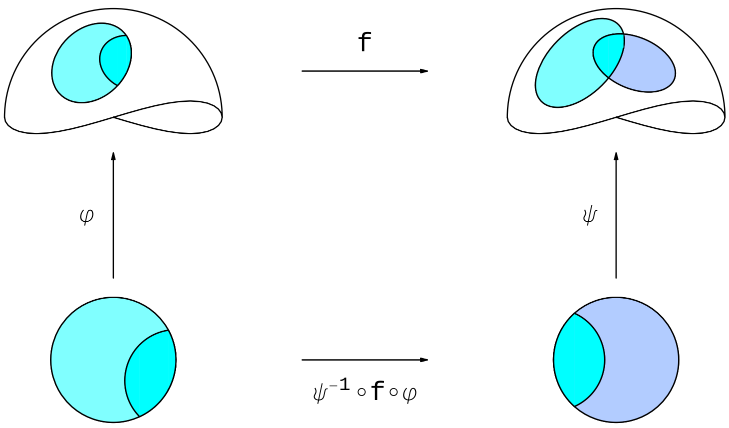
Analysis on Manifolds

Andreas Kriegl
email:andreas.kriegl@univie.ac.at

250096-1, SS 2018, Mon.-Thu. 9^{45}-10^{30}, SR 9



This is the preliminary english version of the script for my lecture course of the same name in the Summer Semester 2018. It was translated from the german original using a pre and post processor (written by myself) for google translate. Due to the limitations of google translate – see the following article by Douglas Hofstadter www.theatlantic.com/.../551570 – heavy corrections by hand had to be done afterwards. However, it is still a rather rough translation which I will try to improve during the semester.

It consists of selected parts of the much more comprehensive differential geometry script (in german), which is also available as a PDF file on http://www.mat.univie.ac.at/∼kriegl/Skripten/diffgeom.pdf.

When choosing the content, I followed the curricula. Accordingly, the following topics should be known, in particular from 'Higher Analysis and Elementary Differential Geometry':

- Curves (see [**81**, 5.5] and [**86**, Kapitel I]), submanifolds of $\mathbb{R}^n$ (see 2.4), partitions of unity (see [**82**, 7.6.2]),
- Transformation formula for multidimensional integrals (see [**82**, 7.5.10]),
- Multilinear forms (see [**82**, 8.2]), differential forms (see [**82**, 8.3]), oriented submanifolds and integration of differential forms (see [**82**, 8.6]), Stokes Theorem (see [**82**, 8.7.3]) and classical integral formulas (see [**82**, 8.1.2,8.1.5,8.1.7]).

And in this lecture should be treated:

- Abstract manifolds,
- Tangential bundle, vector fields and flows, Lie bracket,
- Differential forms, outer derivative and Cartan calculus,
- Integration and the Theorem of Stokes,
- Applications (e.g. symplectic geometry, differential topology).

The structure of the script is thus the following:

In Chapter II, we first recall manifolds as subsets of a Euclidean space, and then introduce them as abstract objects that are obtained by gluing Euclidean spaces.

In Chapter III the concept of derivative is transferred to manifolds. This leads to tangent spaces and tangent mappings and is used to get a notion of sub objects and quotient objects of manifolds.

Ordinary differential equations on manifolds are introduced in Chapter IV. For this, the tangent spaces are merged into a tangent bundle and vector fields are examined as sections of this bundle.

Chapter VI is dedicated to differential forms and their algebraic structure, and also serves as preparation for integration on manifolds in Chapter VII.

At the end of the semester, I will post a detailed list of all the sections treated at https://www.mat.univie.ac.at/∼kriegl/Skripten/2018SS-hist.html.

Of course, the attentive reader will be able to find (typing) errors. I kindly ask to let me know about them (consider the german saying: shared suffering is half the suffering). Future generations of students might appreciate it.

