{\sigma\} \supseteq L$ is a simplicial complex. Obviously $|K_\sigma| \cup \bar\sigma = |K|$ and $|K_\sigma| \cap \bar\sigma = \dot\sigma$. Consider the diagram

$$\begin{tikzcd}
 & {|K|} \arrow[d, dotted, "(3)"] & \\
{|K_\sigma|} \arrow[ur, hook] \arrow[r, dotted, "(1)"] & e^n & \bar\sigma \arrow[ul, hook'] \arrow[l, dotted, "(2)"'] \\
{|L|} \arrow[u, hook] & \dot\sigma \arrow[ul, hook] \arrow[u, dotted, "(1')"] \arrow[ur, hook', "\text{HEP}"'] &
\end{tikzcd}$$

By induction hypothesis we have the required homotopy (1) relative $|L|$ on $|K_\sigma|$. Since $(\bar\sigma, \dot\sigma)$ has HEP by 2.31.1, we may extend its restriction (1') to $\dot\sigma$ to a homotopy (2) on $\bar\sigma$ with initial value f_0. The union of these two homotopies (1) and (2) gives a homotopy h_t rel. $|L|$ indicated by arrow (3) which satisfies $h_0 = f_0$ and $h_1(|K_\sigma|) \subseteq e^n \setminus \{0\}$. By the special case treated above, there is a homotopy $g_t : \bar\sigma \to e^n$ relative $\dot\sigma$ with $g_0 = h_1|_{\bar\sigma} : (\bar\sigma, \dot\sigma) \to (e^n, e^n \setminus \{0\})$ and $g_1(\bar\sigma) \subseteq e^n \setminus \{0\}$. Let $f_1 := h_1|_{|K_\sigma|} \cup g_1$. Then $f_1(|K|) \subseteq e^n \setminus \{0\}$ and $f_0 = h_0 \sim h_1 = h_1|_{|K_\sigma|} \cup g_0 \sim h_1|_{|K_\sigma|} \cup g_1 = f_1$ relative $|L|$. $\square$

3.34 Theorem. Freeing of a cell. [20, 3.3.4] *Let (K, L) be a simplicial pair and let Z be obtained from gluing an n-cell e^n to a space Y and $\dim K < n$. Then every $f : (|K|, |L|) \to (Z, Y)$ is homotopic relative $|L|$ to a mapping $f_1 : |K| \to Y$.*

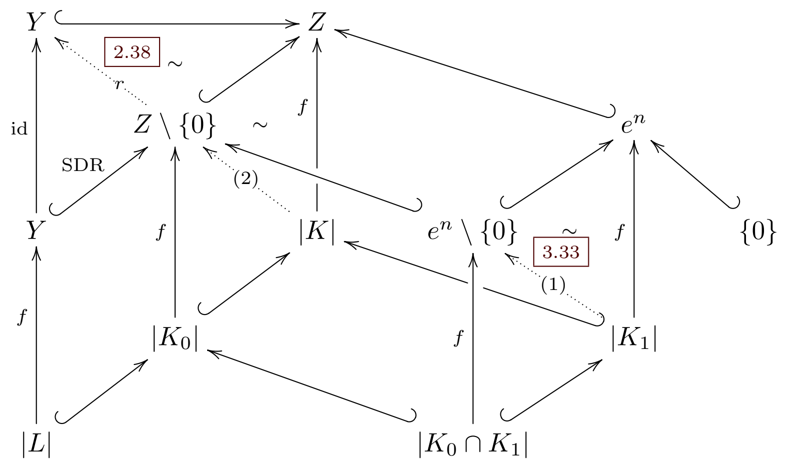


Proof. For $0 \in e^n \subseteq Z$ we consider the subcomplexes

$$K_0 := \Big\{\sigma \in K : f(\bar\sigma) \subseteq Z \setminus \{0\}\Big\} = \Big\{\sigma \in K : \bar\sigma \subseteq f^{-1}(Z \setminus \{0\})\Big\} \supseteq L \text{ and}$$
$$K_1 := \Big\{\sigma \in K : f(\bar\sigma) \subseteq e^n\Big\} = \Big\{\sigma \in K : \bar\sigma \subseteq f^{-1}(e^n)\Big\}$$

By passing to an appropriate iteration (again denoted K) of barycentric subdivisions, we may assume that $K = K_0 \cup K_1$ by 3.26.

Now consider the diagram



By 3.33 there exists a mapping (1) homotopic to $f|_{|K_1|}$ relative $|K_1 \cap K_0|$. Gluing the homotopy with the $f|_{|K_0|}$ gives a homotopy relative $|K_0|$ to a mapping (2). Composing with the retraction r (homotopic to id relative Y) from 2.38 gives the desired mapping $f_1 : |K| \to Y$ homotopic to f relative $|L|$. Note that the triangle on top, as those above (1) and (2) commute only up to homotopy. $\square$

4. CW-Spaces

In this chapter we will generalize the polyhedra to so-called CW-spaces, where the finiteness condition on the number of building blocks is weakend and the boundary of the cells need not be a sphere any more.

Basics

4.1 Definition. [20, 4.1.3] A CW-COMPLEX is a Hausdorff space X together with a partition $\mathcal{X}$ into cells e, such that the following properties hold:

(C1) For every n-cell $e \in \mathcal{X}$ there exists a continuous so-called CHARACTERISTIC MAP $\chi^e : D^n \to X$, which restricts to a homeomorphism from $\mathring{D}^n$ onto e and which maps S^{n-1} into the $n-1$-skeleton X^{n-1} of X, which is defined to be the union of all cells of dimension less than n in $\mathcal{X}$.

(C2) The closure $\bar{e}$ of each cell meets only finitely many cells.

(W) X carries the final topology with respect to $\bar{e}$ for all cells $e \in \mathcal{X}$.

A CW-SPACE is a Hausdorff-space X, which admits a CW-complex $\mathcal{X}$ (which is called CW-DECOMPOSITION of X).

Note that if $\mathcal{X}$ is finite ($\mathcal{X}$ is then called finite CW-complex), then the conditions $(C2)$ and (W) are automatically satisfied.

If $X = X^n \neq X^{n-1}$ then the CW-complex is said to be of dimension n. If $X \neq X^n$ for all n, then it is said to be of infinite dimension.

Note that, since the image $\chi(D^n)$ of the n-ball under a characteristic map is compact, it coincides with $\bar{e}$ and $\chi : D^n \twoheadrightarrow \bar{e}$ is a quotient mapping. So $\dot{e} := \bar{e} \setminus e = \chi(D^n) \setminus \chi(\mathring{D}^n) \subseteq \chi(D^n \setminus \mathring{D}^n) = \chi(S^{n-1})$ and conversely $\chi(S^{n-1}) \subseteq \chi(D^n) = \bar{e}$ and $\chi(S^{n-1}) \subseteq X^{n-1} \subseteq X \setminus e$, thus $\chi(S^{n-1}) = \dot{e}$ and hence in follows that: *The characteristic map χ of each $e \in \mathcal{X}$ is a relative homeomorphism $(D^n, S^{n-1}) \to (\bar{e}, \dot{e})$.*

$$\begin{tikzcd}
\mathring{D}^n \arrow[r, hook] \arrow[d, "\cong"] & D^n \arrow[d, two heads, "\chi^e"] & S^{n-1} \arrow[l, hook'] \arrow[d, two heads] \\
e \arrow[r, hook] & \bar{e} & \dot{e} \arrow[l, hook']
\end{tikzcd}$$

4.2 Example. [20, 4.1.4] *For every simplicial complex K the underlying space $|K|$ is a finite CW-complex*, the cells being the simplices of K and the characteristic maps the inclusions $\bar{e} \subseteq |K|$.

The sphere S^n is a CW-complex with one 0-cell e^0 and one n-cell e^n, in particular *the boundary $\dot{e} = \bar{e} \setminus e$ of an n-cell, needn't be a sphere* in contrast to the situation for simplicial complexes.

The one point union of spheres is a CW-space with one 0-cell and for each sphere a cell of the same dimension.

$S^1 \vee S^2$ can be made in a different way into a CW-complex by taking a point $e^0 \in S^1$ different from the base point. Then $S^1 = e^0 \cup e^1$ and $S^1 \vee S^2 = e^0 \cup e^1 \cup e^2$. But *the boundary $\dot{e}^2$ of the two-cell is not even a union of cells.*

The compact surfaces of genus g are all CW-complexes with one 0-cell and one 2-cell and $2g$ 1-cells (in the orientable case) and g 1-cells (in the non-orientable case), see 1.94.

The projective spaces $\mathbb{P}^n$ are CW-complexes with one cell of each dimension from 0 to n, see 1.95, where F is the characteristic map for the n-cell.

4.3 Definition. [20, 4.1.5] For a subset $\mathcal{Y}$ of a CW-decomposition $\mathcal{X}$ of a space X the underlying space $Y := \bigcup\{e : e \in \mathcal{Y}\}$ is called CW-SUBSPACE and $\mathcal{Y}$ is called CW-SUBCOMPLEX, iff $\mathcal{Y}$ is a CW-decomposition of Y with the trace topology. In this situation (X, Y) is called CW-PAIR.

Let us first characterize finite CW-subcomplexes:

4.4 Lemma. *Let $\mathcal{Y}$ be a finite subset of a CW-decomposition $\mathcal{X}$ of a space X. Then $\mathcal{Y}$ forms a CW-subcomplex iff $Y := \bigcup\{e : e \in \mathcal{Y}\}$ is closed. Cf. 3.13.*

Proof. ($\Rightarrow$) If $\mathcal{Y}$ is a CW-subcomplex, then for every cell $e \in \mathcal{Y}$, there is a characteristic map $\chi : D^n \twoheadrightarrow \bar{e}^Y$. Hence $\bar{e}^Y$ is compact and thus coincides with the closure of e in X, so the finite union $Y = \bigcup\{\bar{e} : e \in \mathcal{Y}\}$ is closed.

($\Leftarrow$) Since Y is closed the characteristic maps for $e \in \mathcal{Y} \subseteq \mathcal{X}$ have values in Y and hence are also characteristic maps with respect to $\mathcal{Y}$. The other properties for a CW-complex are obvious by the first remark in 4.1. □

4.5 Lemma. [20, 4.1.9] *Every compact subset of a CW-complex is contained in some finite subcomplex. In particular a CW-complex is compact iff it is finite.*

Proof. Let X be a CW-complex. We first show that the closure $\bar{e}$ of every cell is contained in a finite subcomplex using induction on the dimension of the cell. Assume this is true for all cells of dimension less than n and let e be an n-cell. By (C2) the boundary $\dot{e}$ meets only finitely many cells, each of dimension less than n. By induction hypotheses each of these cells is contained in some finite subcomplex X_i. Then union of these complexes is again a complex, by 4.4. If we add e itself to this complex, we get the desired finite complex.

Let now K be compact. For every $e \in \mathcal{X}$ with $e \cap K \neq \emptyset$ choose a point x_e in the intersection. Every subset $A \subseteq K_0 := \{x_e : e \cap K \neq \emptyset\} \subseteq K$ is closed, since it meets any $\bar{e}$ only in finitely many points by (C2). Hence K_0 is a discrete compact subset, and thus finite, i.e. K meets only finitely many cells. Since every $\bar{e}$ is contained in a finite subcomplex, we have that K is contained in the finite union of these subcomplexes.

The last statement of the lemma is now obvious. □

4.6 Corollary. *Every CW-complex carries the final topology with respect to its finite subcomplexes and also with respect to its skeletons.*

Proof. Since the closure $\bar{e}$ of every cell e is contained in a finite subcomplex by 4.5 and every finite subcomplex is contained in some skeleton X^n, these families are confinal to $\{\bar{e} : e \in \mathcal{X}\}$. Furthermore, the inclusion of each of its spaces into X is continuous (for the final topology on X induced by the $\bar{e}$ by property (W)). Hence these families induce the same topology. (Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of mappings into a space X, and assume $\mathcal{F}_2$ is confinal to $\mathcal{F}_1$, i.e. for every $f_1 \in F_1$ there is some $f_2 \in \mathcal{F}_2$ and a map h such that $f_1 = f_2 \circ h$. Let X_j denote the space X with the final topology induced by $\mathcal{F}_j$. Then the identity from $X_1 \to X_2$ is continuous, since for every $f_1 \in \mathcal{F}_1$ we can write $\mathrm{id} \circ f_1 = f_2 \circ h$) □

Now we are able to extend 4.4 to infinite subcomplexes.

4.7 Proposition. *Let $\mathcal{X}$ a CW-decomposition of X and let $\mathcal{Y}$ be a subset of $\mathcal{X}$ and $Y := \bigcup\{e : e \in \mathcal{Y}\}$. Then the following statements are equivalent:*

1. *$\mathcal{Y}$ is a CW-subcomplex of $\mathcal{X}$;*
2. *Y is closed in X;*
3. *For every cell $e \in \mathcal{Y}$ we have $\bar{e} \subseteq Y$.*

Proof. (2⇒3) is obvious.
(1⇒3) follows, since the closure $\bar{e}^Y$ in Y is compact and hence equals $\bar{e} := \bar{e}^X$.

For the converse directions we show first that (3) implies:
If $A \subseteq Y$ has closed trace on $\bar{e} := \bar{e}^X$ for each $e \in \mathcal{Y}$, then A is closed in X:
By 4.6 it suffices to show that the trace of A on every finite CW-subcomplex $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed. For each cell $e \in \mathcal{X}_0 \cap \mathcal{Y}$ we have $\bar{e} \subseteq X_0 \cap Y$ by 4.4 and (3). Hence

$$X_0 \cap A = X_0 \cap Y \cap A = \Big(\bigcup_{e \in \mathcal{X}_0 \cap \mathcal{Y}} \bar{e}\Big) \cap A = \bigcup_{e \in \mathcal{X}_0 \cap \mathcal{Y}} (\bar{e} \cap A),$$

which is closed since there are only finitely $e \in \mathcal{X}_0 \cap \mathcal{Y}$.

(3⇒2) by taking $A = Y$ in the previous claim.

(3⇒1) The previous claim shows the condition (W) for $\mathcal{Y}$. The other conditions for being a CW-complex are obvious since $\bar{e}^X = \bar{e}^Y$. □

4.8 Corollary. [20, 4.1.6] *Intersections and unions of CW-complexes are CW-complexes. Connected components and topological disjoint unions of CW-complexes are CW-complexes. If $\mathcal{E} \subseteq \mathcal{X}$ is family of n-cells, then $X^{n-1} \cup \bigcup \mathcal{E}$ is a CW-complex. Each n-cell e is open in X^n.*

Proof. For intersections this follows from (1⇔2) in 4.7. For unions this follows from (1⇔3) in 4.7. The statement on components follows, since $\bar{e}$ is connected and by 4.7 (1⇔3). For topological sums it is obvious. That $X^{n-1} \cup \bigcup \mathcal{E}$ is a CW-complex follows also from $(1 \Leftrightarrow 3)$ in 4.7. In particular, $X^n \setminus e = X^{n-1} \cup \bigcup\{e_1 \neq e : e_1 \text{ an } n\text{-cell in } X^n\}$ is a CW-space, thus it is closed by (1⇔2) in 4.7 and hence e is open in X^n. □

Further constructions of CW-spaces

4.9 Proposition. [20, 4.2.9] *Let X and Y be two CW-complexes. Then $X \times Y$ with cells $e \times f$ for $e \in \mathcal{X}$ and $f \in \mathcal{Y}$ satisfies all properties of a CW-complex, with the possible exception of (W). If X or Y is in addition locally compact, then $X \times Y$ is a CW-complex.*

Proof. Take the product of the characteristic maps in order to obtain a characteristic map for the product cell.

In order to get the property (W) we have to show that the map $\bigsqcup_{e,f} \bar{e} \times \bar{f} \to X \times Y$ is a quotient map. Since it can be rewritten as

$$\bigsqcup_e \bar{e} \times \bigsqcup_f \bar{f} \to \bigsqcup_e \bar{e} \times Y \to X \times Y$$

this follows from 1.33 using compactness of $\bar{e}$ and locally compactness of Y. □

4.10 Proposition. *Let (X, A) be a CW-pair. Then $A \cup X^n$ is obtained from $A \cup X^{n-1}$ by gluing all n-cells contained in $X^n \setminus A$ via the characteristic mappings.*

Proof. Let $\mathcal{E}$ be the set of all n-cells contained in $X \setminus A$ and let characteristic mappings $\chi^e : D^n \twoheadrightarrow \bar{e}$ for every $e \in \mathcal{E}$ be chosen. Let $\chi := \bigsqcup_{e\in\mathcal{E}} \chi^e : \bigsqcup_{e\in\mathcal{E}} D^n \twoheadrightarrow \bigcup_{e\in\mathcal{E}} \bar{e} \subseteq X^n$. We have to show that the rectangle in

$$\begin{tikzcd}
 & & S^{n-1} \arrow[r, hook] \arrow[dll, hook] & D^n \arrow[dl, hook] \arrow[ddl, "\chi^e"] \arrow[ddd, "g^e"] \\
\bigsqcup_e S^{n-1} \arrow[r, hook] \arrow[d, "\chi|_{\bigsqcup_e S^{n-1}}"'] & \bigsqcup_e D^n \arrow[d, "\chi"'] \arrow[ddrr, "\bigsqcup_e g^e"] & & \\
A \cup X^{n-1} \arrow[r, hook] \arrow[drrr, "g^{n-1}"'] & A \cup X^n \arrow[drr, dotted, "!\, g"] & & \\
 & & & Z
\end{tikzcd}$$

is a push-out. So let $g^{n-1} : A \cup X^{n-1} \to Z$ and $g^e : D^n \to Z$ be given, such that $g^{n-1} \circ \chi^e|_{S^{n-1}} = g^e|_{S^{n-1}}$. Then $g : A \cup X^n \to Z$, given by $g|_{A\cup X^{n-1}} = g^{n-1}$ and $g|_e = g^e|_{\mathring{D}^n} \circ (\chi^e|_{\mathring{D}^n})^{-1}$ for $e \in \mathcal{E}$, is the unique mapping making everything commutative. It is continuous by property (W), since on $\bar{e}$ it equals g^{n-1} if $e \subseteq A \cup X^{n-1}$ and composed with the quotient-mapping $\chi^e : D^n \to \bar{e}$ it equals g^e for the remaining e (i.e. $e \in \mathcal{E}$). $\square$

Now we give an inductive description of CW-spaces.

4.11 Theorem. [20, 4.2.2] *A topological space X is a CW-complex iff there are spaces X^n, with X^0 discrete, X^n is formed from X^{n-1} by gluing n-cells and X is the limit of the X^n with respect to the natural inclusions $X^{n-1} \hookrightarrow X^n$.*

Proof. ($\Rightarrow$) We take X^n to be the n-skeleton. Then X carries the final topology with respect to the closed subspaces X^n by 4.6 and X^0 is discrete (see the proof of 4.5). Taking $A := \emptyset$ in 4.10 we get that X^n can be obtained from X^{n-1} by gluing all the n-cells via their corresponding characteristic maps restricted to the boundary spheres.

($\Leftarrow$) We first show by induction that X^n *is a CW-complex, with $n-1$-skeleton X^{n-1} and those cells, which have been glued to X^{n-1} to obtain X^n, as n-cells*: For the discrete space X^0 this is obvious. Since X^n is obtained from X^{n-1} by gluing n-cells we have that X^n is Hausdorff by 1.90 and is as set the disjoint union of the closed subspace X^{n-1}, which is a CW-complex by induction hypothesis, and the homeomorphic image $\bigcup_e e$ of $\bigsqcup_e D^n \setminus \bigsqcup_e S^{n-1} = \bigsqcup_e \mathring{D}^n$.

$$\begin{tikzcd}
\bigsqcup_e S^{n-1} \arrow[r, hook] \arrow[d, "f"'] & \bigsqcup_e D^n \arrow[d, "p"] \\
X^{n-1} \arrow[r, hook] & X^n
\end{tikzcd}$$

As characteristic mappings for the n-cells e we may use $p|_{D^n}$, since it induces a homeomorphism $\mathring{D}^n \to e$ and it maps S^{n-1} to $f(S^{n-1}) \subseteq X^{n-1}$, which is compact and hence contained in a finite subcomplex of X^{n-1} by 4.5. The condition (W) follows, since X^n carries by construction the final topology with respect to X^{n-1} and $p : \bigsqcup D^n \to X^n$, and $\bigsqcup D^n$ carries the final topology with respect to the inclusion of the summands $D^n \subseteq \bigsqcup_e D^n$.

The inductive limit $X := \varinjlim_n X^n$ now obviously satisfies all axioms of a CW-complex – only Hausdorffness is to be checked. So let x, y be different points in X. They lie in some X^n and we find open disjoint neighborhoods U^n and V^n in X^n. We construct open disjoint neighborhoods U^k and V^k in X^k with $k \geq n$ inductively. In fact, take $U^k := U^{k-1} \cup p\Big(r^{-1}(U^{k-1})\Big)$, where $r : \bigsqcup D^k \setminus \{0\} \to X^{k-1}$ is the retraction from 2.38. Then U^k is the image of the open and saturated set $U^{k-1} \sqcup r^{-1}(U^{k-1}) \subseteq X^{k-1} \sqcup \bigsqcup D^k$ and hence open, and $U^k \cap X^{k-1} = U^{k-1}$. Proceeding the same way with V^k gives the required disjoint open sets $U := \bigcup_{k \geq n} U^k$ and $V := \bigcup_{k \geq n} V^k$. $\square$

Example. Gluing a CW-pair to a CW-space does not give a CW-space in general. Consider for example a surjective map $f : S^1 \to S^2$. Then the boundary $\dot{e} = S^1$ of $e := (D^2)^o$ is not contained in any 1-dimensional CW-complex.

So we define:

4.12 Definition. [20, 4.2.4] A continuous map $f : X \to Y$ between CW-complexes is called CELLULAR iff $f(X^n) \subseteq Y^n$ for all n.

4.13 Lemma. *Let $f : X \supseteq A \to Y$ be given and let $Y' \subseteq Y$ and $X' \subseteq X$ be two closed subspaces, such that $f(A \cap X') \subseteq Y'$. Then the canonical mapping $Y' \cup_{f'} X' \to Y \cup_f X$ is a closed embedding, where $f' := f|A'$ with $A' := A \cap X'$.*

Proof.
Consider the commutative diagram:

$$\begin{tikzcd}
A' \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "f'"'] & & X' \arrow[dr, hook] \arrow[dd, "p'|_{X'}"] & \\
& A \arrow[rr, hook] \arrow[dd, "f"] & & X \arrow[dd, "p|_X"] \\
Y' \arrow[rr, hook] \arrow[dr, hook] & & Y' \cup_{f'} X' \arrow[dr, dotted, "\iota"] & \\
& Y \arrow[rr, hook] & & Y \cup_f X
\end{tikzcd}$$

The dashed arrow ι exists by the push-out property of the back side. Since $Y' \cup_{f'} X' = Y' \sqcup p'(X' \setminus A')$ as sets, we get that ι is the inclusion $Y' \sqcup p'(X' \setminus A') \subseteq Y \sqcup p(X \setminus A)$ and hence injective.
Now let $B \subseteq Y' \cup_{f'} X'$ be closed, i.e. $B = B_1 \sqcup p'(B_2)$ with $B_1 \subseteq Y'$ closed and $B_2 \subseteq X' \setminus A'$ such that $(p'|_{X'})^{-1}(B) = (f')^{-1}(B_1) \cup B_2$ is closed in X'.

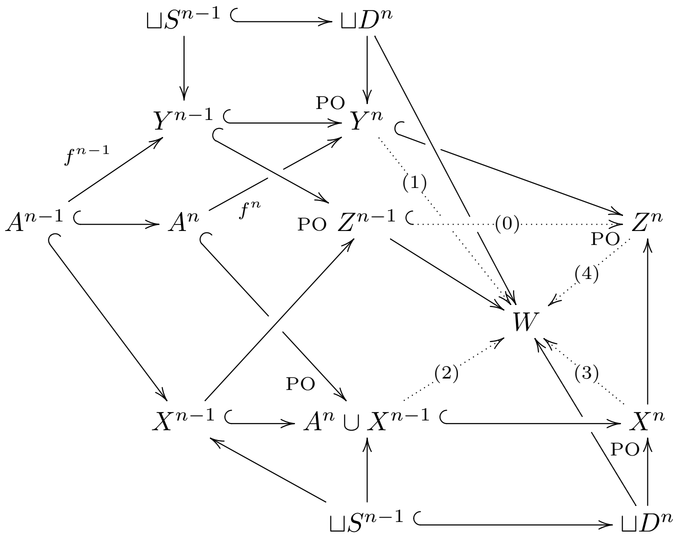
In order to show that $\iota(B) = B_1 \sqcup p(B_2) \subseteq Y' \cup p(X' \setminus A') \subseteq Y \cup p(X \setminus A)$ is closed we only have to show that $f^{-1}(B_1) \cup B_2$ is closed in X, which follows from

$$f^{-1}(B_1) \cup B_2 = \Big(f^{-1}(B_1) \cup (f')^{-1}(B_1)\Big) \cup B_2 = f^{-1}(B_1) \cup \Big((f')^{-1}(B_1) \cup B_2\Big),$$

since $(f')^{-1}(B_1) \cup B_2 \subseteq X' \subseteq X$ is closed and $f^{-1}(B_1) \subseteq A \subseteq X$ is closed. $\square$

4.14 Theorem. [20, 4.2.5] *Let (X, A) be a CW-pair and $f : A \to Y$ a cellular mapping into a CW-complex Y. Then $(Y \cup_f X, Y)$ is a CW-pair with the cells of Y and of $X \setminus A$ as cells.*

Proof. We consider the spaces $Z^n := Y^n \cup_{f_n} X^n$, where $f_n := f|_{A^n}$. Note that $A^n = A \cap X^n$. By 4.13 the Z^n form an increasing sequence of closed subspaces of the Hausdorff space $Z := Y \cup_f X$. Obviously Z^0 is discrete and Z carries the final topology induced by all Z^n. So by 4.11 it remains to show that Z^n can be obtained from Z^{n-1} by gluing all n-cells of Y^n and of $X^n \setminus A^n$. For this we consider the following commutative diagram:



By 4.13 we have the closed embedding (0) and we have to show that Z^n is the push-out of (0) and the union of the two mappings $\sqcup D^n \to Y^n \to Z^n$ and $\sqcup D^n \to X^n \to Z^n$.
So let mappings on all the D^n and on Z^{n-1} into a space W be given whose composites with the arrows from S^{n-1} into these spaces are the same.

Using the push-out property (shown in 4.10) of Y^n, $A^n \cup X^{n-1}$, X^n, and Z^n we get in succession unique maps (1), (2), (3), and (4). The map (4) is then the required unique mapping from $Z^n \to W$. □

4.15 Corollary. [20, 4.2.6] *Let (X, A) be a CW-pair with $A \neq \emptyset$. Then X/A is a CW-space with A as one 0-cell and the image of all cells in $X \setminus A$.*

Proof. $X/A = \{*\} \cup_f X$ by 1.47.1, where $f : A \to \{*\}$ is constant, Now apply 4.14. □

4.16 Corollary. [20, 4.2.8] *Let X be a CW-complex and $n \geq 1$. Then X^n/X^{n-1} is a join of spheres of dimension n, for each n-cell one.*

Proof. By 4.15 X^n/X^{n-1} is a CW-space consisting of one 0-cell and all the n-cells of X. The characteristic mappings for the n-cells of X^n/X^{n-1} into the single 0-cell X^{n-1} have to be constant and hence $X^n/X^{n-1} \cong \bigvee_e S^n$ by 1.97.3. □

4.17 Corollary. [20, 4.2.7] *Let X_i be CW-spaces with base-point $x_i \in X_i^0$. Then the join $\bigvee_i X_i$ is a CW-space.*

Proof. $\bigvee_i X_i = (\bigsqcup_i X_i)/\{x_i : i\}$ is a CW-space by 4.8 and 4.15. □

Homotopy properties

4.18 Theorem. [20, 4.3.2] *For each CW-pair (X, A) we can find a continuous function $u : X \to I$ s.t. $A = u^{-1}(0)$ and $A \hookrightarrow U(A) := u^{-1}(\{t : t < 1\})$ is an SDR. These neighborhoods can be chosen coherently, i.e. $U(A \cap B) = U(A) \cap U(B)$.*
In particular, $A \hookrightarrow X$ is an NDR hence has HEP.

Proof. Let $X^{-1} := \emptyset$. By 4.10 $A \cup X^n$ is obtained by glueing the n-cells e in $X \setminus A$ to $A \cup X^{n-1}$. By 2.38 $A \cup X^{n-1}$ is an SDR in $A \cup X^n \setminus \bigsqcup_e \{0_e\}$. Let the corresponding homotopy relative $A \cup X^{n-1}$ be denoted by h_t^n and the (radial) retraction by $r^n := h_1^n$. Note that $r^n \circ h_t^n = r^n$.

We first define a function $u : X \to [0,1]$ by recursive extension as follows: $u|_{A\cup X^{-1}} = 0$ and let $u_n := u|_{A\cup X^n}$ be given by $u_n|_{A\cup X^{n-1}} = u_{n-1}$ and

$$u_n|_{\bar e} : \chi^e(x) \mapsto \begin{cases} 1 - \|x\|\Big(1 - u_{n-1}\big(\chi^e\big(\frac{x}{\|x\|}\big)\big)\Big) & \text{für } 0 \neq x \in D^n \\ 1 & \text{für } 0 = 0_e \in D^n \end{cases}$$

Then u_n is a well-defined continuous map with $(u_n)^{-1}(0) = A$ and by 4.6 the same holds for u.

Let $U(A) := \{x \in X : u(x) < 1\}$ and $U^n := U(A) \cap (A \cup X^n) = \{x \in A \cup X^n : u_n(x) < 1\}$. Note that the homotopy h_t^n on $A \cup X^n \setminus \bigsqcup_e \{0_e\}$ restricts to a homotopy on U^n with final value $r^n : U^n \to U^{n-1}$, since with every point $x \in U^n = U^{n-1} \cup \{\chi(x) : x \neq 0 \text{ and } \chi(x/\|x\|) \in U^{n-1}\}$ the whole path $\{h_t^n(x) : t \in I\}$ belongs to U^n and $u_n|_{A\cup X^{n-1}} = u_{n-1}$.

By induction on n we construct now homotopies $H_t^n : U^n \to U^n$, by

$$H_t^n := \begin{cases} \mathrm{id} & \text{for } t \leq \frac{1}{n+1}, \\ h_s^n & \text{for } \frac{1}{n+1} \leq t \leq \frac{1}{n} \text{ where } s := n\big(t(n+1) - 1\big) \in [0,1], \\ H_t^{n-1} \circ r^n & \text{for } t \geq \frac{1}{n}. \end{cases}$$

Then H_t^n is well-defined and $H_t^n|_{U^{n-1}} = H_t^{n-1}$, since $H_t^{n-1} = \mathrm{id}$ for $t \leq \frac{1}{n}$ and $h_s^n|_{A\cup X^{n-1}} = \mathrm{id}$. The union $H_t := \bigcup_{n\in\mathbb{N}} H_t^n : U(A) \to U(A)$ is the required deformation relative A and, since $r^n \circ h_s^n = r^n$, $(H_t^{n-1} \circ r^n)(U^n) \subseteq U^{n-1}$, and $r^n|_{U^{n-1}} = \mathrm{id}$, we get by induction

$$\begin{aligned} H_1^n \circ H_t^n &= H_1^{n-1} \circ r^n \circ H_t^n = \\ &= \begin{cases} H_1^{n-1} \circ (r^n \circ h_s^n) = H_1^{n-1} \circ r^n & \text{for } t \leq \frac{1}{n} \\ H_1^{n-1} \circ r^n \circ (H_t^{n-1} \circ r^n) = (H_1^{n-1} \circ H_t^{n-1}) \circ r^n = H_1^{n-1} \circ r^n & \text{for } t \geq \frac{1}{n} \end{cases} \\ &= H_1^n. \end{aligned}$$

Thus $H_1 \circ H_t = H_1 = r^0 \circ r^1 \circ \ldots \circ r^n \circ \ldots : U(A) \to \cdots \to U^{n-1} \to \cdots \to U^0 \to U^{-1} = A$.

In order to show that $A \hookrightarrow X$ is an NDR we consider a new homotopy $\tilde H_t(x) := H_{t\,\max(0,\min(1,2-3u(x)))}(x)$ for all $x \in U(A)$, i.e. $u(x) < 1$. Since $\tilde H_t(x) = x$ for $x \in A$ or $u(x) \geq \frac{2}{3}$ it extends by id to a homotopy of X rel. A. Since $\tilde H_1(x) = H_1(x)$ for $u(x) \leq \frac{1}{3}$ we get the NDR property with $\tilde u(x) := \min\{1, 3u(x)\}$.

By recursive construction, we have $U^n(A \cap B) = U^n(A) \cap U^n(B)$. □

4.19 Corollary. [20, 4.3.3] *Every point x in a CW-complex X has an open neighborhood, of which it is an SDR.*

Proof. Let e be the cell containing x and n its dimension. By 4.18 $A := X^n$ is an NDR of X, so there is a neighborhood $U(A) \subseteq X$ and a homotopy $H_t : U(A) \to U(A)$ between the identity and a retraction $r := H_1 : U(A) \to A$ and we have shown that $r \circ H_t = r$. So we may restrict this homotopy to the open subset $r^{-1}(e) \subseteq U(A)$, showing that e is an SDR in $r^{-1}(e)$. Since x is an SDR in e we obtain the required result by transitivity 2.36.3. □

4.20 Theorem. Cellular approximation. [20, 4.3.4] *For every continuous $f_0 : X \to Y$ between CW-complexes there exists a homotopic cellular mapping. If f_0 is cellular on some CW-subspace A, then the homotopy can be chosen to be rel. A.*

Proof. We recursively extend the constant homotopy on A to a homotopy $h_t^n : A \cup X^n \to Y$ with h_1^n being cellular. For the induction step we use for each n-cell

$e \subseteq X \setminus A$ a characteristic mapping $\chi : D^n \to \bar{e}$. By induction hypothesis we get a mapping $\varphi_0 : (D^n \times \{0\}) \cup (S^{n-1} \times I) \to Y$ given by $f_0 \circ \chi$ on the bottom and by $h_t^{n-1} \circ \chi$ on the mantle $S^{n-1} \times \{t\}$ with $h_1^{n-1} \circ \chi : S^{n-1} \to X^{n-1} \to Y^{n-1} \subseteq Y^n$. Since the domain of φ_0 is a retract in $D^n \times I$ by 2.31.1 and 2.30 we can extend it to a mapping again denoted φ_0 on $D^n \times I$. The image $\varphi_0(D^n \times \{1\})$ is compact and hence contained in a finite CW-complex by 4.5. Let $e^{n_1}, \dots, e^{n_r}$ be its cells of dimensions $n_r \geq \dots \geq n_1 > n$. Then $\varphi_0|_{D^n \times \{1\}} : (D^n \times \{1\}, S^{n-1} \times \{1\}) \to (Y^n \cup e^{n_1} \cup \dots \cup e^{n_r}, Y^n)$ is well defined. Applying now 3.34 r-times we can deform $\varphi_0|_{D^n \times \{1\}}$ successively relative $S^{n-1} \times \{1\}$ so, that its image finally avoids $e^{n_r} \cup \dots \cup e^{n_1}$. Let φ_t be the corresponding homotopy.

We can extend $\varphi_1 : D^n \times \{1\} \to Y^n$ via φ_0 to a continuous mapping on the boundary $(D^n \times I)^{\cdot}$, which is homotopic to $\varphi_0|_{(D^n \times I)^{\cdot}}$ relative $D^n \times \{0\} \cup S^{n-1} \times I$ via φ_t. The pair $(D^n \times I, (D^n \times I)^{\cdot}) \cong (D^{n+1}, S^n)$ is a CW-pair and hence has the HEP by 4.18 and φ_0 lives on $D^n \times I$, so φ_1 can be extended to $D^n \times I$ as well by 2.32.1. Now φ_1 factors over the quotient mapping $\chi \times I$ to a homotopy $t \mapsto h_t^n|_{\bar{e}}$. The union of the $h_t^n|_e$ gives the required h_t^n. □

$$
\begin{tikzcd}
S^{n-1} \arrow[rr, hook] \arrow[dd, "\chi"'] \arrow[dr, "\varphi_0(\_,t)"] & & D^n \arrow[dd, "\chi"] \arrow[dl, "\varphi_1(\_,t)"'] \\
& Y & \\
\dot{e} \arrow[rr, hook] \arrow[ur, "h_t^{n-1}"] & & \bar{e} \arrow[ul, dotted, "h_t^n|_{\bar{e}}"']
\end{tikzcd}
$$

4.21 Corollary. [20, 4.3.5] *Let $f_0, f_1 : X \to Y$ be homotopic and cellular. Then there exists a homotopy $H : X \times I \to Y$ such that $H_t(X^n) \subseteq Y^{n+1}$ for all n.*

Note that the inclusions of the endpoints in I are homotopic and cellular, but every homotopy has to map that point into the 1-skeleton.

Proof. Consider the CW-pair $(X \times I, X \times \dot{I})$ and the given homotopy $f : X \times I \to Y$. Since by assumption its boundary value $f|_{X \times \dot{I}}$ is cellular, we can find another mapping $H : X \times I \to Y$ by 4.20, which is cellular and homotopic to f relative $X \times \dot{I}$. Thus H is the required homotopy, since for $0 < t < 1$ and every n-cell e^n of X the image $H_t(e^n) = H(e^n \times \{t\})$ is contained in $H(e^n \times e^1) \subseteq Y^{n+1}$. □

5. Fundamental Group

Basic properties of the fundamental group

5.1 Definition. [**20**, 5.1.1] A path is a continuous mapping $u : I \to X$. The CONCATENATION $u_0 \cdot u_1$ of two paths u_0 and u_1 is defined by

$$(u_0 \cdot u_1)(t) := \begin{cases} u_0(2t) & \text{for } t < \frac{1}{2} \\ u_1(2t-1) & \text{for } t \geq \frac{1}{2} \end{cases}.$$

It is continuous provided $u_0(1) = u_1(0)$.

The REVERSE PATH $u^{-1} : I \to X$ is given by $u^{-1}(t) := u(1-t)$.

Note that concatenation is not associative and the constant path is not a neutral element. The corresponding identities hold only up to reparametrizations.

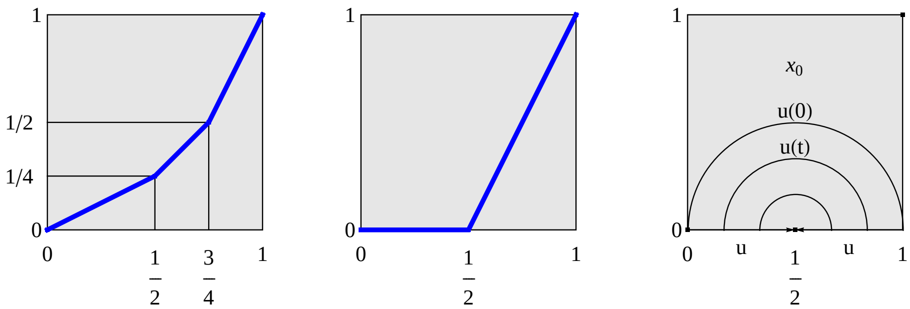
5.2 Lemma. Reparametrization. [**20**, 5.1.5] *Let $u : I \to X$ be a path and $f : I \to I$ be the identity on $\dot{I}$. Then $u \sim u \circ f$ rel. $\dot{I}$.*

Proof. A homotopy is given by $h(t,s) := u(ts + (1-t)f(s))$, see 2.4. $\square$

5.3 Corollary. [**20**, 5.1.6]

1. *Let u, v and w be paths with $u(1) = v(0)$ and $v(1) = w(0)$, then $(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$ rel. $\dot{I}$.*
2. *Let u be path with $x := u(0)$, $y := u(1)$ then $\mathrm{const}_x \cdot u \sim u \sim u \cdot \mathrm{const}_y$ rel. $\dot{I}$.*
3. *Let u be a path with $x := u(0)$ and $y := u(1)$. Then $u \cdot u^{-1} \sim \mathrm{const}_x$ and $u^{-1} \cdot u \sim \mathrm{const}_y$ rel. $\dot{I}$.*

Proof. In (1) and (2) we only have to reparametrize. In (3) we consider the homotopy, which has constant value on each circle with center $(\frac{1}{2}, 0)$. $\square$



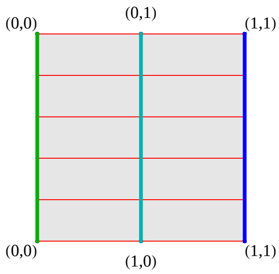
5.4 Definition. [**20**, 5.1.7] Let (X, x_0) be a pointed space. Then the FUNDAMENTAL GROUP (or FIRST HOMOTOPY GROUP) is defined by

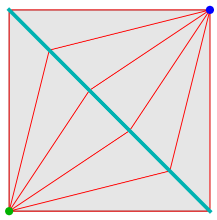
$$\pi_1(X, x_0) := [(I, \dot{I}), (X, \{x_0\})] \cong [(S^1, \{1\}), (X, \{x_0\})],$$

where multiplication is given by $[u] \cdot [w] := [u \cdot w]$, the neutral element is $1_{x_0} := [\mathrm{const}_{x_0}]$ and the inverse to $[u]$ is $[u^{-1}]$. This gives a group by 5.3.

5.5 Lemma. [**20**, 5.1.8] *Let $u : I \to X$ be a path from x_0 to x_1.*
Then $\mathrm{conj}_{[u]} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a group isomorphism, where $\mathrm{conj}_{[u]} : [v] \mapsto [u]^{-1} \cdot [v] \cdot [u] := [u^{-1} \cdot v \cdot u]$. $\square$

5.6 Lemma. [20, 5.1.10] *Let $h : I^2 \to I^2$ be as follows:*





$$(j,t) \mapsto (j,j) \text{ for } t \le \tfrac{1}{2}, j \in \{0,1\}$$

$$(t,0) \mapsto \begin{cases} (2t,0) & \text{for } t \le \frac{1}{2} \\ (1,2t-1) & \text{for } t \ge \frac{1}{2} \end{cases}$$

$$(t,1) \mapsto \begin{cases} (0,2t) & \text{for } t \le \frac{1}{2} \\ (2t-1,1) & \text{for } t \ge \frac{1}{2} \end{cases}$$

and a piecewise affine homeomorphism on the interior, i.e.

$$h(t,s) := \begin{cases} (1-2t)(0,0) + 2t\Big(s(0,1)+(1-s)(1,0)\Big) & \text{for } t \le 1/2 \\ (2-2t)\Big(s(0,1)+(1-s)(1,0)\Big) + (2t-1)(1,1) & \text{for } t \ge 1/2 \end{cases}$$

For continuous $f : (I^2)^{\cdot} \to X$ and $u_j(t) := f(t,j)$ resp. $v_j(s) := f(j,s)$ its values on the 4 edges the following statements are equivalent:

1. *There exists a continuous extension of f to I^2;*
2. *f is 0-homotopic;*
3. *There exists a continuous extension of $f \circ h$ to I^2;*
4. *$u_o \cdot v_1 \sim v_0 \cdot u_1$ rel. $\dot{I}$.*

Proof.
(1 $\Leftrightarrow$ 2) was shown in 2.26.
(3 $\Leftrightarrow$ 4) $f \circ h : (I^2)^{\cdot} \to X$ is the boundary data for the homotopy required in (4).
(1$\Rightarrow$3) Take $\widetilde{f \circ h} := \tilde{f} \circ h$.
(3$\Rightarrow$1) Since $\widetilde{f \circ h}$ is constant on $h^{-1}(s,t)$ for all $(s,t) \in (I^2)^{\cdot}$, it factors over the quotient mapping $h : I^2 \to I^2$ to a continuous extension $\tilde{f} : I^2 \to X$.

$$\begin{tikzcd}
(I^2)^{\cdot} \arrow[r, two heads, "h"] \arrow[d, hook] & (I^2)^{\cdot} \arrow[d, hook] \arrow[dr, "f"] & \\
I^2 \arrow[r, two heads, "h"] \arrow[rr, bend right, "\widetilde{f\circ h}"'] & I^2 \arrow[r, dotted, "\tilde{f}"] & X
\end{tikzcd}$$

□

5.7 Corollary. *Let X be a topological group (monoid) then $\pi_1(X,1)$ is abelian, where 1 denotes the neutral element of X.*

Proof. Consider the map $\tilde{f} : (t,s) \mapsto u(t) \cdot v(s)$ and apply 1$\Rightarrow$4 of 5.6. □

5.8 Proposition. [20, 5.1.12] *Let $V : \pi_1(X,x_0) = [(S^1,\{1\}),(X,\{x_0\})] \to [S^1,X]$ be the mapping forgetting the base-points. Then*

1. *$[u]$ is in the image of V iff $u(1)$ can be connected by a path with x_0.*
2. *V is surjective iff X is path-connected.*
3. *$V(\alpha) = V(\beta)$ iff there exists a $\gamma \in \pi_1(X,x_0)$ with $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$.*
4. *V is injective iff $\pi_1(X,x_0)$ is abelian.*
5. *The 'kernel' $V^{-1}([\mathrm{const}_{x_0}])$ of V is trivial.*

Warning: Since V is not a group-homomorphism, 5 does not contradict 4.

Proof. (1) $[u]$ is in the image of V iff u is homotopic to a base point preserving closed path. Such a homotopy evaluated at the base-point gives a path connecting $u(1)$ with x_0. Conversely, any path v from $u(1)$ to x_0 can be used to give a homotopy between u and a base point preserving path (namely $v^{-1} \cdot u \cdot v$) by 2.32.3 (since $(S^1, \{1\})$ has HEP by 4.18).

(1⇒2) is obvious.

(3) Let $\alpha = [u]$ and $\beta = [v]$. Then $V(\alpha) = V(\beta)$ iff u is homotopic to v.
(⇒) Let h be such a homotopy, $w(s) := h(j, s)$ for $j \in \dot{I}$ and $\gamma := [w]$. Then by (1⇒4) in 5.6 we have $w \cdot v \sim u \cdot w$ rel. $\dot{I}$, i.e. $\gamma \cdot \beta = \alpha \cdot \gamma$ and hence $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$.
(⇐) Let $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ and $\gamma = [w]$. Then $\gamma \cdot \beta = \alpha \cdot \gamma$ and hence $w \cdot u \sim v \cdot w$ rel. $\dot{I}$. Then by (1⇐4) in 5.6 we have $u \sim v$, i.e. $V(\alpha) = V(\beta)$.

(3⇒4)
(⇒) Let $\alpha, \gamma \in \pi_1(X)$ and $\beta := \gamma^{-1} \cdot \alpha \cdot \gamma$. By (3) we have $V(\alpha) = V(\beta)$ and since V is assumed to be injective we get $\alpha = \beta$, i.e. $\gamma \cdot \alpha = \alpha \cdot \gamma$.
(⇐) Conversely, if $V(\alpha) = V(\beta)$, then by (3) there exists a $\gamma \in \pi_1(X)$ with $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma = \alpha$ by commutativity.

(3⇒5) Let $V(\alpha) = [\mathrm{const}_{x_0}] = V(\mathrm{const}_{x_0})$. By (3) there exists a γ with $\alpha = \gamma^{-1} \cdot [\mathrm{const}_{x_0}] \cdot \gamma = \gamma^{-1} \cdot \gamma = 1$. □

5.9 Corollary. [20, 5.1.13] *Let X be path-connected. Then the following statements are equivalent:*

1. $\pi_1(X, x_0) \cong 1$ *for some (any) $x_0 \in X$, i.e. every $u : (S^1, \{1\}) \to (X, \{x_0\})$ is 0-homotopic rel. $\{1\}$;*
2. $[S^1, X] = \{0\}$*, i.e. every $u : S^1 \to X$ is 0-homotopic;*
3. *Any two paths which agree on the endpoints are homotopic rel. $\dot{I}$.*

A path-connected space satisfying these equivalent conditions is called SIMPLY CONNECTED.

Proof. (1⇒2) since $V : \pi^1(X, x_0) \to [S^1, X]$ is onto by 5.8.2.

(2⇒3) Let u_0 and u_1 be two such paths with $u_i(j) = x_j$ for $i, j \in \{0, 1\}$. For $v_j := \mathrm{const}_{x_j}$ the mapping $f : (I^2)^{\cdot} \to X$ given by u_0, v_1, u_1, and v_0 on the 4 edges is by assumption 0-homotopic (i.e. f considered as mapping $S^1 \to X$ is $u_0 \cdot v_1 \cdot u_1^{-1} \cdot v_0^{-1} \sim 0$ by 2), hence f extends to $I^2 \cong C(S^1)$ by 2.26, i.e. to a homotopy $u_0 \sim u_1$ rel. $\dot{I}$.

(3⇒1) is obvious, since then $u \sim \mathrm{const}_{x_0}$ rel. $\dot{I}$. □

Corollary. *Let X be contractible, then X is simply connected.*

Proof. By 2.6.6 we get that $[S^1, X] = \{0\}$ provided X is contractible. □

5.10 Example. [20, 5.1.9] *Let X be a CW-complex without 1-cells, e.g. $X = S^n$ for $n > 1$. Then $\pi_1(X, x_0) = \{1\}$ for each $x_0 \in X^0$.*
In fact, every $u : (I, \dot{I}) \to (X, x_0)$ is by 4.20 homotopic rel. $\dot{I}$ to a cellular mapping v, i.e. $v(I) \subseteq X^1 = X^0$, hence v is constant.

Note that such an X is path-connected iff it has exactly one 0-cell.
(⇒) Let x_0 and x_1 be two 0-cells and u be a path between them. By 4.20 u is homotopic to a cellular and hence constant path rel. $\dot{I}$, since X has no 1-cells. Thus $x_0 = x_1$.
(⇐) Since balls are path-connected each point in X^n can be connected with some point in X^{n-1} and by induction with the unique point in X^0.

5.11 Definition. [20, 5.1.15] Every $f : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\pi_1(f)[u] := [f \circ u]$:
Just use that $u \sim v \Rightarrow f \circ u \sim f \circ v$ and $f \circ (u \cdot v) = (f \circ u) \cdot (f \circ v)$ to get well-definedness and the homomorphy-property.
We will often use the notation f_* as an abbreviation of $\pi_1(f)$.

5.12 Corollary. [20, 5.1.16] π_1 *is a functor from the category of pointed topological spaces to that of groups, i.e. preserves identities and commutativity of diagrams.* □

5.13 Proposition. [20, 5.1.18] π_1 *is homotopy invariant.*
More precisely: If $f \sim g$ *rel.* x_0 *then* $\pi_1(f) = \pi_1(g)$. *If* $f \sim g$ *then* $\pi_1(g) = \operatorname{conj}_{[u]} \circ \pi_1(f)$, *where* u *is the path given by the homotopy at* x_0. *If* $f : X \to Y$ *is a homotopy equivalence then* $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ *is an isomorphism.*

Proof. If $f \sim g$ rel. x_0 and $[v] \in \pi_1(X, x_0)$ then $f \circ v \sim g \circ v$ rel. $\dot{I}$, i.e. $\pi_1(f)[v] = [f \circ v] = [g \circ v] = \pi_1(g)[v]$.

If h is a free homotopy from f to g, then $w(t) := h(t, x_0)$ defines a path from $f(x_0)$ to $g(x_0)$. And applying ($\boxed{1}\Rightarrow\boxed{4}$) in $\boxed{5.6}$ to $(s, t) \mapsto h(t, v(s))$ we get $(f \circ v) \cdot w \sim w \cdot (g \circ v)$ rel. $\dot{I}$, and hence $[f \circ v] \cdot [w] = [(f \circ v) \cdot w] = [w \cdot (g \circ v)] = [w] \cdot [g \circ v]$, i.e. $\pi_1(g)[v] = [g \circ v] = [w]^{-1} \cdot [f \circ v] \cdot [w] = [w]^{-1} \cdot \pi_1(f)[v] \cdot [w] = (\operatorname{conj}_{[w]} \circ \pi_1(f))([v])$.

Let now $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then up to conjugation $\pi_1(f)$ and $\pi_1(g)$ are inverse to each other. □

The fundamental group of the circle

5.15 Proposition. [20, 5.2.2]
The composition $\deg \circ V : \pi_1(S^1, 1) \to [S^1, S^1] \to (\mathbb{Z}, +)$ *is a group isomorphism.*

Proof. By $\boxed{2.15}$ we have that deg is a bijection. Since S^1 is path-connected the map V is surjective by $\boxed{5.8.2}$. Since S^1 is a topological group the map V is also injective by $\boxed{5.7}$ and $\boxed{5.8.4}$.

Remains to show that the composite is a group-homomorphism: Recall that $\deg([u])$ is given by evaluating at 1 the lift $\tilde{u} : \mathbb{R} \to \mathbb{R}$ of the path $u : (S^1, \{1\}) \to (S^1, \{1\})$ with $\tilde{u}(0) = 0$ and $\exp(2\pi i \tilde{u}(t)) = u(\exp(2\pi i t))$. Given $u, v \in \pi_1(S^1, 1)$ with lifts $\tilde{u}$ and $\tilde{v}$, then the lift of $u \cdot v$ is given by

$$t \mapsto \begin{cases} \tilde{u}(2t) & \text{for } t \leq \frac{1}{2} \\ \tilde{u}(1) + \tilde{v}(2t - 1) & \text{for } t \geq \frac{1}{2}. \end{cases} \quad \square$$

5.16 Corollary. [20, 5.2.4] $\pi_1(X, x_0) \cong \mathbb{Z}$ *for every space* X *which is homotopy equivalent* S^1. *In particular this is true for* $\mathbb{C} \setminus \{0\}$, *the Möbius strip, a full torus and the complement of a line in* $\mathbb{R}^3$ *since they all contain* S^1 *as SDR.* □

Constructions from group theory

5.17 Definition. [20, 5.3.1] We will denote with 1 the NEUTRAL ELEMENT in a given group.

A SUBGROUP of a group G is a subset $H \subseteq G$, which is with the restricted group operations itself a group, i.e. $h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$, $h_1^{-1} \in H$, $1 \in H$.

The SUBGROUP $\langle X\rangle_{SG}$ *generated by a subset* $X \subseteq G$ is defined to be the smallest subgroup of G containing X, i.e.

$$\langle X\rangle_{SG} := \bigcap\{H : X \subseteq H \le G\} = \Big\{x_1^{\varepsilon_1}\cdot\dots\cdot x_n^{\varepsilon_n} : x_j \in X, \varepsilon_j \in \{\pm 1\}\Big\}.$$

Given an equivalence relation $\sim$ on G we can form the quotient set $G/\sim$ and have the natural mapping $\pi : G \to G/\sim$. In order that $G/\sim$ carries a group structure, for which π is a homomorphism, i.e. $\pi(x\cdot y) = \pi(x)\cdot\pi(y)$, we need precisely that $\sim$ is a CONGRUENCE RELATION, i.e. $x_1 \sim x_2$, $y_1 \sim y_2 \Rightarrow x_1^{-1} \sim x_2^{-1}$, $x_1\cdot y_1 \sim x_2 \cdot y_2$.

Then $H := \{x : x \sim e\} = \pi^{-1}(e)$ is a NORMAL SUBGROUP (we write $H \triangleleft G$), i.e. is a subgroup such that $g \in G$, $h \in H \Rightarrow g^{-1}hg \in H$. And conversely, for normal subgroups $H \triangleleft G$ we have that $x \sim x\cdot h$ for all $x \in G$ and $h \in H$ defines a congruence relation $\sim$ and $G/H := G/\sim = \{gH : g \in G\}$. This shows, that normal subgroups are exactly the kernels of group homomorphisms. Every surjective group morphism $p : G \to G_1$ is up to an isomorphism $G \to G/\ker p$.

The NORMAL SUBGROUP $\langle X\rangle_{NG}$ *generated by a subset* $X \subseteq G$ is defined to be the smallest normal subgroup of G containing X, i.e.

$$\langle X\rangle_{NG} := \bigcap\{H : X \subseteq H \triangleleft G\} = \Big\{g_1^{-1}y_1g_1\cdot\dots\cdot g_n^{-1}y_ng_n : g_j \in G, y_j \in \langle X\rangle_{SG}\Big\}.$$

5.18 Definition. Let G_i be groups. Then the PRODUCT $\prod_i G_i$ of the $\{G_i : i\}$ is defined to be the solution of the following universal problem:

$$\begin{tikzcd} G_i & & \prod_i G_i \arrow[ll, "\mathrm{pr}_i"'] \\ & H \arrow[ul, "f_i"] \arrow[ur, dotted, "(f_i)_i", "!"'] & \end{tikzcd}$$

A concrete realization of $\prod_i G_i$ is the cartesian product with the component-wise group operations.

5.19 Definition. Let G_i be groups. Then the COPRODUCT (FREE PRODUCT) $\coprod_i G_i$ of is defined to be the solution of the following universal problem:

$$\begin{tikzcd} G_i \arrow[rr, "\mathrm{inj}_i"] \arrow[dr, "f_i"'] & & \coprod_i G_i \arrow[dl, dotted, "(f_i)_i"', "!"] \\ & H & \end{tikzcd}$$

Remark. [20, 5.3.3] A concrete realization of $\coprod_i G_i$ is constructed as follows. Take the set X of all finite sequences of elements of the disjoint union $\bigsqcup_i G_i$. With concatenation of sequences X becomes a monoid, where the empty sequence is the neutral element. Every G_i is injectively mapped into X by mapping g to the sequence with the single entry g. However this injection is not multiplicative and X is not a group. So we consider the congruence relation generated by $(g,h) \sim (gh)$ if g, h belong to the same group and $(1_i) \sim \emptyset$ for the neutral element 1_i of any group G_i. Then $X/\sim$ is a group and the composite $G_i \to X \to X/\sim$ is the required group homomorphism and this object satisfies the universal property of the coproduct.

In every equivalence class of $X/\sim$ we find a unique representative of the form $(g_1,\dots,g_n)$, with $g_j \in G_{i_j}\setminus\{1\}$ and $i_j \neq i_{j+1}$. Since $(g_1,\dots,g_n)$ is just the product of the images of $g_i \in G_i$ we may write this also as $g_1\cdot\dots\cdot g_n$.

5.20 Definition. [20, 5.7.8] Let H, G_1, G_2 be groups and $f_j : H \to G_j$ group homomorphisms. Then the PUSH-OUT $G_1 \coprod_H G_2$ of (f_1, f_2) is a solution of the following universal problem:

$$\begin{tikzcd}
H \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d, "g_2"'] \arrow[ddr, bend left, "k_2"] & \\
G_1 \arrow[r, "g_1"] \arrow[drr, bend right, "k_1"'] & G_1 \coprod_H G_2 \arrow[dr, dotted, "!"] & \\
& & K
\end{tikzcd}$$

It can be constructed as follows:

$$G_1 \coprod_H G_2 := (G_1 \amalg G_2)/N, \text{ where } N := \Big\langle f_1(h) \cdot f_2(h)^{-1} : h \in H \Big\rangle_{NG}$$

and where g_j is given by composing the inclusion $G_j \to G_1 \amalg G_2$ with the natural quotient mapping $G_1 \amalg G_2 \to (G_1 \amalg G_2)/N$.

5.21 Definition. [20, 5.6.3] Let G be a group. Then the ABELIZATION ${}^{ab}G$ of G is an abelian group being solution of the following universal problem:

$$\begin{tikzcd}
G \arrow[rr, "\pi"] \arrow[dr, "f"'] & & {}^{ab}G \arrow[dl, dotted, "\tilde{f}"', "!"] \\
& A &
\end{tikzcd}$$

where A is an arbitrary abelian group.

A realization of ${}^{ab}G$ is given by G/G', where the COMMUTATOR SUBGROUP G' denotes the normal subgroup generated by all COMMUTATORS $[g, h] := ghg^{-1}h^{-1}$. Note that $G' = \{[g_1, h_1] \cdot \dots \cdot [g_n, h_n] : g_j, h_j \in G\}$, since $g[h_1, h_2]g^{-1} = [gh_1g^{-1}, gh_2g^{-1}]$.

Remark. From general categorical results we conclude that the product (and more general limits) in the category of abelian groups is the product (limit) formed in that of all groups. And abelization of a coproduct (more generally a colimit) is just the coproduct (colimit) of the abelizations formed in the category of abelian groups.

5.22 Definition. [20, 5.3.7] Let G_i be abelian groups. Then the COPRODUCT (DIRECT SUM) ${}^{ab}\coprod_i G_i$ of the G_i is defined to be the solution of the following universal problem:

$$\begin{tikzcd}
G_i \arrow[rr, "\mathrm{inj}_i"] \arrow[dr, "f_i"'] & & {}^{ab}\coprod_i G_i \arrow[dl, dotted, "(f_i)_i"', "!"] \\
& H &
\end{tikzcd}$$

where H is an arbitrary abelian group.

Remark. A concrete realization of ${}^{ab}\coprod_i G_i$ is given by those elements of $\prod_i G_i$, for which almost all coordinates are equal to the neutral element.

5.23 Definition. [20, 5.5.3] Let X be a set. Then the FREE GROUP $\mathcal{F}(X)$ is the universal solution to

$$\begin{tikzcd}
X \arrow[rr, "\iota"] \arrow[dr, "f"'] & & \mathcal{F}(X) \arrow[dl, dotted, "\tilde f"', "!"] \\
& H &
\end{tikzcd}$$

where the arrows starting at X are just mappings and $\tilde f$ is a group homomorphism.

Remark. [20, 5.5.2] One has $\mathcal{F}(X) \cong \mathcal{F}(\bigsqcup_{x\in X}\{x\}) \cong \coprod_{x\in X} F(\{x\})$ by a general categorical argument, and $\mathcal{F}(\{*\}) \cong \mathbb{Z}$, as is easily seen.

5.24 Definition. Let X be a set. Then the FREE ABELIAN GROUP ${}^{ab}\mathcal{F}(X)$ is the universal solution to

$$\begin{tikzcd}
X \arrow[rr, "\iota_i"] \arrow[dr, "f"'] & & {}^{ab}\mathcal{F}(X) \arrow[dl, dotted, "\tilde f"', "!"] \\
& A &
\end{tikzcd}$$

where the arrows starting at X are just mappings and $\tilde f$ is a group homomorphism.

Remark. By a general categorical argument we have ${}^{ab}(\mathcal{F}(X)) \cong {}^{ab}\mathcal{F}(X)$. And ${}^{ab}\mathcal{F}(X) \cong {}^{ab}\coprod_x \mathcal{F}(\{x\}) \cong {}^{ab}\coprod_x \mathbb{Z}$, which are just the finite sequences in $\mathbb{Z}^X$.

We will show in $\boxed{9.20}$ the any subgroup of a free abelian group is itself a free abelian group.

5.25 Definition. [20, 5.6.1] Given a set X and a subset $R \subseteq \mathcal{F}(X)$ we define

$$\langle X : R\rangle := \mathcal{F}(X)/\langle R\rangle_{NG}$$

to be the GROUP WITH GENERATORS X AND DEFINING RELATIONS R.
If $G \cong \langle X : R\rangle$, then $\langle X : R\rangle$ is called a REPRESENTATION of the group G.

5.26 Examples. One has $\mathcal{F}(X) = \langle X : \emptyset\rangle$ and $\mathbb{Z}_n := \langle\{x\} : \{x^n\}\rangle$.
More generally, $\coprod_j \langle X_j : R_j\rangle = \langle \bigsqcup X_j : \bigcup_j R_j\rangle$.
Moreover, ${}^{ab}\langle X : R\rangle = \langle X : R \cup \{[x,y] : x,y \in X\}\rangle$

5.27 Remark. [20, 5.8.1] Obviously we have:

1. $\langle X : R\rangle \cong \langle X : R \cup \{r'\}\rangle$ for $r' \in \langle R\rangle_{NG}$.
2. $\langle X : R\rangle \cong \langle X \cup \{a\} : R \cup \{a^{-1}\cdot w\}\rangle$ for $a \notin X$ and $w \in \mathcal{F}(X)$.

These operations are called Tietze operations.

5.28 Theorem. [20, 5.8.2] *Two finite representations $\langle X : R\rangle$ and $\langle Y : S\rangle$ describe isomorphic groups iff there is a finite sequence of Tietze operations converting one description into the other.*

Proof. Let $f : \langle X : R\rangle \xrightarrow{\cong} \langle Y : S\rangle$ be an isomorphism with inverse g.

For each $x \in X$ we choose $\tilde f(x) \in f([x]) \subseteq \mathcal{F}(Y)$ and similarly $\tilde g(y) \in g([y]) \subseteq \mathcal{F}(X)$. By the universal propery we may extend $\tilde f$ and $\tilde g$ to homomorphisms $\tilde f : \mathcal{F}(X) \to \mathcal{F}(Y)$ and $\tilde g : \mathcal{F}(Y) \to \mathcal{F}(X)$. Let

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[dr, dotted, "(1)"'] & \mathcal{F}(X) \arrow[r, two heads] \arrow[d, dotted, "\tilde f\ (2)"] & \langle X : R\rangle \arrow[d, "f", "\cong"'] \\
& \mathcal{F}(Y) \arrow[r, two heads] & \langle Y : S\rangle
\end{tikzcd}$$

$$\tilde S := \{x^{-1}\cdot \tilde f(x) : x \in X\} \subseteq \mathcal{F}(X \sqcup Y) \text{ and } \tilde R := \{y^{-1}\cdot \tilde g(y) : y \in Y\} \subseteq \mathcal{F}(X \sqcup Y).$$

For symmetry reasons it suffices to show that a finite sequence of Tietze-operations of 5.27 applied to $\langle X : R\rangle$ gives $\langle X \sqcup Y : R \cup \tilde{R} \cup S \cup \tilde{S}\rangle$:

Applying the operation 5.27.2 successively for every $y \in Y$ to $\langle X : R\rangle$ obviously yields $\langle X \sqcup Y : R \cup \tilde{R}\rangle$.

For every $y \in \mathcal{F}(Y)$ we have $y^{-1}\cdot\tilde{g}(y) \in \langle\tilde{R}\rangle_{\mathrm{NG}} \subseteq \mathcal{F}(X\sqcup Y)$: In fact, $y = y_1^{\varepsilon_1}\cdot\dots\cdot y_n^{\varepsilon_n}$ for some $y_i \in Y$ and $\varepsilon_i \in \{\pm 1\}$, $y_1^{-1}\cdot\tilde{g}(y_1) \in \tilde{R} \Rightarrow \tilde{g}(y_1)^{-1}\cdot y_1 \in \langle\tilde{R}\rangle \Rightarrow y_1\cdot\tilde{g}(y_1)^{-1} = y_1\cdot(\tilde{g}(y_1)^{-1}\cdot y_1)\cdot y_1^{-1} \in \langle\tilde{R}\rangle_{\mathrm{NG}} \Rightarrow z := y_1^{-\varepsilon_1}\cdot\tilde{g}(y_1)^{\varepsilon_1} \in \langle\tilde{R}\rangle_{\mathrm{NG}} \Rightarrow$

$$\begin{aligned}
y^{-1}\cdot\tilde{g}(y) &= (y_1^{\varepsilon_1}\cdots y_n^{\varepsilon_n})^{-1}\cdot\tilde{g}(y_1)^{\varepsilon_1}\cdots\tilde{g}(y_n)^{\varepsilon_n}\\
&= (y_2^{\varepsilon_1}\cdots y_n^{\varepsilon_n})^{-1}\cdot\underbrace{y_1^{-\varepsilon_1}\cdot\tilde{g}(y_1)^{\varepsilon_1}}_{=z}\cdot\underbrace{\tilde{g}(y_2)^{\varepsilon_2}\cdots\tilde{g}(y_n)^{\varepsilon_n}}_{=:p}\\
&= \underbrace{(y_2^{\varepsilon_2}\cdots y_n^{\varepsilon_n})^{-1}\cdot\tilde{g}(y_2)^{\varepsilon_2}\cdots\tilde{g}(y_n)^{\varepsilon_n}}_{\in\langle\tilde{R}\rangle_{\mathrm{NG}}\text{ by induction hypothesis}}\cdot\underbrace{p^{-1}\cdot z\cdot p}_{\in\langle\tilde{R}\rangle_{\mathrm{NG}}}\in\langle\tilde{R}\rangle_{\mathrm{NG}}.
\end{aligned}$$

For $y \in S$ we have $[\tilde{g}(y)] = g([y]) = g(1) = 1$, i.e. $\tilde{g}(y) \in \langle R\rangle_{\mathrm{NG}}$. Therefore $y = \tilde{g}(y)\cdot(y^{-1}\cdot\tilde{g}(y))^{-1} \in \langle R \cup \tilde{R}\rangle_{\mathrm{NG}}$, i.e. $S \subseteq \langle R\cup\tilde{R}\rangle_{\mathrm{NG}}$.