Andreas Kriegl, Vienna in February 2018

Contents

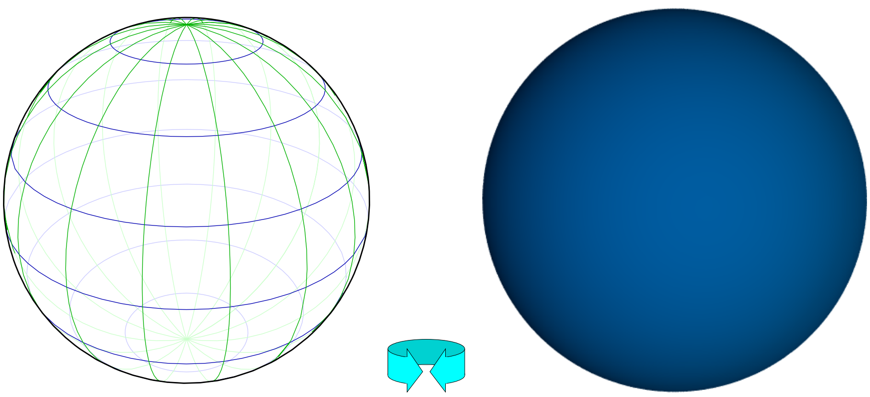
II. Manifolds

In this chapter we introduce the concept of manifolds. We start by playing around with two-dimensional submanifolds of $\mathbb{R}^n$ - so called surfaces -, and we will generalize these in the second section to higher dimensional submanifolds of $\mathbb{R}^n$, and in the third section we will make the examples from the beginning precise. Then we will treat the classical examples of those manifolds, which carry a smooth group structure, so-called Lie-groups. After having introduced the notion of smooth mappings we may turn to abstract manifolds, by which I mean manifolds, which are not embedded into some Euclidean space a priori. After discussing products and disjoint unions of manifolds we come to the question of the abundance of smooth functions on manifolds. In particular, this concerns separation axioms like Hausdorffness, locally compactness, and - most important - paracompactness and the related concept of partitions of unity, which is the main tool for passing from local constructions (like those treated in calculus classes) to global pendants.

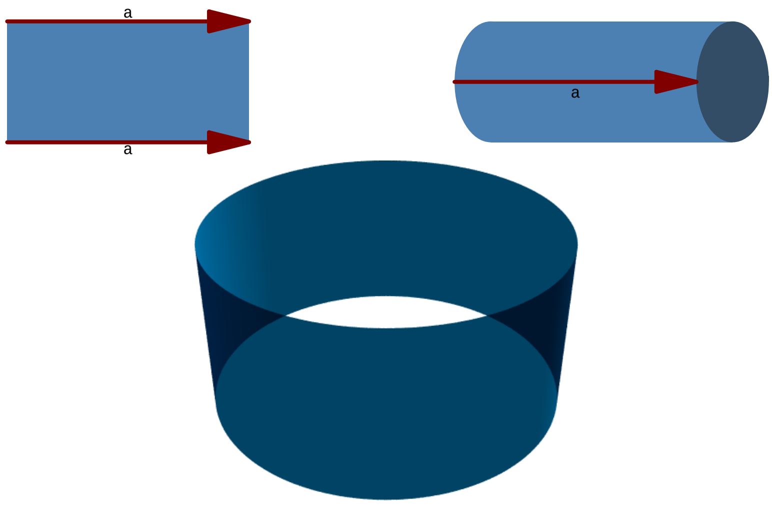
1. Examples of two-dimensional surfaces

For the time being, we want to become playfully acquainted with two-dimensional manifolds in this section. These are objects that look like a disc in the $\mathbb{R}^2$, up to bending and stretching.