For $x \in X$ and $y := \tilde{f}(x)$ we have $[\tilde{g}(y)] = g([y]) = g([\tilde{f}(x)]) = g(f([x])) = [x]$, hence $x^{-1}\cdot\tilde{g}(y) \in \langle R\rangle_{\mathrm{NG}}$ and thus $x^{-1}\cdot\tilde{f}(x) = x^{-1}\cdot\tilde{g}(y)\cdot(y^{-1}\cdot\tilde{g}(y))^{-1} \in \langle R\cup\tilde{R}\rangle_{\mathrm{NG}}$, i.e. $\tilde{S} \subseteq \langle R\cup\tilde{R}\rangle_{\mathrm{NG}}$.

Applying the operation 5.27.1 successively for every $y \in \tilde{S}\cup S$ to $\langle X\sqcup Y : R\cup\tilde{R}\rangle$ yields therefore $\langle X\sqcup Y : R\cup\tilde{R}\cup S\cup\tilde{S}\rangle$. □

Remark. The word problem for finitely presented groups is the problem to determine whether two elements $w, w' \in \mathcal{F}(X)$ define the same element of $\langle X : R\rangle$, or equivalently whether $w^{-1}w' \in \langle R\rangle_{NG}$.

The isomorphy problem is to determine whether two finite group representations describe isomorphic groups.

It has been shown that both problems have no algorithmic solution.

Group descriptions of CW-spaces

5.29 Proposition. [20, 5.2.6] *For pointed spaces (X_i, x_i) we have the following isomorphism $\pi_1(\prod_i X_i, (x_i)_i) \cong \prod_i \pi_1(X_i, x_i)$.*

Proof. Obvious, since $[(Y,y),(\prod_i X_i,(x_i)_i)] \cong \prod_i[(Y,y),(X_i,x_i)]$, by composition with the coordinate projections, and since the concatenation of paths in $\prod_i X_i$ is given component-wise. □

5.30 Proposition. [20, 5.1.21] *Let X_0 be a path component of X and let $x_0 \in X_0$. Then the inclusion of $X_0 \subseteq X$ induces an isomorphism $\pi_1(X_0, x_0) \cong \pi_1(X, x_0)$.*

Proof. Since S^1 and $S^1\times I$ is path-connected, the paths and the homotopies have values in X_0. □

5.31 Proposition. *Let X_α be subspaces of X such that every compact set is contained in some X_α. And for any two of these subspaces there is a third one containing both. Let $x_0 \in X_\alpha$ for all α. Then $\pi_1(X, x_0)$ is the* inductive limit *of all $\pi_1(X_\alpha, x_0)$.*

Proof. Let G be any group and $f_\alpha : \pi_1(X_\alpha) \to G$ be group-homomorphisms, such that for every inclusion $i : X_\alpha \subseteq X_\beta$ we have $f_\beta \circ \pi_1(i) = f_\alpha$. We have to find a unique group-homomorphism $f : \pi_1(X) \to G$, which satisfies $f \circ \pi_1(i) = f_\alpha$ for all inclusions $i : X_\alpha \to X$. Since every closed curve w in X is contained in some X_α, we have to define $f([w]_X) := f_\alpha([w]_{X_\alpha})$. We only have to show that f is well-defined: So let $[w_1]_X = [w_2]_X$ for curves w_1 in X_{α_1} and $w_2 \in X_{\alpha_2}$. The image of the homotopy $w_1 \sim w_2$ is contained in some X_α, which we may assume to contain X_{α_1} and X_{α_2}. Thus $f_{\alpha_1}([w_1]_{X_{\alpha_1}}) = f_\alpha([w_1]_{X_\alpha}) = f_\alpha([w_2]_{X_\alpha}) = f_{\alpha_2}([w_2]_{X_{\alpha_2}})$. □

5.32 Theorem of Seifert and van Kampen. [20, 5.3.11]
Let X be covered by two open path-connected subsets U_1 and U_2 such that $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then

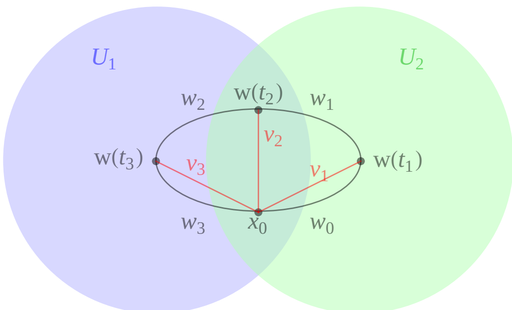
$$\begin{tikzcd}
\pi_1(U_1 \cap U_2, x_0) \arrow[r, "i_*^2"] \arrow[d, "i_*^1"] & \pi_1(U_2, x_0) \arrow[d, "j_*^2"'] \\
\pi_1(U_1, x_0) \arrow[r, "j_*^1"] & \pi_1(X, x_0)
\end{tikzcd}$$

is a push-out, where all arrows are induced by the corresponding inclusions.

Proof. Let $G_j := \pi_1(U_j, x_0)$ für $j \in \{1, 2\}$, $G_0 := \pi_1(U_1 \cap U_2, x_0)$, $G := \pi_1(U_1 \cup U_2, x_0) = \pi_1(X, x_0)$ and $\bar G := (G_1 \amalg G_2)/N$ with $g_i : G_i \to \bar G$ the push-out, where N is the normal subgroup generated by $\{i_*^1([u]) \cdot i_*^2([u])^{-1} : [u] \in G_0\}$. By the universal property of the push-out there exists a unique group-homomorphism $\varphi : \bar G \to G$ with $\varphi \circ g_i = j_*^i$ and we only have to show that it is bijective.

Surjectivity: Let $[w] \in G$. By the Lebesgue-covering lemma applied to $[0, 1]$ we may take n sufficiently large such that for each $0 \le i < n$ we have $w([t_i, t_{i+1}]) \subseteq U_{\varepsilon_i}$ for some $\varepsilon_i \in \{1, 2\}$ and $t_i := \frac{i}{n}$.

Let w_j be the restriction of w to $[t_j, t_{j+1}]$ and let v_i be a path from x_0 to $w(t_i)$ in $U_{\varepsilon_i} \cap U_{\varepsilon_{i-1}}$. We may take v_0 and v_n to be constant x_0. Let $u_i := v_i \cdot w_i \cdot v_{i+1}^{-1}$. Then u_i is a closed path in U_{ε_i} and $w \sim u_0 \cdot \ldots \cdot u_{n-1}$ in X rel. $\dot I$. Let $\bar g_i := g_{\varepsilon_i}([u]_{U_{\varepsilon_i}}) \in \bar G$.



Hence

$$\begin{aligned}
[w]_X &= [u_0]_X \cdot \ldots \cdot [u_{n-1}]_X = j_*^{\varepsilon_0}([u_0]_{U_{\varepsilon_0}}) \cdot \ldots \cdot j_*^{\varepsilon_{n-1}}([u_{n-1}]_{U_{\varepsilon_{n-1}}}) \\
&= \varphi(\bar g_1) \cdot \ldots \cdot \varphi(\bar g_{n-1}) = \varphi(\bar g_1 \cdot \ldots \cdot \bar g_{n-1}) \in \varphi(\bar G).
\end{aligned}$$

Injectivity: Let $z \in \bar G = (G_1 \amalg G_2)/N$ with $\varphi(z) = 1 \in G$. Then we find closed paths u_i in U_{ε_i} for certain $\varepsilon_i \in \{1, 2\}$ with $z = g_{\varepsilon_1}([u_1]_{U_{\varepsilon_1}}) \cdot \ldots \cdot g_{\varepsilon_n}([u_n]_{U_{\varepsilon_n}})$. Since

$$\begin{aligned}
[\mathrm{const}_{x_0}]_X = 1 = \varphi(z) &= \varphi\Big(g_{\varepsilon_1}([u_1]) \cdot \ldots \cdot g_{\varepsilon_n}([u_n])\Big) \\
&= \varphi\big(g_{\varepsilon_1}([u_1])\big) \cdot \ldots \cdot \varphi\big(g_{\varepsilon_n}([u_n])\big) = [u_1]_X \cdot \ldots \cdot [u_n]_X = [u_1 \cdot \ldots \cdot u_n]_X
\end{aligned}$$

there is a homotopy $H : I \times I \to X$ relative $\dot I$ between $u_1 \cdot \ldots \cdot u_n$ and const_{x_0}. We partition $I \times I$ into squares Q, such that $H(Q) \subseteq U_{\varepsilon_Q}$ for certain $\varepsilon_Q \in \{1, 2\}$. We may assume that the resulting partition on the bottom edge $I \times \{0\} \cong I$ is finer than $0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1$. For every vertex k of this partition we choose a curve

v_k connecting x_0 with $H(k)$. If $H(k) \in U_j$ then we may assume that $v_k(I) \subseteq U_j$. If $H(k) = x_0$, we may assume that v_k is constant. For every edge c of such a square Q we define the closed curve $u_c := v_{c(0)} \cdot (H \circ c) \cdot v_{c(1)}^{-1}$ through x_0. Since u_c is contained in some U_j we may consider $[u_c]_{U_j}$ and its image $\bar{c} := g_j([u_c]_{U_j}) \in \bar{G}$. This is well defined, since if u_c is contained in $U_1 \cap U_2$ then $[u_c]_{U_1 \cap U_2}$ is mapped to $[u_c]_{U_j} \in G_j$ for $i \in \{1, 2\}$ and further on to the same element $\bar{c}$ in the push-out $\bar{G}$.

Let now Q be such a square with edges d, r, u, l. Then $d \cdot r \sim l \cdot u$ rel. $\dot{I}$ in Q, hence $u_d \cdot u_r \sim u_l \cdot u_u$ rel. $\dot{I}$ in U_{ε_Q}, i.e. $[u_d] \cdot [u_r] = [u_l] \cdot [u_u]$ in G_{ε_Q} and thus $\bar{d} \cdot \bar{r} = g_{\varepsilon_Q}([u_d]) \cdot g_{\varepsilon_Q}([u_r]) = g_{\varepsilon_Q}([u_l] \cdot [u_u]) = \bar{l} \cdot \bar{u}$ in $\bar{G}$.

$$\begin{tikzcd} \cdot \arrow[r, "u"] & \cdot \\ \cdot \arrow[u, "l"] \arrow[r, "d"'] & \cdot \arrow[u, "r"'] \end{tikzcd}$$

Multiplying in $\bar{G}$ all these equations resulting from one row of squares, gives that the product corresponding to the top line equals in $\bar{G}$ that corresponding to the bottom line, since the inner vertical parts cancel, and those at the boundary are 1. Since the top row represents 1, we get that the same is true for the bottom one. But u_i is homotopic in U_{ε_i} rel. $\dot{I}$ to the concatenation of the corresponding u_c in the bottom row, i.e. $[u_i]_{U_{\varepsilon_i}} = \prod_{c \subseteq [\frac{i-1}{n}, \frac{i}{n}] \times \{0\}} [u_c]_{U_{\varepsilon_i}}$ in G_{ε_i}. Thus $z = \prod_i g_{\varepsilon_i}([u_i]_{U_{\varepsilon_i}}) = \prod_{c \subseteq [0,1] \times \{0\}} g_{\varepsilon_i}([u_c]_{U_{\varepsilon_i}}) = \prod_c \bar{c} = 1$ in $\bar{G}$. □

5.33 Corollary. [20, 5.3.9] [20, 5.3.12] *Let $X = U_1 \cup U_2$ be as in 5.32.*

1. *If $U_1 \cap U_2$ is simply connected, then $\pi_1(U_1 \cup U_2) \cong \pi_1(U_1) \amalg \pi_1(U_2)$.*
2. *If U_1 and U_2 are simply connected, then $U_1 \cup U_2$ is simply connected.*
3. *If U_2 is simply connected, then $\mathrm{incl}_* : \pi_1(U_1) \to \pi_1(U_1 \cup U_2)$ in the push-out square is an epimorphism and its kernel is generated by the image of $\mathrm{incl}_* : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$.*
4. *If U_2 and $U_1 \cap U_2$ are simply connected, then $\pi_1(U_1) \cong \pi_1(U_1 \cup U_2)$.*

Proof.

1 In this situation $N = \{1\}$ and hence $G_1 \amalg G_2$ is the push-out.

2 Here $G_1 \amalg G_2 = \{1\} \amalg \{1\} = \{1\}$ and hence also the push-out.

3 In this situation $G_1 \amalg G_2 = G_1 \amalg \{1\} \cong G_1$ and N is the normal subgroup generated by the image of G_0 in G_1.

4 Here we have $N = \{1\}$ and hence the push-out is isomorphic to G_1. □

5.34 Theorem. [20, 5.4.8] *Let a CW-complex X be the union of two connected CW-subcomplexes A and B. Let $x_0 \in A \cap B$ and $A \cap B$ be connected. Then π_1 maps the push-out square to a push-out.*

Proof. By 4.18 we may choose open neighborhoods $U(A)$, $U(B)$ and $U(A \cap B) = U(A) \cap U(B)$ which contain A, B and $A \cap B$ as SDRs. Then application of 5.32 and of 5.13 gives the result.

$$\begin{tikzcd}
& U(A) \cap U(B) \arrow[rr, hook] \arrow[dd, hook] & & U(A) \arrow[dd, hook] \\
A \cap B \arrow[ur, hook, "\sim"] \arrow[rr, hook, crossing over] \arrow[dd, hook] & & A \arrow[ur, hook, "\sim"] & \\
& U(B) \arrow[rr, hook] & & X \\
B \arrow[ur, hook, "\sim"] \arrow[rr, hook] & & A \cup B \arrow[ur, equal] \arrow[from=uu, hook, crossing over] &
\end{tikzcd}$$

□

5.35 Proposition. [20, 5.4.9] *Let A and B be (connected) CW-complexes. Then $\pi_1(A \vee B, *) \cong \pi_1(A, *) \amalg \pi_1(B, *)$.*

Proof. Since $A \cap B = \{*\}$ in $A \vee B$ and hence simply connected this follows from 5.34 and 5.33.1. $\square$

5.36 Example. By 5.34 we have $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} \amalg \mathbb{Z}$.
However, for spaces being not CW-spaces $\pi_1(A \vee B) \neq \pi_1(A) \amalg \pi_1(B)$ might happen: Take for example for A and B the subset of $\mathbb{R}^2$ formed by infinite many circles tangent at the base point. The closed curve which passes through all those circles alternatingly can not be expressed homotopically as finite product of words in $\pi_1(A)$ and $\pi_1(B)$.

5.37 Proposition. [20, 5.5.9] *Let X_j be a CW-complex with base-point $x_j \in X_j^0$. Then $\pi_1(\bigvee_{j\in J} X_j) \cong \coprod_{j \in J} \pi_1(X_j)$. In particular we have $\pi_1(\bigvee_J S^1) \cong \coprod_J \mathbb{Z} \cong \mathcal{F}(J)$, where the free generators of $\pi_1(\bigvee_j S^1)$ are just the inclusions $\mathrm{inj}_j : S^1 \to \bigvee_J S^1$.*

Proof. This follows from 5.35 by induction for finite J and by 5.31 for general J, since every compact subset is by 4.5 contained in a finite subcomplex of the CW-complex of $\bigvee_{j\in J} X_j$ given by 4.17 and since the coproduct is the inductive limit of its finite subcoproducts. $\square$

5.38 Corollary. [20, 5.4.1] [20, 5.4.2]
Let Y be path-connected with $y_0 \in Y$ and $f : S^{n-1} \to Y$ be continuous. Then the inclusion $Y \subseteq Y \cup_f D^n$ induces an isomorphism $\pi_1(Y, y_0) \to \pi_1(Y \cup_f D^n, y_0)$ if $n \geq 3$ and an epimorphism if $n = 2$. In the later case the kernel is the normal subgroup generated by $[v][f][v^{-1}]$, where v is a path from y_0 to $f(1)$. So

$$\pi_1(Y \cup_f D^2) \cong \pi_1(Y)/\langle \mathrm{conj}_{[v]}([f]) \rangle_{NG}$$

One could say that by gluing D^2 to Y the element $[f] \in \pi_1(Y)$ gets killed.

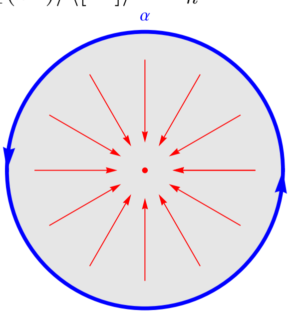
Proof. We take $U := Y \cup_f (D^n \setminus \{0\})$ and $V := e^n$.

$$\begin{tikzcd}
& U\cap V \arrow[rr, hook] \arrow[dd, hook] & & V \arrow[dd, hook] \\
S^{n-1} \arrow[ur, hook, "\sim", "1/2"'] \arrow[dd, "f"] \arrow[dr, phantom, "\sim"] & & \{0\} \arrow[ur, hook, "\sim"] & \\
& U \arrow[rr, hook] & & U\cup V \\
Y \arrow[ur, hook, "\sim"] \arrow[rr, hook] & & Y\cup_f D^n \arrow[ur, equal] &
\end{tikzcd}$$

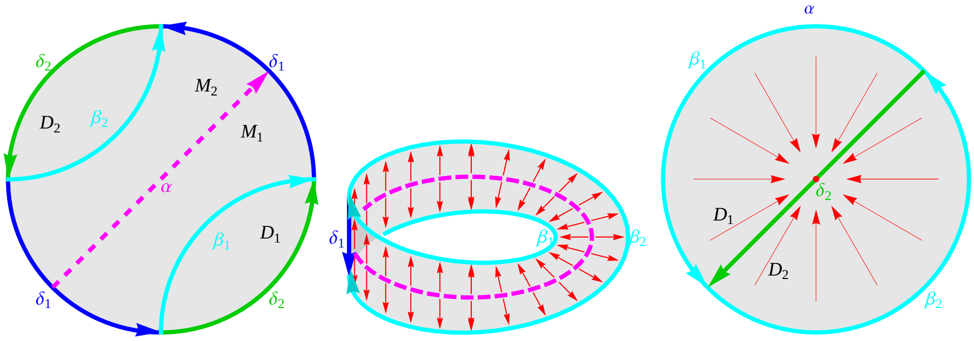
Then $V \sim \{0\}$ and $U \cap V = e^n \setminus \{0\} \sim S^{n-1}$ are simply connected for $n \geq 3$, by 5.10. Thus the inclusion $U \subseteq Y \cup_f D^n$ induces an isomorphism by 5.33.4. Since Y is a SDR of U by 2.38 the inclusion of $Y \to U$ induces an isomorphism by 5.13.

Now for $n = 2$. Again V is simply connected, but $U \cap V \sim S^1$ and hence $\pi_1(U \cap V)$ is the infinite cyclic group generated by the image of a circle of radius say $1/2$. This path is homotopic to $[v][f][v^{-1}]$ in $Y \cup_f D^2$, hence everything follows by 5.33.3. $\square$

5.39 Example. [20, 5.4.4] We have $\pi_1(S^1 \cup_{z^n} D^2) \cong \pi_1(S^1)/\langle [z^n] \rangle \cong \mathbb{Z}_n$.



In particular, $\pi_1(\mathbb{P}^2) = \pi_1(S^1 \cup_{z^2} D^2) \cong \mathbb{Z}_2$.
This can be easily visualized: The top semi-circle α in D^2 has as $\alpha \cdot \alpha$ the full circle, which is contractible to 0. Equally, $\mathbb{P}^2$ is obtained by glueing a 2-cell to the boundary of a Möbius strip and the generator $\alpha \in \pi_1(\mathbb{P}^2)$ is just the middle line on the Möbius strip. Its square $\alpha \cdot \alpha$ is homotopic to the boundary of the Möbius strip which is contractible in the disk.



5.40 Corollary. [20, 5.4.3] [20, 5.4.6] *Let X be a CW-complex and $x_0 \in X^0$. Then $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.*
Moreover, $X^1 \hookrightarrow X$ induces an epimorphism $\pi_1(X^1, x_0) \twoheadrightarrow \pi_1(X, x_0)$ with the normal subgroup generated by $\mathrm{conj}_{[v_e]}[\chi^e|_{S^1}]$ as kernel, where v_e is a path joining x_0 and $\chi^e(1)$ in Y and e runs through all 2-cells in the connected component of x_0 in X.

Proof. If X is a finite CW-complex then this follows from 5.38 by induction. By 4.5 any compact subset of X is contained in a finite subcomplex X_0 hence $\pi_1(X, x_0)$ is the inductive limit of the $\pi_1(X_0, x_0)$ for the finite subcomplexes X_0 containing x_0 by 5.31, hence the result holds in general. □

5.41 Example. [20, 5.4.7] Since $\mathbb{P}^n = \mathbb{P}^2 \cup e^3 \cup \dots \cup e^n$ we have $\pi_1(\mathbb{P}^n) \cong \pi_1(\mathbb{P}^2) \cong \mathbb{Z}_2$.

5.42 Definition. [20, 5.5.11] A CW-complex X with $X = X^1$ is called a GRAPH. A graph is called TREE if it is simply connected.

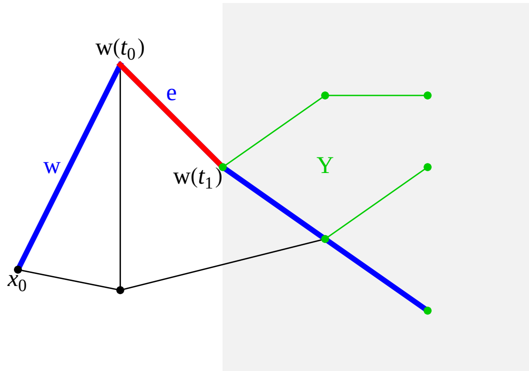
5.43 Lemma. [20, 5.5.12] *A connected graph is a tree iff it is contractible.*

Proof. ($\Rightarrow$) Let X^0 be the 0-skeleton of a tree X. And let $x_0 \in X^0$ be fixed. Every $x \in X^0$ can be connected by a path with x_0, which gives a homotopy $X^0 \to X$. By 4.18 it can be extended to a homotopy $h_t : X \to X$ with $h_0 = \mathrm{id}_X$ and $h_1(X^0) = \{x_0\}$. Let $e \subseteq X$ be a 1-cell with characteristic map $\chi_e : I \cong D^1 \to X$. Then $[h_1 \circ \chi_e] \in \pi_1(X, x_0) = \{1\}$, hence there is a homotopy $k_t^e : (I, \dot I) \to (X, \{x_0\})$ with $k_0^e = h_1 \circ \chi_e$ and $k_1^e(I) = \{x_0\}$. Let $\tilde k_t^e : X^0 \cup e \to X$ be defined by $\tilde k_t^e(X^0) = \{x_0\}$ and $\tilde k_t^e = k_t^e \circ \chi_e^{-1}$ on e. Taking the union of all $\tilde k_t^e$ gives a homotopy $\tilde k_t : X^1 \to X$ between h_1 and the constant map x_0. □

5.44 Lemma. [20, 5.5.13] *Every connected graph X contains a maximal tree. Any maximal tree in X contains all vertices of X.*

Proof. Let $\mathcal{M}$ be the set of trees of X ordered by inclusion. Since the union of any linear ordered subset of $\mathcal{M}$ is a tree (use 4.5), we get by Zorns lemma a maximal tree $Y \subseteq X$.

Let Y be a maximal tree and suppose that there is some $x_0 \in X^0 \setminus Y^0$. Let $w : I \to X$ be a path-connecting x_0 and Y. Let t_1 be minimal in $w^{-1}(Y)$ (hence $w(t_1) \in Y^0$) and $t_0 < t_1$ be maximal in $w^{-1}(X^0 \setminus Y^0)$. Then $w([t_0, t_1])$ is the closure of a 1-cell e and $Y \cup \bar{e}$ is a larger tree, since Y is an SDR of $Y \cup \bar{e}$ by deformation along $\bar{e}$. □



5.45 Corollary. [20, 5.5.17] *Every connected CW-space is homotopy equivalent to a CW-complex with just one 0-cell.*

Proof. The 1-skeleton X^1 is connected since any path with endpoints in X^0 is homotopic to a cellular path $I \to X^1$ by 4.20. For a maximal tree Y in X^1 as constructed in 5.44 we have that $X \twoheadrightarrow X/Y$ is a homotopy equivalence by 2.47 since Y is contractible by 5.43 and (X, Y) has the HEP by 4.18. □

5.46 Proposition. Fundamental group of graphs. [20, 5.5.14]
Let X be a connected graph and $x_0 \in X^0$. Let $Y \subseteq X$ be a maximal tree. For every 0-cell x choose a path v_x in Y connecting x_0 with x. And for every 1-cell $e \subseteq X \setminus Y$ with characteristic mapping $\chi^e : I \cong D^1 \to X^1$ let $s(e) := [v_{\chi^e(0)}][\chi^e][v_{\chi^e(1)}]^{-1} \in \pi_1(X, x_0)$. Then s induces an isomorphism

$$\mathcal{F}(\{e : e \text{ is 1-cell in } X \setminus Y\}) \xrightarrow{\cong} \pi_1(X, x_0),$$

i.e. $\pi_1(X, x_0)$ is the free group generated by $\{s(e) : e$ is 1-cell in $X \setminus Y\}$.

Proof. As in the proof of 5.45 the quotient mapping $p : X \to X/Y$ is a homotopy-equivalence onto a CW-space with just one 0-cell Y. By 4.15 $X/Y \cong \bigvee_e S^1$, where e runs through the (1-)cells in $X \setminus Y$, see also 4.16. Thus $\pi_1(X, x_0) \cong \pi_1(X/Y, y_0) \cong \pi_1(\bigvee_e S^1) \cong \mathcal{F}(\{e : e \text{ is 1-cell in } X \setminus Y\})$ by 5.37. The inverse of this isomorphism is given by $e \mapsto [v_{\chi^e(0)} \cdot \chi^e \cdot v_{\chi^e(1)}^{-1}] = s(e)$. □

5.47 Corollary. [20, 5.5.16] *Let X be a finite connected graph with d_0 vertices and d_1 edges. Then $\pi_1(X)$ is a free group of $1 - d_0 + d_1$ generators.*

Proof. By induction we show that for each $1 \leq n \leq d_0$ there is a tree $Y_n \subseteq X$ with n vertices and $n-1$ edges: Let Y_n for $n < d_0$ be given and choose a point $x_0 \in X^0 \setminus Y_n$ and a path w connecting x_0 with Y_n. Then proceed as in the proof of 5.44 to find an edge $w([t_0, t_1])$ connecting a vertex outside Y_n with one in Y_n. Now $Y_{n+1} = Y_n \cup w([t_0, t_1])$ is the required tree with one more vertex and one more edge.

By 5.46 the result follows, since there are $d_1 - (d_0 - 1)$ many 1-cells not in Y_{d_0}. □

5.48 Theorem. Fundamental group of CW-complexes. [20, 5.6.4]
Let X be a CW-complex with maximal tree Y.

Let generators $s(e^1)$ be constructed for every $e^1 \in X^1 \setminus Y$ as in 5.46.
For every 2-cell $e^2 \in X^2$ define $r(e^2) := [u \cdot \chi_{e^2}|_{S^1} \cdot u^{-1}] \in \pi_1(X^1, x_0)$, where u is a path from x_0 to $\chi_{e^2}(1)$ in X^1 and $\chi_{e^2} : D^2 \to \bar{e}^2$ a characteristic mapping. Then

$$\pi_1(X, x_0) \cong \Big\langle \big\{ s(e^1) : e^1 \text{ is 1-cell in } X^1 \setminus Y \big\} : \big\{ r(e^2) : e^2 \in X^2 \big\} \Big\rangle.$$

Proof. By 5.40 the mapping $\pi_1(X^1, x_0) \to \pi_1(X^2, x_0) \cong \pi_1(X, x_0)$ induced by $X^1 \hookrightarrow X^2 \hookrightarrow X$ is surjective and its kernel is the normal subgroup generated by $r(e^2) = [u \cdot \chi_{e^2}|_{S^1} \cdot u^{-1}] \in \pi_1(X^1)$. Finally, $\pi_1(X^1) \cong \mathcal{F}(\{s(e^1) : e^1 \text{ is 1-cell in } X^1 \setminus Y\})$ by 5.46. $\square$

5.49 Remark. [20, 5.6.5] *For every group representation $\langle S : R \rangle$ there is a 2-dimensional CW-complex X denoted* $\mathrm{CW}(S : R)$ *with $\pi_1(X) \cong \langle S : R \rangle$.*

Proof. Let $X^1 := \bigvee_S S^1$. Every $r \in R \subseteq \mathcal{F}(S) \cong \pi_1(X^1)$ is the homotopy class of a curve mapping $f_r : S^1 \to X^1$ and we glue a 2-cell to X^1 via this mapping. I.e. $X = CW(S : R) := X^1 \bigcup_f (\bigsqcup_{r \in R} D^2)$, where $f := \bigsqcup_{r \in R} f_r$. $\square$

Note that this construction depends on the choice of the $f_r \in [r]$. However, different choices give rise to homotopy equivalent spaces by 2.45. Moreover, they depend on the representation $\langle S : R \rangle$ and not only on its isomorphy class, see the following remark and 5.51.

5.50 Proposition. [20, 5.8.6] *Every connected CW-complex of dimension less or equal to 2 is homotopy equivalent to $CW(S : R)$ for* **some** *representation $\langle S : R \rangle$ of its fundamental group.*

Proof. Choose a maximal tree $Y \subseteq X^1$. Then by the proof of 5.46 we have that X is homotopy equivalent to X/Y, which has as 1-skeleton $\bigvee_S S^1$, where $S := \{e : e \text{ is 1-cell in } X \setminus Y\}$. For every 2-cell e of X/Y (equivalently, of X) we choose a characteristic map χ^e. Thus $X/Y = (\bigvee_S S^1) \cup_{\bigsqcup_e \chi^e|_{S^1}} \bigsqcup_e D^2$. By 2.32.3 we can deform $\chi^e|_{S^1}$ to a base point preserving map $f^e : S^1 \to \bigvee_S S^1$. Hence by 2.45 X/Y is homotopy equivalent to $CW(S : R)$, where $R := \{f^e : e \text{ is 2-cell of } X/Y\}$. $\square$

Remark. Note, that this does **not** solve the isomorphy problem for 2-dimensional CW-complexes: Obviously, $X_1 \sim X_2 \Rightarrow \pi_1(X_1) \cong \pi_1(X_2)$. However, $\pi_1(X_1) \cong \pi_1(X_2) \Rightarrow \exists (S_i, R_i)$ with $\pi_1(X_i) \cong \langle S_i : R_i \rangle$ with $X_i \sim CW(S_i : R_i)$ by 5.50. Despite $\langle S_1 : R_1 \rangle \cong \langle S_2, R_2 \rangle$ it does not follow that $X_1 \sim CW(S_1 : R_1) \sim CW(S_2 : R_2) \sim X_2$, as $\pi_1(S^2) = \{1\} = \pi_1(\{*\})$ with $S^2 \not\sim \{*\}$ by 8.43 and 2.36.1 shows.

The following lemma shows exactly how the homotopy type might change while passing to other representations of the same group (using the Tietze operations of 5.27).

5.51 Lemma. [20, 5.8.7]
We have $CW(S : R \cup \{r\}) \sim CW(S : R) \vee S^2$ for $r \in \langle R \rangle_{NG} \setminus R$ and $CW(S \cup \{s\} : R \cup \{s^{-1} w\}) \sim CW(S : R)$ for $s \notin S$ and $w \in \mathcal{F}(S)$.

This shows that $CW(\langle S : R \rangle) := CW(S : R)$ would not be well-defined.

Proof. Let $X := CW(S : R)$ and $Y := CW(S : R \cup \{r\})$ with $r \in \langle R \rangle_{NG} \setminus R$. Then $Y = X \cup_f D^2$, where $f : S^1 \to \bigvee_S S^1 = X^1 \subseteq X$ is such that $[f] = r \in \pi_1(\bigvee_S S^1) = \mathcal{F}(S)$. Since $r \in \langle R \rangle_{NG}$, we have that $[f]_X = 1 \in \pi_1(X) =$

$\pi_1(\bigvee_S S^1)/\langle R\rangle_{NG}$, hence $f \sim 0$ in X. Thus $Y = X \cup_f D^2 \sim X \cup_0 D^2 = X \vee S^2$ by 2.34.3.

Let $X := CW(S : R)$ and $Y := CW(S \cup \{s\} : R \cup \{s^{-1}w\})$.
Then $Y = (X \vee S^1) \cup_f D^2$, where $f = \sigma^{-1} \cdot \omega$ for the inclusion $\sigma : S^1 \to X \vee S^1$ and $w = [\omega] \in \pi_1(X) \cong \mathcal{F}(S)$. Thus $Y = (X \vee S^1) \cup_f D^2 = (X \cup_0 D^1) \cup_{\sigma^{-1}\cdot\omega} D^2 = X \cup_{f|_{S^1_-}} D^2 \sim X$ and since the lower semi-circle $S^1_- \subseteq D^2$ is an SDR we have that X is also an SDR in Y, by 2.37. $\square$

5.52 Example. [20, 5.7.1]
The fundamental group of the orientable compact surface of genus $g \geq 0$ is

$$\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]\rangle.$$

That of the non-orientable compact surface of genus $g \geq 1$ is

$$\langle \alpha_1, \dots, \alpha_g : \alpha_1^2 \cdot \dots \cdot \alpha_g^2\rangle.$$

Proof. By 1.94 these surfaces are obtained by gluing one 2-cell e to a join $\bigvee S^1$ of $2g$, respectively g, many S^1 and the gluing map is given by $i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \dots$ and $i_1^2 \cdot \dots \cdot i_g^2$, so the homotopy class of the characteristic mapping $\chi^e|_{S^1}$ is $[\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]$ and $\alpha_1^2 \cdot \dots \cdot \alpha_g^2$, respectively. Now apply 5.48 $\square$

5.53 Corollary. [20, 5.7.2] *None of the spaces in 5.52 are homotopy equivalent.*

Proof. The abelization of the fundamental groups are $\mathbb{Z}^{2g}$ and $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. In fact

$$\begin{aligned}
&{}^{ab}\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]\rangle = \\
&\quad = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g], [\alpha_i, \alpha_j], [\beta_i, \beta_j], [\alpha_i, \beta_j]\rangle \\
&\quad \overset{\boxed{5.27.1}}{=\!=} \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : [\alpha_i, \alpha_j], [\beta_i, \beta_j], [\alpha_i, \beta_j]\rangle \\
&\quad = {}^{ab}\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : \emptyset\rangle \\
&\quad = {}^{ab}\mathcal{F}(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g) = \mathbb{Z}^{2g}
\end{aligned}$$

and

$$\begin{aligned}
&{}^{ab}\langle \alpha_1, \dots, \alpha_g : \alpha_1^2 \cdot \dots \cdot \alpha_g^2\rangle \\
&\quad = {}^{ab}\langle \alpha_1, \dots, \alpha_g : (\alpha_1 \cdot \dots \cdot \alpha_g)^2\rangle \\
&\quad \overset{\boxed{5.27.2}}{=\!=} {}^{ab}\langle \alpha_1, \dots, \alpha_g, \alpha : \alpha^2, \alpha^{-1}\alpha_1 \dots \alpha_g\rangle \\
&\quad \overset{\boxed{5.27.2}}{=\!=} {}^{ab}\langle \alpha_1, \dots, \alpha_{g-1}, \alpha : \alpha^2\rangle \\
&\quad = {}^{ab}(\langle \alpha_1, \dots, \alpha_{g-1} : \emptyset\rangle \amalg \langle \alpha : \alpha^2\rangle) \\
&\quad = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2. \quad \square
\end{aligned}$$

Geometric interpretations are the following:

S^2 is simply connected by 5.10 hence π_1 has no generator and no relation.

$S^1 \times S^1$ is a torus. By 5.29 the generators α and β of π_1 are given by $S^1 \times \{1\}$ and $\{1\} \times S^1$, which are a meridian and an equator in the 3-dimensional picture. This can be also seen by gluing the 4 edges of a square as $\alpha\beta\alpha^{-1}\beta^{-1}$. The relation $\alpha\beta = \beta\alpha$ is seen geometrically by taking as homotopy the closed curves given by running through some arc on the equator, then the meridian at that position and then the rest of the equator.

The oriented surface of genus g is obtained by cutting $2g$ holes into the sphere and gluing g cylinders to these holes. Let x_0 be one point on the sphere not contained in the holes. As generators α_j we may take curves through x_0 along some generator $\{x\} \times I$ of the cylinder and as β_i loops around one boundary component $S^1 \times \{0\}$ of the cylinder. Then $\alpha_i \beta_i \alpha_i^{-1}$ describes the loop around the other component and $\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ is a loop around both holes. The product of all these loops is a loop with all holes lying on one side and hence homotopic to a point, cf. 2.36.9

We have discussed the generator α and the relation $\alpha^2 \sim 1$ on $\mathbb{P}^2$ in 5.39.

The non-orientable surface of genus g is obtained from a sphere by cutting g holes and gluing g Möbius-strips. The generators α_j are just conjugates of the middle lines on the Möbius strips. Their squares are homotopic to the boundary circles. And hence the product of all α_i^2 is homotopic to a loop around all holes, which is in turn homotopic to a point.

This shows that beside the sphere, the torus and the projective plane these fundamental groups are not abelian.

6. Covering Maps

We take up the method leading to the calculation $\pi_1(S^1) \cong \mathbb{Z}$ in $\boxed{5.15}$. Basic ingredient was the lifting property of the mapping $t \mapsto \exp(2\pi i t)$, $\mathbb{R} \twoheadrightarrow S^1$, see $\boxed{2.15}$. Its main property can be stated abstractly as follows:

6.1 Definition. Covering maps. [20, 6.1.1] A COVERING MAP $p : Y \to X$ is a surjective continuous map, such that each $x \in X$ has an open neighborhood $U \subseteq X$ for which $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is up to an homeomorphism just the projection $\mathrm{pr} : \bigsqcup_J U \to U$ for some set $J \neq \emptyset$, i.e.

$$\begin{tikzcd}
Y \arrow[d, "p"'] & p^{-1}(U) \arrow[l, hook'] \arrow[d, "p|_{p^{-1}(U)}"] & \bigsqcup_J U \arrow[l, "\cong"'] \arrow[dl, "\mathrm{pr}"] & U \arrow[l, hook', "\mathrm{inj}_j"'] \arrow[dll, bend left, "\mathrm{id}"] \\
X & U \arrow[l, hook'] & &
\end{tikzcd}$$

The images of the summands U in $p^{-1}(U) \subseteq Y$ are called the LEAVES and U is called a TRIVIALIZING NEIGHBORHOOD. The inverse images of points under p are called FIBERS, X is called BASE, and Y TOTAL SPACE.

6.2 Lemma. *Let G be a group acting freely (see $\boxed{1.79}$) on Y.*
Then the action is strictly discontinuous (see $\boxed{1.80}$) if and only if the quotient mapping $\pi : Y \to Y/G$ (see $\boxed{1.77}$) is a covering map.

Proof. ($\Rightarrow$) Since G acts strictly discontinuous we find for each $y \in Y$ a neighborhood V such that: $g \cdot V \cap V \neq \emptyset \Rightarrow g = 1$. Thus $\pi|_V : V \to \pi(V) =: U$ is a bijective quotient mapping (since π is open by the proof of $\boxed{1.79}$) hence a homeomorphism. Furthermore, $\pi^{-1}(U) = G \cdot V = \bigsqcup_{g\in G} g \cdot V$ is open in Y and hence U is open in Y/G.

($\Leftarrow$) If $\pi : Y \to Y/G$ is a convering map, then for every $y \in Y$ there has to exist an open neighborhood $U \subseteq Y/G$ such that $\pi^{-1}(U)$ is a disjoint union of open subsets V homeomorphic via π to U. So $U = \pi(V)$ and $\pi^{-1}(U) = \pi^{-1}(\pi(V)) = G \cdot V$. Suppose $g \cdot V \cap V \neq \emptyset$, i.e. $\exists v \in V$ with $g \cdot v \in V$. Since $\pi(v) = \pi(g \cdot v)$ and π is injective on V we get $v = g \cdot v$ and, since G acts freely, $g = 1$. Thus the action is strictly discontinuous. $\square$

Remark. We used that G acts freely only for ($\Leftarrow$). Otherwise, we could only deduce that g keeps each $v \in V \cap g^{-1}V$ fixed, i.e. g is contained in the isotropy subgroup $G_v := \{g \in G : g \cdot v = v\}$ of v. However, if Y is assumed to be locally connected, we may assume that V is connected and hence $gV = V$ provided $gV \cap V \neq \emptyset$ (since $gV \subseteq \pi^{-1}(\pi V) = \bigsqcup_J V$). Thus $G_v = G_{v'}$ for all $v, v' \in V$ (in fact: $g \in G_v \Rightarrow gv = v \in gV \cap V \Rightarrow gV = V \Rightarrow v' \in gV \cap V \Rightarrow g \in G_{v'}$). Therefore, the family of subsets $\{v \in Y : G_v = H\}$, where H runs through all subgroups of G, forms a partition of Y into open subsets. Thus, if in addition Y is assumed to be connected, then $H := G_v$ is independent on $v \in Y$ (and hence a normal subgroup, see $\boxed{6.16.1}$) and thus the action of G on Y factors to a strictly discontinuous action of $\tilde{G} := G/H$ on Y having the same orbits, i.e. $Y/G = Y/\tilde{G}$.

6.3 Example.

1. Let $Y := \{(\sin(2\pi t), \cos(2\pi t), t) : t \in \mathbb{R}\} \cong \mathbb{R}$ and $p = \mathrm{pr}_{1,2} : Y \to S^1 \subseteq \mathbb{R}^2$. Then p is a covering map: Use 6.2 for $\mathbb{R}/\mathbb{Z} \cong S^1$, see 1.78.2.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\cong", tail, two heads] \arrow[d, "\pi"', two heads] \arrow[dr, "t\mapsto e^{2\pi it}", two heads] & Y \arrow[r, hook] \arrow[d, "p", two heads] & \mathbb{C}\times\mathbb{R} \arrow[d, "\mathrm{pr}"] \\
R/\mathbb{Z} \arrow[r, "\cong"', tail, two heads] & S^1 \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

2. The map $z \mapsto z^n : S^1 \to S^1$ is an n-fold covering map: Use 6.2 for $S^1/\mathbb{Z}_n \cong I/\dot{I} \cong S^1$.
3. The map $S^n \to \mathbb{P}^n$ is a two-fold covering map: Use 6.2 for $S^n/\mathbb{Z}_2 \cong \mathbb{P}^n$, see 1.67 and 1.69.
4. Let $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ be two covering maps, then so is $p_1 \times p_2 : Y_1 \times Y_2 \to X_1 \times X_2$. Examples are $\mathbb{R}^2 \to S^1 \times S^1$, $\mathbb{R}^2 \to \mathbb{R} \times S^1$, and $\mathbb{R} \times S^1 \to S^1 \times S^1$.
5. There is a twofold covering map from $I \times S^1$ to the closed Möbius strip: Use 6.2 for the action of $\mathbb{Z}_2$ on $[-1,1] \times S^1$ given by $(t,\varphi) \mapsto (-t, \varphi + \pi)$, see exercise (1.15).
6. The torus is a two fold covering of Klein's bottle. Use 6.2 for the action of $\mathbb{Z}_2$ on $S^1 \times S^1$ given by $(\varphi, \psi) \mapsto (-\varphi, \psi + \pi)$, see exercise (1.17.3).
7. $\mathbb{Z}_p$ acts freely on S^{2k-1} and the orbit space is the lens space (see 1.81), so we get a covering map $S^{2k-1} \to L(p; q_1, \dots, q_k)$.

6.4 Lemma. [20, 6.1.3] *Let $p : Y \to X$ be a covering map. Then*

1. *The fibers are discrete in Y.*
2. *Every open subset of a trivializing set is trivializing.*
3. *Let $A \subseteq X$. Then $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.*
4. *If $B \subseteq Y$ is connected and $p(B) \subseteq U$ for some trivializing set U, then B is contained in some leaf.*
5. *The mapping p is a surjective local homeomorphism and hence an open quotient mapping.*

Proof. (1) Points in the fiber are separated by the leaves.

(2) and (3) Take the restriction of the characterizing diagram.

(4) B is covered by the leaves. Since each leaf is open, so is the trace on B. Since B is connected only one leaf may hit B, thus B is contained in this leaf.

(5) Obviously the projection is a local homeomorphism. Hence it is open and thus a quotient mapping. □

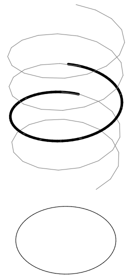
Lemma. *Let X be a connected Hausdorff space and $Y \neq \emptyset$ compact. Then every local homeomorphism $f : Y \to X$ is a covering map.*

Proof. Since f is a local homeomorphism, the fibers $f^{-1}(x)$ are discrete and closed and hence finite since Y is compact.

Let us show next that f is surjective. In fact the image is open in X, since f is a local homeomorphism. It is closed, since Y is compact and X is Hausdorff. Since X is assumed to be connected and $Y \neq \emptyset$ it has to be all of X.

Let $x \in X$. Choose pairwise disjoint neighborhoods V_y for each $y \in f^{-1}(x)$ which are mapped homeomorphically onto a corresponding neighborhood of x. By taking the inverse images of the (finite!) intersection $U := \bigcap_{y \in f^{-1}(x)} f(V_y)$ in the V_y we may assume that the image is the same neighborhood U for all $y \in f^{-1}(x)$. Hence U is trivializing with leaves V_y and thus $p : Y \to X$ is a covering. $\square$

Example.
Not every surjective local homeomorphism is a covering map.



Take for example an open interval $I \subset \mathbb{R}$ of length more than 1. Then the restriction $I \to S^1$ of the covering map from 6.3.1 is not a covering map.

6.5 Definition. Homomorphisms of coverings.

Let $p' : Y \twoheadrightarrow X'$ and $p : Y \twoheadrightarrow X$ be two covering maps with the same total space Y. A HOMOMORPHISM f of these coverings is a map $f : X' \to X$ such that the diagram to the right commutates.

$$\begin{tikzcd} & Y \arrow[dl, "p'"'] \arrow[dr, "p"] & \\ X' \arrow[rr, "f"] & & X \end{tikzcd}$$

Note that such an f exists, iff p factors over p', i.e. the fibers of p' are contained in fibers of p. If such an f exists it is uniquely determined since p' is onto. So we get a category Cov^Y (a quasi-ordering) of all coverings with total space Y.

Conversely, let $p' : Y' \twoheadrightarrow X$ and $p : Y \twoheadrightarrow X$ be two covering maps with the same base space X. A HOMOMORPHISM f of these coverings is a fiber respecting map $f : Y' \to Y$, i.e. the diagram on the right commutates.

$$\begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "p'"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

We denote the set of all homomorphisms from $p' : Y' \twoheadrightarrow X$ to $p : Y \twoheadrightarrow X$ by $\mathrm{Hom}_X(p', p)$. So we get a category Cov_X of all coverings with base space X.

Note that a homomorphism f is nothing else but a lift of $p' : Y' \twoheadrightarrow X$ along $p : Y \twoheadrightarrow X$. The automorphisms f, i.e. invertible homomorphisms $p \to p$, are also called COVERING TRANSFORMATIONS or DECKTRANSFORMATIONS, and we write $\mathrm{Aut}(p)$ for the group formed by them.

6.6 Remark. Unique lifts along covering maps exist locally.
Let $p : (Y, y_0) \to (X, x_0)$ be a covering map and $g : (Z, z_0) \to (X, x_0)$. Take a trivializing neighborhood U of x_0 and let $\tilde U$ be the leaf of p over U which contains y_0. Then $(p|_{\tilde U}) : \tilde U \to U$ is a homeomorphism and hence $(p|_{\tilde U})^{-1} \circ g : Z \supseteq g^{-1}(U) \to \tilde U \subseteq Y$ is a continuous local lift of g.

$$\begin{tikzcd} & & \tilde U \arrow[r, hook] \arrow[dr, "p|_{\tilde U}"', "\cong"] & p^{-1}(U) \arrow[d] \arrow[dr, hook] & \\ W \arrow[r, hook] \arrow[urr, bend left, "\tilde g"] & g^{-1}(U) \arrow[ur, dotted] \arrow[rr, "g|_{g^{-1}(U)}"'] \arrow[dr, hook] & & U \arrow[dr, hook] & Y \arrow[d, "p"] \\ & & Z \arrow[rr, "g"] & & X \end{tikzcd}$$

Let $\tilde g$ be any continuous (local) lift of g with $\tilde g(z_0) = y_0$. Then $W := \tilde g^{-1}(\tilde U) \subseteq \tilde g^{-1}(p^{-1}(U)) = g^{-1}(U)$ is a neighborhood of z_0 and $\tilde g(W) \subseteq \tilde U$, hence $g = p \circ \tilde g$ implies $(p|_{\tilde U})^{-1} \circ g = (p|_{\tilde U})^{-1} \circ p|_{\tilde U} \circ \tilde g = \tilde g$ on W, i.e. locally the lift of $\tilde g$ is unique.

6.7 Lemma. Uniqueness of lifts. [20, 6.2.4]
Let $p : Y \to X$ be a covering map and let Z be connected.
Then any two lifts of a continuous map $g : Z \to X$, which coincide in one point, are equal. In particular, if g is constant so are its lifts.

Proof. Let g^1, g^2 be two lifts of g. Then the set of points $\{z \in Z : g^1(z) = g^2(z)\}$ is clopen: In fact if U^j is the leaf over U containing of $g^j(z)$, then $g^j = (p|_{U^j})^{-1} \circ g$ on the neighborhood $(g^1)^{-1}(U^1) \cap (g^2)^{-1}(U^2)$ of z by 6.12. Hence either $g^1 = g^2$ or $g^1 \neq g^2$ at each point of this neighborhood. □

6.8 Lemma.

Let X locally path-connected and let $q : Z \to Y$ and $p : Y \to X$ be given.

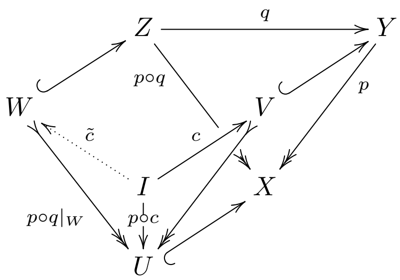
Then the following statements hold:

$$\begin{tikzcd} Z \arrow[rr, "q"] \arrow[dr, "p\circ q"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

1. *If p and $p \circ q$ are coverings and Y is connected, then q is onto.*
2. *If p and $p \circ q$ are coverings and q is onto, then q is a covering.*
3. *If p and q are coverings and X is locally simply-connected, then $p \circ q$ is a covering.*
4. *If q and $p \circ q$ are coverings, then p is a covering.*

Proof. (1) We claim that the image of q is clopen in Y and hence coincides with the connected space Y. For this we consider all leaves $V \subseteq Y$ for p over path-connected open subsets $U \subseteq X$, which are trivializing for p and $p \circ q$. It suffices to show that if such a leaf V meets the image $q(W)$ of a leaf $W \subseteq Z$ over U for $p \circ q$ then it is contained in $q(W)$, since then $q(W) = \bigcup_{V \cap q(W) \neq \emptyset} V$ and $y \in V \setminus q(Z) \Rightarrow V \cap q(Z) = \emptyset$.

So let $w_0 \in W$ be such that $q(w_0) \in V$. Since V has to be path-connected as well, we may connect $q(w_0)$ with any $v \in V$ by a curve c in V. The curve $p \circ c$ has a lift $\tilde{c} = (p \circ q|_W)^{-1} \circ p \circ c$ starting at $w_0 \in (p \circ q)^{-1}(p(c(0)))$ with values in W. By 6.7 the lift $q \circ \tilde{c}$ coincides with c and hence $v = c(1) = q(\tilde{c}(1)) \in q(W)$.



(2) Take a path-connected set $U \subseteq X$ being trivializing for $p \circ q$ and p. Every leaf W of $p \circ q$ over U is mapped by q into some leaf V of p over U: In fact, since the leaves are homeomorphic to U, they are path-connected as well, hence $q(W)$ is completely contained in some leaf V of p over $U = (p \circ q)(W)$ by 6.4.4. Thus $q^{-1}(V)$ is the topological disjoint union of all leaves W of $p \circ q$ over U, which meet $q^{-1}(V)$. Moreover, $q|_W = (p|_V)^{-1} \circ p|_V \circ q|_W = (p|_V)^{-1} \circ (p \circ q)|_W$ is a homeomorphism $W \cong U \cong V$.

(3) Let p and q be coverings, with X locally simply connected. Then the leaves V_j of p over a simply connected trivializing set U are again simply connected, hence are trivializing neighborhoods of q as will be shown in 6.13. Hence $(p \circ q)^{-1}(U) = q^{-1}(p^{-1}(U)) = q^{-1}(\bigsqcup_{j \in J} V_j) = \bigsqcup_{j \in J} q^{-1}(V_j)$ and $q^{-1}(V_j) \cong \bigsqcup_{J_j} V_j$. Since the restriction $(p \circ q)|_W = p|_{V_j} \circ q|_W$ is a homeomorphism $W \cong V_j \cong U$ for each leaf W over V_j, the map $p \circ q$ is a covering as well.

($\boxed{4}$) Let $p \circ q$ and q be coverings. We claim that p is a covering. Let W be a leaf of $p \circ q$ over a path-connected trivializing set U. Since q is an open mapping, $V := q(W)$ is open in Y. Since $(p \circ q)|_W$ is an embedding the same is true for $q|_W$. Thus $q|_W : W \cong V$ is a homeomorphism and consequently also $p|_V = (p \circ q)|_W \circ (q|_W)^{-1} : V \to W \to U$.
We claim that $q(W)$ is a path-component of $p^{-1}(U)$ and hence the distinct ones among these sets form a disjoint partition of $p^{-1}(U)$:

Let $z_0 \in W$ be choosen and let c be a continuous curve in $p^{-1}(U)$ from $q(z_0)$ to some point $y \in p^{-1}(U)$. We have a lift $\tilde{c} := (p \circ q|_W)^{-1} \circ (p \circ c)$ into W of $p \circ c$ with initial value z_0. Then c and $q \circ \tilde{c}$ are two lifts of $p \circ c$ with initial value $q(z_0)$ hence coincide by $\boxed{6.7}$ and thus $y = c(1) = q(\tilde{c}(1)) \in q(W)$.

$$\begin{tikzcd}
W \arrow[r, tail, two heads, "q"] \arrow[ddr, two heads, "p\circ q|_W"'] & V \arrow[r, hook] & p^{-1}(U) \arrow[ddl, two heads, "p"] \\
& I \arrow[ul, dotted, "\tilde{c}"'] \arrow[ur, "c"] \arrow[d, "p\circ c"] & \\
& U &
\end{tikzcd}$$

□

6.9 The category $\mathrm{Cov}^Y_{\mathbf{norm}}$.
We try to get a description of the category Cov^Y of coverings with fixed total space Y. For every group G acting strictly discontinuous on Y (and w.l.o.g. we may assume that $G \subseteq \mathrm{Homeo}(Y)$) we get a covering $\pi : Y \to Y/G$ by $\boxed{6.2}$.

Can we recover G from the covering $\pi : Y \to Y/G$?

Yes: *If Y is connected then* $\mathrm{Aut}(\pi) = G$:
Obviously, $G \subseteq \mathrm{Aut}(\pi)$. Conversely, let $\Phi \in \mathrm{Aut}(\pi)$, i.e. $\pi(y) = \pi(\Phi(y))$ for all $y \in Y$. Choose $y_0 \in Y$, then there is some $g_0 \in G$ with $g_0 \cdot y_0 = \Phi(y_0)$ since the fibers of π are the G-orbits. Since the two mappings Φ and g_0 cover the identity (i.e. are lifts of π along π) and coincide on y_0 they are equal by $\boxed{6.7}$.

Note, that if $G' \leq G$ is a subgroup then $\pi : Y \to Y/G$ factors over $\pi' : Y \to Y/G'$ to a unique mapping $f : Y/G' \to Y/G$, i.e. a homomorphism $\pi' \to \pi$. So we get a functor $\mathrm{Act}_{str.dis.}(Y) \to \mathrm{Cov}^Y$ from the partially ordered set (hence category) $\mathrm{Act}_{str.dis.}(Y)$ of subgroups of $\mathrm{Homeo}(Y)$ for which the action on Y is strictly discontinuous.

Is this functor DENSE, i.e. is every covering mapping $p : Y \to X$ up to isomorphy in the image of this functor? For this we have to find a subgroup $G \leq \mathrm{Homeo}(Y)$ for which the action on Y is strictly discontinuous and such that $p \cong (\pi : Y \to Y/G)$. The natural candidate is $G := \mathrm{Aut}(p)$.
Obviously the action of $\mathrm{Aut}(p)$ on Y is strictly discontinuous, since for any leaf $\tilde{U}$ over some trivializing set U and any g in $\mathrm{Aut}(p)$ we have:
$g(\tilde{U}) \cap \tilde{U} \neq \emptyset$ implies $\exists y \in \tilde{U} : g(y) \in \tilde{U}$. From $p(g(y)) = p(y)$ and since $p|_{\tilde{U}} : \tilde{U} \to U$ is injective we conclude that $g(y) = y = \mathrm{id}(y)$, but then $g = \mathrm{id}$ by $\boxed{6.7}$.
Since $p \circ g = p$ for every $g \in \mathrm{Aut}(p)$, we have that p is constant on the $\mathrm{Aut}(p)$-orbits and hence factors to a mapping $Y/\mathrm{Aut}(p) \to X$, which is by $\boxed{6.8.4}$ a covering map, provided X is *locally path-connected*, and hence a quotient map by $\boxed{6.4.5}$.

$$\begin{tikzcd}
& Y \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
Y/\mathrm{Aut}(p) \arrow[rr, dotted, two heads] & & X
\end{tikzcd}$$

This mapping is injective (and hence a homeomorphism) iff every two points in the same fiber of p are in the same orbit under $\mathrm{Aut}(p)$, i.e. iff $\mathrm{Aut}(p)$ acts transitive on the fibers of p. Such covering maps p are called NORMAL, see $\boxed{6.26}$ for a counter-example. Note that for a group G acting strictly discontinuous on Y the covering $\pi : Y \to Y/G$ is obviously normal. Let $\mathrm{Cov}^Y_{\mathrm{norm}}$ denotes the category of normal coverings with total space Y. Then we have:

6.10 Theorem. [20, 6.5.3] *For path-connected and locally path-connected Y we have an equivalence of categories*

$$\mathrm{Cov}^Y_{norm} \sim \mathrm{Act}_{str.dis.}(Y),$$

i.e. there exists also a functor in the opposite direction and the compositions of these two are up to natural isomorphisms the identity.

Proof. The functor $\mathrm{Act}_{str.dis.}(Y) \to \mathrm{Cov}^Y_{\mathrm{norm}}$ discussed above is given by $\mathrm{Homeo}(Y) \geq G \mapsto (\pi : Y \to Y/G)$ and if $G' \leq G$ then $\pi : Y \to Y/G$ factors over $\pi' : Y \to Y/G'$ to a unique mapping $f : Y/G' \to Y/G$, i.e. a homomorphism $\pi' \to \pi$.

Conversely, every homomorphism $f : \pi' \to \pi$ has to be the unique factorization of $\pi : Y \to Y/G$ and induces an inclusion $G' \subseteq G$: $\Phi \in \mathrm{Aut}(\pi') = G' \Rightarrow \pi' \circ \Phi = \pi'$ $\Rightarrow \pi \circ \Phi = f \circ \pi' \circ \Phi = f \circ \pi' = \pi$, i.e. $\Phi \in \mathrm{Aut}(\pi) = G$. Thus the functor is full and faithfull.

It is a general categorical result, that a full, faithful and dense functor is an equivalence. In fact, an inverse is given by selecting for every object in the range category an inverse image up to an isomorphism and by the full and faithfulness this can be extended to a functor.

We have shown in 6.9 that the functor is dense, hence it induces the desired equivalence of categories. □

We now try to desribe the category Cov_X of coverings with base X in algebraic terms. Since the homomorphisms $p' \to p$ are lifts of p' along p we have to study liftings along coverings in more detail.

6.11 Theorem. Lifting of curves. [20, 6.2.2] [20, 6.2.5] *Let $p : Y \to X$ be a covering. Every path $u : I \to X$ has a unique lift ${}^y\tilde{u}$ with ${}^y\tilde{u}(0) = y$ for given $y \in p^{-1}(u(0))$. Paths homotopic relative their initial value have homotopic lifts.*

In particular we have an action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ given by $[u] : y \mapsto {}^y\tilde{u}(1)$, i.e. the end-point of the lift of u, which starts at y.

The total space Y is path-connected iff X is path-connected and this action is transitive, i.e. for all $y_1, y_2 \in p^{-1}(x_0)$ there exists a $g \in \pi_1(X, x_0)$ with $y_1 \cdot g = y_2$ (equivalently: there exists a $y_0 \in p^{-1}(x_0)$ with $y_0 \cdot \pi_1(X, x_0) = p^{-1}(x_0)$).

Proof. By 6.7 we have to show existence of a lift. By considering a path w as a homotopy being constant in the second factor, it is enough to show that homotopies $h : I \times I \to X$ can be lifted.

For this choose a partition of I^2 into squares $Q_{i,j}$, such that $h(Q_{i,j})$ is contained in a trivializing neighborhood $U_{i,j}$ of X. For each fixed j we construct inductively a lift $\tilde{h}^j$ along $\bigcup_i Q_{i,j}$ with initial value y_0 at the bottom left corner, by taking the leaf $\tilde{U}_{i,j}$ over the trivializing neighborhood of $Q_{i,j}$ which contains the image under $\tilde{h}^j$ of the right bottom corner of $Q_{i-1,j}$. Then $\tilde{h}^j|_{Q_{i,j}}$ can be defined as $(p|_{\tilde{U}_{i,j}})^{-1} \circ h|_{Q_{i,j}}$ and agrees with $\tilde{h}^1|_{Q_{i-1,j}}$ on the vertical edge $Q_{i-1,j} \cap Q_{i,j}$, since this is contained in $\tilde{U}_{i-1,j} \cap \tilde{U}_{i,j}$ by 6.4.4. By induction we can show that these lifts agree on the lines formed by horizontal edges: In fact the image of h on a horizontal edge is contained in the intersection of the trivializing sets containing the image of the square above and below. And since the lifts $\tilde{h}^j$ and $\tilde{h}^{j-1}$ are contained in the respective leaves, and thus in the leaf over the intersection, they have to be equal on the edge. We denote the unique homotopy by ${}^{y_0}\tilde{h}$.

Now suppose h is a homotopy rel. $\dot{I}$ between two paths u_0 and u_1 from x_0 to x_1 and let $y_0 \in p^{-1}(x_0)$. The homotopy $\tilde{h}$ with initail value $\tilde{h}(0, s) = y_0$ has as boundary values the (unique) lifts $\tilde{u}_0$ and $\tilde{u}_1$ with $\tilde{u}_j(0) = y_0$. Since $s \mapsto \tilde{h}(1,s)$ is a lift of the constant path x_1, it has to be constant by 6.7, i.e. $\tilde{h}$ is a homotopy rel. $\dot{I}$.

The lifting property gives us a mapping from $\pi_1(X, x_0)$ into the set of mappings $p^{-1}(x_0) \to p^{-1}(x_0)$ by setting $[u](y) := {}^y\tilde{u}(1)$. This is well defined, since curves u homotopic relative $\dot{I}$ have lifts ${}^y\tilde{u}$ homotopic relative $\dot{I}$ and hence have the same end point in $p^{-1}(x_0)$.

Composition law: The lift of ${}^y\widetilde{u \cdot v}$ is ${}^y\tilde{u} \cdot {}^{y'}\tilde{v}$, where $y' := {}^y\tilde{u}(1)$.

Moreover we have $[u \cdot v](y) = {}^y\widetilde{u \cdot v}(1) = ({}^y\tilde{u} \cdot {}^{y'}\tilde{v})(1) = {}^{y'}\tilde{v}(1) = [v](y') = [v]([u](y))$, where $y' = {}^y\tilde{u}(1) = [u](y)$. Hence, we consider this mapping as a **right** action, i.e. we write $y \cdot [u]$ for $[u](y)$. Then we have $y \cdot ([u] \cdot [v]) = (y \cdot [u]) \cdot [v]$.

In particular, $[u]$ acts on $p^{-1}(x_0)$ as bijection.

Now the statement on path-connectedness:
If Y is path-connected then so is the surjective continuous image X. Furthermore a curve v connecting $y, y' \in p^{-1}(x_0)$ has a closed curve $u := p \circ v$ as image and $v = {}^y\tilde{u}$, so $y \cdot [u] = y'$, i.e. the action is transitive.

Conversely, let $y \in Y$ be arbitrary. Since X is path-connected we have a curve u connecting $p(y)$ with x_0. Its lift ${}^y\tilde{u}$ connects y with $y' := {}^y\tilde{u}(1) \in p^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitive on $p^{-1}(x_0)$ there is a $[u'] \in \pi_1(X, x_0)$ with ${}^{y'}\tilde{u}'(1) = y' \cdot [u'] = y_0$, i.e. the curve ${}^{y'}\tilde{u}'$ connects y' with y_0 and ${}^y\tilde{u} \cdot {}^{y'}\tilde{u}'$ connects y with y_0. □

6.12 Corollary. [20, 6.3.5] *Let X be path-connected. Then the fibers of any covering $p : Y \to X$ can be mapped bijectively onto one another by lifting a curve connecting the foot points.*

Proof. Let $F_0 := p^{-1}(x_0)$, $F_1 := p^{-1}(x_1)$ and let u be a path from x_0 to x_1 then $y \mapsto {}^y\tilde{u}(1)$ defines a mapping $F_0 \to F_1$ and $y \mapsto {}^y\widetilde{u^{-1}}(1)$ a mapping $F_1 \to F_0$ and these mappings are inverse to each other, since the lift of the curve $u \cdot u^{-1} \sim 0$ is 0-homotopic rel. $\dot{I}$ and hence closed. □

6.13 Corollary. *Let X be simply connected and $p : Y \to X$ be a path-connected covering. Then p is a homeomorphism. In particular every simply connected open subset in a locally path-connected base space of a covering is a trivializing neighborhood.*

Proof. Since $\pi_1(X, x_0) = \{1\}$ acts transitively on the fiber $p^{-1}(x_0)$ by 6.11, the fiber has to be single pointed, hence p is injective and thus a homeomorphism.
For the second statement consider a simply connected open subset $U \subseteq X$ and the partition of $p^{-1}(U)$ into (open!) path-connected components $\tilde{U}$. Then $p|_{\tilde{U}} : \tilde{U} \to U$ is a covering map (since every leaf over a path-connected trivialising subset of U is either completely contained in $\tilde{U}$ or in its complement and U is simply connected) and hence a homeomorphism by the first part. □

6.14 General lifting theorem. [20, 6.2.6] *Let Z be path-connected and locally path-connected. Let $p : Y \to X$ be a covering and $g : Z \to X$ continuous. Let $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$ be base points and p and g base point preserving. Then g has a base point preserving lift $\tilde{g}$ iff* $\operatorname{im}(\pi_1(g)) \subseteq \operatorname{im}(\pi_1(p))$.

Proof. ($\Rightarrow$) If $g = p \circ \tilde{g}$ then $\operatorname{im}(\pi_1(g)) = \operatorname{im}(\pi_1(p) \circ \pi_1(\tilde{g})) \subseteq \operatorname{im}(\pi_1(p))$.

($\Leftarrow$) Let $z \in Z$ be arbitrary. Since Z is path-connected we may choose a path u from z_0 to z and take the lift ${}^{y_0}\widetilde{g \circ u}$ and define $\tilde{g}(z) := {}^{y_0}\widetilde{g \circ u}(1)$.

First we have to show that this definition is independent from the choice of u. So let u' be another path from z_0 to z. Then $g \circ (u' \cdot u^{-1}) = (g \circ u') \cdot (g \circ u)^{-1}$ is a closed path through x_0, hence by assumption there exists a closed path v through y_0 with $p \circ v \sim (g \circ u') \cdot (g \circ u)^{-1}$ rel. $\dot{I}$ and hence $(p \circ v) \cdot (g \circ u) \sim (g \circ u')$ rel. $\dot{I}$. Thus ${}^{y_0}\widetilde{g \circ u'}(1) = {}^{y_0}\big((p \circ v) \cdot (g \circ u)\big)^{\sim}(1) = ({}^{y_0}\widetilde{p \circ v} \cdot {}^{y_0}\widetilde{g \circ u})(1) = {}^{y_0}\widetilde{g \circ u}(1)$.

Remains to show that $\tilde{g}$ is continuous. Let $z_1 \in Z$ be fixed and let $\tilde{U}$ be a leaf over a trivializing neighborhood U of $g(z_1)$ containing $\tilde{g}(z_1)$. Let W be a path-connected neighborhood of z_1 with $g(W) \subseteq U$ and let u be a path from z_0 to z_1 and hence $y_1 := \tilde{g}(z_1) = {}^{y_0}\widetilde{g \circ u}(1)$. For every $z \in W$ we can choose a path w_z in W from z_1 to z. Hence $\tilde{g}(z) = {}^{y_0}(g \circ (u \cdot w_z))^{\sim}(1) = ({}^{y_0}\widetilde{g \circ u} \cdot {}^{y_1}\widetilde{g \circ w_z})(1) = {}^{y_1}\widetilde{g \circ w_z}(1)$. But since $g \circ w_z$ is contained in the trivializing neighborhood U and $\tilde{U}$ is the leaf over U containing the lift y_1, we have that ${}^{y_1}\widetilde{g \circ w_z} = (p|_{\tilde{U}})^{-1} \circ g \circ w_z$, and hence $\tilde{g}(z) = {}^{y_1}\widetilde{g \circ w_z}(1) = ((p|_{\tilde{U}})^{-1} \circ g)(z)$ and thus is continuous. □

Thus it is important to determine the image of $\pi_1(p) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

6.15 Proposition. [20, 6.3.1] *Let $p : (Y, y_0) \to (X, x_0)$ be a covering. Then the induced map $\pi_1(p) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective and its image is formed by those $[u] \in \pi_1(X, x_0)$ for which for (some) any representative u the lift ${}^{y_0}\tilde{u}$ is closed, i.e. by those $g \in \pi_1(X, x_0) =: G$ which act trivial on y_0. They form the so called* ISOTROPY SUBGROUP *$G_{y_0} := \{g \in G : y_0 \cdot g = y_0\}$ of G at y_0 with respect to the action of G on $p^{-1}(x_0)$.*

$$\pi_1(p) : \pi_1(Y, y_0) \cong \pi_1(X, x_0)_{y_0} \subseteq \pi_1(X, x_0).$$

Proof. Injectivity: Let $[v] \in \pi_1(Y, y_0)$ be such that $1 = [p \circ v]$, i.e. $p \circ v \sim \mathrm{const}_{x_0}$. By 6.11 we have $v = {}^{y_0}\widetilde{p \circ v} \sim {}^{y_0}\widetilde{\mathrm{const}_{x_0}} = \mathrm{const}_{y_0}$ rel. $\dot{I}$, hence $[v] = 1$.

If some lift v of u is closed, then $\pi_1(p)[v] = [p \circ v] = [u]$, hence $[u] \in \mathrm{im}(\pi_1(p))$. Conversely let $[u] \in \mathrm{im}\, \pi_1(p)$. Then there exists a closed curve v through y_0 with $[p \circ v] = \pi_1(p)[v] = [u]$, hence $u \sim p \circ v$ rel. $\dot{I}$, and so ${}^{y_0}\tilde{u} \sim {}^{y_0}\widetilde{p \circ v} = v$ rel. $\dot{I}$, thus ${}^{y_0}\tilde{u}$ is closed as well. □

In view of 6.11 we study now abstractly given transitive (right) actions of a group G on sets (i.e. discrete spaces) F.

6.16 Lemma. Transitive actions and isotropy subgroups.
*Let G act transitively on F (and on F') from the right. A G-*EQUIVARIANT MAPPING *or G-*HOMOMORPHISM *is a mapping $\varphi : F \to F'$, which satisfies $\varphi(y \cdot g) = \varphi(y) \cdot g$ for all $y \in G$ and $g \in G$. We write $\mathrm{Hom}_G(F, F')$ for the set of all G-homomorphisms $F \to F'$ and $G_y := \{g \in G : y \cdot g = y\}$ for the* ISOTROPY SUBGROUP *of $y \in F$. Then*

1. *We have $G_{y \cdot g} = g^{-1} G_y g$.*
2. *$\{G_y : y \in F\}$ is a conjugacy class of subgroups of G, i.e. an equivalence class of subgroups of H with respect to the relation of being conjugate.*
3. *Let H be a subgroup of G. Then the set $G/H := \{Hg : g \in G\}$ of right classes admits a unique (transitive) right G-action, such that the canonical projection $\pi : G \twoheadrightarrow G/H$, $g \mapsto Hg$ is G-equivariant, where the action of G on G is given by multiplication from the right side.*
4. *For $y \in F$ the G-equivariant mapping $G \twoheadrightarrow F$ given by $g \mapsto y \cdot g$ factors to a G-isomorphism $G/G_y \xrightarrow{\cong} F$.*
5. *For $\varphi \in \mathrm{Hom}_G(F, F')$ we have $G_y \subseteq G_{\varphi(y)}$. Conversely if $y_0 \in F$ and $y_1 \in F'$ satisfy $G_{y_0} \subseteq G_{y_1}$, then there is a unique $\varphi \in \mathrm{Hom}_G(F, F')$ with $\varphi(y_0) = y_1$.*

6. $F \cong_G F' \Leftrightarrow \{G_y : y \in F\} = \{G_{y'} : y' \in F'\}$
$\Leftrightarrow \{G_y : y \in F\} \cap \{G_{y'} : y' \in F'\} \neq \emptyset$.

Note, that we refrain from writing the quotient G/H correctly as $H\backslash G$.

Proof. ($\boxed{1}$) We have $G_{y\cdot g} = g^{-1}G_y g$, since $h \in G_{y\cdot g} \Leftrightarrow y\cdot g\cdot h = y\cdot g \Leftrightarrow y\cdot(ghg^{-1}) = y$, i.e. $ghg^{-1} \in G_y$.

($\boxed{2}$) Since G acts transitively, $\{G_y : y \in F\} = \{G_{y_0\cdot g} = g^{-1}G_{y_0}g : g \in G\}$ is a conjugacy class by ($\boxed{1}$).

($\boxed{3}$) The only possible action of G on G/H such that π is G-equivariant is given by $Hg \cdot g' = \pi(g) \cdot g' := \pi(g \cdot g') = \pi(gg') = Hgg'$. The so defined action makes sense, since $Hg_1 = Hg_2 \Rightarrow g_2 g_1^{-1} \in H \Rightarrow (Hg_1) \cdot g := Hg_1 g = Hg_2 g =: (Hg_2) \cdot g$.

($\boxed{4}$) Consider $\mathrm{ev}_y : G \to F$ given by $g \mapsto y \cdot g$. This G-equivariant mapping has image $y \cdot G = F$, since G acts transitively. Furthermore g' and g have the same image $y \cdot g' = y \cdot g$ iff $g'g^{-1} \in G_y$, so ev_y factors to a G-isomorphism $G/G_y \to F$.

($\boxed{5}$) We have $G_y = \{g : y \cdot g = y\} \subseteq \{g : \varphi(y) \cdot g = \varphi(y \cdot g) = \varphi(y)\} = G_{\varphi(y)}$. Conversely let $G_{y_0} \subseteq G_{y_1}$ and $y \in F$. Since G acts transitively there exists a $g \in G$ with $y = y_0 \cdot g$. Define $\varphi(y) = \varphi(y_0 \cdot g) := \varphi(y_0) \cdot g = y_1 \cdot g$. This definition makes sense, since $y_0 \cdot g' = y_0 \cdot g$ implies $g'g^{-1} \in G_{y_0} \subseteq G_{y_1}$ and hence $y_1 \cdot g' = y_1 \cdot g$. By construction φ is G-equivariant.

($\boxed{6}$) (1⇒2) Let $\varphi : F \to F'$ be a G-equivariant isomorphism. Then $G_y \subseteq G_{\varphi(y)} \subseteq G_{\varphi^{-1}(\varphi(y))} = G_y$ by ($\boxed{5}$).

(1⇐3) By assumption there are $y \in F$ and $y' \in F'$ with $G_y = G_{y'}$ and therefore $F \cong_G G/G_y = G/G_{y'} \cong_G F'$ by $\boxed{4}$. □

6.17 The category $\mathrm{Subgr}(G)$**.**
We use $\boxed{6.16.3}$ for associating to each subgroup $H \leq G$ the transitive action of G on G/H. In order to extend this to a full and faithfull functor, we have to define the morphisms $H \to H'$ between subgroups appropriately:
Let $\varphi \in \mathrm{Hom}_G(G/H, G/H')$ and $y_0 := H \in G/H$. Then $G_{y_0} := \{g \in G : Hg = y_0 \cdot g = y_0 = H\} = H$. By $\boxed{6.16.5}$ φ is uniquely determined by $y_1 := \varphi(y_0) =: H'g_1 \in G/H'$ with $H = G_{y_0} \subseteq G_{y_1} = G_{H'g_1} = g_1^{-1}H'g_1$ by $\boxed{6.16.1}$. So we define

$$\mathrm{Hom}(H, H') := \{g \in G : gH \subseteq H'g\}/H',$$

where H' acts on $\{g : gH \subseteq H'g\}$ by multiplication from the left, since $gH \subseteq H'g$ and $h' \in H'$ implies $h'g\,H \subseteq h'H'\,g = H'\,g = H'\,h'g$.
Then the set $\mathrm{Subgr}(G)$ of subgroups $H \leq G$ and $H''g' \circ H'g := H''g'g$ as composition of these morphisms forms a category:
The composition $H''g' \circ H'g := H''g'g$ is well-defined, since $gH \subseteq H'g$ and $g'H' \subseteq H''g' \Rightarrow g'gH \subseteq g'H'g \subseteq H''g'g$ and since $H''(h''g')(h'g) = H''g'h'g = H''\bar{h}''g'g = H''g'g$ for $\bar{h}'' := g'h'(g')^{-1} \in g'H'(g')^{-1} \subseteq H''$.
The identity on H is given by $H = H\,1$.

Theorem. *We have an equivalence* $\mathrm{Act}_{tr}(G) \sim \mathrm{Subgr}(G)$ *of categories.*

Proof. The functor $\mathrm{Subgr}(G) \to \mathrm{Act}_{\mathrm{tr}}(G)$ is given on morphisms by:

$$\mathrm{Hom}(H, H') \ni H'g_1 \mapsto \varphi(: H\,g \mapsto H'\,g_1 g) \in \mathrm{Hom}_G(G/H, G/H').$$

This is well-defined, since $H\,g = H\,\bar{g} \Rightarrow g_1\bar{g}(g_1 g)^{-1} = g_1\bar{g}g^{-1}g_1^{-1} \in g_1 H g_1^{-1} \subseteq H'$ $\Rightarrow H'\,g_1 g = H'\,g_1\bar{g}$ and since $H'\,(h'g_1)g = H'\,g_1 g$ for $h' \in H'$.

Functorality: $H = H\,1 \mapsto \mathrm{id}_{G/H}$ and the composition $H''g_2 \circ H'g_1 := H''g_2g_1$ is mapped to $H\,g \mapsto H'\,g_1g \mapsto H''\,g_2g_1g$.

The functor is faithfull: $\forall g : Hg \mapsto H'g_1g = H'\bar{g}_1g \Rightarrow H'g_1 = H'\bar{g}_1 \in \mathrm{Hom}(H,H')$.

The functor is full by what we have shown above.

The functor is dense by 6.16.4. $\square$

6.18 Corollary. [20, 6.3.3] *Let G act transitively on F from the right. With $\mathrm{Aut}_G(F)$ we denote the group of all G-equivariant isomorphisms $F \to F$. For a subgroup H of G one denotes with $\mathrm{Norm}_G(H) := \{g \in G : H = g^{-1}Hg\}$, the* NORMALIZER *of H in G, i.e. the largest subgroup of G, which contains H as normal subgroup. Then we have a group isomorphism*

$$\mathrm{Aut}_G(F) \cong \mathrm{Norm}_G(G_{y_0})/G_{y_0}$$

Proof. By 6.16.4 and 6.17 we have

$$\mathrm{Hom}_G(F,F') \cong \mathrm{Hom}_G(G/H,G/H') \cong \mathrm{Hom}(H,H') := \{g : gH \subseteq H'g\}/H',$$

where $H := G_y$ and $H' := G_{y'}$ are isotropy subgroups of G for the action on F and F'. Moreover,

$$
\begin{aligned}
& H'g \in \mathrm{Hom}(H,H') \text{ is an isomorphism}\\
\Leftrightarrow\ & \exists\, Hg' \in \mathrm{Hom}(H',H) : H = Hg' \circ H'g = Hg'g \text{ and } H' = H'g \circ Hg' = H'gg'\\
\Leftrightarrow\ & \exists\, g' \in G : g'H' \subseteq Hg',\ g'g \in H, \text{ and } gg' \in H'\\
\Leftrightarrow\ & \exists\, g' \in G : H \subseteq g^{-1}H'g \subseteq (g'g)^{-1}Hg'g = H,\ g'H' \subseteq Hg',\ g'g \in H, \text{ and } gg' \in H'\\
\Leftrightarrow\ & H = g^{-1}H'g \text{ (and } g' := g^{-1}).
\end{aligned}
$$

Thus $\mathrm{Aut}_G(F) \cong \mathrm{Aut}_G(G/H) \cong \mathrm{Aut}(H) = \{Hg : H = g^{-1}Hg\} = \mathrm{Norm}_G(H)/H$. $\square$

6.19 Corollary. *We have a bijection between the set of isomorphy classes of transitive right actions of G and that of conjugacy classes of subgroups of G.*

Proof. By 6.17 we have a bijection between isomorphy classes of transitive actions and isomorphy classes in $\mathrm{Subgr}(G)$. And by the proof of 6.18 we have that $H'g \in \mathrm{Hom}(H,H')$ is an isomorphism, iff $H = g^{-1}H'g$, i.e. H and H' belong to the same conjugacy class. $\square$

6.20 Corollary. *Let $p : Y \to X$ be a covering with path-connected Y and $x_0 \in X$. The images $\pi_1(p)\big(\pi_1(Y,y)\big)$ for $y \in p^{-1}(x_0)$ form a conjugacy class of subgroups in $\pi_1(X,x_0)$.*

This class is called the CHARACTERISTIC CONJUGACY CLASS of the covering p.

Proof. By 6.15 $\pi_1(p)\big(\pi_1(Y,y)\big) = G_y$ for $G := \pi_1(X,x_0)$ and $y \in F := p^{-1}(x_0)$, and by 6.16.2 $\{G_y : y \in F\}$ is a conjugacy class of subgroups of G. $\square$

6.21 Corollary. *For transitive actions of G on F the following statements are equivalent:*

1. *G_y is normal in G for some (all) $y \in F$;*
2. *$G_y = G_{y'}$ for all $y, y' \in F$;*
3. *The induced action of $G/\bigcap_{y\in F} G_y$ on F is free,*
   i.e. if $g \in G$ has some fixed point $y_0 \in F$ then it acts as identity on F;

4. $\mathrm{Aut}_G(F)$ *acts transitive on* F.

For 3 note that $\bigcap_{y\in F} G_y$ is the kernel of the action $G \to \mathrm{Bij}(F)$ and hence the action factors over $G \twoheadrightarrow G/\bigcap_{y\in F} G_y$.

Proof. (1⇒4) If G_{y_0} is normal in G, then $\mathrm{Norm}_G(G_{y_0}) = G$ and hence $\mathrm{Aut}_G(F) \cong G/G_{y_0}$ by 6.18 which obviously acts transitive, since G does.

(4⇒3) Let $y_0 \cdot g = y_0$ and $y \in F$. Since $\mathrm{Aut}_G(F)$ acts transitive there is an automorphism φ with $y = \varphi(y_0) = \varphi(y_0 \cdot g) = \varphi(y_0) \cdot g = y \cdot g$.

(3⇒2) Let $g \in G_y$, i.e. y is a fixed point of g. Hence g acts as identity, so $g \in G_{y'}$ for all $y' \in F$.

(2⇒1) is obvious, since $G_y = G_{y\cdot g} = g^{-1}G_y g$ by 6.16.1. □

Let us now show that $\mathrm{Cov}_X^{\mathrm{pc}} \to \mathrm{Act}_{\mathrm{tr}}(\pi_1(X,x_0))$ can be extended to a full and faithful functor:

6.22 Proposition. *Let X be locally path-connected. Let $p : Y \to X$ and $p' : Y' \to X$ be two path-connected coverings with typical fibers $F := p^{-1}(x_0)$ and $F' := (p')^{-1}(x_0)$ and $G := \pi_1(X, x_0)$. Then $\mathrm{Hom}_X(p,p') \cong \mathrm{Hom}_G(F,F')$ via $\Phi \mapsto \Phi|_F$.*

Proof. The mapping $\Phi \mapsto \Phi|_F$ is well-defined, i.e. $\Phi|_F$ is a G-homomorphism, since $\Phi(y\cdot[u]) = (\Phi\circ {}^y\tilde u)(1) = {}^{\Phi(y)}\tilde u(1) = \Phi(y)\cdot[u]$. Obviously, this extends $p \mapsto F := p^{-1}(x_0)$ to a functor $\mathrm{Cov}_X^{\mathrm{pc}} \to \mathrm{Act}_{\mathrm{tr}}(\pi_1(X,x_0))$.

It is faithfull, since $\Phi_1|_F = \Phi_2|_F$ implies $\Phi_1(y_0) = \Phi_2(y_0)$ and hence $\Phi_1 = \Phi_2$, by the uniqueness of lifts of p proved in 6.7.

Fullness: Let $\varphi : F \to F'$ be G-equivariant. As $\Phi : Y \to Y'$ we take the lift of $p : Y \to X$ which maps $y_0 \in F$ to $\varphi(y_0) \in F'$. This lift exists by 6.14, since $\pi_1(p)\big(\pi_1(Y,y_0)\big) = G_{y_0} \subseteq G_{\varphi(y_0)} = \pi_1(p')\big(\pi_1(Y',\varphi(y_0))\big)$ by 6.16.5 and since with X also Y is locally path-connected. By 6.16.5 $\Phi|_F = \varphi$, since both are G-equivariant and coincide on y_0. □

6.23 Corollary. [20, 6.3.4] *Two path-connected coverings of a locally path-connected space are isomorphic, iff their characteristic conjugacy classes are the same.*

Proof. $p \cong p' \overset{6.22}{\Longleftrightarrow} F \cong_G F' \overset{6.16.6}{\Longleftrightarrow} \{G_y : y \in F\} = \{G_{y'} : y' \in F'\}$. □

6.24 Corollary. [20, 6.5.5] *Let Y be path-connected and X be locally path-connected. For any covering map $p : Y \to X$ we have*

$$\mathrm{Aut}(p) \cong \mathrm{Aut}_{\pi_1(X,x_0)}\big(p^{-1}(x_0)\big) \cong \mathrm{Norm}\Big(\pi_1(p)\big(\pi_1(Y,y_0)\big)\Big) \;/\; \pi_1(p)\big(\pi_1(Y,y_0)\big).$$

The inverse of this isomorphism is given by mapping $[u] \in \mathrm{Norm}\Big(\pi_1(p)\big(\pi_1(Y,y_0)\big)\Big)$ to the unique covering transformation f which maps y_0 to ${}^{y_0}\tilde u(1)$.

Proof. Since the elements of Aut are just the isomorphisms of an object with itself, this follows directly from 6.22, 6.18, and 6.15. □

6.25 Corollary. Normal coverings. [20, 6.5.8] *For path-connected coverings $p : Y \to X$ of locally path-connected spaces X the following conditions are equivalent:*

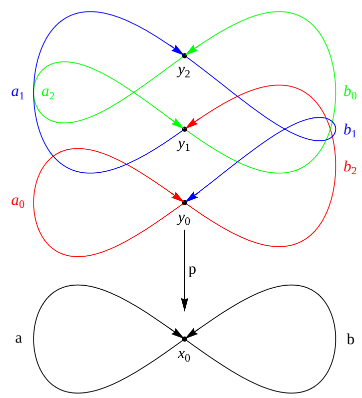
1. *$\pi_1(p)(\pi_1(Y,y))$ is normal in $\pi_1(X,x_0)$ for (some) all y in the fiber over x_0;*
2. *The characteristic conjugacy class of the covering consists of a single group;*

3. *If one lift of a closed path through x_0 is closed, then so are all lifts;*
4. *The covering p is normal, i.e. the group $\operatorname{Aut}(p)$ acts transitive on the fiber over x_0.*

In particular 1–4 is true if $\pi_1(X)$ is abelian or the covering is 2-fold or $\pi_1(Y) = \{1\}$.

Proof. Let $G := \pi_1(X, x_0)$ and $F := p^{-1}(x_0)$. By 6.15 $\pi_1(p)(\pi_1(Y, y)) = G_y$ and by 6.20 the characteristic conjugacy class is $\{G_y : y \in F\}$; the lift with initial value y of a closed curve u through x_0 is closed iff y is a fixed point of $[u]$ acting on F; and the group of covering transformations is $\operatorname{Aut}(p) \cong \operatorname{Aut}_G(F)$ by 6.22. So the result follows from 6.21. □

6.26 Example. [20, 6.1.5]
Since every subgroup of an abelian group is normal and also any subgroup of index two, the simplest non-normal covering could best be found among the 3-fold coverings of $S^1 \vee S^1$.
There is a three-fold covering of $S^1 \vee S^1$, which is not normal.



Proof. Let $\{y_0, y_1, y_2\}$ be the fiber over x_0, let a and b denote parametrizations of the two factors S^1 in $S^1 \vee S^1$ and let a_0, a_1, a_2 be the leaves over a and b_0, b_1, b_2 be the leaves over b. Let b_i be from y_{i+1} to y_{i+2} (mod 3). Let a_0 be a closed path at y_0 and a_1 and a_2 connect y_1 and y_2 in opposite directions.

So a has closed as well as none closed lifts. □

6.27 Corollary. [20, 6.5.6] *If $p : Y \to X$ is a covering with Y simply connected and X locally path-connected, then $\operatorname{Aut}(p) \cong \pi_1(X, x_0)$ as groups and, in particular, p is a normal covering, so $X \cong Y/\operatorname{Aut}(p)$.*

Proof. By 6.24 we have an isomorphism $\operatorname{Aut}(p) \cong \operatorname{Norm}_G(G_{y_0})/G_{y_0}$, $\Phi \leftarrow\!\shortmid G_{y_0} g$, where $g \in G := \pi_1(X, x_0)$, $G_{y_0} = \pi_1(p)(\pi_1(Y, y_0))$, and $\Phi(y_0) = y_0 \cdot g$. By assumption $\pi_1(Y, y_0) = \{1\}$, hence $G_{y_0} = \{1\}$, thus $\operatorname{Norm}_G(G_{y_0}) = G$, and so we have $\operatorname{Norm}_G(G_{y_0})/\{1\} \cong G$.

Moreover, p is normal, since $\operatorname{Aut}(p) \cong \pi_1(X, x_0)$ acts transitively. Hence $X \cong Y/\operatorname{Aut}(p)$ by 6.10. □

6.28 Examples of the fundamental group of orbit spaces. [20, 5.7.5]
We can use 6.27 to calculate $\pi_1(X)$ by finding a covering $p : Y \to X$ with simply connected total space Y (see 6.29) and then determine its automorphism group $\operatorname{Aut}(p) \cong \pi_1(X)$.

In particular, if $X = Y/G$ with simply connected Y and strictly discontinuously acting G, then $\pi_1(X) \cong \operatorname{Aut}(p) = G$ by 6.9. This applies to the examples in 6.3. In particular, we have $\mathbb{Z}$ as group of covering transformations of $\mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}_2$ as group of covering transformations of $S^n \to \mathbb{P}^n \cong S^n/\mathbb{Z}_2$ for $n > 1$. Furthermore, the homotopy group of $L(\frac{q}{p}) \cong S^3/\mathbb{Z}_p$ from 1.81 is $\mathbb{Z}_p$ and that of $M\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cong L(\frac{c}{a})$ from 1.74 (see 1.87) is $\mathbb{Z}_{|a|}$.

6.29 Maximal Covering.

We aim to show that the functor $\mathrm{Cov}_X^{\mathrm{pc}} \to \mathrm{Act}_{\mathrm{tr}}(G)$, where $G := \pi_1(X, x_0)$, is an equivalence of categories. In view of 6.22 it remains to show its denseness.

For this we search for the "maximal" elements first. For transitive actions of G the maximal object is G with the right multiplication on itself, since for every action of G on some F we have G-equivariant mappings $\mathrm{ev}_y : G \to F$, $g \mapsto y \cdot g$, for $y \in F$ by 6.16.3.

The corresponding maximal path-connected covering $p : \tilde{X} \to X$ should thus have as typical fiber $p^{-1}(x_0) = G$ and the action of $G = \pi_1(X, x_0)$ on it should be given by right multiplication. In particular, we must have $G_y = \{1\}$ for all $y \in G$. Choose a base-point $y_0 \in \tilde{X}$ with $x_0 = p(y_0)$. Since $\pi_1(p) : \pi_1(\tilde{X}, y_0) \to G_{y_0} = \{1\}$ is an isomorphism by 6.15, we have that $\tilde{X}$ should be simply connected.

For every point $y \in \tilde{X}$ we find a path v_y from y_0 to y and since $\tilde{X}$ is simply connected the homotopy class $[v_y]$ rel. $\dot{I}$ is well defined.
Let $\dot\sim$ denote temporarily the relation of being 'homotopic relative $\dot{I}$'.

$$\begin{tikzcd}
& C\big((I,\{0\}),(\tilde{X},\{y_0\})\big)/\dot\sim \arrow[dl, "\mathrm{ev}_1"', "\cong"] \arrow[dd, "p\circ \mathrm{ev}_1"] \arrow[dr, "p_*", "\cong"'] & \\
\tilde{X} \arrow[dr, "p"'] & & C\big((I,\{0\}),(X,\{x_0\})\big)/\dot\sim \arrow[dl, "\mathrm{ev}_1"] \\
& X &
\end{tikzcd}$$

Thus $y \mapsto [v_y]$ gives a bijection $\tilde{X} \cong C\big((I,\{0\}),(\tilde{X},\{y_0\})\big)/\dot\sim$ with inverse $\mathrm{ev}_1 : v(1) \leftarrow\!\shortmid [v]$. By the lifting property 6.11, these homotopy classes correspond bijectively to homotopy classes of paths in X starting at x_0.

So we would like to identify $\mathrm{ev}_1 : C\big((I,\{0\}),(X,\{x_0\})\big)/\dot\sim \to X$ as a covering map: Let U be a path-connected neighborhood of $x_1 \in X$. We calculate $\mathrm{ev}_1^{-1}(U)$. Note that $\mathrm{ev}_1^{-1}(x_1) = \{[v] : v \text{ is a path in } X \text{ from } x_0 \text{ to } x_1\}$ and in particular $\mathrm{ev}_1^{-1}(x_0) = \pi_1(X, x_0)$.

$$\begin{aligned}
\mathrm{ev}_1^{-1}(U) &= \{[w] : w(1) \in U\} \qquad (\text{use } w \mathrel{\dot\sim} w \cdot u^{-1} \cdot u \text{ with appropriate } u)\\
&= \Big\{[v]\cdot[u] : v(0) = x_0, v(1) = x_1, u(0) = x_1, u(I) \subseteq U\Big\}\\
&= \Big\{[v]\cdot[u] : [v] \in \mathrm{ev}_1^{-1}(x_1), u(0) = x_1, u(I) \subseteq U\Big\}\\
&= \bigcup_{[v]\in \mathrm{ev}_1^{-1}(x_1)} {}^{[v]}\tilde{U}, \text{ with } {}^{[v]}\tilde{U} := \Big\{[v]\cdot[u] : u(0) = x_1, u(I) \subseteq U\Big\}.
\end{aligned}$$

Since U is path-connected the mapping $\mathrm{ev}_1|_{{}^{[v]}\tilde{U}} : {}^{[v]}\tilde{U} \to U$ is onto. In order that it is injective, we need that $u_0(1) = u_1(1) \Rightarrow [u_0] = [u_1]$, i.e. every closed curve in U through x_1 should be 0-homotopic in X. A space X which has for each of its points a neighborhood with this property is called SEMI-LOCALLY SIMPLY CONNECTED. Note that the closed curves are assumed to be local (i.e. contained in U), whereas the homotopy may leave U. Since any subset of such a set U has the same property, we get for a locally connected semi-locally simply connected space a neighborhood-basis of connected sets U with this property. The cone over the image of $\bigvee_{\mathbb{N}} S^1$ in $\mathbb{R}^2$ discussed in 1.42 gives an example of a contractible (hence simply connected and thus semi-locally simply connected) space which is not locally simply connected.

Note that ${}^{[v_1]}\tilde{U} \cap {}^{[v_2]}\tilde{U} \neq \emptyset$ iff there exist curves u_i with $[v_1]\cdot[u_1] = [v_2]\cdot[u_2]$, where u_i are curves in U from x_1 to the same endpoint. Hence $[u_1] = [u_2]$ by the semi-local simple connectedness and thus $[v_1] = [v_2]$.

For a path-connected, locally path-connected and semi-locally simply connected space X we thus define $\tilde{X}$ to be the set $C\big((I,\{0\}),(X,\{x_0\})\big)/\dot{\sim}$ and $p_1 : \tilde{X} \to X$ by $p_1([u]) := \mathrm{ev}_1(u) = u(1)$. Since for every U as above we want ${}^{[u]}\tilde{U}$ to be a leaf over U, we declare those sets to be open in $\tilde{X}$. In order that these sets form the basis of a topology (cf. [6, 1.1.4]) we have to consider two such neighborhoods U_0 and U_1 and $y \in {}^{y_0}\tilde{U}_0 \cap {}^{y_1}\tilde{U}_1$. Then $p_1(y) \in U_0 \cap U_1$ and hence we can find such a neighborhood $U \subseteq U_0 \cap U_1$ of $p_1(y)$. Then $y \in {}^{y}\tilde{U}$ and ${}^{y}\tilde{U} \subseteq {}^{y_0}\tilde{U}_0 \cap {}^{y_1}\tilde{U}_1$ by construction, since $y \cdot [u] \in {}^{y}\tilde{U}$ and $y =: y_i \cdot [u_i] \in {}^{y_i}\tilde{U}_i$ implies $y \cdot [u] = y_i \cdot [u_i \cdot u] \in {}^{y_i}\tilde{U}_i$. Obviously we have that $p_1|_{{}^{y}\tilde{U}} : {}^{y}\tilde{U} \to U$ is a homeomorphism, and hence $p_1 : \tilde{X} \to X$ is a covering map.

Note that for any path u starting at x_0 we have that $t \mapsto [u_t]$ is the lift along p_1 with starting value $[\mathrm{const}_{x_0}] =: y_0$, where $u_t(s) := u(ts)$. Thus $\tilde{X}$ is path-connected.

Finally $\tilde{X}$ is simply connected: Let v be a closed curve in $\tilde{X}$ through y_0. Then $u := p_1 \circ v$ is a closed curve through x_0 and $v(t) = [s \mapsto u(ts)]$, since both sides are lifts of u with starting point y_0. Hence $[u] = v(1) = v(0) = y_0 = [\mathrm{const}_{x_0}]$. Since homotopies can be lifted, we have $v \sim \mathrm{const}_{y_0}$ rel. $\dot{I}$.

Theorem. Universal covering. [20, 6.6.2]
Let X be path-connected, locally path-connected and semi-locally simply-connected. Then there exists a path-connected, simple-connected covering map $p_1 : \tilde{X} \twoheadrightarrow X$. Every simply connected path-connected covering of X covers any other path-connected covering.

Proof. We have just shown the first part. The other one follows, since we can lift the projection of any simple connected covering by 6.14 and the lift is a covering by 6.8.2. □

6.30 Denseness of $\mathrm{Cov}_X^{\mathbf{pc}} \to \mathrm{Act}_{\mathbf{tr}}(G)$. Let us return to the question of (almost) surjectivity of $\mathrm{Cov}_X^{\mathrm{pc}} \to \mathrm{Act}_{\mathrm{tr}}(G)$, where $G := \pi_1(X,x_0)$. So let G act transitively on F. Then $F \cong G/H$ by 6.16.4, where $H := G_y$ is any isotropy subgroup of this action. Thus we are searching for a covering $p : Y \to X$ with typical fibre $p^{-1}(x_0) \cong F$ and with the action of G on it given by 6.16.3. Thus $\pi_1(p) : \pi_1(Y,y) \to G_y$ should be an isomorphism.

Let $p : Y \to X$ be some path-connected covering of X. By 6.29 the universal covering map $p_1 : \tilde{X} \to X$ lifts to a covering map $\tilde{p_1} : \tilde{X} \to Y$ with $p \circ \tilde{p_1} = p_1$ and $\tilde{p_1}$ is normal by 6.27. Thus $Y \cong \tilde{X}/\operatorname{Aut}(\tilde{p_1})$ and $\operatorname{Aut}(\tilde{p_1}) \cong \pi_1(Y,y)$ by 6.27.

So we define $Y := \tilde{X}/H$ and let $p : Y \to X \cong \tilde{X}/G$ be the unique factorization of p_1 over $\tilde{X} \to \tilde{X}/H$. Then p is a covering map by 6.8.4 with typical fibre $p_1^{-1}(x_0)/H = G/H \cong F$.
The action of $G = \pi_1(X,x_0)$ on G/H is obviously the one induced by the right multiplication of G on $p_1^{-1}(x_0) = G$.

$$\begin{tikzcd}
 & G = p_1^{-1}(x_0) \arrow[r, two heads] \arrow[dl, hook] & G/H = p^{-1}(x_0) \arrow[dl, hook] \arrow[d] \\
\tilde{X} \arrow[r, two heads, "\tilde{p_1}"] \arrow[dr, "p_1"'] & Y := \tilde{X}/H \arrow[d, "p"] & \{G\} = \{x_0\} \arrow[dl, hook] \\
 & X \cong \tilde{X}/G &
\end{tikzcd}$$

6.31 Theorem. [20, 6.6.3] *Let X be path-connected, locally path-connected and semi-locally simply connected. Then we have an equivalence between the category of* *path-connected coverings of X and transitive actions of $G := \pi_1(X, x_0)$.*

$$\mathrm{Cov}_X^{pc} \sim \mathrm{Act}_{tr}(G).$$

Proof. By 6.22 the functor is full and faithful and by 6.30 it is dense. $\square$

6.32 The category $\mathrm{Cov}_X^{\mathbf{pc}}$ is not quasi-ordered.
A QUASI-ORDERING is a relation, which is transitive and reflexive. A category for which each set $\mathrm{Hom}(X', X)$ has at most one element is isomorphic to the quasi-ordering of its objects given by $X' \geq X :\Leftrightarrow \exists f \in \mathrm{Hom}(X', X)$.

Obviously $\mathrm{Hom}(p', p) \cong \mathrm{Hom}_G(F', F)$ may contain more than one element, as the example of any G-action on set F with at least two elements y_0 and y_1 and action on $F' := G$ by right multiplication shows: The mappings $\mathrm{ev}_{y_i} : G \to F$ are different G-homomorphisms by 6.16.4. However, there is an automorphism $\Phi \in \mathrm{Aut}_G(G)$ with $\mathrm{ev}_{y_1} = \mathrm{ev}_{y_0} \circ \Phi$. In fact, choose $g_0 \in G$ with $y_1 = y_0 \cdot g_0$ and let $\Phi(g) := g_0 g$, then $\mathrm{ev}_{y_0}(\Phi(g)) = y_0 \cdot g_0 g = y_1 \cdot g = \mathrm{ev}_{y_1}(g)$.

We give now an example that for two coverings $p : Y \to X$ and $p' : Y' \to X$ there may be more than one element in $\mathrm{Hom}_X(p, p')$ even up to isomorphy.

By 6.31 it is enough to consider the corresponding question for transitive G-actions. For this we will construct subgroups $H \leq H' \leq G$ for which $\mathrm{Norm}_G(H) = H$ and $\mathrm{Norm}_G(H') = H'$ and for which a $g \notin H'$ exists with $gHg^{-1} \subseteq H'$.
Thus $\mathrm{Aut}_G(H) = \{1\}$, $\mathrm{Aut}_G(H') = \{1\}$, and $H' \neq H'g \in \mathrm{Hom}(H, H')$. By 6.17 this gives the corresponding result for transitive actions of G.

Remains to show that H, H', G and g can be found:
So let F be finite, $G := \mathrm{Bij}(F)$ and let $\{F_j : j \in J\}$ be a partition of F in disjoint subsets of different non-zero cardinality. Recall that any finitely generated group appears as fundamental group of some 2-dimensional CW-complex by 5.49.
Then $H := \{g \in G : \forall j \in J : g(F_j) = F_j\}$ is a subgroup with $\mathrm{Norm}_G(H) = H$:
In fact, let $g \notin H$, i.e. there is some j with $g(F_j) \neq F_j$ and let $|F_j|$ be maximal with this property.
There has to exist $y_1, y_2 \in F_j$ such that $g(y_1)$ and $g(y_2)$ are in different sets F_{j_1} and F_{j_2}: Otherwise, there would exist an $i \neq j$ with $F_i \supseteq g(F_j) \cong F_j$, thus $|F_i| \geq |F_j|$ and by the cardinality assumption $|F_i| > |F_j|$. Thus $g(F_i) = F_i \supseteq g(F_j)$ by maximality of j and hence $F_i \supseteq F_j$, a contradiction.
Now take $h \in H$ given by exchanging just y_1 and y_2. Then ghg^{-1} maps $g(y_1)$ to $g(y_2)$, and hence F_{j_1} is not invariant, so $ghg^{-1} \notin H$.

If $F = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $F_1 = \{1\}$, $F_2 = \{2, 3\}$, $F_3 = \{4, 5, 6\}$ and $F_4 = \{7, 8, 9, 10\}$. Let H be given by the partition $\{F_1, F_2, F_3, F_4\}$ and H' be given by $\{F_1 \cup F_2, F_3 \cup F_4\}$ and let $g := (1, 4)(2, 5)(3, 6) \notin H'$. Then $gHg^{-1} \subseteq H'$, since $g^{-1}(F_1 \cup F_2) = F_3$, $g^{-1}(F_3) = F_1 \cup F_2$ and $g^{-1}(F_4) = F_4$, hence $ghg^{-1}(F_3) = gh(F_1 \cup F_2) = g(F_1 \cup F_2) = F_3$, $ghg^{-1}(F_1 \cup F_2) = gh(F_3) = g(F_3) = F_1 \cup F_2$ and $ghg^{-1}(F_4) = F_4$.

Example. Let $p : Y \to X$ and $p' : Y' \to X$ be two coverings. Then there may exist homomorphisms in $\mathrm{Hom}_X(p, p')$ and $\mathrm{Hom}_X(p', p)$ without $p \cong p'$.
In fact we can translate this to transitive actions, resp. subgroups of G. So we need subgroups $H \leq G$ and $H' \leq G$ which are not conjugate, but such that H is contained in some conjugate $g^{-1}H'g$ of H' and conversely. Then $G/H \to G/(g^{-1}H'g) \cong G/H'$ is G-equivariant as is $G/H' \to G/((g')^{-1}Hg') \cong G/H$, but G/H is not isomorphic to G/H'.
In [**12**, p.187] the existence of such groups is shown.

6.33 Example. Threefold coverings. [20, 6.7.3] We now try to identify all 3-fold coverings of $S^1 \vee S^1$ and also those of the torus $S^1 \times S^1$ and of Klein's bottle. For G we have in these cases $\langle\{\alpha,\beta\} : \emptyset\rangle$, $\langle\{\alpha,\beta\} : \{\alpha\beta = \beta\alpha\}\rangle$, and $\langle\{\alpha,\beta\} : \{\alpha^2\beta^2 = 1\}\rangle$.

First we have to determine all transitive actions of $\langle\{\alpha,\beta\} : \emptyset\rangle$ on $\{0,1,2\}$, i.e. group-homomorphisms from the free group with two generators α and β into that group of permutations of $\{0,1,2\}$. We write such permutations in cycle notation, i.e. these are

$$(0),\ (01),\ (02),\ (12),\ (012),\ (021).$$

Where (0) has order 1, (012) and (021) have order 3 and the rest order 2. Let a be the image of α and b that of β. Note, that two actions on $\{0,1,2\}$ are isomorphic if there exists a permutation which conjugates these generators (and hence any element) for one action onto those of the other one.
Up to symmetry we may assume that $\operatorname{ord} a \le \operatorname{ord} b$ for the order of the generators.
If $\operatorname{ord} a = 1$, i.e. $a = (0)$ then $\operatorname{ord} b$ has to be 3 (otherwise the resulting action is not transitive) and the two possible choices are conjugate via (01).
If $\operatorname{ord} a = 2$, then $\operatorname{ord} b$ can be 2, but b has to be different from a (for transitivity) and any two choices $\{a, b\}$ and $\{a', b'\}$ are conjugate via the common element $c \in \{a,b\} \cap \{a',b'\}$; or b can have order 3, and again the choices of b are conjugate by a, and that of a are conjugate by b or b^{-1}.
If $\operatorname{ord} a = 3 = \operatorname{ord} b$, they can be either the same or different.

So we get representatives for all transitive actions with (–) + indicating (none-)normality:

a	b	$S^1 \vee S^1$	$S^1 \times S^1$	Kleins bottle
(0)	(012)	+	+	$\nexists$
(012)	(0)	+	+	$\nexists$
(01)	(02)	–	$\nexists$	–
(01)	(012)	–	$\nexists$	$\nexists$
(012)	(01)	–	$\nexists$	$\nexists$
(012)	(012)	+	+	$\nexists$
(012)	(021)	+	+	+

Note, that the action is normal iff every $g \in G$ acts either fixed-point free or is the identity by 6.21.3. Thus at least both generators a and b have be of order 3 or 1. This excludes the 3 actions in the middle. All other cases are normal, because there the group generated by a and b is $\{(0), (012), (021)\}$ and only the identity (0) has fixed points.

The last two columns are determined by checking $ab = ba$ and $a^2b^2 = 1$.

6.34 Proposition. [20, 6.8.1] *Let $p : Y \to X$ be a covering. Then the following statements are true:*

1. *If X is a CW-complex then so is Y. The cells of Y are the path-components (leaves) of $p^{-1}(e)$ for all cells e of X.*
2. *If X is a manifold so is Y.*
3. *If X is a topological group, so is Y.*

Proof.
(1) The condition (C1) of 4.1 is satisfied:
Let e be a cell of X. Since e is simply connected it is trivializing for the restricted covering $p^{-1}(e) \to e$ by 6.13. Thus each path component $\tilde e$ of $p^{-1}(e)$ is homeomorphic to e via the projection $p|_{\tilde e}$. Since D^n is simply connected we may lift the characteristic map χ^e of e to a characteristic map $\chi^{\tilde e}$ of the lifted cell $\tilde e$ by 6.14.

$$\begin{tikzcd}
& & \tilde e \arrow[r, hook] \arrow[dd, "\cong"', "p|_{\tilde e}"] & Y \arrow[dd, "p"] \\
\mathring{D}^n \arrow[urr, "\cong"] \arrow[r, hook] \arrow[drr, "\cong"'] & D^n \arrow[urr, "\chi^{\tilde e}"] \arrow[drr, "\chi^e"'] & & \\
& & e \arrow[r, hook] & X
\end{tikzcd}$$

The condition (C2) is satisfied: It suffices to show that *every compact subset* $K \subseteq Y$ *meets only finitely many cells* $\tilde e$. Since $p(K)$ is compact it is contained in a finite subcomplex of X by 4.5. So it suffices to check that K meets only finitely many leaves $\tilde e \subseteq p^{-1}(e)$ for each cell e of this subcomplex. Suppose this were not the case. Then choose points $x_i \in K$ contained in different leaves for $i \in \mathbb{N}$ and let $z_i \in (\chi^e)^{-1}(p(x_i))$. The sequence $(z_i)_i$ has an accumulation point z_∞ in D^n. Let U be a trivializing neighborhood of $\chi^e(z_\infty)$ and we may assume that $U \cap e$ is path-connected and all $p(x_i) \in U$. For each i there is a unique leaf, denoted $\tilde U^i$, over U containing x_i, since otherwise two such points are contained in one leaf, and then they can be connected by the lift of a curve in $U \cap e$ and hence would be in one $\tilde e$. Since K is covered by the open sets $\tilde U^i$ together with $Y \setminus \{x_i : i \in \mathbb{N}\}$ we get a contradiction to compactness.

The condition (W) is satisfied:
Since X carries the final topology with respect to the characteristic maps $\chi^e : D^n \to X$ every open subset $U \subseteq X$ carries the final topology with respect to the maps $\chi^e|_{U^e} : U^e \to U$, where $U^e := (\chi^e)^{-1}(U) \subseteq D^n$:
In fact, let $V \subseteq U$ with $(\chi^e|_{U^e})^{-1}(V) \subseteq U^e$ open for all e. Then $(\chi^e)^{-1}(V) = (\chi^e|_{U^e})^{-1}(V)$ is open in D^n and by finality V is open in X and hence in U.

$$\begin{tikzcd}
D^n \arrow[r, "\chi^e"] & X \\
U^e \arrow[u, hook] \arrow[r, "\chi^e|_{U^e}"] & U \arrow[u, hook] \\
(\chi^e)^{-1}(V) \arrow[u, hook] \arrow[r] & V \arrow[u, hook]
\end{tikzcd}$$

Conversely, we claim that *each leaf* $\tilde U$ *over an open trivializing set* $U \subseteq X$ *carries the final topology with respect to* $\chi^{\tilde e} : (\chi^{\tilde e})^{-1}(\tilde U) \to \tilde U$ *for the cells* $\tilde e$ *of* Y*:*
So let $V \subseteq \tilde U$ be such that $(\chi^{\tilde e})^{-1}(V) \subseteq D^n$ is open for all $\tilde e$. We have to show that V is open in $\tilde U$ and since Y carries the final topology with respect to the sets $\tilde U$ it then carries also the final topology with respect to the $\chi^{\tilde e}$.

$$\begin{tikzcd}
D^n \arrow[rrrrrrr, "\chi^{\tilde e}"] \arrow[ddddddd, equal] & & & & & & & Y \arrow[ddddddd, "p"] \\
& (\chi^{\tilde e})^{-1}(p^{-1}(U)) \arrow[ul, hook'] \arrow[rrrrr] \arrow[ddddd, equal] & & & & & p^{-1}(U) \arrow[ur, hook] \arrow[ddddd] & \\
& & (\chi^{\tilde e})^{-1}(\tilde U) \arrow[ul, hook'] \arrow[rrr] \arrow[ddd, hook] & & & \tilde U \arrow[ur, hook] \arrow[ddd, tail, "\cong"'] & & \\
& & & (\chi^{\tilde e})^{-1}(V) \arrow[ul, hook'] \arrow[r] \arrow[d, hook] & V \arrow[ur, hook] \arrow[d, tail] & & & \\
& & & (\chi^e)^{-1}(p(V)) \arrow[r] \arrow[dl, hook'] & p(V) \arrow[dr, hook] & & & \\
& & U^e \arrow[rrr] \arrow[dl, equal] & & & p(\tilde U) \arrow[dr, equal] & & \\
& (\chi^e)^{-1}(U) \arrow[rrrrr] \arrow[dl, hook'] & & & & & U \arrow[dr, hook] & \\
D^n \arrow[rrrrrrr, "\chi^e"] & & & & & & & X
\end{tikzcd}$$

Since $p|_{\tilde U} : \tilde U \to U$ is a homeomorphism, it suffices to show that $p(V)$ is open in U, i.e. $(\chi^e)^{-1}(p(V))$ is open in D^n for all e. This follows from $(\chi^e)^{-1}(p(V)) =$

$\bigcup_{\tilde{e}}(\chi^{\tilde{e}})^{-1}(V)$, which we prove now:
Obviously $(\chi^e)^{-1}(p(V)) = (p\circ\chi^{\tilde{e}})^{-1}(p(V)) = (\chi^{\tilde{e}})^{-1}(p^{-1}(p(V))) \supseteq (\chi^{\tilde{e}})^{-1}(V)$.
Conversely, let $z \in (\chi^e)^{-1}(p(V))$. Consider $c : I \to D^n$ given by $c(t) := (1-t)z$. Then $\chi^e(c(0)) = \chi^e(z) = p(v)$ for some $v \in V \subseteq \tilde{U}$. Let $\tilde{c}$ be the unique local lift into $\tilde{U}$ of $\chi^e \circ c$ with $\tilde{c}(0) = v$. Since $(\chi^e \circ c)(t) \in e$ for all $t > 0$ we have that $\tilde{c}(t)$ has values in some leaf $\tilde{e}$ over e for all $t > 0$ and hence $\tilde{c}(t) = ((p|_{\tilde{e}})^{-1} \circ \chi^e \circ c)(t) = \chi^{\tilde{e}}(c(t))$ for these t. Thus $v = \tilde{c}(0) = \lim_{t\searrow 0}\tilde{c}(t) = \lim_{t\searrow 0}\chi^{\tilde{e}}((1-t)z) = \chi^{\tilde{e}}(z)$, i.e. $z \in (\chi^{\tilde{e}})^{-1}(V)$.

(2) We may take the chart domains to be trivializing sets in X. The leaves can then be used as chart domains of Y.

(3) The group structures $\mu : X \times X \to X$ and $\nu : X \to X$ can be lifted to mappings $Y \times Y \to Y$ and $Y \to Y$: In fact chose $1 \in p^{-1}(1)$. Then $\pi_1(\mu \circ (p \times p))([u_1],[u_2]) = [\mu\circ(p\circ u_1, p\circ u_2)] = [(p\circ u_1)\cdot(p\circ u_2)] = \pi_1(p)[u_1\cdot u_2]$ by the proof of 5.7. Thus $\mu\circ(p\times p)$ has a unique lift to $\tilde{\mu} : Y\times Y \to Y$ by 6.14. Similarly $\pi_1(\nu\circ p)([u]) = [p\circ u]^{-1} = \pi_1(p)[u^{-1}]$. □

6.35 Theorem. [20, 6.9.1] *Every subgroup H of a free group G is free.*
If H has finite index k in G, then $\operatorname{rank}(H) = (\operatorname{rank}(G)-1)\cdot k + 1$.
In particular, there exist subgroups of each finite rank in the free group of rank 2.

Proof. Let G be a free group and H a subgroup of G. By 5.37 G is the fundamental group of a join X of 1-spheres. Since X has a universal covering $\tilde{X} \to X$ by 6.29, there exists also a covering $Y \to X$ with isotropy subgroup H. By 6.34 Y is a graph as well, and hence its homotopy group $\pi(Y) \cong H$ is a free group by 5.46.

If H has finite index k in G, then $\operatorname{rank}(H) - 1 = k\cdot(\operatorname{rank}(G)-1)$ by 5.47, since the fiber of Y is G/H by the proof of 6.30 and hence Y has k-times as many cells of fixed dimension as X.

Let $G := \langle\{a,b\} : \emptyset\rangle$ and $k \geq 1$. Then there exists a unique surjective homomorphism $\varphi : G \to \mathbb{Z}_k$ with $\varphi(a) = 1$ and $\varphi(b) = 0$. Thus $H := \ker\varphi$ has index k in G and hence $\operatorname{rank} H = (2-1)k + 1 = k+1$. □

Some basics on knots

6.36 Definition (knots and their equivalence).
A simple closed curve in $\mathbb{R}^3$ is called a KNOT. We will now describe what it means that two knots are essentially the same. For this we consider two regularly (i.e. smoothly with nowhere vanishing derivative) parameterized simple closed curves $c_j : S^1 \to \mathbb{R}^3$ for $j \in \{0,1\}$. We call them ISOTOPIC if an ISOTOPY h between them exists, i.e. is a smooth homotopy $h : S^1 \times [0,1] \to \mathbb{R}^3$ with $h(t,j) = c_j(t)$ for $j \in \{0,1\}$ and $t \in S^1$ and such that $h(.,s)$ is a simple closed regular curve for each $s \in [0,1]$. This does not seem to be the desired description yet, because if we deform c_0 to c_1, we also have to move the surrounding "air" a bit. Thus we define: c_0 and c_1 are called DIFFEOTOPIC if a DIFFEOTOPY (also called AMBIENT ISOTOPY) H between them exists, that is a smooth homotopy $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $H(c_0(t),1) = c_1(t)$ for all $t \in S^1$, $H(.,0) = \mathrm{id}_{\mathbb{R}^3}$ and such that $H(.,s) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism for all $s \in [0,1]$.

One can show that each isotopy h can be extended to a diffeotopy H, i.e. $h(t,s) = H(h(t,0),s)$.

The definition of isotopy also makes sense for simple closed curves considered as subsets $K_i \subseteq \mathbb{R}^3$ (by forgetting the parametrization). All one has to do is to replace the first condition on H by $H(K_0, 1) = K_1$.

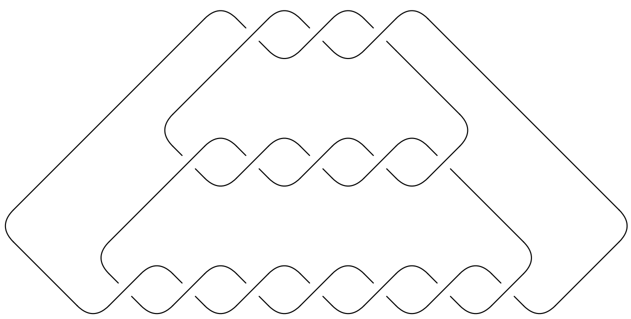
A question that arises is whether the specification of a final value, i.e. a diffeomorphism $H_1 : \mathbb{R}^3 \to \mathbb{R}^3$, is enough to ensure the existence of a diffeotopy H with $H(\_, 1) = H_1$. This can not be true in general, since the identity is orientation-preserving and hence this has also to be true for the the end value H_1. However, under this additional assumption we have:

6.37 Theorem. [2]
Any orientation preserving diffeomorphism can be extended to a diffeotopy.

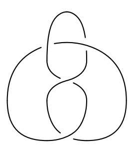
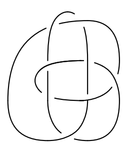
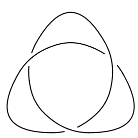
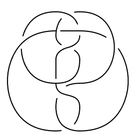
Therefore, one calls two knots EQUIVALENT if there exists a diffeomorphism (or, equivalently, a homeomorphism) $H : \mathbb{R}^3 \to \mathbb{R}^3$, which maps one knot to the other.

Each equivalence class of a knot consists of one or two diffeotopy classes, depending on whether or not it is diffeotopic to its mirror image. A knot is called AMPHICHERIAL if it is diffeotopic to its mirror image.

Each equivalence class of an oriented knot consists of one or two equivalence classes with respect to the relation “oriented-equivalent” (that is, the diffeomorphism has to respect the orientation of the knots), depending on whether or not it is is oriented-equivalent to the reverse knot. In the former case the oriented knot is called INVERTIBLE. The first non-invertible knot was found by [21]:



A table of some knots having or not having the introduced properties is the following:

	invertible	not invertible
amphicheral		
not amphicheral		

How to find out, whether two knots are equivalent? One idea for answering this question is: A knot K is characterized by what it is not, i.e. its complement $\mathbb{R}^3 \setminus K$. However, the validity of that statement was first proved in 1989:

6.38 Theorem. [3]
Two knots are equivalent if and only if their complements are homeomorphic.

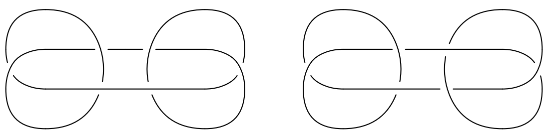
One way to figure out that the two complements are not homeomorphic is to compare their fundamental groups $\pi_1(\mathbb{R}^3 \setminus K)$, the so called KNOT GROUP of K.

An English poem summarizes the most important results about it:

A knot
and another knot
may not be the same,
although
the knot group of the knot
and others knot's knot group
differ not.

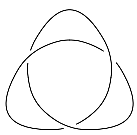
But if
the knot group of a knot
is the knot group
of the not knotted knot
then the knot is not knotted.

That is, two knots with same knot group may but not be equivalent. An example for that situation is the square knot and the granny knot with knot group $\langle\{x, y, z\} : \{xyx = yxy,\, xzx = zxz\}\rangle$:



And if the knot group of a knot is that of the trivial knot $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, namely $\mathbb{Z}$ (since $\mathbb{R}^3 \setminus S^1 \sim S^1 \vee S^2$ by exercise (2.8)), then the knot can be unknotted, i.e. is equivalent to the trivial one. This was demonstrated by [1] using a lemma that was first completely proved in [14].

This leaves us with the problem of calculating the knot group of a knot. This can be done by means of the Wirtinger representation. We explain this method for the example of the simplest non-trivial knot, the trefoil knot:



We envision the knot as a curtain rail and let an infinite long curtain hanging down from it. Let x_0 be some point above the knot. For each part of the knot that lies between two puncture points with the curtain, we choose a loop c_j through x_0, which runs clockwise (viewed in direction of the parametrization of the knot) once around this arc, Now, let any representation of an element of the knot group be given, i.e. a closed loop c in $\mathbb{R}^3 \setminus K$ through x_0. We can deform curve homotopically so that it only transversally intersects the curtain. Then we can go on to deform it homotopically in $\mathbb{R}^3 \setminus K$ by raising the parts between intersection points with the curtain so that they then pass through x_0 and the intersection points stayed unchanged. Now we move the intersection points along the walls of the chambers created by the curtain, so that they eventually coincide with those of the c_j. Then, however, we can align the pieces of the curve between x_0 and the intersection points with the corresponding parts of c_j. So we conclude, that c can be homotopically deformed into a concatenation of curves c_j and reversed c_j. The knot group is thus generated by the c_j and hence a quotient of the free group with the c_j as generators.

We still have to determine the relations that we need to factor out of the free group in order to obtain the knot group. For this we consider small circles running horizontally (counter clockwise when viewed from above) around the edge (say that below c_1) of the chambers of the curtain. Clearly, this circle is homotopic to the constant curve x_0, so it represents the neutral element 1 in the knot group. It is homotopic to $c_1 c_2^{-1} c_1^{-1} c_3$. Thus $c_1 c_2^{-1} c_1^{-1} c_3 \sim 1$ is certainly one of the relations, as well as those that arise through cyclic permutation of the indices 1, 2, 3. They already generate all relations, because any 0-homotopic loop c can be homotopically deformed so that it no longer hits the curtain. When this homotopy meets one of the edges the representation as a word in the letters c_j and c_j^{-1} changes namely, by the corresponding relation just described.

In summary, we have seen that the knot group G of the trefoil knot has as generators c_1, c_2, c_3 and the relations are generated by $c_1 c_2^{-1} c_1^{-1} c_3 \sim 1$, $c_2 c_3^{-1} c_2^{-1} c_1 \sim 1$ and $c_3 c_1^{-1} c_3^{-1} c_2 \sim 1$, i.e.

$$G = \Big\langle \{c_1, c_2, c_3\} : \ \{c_1 c_2^{-1} c_1^{-1} c_3,\ c_2 c_3^{-1} c_2^{-1} c_1,\ c_3 c_1^{-1} c_3^{-1} c_2\} \Big\rangle$$

Now we have to show that G is not the trivial group. In general this is a difficult problem because the word problem for finitely generated groups is not algorithmically solvable. From the first relation we see that $c_3 \sim c_1 c_2 c_1^{-1}$. By inserting this expression for c_3 into the second relation we get

$$G = \Big\langle \{c_1, c_2\} : \big\{c_2 (c_1 c_2^{-1} c_1^{-1}) c_2^{-1} c_1,\ (c_1 c_2 c_1^{-1})(c_2^{-1} c_1^{-1}) c_2\big\} \Big\rangle.$$

Moreover,

$$c_2 c_1 c_2^{-1} c_1^{-1} c_2^{-1} c_1 \sim 1 \quad \Leftrightarrow \quad c_2 c_1 \sim c_1^{-1} c_2 c_1 c_2 \quad \Leftrightarrow \quad c_1 c_2 c_1 \sim c_2 c_1 c_2$$

and the same relation also results from the transformation of the second relation. If we set $x := c_1 c_2 c_1$ and $y := c_1 c_2$, then $c_1 = y^{-1} x$ and $c_2 = c_1^{-1} y = x^{-1} y^2$, so x and y together create this group and the relation translates into $x \sim x^{-1} y^3$. Thus,

$$G = \big\langle \{x, y\} : \{x^2 \sim y^3\} \big\rangle.$$

This group is not $\mathbb{Z}$ because we can specify a surjective group homomorphism $f : G \to \mathcal{S}_3$ in the permutation group $\mathcal{S}_3$ of three elements: $f(x) := (12)$, $f(y) = (123)$. Then $f(x)^2 = (12)^2 = (1) = (123)^3 = f(y)^3$, so f is a well-defined group homomorphism. And since (123) and (12) generate $\mathcal{S}_3$, it is also surjective. Thus G can not be Abelian $((123)(12) = (23) \neq (13) = (12)(123))$ and thus is not isomorphic to $\mathbb{Z}$.

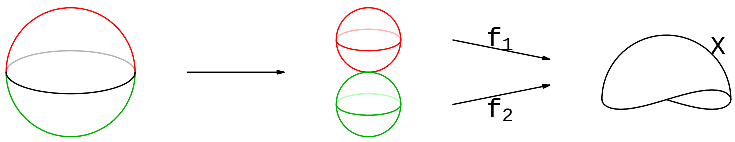
Note, however, that the Abelization G/G' of the knot group, i.e. when one adds the relations $c_i c_j \sim c_j c_i$ to the knot group, is isomorphic to $\mathbb{Z}$, because the generating relations such as e.g. $c_1 c_2^{-1} c_1^{-1} c_3 \sim e$ translate then into $c_2 \sim c_3$. So $G/G' = \langle \{c_1, c_2, c_3\} : \{c_1 \sim c_2 \sim c_3\} \rangle = \langle \{c_1\} : \emptyset \rangle \cong \mathbb{Z}$. That the knot group G of each knot has as Abelization $G/G' \cong \mathbb{Z}$ is due to [**14**].

Since the knot group does not characterize the knots, a number of other invariants have been introduced.

6.39 Higher homotopy groups

In analogy with the fundamental group $\pi_1(X)$ the higher homotopy groups $\pi_k(X)$ are defined as the set of homotopy classes of base point preserving continuous maps $f : S^k \to X$. Thus instead of catching "holes" with "lassos" (i.e., elements of $\pi_1(X)$), one tries to use "nets" (i.e., elements of $\pi_2(X)$) and higher-dimensional analogons.

Two such classes $[f_1]$ and $[f_2]$ can be multiplied by considering the quotient map $S^k \twoheadrightarrow S^k/S^{k-1} \cong S^k \vee S^k$ (where S^{k-1} denotes the equator in S^k containing the base point) and composing it with the union $f_1 \cup f_2 : S^k \vee S^k \to X$.



It can be shown that $\pi_k(X)$ is commutative for $k > 1$.

In contrast to the case $k = 1$ we have no pendant to the Theorem of Seifert and van Kampen. The main tool for calculating higher homotopy groups is the long exact sequence of a (Serre) fibration $F \hookrightarrow Y \xrightarrow{p} X$ (where $F := p^{-1}(Y)$) (see 2.29; for Serre fibrations one requires the homotopy lifting property only for polyhedra as domains):

$$\ldots \to \pi_{k+1}(F) \to \pi_{k+1}(Y) \to \pi_{k+1}(X) \to \pi_k(F) \to \pi_k(Y) \to \pi_k(X) \to \ldots$$

In particular, for covering maps p (and hence discrete F) we have $\pi_k(F) = 0$ for all $k \geq 1$, thus $\pi_k(Y) \cong \pi_k(X)$ for all $k > 1$.

Theorem of J.H.C. Whitehead.
A map $f : (X, x_0) \to (Y, y_0)$ between connected CW-spaces is a homotopy equivalence if and only if the induced homomorphisms $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, y_0)$ are isomorphisms for all $k \geq 0$.

6.40 The homotopy groups of spheres

We have shown in 5.10 that $\pi_1(S^n) = 0$ for $n \geq 2$. More generally $\pi_k(S^n) = 0$ holds for $1 \leq k < n$ by 3.32:

Moreover $\pi_1(S^1) \cong \mathbb{Z}$ generalizes to $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$, where the generator of the group is given by the homotopy class $[\mathrm{id}_{S^n}]$.

We have $\pi_k(S^1) = 0$ for all $k > 1$ since any $\varphi : S^k \to S^1$ can be lifted to the universal (contractible) covering $\mathbb{R} \twoheadrightarrow S^1$ and hence is 0-homotopic. Thus we might be led to expect that $\pi_k(S^n) = \{0\}$ for $k > n \geq 1$.

Surprisingly, this is not the case!
A counter-example is the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$.

6.41 The Hopf fibration $S^3 \to S^2$

It is defined by the following commutative diagram

$$\begin{tikzcd}
(z_1, z_2) \arrow[d, mapsto] \in \mathbb{C}^2 \setminus \{0\} \arrow[d, two heads] & & S^3 \arrow[ll, hook'] \arrow[r, equal, "\sim", "\text{stereogr.proj.}"'] \arrow[d, dotted, two heads, "\text{Hopf fibration}"] & \mathbb{R}^3 \cup \{\infty\} \\
z_2/z_1 \in \mathbb{C} \cup \{\infty\} \arrow[rr, equal, "\sim", "\text{stereogr.proj.}"'] & & S^2 \arrow[r, hook] & \mathbb{C} \times \mathbb{R} = \mathbb{R}^3
\end{tikzcd}$$

Since the inverse to the stereographic projection with pole $p = (0, 0, 1)$ is the map $y \mapsto \frac{2y+(|y|^2-1)p}{|y|^2+1} = \frac{1}{|y|^2+1}(2y, |y|^2 - 1)$, we get the following formula for the Hopf fibration:

$$\begin{aligned}
S^3 \ni (z_1, z_2) \mapsto \frac{1}{|\frac{z_2}{z_1}|^2 + 1}\Big(2\frac{z_2}{z_1}, \big|\frac{z_2}{z_1}\big|^2 - 1\Big) &= \frac{z_1 \bar{z_1}}{|z_1|^2 + |z_2|^2}\Big(2\frac{z_2}{z_1}, \frac{|z_2|^2 - |z_1|^2}{z_1 \bar{z_1}}\Big) \\
&= \big(2 z_2 \bar{z_1}, |z_2|^2 - |z_1|^2\big) \in S^2 \subseteq \mathbb{C} \times \mathbb{R}.
\end{aligned}$$

We consider the inverse images in S^3 of a circle of fixed latitude on the S^2, where θ is the latitude.

$$\begin{aligned}
&(z_1, z_2) \in S^3, \ \left|\frac{z_2}{z_1}\right| = r := \tan\Big(\frac{\pi}{4} + \frac{\theta}{2}\Big) \ \Leftrightarrow \\
&\Leftrightarrow \left\{\begin{matrix} |z_2| = r|z_1| \\ (z_1, z_2) \in S^3 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 = r^2 |z_1|^2 \\ |z_1|^2 + |z_2|^2 = 1 \end{matrix}\right\} \\
&\Leftrightarrow \left\{\begin{matrix} |z_2| = r|z_1| \\ |z_1|^2(1 + r^2) = 1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 = r^2 \dfrac{1}{1 + r^2} \\ |z_1|^2 = \dfrac{1}{1 + r^2} \end{matrix}\right\}
\end{aligned}$$

This corresponds under the stereographic projection $S^3 \to \mathbb{R}^3$ to a torus in $\mathbb{R}^3$ where $A = \sqrt{r^2 + 1}$ and $a = r$.