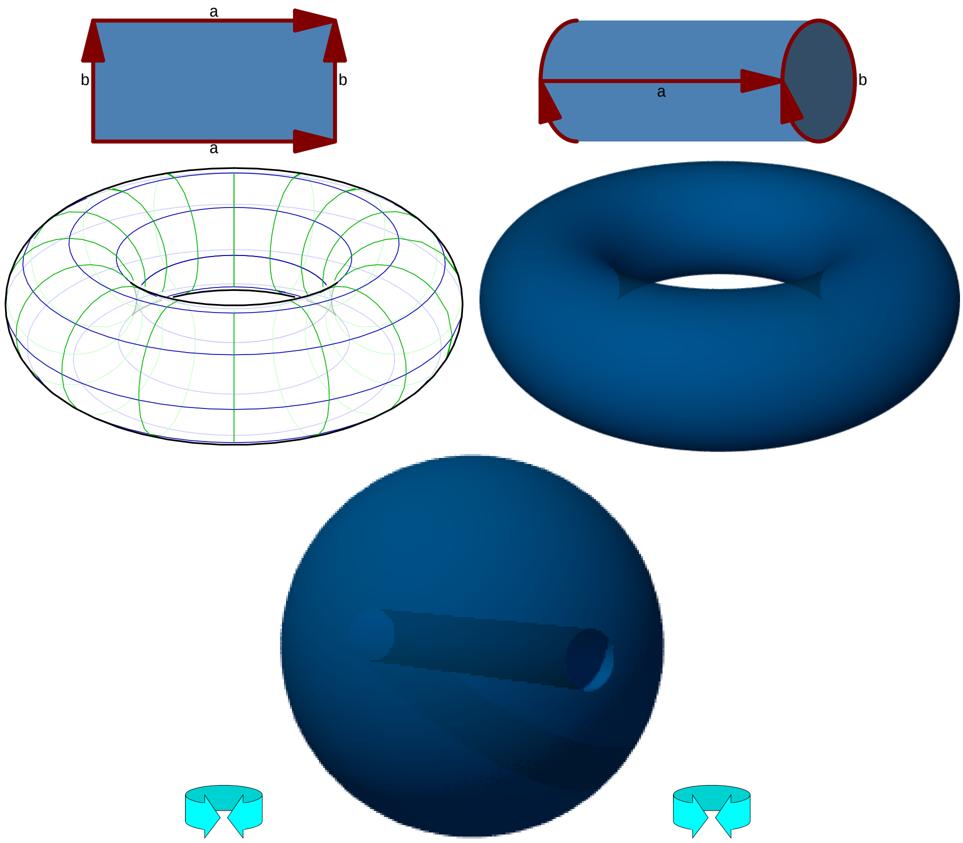
1.1 Examples of orientable surfaces.