Next we consider the inverse image in S^3 of the South Pole of S^2

$$(0, 0, -1) \in S^2 \ \hat{=} \ (r = 0) \in \mathbb{R}^2 \ \hat{=} \ (|z_1| = 1, z_2 = 0) \subset S^3,$$

and of the North Pole of S^2:

$$(0, 0, +1) \in S^2 \ \hat{=} \ (r = \infty) \subset \mathbb{R}^2 \ \hat{=} \ (z_1 = 0, |z_2| = 1) \subset S^3.$$

We claim that in general the inverse image of each point on the S^2 (which is given with respect to the stereographic projection $S^2 \to \mathbb{C}$ by $z_0 \in \mathbb{C}$ with $r := |z_0|$) is a circle in $S^3 \subset \mathbb{R}^4$, which is obtained as intersection of the sphere $S^3 \subset \mathbb{R}^4$ with the plane $z_2 = z_1 z_0$:

$$\left\{\begin{matrix} (z_1, z_2) \in S^3 \\ \dfrac{z_2}{z_1} = z_0 \in \mathbb{C} \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 + |z_1|^2 = 1 \\ z_1 z_0 = z_2 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_1|^2 = \dfrac{1}{1 + r^2} \\ |z_2|^2 = r^2 \dfrac{1}{1 + r^2} \\ z_2 = z_1 z_0 \end{matrix}\right\}$$

In stereographic coordinates, the first two equations in $\mathbb{R}^3$ correspond to the torus $T : z^2 + (\sqrt{x^2 + y^2} - \sqrt{r^2 + 1})^2 = r^2$. Without less of generality let $r = z_0 \in \mathbb{R}$, (otherwise rotate it by $e^{-i\theta}$, which corresponds to a rotation in the (x, y) plane).

$$\text{On } S^3: \left\{\begin{array}{l} z_2 = r z_1 \\ |z_2|^2 = r^2 \dfrac{1}{1+r^2} \\ |z_1|^2 = \dfrac{1}{1+r^2} \end{array}\right\} = \left\{\begin{array}{l} x_2 = r x_1,\ y_2 = r y_1 \\ |z_2|^2 = r^2 \dfrac{1}{1+r^2} \\ |z_1|^2 = \dfrac{1}{1+r^2} \end{array}\right\}$$

$$\text{On } \mathbb{R}^3: \left\{\begin{array}{l} z = r x \\ x^2 + y^2 + z^2 - 1 = 2 r y \\ z^2 + (\sqrt{x^2+y^2} - \sqrt{r^2+1})^2 = r^2 \end{array}\right\}$$

Where we have set $z_1 := x_1 + i\,y_1$, $z_2 := x_2 + i\,y_2$ and used the formulas for stereographic projection:

$$\begin{aligned} x_1 &= \frac{2x}{1+|(x,y,z)|^2} & y_1 &= \frac{2y}{1+|(x,y,z)|^2} \\ x_2 &= \frac{2z}{1+|(x,y,z)|^2} & y_2 &= \frac{|(x,y,z)|^2-1}{1+|(x,y,z)|^2}. \end{aligned}$$

So the inverse image of a point is contained in the union of the two circles obtained from intersecting the torus with the plane $z = rx$. A more carefull investigation shows that it is the circle lying in front of the other with respect to the y-axes.

Let $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$ be the Hopf fibration. Then we have an exact sequence of groups

$$\ldots \to \underbrace{\pi_3(S^1)}_{=0} \to \underbrace{\pi_3(S^3)}_{\cong\mathbb{Z}} \to \pi_3(S^2) \to \underbrace{\pi_2(S^1)}_{=0} \to \ldots$$

Thus $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. The Hopf fibration captures "something high-dimensional" on the 2-sphere.

Suspension Theorem of Hans Freudenthal.
$\pi_k(S^n) = \pi_{k+1}(S^{n+1})$ for $1 \le k < 2n-1$.

With other words, $n \mapsto \pi_{k+n}(S^n)$ is constant for $n \ge k+2$.

It is known that $\pi_{k+n}(S^n)$ is a torsion group for all $0 < k \ne n-1$. Not all homotopy groups of spheres are known. A table of the first groups $\pi_{k+n}(S^n)$ of the low-dimensional spheres is the following, where an entry $\infty^n (p_1)^{n_1} \ldots (p_k)^{n_k}$ denotes the group $\mathbb{Z}^n \oplus (\mathbb{Z}_{p_1})^{n_1} \oplus \cdots \oplus (\mathbb{Z}_{p_k})^{n_k}$ and 1 denotes the trivial group $\{0\}$:

$k\backslash n$	2	3	4	5	6	7	8	9	10
0	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	∞	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
3	2	12	$\infty.12$	24	24	24	24	24	24
4	12	2	2^2	2	1	1	1	1	1
5	2	2	2^2	2	∞	1	1	1	1
6	2	3	24.3	2	2	2	2	2	2
7	3	15	15	30	60	120	$\infty.120$	240	240
8	15	2	2	2	24.2	2^3	2^4	2^3	2^2
9	2	2^2	2^3	2^3	2^3	2^4	2^5	2^4	$\infty.2^3$
10	2^2	12.2	120.12.2	72.2	72.2	24.2	$24^2.2$	24.2	12.2
11	12.2	84.2^2	84.2^3	504.2^2	504.4	504.2	504.2	504.2	504
12	84.2^2	2^2	2^6	2^3	240	1	1	1	12

7. Simplicial Homology

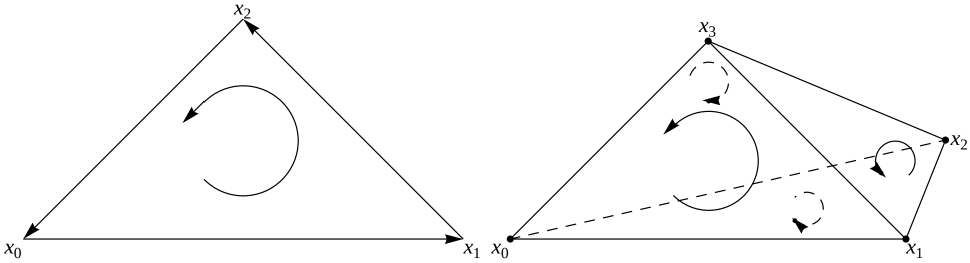
Since it is difficult to calculate within non-abelian groups we try to associate abelian groups to topological spaces. Certainly we could take the Abelization ${}^{ab}\pi_1(X)$ of the fundamental group, but in order to calculate this we can hardly avoid the non-commutative group $\pi_1(X)$ as intermediate step. So we try to find a more direct approach. We start with the most explicitly describable spaces, i.e. the simplicial complexes K. To each closed curve $|\dot\Delta| = S^1 \to |K|$ there is a homotopic simplicial approximation c from some barycentric refinement of $\dot\Delta$ to K by 3.29. Note that any barycentric refinement of $\dot\Delta$ is just a finite sequence of adjacent edges. If we want to get rid of none-commutativity we should consider the curve as formal linear combination $\sum_\sigma n_\sigma \cdot \sigma$ with integer coefficients n_σ of oriented edges σ in K (we dropped those images of edges which are degenerated to some vertex). That the curve is a closed (and connected) curve corresponds to the assumption that every vertex occurs equally often as start and as end point. So we can associate to such a linear combination $c := \sum_\sigma n_\sigma \cdot \sigma$ (a so-called 1-chain) a boundary $\partial\big(\sum_\sigma n_\sigma \cdot \sigma\big) := \sum_\sigma n_\sigma \cdot \partial\sigma$, where $\partial\sigma$ is just $x_1 - x_0$ for σ being the edge from x_0 to x_1. Thus we call a 1-chain c CLOSED iff $\partial c = 0$.

Next we should reformulate what it means that c is 0-homotopic, i.e. that there exists an extension $\tilde c : |\Delta| = D^2 \to |K|$. Again by 3.29 we may assume that $\tilde c$ is simplicial from some barycentric refinement of Δ into K. The image of $\tilde c$ can be viewed as 2-chain, i.e. formal linear combination $\sum n_\sigma \cdot \sigma$ with integer coefficients n_σ of ordered 2-simplices σ of K. Note that an orientation of a triangle induces (or even is) a coherent orientation on the boundary edges. That $\tilde c$ is an extension of c means that the edges of these simplices, which do not belong to c, occur as often with one orientation as with the other. And those which do belong to c occur exactly that many times more often with that orientation than with the other. So we can define the boundary $\partial(\sum_\sigma n_\sigma \cdot \sigma)$ of a linear combination of oriented 2-simplices as $\sum_\sigma n_\sigma \cdot \partial\sigma$, where $\partial\sigma = \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_0\rangle$ for σ being the triangle with vertices x_0, x_1, x_2 in that ordering. Thus c being 0-homotopic seems to correspond to the existence of a 2-chain with boundary c. We call such a 1-chain c EXACT or 0-HOMOLOGUE. The difference between closed and exact 1-chains is an obstruction to simply connectedness of $|K|$. At the same time this easily generalizes to k-chains:

Homology groups

7.1 Definition. Orientation and chain groups. [**20**, 7.1.1] [**20**, 7.1.4]
An ORIENTATION OF A q-SIMPLEX (with $q > 0$) is an equivalence class of linear orderings of its vertices, where two such orderings are called equivalent iff they can be transformed into each other by an even permutation. So if a q-simplex σ has vertices $x_0, \dots, x_q$ then an orientation is fixed by specifying an ordering $x_{\sigma(0)} < \dots < x_{\sigma(q)}$ and two such orderings σ and σ' describe the same oriented simplex iff $\operatorname{sign}(\sigma' \circ \sigma^{-1}) = +1$. We will denote the ordered simplex (i.e. a representant of an oriented simplex σ) with $\langle x_{\sigma(0)}, \dots, x_{\sigma(q)}\rangle$. Let σ^{-1} denote the oriented simplex with the same vertices as σ but the opposite orientation. Warning: A representant for the opposite orientation is only for q congruent to 1 or 2 mod 4 given by the reverse

ordering $x_q < \cdots < x_0$ of an representant $x_0 < \dots x_q$ of the original ordering.



The q^{th}-chain group is the abelian group with all oriented q-simplices σ as generators and $\sigma+\sigma^{-1}=0$ for all these simplices as generating relations. More precisely, it can be described as follows:
Since each simplex is determined by its vertices, a simplicial complex K can be viewed as a finite set of finite subsets (the vertices of its simplices) of some $\mathbb{R}^n$. Temporarily, let

- $K^q := \{\sigma \in K : |\sigma| = q+1\}$ be the set of its q-simplices.
- $\bigcup K := \bigcup_{\sigma\in K} \sigma$ be the set of its vertices.
- $K^{<q>} := \big\{x = (x_0,\dots,x_q) \in (\bigcup K)^{q+1} : x_i \neq x_j \Leftarrow i \neq j\big\}$ be the set of all $q+1$-tuples of distinct vertices.
- $K^{(q)} := \big\{x = (x_0,\dots,x_q) \in K^{<q>} : \{x_0,\dots,x_q\} \in K\big\}$ be the set of all ordered q-simplices in K.
- $K^{[q]} := K^{(q)}/\!\sim$ be the set of oriented q-simplices in K, where $x \sim \rho^*(x) := (x_{\rho(0)},\dots,x_{\rho(i)})$ for each even permutation ρ. Note, that $K^{[0]} = K^{(0)}$.

Then the q^{th}-chain group

$$C_q(K) := {}^{ab}\big\langle K^{[q]}, \{\sigma+\sigma^{-1} : \sigma \in K^{[q]}\}\big\rangle$$

is the abelian group with $K^{[q]}$ as generators and $\sigma^{-1} \sim -\sigma$ for each $\sigma \in K^{[q]}$ as basis of the relations. Note, that for 0-simplices σ^{-1} does not exist, so $C_0(K) := {}^{ab}\langle K^{(0)}\rangle$.

7.2 Lemma. [20, 7.1.5] *By picking an ordering of each simplex we get an isomorphism (depending on the orderings) between $C_q(K)$ and the free abelian group with the (unoriented) q-simplices as generators:*

$$C_q(K) \cong {}^{ab}\langle K^q\rangle.$$

Proof. Let ω be a section to $K^{(q)} \twoheadrightarrow K^q$, i.e. to each set of q vertices, describing a simplex σ in K, we assign an ordering of its elements, and hence an element $\tilde\sigma := [\omega(\sigma)] \in K^{[q]} := K^{(q)}/\!\sim$. Thus $K^{[q]} = \{\tilde\sigma : \sigma \in K^q\} \sqcup \{\tilde\sigma^{-1} : \sigma \in K^q\}$. We claim that ω induces an isomorphism

$${}^{ab}\langle K^q\rangle \to {}^{ab}\langle K^{(q)}\rangle \to {}^{ab}\langle K^{[q]}\rangle \to {}^{ab}\langle K^{[q]} : \{\sigma^{-1}+\sigma : \sigma \in K^{[q]}\}\rangle.$$

$$\begin{tikzcd}
{}^{ab}\langle K^q\rangle \arrow[r, dotted] \arrow[d, hook, dotted, "\tilde\omega"'] & C_q(K) \arrow[l, two heads, dotted, bend left, "\tilde\varphi"] & \\
{}^{ab}\langle K^{(q)}\rangle \arrow[r, two heads] & {}^{ab}\langle K^{[q]}\rangle \arrow[u, two heads] \arrow[ul, two heads, dotted] & \{\sigma^{-1}+\sigma : \sigma \in K^{[q]}\} \arrow[l, hook']
\end{tikzcd}
\qquad
\begin{tikzcd}
K^q \arrow[r, "\mathrm{id}"] \arrow[dr, hook, dotted, "\omega"'] & K^q \arrow[r, hook] & {}^{ab}\langle K^q\rangle \\
 & K^{(q)} \arrow[u, two heads] \arrow[r, two heads] & K^{[q]} \arrow[ul] \arrow[u, dotted, "\varphi"']
\end{tikzcd}$$

For this we consider the map $\varphi : K^{[q]} \to {}^{ab}\langle K^q\rangle$, given by $\tilde\sigma \mapsto \sigma$ and $\tilde\sigma^{-1} \mapsto -\sigma$. This induces a surjective group-homomorphism $\tilde\varphi : {}^{ab}\langle K^{[q]}\rangle \twoheadrightarrow {}^{ab}\langle K^q\rangle$ and, since $\sigma+\sigma^{-1}$ is mapped to 0, it factors over ${}^{ab}\langle K^{[q]}\rangle \twoheadrightarrow C_q(K)$ to an epimorphism $\tilde\varphi : C_q(K) \twoheadrightarrow {}^{ab}\langle K^q\rangle$, This epimorphism is injective, since $[g]$ for $g := \sum_{\sigma\in K^q}(n_{\tilde\sigma}\,\cdot$

$\tilde\sigma + n_{\tilde\sigma^{-1}}\cdot\tilde\sigma^{-1}) \in {}^{ab}\langle K^{[q]}\rangle$ is mapped to $\sum_{\sigma\in K^q}(n_{\tilde\sigma} - n_{\tilde\sigma^{-1}})\cdot\sigma \in {}^{ab}\langle K^q\rangle$ and this vanishes only if $n_{\tilde\sigma} = n_{\tilde\sigma^{-1}}$, i.e. if the image $[g]$ of g in $C_q(K)$ is 0. □

7.3 Definition. Boundary of (oriented) simplices. [20, 7.1.2] [20, 7.1.6] Note that the boundary of ordered simplices can be rewritten as:

$$\begin{aligned}
\partial\langle x_0, x_1\rangle &= x_1 - x_0 = \langle \ulcorner x_0 \urcorner, x_1\rangle + \langle x_0, \ulcorner x_1 \urcorner\rangle^{-1};\\
\partial\langle x_0, x_1, x_2\rangle &= \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_0\rangle\\
&= \langle x_0, x_1, \ulcorner x_2 \urcorner\rangle + \langle \ulcorner x_0 \urcorner, x_1, x_2\rangle + \langle x_0, \ulcorner x_1 \urcorner, x_2\rangle^{-1}\\
&= \langle \ulcorner x_0 \urcorner, x_1, x_2\rangle + \langle x_0, \ulcorner x_1 \urcorner, x_2\rangle^{-1} + \langle x_0, x_1, \ulcorner x_2 \urcorner\rangle,
\end{aligned}$$

where $\ulcorner x_i \urcorner$ indicates that x_i has to be left out. Let σ be the tetrahedron with the natural orientation $x_0 < x_1 < x_2 < x_3$. Its faces should have orientation $\langle x_1, x_2, x_3\rangle$, $\langle x_0, x_2, x_3\rangle^{-1}$, $\langle x_0, x_1, x_3\rangle$ and $\langle x_0, x_1, x_2\rangle^{-1}$.

This leads to the generalized definition:
The ORDERING of the face σ' opposite to the vertex x_j in $\sigma = \langle x_0, \dots, x_q\rangle$ should be given by

$$\sigma' := \langle x_0, \dots, x_{j-1}, \ulcorner x_j \urcorner, x_{j+1}, \dots, x_q\rangle^{(-1)^j}.$$

Let us show that this definition makes sense for oriented simplices. So let τ be a permutation of $\{0, \dots, q\}$. Then $\langle x_{\tau(0)}, \dots, x_{\tau(q)}\rangle = \langle x_0, \dots, x_q\rangle^{\operatorname{sign}\tau}$ and we have to show that

$$\langle x_{\tau(0)}, \dots, \ulcorner x_{\tau(i)} \urcorner, \dots, x_{\tau(q)}\rangle^{(-1)^i} = \langle x_0, \dots, x_{j-1}, \ulcorner x_j \urcorner, x_{j+1}, \dots, x_q\rangle^{(-1)^j \operatorname{sign}\tau}$$

where i is the position of j in $\tau(0), \dots, \tau(q)$, i.e. $i = \tau^{-1}(j)$. Without loss of generality let $i \le j$ (otherwise consider τ^{-1} instead). Consider the permutations of $\{0, \dots, q\}$ given by the function table

$$\begin{array}{ccccccccc}
0 & \dots & i-1 & i & \dots & j-1 & j & j+1 & \dots & q\\
0 & \dots & i-1 & i+1 & \dots & j & i & j+1 & \dots & q\\
\tau(0) & \dots & \tau(i-1) & \tau(i+1) & \dots & \tau(j) & \tau(i) & \tau(j+1) & \dots & \tau(q)
\end{array}$$

The first one is the cyclic permutation $(i, i+1, \dots, j-1, j)$, hence has sign $(-1)^{j-i} = (-1)^{i-j}$, the second one is τ, and the composite leaves $j = \tau(i)$ invariant, has sign $(-1)^{i-j}\cdot\operatorname{sign}\tau$, and as permutation of $\{0, \dots, \ulcorner j \urcorner, \dots, q\}$ induces the identity

$$\langle x_{\tau(0)}, \dots, \ulcorner x_j \urcorner, \dots, x_{\tau(q)}\rangle = \langle x_0, \dots, x_{j-1}, \ulcorner x_j \urcorner, x_{j+1}, \dots, x_q\rangle^{(-1)^{i-j}\operatorname{sign}\tau}.$$

From now on we will use the same notation for ordered and oriented simplices, i.e. $\langle x_0, \dots, x_q\rangle$ will denote an element in $K^{(q)}$ and at the same time its equivalence class in $K^{[q]}$.

For $q > 0$ we define the BOUNDARY OF AN ORIENTED q-SIMPLEX $\sigma = \langle x_0, \dots, x_q\rangle$ to be

$$\partial\sigma := \sum_{j=0}^{q} (-1)^j \langle x_0, \dots, x_{j-1}, \ulcorner x_j \urcorner, x_{j+1}, \dots, x_q\rangle.$$

Extended by linearity and factorization over $\sigma^{-1} \sim -\sigma$ we obtain linear mappings $\partial := \partial_q : C_q(K) \to C_{q-1}(K)$. For $0 \ge q \in \mathbb{Z}$ one puts $C_{q-1}(K) := \{0\}$ and $\partial_q := 0 : C_q(K) \to C_{q-1}(K)$.

7.4 Definition. [20, 7.1.7] [20, 7.1.8] With $Z_q(K) := \operatorname{Ker}(\partial_q)$ we denote the set of CLOSED q-CHAINS or q-CYCLES. With $B_q(K) := \operatorname{Im}(\partial_{q+1})$ we denote the set of EXACT (or 0-HOMOLOGOUS) q-chains (or q-BOUNDARIES). Two q-chains are called HOMOLOGOUS iff their difference is exact.

In particular, $Z_0(K) = C_0(K)$ and $B_{\dim(K)}(K) = \{0\}$.

7.5 Theorem. [20, 7.1.9] $0 = \partial^2 = \partial_q \circ \partial_{q+1}$ *and hence* $B_q \subseteq Z_q$.

Proof. Let $\sigma = \langle x_0, \dots, x_{q+1}\rangle$ with $q \geq 1$. Then

$$\begin{aligned}
\partial\partial\sigma &= \partial \sum_{j=0}^{q+1} (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_{q+1}\rangle \\
&= \sum_{j=0}^{q+1} (-1)^j \Bigg(\sum_{i=0}^{j-1} (-1)^i \langle x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{q+1}\rangle + \\
&\qquad\qquad + \sum_{i=j+1}^{q+1} (-1)^{i-1} \langle x_0, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_{q+1}\rangle \Bigg) \\
&= \sum_{i<j} ((-1)^{i+j} - (-1)^{j+i}) \langle x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{q+1}\rangle \\
&= 0 \quad \square
\end{aligned}$$

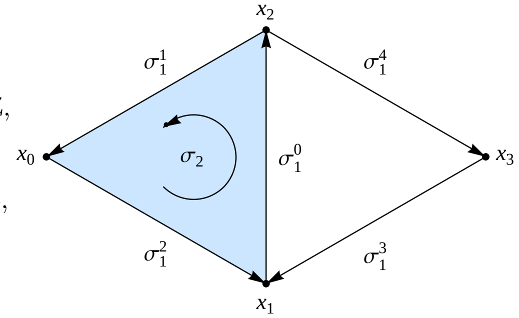
7.6 Definition. Chain complex. [20, 8.3.1]
A CHAIN COMPLEX is a family $(C_q)_{q\in\mathbb{Z}}$ of abelian groups together with group-homomorphisms $\partial_q : C_q \to C_{q-1}$ which satisfy $\partial_q \circ \partial_{q+1} = 0$. Equally, we may consider $C := {}^{ab}\coprod_{q\in\mathbb{Z}} C_q$, which is a $\mathbb{Z}$-graded abelian group and $\partial := {}^{ab}\coprod_{q\in\mathbb{Z}} \partial_q$, which is a graded group homomorphism $C \to C$ of degree -1 and satisfies $\partial^2 = 0$.

7.7 Definition. Homology. [20, 7.1.10]
For a chain complex (C, ∂) we define its HOMOLOGY $H(C, \partial) := \ker \partial / \operatorname{im} \partial$.
This is a $\mathbb{Z}$-graded abelian group with $H(C, \partial) = {}^{ab}\coprod_{q\in\mathbb{Z}} H_q(C, \partial)$, where $H_q(C, \partial) := \ker \partial_q / \operatorname{im} \partial_{q+1}$.

The group $H_q(K) := Z_q(K)/B_q(K)$ is called the q-th HOMOLOGY GROUP of K.

Examples and exact sequences

7.8 Example. [20, 7.2.1] We consider the following simplicial complex K formed by one triangle σ_2 with vertices x_0, x_1, x_2 and edges σ_1^0, σ_1^1, σ_1^2, and one further point x_3 connected by 1-simplices σ_1^3 and σ_1^4 with x_1 and with x_2. We choose orientations as depicted below on each simplex.



The generic chains are of the form:

$$c_0 = \sum_{i=0}^{3} a_i\, x_i \in C_0(K) \quad \text{with } a_i \in \mathbb{Z},$$

$$c_1 = \sum_{i=0}^{4} b_i\, \sigma_1^i \in C_1(K) \quad \text{with } b_i \in \mathbb{Z},$$

$$c_2 = m\, \sigma_2 \in C_2(K) \quad \text{with } m \in \mathbb{Z}.$$

Since $\partial c_2 = m(\sigma_1^0 + \sigma_1^1 + \sigma_1^2) \neq 0$ for $m \neq 0$ the only closed 2-cycle is 0, hence $H_2(K) = 0$.

The boundary $\partial c_1 = (b_1 - b_2)\, x_0 + (b_2 - b_0 + b_3)\, x_1 + (b_0 - b_1 - b_4)\, x_2 + (b_4 - b_3)\, x_3$ vanishes, iff $b_2 = b_1$, $b_4 = b_3$ and $b_0 = b_1 + b_3$. So $Z_1(K)$ is formed by those

$c_1 = b_1(\sigma_1^0+\sigma_1^1+\sigma_1^2)+b_3(\sigma_1^0+\sigma_1^3+\sigma_1^4)$ with $b_1, b_2 \in \mathbb{Z}$ and hence $z_1 := \sigma_1^0+\sigma_1^1+\sigma_1^2$ and $z_1' := \sigma_1^0 + \sigma_1^3 + \sigma_1^4$ form a basis with $\partial\sigma_2 = z_1$. So $B_1(K) = \{m\, z_1 : m \in \mathbb{Z}\}$ und $H_1(K) \cong \mathbb{Z}$.

For the determination of $H_0(K) \cong \mathbb{Z}$ see 7.11.

7.9 Remark. [20, 7.2.2] We have $H_q(K) = 0$ for $q < 0$ and $q > \dim K$. Furthermore, $H_{\dim K}(K) = Z_{\dim K}(K)$ (by 7.4) is a free abelian group as subgroup of $C_{\dim K}(K)$ by 7.2 and 9.20.

7.10 Lemma. [20, 7.2.3] *If $K_1, \dots, K_m$ are the connected components of K, then $C_q(K) \cong \bigoplus_j C_q(K_j)$ and $H_q(K) \cong \bigoplus_j H_q(K_j)$.*

Proof. The subgroup $C(K_i)$ is ∂-invariant. □

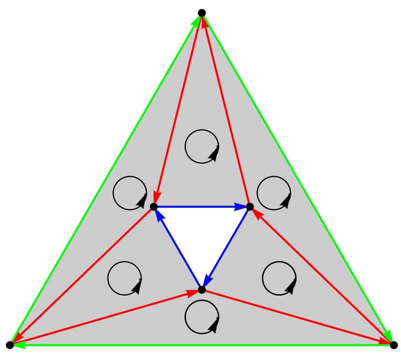
7.11 Lemma. [20, 7.2.4] *$H_0(K)$ is a free abelian group. Generators are given by choosing in each component one point.*

Proof.

$$\begin{tikzcd}
C_1(K) \arrow[r, two heads, "\partial"] & B_0(K) \arrow[r, hook] \arrow[d, equal] & Z_0(K) \arrow[r, two heads] \arrow[d, equal, "7.4"] & H_0(K) \arrow[d, dotted, "\cong"] \\
& \ker(\varepsilon) \arrow[r, hook] & C_0(K) \arrow[r, two heads, "\varepsilon"] & \mathbb{Z}
\end{tikzcd}$$

Because of 7.10 we may assume that K is connected and not empty. Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be the linear map given by $x \mapsto 1$ for all vertices $x \in K$. Obviously ε is surjective. Remains to show that its kernel is $B_0(K)$. Every two vertices x_0 and x_1 are homologous, since there is a 1-chain connecting x_0 with x_1. Thus $c := \sum_x n_x \cdot x$ is homologous to $(\sum_x n_x) \cdot x_0 = \varepsilon(c) \cdot x_0$ and hence $\operatorname{Ker}(\varepsilon) \subseteq B_0$. Conversely let $c = \partial(\sum_\sigma n_\sigma \cdot \sigma) = \sum_\sigma n_\sigma \cdot \partial\sigma$. Since $\varepsilon(\partial\langle x_0, x_1\rangle) = \varepsilon(x_1 - x_0) = 0$ we have the opposite inclusion. □

7.12 Example. The homology of the cylinder $X := S^1 \times I$. [20, 7.2.10]
Note that $S^1 \times I \sim S^1$ and hence we would expect $H_2(X) = 0$ and $H_1(X) = {}^{ab}(\pi_1(S^1)) = \mathbb{Z}$. Let us show that this is in fact true. We consider the triangulation given by 6 triangles. We will show in a later section that the homology does not depend on the triangulation. We orient the triangles in the natural way.

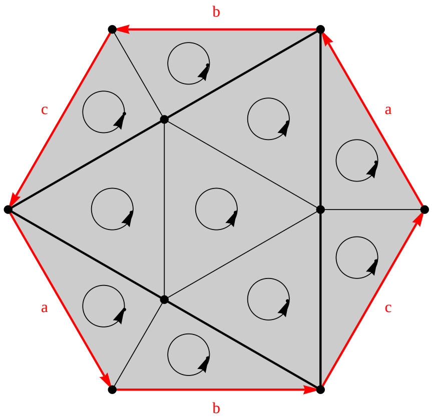


$H_2(X)$: Let $z_2 = \sum_{\dim\sigma=2} n_\sigma \cdot \sigma \in Z_2(X) = H_2(X)$, i.e. $\partial z_2 = 0$. Since those edges, which join the inner boundary with the outer one belong to exactly two 2-simplices, the coefficients of these two simplices have to be equal. So $n := n_\sigma$ is independent on σ.

However $\partial(\sum_\sigma \sigma)$ is the difference of the inner boundary and the outer one, hence not zero, and so $z_2 = n(\sum_\sigma \sigma)$ is a cycle only if $n = 0$, i.e. $H_2(X) = \{0\}$.

$H_1(X)$: Let $[z_1] \in H_1(X)$, i.e. $z_1 = \sum_{\dim\sigma=1} n_\sigma \cdot \sigma \in C_1(X)$ with $\partial z_1 = 0$. Since we may replace z_1 by a homologous chain, it is enough to consider linear combinations of a subset of edges, such that for each triangle at least 2 edges belong to this subset. In particular we can use the 6 interior edges. Since each vertex is a boundary point of exactly two of these edges the corresponding coefficients have to be equal (if we orient them coherently). Thus z_1 is homologous to a multiple of the sum c_1 of theses 6 edges. Hence $Z_1(X)$ is generated by c_1. The only multiple of c_1, which is a boundary, is 0, since the boundary of $\sum_{\dim\sigma=2} n_\sigma\cdot\sigma$ contains $\pm n_\sigma\cdot\sigma_1$, where σ_1 is the none-interior edge of σ. So $H_1(X)\cong\mathbb{Z}$.

7.13 Example. The homology of the projective plane $X := \mathbb{P}^2$.[20, 7.2.14]
We use the triangulation of $\mathbb{P}^2$ by 10 triangles described in 3.9.2. And we take the obvious orientation of all triangles. Note however that on the "boundary edges" these orientations are not coherent.



$H_2(X)$: Let $z_2 = \sum_{\dim\sigma=2} n_\sigma\cdot\sigma \in Z_2(X) = H_2(X)$, i.e. $\partial z_2 = 0$. Since those edges, which belong to the "interior" in the drawing belong to exactly two 2-simplices, the coefficient of these two simplices have to be equal. So $n := n_\sigma$ is independent on σ. However $\partial(\sum_\sigma \sigma)$ is twice the sum $a+b+c$ of the three edges along which we have to glue, and hence is not zero. So $z_2 = n(\sum_\sigma \sigma)$ is a cycle only if $n=0$, i.e. $H_2(X)=\{0\}$.

$H_1(X)$: Let $[z_1]\in H_1(X)$, i.e. $z_1 = \sum_{\dim\sigma=1} n_\sigma\cdot\sigma \in C_1(X)$ with $\partial z_1 = 0$. Now we may replace z_1 by a homologous chain using all edges except the 3 inner most ones and the 3 edges normal to the "boundary". Now consider the vertices on the inner most triangle. Since for each such point exactly two of the remaining edges have it as a boundary point, they have to have the same coefficient, and hence may be replaced by the corresponding "boundary" parts. So z_1 is seen to be homologous to a sum of "boundary" edges. But another argument of the same kind shows that they must occur with the same coefficient. Hence $H_1(X)$ is generated by $a+b+c$. As we have show above $2(a+b+c)$ is the boundary of the sum over all triangles. Whereas $a+b+c$ is not a boundary of some 2-chain $\sum_\sigma n_\sigma\cdot\sigma$, since as before such a chain must have all coefficients equal to say n and hence its boundary is $2n(a+b+c)$. Thus $H_1(\mathbb{P}^2)=\mathbb{Z}_2$, which is no big surprise, since $\pi_1(\mathbb{P}^2)=\mathbb{Z}_2$ by 5.39.

7.14 Definition. Exact Sequences. [20, 8.2.1]
A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups is called EXACT at B iff $\ker g = \operatorname{im} f$. An infinite (or finite) sequence of groups C_q and group homomorphisms $f_q : C_{q+1}\to C_q$ is called exact if it is exact at all (but the end) points.

7.15 Remark. [20, 8.2.2]

1. A sequence $0\to A\xrightarrow{f} B$ is exact iff f is injective.
2. A sequence $A\xrightarrow{f} B\to 0$ is exact iff f is surjective.
3. A sequence $0\to A\xrightarrow{f} B\to 0$ is exact iff f is bijective.
4. Let $A_{q+1}\xrightarrow{f_{q+1}} A_q \xrightarrow{f_q} A_{q-1}\xrightarrow{f_{q-1}} A_{q-2}$ be exact. Then the following statements are equivalent:
   - f_{q+1} is onto;
   - $f_q = 0$;
   - f_{q-1} is injective.

7.16 Lemma. *Let $0 \to C_0 \to C_1 \to \cdots \to C_n \to 0$ be an exact sequence of finitely generated free abelian groups. Then $\sum_{q=0}^{n}(-1)^q \operatorname{rank} C_q = 0$.*

Proof. For a $\mathbb{Z}$-linear map (i.e. abelian group homomorphism) f between free abelian groups we have

$$\operatorname{rank}(\ker f) + \operatorname{rank}(\operatorname{im} f) = \operatorname{rank}(\operatorname{dom} f)$$

by the pendent to the classical formula from linear algebra (use 7.25). Thus taking the alternating sum of all $\operatorname{rank}(\operatorname{dom} f_q)$ gives a telescoping one and hence evaluates to 0. $\square$

7.17 Proposition. [20, 7.2.5] *Let K be a 1-dimensional connected simplicial complex. Then $H_1(K)$ is a free abelian group with $1 - \alpha_0 + \alpha_1$ many generators, where α_i are the number of i-simplices.*

Compare this with the corresponding result 5.47 for fundamental groups.

Proof. Consider the sequence

$$\begin{tikzcd}
0 \arrow[r] & Z_1 \arrow[r, hook] \arrow[d, "\cong"', "\boxed{7.4}"] & C_1 \arrow[r, "\partial"] & Z_0 \arrow[r, two heads] \arrow[d, equal] & H_0 \arrow[r] \arrow[d, "\cong"', "\boxed{7.11}"] & 0 \\
& H_1 & & C_0 \arrow[r, two heads, "\varepsilon"] & \mathbb{Z} &
\end{tikzcd}$$

It is exact by definition and the vertical arrow at H_0 is an isomorphism by 7.11 and hence we get by 7.16 the equation $0 = \operatorname{rank}(H_1) - \alpha_1 + \alpha_0 - 1$ $\square$

7.18 Definition. Cone over simplicial complex. [20, 7.2.6] Let K be a simplicial complex in $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be not contained in the affine subspace generated by all $\sigma \in K$. Let $p \star \langle x_0, \dots, x_q\rangle := \langle p, x_0, \dots, x_q\rangle$ and let $p \star K := K \cup \{p \star \sigma : \sigma \in K\} \cup \{p\}$. It is called the CONE over K with vertex p and is obviously a simplicial complex (see exercise (3.1)). Note that we can extend $p \star (\_)$ to a linear mapping $C_q(K) \to C_q(p \star K)$.

7.19 Proposition. Homology of a cone. [20, 7.2.7]
He have $H_q(p \star K) = \{0\}$ for all $q \neq 0$.

Proof. Let c be a q-chain of K. We claim that

$$\partial(p \star c) = \begin{cases} c - \varepsilon(c)p & \text{if } q = 0 \\ c - p \star \partial c & \text{otherwise.} \end{cases}$$

Note that this shows that any q-chain c (with $q > 0$) is homologous to $p \star \partial c$. In order to show this we may assume that $c = \langle x_0, \dots, x_q\rangle$. For $q = 0$ we have $\partial(p \star c) = \partial\langle p, x_0\rangle = x_0 - p = c - \varepsilon(c)p$. For $q > 0$ we get

$$\begin{aligned}
\partial(p \star c) &= \partial\langle p, x_0, \dots, x_q\rangle \\
&= \langle \widehat{p}, x_0, \dots, x_q\rangle - \sum_{i=0}^{q}(-1)^i \langle p, x_0, \dots, \widehat{x_i}, \dots, x_q\rangle = c - p \star \partial c.
\end{aligned}$$

Now let $c \in Z_q(p \star K)$ for $q > 0$. We have to show that it is a boundary. Clearly c is a combination of simplices of the form $\langle x_0, \dots, x_q\rangle$ and $\langle p, x_0, \dots, x_{q-1}\rangle$, i.e. $c = c_q + p \star c_{q-1}$ with $c_q \in C_q(K)$ and $c_{q-1} \in C_{q-1}(K)$. Hence $c = c_q + p \star c_{q-1} = \partial(p \star c_q) + p \star \partial c_q + p \star c_{q-1}$. So $p \star (\partial c_q + c_{q-1}) \in Z_q$. But, again by the equation above, the boundary of this cone vanishes only if $\partial c_q + c_{q-1} = 0$, hence $c = \partial(p \star c_q) + 0$ is a boundary. $\square$

7.20 Corollary. Homology of a simplex. [20, 7.2.8]
For an n-simplex σ_n let $K(\sigma_n) := \{\tau : \tau \le \sigma_n\}$. Then $K(\sigma_n)$ is a connected simplicial complex of dimension n with $|K(\sigma_n)|$ being an n-ball and we have $H_q(K(\sigma_n)) = 0$ for $q \ne 0$.

Proof. $K(\sigma_n) = x_0 \star K(\sigma_{n-1})$ for $\sigma_n = \langle x_0, \dots, x_n\rangle$ and $\sigma_{n-1} = \langle x_1, \dots, x_n\rangle$. □

7.21 Proposition. Homology of a sphere. [20, 7.2.9]
For an $n+1$-simplex σ_{n+1} let $K(\dot\sigma_{n+1}) := \{\tau : \tau < \sigma_{n+1}\}$. Then $K(\dot\sigma_{n+1})$ is a connected simplicial complex of dimension n with $|K(\dot\sigma_{n+1})|$ being an n-sphere and we have

$$H_q(K(\dot\sigma_{n+1})) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

A generator of $H_n(K(\dot\sigma_{n+1}))$ is $\partial\sigma_{n+1} := \sum_{j=0}^{n+1} (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_{n+1}\rangle$.

Proof. Let $K := K(\dot\sigma_{n+1})$ and $L := K(\sigma_{n+1})$. Then $L \setminus K = \{\sigma_{n+1}\}$ and we have

$$\begin{tikzcd} 0 \arrow[r] & C_{n+1}(L) \arrow[r, tail, "\partial"] & C_n(L) \arrow[r, "\partial"] \arrow[d, equal] & \dots \arrow[r, "\partial"] & C_{q+1}(L) \arrow[r, "\partial"] \arrow[d, equal] & C_q(L) \arrow[r, "\partial"] \arrow[d, equal] & \dots \\ & 0 \arrow[r, "\partial"] & C_n(K) \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & C_{q+1}(K) \arrow[r, "\partial"] & C_q(K) \arrow[r, "\partial"] & \dots \end{tikzcd}$$

By 7.20 the top row is exact (for $q > 0$). Thus we have exactness in the bottom row for all $0 < q < n$. By exactness the arrow $\langle\sigma_{n+1}\rangle \cong C_{n+1}(L) \xrightarrow{\partial} C_n(L)$ is injective, and $H_n(K) = Z_n(K) = Z_n(L) = \partial(C_{n+1}(L)) \cong C_{n+1}(L) = \mathbb{Z}$. □

We will show later that if $|K| \sim |L|$ then $H_q(K) \cong H_q(L)$ for all $q \in \mathbb{Z}$, hence it makes sense to speak about the homology groups of a polyhedra.

7.22 5'Lemma. [20, 8.2.3] *Let*

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\varphi_4"] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\ B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r, "\psi_3"] & B_4 \arrow[r, "\psi_4"] & B_5 \end{tikzcd}$$

be a commutative diagram with exact horizontal rows. If all vertical arrows but the middle one are isomorphisms so is the middle one.

Proof.

(f_3 is injective)

$$\begin{aligned} & f_3a_3 = 0 \Rightarrow 0 = \psi_3 f_3 a_3 = f_4\varphi_3 a_3 \\ & \xRightarrow{f_4 \text{ inj.}} \varphi_3 a_3 = 0 \\ & \xRightarrow{\text{exact at } A_3} \exists a_2 : a_3 = \varphi_2 a_2 \\ & \Rightarrow 0 = f_3 a_3 = f_3\varphi_2 a_2 = \psi_2 f_2 a_2 \\ & \xRightarrow{\text{exact at } B_2} \exists b_1 : f_2 a_2 = \psi_1 b_1 \\ & \xRightarrow{f_1 \text{ surj.}} \exists a_1 : b_1 = f_1 a_1 \\ & \Rightarrow f_2 a_2 = \psi_1 f_1 a_1 = f_2 \varphi_1 a_1 \\ & \xRightarrow{f_2 \text{ inj.}} a_2 = \varphi_1 a_1 \\ & \xRightarrow{\text{exact at } A_2} a_3 = \varphi_2 a_2 = \varphi_2\varphi_1 a_1 = 0 \end{aligned}$$

$$\begin{tikzcd}
a_1 \arrow[r, mapsto, "\varphi_1"] \arrow[d, "f_1"', "\cong"] & a_2 \arrow[r, mapsto, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & a_3 \arrow[r, mapsto, "\varphi_3"] \arrow[d, "f_3"'] & 0 \arrow[d, "f_4"', "\cong"] & \bullet \\
b_1 \arrow[r, mapsto, "\psi_1"] & f_2(a_2) \arrow[r, mapsto, "\psi_2"] & 0 \arrow[r, mapsto, "\psi_3"] & 0 & \bullet
\end{tikzcd}$$

(f_3 is onto) $\quad b_3 \xRightarrow{f_4 \text{ surj.}} \exists a_4 : f_4 a_4 = \psi_3 b_3$

$$\xRightarrow{\text{exact at } B_4} f_5\varphi_4 a_4 = \psi_4 f_4 a_4 = \psi_4\psi_3 b_3 = 0$$
$$\xRightarrow{f_5 \text{ inj.}} \varphi_4 a_4 = 0$$
$$\xRightarrow{\text{exact at } A_4} \exists a_3 : a_4 = \varphi_3 a_3$$
$$\Rightarrow \psi_3 f_3 a_3 = f_4 \varphi_3 a_3 = f_4 a_4 = \psi_3 b_3$$
$$\xRightarrow{\text{exact at } B_3} \exists b_2 : b_3 - f_3 a_3 = \varphi_2 b_2$$
$$\xRightarrow{f_2 \text{ surj.}} \exists a_2 : b_2 = f_2 a_2$$
$$\Rightarrow b_3 = f_3 a_3 + \psi_2 b_2 = f_3 a_3 + \psi_2 f_2 a_2 = f_3(a_3 + \varphi_2 a_2)$$

$$\begin{tikzcd}
\bullet & a_2 \arrow[r, mapsto, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & a_3 \arrow[r, mapsto, "\varphi_3"] \arrow[d, "f_3"'] & a_4 \arrow[r, mapsto, "\varphi_4"] \arrow[d, "f_4"', "\cong"] & \varphi_4 a_4 \arrow[d, "f_5"', "\cong"] \\
\bullet & b_2 \arrow[r, mapsto, "\psi_2"] & b_3 \arrow[r, mapsto, "\psi_3"] & \psi_3 b_3 \arrow[r, mapsto, "\psi_4"] & 0
\end{tikzcd}$$

□

7.23 Remark. Short exact sequences. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called SHORT EXACT.

- We have that the top row in the diagram

$$\begin{tikzcd}
& A_{q+1} \arrow[r, "f_{q+1}"] \arrow[d, two heads] & A_q \arrow[r, "f_q"] \arrow[d, equal] & A_{q-1} & \\
0 \arrow[r] & f_{q+1}(A_{q+1}) \arrow[r, hook] & A_q \arrow[r, two heads] & f_q(A_q) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

  is exact at A_q iff the bottom row is short exact.
- Up to an isomorphism we have the following description of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\cong"'] & B \arrow[r, "p"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\cong"'] & 0 \\
& i(A) \arrow[r, hook] & B \arrow[r, two heads] & B/i(A) &
\end{tikzcd}$$

- The sequence $0 \to \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_m \to 0$ is short exact.
- The sequence $0 \to A \xrightarrow{\text{inj}_1} A \oplus C \xrightarrow{\text{pr}_2} C \to 0$ is short exact.

Lemma. Splitting short exact sequences. [20, 8.2.4] *For a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *the following statements are equivalent:*

1. *There is an (iso)morphism $\varphi : B \to A \oplus C$ such that the diagram is commutative;*

$\Leftrightarrow$ 2. *g has a right inverse ρ;*

$\Leftrightarrow$ 3. *f has a left inverse λ.*

$$\begin{tikzcd}
 & A & & C \arrow[d, "\mathrm{id}"] \arrow[dl, dotted, "\rho"'] & \\
0 \arrow[r] & A \arrow[u, "\mathrm{id}"] \arrow[r, tail, "f"'] \arrow[d, equal] & B \arrow[r, two heads, "g"'] \arrow[ul, dotted, "\lambda"'] \arrow[d, dotted, "\cong"', "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r, tail, "\mathrm{inj}_1"] & A \oplus C \arrow[r, two heads, "\mathrm{pr}_2"] & C \arrow[r] & 0
\end{tikzcd}$$

Under these equivalent conditions the sequence is called SPLITTING.

Proof. (1⇒2) That any morphism $\varphi : B \to A \oplus C$, which makes the diagram commutative, is already an isomorphism follows from 7.22. Thus the morphism $\rho := \varphi^{-1} \circ \mathrm{inj}_2 : c \mapsto \varphi^{-1}(0, c)$ is right inverse to g.

(2⇒3) The morphism $\mathrm{id}_B - \rho \circ g$ has image in $\ker(g)$, hence factors to a morphism $\lambda : B \to A$ over f. Thus $f \circ \lambda \circ f = (\mathrm{id}_B - \rho \circ g) \circ f = f - 0 = f \circ \mathrm{id}$ and so $\lambda \circ f = \mathrm{id}$.

(3⇒1) Define $\psi := (\lambda, g) : B \to A \oplus C$. Then ψ makes the diagram commutative ($\mathrm{pr}_2 \circ \psi = g$ and $\psi \circ f = (\mathrm{id}_A, 0) = \mathrm{inj}_1$). □

7.24 Example. Not every short exact sequence splits. [20, 8.2.5]
The sequence $0 \to \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_m \to 0$ does not split. In fact, every $a \in \mathbb{Z}_m$ has order $\mathrm{ord}(a) \le m < \infty$ but all $0 \neq b \in \mathbb{Z}$ have order $\mathrm{ord}(b) = \infty$, thus 0 is the only $\rho : \mathbb{Z}_2 \to \mathbb{Z}$. Equally, λ as in the lemma cannot exist, since $1 = \lambda(2) = 2\lambda(1)$ has no solution in $\mathbb{Z}$.

7.25 Remark.
If C is free abelian, then any short exact sequence $0 \to A \to B \to C \to 0$ splits: A right inverse to $B \to C$ is given by choosing inverse images of the generators of C.

If $0 \to A \to B \to C \to 0$ is exact and A and C are finitely generated, then so is B. In fact, the generators of A together with inverse images of those of C generate B.

7.26 Definition. Chain-groups as functors. [20, 7.3.1]
Let $\varphi : K \to L$ be a simplicial map between simplicial complexes. Define group homomorphisms $C_q(\varphi) : C_q(K) \to C_q(L)$ by

$$C_q(\varphi) := 0 \text{ for } q < 0 \text{ and for } q > \dim K$$

$$C_q(\varphi)(\langle x_0, \dots, x_q \rangle) := \begin{cases} \langle \varphi(x_0), \dots, \varphi(x_q) \rangle & \text{if } \varphi \text{ is injective on } \{x_0, \dots, x_q\}, \\ 0 & \text{otherwise.} \end{cases}$$

7.27 Definition. Chain mappings. [20, 8.3.4]

Let (C, ∂) and (C', ∂') be two chain complexes. A CHAIN MAPPING is a family of homomorphisms $f_q : C_q \to C'_q$ which commutes with the boundary operators, i.e. $\partial'_q \circ f_q = f_{q-1} \circ \partial_q$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_q \arrow[r, "\partial_q"'] \arrow[d, "f_q"] & C_{q-1} \arrow[r] \arrow[d, "f_{q-1}"] & \cdots \\
\cdots \arrow[r] & C'_q \arrow[r, "\partial'_q"] & C'_{q-1} \arrow[r] & \cdots
\end{tikzcd}$$

7.28 Proposition. C is a functor. [20, 7.3.2] *For every simplicial map $\varphi : K \to L$ the induced map $(C_q(\varphi))_{q \in \mathbb{Z}}$ is a chain mapping.*

Proof. We have to show that $\partial_q(C_q(\varphi)(\sigma)) = C_{q-1}(\varphi)(\partial_q \sigma)$ for every q-simplex $\sigma = \langle x_0, \dots, x_q \rangle$. If all vertices $\varphi(x_j)$ are distinct or are at least two pairs (including the case of a triple) are identical this is obvious. So we may assume that exactly two are the same. By reordering we may assume $\varphi(x_0) = \varphi(x_1)$. Then $C_q(\varphi)(\sigma) = 0$ and hence also $\partial(C_q(\varphi)(\sigma)) = 0$. On the other hand

$$\partial\sigma = \langle \widehat{x_0}, x_1, \dots, x_q\rangle - \langle x_0, \widehat{x_1}, x_2, \dots, x_q\rangle + \sum_{j=2}^{q} (-1)^j \langle x_0, x_1, \dots, \widehat{x_j}, \dots, x_q\rangle.$$

The first two simplices have the same image under $C_{q-1}(\varphi)$ and, since $\varphi(x_0) = \varphi(x_1)$, the other faces are mapped to 0. □

7.29 Lemma. Homology is a functor. [20, 8.3.5]
The chain mappings form a category.
Any chain map f induces homomorphisms $H_q(f) : H_q(C) \to H_q(C')$.

Proof. The first statement is obvious.
Since $f \circ \partial = \partial \circ f$ we have that $f(Z_q(C)) \subseteq Z_q(C') := \ker \partial'_q$ and $f(B_q(C)) \subseteq B_q(C') := \operatorname{im} \partial'_{q+1}$ and hence $H_q(f) : H_q(C) \to H_q(C')$ makes sense:

$$\begin{tikzcd}
0 \arrow[r] & B_q(C) \arrow[r, hook] \arrow[d, "f"] & Z_q(C) \arrow[r, two heads] \arrow[d, "f"] & H_q(C) \arrow[r] \arrow[d, dotted, "H_q(f)"] & 0 \\
0 \arrow[r] & B_q(C') \arrow[r, hook] & Z_q(C') \arrow[r, two heads] & H_q(C') \arrow[r] & 0
\end{tikzcd}$$

□

7.30 Theorem. [20, 8.3.8] *Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence of chain mappings. Then we obtain a long exact sequence in homology:*

$$\dots \xrightarrow{\partial_*} H_q(C') \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \xrightarrow{H_{q-1}(f)} \dots$$

In particular, we can apply this to a chain subcomplex C' of a chain complex C and $C'' := C/C'$: Note that ∂ factors as $\partial'' : C'' \to C''$, via $\partial''(c + C') := \partial c + C'$.

Proof. Consider

$$\begin{tikzcd}
0 \arrow[r] & C'_q \arrow[r, "f"] \arrow[d, "\partial"'] & C_q \arrow[r, "g"] \arrow[d, "\partial"'] & C''_q \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, "f"] & C_{q-1} \arrow[r, "g"] & C''_{q-1} \arrow[r] & 0
\end{tikzcd}$$

Let $\partial_*[z''] := [(f^{-1} \circ \partial \circ g^{-1})(z'')]$ for $z'' \in C''$ with $\partial z'' = 0$.

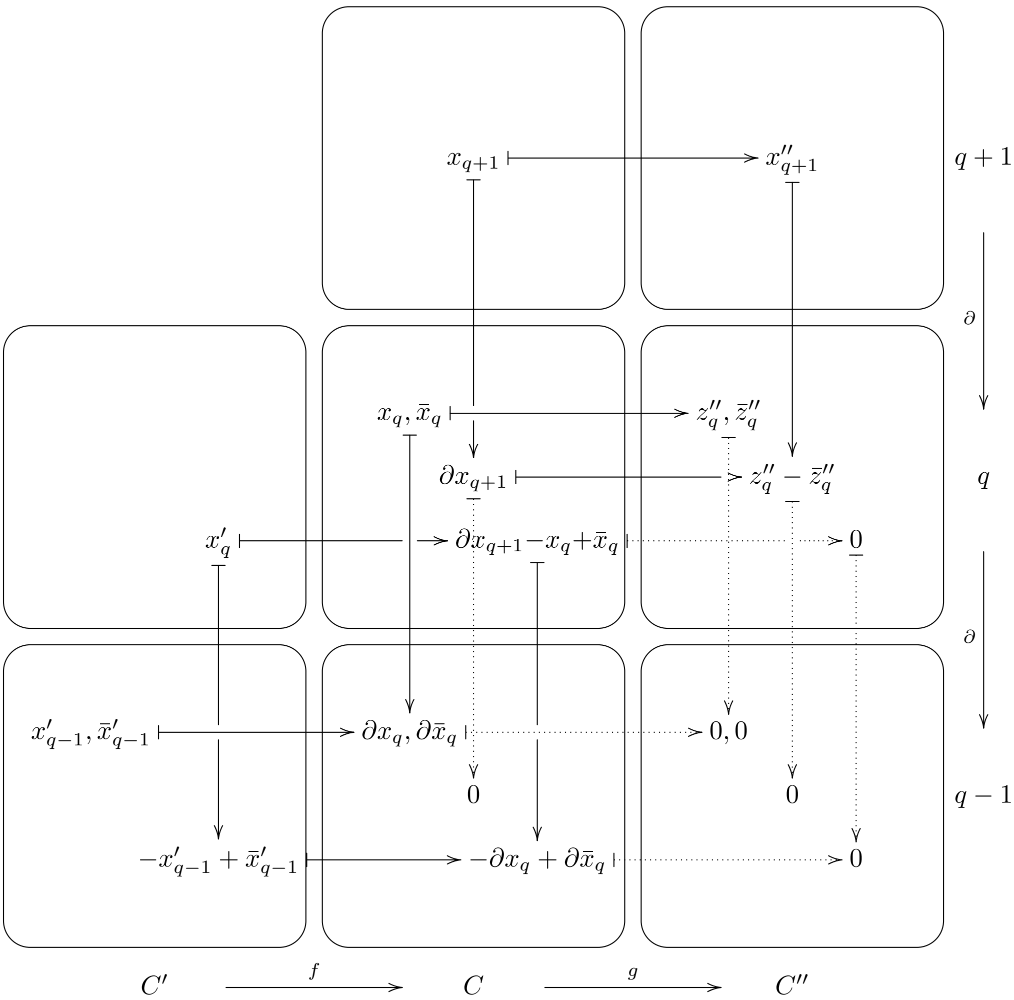
We first show that it is possible to choose elements in the corresponding inverse images and then we will show that the resulting class does not depend on any of the choices.

So let $z''_q \in C''_q$ be a cycle, i.e. $\partial z''_q = 0$. Since g is onto we find $x_q \in C_q$ with $g x_q = z''_q$. Since $g\partial x_q = \partial g x_q = \partial z''_q = 0$, we find $x'_{q-1} \in C'_{q-1}$ with $f x'_{q-1} = \partial x_q$. And hence $x'_{q-1} \in f^{-1}\partial g^{-1} z''_q$.

Furthermore $f\partial x'_{q-1} = \partial f x'_{q-1} = \partial\partial x_q = 0$. Since f is injective we get $\partial x'_{q-1} = 0$ and hence we may form the class $[x'_{q-1}] =: \partial_*[z''_q]$.

$$\begin{tikzcd}
 & x_q \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"'] & z''_q \arrow[d, mapsto, "\partial"'] \\
x'_{q-1} \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\partial"'] & \partial x_q \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"'] & 0 \\
\partial x'_{q-1} \arrow[r, mapsto, "f"] & 0 &
\end{tikzcd}$$

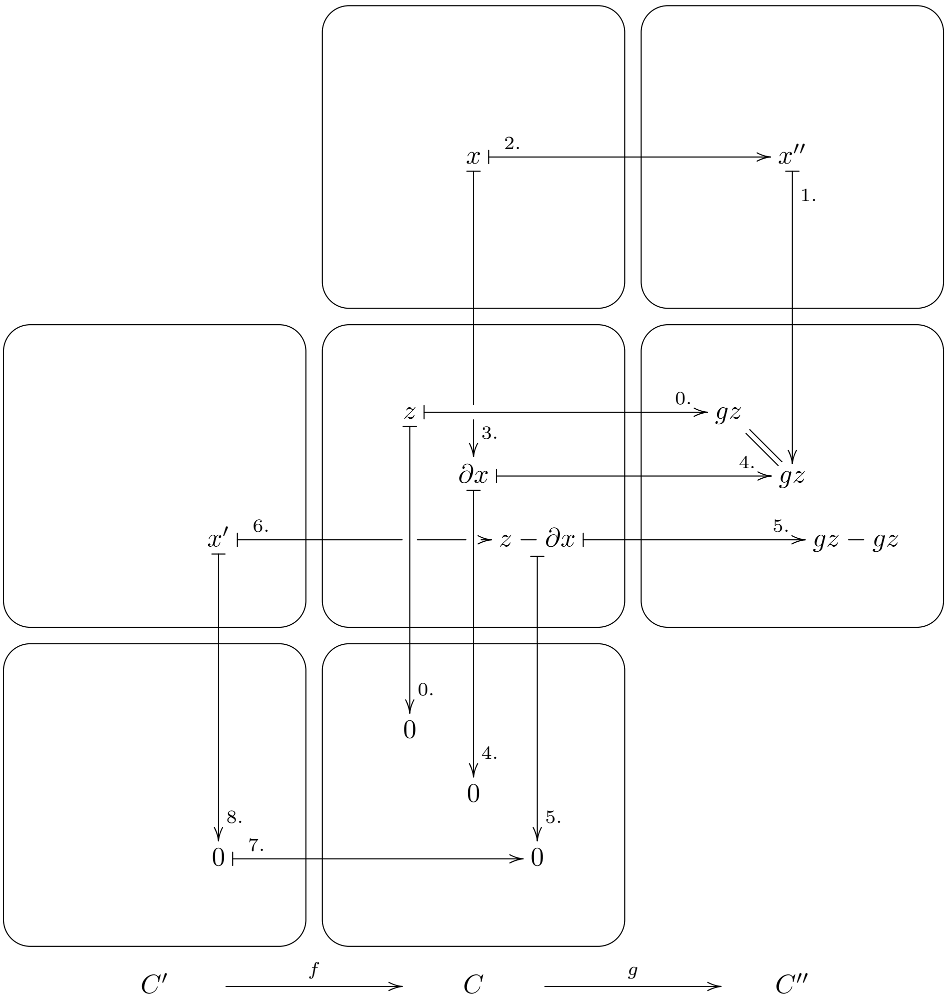
Now the independency from all choices, So let $[z''_q] = [\bar{z}''_q]$, i.e. $\exists x''_{q+1} : \partial x''_{q+1} = z''_q - \bar{z}''_q$. Choose $x_q, \bar{x}_q \in C_q$ as before, so that $g x_q = x''_q$ and $g\bar{x}_q = \bar{x}''_q$. As before choose $x'_{q-1}, \bar{x}'_{q-1} \in C'_{q-1}$ with $f x'_{q-1} = \partial x_q$ and $f\bar{x}'_{q-1} = \partial\bar{x}_q$. We have to show that $[x'_{q-1}] = [\bar{x}'_{q-1}]$. So choose $x_{q+1} \in C_{q+1}$ with $gx_{q+1} = x''_{q+1}$. Then $g\partial x_{q+1} = \partial g x_{q+1} = \partial x''_{q+1} = z''_q - \bar{z}''_q = g(x_q - \bar{x}_q)$, hence there exists an $x'_q \in C_q$ with $fx'_q = \partial x_{q+1} - x_q + \bar{x}_q$. And $f\partial x'_q = \partial f x'_q = \partial(\partial x_{q+1} - x_q + \bar{x}_q) = 0 - \partial x_q + \partial \bar{x}_q = -f(x'_{q-1} - \bar{x}'_{q-1})$. Since f is injective we have $x'_{q-1} = \bar{x}'_{q-1} + \partial x'_q$, i.e. $[x'_{q-1}] = [\bar{x}'_{q-1}]$.



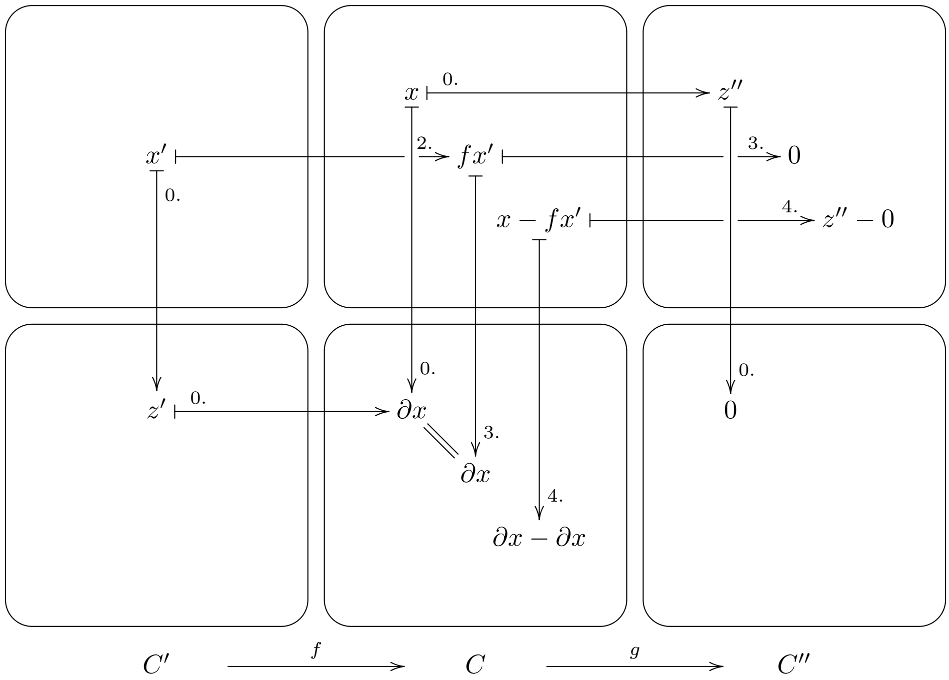
Exactness at $H_q(C')$:
($\subseteq$) $f_*\partial_*[z''] = [ff^{-1}\partial g^{-1} z''] = [\partial g^{-1} z''] = 0$.
($\supseteq$) Let $\partial z' = 0$ and $0 = f_*[z'] = [fz']$, i.e. $\exists x$: $\partial x = fz'$. Then $x'' := gx$ satisfies $\partial x'' = \partial g x = g\partial x = gfz' = 0$ and $\partial_*[x''] = [f^{-1}\partial g^{-1} g x] = [f^{-1}\partial x] = [z']$.

Exactness at $H_q(C)$:
($\subseteq$) since $g \circ f = 0$.
($\supseteq$) Let $\partial z = 0$ with $0 = g_*[z] = [gz]$, i.e. $\exists x''$: $\partial x'' = gz$. Then $\exists x$: $gx = x''$. Hence $gz = \partial x'' = \partial g x = g\partial x \Rightarrow \exists x'$: $fx' = z - \partial x \Rightarrow f\partial x' = \partial f x' = \partial(z - \partial x) = 0 \Rightarrow$

$\partial x' = 0$ and $f_*[x'] = [fx'] = [z - \partial x] = [z]$.



Exactness at $H_q(C'')$:
($\subseteq$) We have $\partial_* g_*[z] = [f^{-1}\partial g^{-1} gz] = [f^{-1}\partial z] = [f^{-1}0] = 0$.
($\supseteq$) Let $\partial z'' = 0$ and $0 = \partial_*[z'']$, i.e. $\exists x'$: $\partial x' = z'$, where $z' \in f^{-1}\partial g^{-1} z''$, i.e. $\exists x$: $gx = z''$ and $fz' = \partial x$. Then $\partial(x - fx') = fz' - f(\partial x') = 0$ and $g(x - fx') = z'' - 0$, i.e. $g_*[x - fx'] = [z'']$.



□

Relative homology

7.31 Definition. Relative homology. [**20**, 7.4.1] Let $K_0 \subseteq K$ be a simplicial subcomplex. Then $C(K_0)$ is a chain subcomplex of $C(K)$ and hence we may form the chain complex $C(K, K_0)$ given by $C_q(K, K_0) := C_q(K)/C_q(K_0)$. Note that by 7.2 we can identify this so-called RELATIVE CHAIN GROUP with the free abelian group (denoted $C_q(K \setminus K_0)$) generated by all q-simplices in $K \setminus K_0$. The boundary operator is given by taking the boundary of $\sum_\sigma k_\sigma \cdot \sigma$ in $C(K)$, but deleting all summands of simplices in $C(K_0)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_q(K_0) & \hookrightarrow & C_q(K) & \twoheadrightarrow & C_q(K,K_0) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & C_{q-1}(K_0) & \hookrightarrow & C_{q-1}(K) & \twoheadrightarrow & C_{q-1}(K,K_0) & \longrightarrow & 0 \end{array}$$

The q-th homology group of $C(K, K_0)$ will be denoted by $H_q(K, K_0)$ and is called the RELATIVE HOMOLOGY of K with respect to K_0.

Using the short exact sequence $0 \to C(K_0) \xrightarrow{i} C(K) \xrightarrow{p} C(K, K_0) \to 0$ we get a long exact sequence in homology by 7.30:

$$\dots \xrightarrow{\partial_*} H_q(K_0) \xrightarrow{H_q(i)} H_q(K) \xrightarrow{H_q(p)} H_q(K, K_0) \xrightarrow{\partial_*} H_{q-1}(K_0) \xrightarrow{H_{q-1}(i)} \dots$$

7.32 Remark. [**20**, 7.4.2]

1. Obviously $C_q(K, K) \cong C_q(\emptyset) = \{0\}$ and hence $H_q(K, K) = \{0\}$.
2. Obviously $C_q(K, \emptyset) = C_q(K)$ and hence $H_q(K, \emptyset) = H_q(K)$.

3. *If K is connected and $K \supseteq K_0 \neq \emptyset$, then $H_0(K, K_0) = \{0\}$*: In fact, let $z \in C_0(K, K_0)$, i.e. $z = \sum_{x \in K \setminus K_0} k_x \cdot x$. Let $x_0 \in K_0$ be chosen fixed. Since K is connected we have $H_0(K) \cong \mathbb{Z}$, $[z] \mapsto \varepsilon(z)$, by 7.11. Thus $z - \varepsilon(z)x_0 \in B_0(K)$, i.e. $\exists c \in C_1(K)$ with $z - \varepsilon(z)x_0 = \partial c$. Then $z = p(z) = p(\varepsilon(z)x_0) + p(\partial c) = \varepsilon(z)0 + \partial(p(c))$, where $p : C_q(K) \to C_q(K, K_0)$ denotes the natural quotient mapping. Thus $[z] = 0 \in H_0(K, K_0)$.

4. Note that in 7.21 we calculated the relative chain complex $C_q(L, K)$, where $L := K(\sigma_n)$ and $K := K(\dot{\sigma}_n)$ and obtained $C_q(L, K) = \{0\}$ for $q \neq n$ and $C_n(L, K) = \langle \sigma_n \rangle \cong \mathbb{Z}$. Hence $H_q(L, K) \cong \{0\}$ for $q \neq 0$ and $H_n(L, K) \cong \mathbb{Z}$.

7.33 Example. [20, 7.4.7] Let M be the Möbius strip with boundary ∂M. We have a triangulation of M in 5 triangles as in 3.9.2. Since ∂M is a 1-sphere $H_1(\partial M) \cong \mathbb{Z}$ by 7.21, where a generator is given by the 1-cycle r formed by the 5-edges of the boundary.

Furthermore $H_1(M) \cong \mathbb{Z}$, where a generator is given by the sum m of the remaining edges: In fact every triangle has two of these edges, so it suffices to consider linear combinations of these edges. Since every vertex belongs to exactly two of theses edges, the coefficients have to be equal.
If a combination of triangles has a multiple of m as boundary (and nothing from r), their coefficients have to be 0, cf. 7.12.

Now consider the following fragment of the long exact homology sequence:

$$\begin{tikzcd}
H_1(\partial M) \arrow[r, "2", "4."'] \arrow[d, equal] & H_1(M) \arrow[r, two heads, "3."'] \arrow[d, equal] & H_1(M, \partial M) \arrow[r, "0", "2."'] & H_0(\partial M) \arrow[r, "\cong", "1."'] \arrow[d, equal] & H_0(M) \arrow[d, equal] \\
\langle [r] \rangle & \langle [m] \rangle & & \langle [x_0] \rangle & \langle [x_0] \rangle
\end{tikzcd}$$

Since $H_0(\partial M) \cong \mathbb{Z} \cong H_0(M)$ by 7.11, where a generator is given by any point x_0 in $\partial M \subseteq M$, we have that the rightmost arrow is a bijection, so the one to the left is 0 and hence the previous one is onto. Remains to calculate the image of $\langle [r] \rangle = H_1(\partial M) \to H_1(M) = \langle [m] \rangle$. For this we consider the sum over all triangles (alternating oriented). It has boundary $2m - r$ and hence $[r]$ is mapped to $2[m]$. Thus $H_1(M, \partial M) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

7.34 Proposition. Homology ladder. [20, 8.3.11] *Let (C, C') and (D, D') be pairs of chain complexes, $C'' := C/C'$, $D'' := D/D'$ and let $f : (C, C') \to (D, D')$ be a chain mapping of pairs. This induces a homomorphism which intertwines with the long exact homology sequences.*

$$\begin{tikzcd}
\dots \arrow[r, "\partial_*"] & H_q(C') \arrow[r, "H_q(i)"] \arrow[d, "f_*"'] & H_q(C) \arrow[r, "H_q(p)"] \arrow[d, "f_*"'] & H_q(C'') \arrow[r, "\partial_*"] \arrow[d, "f_*"'] & H_{q-1}(C'') \arrow[r, "H_{q-1}(g)"] \arrow[d, "f_*"'] & \dots \\
\dots \arrow[r, "\partial_*"] & H_q(D') \arrow[r, "H_q(j)"] & H_q(D) \arrow[r, "H_q(q)"] & H_q(D'') \arrow[r, "\partial_*"] & H_{q-1}(D'') \arrow[r, "H_{q-1}(g)"] & \dots
\end{tikzcd}$$

Proof. The commutativity of all but the rectangle involving ∂_* is obvious. For this remaining one let $z'' \in C''$ be a cycle. We have to show that $f_*\partial_*[z''] = \partial_* f_*[z'']$. So let $z' \in i^{-1}\partial p^{-1} z''$, i.e. $iz' = \partial x$ for some x with $px = z''$. Then $f_*\partial_*[z''] = [fz']$ and we have to show that $j(fz') \in \partial q^{-1} fz''$, which follows from $jfz' = fiz' = f\partial x = \partial fx$ and $q(fx) = f(px) = fz''$. □

$$\begin{tikzcd}
 & x \arrow[r, mapsto, "p"] \arrow[d, mapsto] \arrow[dr, mapsto, "f"] & z'' \arrow[dr, mapsto, "f"] & \\
z' \arrow[r, mapsto, "i"] \arrow[dr, mapsto, "f"] & \partial x \arrow[dr, mapsto, "f"] & fx \arrow[r, mapsto, "q"] \arrow[d, mapsto] & fz'' \\
 & fz' \arrow[r, mapsto, "j"] & \partial fx &
\end{tikzcd}$$

7.35 Corollary. [20, 7.4.6] *Proposition 7.34 applies in particular to simplicial mappings $\varphi : (K, K_0) \to (L, L_0)$ of pairs.* □

7.36 Excision theorem. [20, 7.4.9] *Let K be the union of two subcomplexes K_0 and K_1. Then $(K_1, K_0 \cap K_1) \to (K, K_0)$ induces an isomorphism $H(K_1, K_0 \cap K_1) \to H(K_1 \cup K_0, K_0)$.*

Proof. Note that we have

$$K_1 \setminus (K_0 \cap K_1) = K_1 \setminus K_0 = (K_0 \cup K_1) \setminus K_0$$

and also

$$\begin{tikzcd}
0 \arrow[r] & C_q(K_0 \cap K_1) \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & C_q(K_1) \arrow[r, two heads] \arrow[d, hook, "j_1"'] & C_q(K_1, K_0 \cap K_1) \cong C_q(K_1 \setminus (K_0 \cap K_1)) \arrow[r] \arrow[d, dotted, "\cong"] & 0 \\
0 \arrow[r] & C_q(K_0) \arrow[r, hook, "j_2"] & C_q(K_0 \cup K_1) \arrow[r, two heads] & C_q(K_0 \cup K_1, K_0) \cong C_q((K_0 \cup K_1) \setminus K_0) \arrow[r] & 0
\end{tikzcd}$$

This gives an isomorphism even on the level of chain complexes, as follows from the commutativity of the diagram. □

Let $K := K_0 \cup K_1$ and $U := K \setminus K_1 = K_0 \setminus (K_0 \cap K_1)$ then $K_1 = K \setminus U$ and $K_0 \cap K_1 = K_0 \setminus U$, hence the isomorphism of 7.36 reads $H(K \setminus U, K_0 \setminus U) \cong H(K, K_0)$. Conversely, if (K, K_0) is a pair of simplicial complexes and $U \subseteq K_0$ is such that $K_1 := K \setminus U$ is a simplicial complex, then we get:

7.37 Corollary. [20, 7.4.8] *Let $K_0 \subseteq K$ be a pair of simplicial complexes and $U \subseteq K_0$ a set such that $\forall \tau < \sigma$: $\tau \in U \Rightarrow \sigma \in U$. Then $K_1 := K \setminus U$ and $K_0 \cap K_1 = K_0 \setminus U$ are simplicial complexes and $H(K, K_0) \cong H(K \setminus U, K_0 \setminus U)$.* □

8. Singular Homology

Basics

8.1 Definition. [20, 9.1.1] The STANDARD (CLOSED) q-SIMPLEX Δ_q is the simplex spanned by the standard unit vectors $e_j \in \mathbb{R}^{q+1}$ for $0 \le j \le q$. So

$$\Delta_q := \Big\{(\lambda_0, \dots, \lambda_q) : 0 \le \lambda_j \le 1 : \sum_j \lambda_j = 1\Big\}.$$

8.2 Definition. [20, 9.1.2] For $q \ge 1$ and $0 \le j \le q$ let the FACE-MAP $\delta^j_{q-1} : \Delta_{q-1} \to \Delta_q$ be the unique affine map, which maps e_i to e_i for $i < j$ and to e_{i+1} for $i > j$, i.e.

$$e_0, \dots, e_{q-1} \mapsto e_0, \dots, \widehat{e_j}, \dots, e_q.$$

8.3 Lemma. [20, 9.1.3] *For $q \ge 2$ and $0 \le k < j \le q$ we have $\delta^j_{q-1} \circ \delta^k_{q-2} = \delta^k_{q-1} \circ \delta^{j-1}_{q-2}$.*

Proof. The mapping on the left side has the following effect on the edges:

$$e_0, \dots, e_k, \dots, e_{q-1} \mapsto e_0, \dots, e_k, \dots, \widehat{e_j}, \dots, e_q$$

$$e_0, \dots, e_{q-2} \mapsto e_0, \dots, \widehat{e_k}, \dots, e_{q-1} \mapsto e_0, \dots, \widehat{e_k}, \dots, \widehat{e_j}, \dots, e_q$$

And on the right side:

$$\mapsto e_0, \dots, e_{j-1}, \dots, e_{q-1} \mapsto e_0, \dots, \widehat{e_k}, \dots, e_j, \dots, e_q$$

$$e_0, \dots, e_{q-2} \mapsto e_0, \dots, \widehat{e_{j-1}}, \dots, e_{q-1} \mapsto e_0, \dots, \widehat{e_k}, \dots, \widehat{e_j}, \dots, e_q$$

$\square$

8.4 Definition. [20, 9.1.4] Let X be a topological space. A SINGULAR q-SIMPLEX is a continuous map $\sigma : \Delta_q \to X$. The q-th SINGULAR CHAIN GROUP $S_q(X)$ is the free abelian group generated by all singular q-simplices, i.e.

$$S_q(X) := {}^{\mathrm{ab}}\mathcal{F}(C(\Delta_q, X))$$

Its elements are called SINGULAR q-CHAINS. The boundary operator ∂ is the linear extension of

$$\partial : \sigma \mapsto \sum_{j=0}^{q} (-1)^j \, \sigma \circ \delta^j.$$

By $\boxed{8.3}$ the groups $S_q(X)$ together with ∂ from a chain complex $S(X)$:

$$\begin{aligned}
\partial\partial\sigma &= \partial\Big(\sum_{j=0}^{q} (-1)^j \sigma \circ \delta^j\Big) = \sum_{j=0}^{q} (-1)^j \sum_{k=0}^{q-1} (-1)^k \sigma \circ \delta^j \circ \delta^k \\
&= \sum_{0 \le k < j \le q} (-1)^{j+k} \sigma \circ \delta^j \circ \delta^k + \sum_{0 \le j \le k < q} (-1)^{j+k} \sigma \circ \delta^j \circ \delta^k \\
&\overset{\boxed{8.3}}{=\!=} \sum_{0 \le k < j \le q} (-1)^{j+k} \sigma \circ \delta^k \circ \delta^{j-1} + \sum_{0 \le j < k \le q} (-1)^{j+k-1} \sigma \circ \delta^j \circ \delta^{k-1} = 0. \quad \square
\end{aligned}$$

The q-th SINGULAR HOMOLOGY GROUP $H_q(X)$ is defined to be $H_q(S(X))$. The elements of $B_q(X) := B_q(S(X))$ are called (SINGULAR) q-BOUNDARIES and those of $Z_q(X) := Z_q(S(X))$ are called (SINGULAR) q-CYCLES.

Note that singular 0-simplices can be identified with the points in X and singular 1-simplices with paths in X.

8.5 Definition. [20, 9.1.6] [20, 9.1.8] [20, 9.1.9] Let $f : X \to Y$ be continuous. Then f induces a chain-mapping $f_* := S(f) : S(X) \to S(Y)$ (by $S(f)(\sigma) := f \circ \sigma$ for singular simplices σ) and hence group-homomorphisms

$$f_* := H_q(f) : H_q(X) \to H_q(Y)$$

$$\partial\Big(S(f)(\sigma)\Big) = \partial(f \circ \sigma) = \sum_{j=0}^{q} (-1)^j f \circ \sigma \circ \delta^j = S(f)\Big(\sum_{j=0}^{q} (-1)^j \sigma \circ \delta^j\Big) = S(f)\Big(\partial(\sigma)\Big).$$

So H_q is a functor from continuous maps between topological spaces into group homomorphisms between abelian groups.

8.6 Remark. [20, 9.1.7] The identity $\mathrm{id}_{\Delta_q} : \Delta_q \to \Delta_q$ is a singular q-simplex of Δ_q, which we will denote again by Δ_q. If σ is a singular q-simplex in X, then $S(\sigma)(\Delta_q) = \sigma \circ \mathrm{id}_{\Delta_q} = \sigma$.

We will make use of this several times (e.g. in 8.21, 8.29, and 8.32) in order to construct natural transformations, by defining them first for the standard simplex:
Let F be some functor from topological spaces into groups and $c \in F(\Delta_q)$ be given. Then there is a unique natural transformation $\eta : S_q \to F$, which maps $\Delta_q \in S_q(\Delta_q)$ to $c \in F(\Delta_q)$ given by $\eta_X(\sigma) := F(\sigma)(c)$.

$$\begin{tikzcd}
\Delta_q \arrow[rr, mapsto] \arrow[dd, mapsto] & & c \arrow[dd, mapsto] & \\
& S_q(\Delta_q) \arrow[rr, dotted, "\eta_{\Delta_X}"] \arrow[dd, "\sigma_*"'] & & F(\Delta_q) \arrow[dd, "F(\sigma)"] \\
\sigma \arrow[rr, mapsto] & & \eta_X(\sigma) & \\
& S(X) \arrow[rr, dotted, "\eta_X"] & & F(X)
\end{tikzcd}$$

8.7 Theorem. [20, 9.1.10] *Let $X = \{*\}$ be a single point. Then $H_q(X) = \{0\}$ for $q \neq 0$ and $H_0(X) = S_0(X) \cong \mathbb{Z}$.*

A space X is called ACYCLIC iff it is path-connected and $H_q(X) = \{0\}$ for $q \neq 0$.

Proof. The only singular q-simplex is the constant mapping $\sigma_q : \Delta_q \to \{*\}$. Its boundary is $\partial\sigma_q = \sum_{j=0}^{q}(-1)^j \sigma_q \circ \delta^j = (\sum_{i=0}^{q}(-1)^j)\sigma_{q-1}$. So for even $q > 0$ we have $\partial\sigma_q = \sigma_{q-1}$ and hence $Z_q(X) = \{0\}$. For odd q we have that ($\partial\sigma_q = 0$ and) $\partial\sigma_{q+1} = \sigma_q$, hence $B_q(X) = S_q(X)$. Thus in both cases $H_q(X) := Z_q(X)/B_q(X) = \{0\}$. For $q = 0$ we have $B_0(X) = \{0\}$ and $Z_0(X) = S_0(\{*\}) \cong \mathbb{Z}$, hence $H_0(X) \cong \mathbb{Z}$. □

8.8 Corollary. [20, 9.1.11] *Let $f : X \to Y$ be constant. Then $H_q(f) = 0$ for $q \neq 0$.*

Proof. Obvious, since f factors over a single point. □

8.9 Proposition. [20, 9.1.12] *Let X_j be the path components of X. Then the inclusions of $X_j \to X$ induce an isomorphism ${}^{ab}\coprod_j H_q(X_j) \to H_q(X)$; cf. 7.10.*

Proof. This follows as 7.10: Let σ be a singular simplex of X. Then σ is completely contained in some X_j, hence $C(\Delta_q, X) = \bigsqcup_j C(\Delta_q, X_j)$, thus

$$S_q(X) := {}^{\mathrm{ab}}\mathcal{F}(C(\Delta_q, X)) \cong {}^{ab}\coprod_j {}^{\mathrm{ab}}\mathcal{F}(C(\Delta_q, X_j)) = {}^{ab}\coprod_j S_q(X_j)$$

and this induces an isomorphism of homology groups. □

8.10 Proposition. [20, 9.1.13] *Let X be a topological space. Then $H_0(X)$ is a free abelian group with generators given by choosing one point in each path-component; cf. 7.11.*

Proof. Because of 8.9 we may assume that X is path-connected. The mapping $\varepsilon : Z_0(X) = S_0(X) \to \mathbb{Z}$, $\sum_\sigma n_\sigma \cdot \sigma \mapsto \sum_\sigma n_\sigma$ is onto and as in 7.11 its kernel is just $B_0(X)$, so ε induces an isomorphism $H_0(X) \cong \mathbb{Z}$; cf. 7.11. □

8.11 Corollary. [20, 9.1.14] *Let X and Y be path-connected. Then every continuous mapping $f : X \to Y$ induces an isomorphism $H_0(f) : H_0(X) \to H_0(Y)$.*

Proof. Obvious since the generator is mapped to a generator. □

8.12 Definition. [20, 9.1.15] Let $A \subseteq \mathbb{R}^n$ be convex and $p \in A$ be fixed. For a singular q-simplex $\sigma : \Delta_q \to A$ we define the CONE $p \star \sigma : \Delta_{q+1} = e_0 \star \Delta_q \to A$ by

$$(p \star \sigma)\Big((1-t)e^0 + t\delta^0(x)\Big) := (1-t)p + t\sigma(x) \text{ for } t \in [0,1] \text{ and } x \in \Delta_q.$$

For a q-chain $c = \sum_\sigma n_\sigma \cdot \sigma$ we extend this operation by linearity:

$$p \star c := \sum_\sigma n_\sigma \cdot (p \star \sigma)$$

and obtain a homomorphism $S_q(A) \to S_{q+1}(A)$; cf. 7.18.

8.13 Lemma. [20, 9.1.16] *Let $A \subseteq \mathbb{R}^n$ be convex and $c \in S_q(A)$ then*

$$\partial(p \star c) = \begin{cases} c - \varepsilon(c)\, p & \text{for } q = 0, \\ c - p \star \partial c & \text{for } q > 0, \end{cases}$$

where $\varepsilon\Big(\sum_x n_x \cdot x\Big) = \sum_x n_x$; cf. 7.19.

Proof. It is enough to show this for singular simplices $c = \sigma_q$. For $q = 0$ we have that $p \star \sigma : \Delta_1 \to X$ is a path from p to σ hence $\partial(p \star \sigma) = \sigma - p = \sigma - \varepsilon(\sigma)p$. For $q > 0$ we have $(p \star \sigma) \circ \delta^0 = \sigma$ and $(p \star \sigma) \circ \delta^i = p \star (\sigma \circ \delta^{i-1})$ for $i > 0$ since

$$\begin{aligned}\big((p\star\sigma)\circ\delta^i\big)\big((1-t)e^0 + t\delta^0(x)\big) &= (p\star\sigma)\big((1-t)\delta^0(e^0) + t\delta^i(\delta^0(x))\big)\\ &= (p\star\sigma)\big((1-t)e^0 + t\delta^0(\delta^{i-1}(x))\big) \qquad \text{by } 8.3\\ &= (1-t)p + t\sigma(\delta^{i-1}(x))\\ &= \big(p\star(\sigma\circ\delta^{i-1})\big)\big((1-t)e^0 + t\delta^0(x)\big).\end{aligned}$$

Hence $\partial(p \star \sigma) = (p \star \sigma) \circ \delta^0 + \sum_{i=1}^{q+1}(-1)^i p \star (\sigma \circ \delta^{i-1}) = \sigma - p \star \partial\sigma$. □

8.14 Corollary. [20, 9.1.18] *Let $A \subseteq \mathbb{R}^n$ be convex. Then A is acyclic; cf. 7.19 & 7.20.*

Proof. Let $p \in A$ and $z \in Z_q(A)$ for some $q > 0$. Then $z = \partial(p \star z)$ by 8.13 and hence $Z_q(A) = B_q(A)$, i.e. $H_q(A) = \{0\}$. □

Relative homology

8.15 Definition. [**20**, 9.2.1] Let (X, A) be a pair of spaces. Then we get a pair of chain complexes $(S(X), S(A))$ and hence a short exact sequence

$$0 \to S(A) \to S(X) \twoheadrightarrow S(X, A) \to 0,$$

where $S_q(X, A) := S_q(X)/S_q(A)$. Its elements are called RELATIVE SINGULAR q-CHAINS. **But unlike 7.31 we can not identify them with formal linear combinations of simplices in $X \setminus A$.**

8.16 Remark. [**20**, 9.2.3] However, as in 7.31 we get a long exact sequence in homology

$$\cdots \to H_{q+1}(X, A) \xrightarrow{\partial_*} H_q(A) \to H_q(X) \to H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \to \cdots,$$

where $H_q(X, A) := H_q(S(X, A))$. Note that $z \in S_q(X)$ with $\partial z \in S_{q-1}(A)$ describe cycles $z + S_q(A)$ in $S_q(X, A)$ (since $\partial(c + S_q(A)) := \partial c + S_{q-1}(A)$) and hence classes $[z + S_q(A)] \in H_q(X, A)$.
In particular, for acyclic A and injective $H_0(A) \to H_0(X)$ we get $H_q(X) \cong H_q(X, A)$ for all $q \neq 0$.
For a continuous mapping of pairs $(X, A) \to (Y, B)$ we get a homology ladder by 7.34.

8.17 Remark. [**20**, 9.2.2] As in 7.32 we get

1. $H_q(X, X) = \{0\}$,
2. $H_q(X, \emptyset) \cong H_q(X)$, and
3. $H_0(X, A) = \{0\}$ for path-connected X and $A \neq \emptyset$.

8.18 Remark. [**20**, 9.2.4] Using the long exact homology sequence

$$\cdots \to H_{q+1}(X, A) \to H_q(A) \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \cdots,$$

we obtain:

1. Let $A \subseteq X$ be such that $H_q(A) \to H_q(X)$ is injective for all q. Then we get short exact sequences $0 \to H_q(A) \to H_q(X) \to H_q(X, A) \to 0$, where $H_q(X, A) \to H_{q-1}(A)$ is 0, since the next one in the long exact sequence is assumed to be injective.
2. Let $A \subseteq X$ be a retract (i.e. has a left inverse). Then by functorality $H_q(A) \to H_q(X)$ is a retract and hence by 1 we have (splitting) short exact sequences, i.e. $H_q(X) \cong H_q(A) \oplus H_q(X, A)$.
3. Let $x_0 \in X$. The constant mapping $X \to \{x_0\}$ is a retraction, hence $H_q(X) \cong H_q(\{x_0\}) \oplus H_q(X, \{x_0\})$ by 2. By 8.7 we have that $H_q(\{x_0\}) = \{0\}$ for $q \neq 0$ and $H_0(\{x_0\}) = \mathbb{Z}$, hence $H_q(X, \{x_0\}) \cong H_q(X)$ for $q > 0$ and $0 \to \mathbb{Z} \to H_0(X) \to H_0(X, \{x_0\}) \to 0$ is splitting exact.
4. Let $f : (X, A) \to (Y, B)$ be such that $f_* : H_q(A) \to H_q(B)$ and $f_* : H_q(X) \to H_q(Y)$ are isomorphisms for all q. Then the same is true for $f_* : H_q(X, A) \to H_q(Y, B)$ by the 5'Lemma applied to the homology ladder of 7.34.

8.19 Theorem. Exact homology sequence of a triple. [**20**, 9.2.5]
Let $B \subseteq A \subseteq X$. Then we get a long exact homology sequence

$$\cdots \to H_{q+1}(X, A) \xrightarrow{\partial_*} H_q(A, B) \to H_q(X, B) \to H_q(X, A) \to \cdots$$

The operator ∂_* can also be described by $[z]_{(X,A)} \mapsto [\partial z]_{(A,B)}$ for $z \in S_q(X)$ with $\partial z \in S_{q-1}(A)$ or as composition $H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A) \to H_q(A,B)$.

Note, that $B := \emptyset$ gives us the long exact sequence of 8.16.

Proof. We have a short sequence

$$0 \to S(A,B) \to S(X,B) \to S(X,A) \to 0.$$

given by

$$\begin{tikzcd}
 & S(B) \arrow[r, equal] \arrow[d, hook] & S(B) \arrow[r, hook] \arrow[d, hook] & S(A) \arrow[d, hook] & \\
 & S(A) \arrow[r, hook] \arrow[d] & S(X) \arrow[r, equal] \arrow[d] & S(X) \arrow[d] & \\
0 \arrow[r] & S(A,B) \arrow[r, hook] \arrow[d] & S(X,B) \arrow[r, two heads] \arrow[d] & S(X,A) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom row is exact at $S(X,A)$ and also at $S(A,B)$: In fact for $\dot a \in S(A,B)$ let the image in $S(X,B)$ be 0. Then $a = b \in S(B)$ and hence $\dot a = 0$ in $S(A,B)$.

It is also exact at $S(X,B)$, since for $\dot x \in S(X,B)$ which is mapped to 0 in $S(X,A)$ the image $x \in S(X)$ is an $a \in S(A)$ and hence satisfies $\dot a$ is mapped to $\dot x$ (see below).

$$\begin{tikzcd}
b \arrow[r, equal] \arrow[d, mapsto, "4."] & \exists b \arrow[d, mapsto, "3."] \\
\exists a \arrow[r, mapsto, "2."] \arrow[d, mapsto, "1."] & a \arrow[d, mapsto, "2."] \\
\dot a \arrow[r, mapsto, "0."] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \exists a \arrow[d, mapsto, "3."] \\
a \arrow[r, mapsto, "4."] \arrow[d, mapsto, "4."] & \exists x \arrow[r, equal] \arrow[d, mapsto, "1."] & x \arrow[d, mapsto, "2."] \\
\dot a \arrow[r, mapsto, "5."] & \dot x \arrow[r, mapsto, "0."] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & z + S_q(B) \arrow[r, mapsto] \arrow[d, mapsto, "\partial"] & z + S_q(A) \arrow[d, mapsto, "\partial"] \\
\partial z + S_{q-1}(B) \arrow[r, mapsto] & \partial z + S_{q-1}(B) \arrow[r, mapsto] & 0 + S_{q-1}(A)
\end{tikzcd}$$

So this short exact sequence induces a long exact sequence in homology by 7.30. The boundary operator maps the class $[z + S(A)]$ with $\partial z \in S(A)$ to $[\partial z + S(B)]$ by construction 7.30 (see the diagram above).

This is precisely the image of value of the boundary operator $[\partial z]$ for the pair (X,A) under the natural map $H(A) \to H(A,B)$:

$$\begin{tikzcd}
 & z \arrow[r, mapsto] \arrow[d, mapsto] & z + S_q(A) \\
\partial z \arrow[r, mapsto] & \partial z &
\end{tikzcd}$$

□

Homotopy Theorem

We are going to prove now that homotopic mappings induce identical mappings in homology. For this we consider first a homotopy, which is as free and as natural as possible, i.e. the homotopy given by $\mathrm{ins}_t : X \to X \times I$, $x \mapsto (x,t)$. We have to show that ins_0 and ins_1 induce the same mapping in homology. So the images of a cycle should differ only by a boundary. Let $\sigma : \Delta_q \to X$ be a singular simplex. Then we may consider the cylinder $\sigma(\Delta_q) \times I$ over $\sigma(\Delta_q)$. It seems clear, that we can triangulate $\Delta_q \times I$ (cf. exercise (3.2)). The image of the corresponding chain c_{q+1} under $\sigma \times I$ gives then a $q+1$-chain in $X \times I$, whose boundary consists of the parts $\sigma \times \{1\} = \mathrm{ins}_1 \circ \sigma$ and $\sigma \times \{0\} = \mathrm{ins}_0 \circ \sigma$ and a triangulation of $(\sigma \times I)_* \partial c_q$. Note that it would have been easier here, if we had defined the singular homology by using squares instead of triangles, since it is not so clear how to describe an explicit triangulation of $\Delta_q \times I$, in fact we will show the existence of c_{q+1} by induction in lemma 8.21.

We make use of the following

8.20 Definition. [20, 8.4.6] Let $R, S : \mathcal{X} \to \mathcal{Y}$ be two functors. A natural transformation $\varphi : R \to S$ is a family consisting of $\mathcal{Y}$-morphisms $\varphi_X : R(X) \to S(X)$ for each object $X \in \mathcal{X}$ such that for every $\mathcal{X}$-morphism $f : X \to X'$ the following diagram commutes:

$$\begin{tikzcd} R(X) \arrow[r, "\varphi_X"] \arrow[d, "R(f)"'] & S(X) \arrow[d, "S(f)"] \\ R(X') \arrow[r, "\varphi_{X'}"'] & S(X') \end{tikzcd}$$

8.21 Lemma. [20, 9.3.7] *Let $\varphi_0, \varphi_1 : S(\_) \to S(\_ \times I)$ be two natural transformations and assume furthermore that $H_0(\varphi_0) = H_0(\varphi_1) : H_0(\{*\}) \to H_0(\{*\} \times I)$. Then φ_0 and φ_1 are chain homotopic (see 8.22), i.e. there exists $\mathcal{Z} = (\mathcal{Z}_q)_q$ with homomorphisms $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$ such that $\partial \mathcal{Z}_q + \mathcal{Z}_{q-1}\partial = \varphi_1 - \varphi_0$ on $S_q(X)$.*

Proof. We construct $\mathcal{Z}_q$ by induction on q:
For $q < 0$ let $\mathcal{Z}_q := 0$. Now let $\mathcal{Z}_j$ for all $j < q$ be already constructed. Consider the natural transformation $\varphi := \varphi_1 - \varphi_0$. We first treat the case $X := \Delta_q$. In particular, we have to find for $\sigma := \mathrm{id}_{\Delta_q} = \Delta_q \in S_q(X)$ an element $\mathcal{Z}_q(\Delta_q) =: c_{q+1} \in S_{q+1}(\Delta_q \times I)$ with $\partial c_{q+1} = \varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q$. For $q = 0$ this follows from the assumption $[\varphi(\Delta_0)] = 0 \in H_0(\Delta_0 \times I)$. For $q > 0$ we can use that $S(\Delta_q \times I)$ is acyclic by 8.14, since $\Delta_q \times I$ is a convex subset of $\mathbb{R}^{q+2}$. So we only have to show that the right side is a cycle. In fact, by induction hypothesis (applied to $\partial\Delta_q$) we have

$$\partial\big(\varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q\big) = \varphi\partial\Delta_q - \big(\varphi - \mathcal{Z}_{q-2}\partial\big)\partial\Delta_q = \varphi\partial\Delta_q - \big(\varphi\partial\Delta_q - \mathcal{Z}_{q-2}\partial\partial\Delta_q\big) = 0.$$

Now we extend $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$ by naturality to the case of a general X: I.e. for $\sigma : \Delta_q \to X$ we define $\mathcal{Z}_q(\sigma) := S_{q+1}(\sigma \times I)(c_{q+1})$.

$$\begin{tikzcd} \Delta_q \arrow[r, mapsto, "\mathcal{Z}_q"'] \arrow[d, mapsto, "\sigma_*"] & c_{q+1} \arrow[d, mapsto, "(\sigma\times I)_*"] \\ S_q(X) \arrow[r, "\mathcal{Z}_q"'] \arrow[d, "f_*"] & S_q(X \times I) \arrow[d, "(f\times I)_*"] \\ S_q(Y) \arrow[r, "\mathcal{Z}_q"'] & S_q(Y \times I) \end{tikzcd}$$

Then $\mathcal{Z}_q$ is in fact natural, since $S_{q+1}(f \times I)\mathcal{Z}_q(\sigma) = S_{q+1}(f \times I)S_{q+1}(\sigma \times I)(c_{q+1})$ and $\mathcal{Z}_q S_q(f)(\sigma) = \mathcal{Z}_q(f\sigma) = S_{q+1}(f\sigma \times I)(c_{q+1})$ and $(f \times I) \circ (\sigma \times I) = (f \circ \sigma) \times I$.

Furthermore $\mathcal{Z}_q$ is also a chain homotopy, since

$$\begin{aligned} \partial\mathcal{Z}_q(\sigma) &= \partial S_{q+1}(\sigma \times I)(c_{q+1}) = S_q(\sigma \times I)\partial c_{q+1} = S_q(\sigma \times I)(\varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q) \\ &= \varphi S_q(\sigma)\Delta_q - \mathcal{Z}_{q-1}\partial S_q(\sigma)\Delta_q = \varphi(\sigma) - \mathcal{Z}_{q-1}\partial(\sigma). \quad \square \end{aligned}$$

8.22 Definition. [20, 8.3.12] [20, 8.3.15] Two chain mappings $\varphi, \psi : C \to C'$ are called (CHAIN) HOMOTOPIC and we write $\varphi \sim \psi$ if there exists a graded group-homomorphism $\mathcal{Z} : C \to C'$ of degree 1 (i.e. a family $\mathcal{Z} = (\mathcal{Z}_q)_{q\in\mathbb{Z}}$ of group homomorphisms $\mathcal{Z}_q : C_q \to C'_{q+1}$) with $\varphi - \psi = \partial\mathcal{Z} + \mathcal{Z}\partial$.

8.23 Proposition. [20, 8.3.13]
Let $\varphi \sim \psi : C \to C'$. Then $H(\varphi) = H(\psi) : H(C) \to H(C')$.

Proof. Let $[c] \in H(C)$, i.e. $\partial c = 0$, then $H(\varphi)[c] - H(\psi)[c] = [(\varphi - \psi)c] = [\partial\mathcal{Z}c + \mathcal{Z}\partial c] = [\partial\mathcal{Z}c] = 0$. □

8.24 Proposition. [20, 8.3.14] *Chain homotopies are compatible with compositions and chain homotopic is an equivalence relation.*

Proof. Clearly, for $\varphi \sim \psi$ we have $\chi \circ \varphi \sim \chi \circ \psi$ (since $\chi(\varphi - \psi) = \chi(\partial\mathcal{Z} + \mathcal{Z}\partial) = \partial\chi\mathcal{Z} + \chi\mathcal{Z}\partial$) and similarly $\varphi\circ\chi \sim \psi\circ\chi$ and being chain homotopic is transitive. □

8.25 Theorem. [20, 9.3.1]
Let $f \sim g : (X, A) \to (Y, B)$. Then $f_* = g_* : H_q(X, A) \to H_q(Y, B)$.

Proof. By 8.21 we have that the chain mappings induced by the inclusions $\mathrm{ins}_j : X \to X \times I$ are chain homotopic to each other for $j \in \{0, 1\}$ by a chain homotopy $\mathcal{Z}$. Let h be a homotopy of pairs between f and g, i.e. $f = h \circ \mathrm{ins}_0$ and $g = h \circ \mathrm{ins}_1$. By 8.24 the composite $h\circ\mathcal{Z}$ is a chain homotopy $S(f) \sim S(g) : S(X) \to S(Y)$ and its restriction is a chain homotopy $S(f) \sim S(g) : S(A) \to S(B)$, since the construction is natural. Thus $S(f) \sim S(g) : S(X, A) \to S(X, B)$, since $S(f)$, $S(g)$, ∂, and $h \circ \mathcal{Z}$ are given on the relative singular chains by their value on representants. By 8.23 we have that $H(f) = H(g) : H(X, A) \to H(X, B)$. □

8.26 Corollary. [20, 9.3.2]
Let $f \sim g : X \to Y$. Then $f_* = g_* : H_q(X) \to H_q(Y)$.

Proof. Obvious, since $H_q(X, \emptyset) \cong H_q(X)$ naturally. □

8.27 Corollary. [20, 9.3.3] *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for all q. In particular, all contractible spaces are acyclic.*

Proof. Obvious by functoriality and 8.26 since an inverse g up to homotopy induces an inverse $H(g)$ of $H(f)$. □

8.28 Corollary. [20, 9.3.4] [20, 9.3.5] [20, 9.3.6]

1. *Let $A \subseteq X$ be a DR. Then $H_q(A) \to H_q(X)$ is an isomorphism and hence $H_q(X, A) = \{0\}$ for all q, cf. 8.18.2.*
2. *Let $B \subseteq A \subseteq X$ and A be a DR of X. Then $H_q(A, B) \to H_q(X, B)$ is an isomorphism.*
3. *Let $B \subseteq A \subseteq X$ and B be a DR of A. Then $H_q(X, B) \to H_q(X, A)$ is an isomorphism.*

Proof. The first part follows as special case from 8.27 and from 8.16, the long exact homology sequence of a pair. The other two cases then follow by using 8.19, the long exact homology sequence of a triple. □

Excision Theorem

In order to prove the excision theorem for the singular homology we need the barycentric refinement for singular simplices, since a singular simplex in X need neither be contained in $S(U)$ nor in $S(V)$ for a given covering $\{U, V\}$ of X.

8.29 Definition. [20, 9.4.1] For the standard q-simplex Δ_q we define the BARYCENTRIC CHAIN $B(\Delta_q) \in S_q(\Delta_q)$ recursively by (cf. 3.24)

$$B(\Delta_0) := \Delta_0$$

$$B(\Delta_q) := \widehat{\Delta_q} \star \sum_{j=0}^{q} (-1)^j S(\delta^j)\big(B(\Delta_{q-1})\big) \text{ for } q \geq 1,$$

where $\widehat{\Delta_q} := \frac{1}{q+1} \sum_{j=0}^{q} e^j$ is the barycenter. Next we define in a natural way

$$B(\sigma) = B\big(S(\sigma)(\Delta_q)\big) := S(\sigma)B(\Delta_q) \text{ for } \sigma : \Delta_q \to X$$

and extend it linearly to $B : S_q(X) \to S_q(X)$ by setting

$$B\Big(\sum_\sigma n_\sigma \cdot \sigma\Big) := \sum_\sigma n_\sigma B(\sigma).$$

Note that the recursion formula for $B(\Delta_q)$ can be rewritten as

$$B\Delta_q = \widehat{\Delta_q} \star B\partial\Delta_q.$$

8.30 Proposition. [20, 9.4.2] *The barycentric refinement is a natural chain mapping $B : S(\_) \to S(\_)$ with $B \sim \mathrm{id}$.*

Proof. Let us first show naturality: So let $f : X \to Y$ be continuous. Then

$$(f_*B)\sigma = (f_*\sigma_*B)\Delta_q = (f \circ \sigma)_* B\Delta_q = B(f \circ \sigma) = (Bf_*)\sigma.$$

Next we prove that it is a chain mapping, i.e. $\partial B = B\partial$. On $S_q(X)$ with $q \leq 0$ this is obvious. Now we use induction for $q > 0$:

$$\begin{aligned}\partial B\sigma &= \partial\sigma_* B\Delta_q = \sigma_*\partial B\Delta_q = \sigma_*\partial\Big(\widehat{\Delta_q} \star B\partial\Delta_q\Big)\\ &\overset{\boxed{8.13}}{=\!=} \sigma_*\Big(B\partial\Delta_q - \widehat{\Delta_q} \star \partial B\partial\Delta_q\Big) \overset{\text{I.Hyp.}}{=\!=} B\sigma_*\partial(\Delta_q) - \sigma_*\Big(\widehat{\Delta_q} \star B\partial\partial(\Delta_q)\Big)\\ &= B\partial\sigma_*(\Delta_q) - 0 = B\partial\sigma.\end{aligned}$$

Finally we prove the existence of a chain homotopy $\mathrm{id} \sim B : S \to S$. Let $i = \mathrm{ins}_0 : X \to X \times I$ be given by $x \mapsto (x, 0)$ and $p = \mathrm{pr}_1 : X \times I \to X$ given by $(x, t) \mapsto x$ then $S(p) \circ S(i) = \mathrm{id}$. Since $B|_{S_0} = \mathrm{id}$ we have a chain homotopy $S(i) \circ B \sim S(i)$ by 8.21. Composing with $S(p)$ gives a chain homotopy $B = S(p) \circ S(i) \circ B \sim S(p) \circ S(i) = \mathrm{id}$ by 8.24. □

8.31 Corollary. [20, 9.4.3] *Let $A \subseteq X$. Then $B_* = \mathrm{id} : H(X, A) \to H(X, A)$.*

By iteration we get the corresponding result for $B^r := \overset{r \text{ times}}{B \circ \ldots \circ B}$.

Proof. Let $\alpha \in H_q(X, A)$ be given, i.e. $\alpha = [z + S_q(A)]$ for a $z \in S_q(X)$ with $\partial z \in S_{q-1}(A)$. By 8.30 $B \sim \mathrm{id}$. Let $(\mathcal{Z}_q : S_q(\_) \to S_{q+1}(\_))_q$ be a corresponding natural chain homotopy. Then $Bz - z = \partial\mathcal{Z}_q z + \mathcal{Z}_{q-1}\partial z \in \partial\mathcal{Z}_q z + S_q(A)$, i.e. Bz is homologous to z relative A and, furthermore, $\partial Bz \in \partial z + 0 + \partial \mathcal{Z}_{q-1}\partial z \in S_{q-1}(A)$, so Bz is a cycle relative A, i.e. $\alpha = [z + S_q(A)] = [Bz + S_q(A)] = B_*(\alpha)$. $\square$

8.32 Lemma. [20, 9.4.4] *Let X be the union of two open subsets U and V. Then for every $c \in S_q(X)$ there is an $r > 0$ with $B^r c \in S_q(U) + S_q(V) \subseteq S_q(X)$.*

Proof. It is enough to show this for c being a singular simplex $\sigma : \Delta_q \to X$. The sets $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$ form an open covering of Δ_q. Let λ be the Lebesgue number for this covering, i.e. all subsets of Δ_q of diameter less than λ belong to one of the two sets. Since $B^r(\Delta_q)$ is a finite linear combination of singular simplices, whose image are closed simplices of the r-th barycentric refinement of $K := \{\tau : \tau \leq \Delta_q\}$, we have by 3.26 that for sufficiently large r each summand of $B^r(\Delta_q)$ has image in $\sigma^{-1}(U)$ or in $\sigma^{-1}(V)$. Hence $B^r(\sigma) = B^r\big(S(\sigma)(\Delta_q)\big) = S(\sigma)B^r(\Delta_q)$ is a sum of summands in $S_q(U)$ and in $S_q(V)$. $\square$

8.33 Excision theorem. [20, 9.4.5]
Let $X_j \subseteq X$ for $j \in \{1, 2\}$ such that the interiors $\mathring{X}_j$ cover X.
Then the inclusion $i_* : (X_2, X_2 \cap X_1) \to (X_2 \cup X_1, X_1)$ induces isomorphisms $H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$ for all q.
In particular this applies to $X_1 := Y \subseteq X$ and $X_2 := X \setminus Z$ for subsets Z and Y satisfying $\bar{Z} \subseteq \mathring{Y}$ and so gives isomorphisms $H_q(X \setminus Z, Y \setminus Z) \to H_q(X, Y)$.

Proof. We have to show that $i_* : H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$ is bijective.

i_* **is onto:** Let $\beta \in H_q(X_2 \cup X_1, X_1)$, i.e. $\beta = [z + S_q(X_1)]$ for some $z \in S_q(X)$ with $\partial z \in S_{q-1}(X_1)$. By 8.32 there exists an $r > 0$ and $u_j \in S_q(\mathring{X}_j)$ such that $z \sim B^r z = u_1 + u_2 \sim u_2$ relative X_1 by 8.31. We have $\partial u_2 \in S_{q-1}(X_2)$ and $\partial u_2 = \partial B^r z - \partial u_1 = B^r \partial z - \partial u_1 \in S_{q-1}(X_1)$, hence $\partial u_2 \in S_{q-1}(X_1 \cap X_2)$. So $\alpha := [u_2 + S_q(X_2 \cap X_1)] \in H_q(X_2, X_2 \cap X_1)$ and it is mapped by i_* to β.

i_* **is injective:** Let $\alpha \in H_q(X_2, X_2 \cap X_1)$ be such that $i_*\alpha = 0$. Then $\alpha = [x_2 + S_q(X_2 \cap X_1)]$ for some $x_2 \in S_q(X_2)$ and since $0 = i_*\alpha = [x_2 + S_q(X_1)] \in H_q(X, X_1)$ we have a $(q+1)$-chain c in X and a q-chain x_1 in X_1 with $\partial c = x_2 + x_1$. Again by 8.32 there is an $r > 0$ such that $B^r c = u_1 + u_2$ with $u_j \in S_q(\mathring{X}_j)$. Hence $\partial u_1 + \partial u_2 = \partial B^r c = B^r \partial c = B^r(x_2 + x_1)$. So $a := B^r x_2 - \partial u_2 = \partial u_1 - B^r x_1$ is a chain in $X_1 \cap X_2$ and $x_2 \sim B^r x_2 = \partial u_2 + a$ by 8.31, i.e. $\alpha = [x_2 + S_q(X_2 \cap X_1)] = [\partial u_2 + a + S_q(X_2 \cap X_1)] = 0$, since $a \in S_q(X_2 \cap X_1)$ and $u_2 \in S_q(X_2)$.

The alternative description is valid, since the interiors of $X_1 := Y$ and $X_2 := X \setminus Z$ cover X iff $\mathring{Y} = \mathring{X}_1 \supseteq X \setminus \mathring{X}_2 = X \setminus (X \setminus \bar{Z}) = \bar{Z}$. Obviously $Y \setminus Z = X_1 \cap X_2$. $\square$

8.34 Corollary. [20, 9.4.6] [20, 9.4.7] *Let (X, A) be a CW-pair. Then the quotient map $p : (X, A) \to (X/A, A/A)$ induces an isomorphism in homology for all q and, in particular, hence $H_q(X, A) \cong H_q(X/A)$ for all $q \neq 0$.*

Proof. By 4.18 we have an open neighborhood U of A in X, of which A is an SDR. Let $p : X \to X/A =: Y$ be the quotient mapping and let $V := p(U) \subseteq X/A =: Y$ and $y := A/A \in X/A$. Since U is saturated its image $V \subseteq Y$ is open and $p(A) = \{y\}$ is an SDR in V. Now consider

$$\begin{tikzcd}
H_q(X,A) \arrow[r, "\cong", "8.28.3"'] \arrow[d, "p_*"', "{\cong,\ 3.}"] & H_q(X,U) \arrow[d, "p_*"', "{\cong,\ 2.}"] & H_q(X\setminus A, U\setminus A) \arrow[l, "\cong"', "8.33"] \arrow[d, "p_*"', "{\cong\ 1.34,\ 1.}"] \\
H_q(Y,\{y\}) \arrow[r, "\cong", "8.28.3"'] & H_q(Y,V) & H_q(Y\setminus\{y\}, V\setminus\{y\}) \arrow[l, "\cong"', "8.33"]
\end{tikzcd}$$

By 1.34 we have that $p : (X, A) \to (Y, \{y\})$ is a relative homeomorphism, so the vertical arrow on the right side is induced by an isomorphism of pairs and hence is an isomorphism. The horizontal arrows on the right side are isomorphisms by the excision theorem 8.33. Hence the vertical arrow in the middle is an isomorphism. By 8.28.3 the horizontal arrows on the left are isomorphisms, hence also the vertical arrow on the left.

Finally, by 8.18.3 we have $H_q(Y, \{y\}) \cong H_q(Y)$ for $q > 0$. □

8.35 Corollary. [20, 9.4.8] *Let $f : (X, A) \to (Y, B)$ be a relative homeomorphism of CW-pairs. Assume furthermore that $X \setminus A$ contains only finitely many cells or $f : X \to Y$ is a quotient mapping. Then $f_* : H_q(X, A) \to H_q(Y, B)$ is an isomorphism for all q.*

Proof. By 1.34 we have an induced continuous bijective mapping $\tilde f : X/A \to Y/B$ making the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & B \arrow[d, hook] \\
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
X/A \arrow[r, "\tilde f", "{\cong,\ 1.}"'] & Y/B
\end{tikzcd}
\qquad
\begin{tikzcd}
H_q(X,A) \arrow[r, "f_*", "{\cong,\ 4.}"'] \arrow[d, "p_*"', "8.34"] & H_q(Y,B) \arrow[d, "q_*"', "8.34"] \\
H_q(X/A, A/A) \arrow[r, "{\cong,\ 3.}"'] & H_q(Y/B, B/B) \\
H_q(X/A) \arrow[r, "\tilde f_*", "{\cong,\ 2.}"'] \arrow[u, "8.18.3"'] & H_q(Y/B) \arrow[u, "8.18.3"']
\end{tikzcd}$$

That this bijection is a homeomorphism follows in case $X \setminus A$ has only finitely many cells since then X/A is compact by 4.15 and 4.5, and in the case where $f : X \to Y$ is a quotient map then so is $X \to Y \to Y/B$ and hence also $X/A \to Y/B$. Thus $\tilde f_* : H_q(X/A) \cong H_q(Y/B)$ and by 8.18.3 (and 8.18.4) the horizontal arrow in the middle on the right is an isomorphism. By 4.15 both X/A and Y/B are CW-spaces thus by 8.34 the vertical down-arrows on the right are isomorphisms as well, so the same has to be true for the top horizontal arrow on the right. □

8.36 Proposition. [20, 9.4.9] *Let X_j be CW-complexes with 0-cells $x_j \in X_j$ as base-points. Then we have natural isomorphisms ${}^{ab}\coprod_j H_q(X_j) \cong H_q(\bigvee_j X_j)$ for $q \neq 0$.*

Proof. We have $\bigvee_j X_j := \bigsqcup_j X_j/A$, where $A := \{x_j : j \in J\}$ with $H_q(A) = {}^{ab}\coprod_j H_q(\{x_j\}) = 0$ by 8.9 and 8.7 for $q \neq 0$. Furthermore, $H_0(A) \to H_0(\bigsqcup_j X_j)$ is injective by 8.18.2, so

$$H_q\Big(\bigvee_j X_j\Big) \overset{8.34}{\cong} H_q\Big(\bigsqcup_j X_j, A\Big) \overset{8.16}{\cong} H_q\Big(\bigsqcup_j X_j\Big) \overset{8.9}{\cong} {}^{ab}\coprod_j H_q(X_j). \quad \square$$

8.37 Proposition. Mayer-Vietoris sequence. [20, 9.4.10] *Let $X = X_1 \cup X_2$, where the $X_j \subseteq X$ are open. Then there is a long exact sequence*

$$\cdots \to H_q(X_1 \cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(X_1 \cap X_2) \to \cdots.$$

Proof. Let $S := S(X)$, $S_1 := S(X_1) \subseteq S(X)$ and $S_2 := S(X_2) \subseteq S(X)$. Then $S(X_1 \cap X_2) = S_1 \cap S_2$. Let $S_1 + S_2$ be the chain complex which has the subgroup

of S generated by S_1 and S_2 in every dimension.
We claim that the following short sequence

$$0 \to S_1/(S_1 \cap S_2) \to S/S_2 \to S/(S_1 + S_2) \to 0$$

is exact:

$$\begin{tikzcd}
S_1 \cap S_2 \arrow[r, hook] \arrow[d, hook] & S_2 \arrow[r, equal] \arrow[d, hook] & S_2 \arrow[d, hook] & \\
S_1 \arrow[r, hook] \arrow[d] & S_1 + S_2 \arrow[r, hook] \arrow[d] & S \arrow[r, two heads] \arrow[d] & S/(S_1+S_2) \arrow[d, equal] \\
S_1/S_1 \cap S_2 \arrow[r, equal, "\sim"] & (S_1+S_2)/S_2 \arrow[r, hook] & S/S_2 \arrow[r, two heads] & S/(S_1+S_2)
\end{tikzcd}$$

In fact, by the first isomorphy theorem we have $S_1/(S_1 \cap S_2) \cong (S_1 + S_2)/S_2$ and hence the inclusion $S_1 + S_2 \subseteq S$ induces an injection $S_1/(S_1 \cap S_2) \to S/S_2$. The quotient of it is $(S/S_2)/((S_1 + S_2)/S_2) \cong S/(S_1 + S_2)$ by the second isomorphy theorem, which proves the claim.

By the excision theorem 8.33 we have that the inclusion $(S_1, S_1 \cap S_2) \hookrightarrow (S, S_2)$ induces an isomorphism $H(S_1/(S_1 \cap S_2)) =: H(X_1, X_1 \cap X_2) \to H(X_1 \cup X_2, X_2) := H(S/S_2)$. Hence the long exact homology sequence 7.30 gives $H(S/(S_1+S_2)) = 0$.

If we consider now the short exact sequence

$$0 \to S_1 + S_2 \to S \to S/(S_1 + S_2) \to 0$$

then we deduce from the long exact homology sequence 7.30 that $H(S_1 + S_2) \to H(S)$ is an isomorphism.

Now consider the sequence

$$0 \to S_1 \cap S_2 \to S_1 \oplus S_2 \to S_1 + S_2 \to 0,$$

where the inclusion is given by $c \mapsto (c, -c)$ and the projection by $(c_1, c_2) \mapsto c_1 + c_2$. This is obviously short exact, since (c_1, c_2) is mapped to 0 iff $c_1 + c_2 = 0$, i.e. $c := c_1 = -c_2 \in S_1 \cap S_2$ is mapped to (c_1, c_2). So we get a long exact homology sequence 7.30, where we may replace $H(S_1 + S_2)$ by $H(S) =: H(X)$ by what we said above.

Note that the boundary operator is $[z] \mapsto [\partial z_1] = [-\partial z_2]$, where $B^r z = z_1 + z_2$. $\square$

8.38 Remark. [20, 9.4.12]

(1) Instead of openness of X_1 and X_2 it is enough to assume in 8.37 that there are open neighborhoods of X_1 and X_2 which have X_1 and X_2 and their intersection has $X_1 \cap X_2$ as DRs. In particular this applies to CW-subspaces X_i of a CW-complex X by 4.18.

(2) Let $X_1 \cap X_2$ be acyclic. Then the Mayer-Vietoris sequence gives $H_q(X) \cong H_q(X_1) \oplus H_q(X_2)$ for $q \neq 0$. In fact only the case $q = 1$ needs some argument: We have the exact sequence

$$\begin{tikzcd}
0 = H_1(X_1 \cap X_2) \arrow[r] & H_1(X_1) \oplus H_1(X_2) \arrow[r] & H_1(X) \arrow[dll, "0"'] & \\
\mathbb{Z} = H_0(X_1 \cap X_2) \arrow[r, tail] & H_0(X_1) \oplus H_0(X_2) \arrow[r] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

and the mapping $H_0(X_1 \cap X_2) \to H_0(X_1) \oplus H_0(X_2)$ is injective, since the generator is mapped to a generator of $H_0(X_1)$ and of $H_0(X_2)$.

(3) Let X_1 and X_2 be acyclic. Then we have $H_q(X_1 \cap X_2) \cong H_{q+1}(X)$ for $q > 0$ and furthermore $H_1(X)$ is free abelian and if $H_0(X_1 \cap X_2) \cong \mathbb{Z}^k$ with $k \neq 0$ then

$$\begin{tikzcd}
H_1(X_1) \oplus H_1(X_2) \arrow[r] \arrow[d, equal] & H_1(X) \arrow[r, tail] & H_0(X_1 \cap X_2) \arrow[r] \arrow[d, equal] & H_0(X_1) \oplus H_0(X_2) \arrow[r] \arrow[d, equal] & H_0(X) \arrow[r] \arrow[d, equal] & 0 \\
0 & & \mathbb{Z}^k & \mathbb{Z}^2 & \mathbb{Z} &
\end{tikzcd}$$

gives $H_1(X) \cong \mathbb{Z}^{k-1}$ via the rank formula $\operatorname{rank}(\ker f) + \operatorname{rank}(\operatorname{im} f) = \operatorname{rank}(\operatorname{dom} f)$, where we used that $X = X_1 \cup X_2$ is connected being the union of two connected not disjoint sets.

(4) Consider the covering $S^n = D^n_+ \cup D^n_-$. By 1 we get a long exact Mayer-Vietoris sequence. And since D^n_+ and D^n_- are convex, they are acyclic by 8.14. So $H_q(S^n) \cong H_{q-1}(D^n_+ \cap D^n_-) = H_{q-1}(S^{n-1})$ for $q > 1$ and $n > 0$ by 3. Inductively we hence get $H_q(S^n) \cong H_{q-n}(S^0) = \{0\}$ for $q > n$, since S^0 is discrete. And for $0 < q < n$ we get $H_q(S^n) \cong H_1(S^{n-q+1}) = \{0\}$, since

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^{n-q+1}) \arrow[r, tail] & H_0(S^{n-q}) \arrow[r] \arrow[d, equal] & H_0(D^{n-q}_+) \oplus H_0(D^{n-q}_-) \arrow[r] \arrow[d, equal] & H_0(S^{n-q+1}) \arrow[r] \arrow[d, equal] & 0 \\
& & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

and $H_n(S^n) \cong H_1(S^1) \cong \mathbb{Z}$, since

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^1) \arrow[r, tail] & H_0(S^0) \arrow[r] \arrow[d, equal] & H_0(D^0_+) \oplus H_0(D^0_-) \arrow[r] \arrow[d, equal] & H_0(S^1) \arrow[r] \arrow[d, equal] & 0 \\
& & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

Homology of balls, spheres and their complements

8.39 Proposition. [20, 9.5.1] *Let $n \geq 0$. Then*

$$H_q(\Delta_n, \dot{\Delta}_n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

The generator in $H_n(\Delta_n, \dot{\Delta}_n)$ will be denoted $[\Delta_n]$ and is given by the relative homology class of the singular simplex $\mathrm{id}_{\Delta_n} : \Delta_n \to \Delta_n$. Cf. 7.32.4.

Proof. We prove this by induction on n:

$(n = 0)$ $H_q(\Delta_0, \dot{\Delta}_0) = H_q(\{1\}, \emptyset) \overset{8.17.2}{=\!=\!=} H_q(\{*\})$.

$(n > 0)$ We consider Δ_{n-1} as face opposite to e_n in Δ_n and let $A_n := \dot{\Delta}_n \setminus \mathring{\Delta}_{n-1}$.

Since A_n is a DR of Δ_n, we get $H_q(\Delta_n, \dot{\Delta}_n) \cong H_{q-1}(\dot{\Delta}_n, A_n)$ from 8.19 for the triple $A_n \subseteq \dot{\Delta}_n \subseteq \Delta_n$. Since $\Delta_{n-1} \setminus \dot{\Delta}_{n-1} = \dot{\Delta}_n \setminus A_n$ we get from 8.35 that the inclusion induces an isomorphism $H_{q-1}(\Delta_{n-1}, \dot{\Delta}_{n-1}) \cong H_{q-1}(\dot{\Delta}_n, A_n)$. Hence $H_q(\Delta_n, \dot{\Delta}_n) \cong H_{q-1}(\Delta_{n-1}, \dot{\Delta}_{n-1})$ and by recursion we finally reach $H_{q-n}(\Delta_0, \dot{\Delta}_0)$ – which we calculated above – in case $q \geq n$, and $H_0(\Delta_{n-q}, \dot{\Delta}_{n-q}) = 0$ by 8.17.3 in case $q < n$, since Δ_{n-q} is connected and $\dot{\Delta}_{n-q} \neq \emptyset$.

$$\begin{tikzcd}
H_n(\Delta_n, \dot{\Delta}_n) \arrow[d, tail, two heads, "8.19"] \\
H_{n-1}(\dot{\Delta}_n, A_n) \\
H_{n-1}(\Delta_{n-1}, \dot{\Delta}_{n-1}) \arrow[u, tail, two heads, "8.35"]
\end{tikzcd}$$

Let $[\Delta_n]$ denote the relative homology class in $H_n(\Delta_n, \dot\Delta_n)$ of $\mathrm{id}_{\Delta_n} : \Delta_n \to \Delta_n$. Then its image in $H_{n-1}(\dot\Delta_n, A_n)$ ist given by $[\partial \mathrm{id}_{\Delta_n} + S_{n-1}(A_n)]$ which equals the image $[\mathrm{id}_{\Delta_{n-1}} + S_{n-1}(A_n)]$ of $[\Delta_{n-1}] \in H_{n-1}(\Delta_{n-1}, \dot\Delta_{n-1})$. Obviously $[\Delta_0]$ is the generator of $H_0(\Delta_0, \dot\Delta_0) = H_0(\{1\})$. □

8.40 Corollary. [20, 9.5.2] *For $n \geq 0$ we have*

$$H_q(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by $[D^n]$. It is given by the relative homology class of a homeomorphism $\Delta_n \to D^n$. □

8.41 Corollary. [20, 9.5.3] *For $n > 0$ we have*

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \text{ or } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by $[S^n]$. It is given by $[S^n] = \partial_*([D^{n+1}]) = [\partial D^{n+1}]$.

This gives a different proof from 8.38.4

Proof. For $q > 0$ consider the homology sequence of the pair $S^n \subseteq D^{n+1}$:

$$\begin{array}{ccccccc} H_{q+1}(D^{n+1}) & \longrightarrow & H_{q+1}(D^{n+1}, S^n) & \xrightarrow{\cong} & H_q(S^n) & \longrightarrow & H_q(D^{n+1}) \\ \| \; 8.14 & & & & & & 8.14 \; \| \\ 0 & & & & & & 0 \end{array}$$

□

8.42 Corollary. [20, 9.5.6] *By 8.36 we have $H_q(\bigvee_j S^n) = 0$ for $q \notin \{0, n\}$ and $H_n(\bigvee_j S^n) \cong {}^{ab}\coprod_j \mathbb{Z}$ and the generators are $(\mathrm{inj}_j)_*[S^n]$.* □

We now prove the following (strengthend) part of

Proposition 1.20. *Let $m > n \geq 0$. Then $\mathbb{R}^m \not\cong \mathbb{R}^n$ and $S^m \not\sim S^n$.*

We have "proved" this by applying the theorem 1.19 of the invariance of domains.

Proof of 1.20 for $\mathbb{R}^n$ and S^n. By 8.39 we have $H_m(S^m) \cong \mathbb{Z}$ but $H_m(S^n) = \{0\}$, so $S^m \not\sim S^n$. Assume $\mathbb{R}^m \cong \mathbb{R}^n$ then $S^{m-1} \sim \mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\} \sim S^{n-1}$ for $n > 0$, hence $m = n$. For $n = 0$ we get the result since $\mathbb{R}^0 = \{0\}$ is compact. □

8.43 Proposition. [20, 11.1.1] *The sphere S^n is not contractible and is not a retract in D^{n+1} for $n \geq 0$.*

Proof. Since $H_n(S^n) \cong \mathbb{Z} \not\cong \{0\} = H_n(\{*\})$ the first statement is clear. And the second follows, since retracts of contractible spaces are contractible. In fact let $h_t : X \to X$ be a contraction and let $i : A \to X$ have a left inverse $p : X \to A$. Then $p \circ h_t \circ i : A \to A$ is a contraction of A. □

8.44 Corollary. Brouwers fixed point theorem. [20, 11.1.2]
Every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. Otherwise we can define a retraction as in 2.21. □

8.45 Proposition. [20, 11.7.1] *Let $B \subseteq S^n$ be a ball. Then $S^n \setminus B$ is acyclic.*

Proof. Induction on $r := \dim B$.
($r = 0$) Then B is a point and hence $S^n \setminus B \cong \mathbb{R}^n$ is contractible and thus acyclic.
($r + 1$) Let $z \in Z_q(S^n \setminus B)$ for $q > 0$ and $z := x - y \in Z_0(S^n \setminus B)$ for $q = 0$ with $x, y \in S^n \setminus B$. We have to show that $\exists b \in S_{q+1}(S^n \setminus B)$ with $\partial b = z$.

Consider a homeomorphisms $f : I^{r+1} = I^r \times I \cong B$. Then $B_t := f(I^r \times \{t\})$ is an r-ball. Thus by induction hypothesis there are $b_t \in S_{q+1}(S^n \setminus B_t)$ with $\partial b_t = z$ considered as element in $S_q(S^n \setminus B_t) \hookleftarrow S_q(S^n \setminus B)$. Since the image of b_t is disjoint to B_t, we can choose an open neighborhood V_t of t such that $I^r \times V_t \subseteq f^{-1}(S^n \setminus \operatorname{im}(b_t))$. Using compactness we find a partition of $0 = t_0 < t_1 < \cdots < t_N = 1$ of I into finitely many intervals $I_j := [t_j, t_{j+1}]$ such that for each $0 \le j < N$ there exists a t with $I_j \subset V_t$. Let $b_j := b_t \in S_{q+1}(Y_j)$ where Y_j is the open subset $S^n \setminus f(I^r \times I_j)$ and let $X_j := \bigcap_{i<j} Y_i = S^n \setminus f(I^r \times [0, t_j])$. Then $X_j \cap Y_j = X_{j+1}$ and $X_j \cup Y_j = S^n \setminus \big(f(I^r \times [0, t_j]) \cap f(I^r \times [t_j, t_{j+1}])\big) = S^n \setminus f(I^r \times \{t_j\})$.

We now show by induction on j that $[z] = 0$ in $H_q(X_j)$. For ($j = 0$) nothing is to be shown, since $X_0 = S^n$ and $z \in Z_q(S^n \setminus B) \subseteq Z_q(S^n \setminus \{*\}) \cong Z_q(\mathbb{R}^n) = B_q(\mathbb{R}^n) \subseteq B_q(S^n)$. For ($j + 1$) we apply the Mayer-Vietoris sequence 8.37 to the open sets X_j and Y_j:

$$\begin{array}{ccccc}
H_{q+1}(\overbrace{X_j \cup Y_j}^{S^n \setminus f(I^r \times \{t_j\})}) & \longrightarrow & H_q(\overbrace{X_j \cap Y_j}^{X_{j+1}}) & \rightarrowtail & H_q(X_j) \oplus H_q(Y_j) \\
{\scriptstyle \text{ind. on } r}\big\| & & & & \\
0 & & & &
\end{array}$$

The image of $[z] \in H_q(X_{j+1})$ in $H_q(X_j) \oplus H_q(Y_j)$ is zero, since the first component is $[z] = 0 \in H_q(X_j)$ by induction hypothesis on j, and the second component $[z] = [\partial b_j] = 0 \in H_q(Y_j)$. Since the group on the left side is zero, the arrow on the right is injective and we get $[z] = 0 \in H_q(X_{j+1})$.

Since $X_N = S^n \setminus B$, we are done. □

8.46 Theorem. [20, 11.7.4] *Let $S \subseteq S^n$ be an r-sphere with $0 \le r < n$ and $n \ge 2$. Then*

$$H_q(S^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } r = n-1 \text{ and } q = 0 \\ \mathbb{Z} & \text{for } r < n-1 \text{ and } q \in \{0, n-1-r\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Induction on r:
($r = 0$) Then $S \cong S^0 = \{-1, +1\}$ and $S^n \setminus S \sim \mathbb{R}^n \setminus \{0\} \sim S^{n-1}$, so the result follows from 8.38.4 or 8.41.
($r > 0$) We have $S^r = D^r_- \cup D^r_+$ and $B_\pm := f(D^r_\pm)$ are r-balls and $S' := f(S^{r-1})$ is an $(r-1)$-sphere. By 8.45 $S^n \setminus B_\pm$ are acyclic and since $S^n \setminus S' = (S^n \setminus B_+) \cup (S^n \setminus B_-)$ and $S^n \setminus S = (S^n \setminus B_+) \cap (S^n \setminus B_-)$ we get by 8.38.3 that $H_q(S^n \setminus S) \cong H_{q+1}(S^n \setminus S')$ for $q > 0$ and $H_0(S^n \setminus S) \cong H_1(S^n \setminus S') \oplus \mathbb{Z}$. By recursion we finally arrive at $H_{q+r}(S^n \setminus \{\pm 1\}) = H_{q+r}(S^{n-1})$, which we treated before. □

8.47 Proposition. [20, 11.7.2] [20, 11.7.5] *Let $n \ge 2$.*
If $B \subseteq \mathbb{R}^n$ is a ball, then

$$H_q(\mathbb{R}^n \setminus B) = \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n-1\} \\ 0 & \text{otherwise.} \end{cases}$$

If $S \subseteq \mathbb{R}^n$ is an r-sphere with $0 \le r < n$, then

$$H_q(\mathbb{R}^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \textit{for } (r = n-1, q = 0) \textit{ or } (r = 0, q = n-1) \\ \mathbb{Z} & \textit{for } (r < n-1 \ne q \in \{0, n-1-r\}) \textit{ or } (r \ne 0, q = n-1) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $A \subseteq \mathbb{R}^n \cong S^n \setminus \{*\} \subset S^n$ be compact. The long exact homology sequence 8.16 of the pair $(S^n \setminus A, \mathbb{R}^n \setminus A)$ gives

$$\to H_{q+1}(S^n \setminus A, \mathbb{R}^n \setminus A) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \to H_q(S^n \setminus A) \to H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \to$$

By the excision theorem 8.33 applied to $A \subseteq \mathbb{R}^n \subseteq S^n$ we get $H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \cong H_q(S^n, \mathbb{R}^n)$, which is isomorphic by 8.28.3 to $H_q(S^n, \{*\})$, since $\mathbb{R}^n$ is contractible. By 8.18.3 this homology group equals $H_q(S^n)$ for $q > 0$ and by 8.17.3 it is 0 for $q = 0$, since S^n is path-connected, i.e.

$$H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \overset{\boxed{8.33}}{=\!=} H_q(S^n, \mathbb{R}^n) \overset{\boxed{8.28.3}}{=\!=} H_q(S^n, \{*\}) \overset{\boxed{8.18.3}}{=\!=} \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence from above thus is

$$\ldots \to H_{q+1}(S^n, \{*\}) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \to H_q(S^n \setminus A) \to H_q(S^n, \{*\}) \to \ldots$$

In particular, $H_q(\mathbb{R}^n \setminus A) \cong H_q(S^n \setminus A)$ for $q \notin \{n-1, n\}$ and by 8.45 and 8.46 for A a sphere or ball the sequence is near $q = n-1$:

$$0 \to H_n(\mathbb{R}^n \setminus A) \to 0 \to \mathbb{Z} \to H_{n-1}(\mathbb{R}^n \setminus A) \to H_{n-1}(S^n \setminus A) \to 0,$$

This gives $H_n(\mathbb{R}^n \setminus A) = 0 = H_n(S^n \setminus A)$ and $H_{n-1}(\mathbb{R}^n \setminus A) \cong \mathbb{Z} \oplus H_{n-1}(S^n \setminus A)$, from which the claimed result follows. □

8.48 Corollary. Jordan's separation theorem generalized). [20, 11.7.6] [20, 11.7.7] *Let $X \in \{\mathbb{R}^n, S^n\}$ with $n \ge 2$. For any r-sphere S with $r < n-1$ we have that $X \setminus S$ is connected (i.e. we cannot cut X into two pieces along such a sphere). If S is an $n-1$-sphere then $X \setminus S$ has exactly two components, both of which have S as boundary. If $X = S^n$ then the components are acyclic.*

Proof. For spheres S of dimension $r < n-1$ the result follows from 8.46 and 8.47 since $H_0(X \setminus S) \cong \mathbb{Z}$ in these cases.