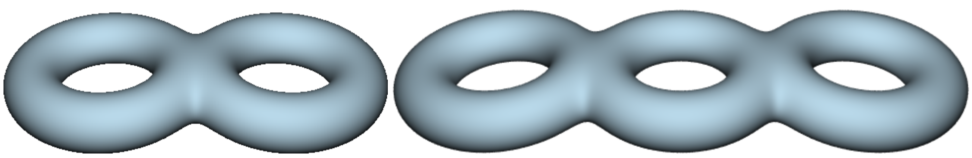
Sphere



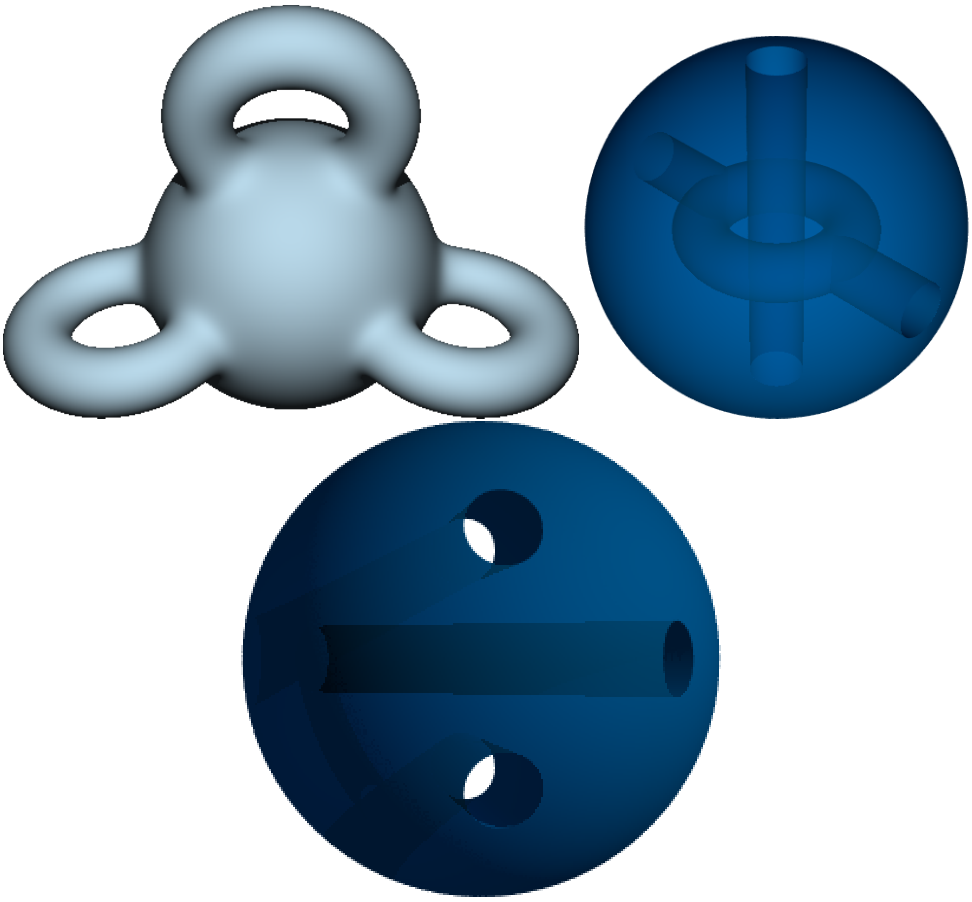
Cylinder



Torus



Orientable compact surfaces of genus 2 and 3



Orientable compact surface of genus 3

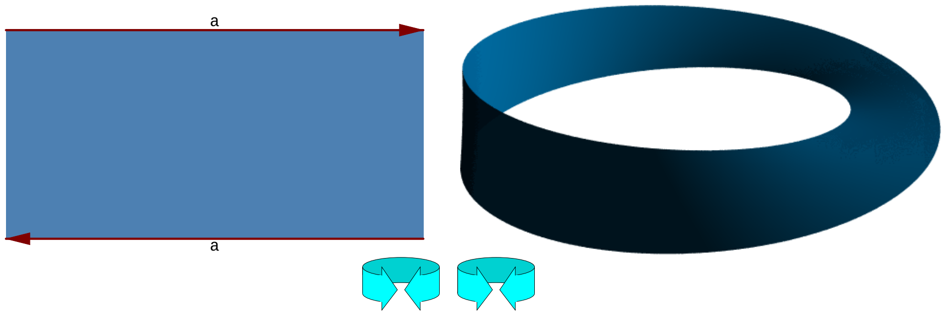
1.2 Classification theorem for orientable surfaces.

Each compact, connected surface in $\mathbb{R}^3$ *is homeomorphic to a surface of some genus* $g \geq 0$*, i.e. arises from the sphere by glueing* g *cylinders to it.*

Without proof, see, e.g. [65, 9.3.5]. We give some evidence for that in 1.4.

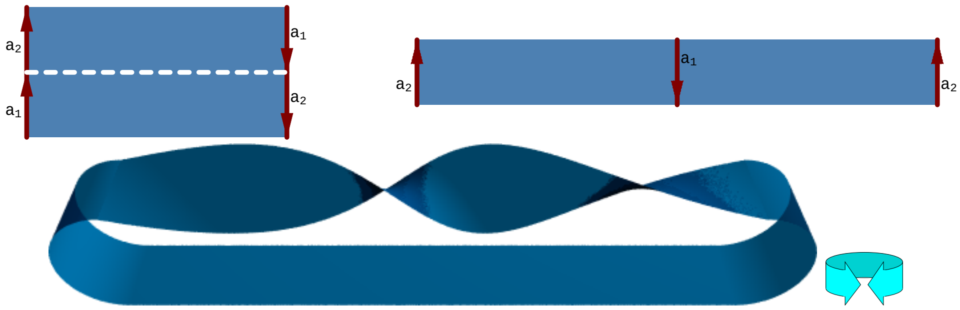
1.3 Examples of non-orientable surfaces.

Examples of two-dimensional, connected, non-orientable surfaces:



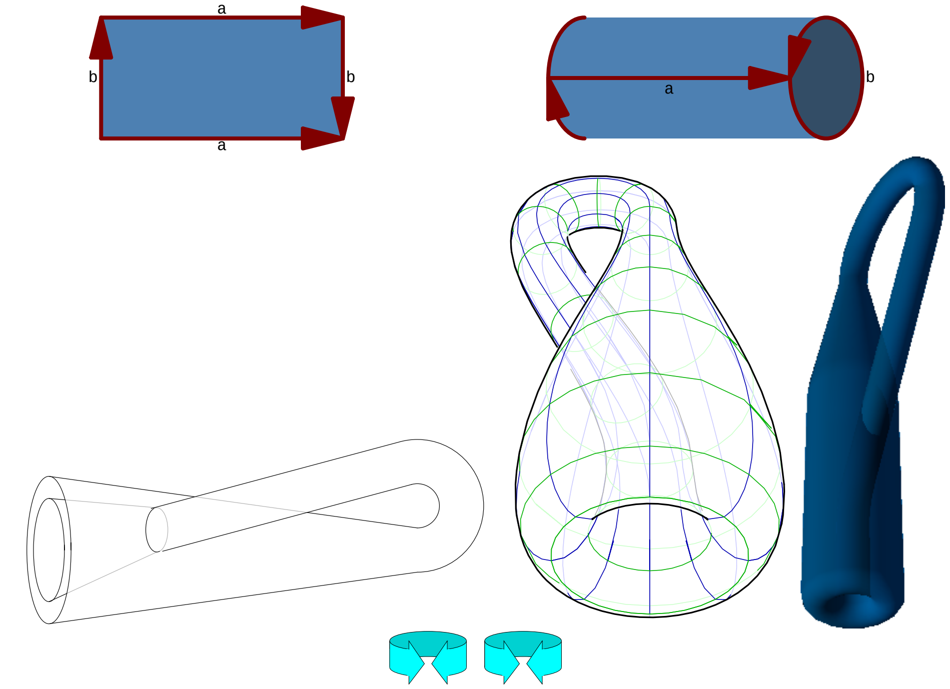
Möbius strip

If you cut the Möbius strip lengthwise, you get a doubly twisted ribbon, which can be untwisted in $\mathbb{R}^4$ (see 3.10).



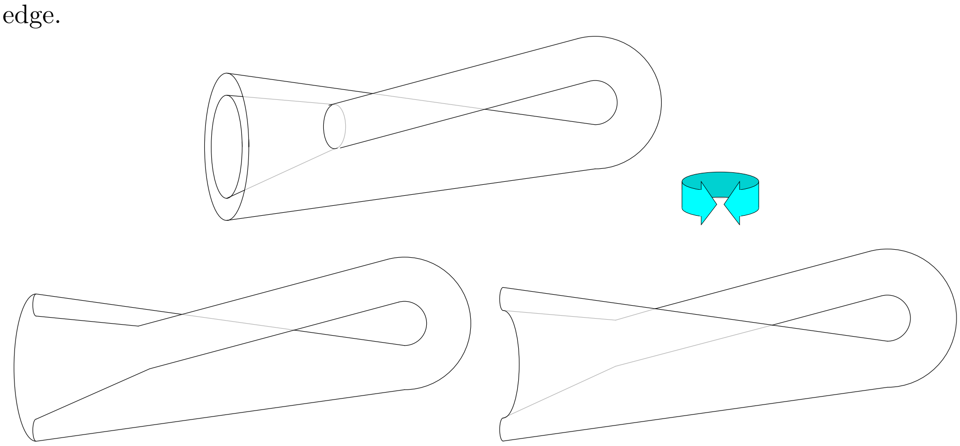
2-fold twisted Möbius strip

Examples of two-dimensional, connected, compact, non-orientable surfaces:



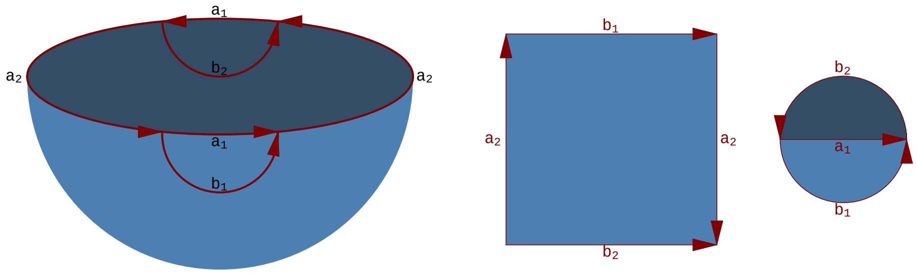
Klein bottle

This is called the Klein bottle, which can be realized in $\mathbb{R}^4$ without self-intersections and which can also be obtained by gluing two Möbius strips along their boundary edge.



Klein's bottle dissected

Another example is the PROJECTIVE PLANE $\mathbb{P}^2$, which is the set of all lines through the zero point in $\mathbb{R}^3$. One can obtain the projective plane from the sphere in the following way: The antipodal points on the sphere generate the same line and hence must be identified with each other. To do this, we stick the northern hemisphere antipodally to the southern one. We still have to identify opposite points on the equator. For this we deform the hemisphere to a disc, from which we cut out a semicircle on both sides and, after gluing the antipodal points at the equator, we get a Möbius strip and a disc. Now you just have to glue the edge of the disc to the boundary of the Möbius strip.

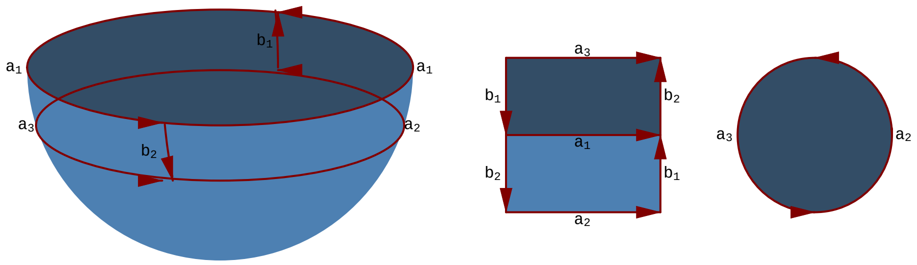


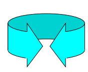
Projective plane

You can imagine this in three ways:

1) Draw the Möbius strip and glue the disk (with self-intersection).

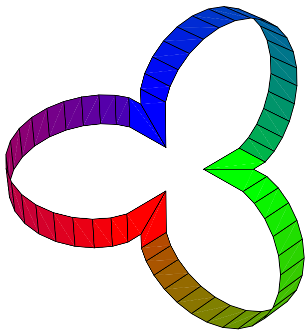
2) Draw the disk and glue the Möbius strip (with self-intersection). This is also called the CROSS CAP.

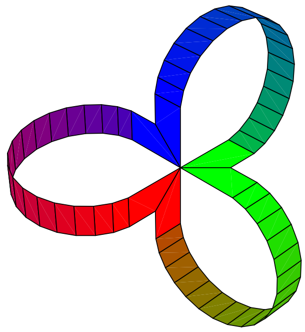


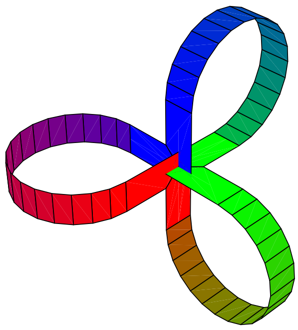


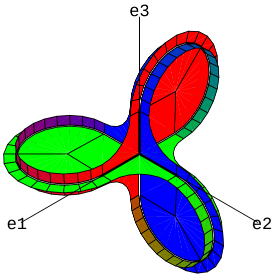
Cross cap

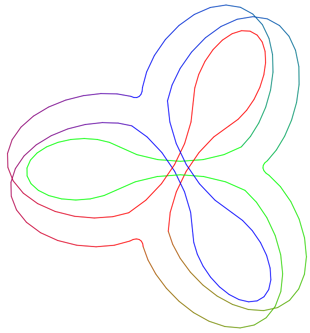
3) Again, we glue a Möbius strip (three-fold twisted and self-intersected) to a disc. This is also called BOY'S SURFACE.