If S is a sphere of dimension $n-1$, then $H_0(X \setminus S) \cong \mathbb{Z}^2$ by 8.46 and 8.47. Hence $X \setminus S$ has two components, say U and V.

That for $X = S^n$ the components U and V are acyclic follows from $H_q(U) \oplus H_q(V) \cong H_q(X \setminus S) = \{0\}$ for $q \ne 0$.

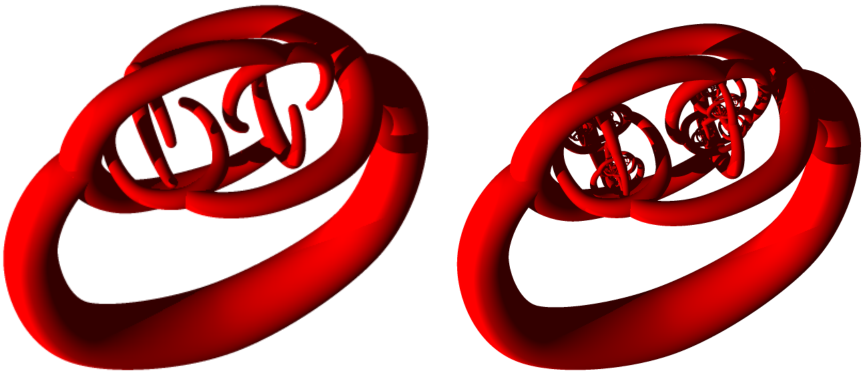
($\dot U \subseteq S$) In fact $\dot U \cap U = \emptyset$, since U is open and thus $\dot U = \bar U \setminus \mathring{U} = \bar U \setminus U$. From $U \subseteq X \setminus V$ we get $\bar U \subseteq \overline{X \setminus V} = X \setminus V$ since V is open. So $\dot U = \bar U \setminus U \subseteq (X \setminus V) \setminus U = X \setminus (U \cup V) = S$.

($S \subseteq \dot U$) Let $x \in S$ and W be a neighborhood of $x \in X$. Choose $n-1$-balls B and B' with $S = B \cup B'$ and such that $x \in B \subseteq W$. Let c be a path in $\mathbb{R}^n$ from U to V, which avoids $B' \subseteq S$ (this is possible by 8.47 since $X \setminus B'$ is path-connected). Let $t_0 := \sup\{t : c(t) \in U\}$. Hence $y := c(t_0) \in \bar U \setminus U = \dot U \subseteq S = B \cup B'$. Hence $y \in B \subseteq W$ and so $W \cap \dot U$ contains y and is not empty, hence $x \in \dot U$. □

8.49 Remark. [20, 11.7.8] For dimension 2 we have Schönflies's theorem (see [13, §9]): *For every Jordan curve in S^2, i.e. injective continuous mapping $c : S^1 \to S^2$,*

there exists a homeomorphism $f : S^2 \cong S^2$ *with* $f|_{S^1} = c$.
Thus up to a homeomorphism a Jordan-curve looks like the equator $S^1 \subseteq S^2$.

In dimension greater than 2, Alexanders horned sphere is a counterexample: One component of the complement is not simply connected. This gives at the same time an example of an open subset $U \subseteq S^3$, which is homologically trivial (i.e. acyclic) but not homotopy-theoretical ($\pi_1(U) \neq 0$).

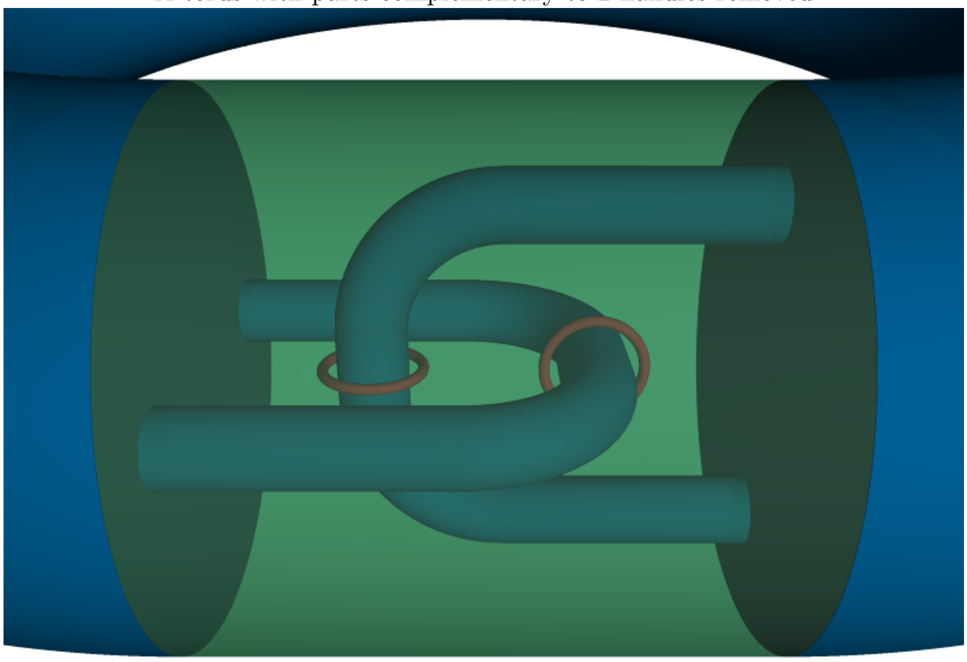


The third and the final step in constructing the horned sphere

A sphere with 4 horns attached

A sphere with 8 more horns attached

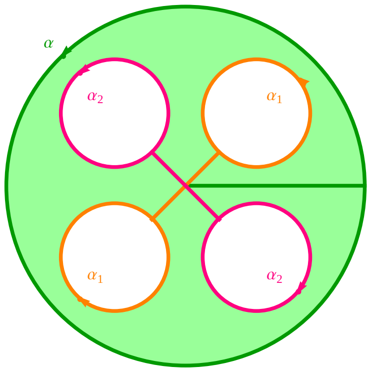
A torus with parts complementary to 2 handles removed



The generators of the fundamental group of the removed part

Let U_n be the outer component of the complement of the sphere with 2^n-handles constructed in the n-th step. The outer component of Alexanders horned sphere is then the union $U_\infty = \bigcup_{n\in\mathbb{N}} U_n$ and each of its compact subsets is contained in U_n for some n. By 5.31 we have that $\pi_1(U_\infty) = \varinjlim_n \pi_1(U_n)$ is the injective limit. We determine $\pi_1(U_n)$ recursively:
By 1.73 the complement U_0 of a filled torus in S^3 is an open torus $\mathring{D}^2 \times S^1 \sim S^1$ and hence its fundamental group $\pi_1(U_0) \cong \mathbb{Z}$, where a generator α is given by an enlarged meridian of the original torus. The inclusion $U_n \hookrightarrow \bar U_n$ induces an isomorphism of the fundamantal groups, and $\bar U_1$ ist the union of $\bar U_0$ and the closure $\bar Z_1$ of the part Z_1, which we remove from the torus in the first step. Note that $\bar Z_1 \cong (D^2 \setminus (\mathring{D}^1_0 \sqcup \mathring{D}^2_1)) \times I \sim S^1 \vee S^1$ (cf. 1.65.4 and 2.36.9). Let α_1 and α_2 be the generators (i.e. loops along the two handles) of $\pi_1(\bar Z_1) \cong \pi_1(Z_1)$.

The intersection $A_1 := \bar{U}_0 \cap \bar{Z}_1 \cong S^1 \times I \sim S^1$ has also fundamental group $\mathbb{Z}$ and its generator $S^1 \times \{\frac{1}{2}\}$ (denoted α) is mapped by the inclusion $\iota_0 : A_1 \hookrightarrow \bar{U}_0$ to the generator α of $\pi_1(\bar{U}_0)$, i.e. $\pi_1(\iota_0) : \pi_1(A_1) \to \pi_1(U_0)$ is an isomorphism.
By the theorem 5.33 of Seifert and von Kampen the pushout is $\pi_1(\bar{U}_1)$ and thus $\pi_1(\bar{Z}_1) \to \pi_1(\bar{U}_1)$ is an isomorphism as well. The inclusion $\iota_1 : A_1 \to \bar{Z}_1$ maps α to the commutator $[\alpha_1, \alpha_2]$ (look at the plane through $\alpha = S^1 \times \{\frac{1}{2}\} \subseteq \bar{Z}_1$, it has 4 holes with boundary parametrized by α_1, α_2, α_1^{-1}, and α_2^{-1}, cf. 2.36.9). Hence the same is true for $\mathbb{Z} \cong \pi_1(U_0) \to \pi_1(U_1) = \mathbb{Z} \amalg \mathbb{Z}$.



$$\begin{tikzcd}
\alpha \in \mathbb{Z} \arrow[r, equal] \arrow[d, mapsto] & \pi_1(S^1) \cong \pi_1(A_1) \arrow[r, tail, two heads, "\cong"] \arrow[d] & \pi_1(\bar{U}_0) \cong \pi_1(U_0) \arrow[d] \\
{[\alpha_1, \alpha_2]} \in \mathbb{Z} \coprod \mathbb{Z} \arrow[r, equal] & \pi_1(S^1 \vee S^1) \cong \pi_1(\bar{Z}_1) \arrow[r, tail, two heads, "\cong"] & \pi_1(\bar{U}_1) \cong \pi_1(U_1)
\end{tikzcd}$$

Using analogous arguments we obtain that $\pi_1(U_n)$ is the free group with 2^n-many generators α_i^n with $0 \leq i < 2^n$ and the inclusion $U_{n-1} \hookrightarrow U_n$ maps $\alpha_i^{n-1} \mapsto [\alpha_{2i}^n, \alpha_{2i+1}^n]$. Thus the set $\pi_1(U_\infty)$ is the union of these free groups and hence U_∞ is not simply connected. Note however, that the Abelisation of $\pi_1(U_\infty)$ is obviously trivial.

1.19 Corollary. Invariance of the domain.
Let $X, Y \subseteq \mathbb{R}^n$ be homeomorphic. If X is open then so is Y.

Proof. Take $x \in X$ and $y := f(x) \in Y$. By assumption there is a ball $B := \{z : |z - x| \leq r\} \subseteq X$. Let $S := \partial B$. Then $\mathbb{R}^n \setminus f(S) = (\mathbb{R}^n \setminus f(B)) \cup (f(B) \setminus f(S))$. The first part is connected by 8.47 and the second one coincides with $f(B \setminus S) \cong B \setminus S = \mathring{D}^n$ and hence is connected as well. Thus they are the path components of the open set $\mathbb{R}^n \setminus f(S)$ and hence are open in $\mathbb{R}^n$. So the component $f(B \setminus S) \subseteq f(B) \subseteq f(X) = Y$ is an open neighborhood of y in $\mathbb{R}^n$, and thus Y is open. $\square$

Cellular Homology

8.50 Proposition. [20, 9.6.1] *Let X be a CW-complex.*
Then $H_p(X^q, X^{q-1}) = 0$ for $p \neq q$.

Proof. For $q = 0$ we have $H_p(X^q, X^{q-1}) = H_p(X^0, \emptyset) = H_p(X^0) = 0$ by 8.17.2, 8.7 and 8.9.

So let $q > 0$. For $p = 0$ we have $H_0(X^{q-1}) \twoheadrightarrow H_0(X^q) \xrightarrow{0} H_0(X^q, X^{q-1}) \to 0$, where the first mapping is onto (since each component of X^q meets X^{q-1}) and so the second one is 0.
Now let $p \neq 0$. By 8.34 we have $H_p(X^q, X^{q-1}) \cong H_p(X^q / X^{q-1})$ and so the result follows from 8.42, since $X^q / X^{q-1} \cong \bigvee S^q$ by 4.16. $\square$

8.51 Corollary. [20, 9.6.2] *The inclusions induce an epimorphism $H_q(X^q) \twoheadrightarrow H_q(X)$ and an isomorphism $H_q(X^{q+1}) \xrightarrow{\cong} H_q(X)$.*

Proof. By 8.50 and

$$H_{p+1}(X^q, X^{q-1}) \to H_p(X^{q-1}) \to H_p(X^q) \to H_p(X^q, X^{q-1})$$

the first arrow in the sequence

$$H_q(X^q) \twoheadrightarrow H_q(X^{q+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_q(X^{q+j}) \to H_q(X)$$

is onto and all others but the last one are isomorphisms. So we have the result for finite CW-complexes. In the general case we use that every singular simplex lies in some X^p by 4.5, hence $H_q(X^{q+1}) \to H_q(X)$ is surjective. Similar one shows injectivity, since $[z] = 0 \in H_q(X)$ implies $z = \partial c$ for some $c \in S_{q-1}(X) = \bigcup_p S_{q-1}(X^p)$, hence $c \in S_{q-1}(X^p)$ for some p and thus $[z] = 0 \in H_q(X^p)$. □

8.52 Corollary. [20, 9.6.3] *Let X be a CW-space without q-cells. Then $H_q(X) = 0$. In particular $H_q(X) = 0$ for $q > \dim X$.*

Proof. From the homology sequence

$$H_{q+1}(X^p, X^{p-1}) \to H_q(X^{p-1}) \to H_q(X^p) \to H_q(X^p, X^{p-1})$$

for $q > p$ and 8.50 we deduce $H_q(X^{q-1}) \cong \ldots \cong H_q(X^{-1}) = 0$. By assumption $X^q = X^{q-1}$ and hence $H_q(X^q, X^{q-1}) = 0$. So we get the surjectivity of $H_q(X^{q-1}) \twoheadrightarrow H_q(X^q)$ and thus $H_q(X^q) = 0$ as well. Now the result follows since $H_q(X^q) \twoheadrightarrow H_q(X)$ is onto by 8.51. □

8.53 Definition. [20, 9.6.4] The q-th CELLULAR CHAIN GROUP of a CW-complex X is defined as

$$C_q(X) := H_q(X^q, X^{q-1}),$$

and its elements are called CELLULAR q-CHAINS. For every q-cell e in X with characteristic map $\chi^e : (D^q, S^{q-1}) \to (X^q, X^{q-1})$ we define a so-called orientation $\chi^e_*([D^q]) \in C_q(X)$ as the image of $\chi^e_* : H_q(D^q, S^{q-1}) \cong \mathbb{Z} \to H_q(X^q, X^{q-1})$, where $[D^q]$ denotes the generator in $H_q(D^q, S^{q-1})$ induced from a homeomorphism $\Delta^q \to D^q$, see 8.40.

Lemma. *For every cell there are exactly two orientations, which differ only by their sign. And $C_q(X)$ is a free abelian group generated by a selection of orientations for each q-cell.*

Proof. Let χ_1 and χ_2 be two characteristic mappings for e. We can consider them as relative homeomorphisms $\chi_j : (D^q, S^{q-1}) \to (X^{q-1} \cup e, X^{q-1})$. By 8.35 they induce isomorphisms in homology. Hence $H_q(\chi_1)[D^q] = \pm H_q(\chi_2)[D^q]$, since the generator in $H_q(X^{q-1} \cup e, X^{q-1})$ has to correspond to a generator in $H_q(D^q, S^{q-1})$, and the only ones are $\pm[D^q]$.

Obviously $C_0(X) = H_0(X^0, \emptyset) = H_0(X^0)$ is free abelian generated by the set X^0.

For $q > 0$ the projection $p : (X^q, X^{q-1}) \to (Y, \{y_0\}) := (X^q/X^{q-1}, X^{q-1}/X^{q-1})$ induces by 8.34 an isomorphism $p_* : C_q(X) := H_q(X^q, X^{q-1}) \to H_q(Y, \{y_0\})$. Since Y is a join of q-spheres we have that $p_* \chi^e_*[D^q]$ form a basis in the free abelian group $H_q(Y, \{y_0\})$, as follows from 8.42: In fact, consider the following commutative diagram and the induced one in homology:

$$\begin{tikzcd} (D^q, S^{q-1}) \arrow[r, "\chi^e"] \arrow[d, "h"] & (X^q, X^{q-1}) \arrow[d, "p"] \\ (S^q, \{*\}) \arrow[r] & (X^q/X^{q-1}, \{*\}) \end{tikzcd} \qquad \begin{tikzcd} H_q(D^q, S^{q-1}) \arrow[r, "\chi^e_*"] \arrow[d, "h_*", "\cong"'] & H_q(X^q, X^{q-1}) = C_q(X) \arrow[d, "p_*", "\cong"'] \\ \mathbb{Z} = H_q(S^q) \arrow[r, "\boxed{8.42}"] & H_q(\bigvee S^q) = {}^{\mathrm{ab}}\coprod \mathbb{Z} \end{tikzcd}$$

The vertical arrows are isomorphisms in homology by 8.34 and the bottom arrow maps the generator $[S^q] \in H_q(S^q) \cong H_q(S^q, \{*\})$ to the corresponding generator in $H_q(X^q/X^{q-1}) \cong H_q(X^q/X^{q-1}, \{*\})$ by 8.42. □

8.54 Definition. [20, 9.6.6] Using the long exact sequences for the pairs (X^{q+1}, X^q) and (X^q, X^{q-1}) we have

$$\begin{tikzcd} & C_{q+1}(X) \arrow[d, equal] & & & \\ \dots \arrow[r, "j_*"] & H_{q+1}(X^{q+1}, X^q) \arrow[r, "\partial_*"] & H_q(X^q) \arrow[r] \arrow[d, equal] & H_q(X^{q+1}) \arrow[r] & \dots \\ \dots \arrow[r] & H_q(X^{q-1}) \arrow[r] \arrow[d, equal, "\boxed{8.52}"'] & H_q(X^q) \arrow[r, hook, "j_*"] & H_q(X^q, X^{q-1}) \arrow[r, "\partial_*"] \arrow[d, equal] & \dots \\ & 0 & & C_q(X) & \end{tikzcd}$$

Let $\partial := j_* \circ \partial_* : C_{q+1}(X) \to H_q(X^q) \to C_q(X)$. We have $\partial^2 = 0$ by the exactness of the second sequence at $H_q(X^q, X^{q-1})$ and thus we obtain a chain complex. Its homology $H(C(X))$ is called CELLULAR HOMOLOGY of the CW-complex X.
For any $q+1$-cell e with characteristic map χ^e and $q > 0$ we get $\partial(\chi^e_*[D^{q+1}]) = j_*\partial_*\chi^e_*[D^{q+1}] \overset{(\dagger)}{=} j_*(\chi^e|_{S^q})_*\partial_*[D^{q+1}] = j_*(\chi^e|_{S^q})_*[\partial D^{q+1}] = j_*(\chi^e|_{S^q})_*[S^q]$, where for ($\dagger$) we used the homology ladder

$$\begin{tikzcd} {[D^{q+1}] \in H_{q+1}(D^{q+1}, S^q)} \arrow[r, "\chi^e_*"] \arrow[d, "\partial_*"] & H_{q+1}(X^{q+1}, X^q) \arrow[r, equal] \arrow[d, "\partial_*"] & C_{q+1}(X) \arrow[d, "\partial"] \\ {[S^q] \in H_q(S^q)} \arrow[r, "(\chi^e|_{S^q})_*"] & H_q(X^q) \arrow[r, "j_*"] & C_q(X). \end{tikzcd}$$

Example. Despite looking rather complicated the cellular homology is often easy to calculate. Take for example the cell decompostion of the sphere $S^n = e_0 \cup e_n$. Thus

$$C_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the cellular boundary operator ∂ is 0 for $n > 1$ and hence

$$H_q(C(X)) = C_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Singular versus cellular homology

8.55 Proposition. [20, 9.6.9] [20, 9.6.11] *The homomorphism $j_* : H_q(X^q) \rightarrowtail H_q(X^q, X^{q-1})$ is injective and maps onto the q-th cellular cycles.*
The map $i_* : H_q(X^q) \twoheadrightarrow H_q(X)$ is onto and its kernel is mapped by j_* onto the q-th

cellular boundaries.
Thus one obtains isomorphisms

$$j_*: H_q(X) \xrightarrow{\cong} H_q(C(X)),$$

which are natural for cellular mappings.

Proof. From the exact sequence

$$0 \overset{\boxed{8.52}}{=\!=\!=} H_q(X^{q-1}) \to H_q(X^q) \xrightarrow{j_*} H_q(X^q, X^{q-1}) =: C_q(X)$$

we deduce that j_* is injective and hence $\mathrm{Ker}(\partial) := \mathrm{Ker}(j_*\partial_*) = \mathrm{Ker}(\partial_*) = \mathrm{Im}(j_*)$, which proves the first statement.

From the exact homology sequence $\boxed{8.16}$ of the pair (X, X^{q+1})

$$\begin{tikzcd}
H_{q+1}(X^{q+1}) \arrow[r, two heads, "" ', "\boxed{8.51}"'] & H_{q+1}(X) \arrow[r, "0"] & H_{q+1}(X, X^{q+1}) \arrow[r, "0"] \arrow[d, equal] & H_q(X^{q+1}) \arrow[r, tail, "\cong", "\boxed{8.51}"'] & H_q(X) \\
& & 0 & &
\end{tikzcd}$$

we get $H_{q+1}(X, X^{q+1}) = 0$.

By the exact homology sequence $\boxed{8.19}$ for the triple $X^q \subseteq X^{q+1} \subseteq X$

$$\begin{tikzcd}
& & 0 \arrow[d, equal] \\
H_{q+1}(X^{q+1}, X^q) \arrow[r, two heads] & H_{q+1}(X, X^q) \arrow[r] & H_{q+1}(X, X^{q+1})
\end{tikzcd}$$

we get that $H_{q+1}(X^{q+1}, X^q) \twoheadrightarrow H_{q+1}(X, X^q)$ is onto. The q-th cellular boundaries are the image of the top row in

$$\begin{tikzcd}
H_{q+1}(X^{q+1}, X^q) \arrow[r, "\partial_*"] \arrow[d, two heads] & H_q(X^q) \arrow[r, "j_*"] \arrow[d, equal] & H_q(X^q, X^{q-1}) \\
H_{q+1}(X, X^q) \arrow[r, "\partial_*"] & H_q(X^q) \arrow[r, two heads, "i_*", "\boxed{8.51}"'] & H_q(X)
\end{tikzcd}$$

Since the rectangle commutes by naturality of ∂_* and since $\mathrm{Im}\,\partial_* = \mathrm{Ker}\,i_*$ we get

$$\mathrm{Im}(\partial) := \mathrm{Im}(j_*\partial_*) = j_*(\mathrm{Im}\,\partial_*) = j_*(\mathrm{Ker}\,i_*),$$

i.e. the q-th cellular boundaries are the image of $\mathrm{Ker}\,i_*$ under j_*. Now we get the desired natural isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,i_* \arrow[r, hook] \arrow[ddd, "j_*"', "{\cong,\ 2.}"] & H_q(X^q) \arrow[rr, two heads, "i_*"] \arrow[dr, "j_*"] \arrow[ddd, "j_*"', "{\cong,\ 1.}"] & & H_q(X) \arrow[r] \arrow[ddd, dotted, "j_*"', "{\cong,\ 3.}"] & 0 \\
& & & H_q(X^q, X^{q-1}) \arrow[d, equal] & & \\
& & & C_q(X) & & \\
0 \arrow[r] & \mathrm{Im}\,\partial_{q+1} \arrow[r, hook] & \mathrm{Ker}\,\partial_q \arrow[rr, two heads] \arrow[ur, hook] & & H_q(C(X)) \arrow[r] & 0
\end{tikzcd}$$

$\square$

8.56 Proposition. [20, 9.6.10] *For $q \geq 1$ we have that in the short exact sequence*

$$0 \to \mathrm{Ker}(i_*) \hookrightarrow H_q(X^q) \overset{i_*}{\twoheadrightarrow} H_q(X) \to 0$$

$H_q(X^q)$ is free abelian and $\mathrm{Ker}(i_*)$ is generated by the $H_q(\chi^e)[S^q]$, where $\chi^e : S^q \to X^q$ are chosen gluing maps for each $q+1$-cell e in X.

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} i_* \arrow[r, hook] \arrow[d, equal] & H_q(X^q) \arrow[r, two heads, "i_*", "8.51"'] \arrow[dd, equal] & H_q(X) \arrow[r] & 0 \\
& \operatorname{Bild} \partial_* \arrow[dr, hook] & & & \\
\dots \arrow[r] & C_{q+1}(X) \arrow[r, "\partial_*"'] \arrow[u, two heads, "\partial_*"] & H_q(X^q) \arrow[r, two heads] & H_q(X^{q+1}) \arrow[r] \arrow[uu, "\cong"', "8.51"] & \dots
\end{tikzcd}$$

By 8.55 we have that $H_q(X^q) \cong \operatorname{Ker} \partial_q \subseteq C_q(X)$ and hence is free abelian by 9.20. Furthermore $H_q(X^{q+1}) \cong H_q(X)$ by 8.51, and hence the kernel of $i_* : H_q(X^q) \to H_q(X)$ equals the kernel of $H_q(X^q) \to H_q(X^{q+1}) \cong H_q(X)$, and equals the image of $\partial_* : C_{q+1}(X) := H_{q+1}(X^{q+1}, X^q) \to H_q(X^q)$ by the homology sequence of the pair (X^{q+1}, X^q). By 8.53 we have that $C_{q+1}(X)$ is the free abelian group generated by $\chi^e_*[D^{q+1}]$, where $\chi^e : (D^{q+1}, S^q) \to (X^{q+1}, X^q)$ are chosen characteristic maps for each $q+1$-cells e in X. By 8.54 we have that $\partial_*(\chi^e_*[D^{q+1}]) = \chi^e_*[S^q]$. □

8.57 Proposition. [20, 9.9.10] *For the projective spaces we have*

$$H_q(\mathbb{P}^n(\mathbb{C})) \cong \begin{cases} \mathbb{Z} & \textit{for } q = 0, 2, \dots, 2n \\ 0 & \textit{otherwise} \end{cases}$$

and

$$H_q(\mathbb{P}^n(\mathbb{H})) \cong \begin{cases} \mathbb{Z} & \textit{for } q = 0, 4, \dots, 4n \\ 0 & \textit{otherwise.} \end{cases}$$

For the homology of the real projective spaces $\mathbb{P}^n(\mathbb{R})$ see 8.67.

Proof. By 1.95 there are no cells in all but the dimensions divisible by 2 (resp. 4), thus the boundary operator of the cellular homology is 0 (since either domain or codomain is zero) and hence the homology coincides with the cellular chain complex. □

Simplicial versus singular homology

We are going to show now that the singular homology of a singular complex K is naturally isomorphic to the homology of the associated CW-space $|K|$. The idea behind this isomorphism is very easy: To a given simplex $\sigma = \langle x_0, \dots, x_q \rangle \in K$ one associates the affine singular simplex $\bar\sigma : \Delta_q \to |K|$, which maps $e_j \mapsto x_j$ for all $0 \le j \le q$. We will show that this induces an isomorphism $H_q(K) \to H_q(|K|)$, $[\sigma] \mapsto [\bar\sigma]$. In order that it is well defined, we have to show that an even permutation of the vertices does not change the homology class of $\bar\sigma$. We do this in the following

8.58 Lemma. [20, 9.7.1] *Let τ be a permutation of $\{0, \dots, q\}$. Then τ induces an affine mapping $\tau : (\Delta_q, \dot\Delta_q) \to (\Delta_q, \dot\Delta_q)$, with $H_q(\tau)[\Delta_q] = \operatorname{sign}(\tau)[\Delta_q] \in H_q(\Delta_q, \dot\Delta_q)$.*

Proof. Since any permutation is a product of transpositions, we may assume that τ is a transposition, say $(0,1)$. Let an affine $\sigma : \Delta_{q+1} \to \Delta_q$ be defined by $e_0 \mapsto e_1$ and $e_i \mapsto e_{i-1}$ for all $i > 0$. The boundary of this singular $q+1$-simplex in Δ_q is

$\partial\sigma = \sigma\circ\delta^0+\sum_{i\notin\{0,2\}}(-1)^i\sigma\circ\delta^i+\sigma\circ\delta^2 = \mathrm{id}_{\Delta_q}+c+\tau$ for $c := \sum_{i\notin\{0,2\}}(-1)^i\sigma\circ\delta^i \in S_q(\dot\Delta_q)$. Hence $\tau_*[\Delta_q] = [\tau] = -[\Delta_q] \in H_q(\Delta_q, \dot\Delta_q)$. □

Although this lemma shows that the mapping $H_q(K) \to H_q(|K|)$ is well-defined, it is not so obvious that it is an isomorphism, since there are a lot more singular simplices in $|K|$ then just the simplices of K. So we will make a little detour via the cellular homology.

8.59 Definition. [20, 9.7.2] Let $\sigma = \langle x_0, \dots, x_q\rangle$ be an oriented q-simplex of a simplicial complex K. This induces an affine mapping $\tilde\sigma : (\Delta_q, \dot\Delta_q) \to (|K|^q, |K|^{q-1})$, which can be considered as characteristic mapping for $\sigma \subseteq |K|$. Hence we get a mapping

$$\Phi : C_q(K) \to C_q(|K|) := H_q(|K|^q, |K|^{q-1}), \quad \sigma \mapsto \tilde\sigma_*[\Delta_q] = [\tilde\sigma].$$

Recall that $C_q(K) := {}^{ab}\big\langle K^{[q]}, \{\sigma+\sigma^{-1} : \sigma\in K^{[q]}\}\big\rangle$ by 7.1. Note that $\tilde\sigma$ depends on the chosen ordering of the vertices. Nevertheless, Φ is well-defined (i.e. depends no longer on the ordering but only on the orientation) by 8.58 and since we may identify $C_q(K)$ with the free abelian group generated by the simplices with some fixed orientation by 7.2.

8.60 Theorem. [20, 9.7.3]
The mapping $\Phi : \sigma \mapsto [\tilde\sigma]$ defines a natural isomorphism $C(\_) \xrightarrow{\cong} C(|\_|)$.

Proof. That $\Phi_K : C(K) \to C(|K|)$ is an isomorphism is clear, since the free generators σ (see 7.2) are mapped to the free generators $[\tilde\sigma]$ (see 8.53).

It is natural for simplicial mappings $\psi : K \to L$. In fact take a simplex $\sigma = \langle x_0, \dots, x_q\rangle \in K$. If ψ is injective on the vertices x_j of σ, then

$$\Phi\psi\sigma = \Phi\langle\psi(x_0), \dots, \psi(x_q)\rangle = [\langle\psi(x_0), \dots, \psi(x_q)\rangle^\sim] = [|\psi|\circ\tilde\sigma] = |\psi|_*[\tilde\sigma] = |\psi|_*\Phi\sigma.$$

In the other case $\psi\sigma = 0$, hence $\Phi\psi\sigma = 0$ and $|\psi|_*\Phi\sigma = [|\psi|\circ\tilde\sigma]$, but $|\psi|\circ\tilde\sigma$ has values in $|L|^{q-1}$, hence $[|\psi|\circ\tilde\sigma] = 0 \in H_q(|L|^q, |L|^{q-1})$.

Let us show that it is a chain mapping. For $\sigma = \langle x_0, \dots, x_q\rangle$ we have

$$\partial\Phi\sigma = j_*\partial_*[\tilde\sigma] = j_*[\partial\tilde\sigma] = [\partial\tilde\sigma] = \Big[\sum_j(-1)^j\tilde\sigma\circ\delta^j\Big] \quad \text{and}$$

$$\Phi\partial\sigma = \Phi\Big(\sum_j(-1)^j\langle x_0, \dots, \widehat{x_j}, \dots, x_q\rangle\Big) = \Big[\sum_j(-1)^j\tilde\sigma\circ\delta^j\Big]$$

So $\partial\Phi = \Phi\partial$. □

8.61 Corollary. [20, 9.7.4] *Let K be a simplicial complex. Then we have natural isomorphisms $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \xleftarrow{j_*} H_q(|K|)$, from the simplicial over the cellular to the singular homology.*

Proof. This follows by composing the isomorphisms in 8.60 and 8.55. □

Let us now come back to the description of the isomorphism $H(K) \cong H(|K|)$ indicated in the introduction to this section.

8.62 Proposition. [20, 9.7.7] *The isomorphism $H(K) \cong H(|K|)$ between simplicial and singular homology can be described as follows: Choose a linear ordering of the vertices of K, and then map a simplex $\sigma = \langle x_0, \dots, x_q\rangle$ with $x_0 < \dots < x_q$ to $\tilde\sigma$, which is just σ considered as affine map $\Delta_q \to |K|$, $e_j \mapsto x_j$.*

Proof. We consider the following commutative diagram and take $\alpha \in H_q(K)$: It can be represented by a simplicial cycle $z := \sum_\sigma n_\sigma \sigma \in Z_q(K) \subseteq C_q(K)$. On the other hand we can consider the singular q-chain $\tilde z := \sum_\sigma n_\sigma \tilde\sigma \in S_q(|K|^q)$, since the image of $\tilde\sigma$ is the closure of the simplex σ and hence contained in $|K|^q$. This singular chain is a cycle, since $\partial\tilde z = \sum_\sigma n_\sigma \partial\tilde\sigma \overset{!}{=} \sum_\sigma n_\sigma \widetilde{\partial\sigma} = (\partial(\sum_\sigma n_\sigma \sigma))^\sim = \widetilde{\partial z} = \tilde 0 = 0$ and hence we may consider $\beta := [\tilde z] \in H_q(|K|^q)$, i.e. $i_*(\beta) = [\tilde z] \in H_q(|K|)$.

$$
\begin{tikzcd}
Z_q(|K|^q) \arrow[r, two heads] & H_q(|K|^q) \arrow[r, two heads, "i_*"] \arrow[dl, "j_*"'] \arrow[dd, "j_*"', "\cong\ \boxed{8.55}"] & H_q(|K|) \arrow[dd, "j_*"', "\cong"] \\
H_q(|K|^q, |K|^{q-1}) \arrow[d, equal] & & \\
C_q(|K|) & \operatorname{Ker}\partial_q \arrow[r, two heads] \arrow[ul] & H_q(C(|K|)) \\
C_q(K) \arrow[u, "\Phi", "\cong\ \boxed{8.60}"'] & Z_q(K) \arrow[l] \arrow[r, two heads] \arrow[u, "\Phi", "\cong\ \boxed{8.60}"'] & H_q(K) \arrow[u, "\Phi_*", "\cong"']
\end{tikzcd}
$$

Note that $\Phi(z) = \sum_\sigma n_\sigma \Phi(\sigma) = \sum_\sigma n_\sigma [\tilde\sigma] = [\sum_\sigma n_\sigma \tilde\sigma] = [\tilde z] = j_*(\beta) \in C_q(|K|)$. Thus the composition of isomorphisms $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \xleftarrow{j_*} H_q(|K|)$ maps $\alpha = [z] \mapsto [\Phi(z)] \mapsto i_* j_*^{-1}[\Phi(z)] = i_*(\beta) = [\tilde z] \in H_q(|K|)$. Note that $[z]$ denotes classes in various homology groups during this calculation. □

Fundamental group versus first homology group

8.63 Proposition. [20, 9.8.1] *There is a natural homomorphism $h_1 : \pi_1(X, x_0) \to H_1(X)$ given by $[\varphi] \mapsto \varphi_*[S^1] = [\varphi]$, where for the last equality $\varphi : (S^1, 1) \to (X, x_0)$ is considered as singular chain $\dot\Delta_2 \cong S^1 \to X$.*
If X is path-connected then this homomorphism is surjective and its kernel is just the commutator subgroup. Thus $H_1(X) \cong {}^{ab}\pi_1(X, x_0)$, the abelization of $\pi_1(X, x_0)$.

Proof. That h is natural is clear. Let us show that it is a homomorphism: So let two closed curves φ, ψ considered as maps $(S^1, 1) \to (X, x_0)$ be given. The corresponding paths $I \to X$ are obtained by composing them with $t \mapsto e^{2\pi i t}$, $I \to S^1$. Hence $\varphi \cdot \psi := (\varphi, \psi) \circ \nu : (S^1, 1) \to (S^1, 1) \vee (S^1, 1) \to (X, x_0)$, where $\nu : S^1 \to S^1 \vee S^1$ is given by $t \mapsto (e^{2\pi i 2t}, 1) \in S^1 \vee S^1 \subseteq S^1 \times S^1$ for $2t \le 1$ and $t \mapsto (1, e^{2\pi i(2t-1)}) \in S^1 \vee S^1$ for $2t \ge 1$. In order to determine $\nu_* : H_1(S^1) \to H_1(S^1 \vee S^1)$ we consider the relative homeomorphism $\sigma : (\Delta_1, \dot\Delta_1) \to (S^1, \{1\})$ given by $(1-t)e_0 + t e_1 \mapsto e^{2\pi i t}$. It induces an isomorphism $\sigma_* : \mathbb{Z} \cong H_1(\Delta_1, \dot\Delta_1) \to H_1(S^1, \{1\}) \cong H_1(S^1) \cong \mathbb{Z}$, with $\sigma_* : [\Delta_1] \mapsto [\sigma \circ \mathrm{id}_{\Delta_1}] = [\sigma] = [S^1]$ for the generators (by 8.39 for $[\Delta_1]$; by 8.41 and 8.40 for $[S^1]$). Using the barycentric refinement $B\sigma = \sigma_*(B\Delta_1)$ (see 8.29) gives

$$
\nu_*[S^1] = \nu_*[\sigma] \overset{\boxed{8.31}}{=\!=} \nu_*[B\sigma] = \underbrace{[\mathrm{inj}_1 \circ \sigma] + [\mathrm{inj}_2 \circ \sigma]}_{\in H_1(S^1 \vee S^1)} \overset{\boxed{8.36}}{=\!=} \underbrace{[S^1] \oplus [S^1]}_{\in H_1(S^1) \oplus H_1(S^1)}, \text{ thus}
$$

$$
\begin{aligned}
h_1([\varphi] \cdot [\psi]) &= h_1([\varphi \cdot \psi]) = h_1([(\varphi, \psi) \circ \nu]) = ((\varphi, \psi) \circ \nu)_*[S^1] = (\varphi, \psi)_* \nu_*[S^1] \\
&= (\varphi, \psi)_* \Big([S^1] \oplus [S^1]\Big) = \varphi_*[S^1] + \psi_*[S^1] = h_1[\varphi] + h_1[\psi].
\end{aligned}
$$

Although the theorem is valid for arbitrary path-connected topological spaces, see [15, IV.3.8], we give the proof only for connected CW-complexes X. Since π_1 and H_1 do not depend on cells of dimension greater then 2 by 5.40 and 8.51, we may assume $\dim X \le 2$. The theorem is invariant under homotopy equivalences, hence we may assume by 5.45 that X has exactly one 0-cell and that this cell is x_0. So

X^1 is a one-point union of 1-cells and X is obtained by gluing 2-cells e via maps $f^e : S^1 \to X^1$. By 2.32.3 and 2.45 we may assume that $f^e(1) = x_0$.

Now consider the diagram below.

$$\begin{tikzcd}
 & \pi_1(X^1,x_0)' \arrow[r] \arrow[d, hook, "5.24"] & \pi_1(X,x_0)' \arrow[d, hook] \\
N \arrow[r, hook] \arrow[d, "h_1|_N"'] & \pi_1(X^1,x_0) \arrow[r, two heads, "i_*", "5.48"'] \arrow[d, "h_1"', "5.37,\ 8.42"] & \pi_1(X,x_0) \arrow[d, "h_1"] \\
U \arrow[r, hook] & H_1(X^1) \arrow[r, two heads, "i_*", "8.56"'] & H_1(X)
\end{tikzcd}$$

By 5.48 the top i_* is onto and its kernel N is the normal subgroup generated by the $[f^e]$. By 8.56 the bottom i_* is onto and its kernel U is the subgroup generated by the $(f^e)_*[S^1] =: h_1(f^e)$. By 5.37 and 8.42 the two spaces in the middle are free resp. free abelian, with the corresponding generators, and by 5.24 we know that the abelization of a free group is the free abelian group.

So we have that the result is true for X^1. Furthermore $h_1(N) = U$, since the generators of N are mapped to those of U. By diagram chasing the general result follows: Let $G := \pi_1(X^1, x_0)$. The homomorphism $h_1 : \pi_1(X, x_0) \twoheadrightarrow H_1(X)$ is obviously surjective and its kernel is given by all gN, for which $0 = h_1(gN) = h_1(g)U$, i.e. $h_1(g) \in U$. Again by surjectivity of $h_1 : N \twoheadrightarrow U$ we have an $n \in N$ with $h_1(n) = h_1(g)$, i.e. $gn^{-1} \in \ker(h_1) = G'$. So $gN \in G'/N = (G/N)'$. The converse inclusion $(G/N)' \subseteq \ker(h_1)$ is clear, since $H_1(X)$ is abelian. □

8.64 Corollary. [20, 9.8.2] *For the closed orientable surface X of genus g we have $H_1(X) \cong \mathbb{Z}^{2g}$, for the non-orientable one we have $H_1(X) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, and for the projective spaces we have $H_1(\mathbb{P}^n) \cong \mathbb{Z}_2$ for $2 \le n \le \infty$.*

Proof. Use the formulas given in the proof of 5.53 and in 5.41. □

8.65 Proposition. [20, 9.9.2] *For continuous $f : (S^1, 1) \to (S^1, 1)$ the induced homomorphism $f_* : H_1(S^1) \to H_1(S^1)$ is given by $[S^1] \mapsto \deg(f) \cdot [S^1]$.*

Proof. For $[c] \in \pi_1(S^1, 1)$ we have $\deg(f \circ c) = \deg(f) \cdot \deg(c)$ by 2.15.3 and $\deg : \pi_1(S^1, 1) \cong \mathbb{Z}$ is an isomorphism by 5.15, thus $\pi_1(f)$ acts by multiplication with $\deg(f)$ and using the naturality of h_1, gives the same result for $H_1(f)$.

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "\deg(f)\cdot"'] \arrow[dr, phantom, "5.15"] & \pi_1(S^1,1) \arrow[l, "\cong"', "\deg"] \arrow[r, "\cong", "h_1"'] \arrow[d, "\pi_1(f)"'] \arrow[dr, phantom, "8.63"] & H_1(S^1) \arrow[d, "H_1(f)"] \\
\mathbb{Z} & \pi_1(S^1,1) \arrow[l, "\cong"', "\deg"] \arrow[r, "\cong", "h_1"'] & H_1(S^1)
\end{tikzcd}$$

For a direct proof see [20, 9.5.5] and 2.15. □

8.66 Proposition. [20, 9.9.9]
The homology of the closed orientable surface of genus g is:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

and that for the non-orientable one of genus g is:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This time we calculate the cellular homology. Recall that in both cases X can be described as the CW-complex obtained by gluing one 2-cell e^2 to a join of circles S^1 along a map $f : S^1 \to \bigvee^k S^1$ of the form $i_{m_1}^{n_1} \cdot \dots \cdot i_{m_l}^{n_l}$ (see 1.94). Thus the non-vanishing cellular chain groups are $C_0(X) \cong \mathbb{Z}$, $C_1(X) \cong \mathbb{Z}^k$, and $C_2(X) \cong \mathbb{Z}$ with generators given by the base-point e^0, the 1-cells $e_1^1, \dots, e_k^1$, and one 2-cell e^2 with chosen orientations, by 8.53. More precisely, the generators are $\tilde e := (\chi^e)_*[D^q] \in C_q(X) = H_q(X^q, X^{q-1})$ for the generator $[D^q] \in H_q(D^q, S^{q-1})$ and a characteristic maps χ^e for each q-cell e. As in the proof of 8.63 for ν_* and using 8.65 one shows that

$$f_*[S^1] = (i_{m_1}^{n_1})_*[S^1] + \dots + (i_{m_l}^{n_l})_*[S^1] = n_1 \cdot (i_{m_1})_*[S^1] + \dots + n_l \cdot (i_{m_l})_*[S^1].$$

Hence $\partial(\tilde e^2) = \partial((\chi^{e^2})_*[D^2]) = j_*(\chi^{e^2}|_{S^1})_*[S^1] = j_* f_*[S^1] = n_1 \tilde e_{m_1}^1 + \dots + n_l \tilde e_{m_l}^1$ by 8.54 and since $\tilde e_m^1 = (\chi^{e_m^1})_*[D^1] = j_*(i_m)_*[S^1]$):

$$\begin{tikzcd} S^0 \arrow[r, hook] \arrow[d, "\mathrm{const}"] & D^1 \arrow[r, two heads] \arrow[d, "\chi^{e_m^1}"] & S^1 \arrow[d, hook, "i_m"] \\ X^0 \arrow[r, hook] & X^1 \arrow[r, equal] & \bigvee_k S^1 \end{tikzcd}$$

$$(\chi^{e_m^1})_* : H_1(D^1, S^0) \cong H_1(S^1) \xrightarrow{(i_m)_*} H_1(X^1) \xrightarrow{j_*} H_1(X^1, X^0) =: C_1(X)$$

For the boundary of the 1-cells we get

$$\begin{aligned} \partial(\tilde e_j^1) &= j_* \partial_* (\chi^{e_j^1})_*[D^1] = j_*(\chi^{e_j^1}|_{S^0})_* \partial_*[D^1] = j_*(\chi^{e_j^1}|_{S^0})_*[\partial D^1] \\ &= j_* \, \mathrm{const}_*[(+1) - (-1)] = \tilde e^0 - \tilde e^0 = 0. \end{aligned}$$

In case of the oriented closed surface X of genus g (where $k = 2g$) we thus have $\partial \tilde e^2 = \tilde e_1^1 + \tilde e_2^1 - \tilde e_1^1 - \tilde e_2^1 + \dots = 0$, hence $H_q(X) \cong H_q(C(X)) = C_q(X)$ is as claimed.

In case of a non-orientable surfaces X of genus g (where $k = g$) we have $\partial \tilde e^2 = 2\tilde e_1^1 + \dots + 2\tilde e_g^1$, which shows that $H_2(X) = \operatorname{Ker} \partial_2 = \{0\}$ and

$$\begin{aligned} H_1(X) &= \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 = \mathbb{Z}^g / 2\mathbb{Z}(\tilde e_1^1 + \dots + \tilde e_g^1) \\ &= {}^{ab}\Big\langle \{\tilde e_1^1, \dots, \tilde e_g^1\} : \{2(\tilde e_1^1 + \dots + \tilde e_g^1) = 0\} \Big\rangle \\ &\overset{5.27.2}{=\!=} {}^{ab}\Big\langle \{\tilde e_1^1, \dots, \tilde e_g^1, x\} : \{x = \tilde e_1^1 + \dots + \tilde e_g^1,\ 2x = 0\} \Big\rangle \\ &\overset{5.27.2}{=\!=} {}^{ab}\Big\langle \{\tilde e_1^1, \dots, \tilde e_{g-1}^1, x\} : \{2x = 0\} \Big\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2. \end{aligned}$$

□

8.67 Proposition. [20, 9.9.14] *For the real projective spaces we have*

$$H_q(\mathbb{P}^n(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \text{ or } q = n \text{ odd}, \\ \mathbb{Z}_2 & \text{for } 0 < q < n \text{ with } q \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The idea is to consider the CW-decomposition of S^n compatible with the equivalence relation $x \sim -x$, which gives $\mathbb{P}^n = S^n/\sim$ (see 1.95). For this we consider the spheres $S^0 \subset S^1 \subset \dots \subset S^n$ and the cells $\{x \in S^q : \pm x_{q+1} > 0\}$ with characteristic map $f_\pm^q : x \mapsto (x, \pm\sqrt{1 - |x|^2})$. They form a cell decomposition of S^n and hence $\tilde e_\pm^q := (f_\pm^q)_*[D^q]$ is a basis in $C_q(S^n)$ by 8.53. We have the reflection $r : D^q \to D^q$, $x \mapsto -x$ and may consider it as mapping $r : (S^q, S^{q-1}) \to (S^q, S^{q-1})$ to obtain an homomorphism $r_* : C_q(S^n) \to C_q(S^n)$ and also $r_* : H_q(D^q, S^{q-1}) \to H_q(D^q, S^{q-1})$.

Claim: $r_*\tilde{e}^q_+ = (-1)^q \tilde{e}^q_-$:
Note that $r_*[D^q] = (-1)^q[D^q] \in H_q(D^q, S^{q-1})$ which is obvious for $q = 1$ and follows by induction for $q \geq 2$:
A reflection r on a hyperplane A induces $-\mathrm{id}$ on $H_q(D^q, S^{q-1}) \cong H_{q-1}(S^{q-1})$, since we may choose a q-simplex σ with edges x_0, x_1 symmetric with respect to A, $x_2, \dots, x_q \in A$ and 0 in its interior. Let $\tilde{\sigma} : \Delta_q \to \langle x_0, \dots, x_q\rangle$ the associated (see 8.59) singular simplex and $p : \mathbb{R}^q \setminus \{0\} \to S^{q-1}$ the retraction $x \mapsto x/\|x\|$. Then $p \circ \tilde{\sigma} : \dot{\Delta}_q \to S^{q-1}$ is a homeomorphism and $r \circ p \circ \tilde{\sigma} = p \circ r \circ \tilde{\sigma} = p \circ \tilde{\sigma} \circ \rho|_{\dot{\Delta}_q}$, where ρ is the affine isomorphism, which exchanges e_0 and e_1. Hence $r_* = -\mathrm{id}$ since $(\rho|_{\dot{\Delta}_q})_* = -\mathrm{id}$ by 8.58.

Since $r \circ f^q_+ = f^q_- \circ r$ we thus get

$$r_*\tilde{e}^q_+ = r_*(f^q_+)_*[D^q] = (f^q_-)_* r_*[D^q] = (-1)^q (f^q_-)_*[D^q] = (-1)^q \tilde{e}^q_-.$$

Claim: $\partial\tilde{e}^{q+1}_- = \partial\tilde{e}^{q+1}_+ = \pm(\tilde{e}^q_+ - \tilde{e}^q_-)$:
Since $f^{q+1}_\pm|_{S^q} = \mathrm{id}$ we get $\partial\tilde{e}^{q+1}_\pm = \partial(f^{q+1}_\pm)_*[D^q] = j_*(f^{q+1}_\pm|_{S^q})_*[S^q] = j_*[S^q]$ by 8.54. Using 8.41 we consider the long exact sequence of the pair for $q > 1$:

$$\begin{tikzcd}
H_q(S^{q-1}) \arrow[r] \arrow[d, equal] & H_q(S^q) \arrow[r, tail, "j_*"] \arrow[d, equal] & H_q(S^q, S^{q-1}) \arrow[r, two heads, "\partial_*"] \arrow[d, equal] & H_{q-1}(S^{q-1}) \arrow[r] \arrow[d, equal] & H_{q-1}(S^q) \arrow[d, equal] \\
0 & {}^{ab}\langle\{[S^q]\}\rangle \cong \mathbb{Z} & {}^{ab}\langle\{\tilde{e}^q_+, \tilde{e}^q_-\}\rangle \cong \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & 0
\end{tikzcd}$$

So $\partial_* \neq 0$ since it is onto and in particular applied to the generators $\tilde{e}^q_\pm$ we have $\partial_*\tilde{e}^q_\pm = \partial_*(f^q_\pm)_*[D^q] = (f^q_\pm|_{S^{q-1}})_*[S^{q-1}] = \mathrm{id}[S^{q-1}]$ by 8.19, hence $\partial_*\tilde{e}^q_- = \partial_*\tilde{e}^q_+ \neq 0$. So $\mathbb{Z} \cdot (\tilde{e}^q_+ - \tilde{e}^q_-) = \operatorname{Ker}\partial_* = \operatorname{Im} j_* = \mathbb{Z} \cdot j_*[S^q]$. Thus $j_*[S^q] = \pm(\tilde{e}^q_+ - \tilde{e}^q_-)$.

The projective space $\mathbb{P}^n = S^n/\sim$ is a CW-complex with cells $p(f^q_+(D^q)) = p(f^q_-(D^q))$ and with characteristic mappings $p \circ f^q_+ : D^q \to \mathbb{P}^q$. Hence the generator of $C_q(\mathbb{P}^n)$ is given by $p_*(\tilde{e}^q_+) = (p \circ f^q_+)_*[D^q] =: \tilde{e}^q$. Since $p \circ r = p$ we have by the first claim that $p_*(\tilde{e}^q_-) = (-1)^q p_*(r_*\tilde{e}^q_+) = (-1)^q p_*(\tilde{e}^q_+) = (-1)^q \tilde{e}^q$.
For $0 < q \leq n$ we get by the second claim that

$$\begin{aligned}
\partial\tilde{e}^q &= \partial p_*(\tilde{e}^q_+) = p_*\partial(\tilde{e}^q_+) = \pm p_*(\tilde{e}^{q-1}_+ - \tilde{e}^{q-1}_-) \\
&= \pm(1 - (-1)^{q-1})\tilde{e}^{q-1} = \begin{cases} 0 & \text{for odd } q. \\ \pm 2\tilde{e}^{q-1} & \text{for even } q. \end{cases}
\end{aligned}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & C_n(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & \dots \arrow[r, "\partial"] & C_4(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & C_3(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & C_2(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & C_1(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & C_0(\mathbb{P}^n) \arrow[r, "\partial"] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \dots \arrow[r] & \mathbb{Z} \arrow[r, "\pm 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\pm 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\partial"] & 0
\end{tikzcd}$$

$$H_q: \quad \dots \quad 0 \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z}_2 \quad \mathbb{Z}$$

Thus for even $q > 0$ we have no non-trivial cycle in $C_q(\mathbb{P}^n)$ and for odd $q > 0$ we have that $\tilde{e}^q$ is a cycle and $2\tilde{e}^q = \pm\partial\tilde{e}^{q+1}$ is a boundary for $q < n$. So the claimed homology follows. □

9. Cohomology

▷ **9.1 Definition.** Roughly speaking cohomology is the dual construction to homology. Let

$$\cdots \to C_q \xrightarrow{\partial} C_{q-1} \to \cdots$$

be a chain complex and G be an abelian group. Then

$$\cdots \leftarrow \operatorname{Hom}(C_q, G) \xleftarrow{\partial^*} \operatorname{Hom}(C_{q-1}, G) \leftarrow \cdots$$

defines another chain complex $C^{-q} := \operatorname{Hom}(C_q, G)$ and hence we may consider its homology $H(C^*)$ and we call $H^q(C;G) := H_{-q}(C^*) = H_{-q}(\operatorname{Hom}(C_{-*}, G))$ the COHOMOLOGY of C with coefficients in G.

▽ In particular, we have

- the simplicial cohomology groups $H^q(K;G)$ of simplicial complexes;
- the singular cohomology groups $H^q(X;G)$ of topological spaces X; and
- the cellular cohomology groups $H^q(C(X);G)$ of CW-spaces X.

Note that $\operatorname{Hom}(\_, G) : \underline{\text{A-Gru}} \to \underline{\text{A-Gru}}$ is a contravariant functor which maps $f : C \to C'$ to $f^* : \operatorname{Hom}(C', G) \to \operatorname{Hom}(C, G)$ defined by $f^*(g) := g \circ f$. Hence we better use $\underline{\text{A-Gru}}^{op}$ (the category $\underline{\text{A-Gru}}$ but with all arrows reversed) as its domain to get a covariant functor.

Since $\operatorname{Hom}(\_, G)$ is additiv (i.e. $(f_1 + f_2)^* = f_1^* + f_2^* : (g \mapsto f_1 \circ g + f_2 \circ g)$) it preserves the biproduct $C_1 \oplus C_2$ (see [**9**, 3.27],), which is completely described by the projections pr_i and the injections inj_i with $\operatorname{pr}_i \circ \operatorname{inj}_j = \delta_{i,j}$. Thus this Hom-functor also preserves splitting exact sequences.

▷ **9.2 Remark.**
A more naive dual construction would be to consider $\operatorname{Hom}(H_q(C), G)$ and leads to the question: *Do these two constructions coincide?*

We get mappings $h : H^q(C;G) \to \operatorname{Hom}(H_q(C), G)$ defined by $[\varphi] \mapsto \widetilde{\varphi|_{Z_q}}$, where $\varphi \in \operatorname{Hom}(C_q, G)$ with $0 = \partial^*(\varphi) = \varphi \circ \partial : C_{q+1} \to C_q \to G$, i.e. $\varphi|_{B_q} = 0$. Hence $\varphi|_{Z_q} : Z_q \to G$ factors over $Z_q \twoheadrightarrow H_q(C) = Z_q/B_q$ and thus defines an element $h([\varphi]) := \widetilde{\varphi|_{Z_q}} \in \operatorname{Hom}(H_q(C), G)$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, "\partial"] \arrow[dd, two heads, "\partial"'] \arrow[dr, "0"] & C_q \arrow[r, "\partial"] \arrow[d, "\varphi"] & C_{q-1} \arrow[r] & \cdots \\
& & G & & \\
0 \arrow[r] & B_q(C) \arrow[r, hook, "i"] & Z_q(C) \arrow[r, two heads] \arrow[uu, hook, bend right] \arrow[u, dotted, "\varphi|_{Z_q}"', "1."] & H_q(C) \arrow[r] \arrow[ul, dotted, "\widetilde{\varphi|_{Z_q}}"', "2."] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \underset{[\varphi]}{H^q(C;G)} \arrow[l] & Z^q(C;G) \arrow[l, two heads] \arrow[d, hook] & B^q(C;G) \arrow[l, hook', "\varphi"'] & 0 \arrow[l] \\
\cdots & \underset{0}{\operatorname{Hom}(C_{q+1}, G)} \arrow[l] & \operatorname{Hom}(C_q, G) \arrow[l, "\partial^*"'] \arrow[d] & \underset{\varphi}{\operatorname{Hom}(C_{q-1}, G)} \arrow[l, "\partial^*"'] \arrow[u, two heads, "\partial^*"] & \cdots \arrow[l] \\
& \underset{0}{\operatorname{Hom}(B_q(C), G)} \arrow[u, two heads, "\partial^*"] & \underset{\varphi|_{Z_q}}{\operatorname{Hom}(Z_q(C), G)} \arrow[l, "i^*"'] & \underset{\widetilde{\varphi|_{Z_q}}}{\operatorname{Hom}(H_q(C), G)} \arrow[l, tail] & 0 \arrow[l]
\end{tikzcd}$$

Let $\cdots \to C_q \xrightarrow{\partial} C_{q-1} \to \cdots$ be a chain complex of free abelian groups. Consider its cycle subgroups $Z_q \subseteq C_q$ and boundary subgroups $B_q \subseteq Z_q$, i.e. the short exact and splitting (since B_{q-1} is free abelian) sequence

$$0 \longrightarrow Z_q \overset{j}{\hookrightarrow} C_q \overset{\partial}{\twoheadrightarrow} B_{q-1} \longrightarrow 0$$

For an abelian group G we apply the functor $\operatorname{Hom}(\_, G)$ to this sequence and obtain a short exact(!) sequence of chain complexes, where the boundary operator in the middle is given by ∂^* and the others are 0.

$$\begin{tikzcd} 0 & \operatorname{Hom}(Z_{-*}, G) \arrow[l] \arrow[loop below, "0"] & \operatorname{Hom}(C_{-*}, G) \arrow[l, "j^*"'] \arrow[loop below, "\partial^*"] & \operatorname{Hom}(B_{1-*}, G) \arrow[l, "\partial^*"'] \arrow[loop below, "0"] & 0 \arrow[l] \end{tikzcd}$$

Applying the homology functor H_q gives a long exact sequence for the cohomology groups $H^{-q}(C; G) := H_q(\operatorname{Hom}(C_{-*}, G))$, etc.:

$$\begin{tikzcd} \cdots & H^q(B;G) \arrow[l] \arrow[d, equal] & H^q(Z;G) \arrow[l] \arrow[d, equal] & H^q(C;G) \arrow[l, "(j^*)_*"'] & H^{q-1}(B;G) \arrow[l, "(\partial^*)_*"'] \arrow[d, equal] & H^{q-1}(Z;G) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\ & \operatorname{Hom}(B_q, G) & \operatorname{Hom}(Z_q, G) \arrow[l, "i^*"'] & & \operatorname{Hom}(B_{q-1}, G) & \operatorname{Hom}(Z_{q-1}, G) \arrow[l, "i^*"'] & \end{tikzcd}$$

Since the boundary operator on $\operatorname{Hom}(Z_{-*}, G)$ and on $\operatorname{Hom}(B_{-*}, G)$ is 0, we have $H^q(Z;G) = \operatorname{Hom}(Z_q, G)$ and $H^q(B;G) = \operatorname{Hom}(B_q, G)$. Moreover the connecting homomorphism $H^q(Z;G) \to H^q(B;G)$ is i^*, where $i : B_q \hookrightarrow Z_q$ denotes the inclusion: Let $\varphi \in \operatorname{Hom}(Z_q, G)$ and $\tilde\varphi \in \operatorname{Hom}(C_q, G)$ with $\tilde\varphi|_{Z_q} = j^*(\tilde\varphi) = \varphi$ (exists, since the short exact sequence $Z_q \to C_q \to B_{q-1}$ splits). Hence for the connecting homomorphism $[\varphi] \mapsto [(\partial^*)^{-1}\partial^*(j^*)^{-1}\varphi] = [(\partial^*)^{-1}\partial^*\tilde\varphi] = [i^*\varphi]$, since $\partial^*(i^*(\varphi)) = \partial^*(\varphi|_{B_q}) = \varphi \circ \partial = \tilde\varphi \circ \partial = \partial^*(\tilde\varphi)$. Now consider the short exact sequence

$$0 \longrightarrow B_q \underset{i}{\hookrightarrow} Z_q \twoheadrightarrow H_q(C) \longrightarrow 0$$

Let us assume for the moment that applying $\operatorname{Hom}(\_, G)$ gives again a short exact sequence (e.g. if $H_q(C)$ is free abelian (or, more general, a projective module), since then the sequence splits and so also its image under the additive functor $\operatorname{Hom}(\_, G)$)

$$0 \longleftarrow \operatorname{Hom}(B_q, G) \xleftarrow{i^*} \operatorname{Hom}(Z_q, G) \longleftarrow \operatorname{Hom}(H_q(C), G) \longleftarrow 0$$

In particular i^* is onto, hence $(\partial^*)_* = 0$ and thus $(j^*)_*$ is injective and its image is $\operatorname{Ker}(i^*) = \{\varphi \in \operatorname{Hom}(Z_q, G) : \varphi|_{B_q} = 0\} \cong \operatorname{Hom}(H_q(C), G)$, i.e.

$$(j^*)_* = h : H^q(C; G) \cong \operatorname{Hom}(H_q(C), G).$$

▽

9.3 Example.[20, 13.1.2]

1. $\operatorname{Hom}(\mathbb{Z}, G) \cong G$ via $\varphi \mapsto \varphi(1)$.
2. $\operatorname{Hom}(\mathbb{Z}_n, G) = \{g \in G : ng = 0\}$ via $\varphi \mapsto g := \varphi(1)$, since $0 = \varphi(0) = \varphi(n) = ng$.
3. $\operatorname{Hom}(\mathbb{Z}_n, G) = 0$ if G is torsion free by $\boxed{1}$.
4. $\operatorname{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$ by $\boxed{1}$.
5. $\operatorname{Hom}(\_, G)$ is additive.

Let us now check, whether $\mathrm{Hom}(\_, G)$ preserves also short exact sequences (which are not assumed to be splitting).

▷ **9.4 Proposition.**[20, 13.1.5] *If* $0 \leftarrow C \xleftarrow{p} B \xleftarrow{i} A$ *is exact, then*

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{p^*} \mathrm{Hom}(B, G) \xrightarrow{i^*} \mathrm{Hom}(A, G)$$

is also exact, i.e. $\mathrm{Hom}(\_, G)$ *is a* LEFT EXACT FUNCTOR.

Proof. (p^* is injective) Let $0 = p^*(\varphi) = \varphi \circ p$. Then $\varphi = 0$, since p is onto.

($\ker(i^*) = \mathrm{im}(p^*)$) Let $0 = i^*(\varphi) = \varphi \circ i$, i.e. φ vanishes on $\mathrm{im}(i) = \ker(p)$ and hence factors to a $\tilde{\varphi} : C \to G$ with $\varphi = \tilde{\varphi} \circ p = p^*(\tilde{\varphi})$. The converse inclusion is obvious by $p \circ i = 0$. □

9.5 Remark. Exactness at $\mathrm{Hom}(A, G)$ would mean that $i^* : \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is onto for injective $i : A \to B$, i.e. every homomorphism $\varphi : A \to G$ must have an extension to B. An abelian group G having this property for arbitrary monomorphisms $A \hookrightarrow B$ is called INJECTIVE. Thus the arguments in 9.2 hold for injective G even if $H_q(C)$ is not free abelian.

9.6 Example. $\mathbb{Z}_2$ is not injective.[20, 13.1.4]

The exact sequence $0 \to 2\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Z} \overset{\pi}{\twoheadrightarrow} \mathbb{Z}_2 \to 0$ is mapped to

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \arrow[r, tail, "\pi^*"] \arrow[d, equal, "\boxed{9.3.4}"'] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) \arrow[r, "i^*"] \arrow[d, equal, "\boxed{9.3.1}"'] & \mathrm{Hom}(2\mathbb{Z}, \mathbb{Z}_2) \arrow[d, equal] \\
 & \mathbb{Z}_2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2
\end{tikzcd}$$

9.7 Definition. A LEFT MODULE over a ring R is an abelian group M together with a multiplication $\cdot : R \times M \to M$ which satisfies the distributive laws

$$r \cdot (x + y) = r \cdot x + r \cdot y \quad \text{and} \quad (r + s) \cdot x = r \cdot x + s \cdot y$$

the "associativity" law

$$r \cdot (s \cdot x) = (r\, s) \cdot x$$

and the unit $1 \in R$ acts as identity $1 \cdot x = x$.
The $\mathbb{Z}$-modules are exactly the abelian groups.

A left module M over a ring R is called INJECTIVE iff any short exact sequence $0 \to M \to B \to C \to 0$ of left R-modules splits, or, equivalently, if $i : A \to B$ is an injective module homomorphism then every module homomorphism $f : A \to M$ extends to B (i.e. $\mathrm{Hom}(i, M)$ is onto):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[dl, dotted] \\
 & M &
\end{tikzcd}$$

▽ (⇒)

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "i_2"] & & \\
0 \arrow[r] & M \arrow[r, tail, "i_1"] \arrow[d, "\mathrm{id}_M"'] & PO \arrow[r, hook] \arrow[dl, dotted] & PO/M \arrow[r] & 0 \\
 & M & & &
\end{tikzcd}$$

Note, that the push-out of a mono is a mono: In fact, let $0 = i_1(m) = [m \oplus 0]$. Then $m \oplus 0 = f(a) \oplus i(-a)$ for some $a \in A$, hence $a = 0$ (since i is injective) and thus $m = f(0) = 0$.

A left module M over a ring R is called PROJECTIVE iff any short exact sequence $0 \to A \to B \to M \to 0$ of left R-modules splits, or, equivalently, if $p : B \to C$ is a surjective module homomorphism, then for every module homomorphism $f : M \to B$ lifts to C (i.e. $\mathrm{Hom}(M, p)$ is onto):

$$\begin{tikzcd}
0 & C \arrow[l] & B \arrow[l, two heads, "p"'] \\
& M \arrow[u, "f"] \arrow[ur, dotted] &
\end{tikzcd}$$

($\Rightarrow$)

$$\begin{tikzcd}
0 & C \arrow[l] & B \arrow[l, two heads, "p"'] & & \\
0 & M \arrow[l] \arrow[u, "f"] & M \times_C B \arrow[l, two heads, "\mathrm{pr}_1"'] \arrow[u, "\mathrm{pr}_2"'] & \ker \mathrm{pr}_1 \arrow[l, hook'] & 0 \arrow[l] \\
& M \arrow[u, "\mathrm{id}_M"] \arrow[ur, dotted] & & &
\end{tikzcd}$$

9.8 Lemma. Stability of projective and injective objects.

- *Coproducts and direct summands of projective objects are projective.*
- *Products and direct summands of injective objects are injective.*

Proof.

$$\begin{tikzcd}
B \arrow[r, two heads] & C \arrow[r] & 0 \\
M_i \arrow[u, dotted, "1."] \arrow[r, "\mathrm{inj}_i"'] & \coprod_i M_i \arrow[ul, dotted, "2."] \arrow[u, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, two heads] & C \arrow[r] & 0 \\
N \arrow[u, dotted, "1."] \arrow[rr, bend right] & M \arrow[l, tail] \arrow[ul, dotted, "2."] \arrow[u, "f"'] \arrow[r, "\mathrm{id}_M"] & M \arrow[ul, "f"']
\end{tikzcd}$$

□

9.9 Lemma. *A module is projective, iff it is a direct summand in a free module. An abelian group is projective if and only if it is free abelian.*

Proof. ($\Leftarrow$) By 9.8 it is enough to show show this for a free module $M := \mathcal{F}(X)$. Let $p : C \to B$ be onto and $f : \mathcal{F}(X) = M \to B$ a homomorphism. Then we define $\tilde{f} : M \to C$ by sending each generator $x \in X$ to a chosen inverse image in $p^{-1}(f(x))$.

($\Rightarrow$) Since every module M is the quotient of a (the) free module (${}^{ab}\mathcal{F}(M)$) we may lift the identity on M, hence M is a direct summand of a free module. And for abelian groups it is itself free by 9.20. □

9.10 Example. Projective modules are not always free:
Let $R := \mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $\mathbb{Z}_2$ is a projective R-module but not free.

9.11 Definition. An abelian group A is called DIVISIBLE, iff for every $0 < n \in \mathbb{N}$ and $g \in A$ there exists an $x \in A$ mit $n \cdot x = g$.

Examples are: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_{p^\infty} := \varinjlim_{k \in \mathbb{N}} \mathbb{Z}_{p^k} \cong \{e^{2\pi j/p^k} : j, k \in \mathbb{N}\}$, where the connecting mappings $\mathbb{Z}_{p^k} \to \mathbb{Z}_{p^{k+1}}$ are given by multiplication with p.

9.12 Lemma. *An abelian group is injective if and only if it is divisible.*

Proof. ($\Leftarrow$) Let A be a subgroup of B and $f : A \to M$ be a homomorphism. Consider the set $\mathcal{S} := \{(g, C) : A \subseteq C \leq B, g : C \to M, g|_A = f\}$ of all partial extensions of f ordered by componentwise inclusion. Obviously any linearly ordered subset $\mathcal{S}_0 \subseteq \mathcal{S}$ has an upper bound given by the componentwise union $(\bigcup_{(g,C)\in\mathcal{S}_0} g, \bigcup_{(g,C)\in\mathcal{S}_0} C)$. By Zorns Lemma we have a maximal element $(\tilde{f}, \tilde{A})$. Suppose $\tilde{A} \neq C$ and take $g \in B \setminus C$. If $g + C$ has infinite order in B/C, then $\tilde{f}$ can be extended to $\langle C \cup \{g\}\rangle \cong C \oplus \langle g\rangle$ by $\tilde{f}(c+kg) := f(c)$, contradicting maximality. Otherwise let n be minimal with $ng \in C$. Since M is divisible there exists $x \in M$ with $nx = f(ng)$ so we can extend f to $C + \langle g\rangle$ by $\tilde{f}(c+kg) := f(c) + k\,x$, again a contradiction.

($\Rightarrow$) Let $0 < n \in \mathbb{N}$ and $g \in M$. Consider the inclusion $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ and $f : n\mathbb{Z} \to M$ given by $n \mapsto g$. By injectivity of M we have an extension $\tilde{f} : \mathbb{Z} \to M$ and then $x := \tilde{f}(1)$ solves $nx = n\tilde{f}(1) = \tilde{f}(n) = f(n) = g$. □

9.13 Remark. One can show that the divisible abelian groups are exactly the direct sums of $\mathbb{Q}$ and the $\mathbb{Z}_{p^\infty}$.

▷ **9.14 Remark.** In order to generalize the arguments in 9.2 we need an exact sequence

$$0 \longrightarrow \operatorname{Hom}(H_q(C), M) \longrightarrow \operatorname{Hom}(Z_q, M) \longrightarrow \operatorname{Hom}(B_q, M) \longrightarrow\ ?\ \longrightarrow \cdots$$

For injective M we can replace '?' by 0. So we try to 'approximate' a general module M by injective modules, i.e. we try to find an exact sequence of the form $0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$, where all I_j are injective modules, a so called INJECTIVE RESOLUTION of M.

For the induction step we need:

9.15 Proposition. *Every module is submodule of injective module.*

▽ **Proof.** For abelian groups injectivity is equivalent to divisibility by 9.12. Any abelian group A is quotient of a free group, i.e. a coproduct of copies of $\mathbb{Z}$ which embeds in the divisible group given by the corresponding coproduct of $\mathbb{Q}$. Taking the push-out shows that A is a subgroup of a (divisible) quotient of a divisible group.

$$\begin{tikzcd} \coprod \mathbb{Z} \arrow[r, hook] \arrow[d, two heads] & \coprod \mathbb{Q} \arrow[d, two heads] \\ A \arrow[r, tail] & PO \end{tikzcd}$$

For every R-module N there is a R-module structure on $\operatorname{Hom}_{\mathbb{Z}}(R, N)$ given by $r \cdot \varphi : r' \mapsto \varphi(r'r)$. We have $\operatorname{Hom}_{\mathbb{Z}}(N, D) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(R, D))$: We map $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(N, D)$ to $\tilde{\varphi} : x \mapsto (r' \mapsto \varphi(r'x))$. We have $\tilde{\varphi} \in \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(R, D))$, since

$$\tilde{\varphi}(rx)(r') = \varphi(r'rx) = \tilde{\varphi}(x)(r'r) = (r \cdot \tilde{\varphi}(x))(r').$$

Conversely, $\tilde{\varphi} \mapsto \varphi := \operatorname{ev}_1 \circ \tilde{\varphi}$.

If $M \rightarrowtail D$ is a group-monomorphism into a divisible(=injective) abelian group D. Then the corresponding R-module homomorphism $M \to \operatorname{Hom}_{\mathbb{Z}}(R, D)$ is obviously a monomorphism (we assume that R is a ring with unit) and $\operatorname{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module. □

▷ **9.16 Corollary.** *Every module has an injective resolution.*

▽ **Proof.** By 9.15 we find for every module M an injective module I_0 and an embedding $M \hookrightarrow I_0$. Now proceed recursively by chossing an embedding of the $I_k/\operatorname{im}(I_{k-1} \to I_k)$ into an injective module I_{k+1}. □

▷ **9.17 Lemma.** *Every module has a* PROJECTIVE RESOLUTION.

▽ **Proof.** Let M be a module. Then M is quotient of the free module $P_0 = \coprod_M R$. Consider the kernel K_0 of this quotient map $\pi : P_0 \twoheadrightarrow M$. If $R = \mathbb{Z}$, i.e. M is just an abelian group (i.e. $R = \mathbb{Z}$), then K_0 is free as well by 9.20 and we found the projective resolution $0 \to K_0 \to P_0 \to M \to 0$. For general R we find a free module P_1 which has K_0 as quotient. Recursively we get an exact sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0. \qquad \square$$

9.18 Lemma. *Let $P \to M \to 0$ be a projective resolution, $X \to N \to 0$ an arbitrary resolution (i.e. exact sequence), and $f : M \to N$ a homomorphism. Then there exists a homomorphism $\tilde f : P \to X$ of chain-complexes, which extends f and which is unique up to chain homotopies:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial"] \arrow[d, dotted, "\tilde f_{n+1}"] & P_n \arrow[r, "\partial"] \arrow[d, dotted, "\tilde f_n"] \arrow[dl, dashed, "s_n"'] & P_{n-1} \arrow[r] \arrow[d, dotted, "\tilde f_{n-1}"] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dotted, "\tilde f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & X_{n+1} \arrow[r, "\partial"'] & X_n \arrow[r, "\partial"'] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

The following proof shows, that we don't need that $P \to M \to 0$ is exact and that $X_0 \to N$ is onto, P being a chain complex, X an exact sequence, and f mapping the image of $P_0 \to M$ into that of $X_1 \to N$ suffices.

Proof. Existence: Since P_0 is projective and $X_0 \to N$ is onto, we have a lift $\tilde f_0 : P_0 \to X_0$ of $f \circ \pi : P_0 \to M \to N$ and recursively we get lifts $\tilde f_n : P_n \to X_n$ since $P_{n+1} \to P_n \xrightarrow{\tilde f_n} X_n \to X_{n-1}$ is 0 hence has values in $\ker(X_n \to X_{n-1}) = \operatorname{im}(X_{n+1} \to X_n)$ and by projectivity of P_{n+1} has a lift $\tilde f_{n+1} : P_{n+1} \to X_{n+1}$.

Uniqueness: Let $\tilde g$ we another lift of f. Then $\tilde f_0 - \tilde g_0$ has values in the kernel of $X_0 \to N$ and hence has a lift $s_0 : P_0 \to X_1$. Recursively we get $s_n : P_n \to X_{n+1}$ with $\partial s_n + s_{n-1}\partial = \tilde f_n - \tilde g_n$: Since $\partial(\tilde f_n - \tilde g_n - s_{n-1}\partial) = (\tilde f_{n-1} - \tilde g_{n-1} - \partial s_{n-1})\partial = s_{n-2}\partial^2 = 0$ there exists a lift $s_n : P_n \to X_{n+1}$ with $\partial s_n = \tilde f_n - \tilde g_n - s_{n-1}\partial$. $\square$

9.19 Lemma. *Let $0 \to M \to I$ be an injective resolution, $0 \to N \to X$ an arbitrary resolution (i.e. exact sequence), and $f : N \to M$ a homomorphism. Then there exists a homomorphism $\tilde f : X \to I$ of cochain-complexes, which extends f and which is unique up to chain homotopies:*

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\partial"] \arrow[d, "f"'] & X_0 \arrow[r, "\partial"] \arrow[d, dotted, "\tilde f_0"'] & \cdots \arrow[r, "\partial"] & X_n \arrow[r, "\partial"] \arrow[d, dotted, "\tilde f_n"'] & \cdots \\
0 \arrow[r] & M \arrow[r, "\partial"] & I_0 \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & I_n \arrow[r, "\partial"] & {}
\end{tikzcd}$$

Proof. Existence: Since I_0 is injective and $N \to X_0$ is injective, we have an extension $\tilde f_0 : X_0 \to I_0$ of $N \to M \to I_0$ and recursively we get extensions $\tilde f_n : X_n \to I_n$ since $X_{n-2} \to X_{n-1} \xrightarrow{f_{n-1}} I_{n-1} \to I_n$ is 0 hence factors over $\operatorname{im}(X_{n-1} \to X_n) \cong X_{n-1}/\ker(X_{n-2} \to X_{n-1})$.

Uniqueness: Let $\tilde f$ be another extension of f. Then $\tilde f_0 - \tilde f'_0$ vanishes on the image of $N \to X_0$ and factors over $X_0/\ker(X_0 \to X_1) \cong \operatorname{im}(X_0 \to X_1)$. By injectivity of I_0 we get an extension $s_0 : X_1 \to I_0$. Recursively we get $s_k : X_{k+1} \to I_k$ with $\partial s_k + s_{k-1}\partial = \tilde f_k - \tilde f'_k$: Since $\partial(\tilde f_{k+1} - \tilde f'_{k+1} - s_k\partial) = 0$ there exists a $s_{k+1} : X_{k+2} \to I_{k+1}$ with $\partial s_{k+1} = \tilde f_{k+1} - \tilde f'_{k+1} - s_k\partial$. $\square$

▷ **9.20 Proposition.** *Every subgroup of a free abelian group is free abelian. More generally, every submodule of a free module over a PID is free. Thus we find in this situation a projective resolution of the form:*

$$0 \to P_1 \to P_0 \to M \to 0.$$

▽ **Proof.** Let H be a submodule of a free module $G := \coprod_J R$. For every subset $\Lambda \subseteq J$ we consider $H_\Lambda := H \cap \coprod_\Lambda R$. Let

$$\mathcal{S} := \Big\{(\Lambda, B) : \Lambda \subseteq J, H_\Lambda \text{ is free with generators } B \subseteq H_\Lambda\Big\}$$

and define a partial ordering $(\Lambda, B) \preceq (\Lambda', B') :\Leftrightarrow \Lambda \subseteq \Lambda'$ and $B \subseteq B'$.

For every linearly ordered subset $\mathcal{S}_0 \subseteq \mathcal{S}$ let $(\Lambda_\infty, B_\infty) := (\bigcup_{(\Lambda,B)\in\mathcal{S}_0} \Lambda, \bigcup_{(\Lambda,B)\in\mathcal{S}_0} B)$. Then B_∞ are free generators of

$$H_{\Lambda_\infty} = H \cap \coprod_{\Lambda_\infty} R = H \cap \Big(\bigcup_{(\Lambda,B)\in\mathcal{S}_0} \coprod_\Lambda R\Big) = \bigcup_{(\Lambda,B)\in\mathcal{S}_0} H \cap \coprod_\Lambda R = \bigcup_{(\Lambda,B)\in\mathcal{S}_0} H_\Lambda$$

(i.e. $\coprod_{B_\infty} R \to H_{\Lambda_\infty}$, $(\lambda_b) \mapsto \sum_b \lambda_b\, b$ is an isomorphism). Hence $(\Lambda_\infty, B_\infty)$ is an upper bound for $\mathcal{S}_0$. Thus by Zorns Lemma there exists a maximal element (Λ_0, B_0) of $\mathcal{S}$. Remains to show that $\Lambda_0 = J$. Otherwise choose $j \in J \setminus \Lambda_0$ and consider $\Lambda_1 := \{j\} \cup \Lambda_0$. Then $\coprod_{\Lambda_1} R = R \oplus \coprod_{\Lambda_0} R$ and since $H_{\Lambda_0} = H_{\Lambda_1} \cap \coprod_{\Lambda_0} R$ the inclusion $H_{\Lambda_1} \hookrightarrow \coprod_{\Lambda_1} R$ induces an injection $H_{\Lambda_1}/H_{\Lambda_0} \rightarrowtail \coprod_{\Lambda_1} R/\coprod_{\Lambda_0} R \cong R$. Since R is a PID there exists an $r \in R$ with $H_{\Lambda_1}/H_{\Lambda_0} \cong Rr \cong R$ and hence $H_{\Lambda_0} \oplus R \cong H_{\Lambda_1}$ since R is a free R-module. Let b_1 be the image of $(0,1)$ in H_{Λ_1}. Then $B_1 := B_0 \sqcup \{b_1\}$ are free generators of H_{Λ_1}, a contradiction to maximality. □

9.21 Double complex lemma. *Let $(C^{i,j})_{i,j\geq 0}$ be a double complex, i.e. we have given boundary operators $\partial_v : C^{i,j} \to C^{i+1,j}$ and $\partial_h : C^{i,j} \to C^{i,j+1}$ which satisfy $\partial_v^2 = 0$, $\partial_h^2 = 0$, and $\partial_h \circ \partial_v + \partial_v \circ \partial_h = 0$. Let $C^{-1,j} := \operatorname{Ker}(\partial_v : C^{0,j} \to C^{1,j})$ and $C^{i,-1} := \operatorname{Ker}(\partial_h : C^{i,0} \to C^{i,1})$ and $C^n := {}^{ab}\coprod_{i+j=n} C^{i,j}$ with $\partial : C^n \to C^{n+1}$ be given by $\partial_h + \partial_v$. Then $C^{-1,*}$, $C^{*,-1}$ and C^* are cochain complexes and $H^k(C^{*,-1}) \cong H^k(C^*) \cong H^k(C^{-1,*})$.*

Note that instead of anti-commutativity $\partial_h \circ \partial_v + \partial_v \circ \partial_h = 0$ we could assume commutativity $\partial_h \circ \partial_v = \partial_v \circ \partial_h$ if we replace $\partial_h^{i,j} : C^{i,j} \to C^{i,j+1}$ by $(-1)^i \partial_h^{i,j}$.

Proof. By symmetry it suffices to show $H^k(C^*) \cong H^k(C^{*,-1})$: Define a natural homomorphism $\varphi : H^k(C^{*,-1}) \to H^k(C^*)$ by $[a^{k,0}] \mapsto [a^{k,0} \oplus 0 \oplus \dots \oplus 0]$. Conversely let $x = [a^0 \oplus \dots \oplus a^k] \in H^k(C^*)$ with $a^i \in C^{k-i,i}$. We claim that if $a_{i+1} = \dots = a_k = 0$ for some $i > 0$ then we may also assume that $a^i = 0$: Then $\partial_h(a^i) = \operatorname{pr}_{k-i,i+1}(\partial(x)) = \operatorname{pr}_{k-i,i+1}(0) = 0$ and by exactness of the $(k-i)$-th row, there exists an $e \in C^{k-i,i-1}$ with $\partial_h(e) = a^i$. Then

$$\begin{aligned}
[a^0 \oplus \dots \oplus a^i \oplus 0 \oplus \dots \oplus 0] &- [a^0 \oplus \dots \oplus (a^{i-1} - \partial_v(e)) \oplus 0 \oplus \dots \oplus 0] \\
&= [\dots \oplus 0 \oplus \partial_v(e) \oplus a^i \oplus 0 \oplus \dots] \\
&= [\dots \oplus 0 \oplus \partial_v(e) \oplus \partial_h(e) \oplus 0 \oplus \dots] \\
&= [\partial(\dots \oplus 0 \oplus e \oplus 0 \oplus \dots)],
\end{aligned}$$

i.e. $[a^0 \oplus \dots \oplus a^i \oplus 0 \oplus \dots \oplus 0] = [a^0 \oplus \dots \oplus (a^{i-1} - \partial_v(e)) \oplus 0 \oplus \dots \oplus 0]$. It is easy to check that this gives the required isomorphism. □

9.22 Lemma. *The functor* $\operatorname{Hom}_R(M, \_) : \underline{R\text{-}Mod} \to \underline{R\text{-}Mod}$ *is left exact.*

Proof. Let $0 \to N' \xrightarrow{i} N \xrightarrow{p} N'' \to 0$ be a short exact sequence and consider the sequence

$$0 \to \mathrm{Hom}_R(M, N') \xrightarrow{i_*} \mathrm{Hom}_R(M, N) \xrightarrow{p_*} \mathrm{Hom}_R(M, N'').$$

It is exact at $\mathrm{Hom}_R(M, N')$, since i_* is obviously injective.

It is exact at $\mathrm{Hom}_R(M, N)$, since $\varphi \in \mathrm{Hom}_R(M, N)$ is in $\ker(p_*) \Leftrightarrow 0 = p_*(\varphi) = p \circ \varphi$ $\Leftrightarrow \mathrm{im}(\varphi) \subseteq \ker(p) = \mathrm{im}(i) \Leftrightarrow \varphi$ factors to a homomorphism $\tilde{\varphi} : M \to N'$ over $i : M' \to M \Leftrightarrow \varphi \in \mathrm{im}(i^*)$. □

In general, p_* will not be onto, since this would mean, that every homomorphism $\varphi : M \to N''$ can be lifted along $p : N \to N''$ to a morphism $\tilde{\varphi} : M \to N$.

▷ **9.23 Theorem.** *There are functors $\mathrm{Ext}^n_R : \underline{R\text{-}Mod}^{op} \times \underline{R\text{-}Mod} \to \underline{AGru}$ for $n \in \mathbb{Z}$ (called the* RIGHT-DERIVED FUNCTORS *of* Hom*) and natural transformations δ such that:*

1. $\mathrm{Ext}^n_R(M, N) = 0$ *for* $n < 0$.
2. $\mathrm{Ext}^0_R \cong \mathrm{Hom}$.
3. $\mathrm{Ext}^n_R(M, N) = 0$ *for all* $n > 0$ *if* M *is projective or* N *is injective.*
4. *For every short exact sequence* $0 \to M' \to M \to M'' \to 0$ *there is a long exact sequence*

$$\cdots \to \mathrm{Ext}^n_R(M'', N) \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M', N) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_R(M'', N) \to \cdots .$$

For every short exact sequence $0 \to N' \to N \to N'' \to 0$ *there is a long exact sequence*

$$\cdots \to \mathrm{Ext}^n_R(M, N') \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M, N'') \xrightarrow{\delta} \mathrm{Ext}^{n+1}_R(M, N') \to \cdots .$$

For fixed N the functor $\mathrm{Ext}^*_R(\_, N)$ together with the natural transformation δ is up to isomorphisms uniquely determined by 1*-*4*. And similarly for each fixed M.*

▽

Proof.
(1) By 9.16 there is an injective resolution I of N:

$$0 \to N \to I_0 \to I_1 \to I_2 \to \cdots$$

Applying $\mathrm{Hom}_R(M, \_)$ to I (only!) gives a cochain complex

$$0 \to \mathrm{Hom}_R(M, I_0) \to \mathrm{Hom}_R(M, I_1) \to \mathrm{Hom}_R(M, I_2) \to \cdots$$

and we define $\mathrm{Ext}^k_R(M, N) := H^k(\mathrm{Hom}_R(M, I_*)) := H_{-k}(\mathrm{Hom}_R(I_{-*}))$.

By 9.19 and 8.23 the groups $\mathrm{Ext}^p_R(M, N)$ are independent on the injective resolution of N.

(2) By definition $\mathrm{Ext}^0_R(M, N)$ is just the kernel of $\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, I_1)$ and by left exactness in 9.22 the sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, I_0) \to \mathrm{Hom}_R(M, I_1) \to \cdots$$

is exact, hence this kernel is isomorphic to $\mathrm{Hom}_R(M, N)$.

(3) If N is injective then we may take $I_0 := N$ and $I_k := 0$ for $k > 0$ as injective resolution. Hence $\mathrm{Hom}_R(M, I_k) = 0$ and thus also $\mathrm{Ext}^k_R(M, N) = H^k(\mathrm{Hom}_R(M, I)) = 0$ for $k > 0$.

4) Let $0 \leftarrow M'' \leftarrow M \leftarrow M' \leftarrow 0$ be short exact and I be an injective resolution of N. Since I_k is injective we have short exact sequences

$$0 \to \mathrm{Hom}_R(M'', I_k) \to \mathrm{Hom}_R(M, I_k) \to \mathrm{Hom}_R(M', I_k) \to 0$$

and this gives a short exact sequence of cochain complexes since Hom_R is a bifunctor:

$$0 \to \mathrm{Hom}_R(M'', I) \to \mathrm{Hom}_R(M, I) \to \mathrm{Hom}_R(M', I) \to 0$$

By 7.30 we get a long exact sequence in (co)homology:

$$\cdots \to \mathrm{Ext}_R^k(M'', N) \to \mathrm{Ext}_R^k(M, N) \to \mathrm{Ext}_R^k(M', N) \xrightarrow{\delta} \mathrm{Ext}_R^{k+1}(M'', N) \to \cdots.$$

(Projective construction) Alternatively we could use a projective resolution P of M instead of an injective resolution I of N in order to define $\mathrm{Ext}_R^k(M, N)$ as $H^k(\mathrm{Hom}_R(P_*, N))$. That this gives naturally isomorphic functors to those defined before is seen as follows: Consider the double-complex $(\mathrm{Hom}_R(P_i, I_j))_{i,j}$. Since $\mathrm{Hom}_R(\_, I_j)$ and $\mathrm{Hom}_R(P_i, \_)$ are left-exact, the complex $C^{*,-1}$ is $\mathrm{Hom}_R(M, I_*)$ and $C^{-1,*}$ is $\mathrm{Hom}_R(P_*, N)$ (cf. 2). Thus by 9.21 the two definitions are isomorphic.

In particular this shows that 3 is valued for projective M and the second long exact sequence in 4 holds as well.

(Uniqueness) We proceed by induction on k. For $k \le 0$ we have uniqueness by (1) and (2). So we assume that we have two sequences of functors Ext_R^*, which are naturally isomorphic till order k, and we have natural connecting morphisms. Then a diagram chase starting at a short exact sequence $0 \to M' \to M \to M'' \to 0$ with free M shows that they are also isomorphic in order $k+1$ on M'':

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & \mathrm{Ext}_R^k(M', N) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Ext}_R^{k+1}(M'', N) \arrow[r] \arrow[d, dotted, "\cong"] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \overline{\mathrm{Ext}}_R^k(M', N) \arrow[r, "\cong"] & \overline{\mathrm{Ext}}_R^{k+1}(M'', N) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

□

▷ **9.24 Lemma.** $\mathrm{Ext}_R^k(M, N) = 0$ *for* $k \ge 2$, *arbitrary* M *and* N, *and any PID* R *(in particular, for* $R := \mathbb{Z}$*).*

▽ **Proof.** By 9.20 we may use a projective resolution P with $P_k = 0$ for all $k \ge 2$. Hence $\mathrm{Hom}(P_k, N) = 0$ and thus also $\mathrm{Ext}^k(M, N)$. □

▷ **9.25 Lemma.** *A module* N *is injective* $\Leftrightarrow$ $\mathrm{Ext}_R^k(M, N) = 0$ *for all* M *and* $k = 1$ *(or all* $k \ge 1$*).*

▽ **Proof.** N injective $\Rightarrow 0 \to N \to N \to 0$ is an injective resolution $\Rightarrow \mathrm{Hom}(M, I_k) = 0$ for $k \ge 1 \Rightarrow \mathrm{Ext}^k(M, N) = 0$ for $k \ge 1 \Rightarrow \mathrm{Ext}^1(M, N) = 0 \Rightarrow \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to 0$ is exact for short exact sequences $0 \to M' \to M \to M'' \to 0$, i.e. N is injective. □

▷ **9.26 Lemma.** *A module* M *is projective* $\Leftrightarrow$ $\mathrm{Ext}_R^k(M, N) = 0$ *for all* N *and* $k = 1$ *(or all* $k \ge 1$*).*

▽ **Proof.** M projective $\Rightarrow 0 \to M \to M \to 0$ is a projective resolution $\Rightarrow \mathrm{Hom}(P_k, N) = 0$ for $k \ge 1 \Rightarrow \mathrm{Ext}^k(M, N) = 0$ for $k \ge 1 \Rightarrow \mathrm{Ext}^1(M, N) = 0 \Rightarrow \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0$ is exact for short exact sequences $0 \to M' \to M \to M'' \to 0$, i.e. M is projective. □

▷ **9.27 Lemma.** *A ring R is* SEMISIMPLE *(i.e. every short exact sequence of R-moduls splits, equivalently, is semisimple as module over itself)* $\Leftrightarrow$ $\operatorname{Ext}^k_R(M,N)=0$ *for all R-modules M and N and $k=1$ (or even all $k\geq 1$).*

▽ **Proof.** R semisimple iff every short exact sequence splits, i.e. every R-module N is injective. By 9.26 this is equivalent to $\operatorname{Ext}^k_R(M,N)=0$ for $k=1$ (or even all $k\geq 1$). □

9.28 Remark. Is every abelian group A with $\operatorname{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z})=0$ free abelian? This is undecideable in ZFC by [**16, 17, 18**]

▷ **9.29 Examples.**

- $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_q,\mathbb{Z})\cong\mathbb{Z}_q$. The exact sequence $0\to q\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}_q\to 0$ is a projective resolution, hence

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_q,G) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},G) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathbb{Z}}(q\mathbb{Z},G) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_q,G) \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z},G) \arrow[d, equal] \\
& G \arrow[r, "q\cdot"] & G & & 0
\end{tikzcd}$$

  is exact and thus $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_q,G)\cong G/qG$.

- For $R:=\mathbb{Z}_{p^2}$ is $\operatorname{Ext}^k_R(\mathbb{Z}_p,\mathbb{Z}_p)=\mathbb{Z}_p$ for all $k\geq 0$:
  A projective resolution P is $\cdots\to R\to R\to R\twoheadrightarrow\mathbb{Z}_p\to 0$, where $\partial:R\to R$ is given by $1+p^2\mathbb{Z}\mapsto p+p^2\mathbb{Z}$, i.e. $[k]\mapsto[p\,k]$. Then $\operatorname{Hom}_R(P_k,N)=\operatorname{Hom}_R(R,N)=N$ and $\partial^*=p:N\to N$, i.e. $\partial^*=0$ for $N:=\mathbb{Z}_p$. Thus $\operatorname{Ext}^k_R(\mathbb{Z}_p,\mathbb{Z}_p)=\operatorname{Hom}_R(R,\mathbb{Z}_p)=\mathbb{Z}_p$.

9.30 Universal coefficient theorem for cohomology.
Let R be a PID, C a free chain complex over R, and M an R-module. Then there are splitting natural short exact sequences:

$$0\to\operatorname{Ext}^1_R(H_{q-1}(C),M)\to H^q(C,M)\to\operatorname{Hom}(H_q(C),M)\to 0$$

▽ **Proof.** We proceed as in 9.2:

$$\begin{tikzcd}
0 \arrow[r] & Z_q \arrow[r, hook, "j"'] & C_q \arrow[l, dotted, bend right, "p"'] \arrow[r, two heads, "\partial"'] & B_{q-1} \arrow[l, dotted, bend right, "\boxed{9.20}"'] \arrow[r] & 0 \\
0 & \mathrm{Hom}(Z_q, M) \arrow[l] \arrow[r, dotted, bend left, "p^*"] & \mathrm{Hom}(C_q, M) \arrow[l, tail, "j^*"] & \mathrm{Hom}(B_{q-1}, M) \arrow[l, tail, "\partial^*"] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & H^q(B,M) \arrow[l, "j^*"'] \arrow[d, equal] & H^q(Z,M) \arrow[l, "\delta^*"'] \arrow[d, equal] & H^q(C,M) \arrow[l, "(j^*)_*"'] \arrow[dd, equal] & H^{q-1}(B,M) \arrow[l, "(\partial^*)_*"'] \arrow[d, equal] & H^{q-1}(Z,M) \arrow[l, "\delta^*"'] \arrow[d, equal] & \cdots \arrow[l] \\
& \mathrm{Hom}(B_q, M) & \mathrm{Hom}(Z_q, M) \arrow[l, "i^*"'] & & \mathrm{Hom}(B_{q-1}, M) \arrow[d, two heads] & \mathrm{Hom}(Z_{q-1}, M) \arrow[l, tail, "i^*"'] & \\
& 0 & \mathrm{Ker}\, i^* \arrow[l] \arrow[u, hook] \arrow[d, equal] & H^q(C,M) \arrow[l, dotted, two heads] & \mathrm{Cok}\, i^* \arrow[l, dotted, tail] \arrow[d, equal] & 0 \arrow[l] & \\
& & \mathrm{Hom}(H_q, M) & & \mathrm{Ext}^1(H_{q-1}, M) & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \mathrm{Ext}^1(H_{q-1}, M) \arrow[l] & \mathrm{Hom}(B_q, M) \arrow[l, tail] & & \mathrm{Hom}(Z_q, M) \arrow[ll, "i^*"'] \arrow[dl, two heads, "i^*"] & \mathrm{Hom}(H_q, M) \arrow[l, tail] & 0 \arrow[l] \\
& & & \mathrm{Bild}(i^*) \arrow[ul, hook] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & B_q \arrow[r, hook, "i"'] & Z_q \arrow[r, two heads, "\pi"] & H_q \arrow[r] & 0
\end{tikzcd}$$

A splitting for the sequence is given by $\mathrm{Hom}(H_q, M) \ni \varphi \mapsto [\varphi \circ \pi \circ p] \in H^q(C, M)$. □

▷ **9.31 Proposition. Ext^1 via extensions.** $\mathrm{Ext}^1(M, N) \cong \mathrm{Ext}(M, N)$, *the set of isomorphy classes of extensions of M with N.*

▽ **Proof.** Let $\xi : 0 \to A \to B \to C \to 0$ be short exact. Then $0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(A, A) \to \mathrm{Ext}^1(C, A) \to \cdots$ is exact by 9.23.4 and we may consider the image (denoted $\Psi(\xi)$) of $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ in $\mathrm{Ext}^1(C, A)$.

(Ψ is well-defined) Two extensions $A \to B \to C$ and $A \to B' \to C$ are called EQUIVALENT, if a homomorphism (hence isomorphism by 7.22) $\varphi : B \to B'$ exists, such that

$$\begin{tikzcd}
0 & C \arrow[l] \arrow[d, equal] & B \arrow[l] \arrow[d, dotted, "\varphi"'] & A \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\
0 & C \arrow[l] & B' \arrow[l] & A \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The long exact sequence for Ext^* is natural

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(C, A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(B, A) \arrow[r] & \mathrm{Hom}(A, A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1(C, A) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(C, A) \arrow[r] & \mathrm{Hom}(B', A) \arrow[r] \arrow[u, "\varphi^*"] & \mathrm{Hom}(A, A) \arrow[r] & \mathrm{Ext}^1(C, A) \arrow[r] & \cdots
\end{tikzcd}$$

hence the images of id_A in $\mathrm{Ext}^1(C, A)$ are the same.

(Ψ is onto) Let $0 \to R \to P \to C \to 0$ be a short exact sequence with projective P. Then $0 \to \mathrm{Hom}(C,A) \to \mathrm{Hom}(P,A) \to \mathrm{Hom}(R,A) \to \mathrm{Ext}^1(C,A) \to \mathrm{Ext}^1(P,A) = 0$ is exact. So for $\psi \in \mathrm{Ext}^1(C,A)$ there exists an inverse image $\varphi : R \to A$. Let $B =: \varphi_*(P)$ be the push-out of $R \to P$ and φ. We get obvious morphisms to make the following diagram commutative with exact rows:

$$\begin{tikzcd}
0 & C \arrow[l] & B \arrow[l, dotted] & A \arrow[l, dotted] & 0 \arrow[l] \\
0 & C \arrow[l] \arrow[u, equal] & P \arrow[l] \arrow[u, dotted] & R \arrow[l] \arrow[u, "\varphi"'] & 0 \arrow[l]
\end{tikzcd}$$

That it is exact at A follows by this property of the push-out (see 9.7) and exactness at $\varphi_*(P)$ can be seen from its construction:

$$\begin{tikzcd}
0 & C \arrow[l] & & \varphi_*(P) \arrow[ll, two heads, dotted, "g'"'] & A \arrow[l, tail, dotted, "f'"'] \arrow[dl, "\mathrm{inj}_2"] & 0 \arrow[l] \\
& & P \arrow[dl, "g"] & P \oplus A \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[u, two heads, "\pi"] & & \\
0 & C \arrow[l] \arrow[uu, equal] & & P \arrow[ll, two heads, "g"] \arrow[u, "\mathrm{inj}_1"'] & R \arrow[l, tail, "f"] \arrow[uu, "\varphi"'] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{aligned}
\ker g' &= \{p \oplus a + \ker \pi : g(p) = 0\} = \{p \oplus a + \ker \pi : p \in f(R)\} \\
&= \{f(r) \oplus a : r \in R, a \in A\} = \{0 \oplus (a + \varphi(r)) + \ker \pi : r \in R, a \in A\} = f'(A)
\end{aligned}$$

From this we get:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C,A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(B,A) \arrow[r] \arrow[d] & \mathrm{Hom}(A,A) \arrow[r] \arrow[d, "\varphi^*"'] & \mathrm{Ext}^1(C,A) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(C,A) \arrow[r] & \mathrm{Hom}(P,A) \arrow[r] & \mathrm{Hom}(R,A) \arrow[r] & \mathrm{Ext}^1(C,A) \arrow[r] & \cdots
\end{tikzcd}$$

And hence $\Psi(A \to B \to C)$ is by definition the image of id_A in $\mathrm{Ext}^1(C,A)$ and this is also the image ψ of $\varphi^*(\mathrm{id}_A) = \varphi$.

(Ψ is injective) Let the image of two extensions $A \to B \to C$ and $A \to B' \to C$ be the same and let $P \to C \to 0$ be a projective resolution of C. By 9.18 we get morphisms

$$\begin{tikzcd}
0 & C \arrow[l] & B \arrow[l, two heads, "g"'] & A \arrow[l, tail, "f"'] & 0 \arrow[l] \\
0 & C \arrow[l] \arrow[u, equal] \arrow[d, equal] & P_0 \arrow[l, two heads] \arrow[u, dotted, two heads, "\varphi"] \arrow[d, dotted, two heads, "\varphi'"] & P_1 \arrow[l, "\delta"] \arrow[u, dotted, "\psi"'] \arrow[d, dotted, "\psi'"'] & \cdots \arrow[l] \\
0 & C \arrow[l] & B' \arrow[l, two heads] & A \arrow[l, tail] & 0 \arrow[l]
\end{tikzcd}$$

and by taking P_0 sufficiently large (i.e. a free P_0 wuch that $P_0 \to C \oplus B \oplus B'$ onto), we may assume that φ and φ' are onto. By replacing P_1 with $R_1 := \ker \delta$, we may assume that $C \leftarrow P_0 \leftarrow R_1$ is short exact.

Now consider

$$\begin{tikzcd}
& \mathrm{Hom}(A,A) \arrow[r] \arrow[d, shift right, "(\psi)^*"'] \arrow[d, shift left, "(\psi')^*"] & \mathrm{Ext}^1(C,A) \arrow[d, equal] & \\
\mathrm{Hom}(P_0,A) \arrow[r, "\delta^*"] & \mathrm{Hom}(P_1,A) \arrow[r, two heads] & \mathrm{Ext}^1(C,A) \arrow[r] & \mathrm{Ext}^1(P_0,A)=0
\end{tikzcd}$$

By assumption the images of id_A in $\mathrm{Ext}^1(C,A)$ are the same, hence also the images of $(\psi)^*(\mathrm{id}_A)=\psi$ and $(\psi')^*(\mathrm{id}_A)=\psi'$ in $\mathrm{Ext}^1(C,A)$, i.e. $\psi'-\psi\in\ker(\mathrm{Hom}(P_1,A)\to\mathrm{Ext}^1(C,A))=\mathrm{im}(\delta^*)$. Thus there exists a $\chi\in\mathrm{Hom}(P_0,A)$ with $\psi'-\psi=\delta^*(\chi)=\chi\circ\delta$. If we replace φ by $\bar\varphi:=\varphi+f\circ\chi\in\mathrm{Hom}(P_0,B)$ and ψ by $\bar\psi=\psi+\chi\circ\delta=\psi'\in\mathrm{Hom}(P_1,A)$ we get the commutative diagram:

$$\begin{tikzcd}
0 & C \arrow[l] & B \arrow[l, two heads, "g"'] & A \arrow[l, tail, "f"'] & 0 \arrow[l] \\
0 & C \arrow[l] \arrow[u, equal] & P_0 \arrow[l, two heads] \arrow[u, "\bar\varphi"] \arrow[ur, dashed, "\chi"] & P_1 \arrow[l, "\delta"'] \arrow[u, "\psi'=\bar\psi"'] &
\end{tikzcd}$$

We have $\ker\varphi'=\ker\bar\varphi$: In fact $p_0\in\ker\varphi'\Rightarrow g(\varphi'(p_0))=0$, i.e. $p_0=\delta(p_1)$ for some $p_1\in P_1$. So $0=\varphi'(\delta(p_1))=(f\circ\psi')(p_1)\Leftrightarrow 0=\psi'(p_1)=\bar\psi(p_1)\Leftrightarrow 0=(f'\circ\bar\psi)(p_1)=\bar\varphi(\delta(p_1))$.
Furthermore, φ', φ and thus $\bar\varphi$ are onto: In fact, ψ (and equally ψ') is onto, since for $a\in A$ we get $p_0\in P_0$ with $\varphi(p_0)=f(a)$ and hence $0=g(f(a))=g(\varphi(p_0))$, so $p_0\in\mathrm{im}\,\delta$, i.e. $\exists\ p_1\in P_1$: $\delta(p_1)=p_0$, hence $f(a)=\varphi(p_0)=\varphi(\delta(p_1))=f(\psi(p_1))$, and so $a=\psi(p_1)$. Now let $\varphi(p_0)=b$ and choose p_1 with $\bar\psi(p_1)=\psi'(p_1)=-\chi(p_0)$. Then $\bar\varphi(p_0+\delta p_1)=\bar\varphi(p_0)+\bar\varphi(\delta(p_1))=\varphi(p_0)+f(\chi(p_0))+f(\bar\psi(p_1))=b$. So we get a morphism between $B\cong P_0/\ker\bar\varphi$ and $B'\cong P_0/\ker\varphi'$ which induces on A and on C the identity. Thus the two extensions are equivalent. $\square$

9.32 Definition. Ext **as** $\underline{AGru}$**-valued functor.** The functorial properties of Ext are:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, dotted] \arrow[u, equal] & \gamma^*(B) \arrow[r, dotted] \arrow[u, dotted] & C' \arrow[r] \arrow[u, "\gamma"'] & 0
\end{tikzcd}$$

where $\gamma^*(B)$ denotes the pull-back and $\mathrm{Ext}(A,\gamma):\mathrm{Ext}(C,A)\to\mathrm{Ext}(C',A)$ maps $A\to B\to C$ to $A\to\gamma^*(B)\to C$. Similarly,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r, dotted] & \alpha_*(B) \arrow[r, dotted] & C \arrow[r] & 0
\end{tikzcd}$$

where $\alpha_*(B)$ denotes the push-out and $\mathrm{Ext}(\alpha,C):\mathrm{Ext}(C,A)\to\mathrm{Ext}(C,A')$ maps $A\to B\to C$ to $A\to\alpha_*(B)\to C$.
In fact, let $\psi\in\mathrm{Ext}^1(C,A)$ correspond to $\xi:A\to B=\varphi^*(P)\to C$, where $0\to R\to P\to C\to 0$ is short exact with projective P and φ an inverse image of ψ with respect to $\mathrm{Hom}(R,A)\twoheadrightarrow\mathrm{Ext}^1(C,A)$. By naturality $\mathrm{Ext}^1(C,\alpha)(\psi)$ is the image of $\alpha_*(\varphi)=\alpha\circ\varphi$ with respect to $\mathrm{Hom}(R,A')\twoheadrightarrow\mathrm{Ext}^1(C,A')$ and the corresponding short exact sequence $\mathrm{Ext}(C,\alpha)(\xi)$ is the pushout $(\alpha\circ\varphi)_*(P)=\alpha_*(\varphi_*(P))=\alpha_*(B)$.

That $\mathrm{Ext} : \underline{R\text{-}Mod}^{\mathrm{op}} \times \underline{R\text{-}Mod} \to \underline{Set}$ is a bifunctor follows also from

$$\begin{tikzcd}
 & A \arrow[rrr] \arrow[dr, "\alpha"] \arrow[ddl, equal] & & & B \arrow[rrr] \arrow[dr] & & & C \arrow[dr, equal] & \\
 & & A' \arrow[rrr] \arrow[ddl, equal] & & & \alpha_* B \arrow[rrr] & & & C \\
A \arrow[rrr] \arrow[dr, "\alpha"] & & & \gamma^* B \arrow[rrr] \arrow[d] \arrow[uur] & & & C' \arrow[uur, "\gamma"] \arrow[dr, equal] & & \\
 & A' \arrow[rr] & & \alpha_*\gamma^* B \arrow[r, dotted, "\cong"'] \arrow[uurr, dotted] & \gamma^*\alpha_* B \arrow[rrr] \arrow[uur] & & & C' \arrow[uur, "\gamma"] &
\end{tikzcd}$$

where the morphism $\alpha_*\gamma^* B \to \gamma^*\alpha_* B$ is obtained by the universal properties and it is an isomorphism by the 5'Lemma 7.22.

The group structure on $\mathrm{Ext}(C, A)$ induced by the bijection of 9.31 is given by the Baer-sum of extensions, which can be defined as follows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_2"] & B_2 \arrow[r, "g_2"] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r, "f_1 \oplus f_2"] & B_1 \oplus B_2 \arrow[r, "g_1 \oplus g_2"] & C \oplus C \arrow[r] & 0 \\
0 \arrow[r] & A \oplus A \arrow[r, dotted] \arrow[u, equal] \arrow[d, "\Sigma"'] & \Delta^*(B_1 \oplus B_2)) \arrow[r, dotted] \arrow[u, dotted] \arrow[d, dotted] & C \arrow[r] \arrow[u, "\Delta"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, dotted] & \Sigma_*(\Delta^*(B_1 \oplus B_2) \arrow[r, dotted] & C \arrow[r] & 0
\end{tikzcd}$$

or, equivalently, by

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\Sigma"] & B_1 \oplus B_2 \arrow[r] \arrow[d, dotted] & C \oplus C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, dotted] & \Sigma_*(B_1 \oplus B_2) \arrow[r, dotted] & C \oplus C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, dotted] \arrow[u, equal] & \Delta^*(\Sigma_*(B_1 \oplus B_2)) \arrow[r, dotted] \arrow[u, dotted] & C \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

For this note, that the addition on $\mathrm{Hom}(M, I)$ can be described by

$$+ : \mathrm{Hom}(M, I) \times \mathrm{Hom}(M, I) \to \mathrm{Hom}(M \times M, I \times I) \xrightarrow{\mathrm{Hom}(\Delta,\Sigma)} \mathrm{Hom}(M, I),$$

where $\Delta : M \to M \times M$ is given by $x \mapsto (x, x)$ and $\Sigma : I \times I \to I$ by $(x_1, x_2) \mapsto x_1 + x_2$. Thus addition on $\mathrm{Ext}^1(M, N)$ is also the composite

$$\mathrm{Ext}^1(M, N) \times \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M \times M, N \times N) \xrightarrow{\mathrm{Ext}^1(\Delta,\Sigma)} \mathrm{Ext}^1(M, N).$$

On Ext this sends two extensions $A \to B_1 \to C$ and $A \to B_2 \to C$ first to $A \oplus A \to B_1 \oplus B_2 \to C \oplus C$ and then to $\Delta^*(\Sigma_*(B_1 \oplus B_2))$.

The Baer-sum can also be constructed by taking the pull-back PB of g_1 and g_2 and then the coequalizer of $(f_1, 0), (0, f_2) : A \to PB$:

$$\begin{tikzcd}
 & & B_1 \arrow[dr, "g_1"] & \\
A \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & PB \arrow[ur] \arrow[dr] \arrow[r, two heads] & \mathrm{CoEqu} \arrow[r, dotted] & C \\
 & & B_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

This follows since the following two types of cones correspond to each other

$$\begin{tikzcd}
B_1 \oplus B_2 \arrow[r, "g_1 \oplus g_2"] & C \oplus C \\
X \arrow[u, dotted, "(h_1, h_2)"] \arrow[r, dotted, "k"'] & C \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "g_1"] & C \\
X \arrow[u, dotted, "h_1"] \arrow[ur, dotted, "k"'] \arrow[r, dotted, "h_2"'] & B_2 \arrow[u, "g_2"']
\end{tikzcd}$$

and also the following two types of cocones

$$\begin{tikzcd}
A \oplus A \arrow[r, "(f_1, f_2)"] \arrow[d, "\Sigma"] & B \arrow[d, dotted, "h_2"] \\
A \arrow[r, dotted, "h_1"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[dr, dotted, bend right, "h_1"'] & B \arrow[d, dotted, "h_2"] \\
 & X
\end{tikzcd}$$

9.33 Definition. Group-Cohomology. Let G be a (not necessarily abelian) group and M a G-module, i.e. and abelian group together with an action (i.e. group-homomorphism) $G \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$. Then we are interested in the submodule $M^G := \{x \in M : g \cdot x = x \ \forall g \in G\}$ of joint fixed points (i.e. the G-invariant elements). We can extend the group-action of G on M to a ring-action of the group ring $\mathbb{Z}[G]$ of G on M, i.e. the free abelian group with G as set of generators and with convolution as ring-multiplication

$$(x \star y)(g) := \sum_{hk=g} x(h) y(k) = \sum_{h \in G} x(h)\, y(h^{-1}x),$$

by

$$x \cdot m := \sum_{g \in G} x(g)\, g \cdot m.$$

Thus G-group-modules are in 1-1 correspondence with $\mathbb{Z}[G]$-ring-modules.

We have $M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$, where we consider $\mathbb{Z}$ as trivial $\mathbb{Z}[G]$-module: In fact, $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) \Leftrightarrow \forall g \in G : \varphi(k) = \varphi(g \cdot k) = g \cdot \varphi(k)$, i.e. $k\, \varphi(1) = \varphi(k) \in M^G$.

Thus $M \mapsto M^G$ is a left-exact functor $\underline{G\text{-}Mod} \to \underline{AGru}$ and we define the cohomology of G with coefficients in M as

$$H^k(G, M) := \mathrm{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

In particular, we have $H^0(G, M) = M^G$ and $H^1(G, M) = \mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$, the group of isomorphy classes of module extensions of $\mathbb{Z}$ with M.

In general we can use the projective resolution

$$\cdots \to \mathbb{Z}[G^{n+1}] \xrightarrow{\partial} \mathbb{Z}[G^n] \to \cdots \to \mathbb{Z}[G] \xrightarrow{\Sigma} \mathbb{Z} \to 0,$$

where the action of G on the generators of $\mathbb{Z}[G^{n+1}]$ is given by

$$g \cdot (g_0, \ldots, g_n) := (gg_0, \ldots, gg_n)$$

and ∂ is given by

$$\partial(g_0,\dots,g_n) := \sum_{i=0}^{n}(-1)^i(g_0,\dots,\widehat{g_i},\dots,g_n).$$

Thus $\operatorname{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z},M)$ is defined as the cohomology of

$$\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^*],M) = \Big\{\varphi\in M^{G^*} : \varphi(gg_0,\dots,gg_n) = g\cdot\varphi(g_0,\dots,g_n)\Big\}$$

with respect to the coboundary operator

$$\begin{aligned}\partial^* &: \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^n],M)\to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}],M),\\ \partial^*\varphi(g_0,\dots,g_n) &= \sum_{i=0}^{n}(-1)^i\varphi(g_0,\dots,\widehat{g_i},\dots,g_n).\end{aligned}$$

By the defining relation for $\varphi\in\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^*],M)$ it is enough to know

$$\bar\varphi(g_1,\dots,g_n) := \varphi(1,g_1,g_1g_2,\dots,g_1\dots g_n),$$

since

$$\begin{aligned}\varphi(g_0,g_1,\dots,g_n) &= g_0\cdot\varphi(1,g_0^{-1}g_1,g_0^{-1}g_2 = g_0^{-1}g_1g_1^{-1}g_2,\dots,g_0^{-1}g_n)\\ &= g_0\cdot\bar\varphi(g_0^{-1}g_1,\dots,g_{n-1}^{-1}g_n).\end{aligned}$$

The coboundary operator than takes the form

$$\begin{aligned}\partial^*\bar\varphi(g_1,\dots,g_n) &= \partial^*\varphi(1,g_1,\dots,g_1\cdots g_n)\\ &= \varphi(g_1,g_1g_2,\dots,g_1\cdots g_n) + \sum_{i=1}^{n}(-1)^i\varphi(1,g_1,\dots,\widehat{g_1\cdots g_i},\dots,g_1\cdots g_n)\\ &= g_1\cdot\varphi(1,g_2,\dots,g_2\cdots g_n) + \sum_{i=1}^{n}(-1)^i\varphi(1,g_1,\dots,\widehat{g_1\cdots g_i},\dots,g_1\cdots g_n)\\ &= g_1\cdot\bar\varphi(g_2,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i\bar\varphi(g_1,\dots,g_ig_{i+1},\dots,g_n) + (-1)^n\bar\varphi(g_1,\dots,g_{n-1}).\end{aligned}$$

Let us now determine $H^2(G,M)$:

$$\begin{tikzcd}\cdots \arrow[r] & \operatorname{Hom}_G(\mathbb{Z}[G^2],M) \arrow[r, "\partial^*"] \arrow[d, equal] & \operatorname{Hom}_G(\mathbb{Z}[G^3],M) \arrow[r, "\partial^*"] \arrow[d, equal] & \operatorname{Hom}_G(\mathbb{Z}[G^4],M) \arrow[r] \arrow[d, equal] & \cdots\\ & M^G & M^{G^2} & M^{G^3} & \end{tikzcd}$$

Thus

$$\begin{aligned}&H^2(G,M)\cong Z^2(G,M)/B^2(G,M),\text{ where}\\ &Z^2(G,M) = \{\bar\varphi : G^2\to M : g_1\cdot\bar\varphi(g_2,g_3) - \bar\varphi(g_1g_2,g_3) + \bar\varphi(g_1,g_2g_3) - \bar\varphi(g_1,g_2)\}\\ &B^2(G,M) = \{(g_1,g_2)\mapsto g_1\cdot\bar\psi(g_2) - \bar\psi(g_1g_2) + \bar\psi(g_1) : \bar\psi : G\to M\}\end{aligned}$$

9.34 Group extensions

We consider (equivalence classes of) short exact sequenes $N \xrightarrow{i} H \xrightarrow{p} G$ of (not necessarily abelian) groups. By choosing an inverse image $s(g)\in p^{-1}(g)$ for every $g \in G$ we get a mapping $s : H \leftarrow G$ right inverse to p. Using this we have $H \cong N \times G$ via

$$\begin{aligned} H \leftarrow N \times G, &\quad i(n) \cdot s(g) \leftarrow\!\shortmid (n, g) \\ H \to N \times G, &\quad h \mapsto (h \cdot s(p(h))^{-1}, p(h)) \quad \text{und} \end{aligned}$$

The group multiplication on $N \times G$ induced from H is thus given by

$$(n_1, g_1) \cdot (n_2, g_2) = (i(n_1) \cdot s(g_1) \cdot i(n_2) \cdot s(g_2) \cdot s(g_1 \cdot g_2)^{-1}, g_1 \cdot g_2).$$

If we put

$$\begin{aligned} c : G \times G \to N, &\quad i(c(g, g'))) := s(g) \cdot s(g') \cdot s(g \cdot g')^{-1} \text{ und} \\ \rho : G \to \operatorname{Aut}(N), &\quad i(\rho(g)(n)) := s(g) \cdot i(n) \cdot s(g)^{-1}, \end{aligned}$$

then the multiplication is given by

$$(n, g) \cdot (n', g') = (n \cdot \rho(g)(n') \cdot c(g, g'), g \cdot g').$$

Let $s' : H \leftarrow G$ be another section of $p : H \to G$. Then there exists a uniquely determined mapping $\tau : G \to N$ with $s'(g) = i(\tau(g) \cdot s(g)$. The corresponding mappings $c' : G \times G \to N$ and $\rho' : G \to \operatorname{Aut}(N)$ is then given by

$$\begin{aligned} \rho'(g)(n) &= \tau(g) \cdot \rho(g)(n) \cdot \tau(g)^{-1} \\ c'(g, g') &= \tau(g) \cdot \rho(g)(\tau(g')) \cdot c(g, g') \cdot \tau(g \cdot g')^{-1}. \end{aligned}$$

Let $N \to H' \to G$ be another extension, which is isomorphic via $\varphi : H \to H'$. Then φ can be described as

$$\varphi : (n, g) \mapsto (n \cdot \tau(g), g), \quad N \times G \cong H \to H' \cong N \times G,$$

where we use the section $s' := \varphi \circ s$ for the second extension.

9.35 abelian extensions

Let us now restrict to the case, where N is abelian and we write it additively. Then we get an action ρ of G on N defined by

$$\rho(p(h))(n) := i^{-1}(h \cdot i(n) \cdot h^{-1}).$$

$$\begin{tikzcd} N \arrow[r, hook, "i"] & H \arrow[r, two heads, "p"] \arrow[d, "\mathrm{conj}"'] & G \arrow[dl, dotted, "\rho"] \\ & \operatorname{Aut}(N) & \end{tikzcd}$$

With other words, the previously defined ρ does not depend on the section s:

In fact, $p(h) = p(h')$ implies $h^{-1} \cdot h' = i(n')$ for some $n' \in N$, hence $h' \cdot i(n) \cdot (h')^{-1} = h \cdot i(n') \cdot i(n) \cdot i(n')^{-1} \cdot h^{-1} = h \cdot i(n' + n - n') \cdot h^{-1}$. Thus the definition of ρ gives a well-defined representation (turning N into a G-module), since $\mathrm{conj} : H \to \operatorname{Aut}(N)$ is one. Let now $s : H \leftarrow G$ be any section. Then the group multiplication on $N \times G$ is given by

$$(n_1, g_1) \cdot (n_2, g_2) = (n_1 + \rho(g_1)(n_2) + c(g_1, g_2), g_1 \cdot g_2),$$

where $c : G \times G \to N$ is defined by

$$c(g_1, g_2) := i^{-1}\Big(s(g_1) \cdot s(g_2) \cdot s(g_1 \cdot g_2)^{-1}\Big).$$

The two sides of the associativity law are:

$$\begin{aligned}
&((n_1,g_1)\cdot(n_2,g_2))\cdot(n_3,g_3) = \\
&\qquad = (n_1+\rho(g_1)(n_2)+c(g_1,g_2), g_1\cdot g_2)\cdot(n_3,g_3)\\
&\qquad = (n_1+\rho(g_1)(n_2)+c(g_1,g_2)+\rho(g_1\cdot g_2)(n_3)+c(g_1\cdot g_2,g_3), g_1\cdot g_2\cdot g_3)\\
&(n_1,g_1)\cdot((n_2,g_2)\cdot(n_3,g_3)) = \\
&\qquad = (n_1,g_1)\cdot(n_2+\rho(g_2)(n_3)+c(g_2,g_3), g_2\cdot g_3)\\
&\qquad = (n_1+\rho(g_1)\Big(n_2+\rho(g_2)(n_3)+c(g_2,g_3)\Big)+c(g_1,g_2\cdot g_3), g_1\cdot g_2\cdot g_3)
\end{aligned}$$

Thus c (together with ρ) gives an associative structure if and only if (using commutativity of N) the following cocycle-equation is satisfied:

$$c(g_1,g_2)+c(g_1\cdot g_2,g_3)=\rho(g_1)\Big(c(g_2,g_3)\Big)+c(g_1,g_2\cdot g_3),$$

i.e.

$$\partial_\rho c(g_1,g_2,g_3):=\rho(g_1)\Big(c(g_2,g_3)\Big)-c(g_1\cdot g_2,g_3)+c(g_1,g_2\cdot g_3)-c(g_1,g_2)=0.$$

Since we may assume $s(1)=1$ (by replacing s by $s'(g):=s(g)\cdot s(1)^{-1}$), we get $i(c(1,1))=s(1)=1=i(0)$ and further more:

$$\begin{aligned}
0&=\partial_\rho c(1,1,g)=\rho(1)(c(1,g))-c(1,g)+c(1,g)-c(1,1)=c(1,g)\\
0&=\partial_\rho c(g,1,1)=\rho(g)(c(1,1))-c(g,1)+c(g,1)-c(g,1)=-c(g,1)\\
0&=\partial_\rho c(g,g^{-1},g)=\rho(g)(c(g^{-1},g))-c(1,g)+c(g,1)-c(g,g^{-1})m\\
&=\rho(g)(c(g^{-1},g))-c(g,g^{-1})
\end{aligned}$$

Thus a mapping $c:G\times G\to N$, which satisfies this cocycle equality and $c(1,1)=00$, defines a group structure on $H:=N\times G$ by

$$\begin{aligned}
(n,g)\cdot(n',g')&:=(n+\rho(g)(n')+c(g,g'), g\cdot g')\\
(n,g)^{-1}&=(-c(g^{-1},g)+\rho(g^{-1})(n^{-1}), g^{-1})
\end{aligned}$$

such that $1\to N\xrightarrow{i} H\xrightarrow{p} G\to 1$ is an abelian extension, where $i:N\to H$ is given by $n\mapsto(n,1)$ and p by $(n,h)\mapsto h$. Furthermore the section $s:G\to N\times G$ is given by $h\mapsto(1,h)$ and satisfies

$$s(g)\cdot s(g')\cdot s(g\cdot g')^{-1}=(1,g)\cdot(1,g')\cdot(1,g\cdot g')^{-1}=(c(g,g'),1).$$

9.36 Isomorphy classes of abelian extensions

The question arises, which cocycles c give isomorphic extensions (with the same action ρ). Let first s' be another section (with $s'(1)=1$). Then $s'(g)=i(\tau(g))\cdot s(g)$ for a mapping $\tau:G\to N$, with $\tau(1)=1$. The following direct calculation for the associated cocycles c and c' yields

$$\begin{aligned}
i\big(c'(g,g')\big)&=s'(g)\cdot s'(g')\cdot s'(g\cdot g')^{-1}\\
&=i(\tau(g))\cdot s(g)\cdot i(\tau(g')))\cdot s(g')\cdot s(g\cdot g')^{-1}\cdot i(\tau(g\cdot g'))^{-1}\\
&=i(\tau(g))\cdot s(g)\cdot i(\tau(g'))\cdot s(g)^{-1}\cdot s(g)\cdot s(g')\cdot s(g\cdot g')^{-1}\cdot i(\tau(g\cdot g'))^{-1}\\
&=i(\tau(g))\cdot i(\rho(g)(\tau(g')))\cdot i(c(g,g'))\cdot i(\tau(g\cdot g'))^{-1}\\
&=i\Big(\tau(g)+\rho(g)(\tau(g'))+c(g,g')-\tau(g\cdot g')\Big)\\
&=i\big(\partial_\rho\tau(g,g')+c(g,g')\big),
\end{aligned}$$

where $\partial_\rho\tau(g,g'):=\rho(g)\Big(\tau(g')\Big)-\tau(g\cdot g')+\tau(g)$.

Let now $\varphi : H' \to H$ be an ismomorphism of groups, such that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & H \arrow[r, "p"] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & N \arrow[r, "i'"] & H' \arrow[r, "p'"] \arrow[u, dotted, "\varphi", "\cong"'] & G \arrow[r] & 1 \end{tikzcd}$$

where $H = N\times G$ with the group structure induced by the cocycle c and $H' = N\times G$ with that induced by the cocycle c'. We get two sections s and $\varphi\circ s'$ for $p : H \to G$, and thus a $\tau : H \to N$ with

$$\varphi(s'(g)) = i(\tau(g)) \cdot s(g).$$

For the cocycles a short calculation yields:

$$c'(g, g') = \partial_\rho \tau(g, g') + c(g, g').$$

Conversely, any $\tau : H \to N$ induces an isomorphism $\varphi : H' \to H$ of groups by $\varphi(n, g) := (n + \tau(g), g)$, since

$$\begin{aligned} \varphi(n, g) \cdot \varphi(n, g') &= (n + \tau(g) + \rho(g)(n' + \tau(g')) + c(g, g'), g \cdot g') \\ &= (n + \tau(g) + \rho(g)(n') + \rho(g)(\tau(g')) + c'(g, g') - \partial_\rho \tau(g, g'), g \cdot g') \\ &= \big((n + \rho(g)(n')) + c'(g, g') + \rho(g)(\tau(g')) - \partial_\rho \tau(g, g') + \tau(g), g \cdot g'\big) \\ &= \varphi(n + \rho(g)(n') + c'(g, g'), g \cdot g') = \varphi((n, g) \cdot (n', g')) \end{aligned}$$

Thus we obtained:

9.37 Theorem. *Isomorphy classes of abelian extensions with respect to a representation $\rho : G \to \mathrm{Aut}(N)$ are in bijective correspondance to the second group cohomology*

$$H^2(G, N) \cong \{c \in N^{G\times G} : \partial_\rho c = 0\}/\{\partial_\rho \tau : \tau \in N^G\}. \quad \square$$

Note, that the conditions $c(1, 1) = 1$ and $\tau(1) = 1$ can be dropped (see [10, A.6]).

Applications to cohomology of spaces

▷ **9.39 Remark.** Let $C' \to C \to C''$ be a splitting short exact sequence of chain complexes. Then $\mathrm{Hom}(C'', G) \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C', G)$ is also a splitting short exact sequence of cochain complexes. Hence the corresponding homologies (i.e. cohomologies of the original chain complexes) form a long exact sequence

$$\dots \to H^q(C'', G) \to H^q(C, G) \to H^q(C', G) \to H^{q+1}(C'', G) \to \dots.$$

In particular, we get the following corollaries:

9.40 Corollary. [20, 13.5.7] *For a pair (X, A) of spaces*

$$\dots \xrightarrow{\delta^*} H^q(X, A; G) \xrightarrow{j^*} H^q(A; G) \xrightarrow{i^*} H^q(X; G) \xrightarrow{\delta^*} H^{q+1}(X, A; G) \xrightarrow{j^*} \dots$$

is an exact sequence. $\square$

9.41 Corollary. [20, 13.5.8] *For a triple (X, A, B) of spaces*

$$\dots \xrightarrow{\delta^*} H^q(X, A; G) \xrightarrow{j^*} H^q(X, B; G) \xrightarrow{i^*} H^q(A, B; G) \xrightarrow{\delta^*} H^{q+1}(X, A; G) \xrightarrow{j^*} \dots$$

is an exact sequence.

9.42 Proposition. [20, 13.5.9]
Let $f, g : C \to C'$ be chain-homotopic. Then $f^* = g^* : H^q(C'; G) \to H^q(C; G)$. In particular, if f is a chain-homotopy equivalence, then f^* is an isomorphism.

Proof. If we dualize, the dual of the chain-homotopy gives a chain-homotopy between f^* and g^* and hence induce the same mapping in the homology of the dual complexes. □

9.43 Corollary. [20, 13.5.10]
If $f \sim g : (X, A) \to (Y, B)$, then $f^* = g^* : H^q(Y, B; G) \to H^q(X, A; G)$. In particular, if f is a homotopy equivalence, then f^* is an isomorphism. □

▽

A carefull analysis of 8.32 shows the following

9.44 Proposition. [19, 4.4.14] *If X is union of the interior of two subsets X_1 and X_2, then the inclusion is a chain equivalence $S(X_1) + S(X_2) \sim S(X)$.*

Thus $H^q(X) \cong H^q(S(X_1) + S(X_2))$ in such a situation.

9.45 3×3**-Lemma.** *If the top two rows and all columns in the following diagram a short exact, then so is the bottom row.*

$$\begin{tikzcd}
A_1 \arrow[r, tail, "f"] \arrow[d, tail, "\partial"] & B_1 \arrow[r, two heads, "g"] \arrow[d, tail, "\partial"] & C_1 \arrow[d, tail, "\partial"] \\
A_2 \arrow[r, tail, "f"] \arrow[d, "\partial"] & B_2 \arrow[r, two heads, "g"] \arrow[d, "\partial"] & C_2 \arrow[d, "\partial"] \\
A_3 \arrow[r, "f"] & B_3 \arrow[r, "g"] & C_3
\end{tikzcd}$$

Proof. (Exact at A_3) Let $a_3 \in A_3$ with $fa_3 = 0$. Choose $a_2 \in A_2$ with $\partial a_2 = a_3$. Then $\partial f a_2 = f\partial a_2 = fa_3 = 0$ hence there exists $b_1 \in B_1$ with $\partial b_1 = fa_2$. Since $\partial g b_1 = g\partial b_1 = gfa_2 = 0$ also $gb_1 = 0$, hence there exists $a_1 \in A_1$ with $fa_1 = b_1$. Then $f\partial a_1 = \partial f a_1 = \partial b_1 = fa_2$, hence $a_2 = \partial a_1$ and thus $a_3 = \partial a_2 = \partial^2 a_1 = 0$.

(Exact at B_3) Let $b_3 \in B_3$ with $gb_3 = 0$. Choose $b_2 \in B_2$ with $\partial_2 b_2 = b_3$. Since $\partial g b_2 = g\partial b_2 = gb_3 = 0$ there exists $c_1 \in C_1$ with $\partial c_1 = gb_2$ and there exists $b_1 \in B_1$ with $gb_1 = c_1$. Then $g\partial b_1 = \partial g b_1 = \partial c_1 = gb_2$. Thus we find $a_2 \in A_2$ with $fa_2 = b_2 - \partial b_1$ and thus $f\partial a_2 = \partial f a_2 = \partial b_2 - 0 = b_3$.
The converse is obvious, since $gf\partial = \partial g f = 0 : A_2 \to C_3$ and ∂ is onto.

(Exact at C_3) is obvious, since $B_2 \xrightarrow{g} C_2 \xrightarrow{\partial} C_3$ is onto. □

9.46 Relative Mayer-Vietoris sequence. [19, 5.4.9] *Let $X_i \subseteq X$ and $A_i \subseteq X_i$ with $S(X_1) + S(X_2) \hookrightarrow S(X_1 \cup X_2)$ and $S(A_1) + S(A_2) \hookrightarrow S(A_1 \cup A_2)$ inducing isomorphisms in cohomology. For any R-module G he have the exact sequence:*

$$\begin{aligned}
\dots \to H^q(X_1 \cup X_2, A_1 \cup A_2; G) \to H^q(X_1, A_1; G) \oplus H^q(X_2, A_2; G) \to \\
\to H^q(X_1 \cap X_2, A_1 \cap A_2; G) \to H^{q+1}(X_1 \cup X_2, A_1 \cup A_2; G) \to \cdots .
\end{aligned}$$

Proof. The first 2 rows in the following diagram are short exact (see 8.37) and by definition also all columns, thus the third row is short exact as well

$$\begin{tikzcd} S(A_1 \cap A_2) \arrow[r, tail] \arrow[d, tail] & S(A_1) \oplus S(A_2) \arrow[r, two heads] \arrow[d, tail] & S(A_1) + S(A_2) \arrow[d, tail] \\ S(X_1 \cap X_2) \arrow[r, tail] \arrow[d, two heads] & S(X_1) \oplus S(X_2) \arrow[r, two heads] \arrow[d, two heads] & S(X_1) + S(X_2) \arrow[d, two heads] \\ S(X_1 \cap X_2, A_1 \cap A_2) \arrow[r, tail] & \bigoplus_{i=1}^2 S(X_i, A_i) \arrow[r, two heads] & (S(X_1) + S(X_2))/(S(A_1) + S(A_2)) \end{tikzcd}$$

So we get a long exact sequence in cohomology, and by the 5'Lemma 7.22 applied to the long exact cohomology sequences induced by the following short exact sequences

$$\begin{tikzcd} S(A_1) + S(A_2) \arrow[r, hook] \arrow[d, hook] & S(X_1) + S(X_2) \arrow[r, two heads] \arrow[d, hook] & (S(X_1) + S(X_2))/(S(A_1) + S(A_2)) \arrow[d] \\ S(A_1 \cup A_2) \arrow[r, hook] & S(X_1 \cup X_2) \arrow[r, two heads] & S(X_1 \cup X_2)/S(A_1 \cup A_2) \end{tikzcd}$$

the mapping $(S(X_1)+S(X_2))/(S(A_1)+S(A_2)) \to S(X_1 \cup X_2)/S(A_1 \cup A_2)$ induces an isomorphism in homology and so we get the claimed exact sequence. □

▷ **9.47 Corollary.** *If X is the union of the interiors of X_1 and X_2 and $A_1 \cup A_2$ is the union of the interiors of A_1 and A_2 then we have the relative Mayer-Vietoris sequence in cohomology.* □

9.48 Remark. The relative Mayer-Vietoris sequence 9.46 implies the exact sequence of a triple (and a pair). In fact, given a triple (X, A, B), then we can apply 9.46 to the pairs (X, B) and (A, A).

9.49 Corollary. Excision theorem. [20, 13.5.12] *Let $U \subseteq A \subseteq X$ with $\bar{U} \subseteq \mathring{A}$. Then $i : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism*

$$i^* : H^q(X, A; G) \cong H^q(X \setminus U, A \setminus U; G).$$

▽ **Proof for PIDs.** We use the equivalent description as in 8.33. By the excision theorem 8.33 for homology the inclusion $i_* : (X_2, X_2 \cap X_1) \to (X_2 \cup X_1, X_1)$ induces isomorphisms $H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$ for all q. Using now the universal coefficient theorem 9.30 gives

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathrm{Ext}^1_R(H_{q-1}(X_2, X_2 \cap X_1), G) \arrow[r, "\cong"'] \arrow[d] & \mathrm{Ext}^1_R(H_{q-1}(X_2 \cup X_1, X_1), G) \arrow[d] \\ H^q(X_2, X_2 \cap X_1; G) \arrow[r] \arrow[d] & H^q(X_2 \cup X_1, X_1; G) \arrow[d] \\ \mathrm{Hom}(H_q(X_2, X_2 \cap X_1), G) \arrow[r, "\cong"'] \arrow[d] & \mathrm{Hom}(H_q(X_2 \cup X_1, X_1), G) \arrow[d] \\ 0 & 0 \end{tikzcd}$$

and the 5'Lemma 7.22 yields the result. □

General proof. We use again the equivalent description as in 8.33. Let $A_1 := X_1$ and $A_2 := X_1 \cap X_2$ then $A_1 \cup A_2 = X_1$ and $A_1 \cap A_2 = X_1 \cap X_2$, hence the relative Mayer-Vietoris sequence 9.46 gives:

$$\cdots \to 0 \to H^q(X_1 \cup X_2, X_1; G) \to H^q(X_2, X_1 \cap X_2; G) \to 0 \to \cdots \qquad \square$$

▷ **9.50 Example.** By 8.41 we have

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \text{ or } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

and thus by the universal coefficient theorem 9.30 (since $\mathbb{Z}$ is projective)

$$H^q(S^n; G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_q(S^n), G) \cong \begin{cases} G & \text{for } q = n \text{ or } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Analogous results follow for the cohomology of $S^n \setminus S$, $S^n \setminus B$, $\mathbb{R}^n \setminus S$, $\mathbb{R}^n \setminus B$, F_g, $\mathbb{P}^n(\mathbb{C})$, and of $\mathbb{P}^n(\mathbb{H})$ for r-spheres S, r-Balls B, and the orientable closed surfaces F_g of genus g, see 8.45, 8.46, 8.47, 8.66, and 8.57. In these cases one only has to replace all $\mathbb{Z}^k$ in the homology groups by G^k and obtains the corresponding cohomology groups.

9.51 Example. [20, 13.6.9] For the none-orientable closed surface X of genus g we got in 8.66

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence by the universal coefficient theorem 9.30 and 9.3 and 9.29

$$\begin{aligned} H^0(X; G) &\cong \mathrm{Hom}(H_0(X), G) \oplus \mathrm{Ext}^1(H_{-1}(X), G) \\ &\cong \mathrm{Hom}(\mathbb{Z}, G) \oplus \mathrm{Ext}^1(0, G) \cong G \\ H^1(X; G) &\cong \mathrm{Hom}(H_1(X), G) \oplus \mathrm{Ext}^1(H_0(X), G) \\ &\cong \mathrm{Hom}(\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, G) \oplus \mathrm{Ext}^1(\mathbb{Z}, G) = G^{g-1} \oplus \{g \in G : 2g = 0\} \\ H^2(X; G) &\cong \mathrm{Hom}(H_2(X), G) \oplus \mathrm{Ext}^1(H_1(X), G) \\ &\cong \mathrm{Hom}(0, G) \oplus \mathrm{Ext}^1(\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, G) = G/2G \end{aligned}$$

In particular, for $g = 1$ we have

$H_q(\mathbb{P}^2)$	$q=0$	$q=1$	$q=2$
$G=\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0

$H^q(\mathbb{P}^2; G)$	$q=0$	$q=1$	$q=2$
$G=\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$G=\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$
$G=\mathbb{R}$	$\mathbb{R}$	0	0

9.52 Example. By 8.67 we have for the real projective spaces

$$H_q(\mathbb{P}^n(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \text{ or } q = n \text{ odd}, \\ \mathbb{Z}_2 & \text{for } 0 < q < n \text{ with } q \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by the universal coefficient theorem 9.30 and 9.3 and 9.29 we get

$$H^q(\mathbb{P}^n; G) \cong \mathrm{Hom}(H_q(\mathbb{P}^n), G) \oplus \mathrm{Ext}^1(H_{q-1}(\mathbb{P}^n), G)$$

$$\cong \begin{cases} \mathrm{Hom}(\mathbb{Z}, G) \oplus \mathrm{Ext}^1(0, G) \cong G & \text{for } q = 0, \\ \mathrm{Hom}(\mathbb{Z}_2, G) \oplus \mathrm{Ext}^1(\mathbb{Z}, G) \cong \{g \in G : 2g = 0\} & \text{for } q = 1, \\ \mathrm{Hom}(\mathbb{Z}_2, G) \oplus \mathrm{Ext}^1(0, G) \cong \{g \in G : 2g = 0\} & \text{for odd } 1 < q < n, \\ \mathrm{Hom}(0, G) \oplus \mathrm{Ext}^1(\mathbb{Z}_2, G) \cong G/2G & \text{for even } 0 < q \le n, \\ \mathrm{Hom}(\mathbb{Z}, G) \oplus \mathrm{Ext}^1(0, G) \cong G & \text{for odd } q = n \end{cases}$$

In particular, $H^q(\mathbb{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for all $0 \le q \le n$, whereas $H^q(\mathbb{P}^n;\mathbb{R}) = 0$ for $0 < q \neq n$ and for even $q = n$.

▽

▷ **9.53 Definition. Cup-product.** [20, 15.2.3]
Although cohomology can be calculated in principle from the homology by the universal coefficient theorem 9.30, cohomolgy has the advange of additional algebraic structure. Let R be a commutative ring with unit. Elements $\varphi \in H^q(X;R)$ are represented by homomorphisms $f : S_n(X) \to R$. For such cochains $f : S_p(X) \to R$ and $g : S_q(X) \to R$ one defines the CUP-PRODUCT

$$f \cup g : \sigma \mapsto f(\sigma \circ \iota_{o,\dots,p}) \cdot g(\sigma \circ \iota_{p,\dots,p+q}),$$

where $\sigma : \Delta^{p+q} \to X$ is any singular $(p+q)$-simplex and $\iota_{0,\dots,p} : \Delta^p \hookrightarrow \Delta^{p+q}$ (resp. $\iota_{p,\dots,p+q} : \Delta^q \hookrightarrow \Delta^{p+q}$) denotes the canonical embedding onto the 'front'-side (resp. 'back'-side). This operation satisfies the Leibiz-rule

$$\partial^*(f \cup g) = \partial^* f \cup g + (-1)^p f \cup \partial^* g$$

and hence induces a welldefined mapping

$$\cup : H^p(X;R) \times H^q(X;R) \to H^{p+q}(X,R).$$

which turns $H^*(X;R)$ into a graduated commutative ring, i.e. we have

▽

commutativity: $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$.
distributivity: $(\alpha + \alpha') \cup \beta = \alpha \cup \beta + \alpha' \cup \beta$.
homogeneity: $(r\alpha) \cup \beta = r(\alpha \cup \beta)$ for $r \in R$.
associativity: $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.
neutral element: $1_x \cup \alpha = \alpha$.
naturality: $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$ for $f : X' \to X$.

This additional algebraic structure is a main advantage of the cohomology over the homology.

▷ **9.54 Example.** [20, 15.3.6.c] One can show

$$H^*(X \vee Y) \cong H^*(X) \times H^*(Y)$$

and

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

as rings with respect to the cup-product $\cup$, where

$$H^*(X) \otimes H^*(Y) = \Big(\sum_{p+q=n} H^p(X) \otimes H^q(Y) \Big)_{n \in \mathbb{N}}$$

and the product is defined component-wise. Thus the spaces $S^m \vee S^n \vee S^{m+n}$ and $S^m \times S^n$ for $m > n \ge 1$ have isomorphic fundamental groups (by 5.37 and 5.29), homology groups (by 8.36 and 10.33) and cohomology groups.

$$\begin{aligned}\pi_1(S^m \vee S^n \vee S^{m+n}) &\cong \pi_1(S^m) \amalg \pi_1(S^n) \amalg \pi_1(S^{m+n}) \cong \pi_1(S^n) \\ &\cong \pi_1(S^m) \times \pi_1(S^n) \cong \pi_1(S^m \times S^n)\end{aligned}$$

$$H_k(S^m \vee S^n \vee S^{m+n}) \cong H_k(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & \text{for } k \in \{0, n, m, m+n\} \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(S^m \vee S^n \vee S^{m+n}; G) \cong H^k(S^m \times S^n; G) \cong \begin{cases} G & \text{for } k \in \{0, n, m, m+n\} \\ 0 & \text{otherwise} \end{cases}$$

However, the cohomology ring of the first space is trivial, whereas that of the second is not.

▽

9.55 Example. [20, 15.5.2] One can show, that the cohomology ring of $\mathbb{P}^n(\mathbb{C})$ is isomorphic to $\mathbb{Z}[x]/\langle x^{n+1}\rangle$, where x corresponds to the generator in $H^2(\mathbb{P}^n(\mathbb{C})) \cong \mathbb{Z}$ (by 9.50). Moreover, $H^*(\mathbb{P}^\infty(\mathbb{C})) \cong \mathbb{Z}[x]$.

9.56 Example. [20, 15.5.4] One can show that the cohomology ring of $\mathbb{P}^n(\mathbb{R})$ with coefficients in $\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}[x]/\langle x^{n+1}\rangle$, where x corresponds to the generator in $H^1(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}_2) \cong \mathbb{Z}_2$ (by 9.52). Moreover, $H^*(\mathbb{P}^\infty(\mathbb{R}), \mathbb{Z}_2) \cong \mathbb{Z}[x]$.

9.57 Lemma. [20, 15.5.8] *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be continuous with $n > m \geq 1$. Then $\pi_1(f) : \pi_1(\mathbb{P}^n) \to \pi_1(\mathbb{P}^m)$ is trivial.*

Proof. For $m = 1$ this is obvious, since $\pi_1(\mathbb{P}^n) \cong \mathbb{Z}_2$ and $\pi_1(\mathbb{P}^1) \cong \mathbb{Z}$. So let $m > 1$ and $k \in \{m, n\}$. Then $\mathbb{Z}_2 \cong \pi_1(\mathbb{P}^k) \cong H_1(\mathbb{P}^k) \cong \operatorname{Hom}(H_1(\mathbb{P}^k), \mathbb{Z}_2) \cong H^1(\mathbb{P}^k; \mathbb{Z}_2)$. Thus it remains to show that $f^* : H^1(\mathbb{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{P}^n; \mathbb{Z}_2)$ is trivial. Otherwise, $f^*(\alpha) = \beta \neq 0$, where β and α are the non-zero elements in $\mathbb{Z}_2$. By 9.56 the n-fold cup-products are $\alpha \cup \ldots \cup \alpha = 0$, whereas $f^*(\alpha \cup \ldots \cup \alpha) = \beta \cup \ldots \cup \beta \neq 0$, a contradiction. □

9.58 Lemma. [20, 15.5.9] *There exists no continuous $g : S^n \to S^m$ for $n > m \geq 1$ with $g(-x) = -g(x)$ for all x.*

Proof. Otherwise, g would induce a continuous $\bar{g} : \mathbb{P}^n \to \mathbb{P}^m$. By 9.57 $\pi_1(\bar{g}) : \pi_1(\mathbb{P}^n) \to \pi_1(\mathbb{P}^m)$ is trivial, hence $\bar{g}$ has a lift $\tilde{g} : \mathbb{P}^n \to S^m$ along $p : S^m \to \mathbb{P}^m$. For fixed $x \in S^n$ either $(\tilde{g} \circ p)(x) = g(x)$ or $(\tilde{g} \circ p)(x) = -g(x)$. In the second case $(\tilde{g} \circ p)(-x) = (\tilde{g} \circ p)(x) = -g(x) = g(-x)$ and thus in both cases $\tilde{g} \circ p = g$ by 6.7. Since $p(x) = p(-x)$ we get $g(x) = g(-x) = -g(x) \in S^m$, a contradiction. □

▷ **9.59 Theorem of Borsuk-Ulam.** [20, 15.5.10] *For each continuous $f : S^n \to \mathbb{R}^n$ exists an $x \in S^n$ with $f(x) = f(-x)$. In particular, there is no embedding $S^n \hookrightarrow \mathbb{R}^n$.*

This generalizes 2.27.

▽ **Proof.** Otherwise, consider $g : x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ which is a continuous map $S^n \to S^{n-1}$ with $g(-x) = -g(x)$ for all x. Since S^0 is discrete, this is impossible in the case $n = 1$ and for $n > 1$ it is impossible by 9.58. □

10. Homology with Coefficients

In this section G is a fixed abelian group or more generally, an R-module. We are particularly interested in the cases $G = \mathbb{Z}$, $G = \mathbb{Z}_2$, $G = \mathbb{Q}$ or $G = \mathbb{R}$. The chain groups we considered so far are free abelian groups, i.e. its elements were formal linear combinations with coefficients in $\mathbb{Z}$, and we will replace $\mathbb{Z}$ by the group G now. Since the boundary operator ∂ was defined on the generators and extended $\mathbb{Z}$-linearly to the chain groups it is well defined for this modified chain groups as well and hence we can consider its homology. An advantage of using $G = \mathbb{Z}_2$ is, that we get rid of signs. And with $G = \mathbb{Q}$ or $G = \mathbb{R}$ we will get rid of torsion elements.

In order to make this process as natural as possible we have to consider tensor products and for their (categorical) construction coseparators are helpfull:

10.1 Definition. A R-modules S is called COSEPARATOR iff $\operatorname{Hom}_R(\_, S)$ is faithfull, i.e. $f : M \to M'$ with $\operatorname{Hom}(f, S) = 0$ implies $f = 0$.

10.2 Lemma. *The category <u>R-Mod</u> of R-modules has a* COSEPARATOR.

Proof. Note, that S is a coseparator iff for every $0 \neq a \in A$ we find a $\varphi \in \operatorname{Hom}(A, S)$ with $\varphi(a) \neq 0$:
($\Leftarrow$) Let $0 \neq f : A' \to A$. Then there exists an $a' \in A'$ with $a := f(a') \neq 0$, so by assumption we find $\varphi \in \operatorname{Hom}(A, S)$ with $\operatorname{Hom}(f, S)(\varphi) = f^*(\varphi) = \varphi \circ f$ not vanishing on a', i.e. $\operatorname{Hom}(f, S) \neq 0$.
($\Rightarrow$) Let $0 \neq a \in A$ and consider $f : R \to A$, $r \mapsto ra$. Then $f \neq 0$, thus there is a $\varphi \in \operatorname{Hom}(A, S)$ with $0 \neq f^*(\varphi) = \varphi \circ f$, i.e. $0 \neq \varphi(f(1)) = \varphi(a)$.

$\mathbb{Q}/\mathbb{Z}$ is an injective coseparator for <u>AGru</u>: Let A be an abelian group and $0 \neq a \in A$. Consider $\varphi : \mathbb{Z} \to A$ given by $\varphi(k) = k \cdot a$ and its kernel $\operatorname{Ker}\varphi := \{k \in \mathbb{Z} : k \cdot a = 0\} = \mathbb{Z} \cdot \operatorname{ord}(a)$. Then $\varphi(\mathbb{Z}) \cong \mathbb{Z}/\operatorname{Ker}\varphi = \mathbb{Z}_{\operatorname{ord}(a)}$ and $\mathbb{Z}_{\operatorname{ord}(a)}$ embeds into $\mathbb{Q}/\mathbb{Z}$ by $\iota : [k] \mapsto [\frac{k}{\operatorname{ord}(a)}]$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible(=injective) ι can be extended along $\mathbb{Z}_{\operatorname{ord}(a)} \hookrightarrow A$ to obtain a homomorphism $\tilde\iota : A \to \mathbb{Q}/\mathbb{Z}$ with $\tilde\iota(a) = \tilde\iota(\varphi(1)) = \tilde\iota((j \circ \pi)(1)) = \iota(\pi(1)) \neq 0$.

$$\begin{tikzcd}
\mathbb{Z}\cdot\operatorname{ord}(a) \arrow[r, hook] & \mathbb{Z} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & A \arrow[dr, dotted, "\tilde\iota"] & \\
& & \mathbb{Z}_{\operatorname{ord}(a)} \arrow[u, hook, "j"] \arrow[r, tail, "\iota"'] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

$\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective coseparator for <u>R-Mod</u>:
Let $0 \neq b_0 \in B$ and $\varphi : B \to \mathbb{Q}/\mathbb{Z}$ a homomorphism of groups with $\varphi(b_0) \neq 0$. By the proof of 9.15 we have $\operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_R(B, \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$ and the corresponding R-module homomorphism $\tilde\varphi : B \to \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ satisfies $\tilde\varphi(b)(1) = \varphi(b) \neq 0$, i.e. $\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is a coseparator for R-modules. □

Remark. It follows that the category *<u>R-Mod</u>* of R-modules is cocomplete, i.e. arbitrary colimits exist, since every complete, local-small (every object has only a set of non-equivalent subobjects) category which has a coseparator is cocomplete, see [**9**, 3.37].

▷ **10.3 Corollary.** *For any left R-module M the Hom-functor* $\operatorname{Hom}_{\mathbb{Z}}(M, \_) :$ *<u>AGru</u>* $\to$ *<u>Mod-R</u> is a right adjoint, i.e. there exists a functor denoted* $\_ \otimes_R M$: *<u>Mod-R</u>* $\to$ *<u>AGru</u> such that there are natural isomorphisms*

$$\operatorname{Hom}_{\mathbb{Z}}(N \otimes_R M, G) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, G)).$$

An explicit construction of $N \otimes_R M$ is the following: Take the free abelian group generated by $N \times M$ and factor out the subgroup generated by all the elements $(x + x', y) - (x, y) - (x', y)$, $(x, y + y') - (x, y) - (x, y')$, and $(x \cdot r, y) - (x, r \cdot y)$ for $x, x' \in N$, $y, y' \in M$, and $r \in R$.

▽ **Proof.** The right R-Module structure on $\operatorname{Hom}_{\mathbb{Z}}(M, G)$ is given by $(\varphi \cdot r)(x) = \varphi(r \cdot x)$. This functor has all the properties required for the Special Adjoint Functor Theorem (see [9, 4.27]), i.e. is continuous, Mod-R $\cong$ R^{op}-*Mod* is complete (products are the cartesian product with component-wise operations, kernels are the zero-sets as submodules), is locally small (i.e. there is only a set of submodules for any given module), and has a coseparator. Thus it has a left adjoint $\_ \otimes_R M$: *Mod-R* $\to$ *AGru*. $\square$

10.4 Remark. Note, that $\varphi \in \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, G)) \Leftrightarrow \hat\varphi(x \cdot r, y) := \varphi(x \cdot r)(y) = (\varphi(x) \cdot r)(y) = \varphi(x)(r \cdot y) = \hat\varphi(x, r \cdot y)$ and is additive in both variables separately. Let us denote the set of these $\hat\varphi$ by

$$\begin{aligned}\operatorname{Bilin}_R(N, M; G) := \{\psi \in G^{N \times M} :& \psi(n\, r, m) = \psi(n, r\, m),\\ & \psi(n + n', m) = \psi(n, m) + \psi(n', m),\\ & \psi(n, m + m') = \psi(n, m) + \psi(n, m')\}\end{aligned}$$

If we take $G := N \otimes_R M$, then $\operatorname{id}_{N \otimes_R M}$ corresponds to such a mapping $\hat\varphi : N \times M \to N \otimes_R M$ denoted $\otimes$. Thus $xr \otimes y = x \otimes ry$. Moreover, the bijection $\operatorname{Hom}_{\mathbb{Z}}(N \otimes_R M, G) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, G)) \cong \operatorname{Bilin}_R(M, N; G))$ is given by $\varphi \mapsto \varphi \circ \otimes (: N \times M \to N \otimes_R M \to G)$, as chaseing $\operatorname{id}_{N \otimes_R M}$ through the following diagram shows

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{Z}}(N \otimes_R M, N \otimes_R M) \arrow[r, tail, two heads, "\cong"] \arrow[d, "\varphi_*"] & \operatorname{Bilin}_R(N, M; N \otimes_R M) \arrow[d, "\varphi_*"] \\
\operatorname{Hom}_{\mathbb{Z}}(N \otimes_R M, G) \arrow[r, tail, two heads, "\cong"] & \operatorname{Bilin}_R(N, M; G)
\end{tikzcd}$$

Consequently the abelian group $N \otimes_R M$ is generated generated by $\{x \otimes y : x \in N, y \in M\}$.

10.5 Lemma. *If* $\operatorname{Hom}(A', \_) \cong \operatorname{Hom}(A, \_)$, *then* $A \cong A'$.

Proof. Let $\varphi_B : \operatorname{Hom}(A', B) \to \operatorname{Hom}(A, B)$ be the natural isomorphism. Define $f := \varphi_{A'}(\operatorname{id}_{A'}) \in \operatorname{Hom}(A, A')$ and $g := \varphi_A^{-1}(\operatorname{id}_A) \in \operatorname{Hom}(A', A)$ and consider in the following diagrams

$$\begin{tikzcd}
\operatorname{Hom}(A', A') \arrow[r, tail, two heads, "\varphi_{A'}"] & \operatorname{Hom}(A, A') \\
\operatorname{Hom}(A', A) \arrow[u, "f_*"] \arrow[r, tail, two heads, "\varphi_A"] & \operatorname{Hom}(A, A) \arrow[u, "f_*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Hom}(A, A) & \operatorname{Hom}(A', A) \arrow[l, tail, "\varphi_A"'] \\
\operatorname{Hom}(A, A') \arrow[u, "g_*"] & \operatorname{Hom}(A', A') \arrow[l, tail, "\varphi_{A'}"'] \arrow[u, "g_*"]
\end{tikzcd}$$

the image of $g \in \operatorname{Hom}(A', A)$ (resp. $f \in \operatorname{Hom}(A, A')$) to conclude that $g \circ f = \operatorname{id}_A$ and $f \circ g = \operatorname{id}_{A'}$. $\square$

10.6 Lemma. *If for a sequence* $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *the dual sequences* $0 \to \operatorname{Hom}(M'', G) \xrightarrow{g^*} \operatorname{Hom}(M, G) \xrightarrow{f^*} \operatorname{Hom}(M', G)$ *are exact for every* G, *then the original sequence is exact.*

Proof. (Exact at M'') Take $G := M''/g(M)$ and $p : M'' \twoheadrightarrow G$ the canonical quotient mapping. Then $g^*(p) = p \circ g = 0$ and by assumption $p = 0$, i.e. $0 = G = M''/g(M)$. Thus g is onto.

(Exact at M) Take as $G := M/f(M)$ and consider the canonical projection $p : M \twoheadrightarrow G$. Note that $\ker(f^*) = \{\varphi \in \mathrm{Hom}(M,G) : \varphi|_{f(M)} = 0\}$ and $\mathrm{im}(g^*) = \{g^*(\psi) = \psi \circ g : \psi \in \mathrm{Hom}(M'',G)\} = \{\varphi \in \mathrm{Hom}(M,G) : \varphi \text{ factors over } g\} = \{\varphi \in \mathrm{Hom}(M,G) : \varphi|_{\ker g} = 0\}$. Thus $p \in \ker(f^*) = \mathrm{im}(g^*)$, i.e. $p(\ker g) = \{0\}$. Hence $\ker g \subseteq \mathrm{im}(f)$. Conversely, take $G = M''$. then $0 = f^*(g^*(\mathrm{id}_{M''})) = g \circ f$. □

▷ **10.7 Corollary.** *We have natural isomorphisms $R \otimes_R M \cong M$ and $\_ \otimes_R M$ commutes with colimits and is right-exact.*

▽ **Proof.** Since $\mathrm{Hom}(R \otimes_R M, G) \cong \mathrm{Hom}_R(R, \mathrm{Hom}(M,G)) \cong \mathrm{Hom}(M,G)$, it follows from 10.5 that $R \otimes_R M \cong M$.
As left adjoint $\_ \otimes_R M$ commutes with colimits.
Let now $N' \to N \to N'' \to 0$ be exact. Then

$$\mathrm{Hom}_R(N',P) \leftarrow \mathrm{Hom}_R(N,P) \leftarrow \mathrm{Hom}_R(N'',P) \leftarrow 0$$

is exact and in particular for $P := \mathrm{Hom}(M,G)$. Thus

$$\mathrm{Hom}(N' \otimes_R M, G) \leftarrow \mathrm{Hom}(N \otimes_R M, G) \leftarrow \mathrm{Hom}(N'' \otimes_R M, G) \leftarrow 0$$

is exact, and by 10.6 the sequence

$$N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$$

is exact. □

10.8 Remark. Note, that $\otimes_R$ is also a covariant functor in the second variable, since $\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(\_, G))$ and $\mathrm{Hom}_{\mathbb{Z}}(\_, G)$ are contravariant functors $\underline{R\text{-}Mod} \to \underline{AGru}$.

▷ **10.9 Definition.** An R-module M is called FLAT, iff for every monomorphism $\alpha : A \to A'$ of right R-modules the tensor product $\alpha \otimes_R M : A \otimes_R M \to A' \otimes_R M$ is injective, i.e. $\_ \otimes_R M$ is (left) exact.

▽

10.10 Proposition. *Coproducts and direct summands of flat modules are flat. Every projective module is flat and every flat module over an integral domain is torsion-free.*

Proof. The statement on coproducts follows, since the tensor product commutes with coproducts, and a coproduct (as subspace of the product) of monomorphisms is a monomorphism.

Let $M' \hookrightarrow M$ be a direct summand of a flat module and $A' \to A$ be injective. Then $A \otimes_R M' \to A \otimes_R M$ and $A' \otimes_R M' \to A' \otimes M$ are sections and $A' \otimes_R M \to A \otimes_R M$ is injective, thus also $A' \otimes_R M' \to A \otimes_R M'$.

$$\begin{array}{ccc} A' \otimes_R M & \rightarrowtail & A \otimes_R M \\ \uparrow & & \uparrow \\ A' \otimes_R M' & \longrightarrow & A \otimes_R M' \end{array}$$

Since every projective module is a direct summand in a free module it suffices to show that R itself is flat, which is obvious, since $A \otimes_R R \cong A$.

Let now M be a flat module and assume it is not torsion free, so there is $0 \neq a \in M$ and $0 \neq r \in R$ with $ra = 0$. Consider $\alpha : R \to R$ given by $r' \mapsto r'r$, which is a monomorphism, since R is an integral domain. Since M is flat, $\alpha \otimes_R M : R \otimes_R M \to R \otimes_R M$ is injective. Since $(\alpha \otimes_R M)(1 \otimes a) = r \otimes a = 1 \otimes ra = 0$ it follows $a = 1 \otimes a = 0$, a contradiction. □

10.11 Lemma. *An R-module M is flat if and only if for every ideal $0 \neq I \triangleleft R$ the canonical mapping $I \otimes_R M \to R \otimes_R M \cong M$ is injective.*
In particular, every module over a field R is flat.

Proof. ($\Rightarrow$) Since $I \hookrightarrow R$ is injective and M is flat, also $I \otimes_R M \to R \otimes_R M \cong M$ is injective.

($\Leftarrow$) Let $N' \hookrightarrow N$ be a submodule. Since every module is the inductive limit of its finitely generated submodules F and $\_ \otimes_R M$ commutes with colimits it is enough to consider finitely generated N ($N' = \bigcup_F N' \cap F$). So we have an epimorphism $R^n \twoheadrightarrow N$ for some finite n. Let K denote its kernel and let P be the pull-back of $R^n \twoheadrightarrow N$ and $N' \hookrightarrow N$. Then $N' \cong P/K$ and applying $\_ \otimes_R M$ to both short exact sequences gives

$$\begin{tikzcd}
0 \arrow[r] & K \otimes_R M \arrow[r, hook] & R^n \otimes_R M \arrow[r, two heads] & N \otimes_R M \arrow[r] & 0 \\
& K \otimes_R M \arrow[u, equal] \arrow[r] & P \otimes_R M \arrow[u, tail] \arrow[r, two heads] & N' \otimes_R M \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

It follows, that $N' \otimes_R M \to N \otimes_R M$ is injective, provided we can show that $K \otimes_R M \to R^n \otimes_R M$ is injective for every submodule $K \subseteq \mathbb{K}^n$, which we prove now by induction on n.

(n=1) Then $K \hookrightarrow R$ is an ideal, hence by assumption $K \otimes_R M \to R \otimes_R M$ is injective.

(n+1) We consider

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, hook] & R \times R^n \arrow[r, two heads] \arrow[l, dotted, bend left] & R^n \arrow[r] & 0 \\
0 \arrow[r] & K \cap R \arrow[u, hook] \arrow[r, hook] & K \arrow[u, hook] \arrow[r, two heads] & K/(K \cap R) \arrow[u, tail] \arrow[r] & 0
\end{tikzcd}$$

and apply $\_ \otimes_R M$ to obtain

$$\begin{tikzcd}
0 \arrow[r] & R \otimes_R M \arrow[r, hook] & R^{n+1} \otimes_R M \arrow[r, two heads] & R^n \otimes_R M \arrow[r] & 0 \\
& (K \cap R) \otimes_R M \arrow[u, tail] \arrow[r] & K \otimes_R M \arrow[u] \arrow[r, two heads] & K/(K \cap R) \otimes_R M \arrow[u, tail] \arrow[r] & 0
\end{tikzcd}$$

Thus also the vertical arrow in the middle is injective. □

10.12 Proposition. *If R is a PID. Then every torsion-free R-module is flat.*

Proof.
Since R is a PID, every ideal $0 \neq I \triangleleft R$ is of the form $I = Rr$ for some $0 \neq r \in R$. Since M is torsion-free, the mapping $r : R \to I$, $r' \mapsto r'r$, is an isomorphism, hence $I \otimes_R M \to R \otimes_R M$ is an isomorphism. By 10.11 this implies that M is flat. □

$$\begin{tikzcd}
I \otimes_R M \arrow[r, "\iota \otimes_R M"'] & R \otimes_R M \\
R \otimes_R M \arrow[u, "r \otimes_R M"] & M \arrow[l, "\cong"'] \arrow[u, "\cong"']
\end{tikzcd}$$

10.13 Example. The torsion-free(=flat) group $\mathbb{Q}$ is not free(=projective): It is divisible, whereas free abelian groups are not, since their generators cannot be divided by $n > 1$.

10.14 Lemma. *Let R be a PID and M a finitely generated torsion-free R-module. Then M is a free module.*

Proof. Let S be a finite set of generator for M. We find a maximal subset $S_0 \subseteq S$ such that $M_0 := \langle S_0 \rangle$ is a free submodule. If $x \in S \setminus S_0$ then we find $0 \neq r_x \in R$ and $r_s \in R$ for $s \in S_0$ such that $r_x x + \sum_{s\in S_0} r_s s = 0$, i.e. M/M_0 is a torsion module. Now let $r := \prod_{x\in S\setminus S_0} r_x \neq 0$, since R is an integral domain. Since M_0 is free and $rM \subseteq M_0$ we have that rM is free by 9.20. Since M is torsion free the multiplication map $r : M \to rM$ is a isomorphism, hence M is free. $\square$

10.15 Corollary. *If M is an (R, S)-bimodule (i.e. an abelian group with left R-action and a right S-action, which commute with each other) and G is an right S-module, then $\mathrm{Hom}_S(M, G)$ is a right R-submodule of $\mathrm{Hom}_{\mathbb{Z}}(M, G)$ and $N \otimes_R M$ is a right S-module and we have natural isomorphisms*

$$\mathrm{Hom}_S(N \otimes_R M, G) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, G)). \quad \square$$

If, in particular, R is a commutative ring, then every R-module is also an (R, R)-bimodule, where the two actions coincide. Thus $N \otimes_R M$ is itself an R-module with

$$\mathrm{Hom}_R(N \otimes_R M, G) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, G)).$$

10.16 Corollary. *For commutative rings R we have $M \otimes_R N \cong N \otimes_R M$ and $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$.*

Proof. The first isomorphism follows using 10.5 from

$$\begin{aligned}\mathrm{Hom}(M \otimes_R N, G) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, G)) \\ &\cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, G)) \cong \mathrm{Hom}_R(N \otimes_R M, G),\end{aligned}$$

via $f \mapsto \tilde{f}$, where $\tilde{f}(y)(x) := f(x)(y)$. And the second one follows from

$$\begin{aligned}\mathrm{Hom}_R((M \otimes_R N) \otimes_R P, G) &\cong \mathrm{Hom}_R(M \otimes_R N, \mathrm{Hom}_R(P, G)) \\ &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, \mathrm{Hom}_R(P, G))) \\ &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N \otimes_R P, G)) \\ &\cong \mathrm{Hom}_R(M \otimes_R (N \otimes_R P), G). \quad \square\end{aligned}$$

10.17 Example.[20, 10.2.4]

1. $A \otimes \mathbb{Z}_m = A/mA$ and hence $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{\gcd(m,n)}$:
   $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, G) = \{g \in G : mg = 0\}$ by 9.3.1, hence $\mathrm{Hom}(A \otimes \mathbb{Z}_m, G) \cong \mathrm{Hom}(A, \{g : mg = 0\}) = \mathrm{Hom}(A/mA, G)$.
2. $A \otimes \mathbb{R} = 0$ if A is a torsion group, i.e. all elements in A have finite order: Let $\varphi \in \mathrm{Bilin}(A, \mathbb{R}; G) \cong \mathrm{Hom}(A, \mathrm{Hom}(\mathbb{R}, G))$. Then $\varphi = 0$, since $\varphi(a, b) = \varphi(a, r\,\frac{b}{r}) = \varphi(a\,r, \frac{b}{r}) = \varphi(0, \frac{b}{r}) = 0$.
3. $\mathbb{Z}_2$ is not a flat abelian group:
   $\mathbb{Z}_2 \otimes \mathbb{Z} = \mathbb{Z}_2$ is not a subgroup of $\mathbb{Z}_2 \otimes \mathbb{R} = 0$ although $\mathbb{Z} \hookrightarrow \mathbb{R}$ is one.
4. $({}^{ab}\coprod_J \mathbb{Z}) \otimes B \cong {}^{ab}\coprod_J B$, by 10.7. In particular, the tensor product of two free rings with p and q many generators is a free ring with $p \cdot q$ many generators.

▷ **10.18 Definition. Homology with coefficients.** [20, 10.5.1] Let (X, A) be a pair of spaces and G be an abelian group. Then the q-th homology of (X, A) with coefficients in G is defined as the abelian group

$$H_q(X, A; G) := H_q(S(X, A) \otimes_{\mathbb{Z}} G)$$

If G is even a right R-module over some ring R, then $S(X,A) \otimes_{\mathbb{Z}} G$ is a chain complex of right R-modules and hence $H_q(X,A;G)$ are also right R-modules.

Again the question arises what $H_q(X;G)$ has to do with $H_q(X) \otimes G$.

10.19 Universal coefficient theorem for homology with flat coefficients. *Let C be a chain complex of right R-modules and M a flat left R-module. Then we have a natural isomorphism*

$$H_q(C) \otimes_R M \cong H_q(C \otimes_R M).$$

Proof. We proceed analogous to 9.2 and the proof of 9.30. We apply $\_ \otimes_R M$ to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z_q \arrow[r, hook, "j"] & C_q \arrow[r, two heads, "\partial"] & B_{q-1} \arrow[r] & 0 \end{tikzcd}$$

and obtain the short exact sequence (of chain-complexes)

$$\begin{tikzcd} 0 \arrow[r] & Z_q \otimes M \arrow[r, hook, "j\otimes M"] & C_q \otimes M \arrow[r, two heads, "\partial\otimes M"] & B_{q-1} \otimes M \arrow[r] & 0 \end{tikzcd}$$

which gives by 7.30 a long exact sequence in homology

$$\begin{tikzcd} \cdots \arrow[r, "\delta_*"] & H_q(Z\otimes M) \arrow[r, "(j\otimes M)_*"] \arrow[d, equal] & H_q(C\otimes M) \arrow[r, "(\partial\otimes M)_*"] & H_{q-1}(B\otimes M) \arrow[r, "\delta_*"] \arrow[d, equal] & H_{q-1}(Z\otimes M) \arrow[r] \arrow[d, equal] & \cdots \\ & Z_q \otimes M & & B_{q-1}\otimes M \arrow[r, "i\otimes M"] & Z_{q-1}\otimes M & \end{tikzcd}$$

The identities hold, since the boundary operator on Z and on B and hence on $Z \otimes_R M$ and $B \otimes_R M$ is 0. The rectangle commutes (i.e. the connecting homomorphism δ_* is $i \otimes_R M$), since $(\partial \otimes_R M) \circ (\partial \otimes_R M)^{-1} : B_{q-1} \otimes_R M \to C_{q-1} \otimes_R M$ is just the composite $B_{q-1} \otimes_R M \xrightarrow{i\otimes_R M} Z_{q-1} \otimes_R M \xrightarrow{j\otimes_R M} C_{q-1} \otimes_R M$.
Now consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B_q \arrow[r, hook, "i"'] & Z_q \arrow[r, two heads] & H_q(C) \arrow[r] & 0 \end{tikzcd}$$

Taking the tensor product with the flat module M yields the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B_q \otimes M \arrow[r, hook, "i\otimes M"'] & Z_q \otimes M \arrow[r, two heads] & H_q(C) \otimes M \arrow[r] & 0 \end{tikzcd}$$

In particular, $i \otimes_R M = \delta_*$ is injective, so $(\partial \otimes M)_* = 0$ and $(j \otimes M)_*$ is onto. The kernel of $(j \otimes M)_*$ is the image of $\delta_* = i \otimes_R M$, i.e. the kernel of the epimorphism $Z_q \otimes_R M \twoheadrightarrow H_q(C) \otimes_R M$. Hence $(j \otimes M)_*$ factors to an isomorphism $H_q(C) \otimes_R M \to H_q(C \otimes_R M)$. □

$$\begin{tikzcd} H_q(B\otimes M) \arrow[r, "\delta_*"] \arrow[d, equal] & H_q(Z\otimes M) \arrow[r, two heads, "(j\otimes M)_*"] \arrow[d, equal] & H_q(C\otimes M) \arrow[r, "0"] & H_{q-1}(B\otimes M) \arrow[r, tail, "\delta_*"] \arrow[d, equal] & H_{q-1}(Z\otimes M) \arrow[d, equal] \\ B_q\otimes M \arrow[r, tail, "i\otimes M"] & Z_q\otimes M \arrow[r, two heads] & H_q(C)\otimes M \arrow[u, dotted, "\cong"'] & B_{q-1}\otimes M \arrow[r, tail, "i\otimes M"] & Z_{q-1}\otimes M \end{tikzcd}$$

10.20 Corollary. *Let (X,A) be a pair of spaces and G be a torsion-free group. Then we have a natural isomorphism*

$$H_q(X,A) \otimes_{\mathbb{Z}} G \cong H_q(X,A;G). \quad \square$$

10.21 Theorem. *There are functors Tor_n^R : Mod-R × R-Mod → AGru and natural transformations such that*

1. $\mathrm{Tor}_n^R(N,M) = 0$ *for* $n < 0$.

2. $\operatorname{Tor}_0^R(N, M) \cong N \otimes_R M$.
3. $\operatorname{Tor}_n^R(N, M) = 0$ *for all* $n > 0$ *if* N *or* M *is projective.*
4. *For every short exact sequence* $0 \to N' \to N \to N'' \to 0$ *in* <u>*Mod-R*</u> *there is a long exact sequence in* <u>*AGru*</u>

$$\cdots \to \operatorname{Tor}_n^R(N', M) \to \operatorname{Tor}_n^R(N, M) \to \operatorname{Tor}_n^R(N'', M) \xrightarrow{\delta} \operatorname{Tor}_{n-1}^R(N', M) \to \cdots.$$

For every short exact sequence $0 \to M' \to M \to M'' \to 0$ *in* <u>*R-Mod*</u> *there is a long exact sequence in* <u>*AGru*</u>

$$\cdots \to \operatorname{Tor}_n^R(N, M') \to \operatorname{Tor}_n^R(N, M) \to \operatorname{Tor}_n^R(N, M'') \xrightarrow{\delta} \operatorname{Tor}_{n-1}^R(N, M') \to \cdots.$$

For fixed M *the functor* $\operatorname{Tor}_*^R(\_, M)$ *together with the natural transformation* δ *is up to isomorphisms uniquely determined by* 1-4. *And similarly for each fixed* N.

▽ **Proof.** We consider a projective resolution $P \to M \to 0$ and the induced chain complex

$$\cdots \to N \otimes_R P_2 \to N \otimes_R P_1 \to N \otimes_R P_0 \to 0$$

Then $\operatorname{Tor}_n^R(N, M)$ is defined as its homology, i.e. $\operatorname{Tor}_n(N, M) := H_n(N \otimes P_*)$. Now proceed as in the proof of 9.23:

(1) is obvious by definition.

(2) By definition $\operatorname{Tor}_0^R(N, M)$ is just the cokernel of $N \otimes_R P_1 \to N \otimes_R P_0$, i.e. the group $N \otimes_R P_0$ modulo the image of $N \otimes_R P_1 \to N \otimes_R P_0$ and by right exactness the sequence $N \otimes_R P_1 \to N \otimes_R P_0 \twoheadrightarrow N \otimes_R M \to 0$ is exact, hence this cokernel is isomorphic to $N \otimes_R M$.

(3) If M is projective, then we may take $P_0 = M$ and $P_k = 0$ for all $k > 0$, hence $N \otimes P_k = 0$ and thus also $\operatorname{Tor}_k^R(N, M) = H_k(N \otimes_R P) = 0$ for these k.

(4) Let $0 \to N' \to N \to N'' \to 0$ be short exact and P be a projective resolution of M. So we have short exact sequences

$$0 \to N' \otimes_R P_k \to N \otimes_R P_k \to N'' \otimes_R P_k \to 0$$

and this gives a short exact sequence of cochain complexes since $\otimes_R$ is a bifunctor:

$$0 \to N' \otimes_R P \to N \otimes_R P \to N'' \otimes_R P \to 0$$

By 7.30 we get a long exact sequence in homology:

$$\cdots \to \operatorname{Tor}_k^R(N', M) \to \operatorname{Tor}_k^R(N, M) \to \operatorname{Tor}_k^R(N'', M) \xrightarrow{\delta} \operatorname{Tor}_{k-1}^R(N', M) \to \cdots.$$

Again by the Double Complex Lemma 9.21 it does not matter whether we take a projective resolution of N or of M for the definition of $N \otimes_R M$. So also the second long exact sequence of 4 holds.

Uniqueness follows the same way as in the proof of 9.23. □

10.22 Lemma. *For commutative rings* R *the functor* Tor_1 *is commutative, associative and preserves colimits.*

Proof. This follows from the same properties 10.16 of the tensor product. □

10.23 Lemma. *Let* R *be a PID (e.g.* $R = \mathbb{Z}$*). Then* $\operatorname{Tor}_k^R(N, M) = 0$ *for arbitrary* M *and* N *and all* $k \geq 2$.

Proof. By 9.20 we have a projective resolution P of M with $P_k = 0$ for all $k \geq 2$. Hence $N \otimes_R P_k = 0$ and thus also $\operatorname{Tor}_k(N, M) := H_k(N \otimes_R P) = 0$ for those k. □

10.24 Remark.[20, 10.3.6]

1. A module M is flat iff $\operatorname{Tor}_1^R(N,M)=0$ for all N:
   ($\Leftarrow$) obvious by 10.21.4 and 10.21.2.
   ($\Rightarrow$) Let $0\to Q\to P\to N\to 0$ be short exact with free P. By 10.21.4 we have the exact sequence
   $$0=\operatorname{Tor}_1^R(P,M)\to \operatorname{Tor}_1^R(N,M)\to Q\otimes_R M\rightarrowtail P\otimes_R M$$
   with $\operatorname{Tor}_1^R(P,M)=0$ by 10.21.3 since P is free and with $Q\otimes_R M\to P\otimes_R M$ injective since M is flat. Thus $\operatorname{Tor}_1^R(N,M)=0$.
2. $\operatorname{Tor}_1(A,\mathbb{Z}_n)\cong\{a\in A: na=0\}$: Consider the short exact sequence $n\mathbb{Z}\hookrightarrow\mathbb{Z}\twoheadrightarrow\mathbb{Z}_n$ leading to the long exact sequence
   $$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Tor}_1(A,\mathbb{Z}) \arrow[r] \arrow[d, equal, "10.21.3"'] & \operatorname{Tor}_1(A,\mathbb{Z}_n) \arrow[r, tail] \arrow[d, equal] & A\otimes n\mathbb{Z} \arrow[r] \arrow[d, equal] & A\otimes\mathbb{Z} \arrow[r] \arrow[d, equal] & A\otimes\mathbb{Z}_n \arrow[r] \arrow[d, equal, "10.17.1"] & 0 \\ & 0 & \{a: na=0\} & A \arrow[r, "n"] & A & A/nA & \end{tikzcd}$$
   More generally, $\operatorname{Tor}_1^R(R/(Rr),M)\cong\{x\in M: rx=0\}$ provided R is commutative and r not a zero divisor: Again $R\xrightarrow{r}R\to R/(Rr)$ is short exact, hence we have the exact sequence
   $$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Tor}_1^R(R,M) \arrow[r] \arrow[d, equal, "10.21.3"'] & \operatorname{Tor}_1^R(R/(Rr),M) \arrow[r, tail] \arrow[d, equal] & R\otimes_R M \arrow[r, "r"] \arrow[d, equal] & R\otimes_R M \arrow[r] \arrow[d, equal] & \cdots \\ & 0 & \{x\in M: r\,x=0\} & M \arrow[r, "r"] & M & \end{tikzcd}$$
3. $\operatorname{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n)=\mathbb{Z}_{\gcd(m,n)}$:
   Again by 2 we have $\operatorname{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n)=\{a\in\mathbb{Z}_m: na=0\}\cong\mathbb{Z}_{\gcd(m,n)}$.

10.25 Lemma. *Let R be a PID. Then $\operatorname{Tor}_1^R(A,B)=\operatorname{Tor}_1^R(\operatorname{Tor}(A),\operatorname{Tor}(B))$, where $\operatorname{Tor}(G)$ denotes the torsion submodule of G.*

This motivates the notation Tor_1, which is also called the TORSION PRODUCT .

Proof. Consider the short exact sequence $\operatorname{Tor}(B)\hookrightarrow B\twoheadrightarrow B/\operatorname{Tor}(B)$. Since $B/\operatorname{Tor}(B)$ is torsion-free we get $\operatorname{Tor}_1^R(A,B/\operatorname{Tor}(B))=0$ by 10.12. So get an exact sequence

$$\begin{tikzcd} \operatorname{Tor}_2^R(A,B/\operatorname{Tor}(B)) \arrow[r] \arrow[d, equal, "10.26"'] & \operatorname{Tor}_1^R(A,\operatorname{Tor}(B)) \arrow[r] & \operatorname{Tor}_1^R(A,B) \arrow[r] & \operatorname{Tor}_1^R(A,B/\operatorname{Tor}(B)) \arrow[d, equal, "10.12"'] \\ 0 & & & 0 \end{tikzcd}$$

and hence an isomorphism $\operatorname{Tor}_1^R(A,\operatorname{Tor}(B))\cong\operatorname{Tor}_1^R(A,B)$. Now use the symmetry of Tor_1^R. $\square$

▷ **10.26 Künneth theorem.** [20, 12.3.3] *Let R be a PID and C a chain complex of free (or at least flat) modules and C' be any chain complex. Then we have natural short exact sequences*

$$\coprod_{p+q=n} H_p(C)\otimes_R H_q(C') \rightarrowtail H_n(C\otimes_R C') \twoheadrightarrow \coprod_{p+q=n-1}\operatorname{Tor}_1^R(H_p(C),H_q(C')).$$

If C and C' are free, then the sequences split.

The tensor product of chain complexes has $\coprod_{p+q=n} C_p\otimes C'_q$ as n-th component $(C\otimes_R C')_n$ by definition and the boundary operator is given by $\partial(c\otimes c'):=\partial c\otimes$

$c' + (-1)^p c \otimes \partial c'$ for $c \in C_p$ and $c' \in C_q$.
We will also use the abbreviation

$$\mathrm{Tor}_1^R(H_p(C), H_q(C'))_{n-1} := \coprod_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(C')).$$

▽ **Proof.** Again we start with the short exact (and, in case C_{p-1} and hence B_{p-1} is free, splitting) sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_p \arrow[r, hook, "j"] & C_p \arrow[r, two heads, "\partial"] & B_{p-1} \arrow[r] & 0 \end{tikzcd}$$

Tensoring with C'_q and takeing direct the sums over $p+q=n$ gives short exact sequences (If C_{p-1} is flat(=torsion-free) then also B_{p-1} hence $\mathrm{Tor}_1^R(B_{p-1}, C'_q) = 0$) of chain complexes (where $(\bar{B})_p := B_{p-1}$) by 10.21.4:

$$\begin{tikzcd} 0 \arrow[r] & Z \otimes C' \arrow[r, hook, "j\otimes C'"] & C \otimes C' \arrow[r, two heads, "\partial\otimes C'"] & \bar{B} \otimes C' \arrow[r] & 0. \end{tikzcd}$$

By 7.30 we get the long exact sequence in homology:

$$\begin{tikzcd} \cdots \arrow[r, "\delta_*"] & H_n(Z \otimes C') \arrow[r, "(j\otimes C')_*"] \arrow[d, equal] & H_n(C \otimes C') \arrow[r, "(\partial\otimes C')_*"] & H_{n-1}(B \otimes C') \arrow[r, "\delta_*"] \arrow[d, equal] & H_{n-1}(Z \otimes C') \arrow[r] \arrow[d, equal] & \cdots \\ & (Z \otimes H(C'))_n & & (\bar{B} \otimes H(C'))_n \arrow[r, "i\otimes H'"] & (Z \otimes H(C'))_{n-1} \end{tikzcd}$$

The identities follow from 10.19 by taking direct sums, since the boundary operator on Z and on B is 0. The rectangle commutes by summing up the corresponding rectangles in the proof of 10.19. Again we consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B_p(C) \arrow[r, hook, "i"'] & Z_p(C) \arrow[r, two heads] & H_p(C) \arrow[r] & 0 \end{tikzcd}$$

Taking the tensor product with $H'_q := H_q(C')$ yields the exact sequence (since Z_{p-1} is flat)

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1(H, H')_{n-1} \arrow[r, tail] & B_p \otimes H'_q \arrow[r, "i\otimes H'_q"] \arrow[dr, two heads] & Z_p \otimes H'_q \arrow[r, two heads] & H_p \otimes H'_q \arrow[r] & 0 \\ & & & \mathrm{im}(i \otimes H'_q) \arrow[u, hook] \end{tikzcd}$$

and by summing over $p+q=n$ we get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \coprod_{p+q=n-1} \mathrm{Tor}_1(H_p, H'_q) \arrow[r, tail] & (B \otimes H')_n \arrow[r, "i\otimes H'"] \arrow[dr, two heads] & (Z \otimes H')_n \arrow[r, two heads] & (H \otimes H')_n \arrow[r] & 0 \\ & & & \mathrm{im}((i \otimes H')_n) \arrow[u, hook] \end{tikzcd}$$

In particular, $\mathrm{im}((\partial \otimes_R C')_*) = \ker(\delta_*) = \ker(i \otimes_R H') \cong \coprod_{p+q=n-1} \mathrm{Tor}_1(H_p, H'_q)$.

On the other hand the kernel of $(j \otimes_R C')_*$ is the image of $\delta_* = i \otimes_R H'$, i.e. the kernel of the epimorphism $(Z \otimes_R H')_n \twoheadrightarrow (H \otimes_R H')_n$. Thus $(j \otimes_R C')_*$ factors over

$(H \otimes_R H')_n$ to yield a monomorphism with the kernel of $(\partial \otimes C')_*$ as image:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(\bar B \otimes C') \arrow[r, "\delta_*"'] \arrow[d, equal] & H_n(Z \otimes C') \arrow[r, "j_*"'] \arrow[d, equal] & H_n(C \otimes C') \arrow[r, "\partial_*"'] \arrow[dd, equal] & H_n(\bar B \otimes C') \arrow[r, "\delta_*"'] \arrow[d, equal] & H_{n-1}(Z \otimes C') \arrow[r] \arrow[d, equal] & \cdots \\
& (B \otimes H')_n \arrow[r, "i \otimes H'"'] & (Z \otimes H')_n \arrow[d] & & (B \otimes H')_{n-1} \arrow[r, "i \otimes H'"'] & (Z \otimes H')_{n-1} & \\
& 0 \arrow[r] & \mathrm{Cok}(i \otimes H') \arrow[r, tail] \arrow[d, equal] & H_n(C \otimes C') \arrow[r, dotted, two heads] & \mathrm{Ker}(i \otimes H') \arrow[r] \arrow[u, hook] \arrow[d, equal] & 0 & \\
& & (H \otimes H')_n & & \mathrm{Tor}_1(H \otimes H')_{n-1} & &
\end{tikzcd}$$

If both chain complexes are free, then we have retractions $r : C_p \to Z_p$ and $r' : C'_q \to Z'_q$. The homomorphism $r \otimes r' : (C \otimes_R C')_n \to (H(C) \otimes_R H(C'))_n$ maps the boundaries of $(C \otimes C')_n$ to 0, hence induces a homomorphism $H_n(C \otimes C') \to (H(C) \otimes H(C'))_n$, which is obviously inverse to the monomorphism $(H(C) \otimes H(C'))_n \to H_n(C \otimes C')$ constructed above. $\square$

As a special case of 10.26 we obtain:

▷ **10.28 Universal coefficient theorem for homology of chain complexes.** [20, 10.4.6]
Let C be a free chain complex and M be a module over a PID R. There there is a splitting natural short exact sequence

$$H_q(C) \otimes_R M \rightarrowtail H_q(C \otimes_R M) \twoheadrightarrow \mathrm{Tor}_1^R(H_{q-1}(C), M)$$

▽ **Proof.** Let another chain complex C' be defined by $C'_0 := M$ and $C'_q = 0$ for all $q \neq 0$. By the Künneth-Theorem 10.26 we have the short exact sequence

$$\begin{tikzcd}
\coprod\limits_{p+q=n} H_p(C) \otimes_R H_q(C') \arrow[r, tail] \arrow[d, equal] & H_n(C \otimes_R C') \arrow[r, two heads] \arrow[d, equal] & \coprod\limits_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(C')) \arrow[d, equal] \\
H_n(C) \otimes_R M & H_n(C \otimes_R M) & \mathrm{Tor}_1^R(H_{n-1}(C), M)
\end{tikzcd}$$

Since B_{n-1} is free we get a right inverse $s : B_{n-1} \to C_n$ for ∂. This induces a morphism $B_{n-1} \otimes_R M \to C_n \otimes_R M$, which maps the kernel $\mathrm{Tor}_1^R(H_{n-1}, M)$ of $i \otimes_R M : B_{n-1} \otimes_R M \to Z_{n-1} \otimes_R M$ into $Z_n(C \otimes_R M)$, and thus defines a section for $H_n(C \otimes_R M) \twoheadrightarrow \mathrm{Tor}_1(H_{n-1}(C), M)$.

$$\begin{tikzcd}
\ker(\partial \otimes M) \arrow[d, hook] & \ker(i \otimes M) \arrow[l, dotted] \arrow[d, hook] \\
C_n \otimes M \arrow[r, "\partial \otimes M"] \arrow[d, "\partial \otimes M"'] & B_{n-1} \otimes M \arrow[l, dotted, bend left, "s \otimes M"] \arrow[d, "i \otimes M"] \\
C_{n-1} \otimes M & Z_{n-1} \otimes M \arrow[l, dotted, "\boxed{10.19}", "j \otimes M"']
\end{tikzcd}$$

$\square$

▷ **10.29 Universal coefficient theorem for homology of spaces.** [20, 10.5.3]
Let (X, A) be a pair of spaces and G be an abelian group. Then we have splitting short exact sequences

$$H_q(X, A) \otimes_{\mathbb{Z}} G \rightarrowtail H_q(X, A; G) \twoheadrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{q-1}(X, A), G) \qquad \square$$

▽

10.30 Example.

1. If $H_{q-1}(X)$ is free (or at least torsion-free), then $\operatorname{Tor}_1(H_{q-1}(X),G) = 0$ and hence $H_n(X)\otimes G \cong H_n(X;G)$. In particular, we get easily $H_0(X;G)$, $H_q(D^n,S^{n-1};G)$, $H_q(S^n;G)$, $H_q(F_g;G)$, $H_q(\mathbb{P}^n(\mathbb{C});G)$, etc..
2. $H_q(\mathbb{P}^n;\mathbb{Z}_2)\cong\mathbb{Z}_2$ for $0\le q\le n$:
   By 8.67 we have $H_q(\mathbb{P}^n)\in\{\mathbb{Z},\mathbb{Z}_2,0\}$ and hence $H_q(\mathbb{P}^n)\otimes\mathbb{Z}_2\in\{\mathbb{Z}_2,\mathbb{Z}_2,0\}$ and $\operatorname{Tor}_1(H_q(\mathbb{P}^n),\mathbb{Z}_2)\in\{0,\mathbb{Z}_2,0\}$. Thus $H_q(\mathbb{P}^n;\mathbb{Z}_2)\cong\mathbb{Z}_2$ for $0\le q\le n$.

▷ **10.31 Proposition.** [20, 10.5.5]
The homotopy theorem, the relative Mayer-Vietoris sequence and their consequences (like the excision theorem and the exact sequence for a pair and a triple) hold also for the homology with coefficients.

▽ **Proof.** The homotopy theorem 8.28 carries over, since a homotopy between mappings $(X,A)\to(Y,B)$ induces a chain homotopy for the correspoding chain mappings $S(X,A)\to S(Y,B)$ and tensoring with G gives a chain homotopy for the chain mappings $S(X,A)\otimes_{\mathbb{Z}}G\to S(Y,B)\otimes_{\mathbb{Z}}G$. By 8.23 this induces the identity in the homology (with coefficients).

The relative Mayer-Vietoris sequences (and its consequences) is shown as for the cohomology in 9.46, since all chain complexes considered there consist of free abelian groups, hence the corresponding short exact sequences are splitting and thus are also short exact after tensoring with G. Hence we have the corresponding long exact sequences also in homology with coefficients by 7.30. □

▷ **10.32 Eilenberg-Zilber theorem.** [20, 12.2.6]
There is a natural equivalence of chain complexes $S(X\times Y)\sim S(X)\otimes_{\mathbb{Z}}S(Y)$.

▽ **Proof.** ($\leftarrow$) Let first $X=\Delta_p$ und $Y=\Delta_q$. For $n=0$ we define $\varphi_0:(S(X)\otimes S(Y))_0\to S(X\times Y)_0$ by $\varphi(x\otimes y):=(x,y)$ for $x\in X$ und $y\in Y$. By 9.18 this can be extended to a chain mapping $\varphi: S(\Delta_p)\otimes S(\Delta_q)\to S(\Delta_p\times\Delta_q)$. For arbirary X and Y define φ by $\varphi(\sigma\otimes\tau):=(\sigma\times\tau)_*(\varphi(\Delta_p\otimes\Delta_q))$

($\to$) For $X=\Delta_p$, $Y=\Delta_q$, and $(n=0)$ we define $\psi_0: S(X\times Y)_0\to(S(X)\otimes S(Y))_0$ by $\psi_0(x,y):=x\otimes y$. By the Künneth-Theorem 10.26 we have that $H_n(S(\Delta_p)\otimes S(\Delta_q))=0$ for all $n>0$. For singular 1-simplices σ and τ with $\partial\sigma=: x_1-x_0$ and $\partial\tau=: y_1-y_0$ we have that $(\sigma,\tau):\Delta_1\to X\times Y$ is a singular 1-simplex with boundary $c=(x_1,y_1)-(x_0,y_0)$. Since $\psi_0(c)=x_1\otimes y_1-x_0\otimes y_0=\partial(\sigma\otimes y_1+x_0\otimes\tau)$ we can extend ψ by 9.18 to a chain mapping $\psi: S(X\times Y)\to S(X)\otimes S(Y)$. For arbirary X and Y we define ψ by $\psi(\sigma,\tau):=(\sigma\times\tau)_*(\psi(\Delta_p,\Delta_q))$.

In dimension 0 obviously $\varphi\circ\psi=\mathrm{id}$ and $\psi\circ\varphi=\mathrm{id}$. By 9.18 we get a chain homotopies $\varphi\circ\psi\sim\mathrm{id}$ and $\psi\circ\varphi\sim\mathrm{id}$ for $X=\Delta_p$ and $Y=\Delta_q$. By naturality they can be extended to arbitrary X and Y. □

Using 9.18 one can easily show that ψ is uniqely determined up to chain homotopies and hence the induced isomorphism of homologies is uniquely determined. In particular, one can use

$$\psi(\sigma:\Delta_n\to X\times Y):=\sum_{p+q=n}(\mathrm{pr}_1\circ\sigma\circ\iota_{0,\dots,p})\otimes(\mathrm{pr}_2\circ\sigma\circ\iota_{p,\dots,p+q})\in(S(X)\otimes S(Y))_n.$$

▷ **10.33 Corollary. Künneth theorem for spaces.** [20, 12.4.3]
We have a splitting short exact sequence

$$(H(X)\otimes_{\mathbb{Z}}H(Y))_n \rightarrowtail H_n(X\times Y)\twoheadrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H(X),H(Y))_{n-1}$$

Proof. By the Künneth-Theorem 10.26 we have the splitting short exact sequence

$$\coprod_{p+q=n} H_p(X)\otimes H_q(Y) \rightarrowtail H_n(S(X)\otimes S(Y)) \twoheadrightarrow \coprod_{p+q=n-1} \mathrm{Tor}_1(H_p(X),H_q(Y)).$$

By the Eilenberg-Zilber Theorem 10.32 we have $S(X)\otimes S(Y)\sim S(X\otimes Y)$. Hence $H_n(S(X)\otimes S(Y))\cong H_n(S(X\times Y))=H_n(X\times Y)$ by 8.23. □

10.35 Corollary. [20, 12.5.5] *Let R be a field, then*

$$H_*(X;R)\otimes H_*(Y;R)\cong H_*(X\times Y;R).$$

Proof. Since $C:=S(X)\otimes_{\mathbb{Z}} R$ is a chain complex of R-modules, the $H_p(X;R):=H_p(C)$ are R-modules. Since R is a field, all R-modules are flat by 10.11, hence $\mathrm{Tor}_1^R(H_p(C),H_q(C'))=0$. By the Künneth-Theorem 10.26 $H(X,R)\otimes_R H(Y,R)=H(C)\otimes_R H(C')\cong H(C\otimes_R C')$, so it remains to show that

$$C\otimes_R C'=(S(X)\otimes_{\mathbb{Z}}R)\otimes_R(S(X')\otimes_{\mathbb{Z}}R)\cong(S(X)\otimes_{\mathbb{Z}}S(X'))\otimes_{\mathbb{Z}}R\sim S(X\times X')\otimes_{\mathbb{Z}}R,$$

which is obvious via $(s\otimes r)\otimes(s'\otimes r')\mapsto(s\otimes s')\otimes rr'$ with inverse mapping $(s\otimes 1)\otimes(s'\otimes r)\leftarrow\!\shortmid (s\otimes s')\otimes r$. □

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Index