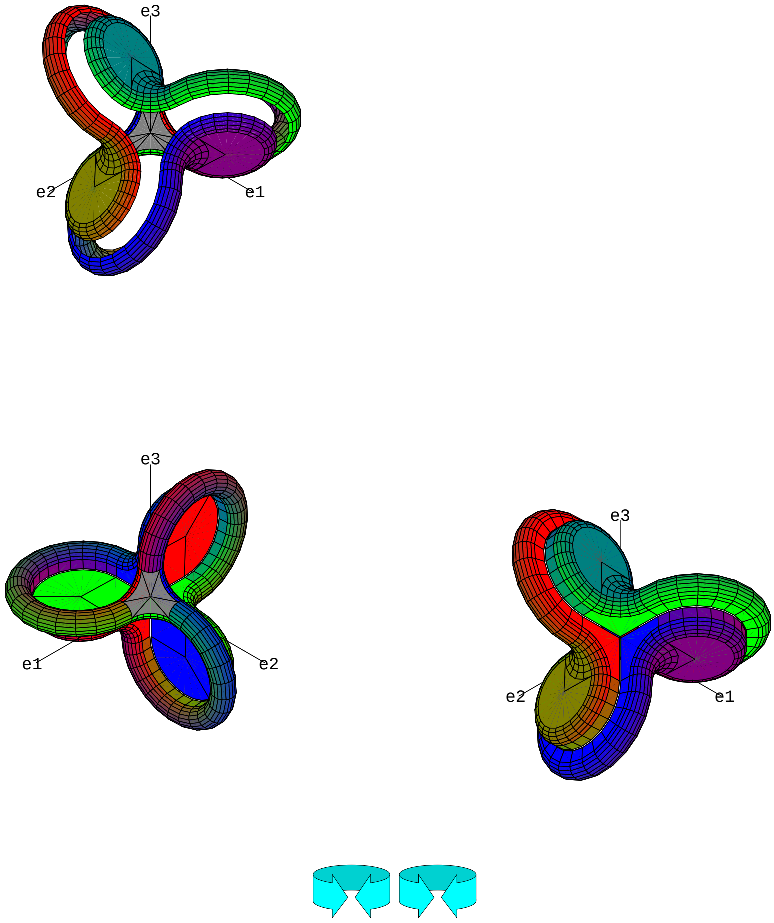







e3
e1
e2



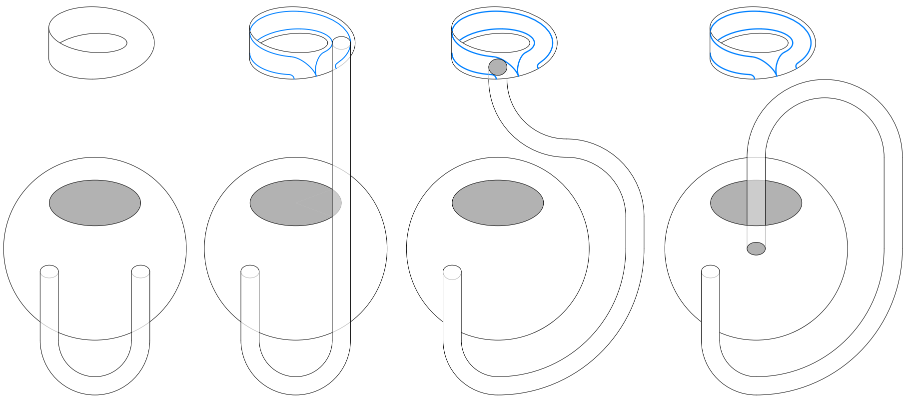


Construction of Boys surface

1.4 Classification theorem for non-orientable surfaces.

Each non-orientable, connected, compact surface arises from a sphere by glueing a finite number of (≥ 1*) cross-caps to it. The number of glued cross-caps is called the genus of the surface.*

Without proof, see, e.g. [**65**, 9.3.10]. An evidential proof for this and for 1.2 uses surgery as follows: Try to find a simply closed curve on the surface M, which does not section M in two parts, and widen this curve to a band, i.e. a rectangle with one pair of parallel sides glued together. Depending on whether this gluing involves a twist or not, it is a Möbius strip or a cylinder. We remove this band and glue one or two disks to the sectioning circle(s) and get a new surface M'. Conversely, M is obtained from M' by gluing a cross cap or a handle to it. We continue this process until the resulting surface decomposes into two parts along each simple closed curve. One has to convince oneself, that this surface is then homeomorphic to the sphere: Each such curve can be extended to a cylinder, and if one glues discs to the complement of this cylinder, then the two smaller resulting surfaces have the same property. So M can be obtained from a sphere by gluing handles and cross-caps to it. However, it is also not obvious that the above process really stops after finitely many steps. Furthermore, it remains to show that it suffices to glue only handles or only cross-caps. It suffices to show that if you cut a hole in a torus and glue a Möbius strip to it, then this is the same as cutting a hole in a Klein bottle and gluing a Möbius strip to it. This is shown by the following drawing:



Transforming of a torus into Klein's Bottle

2. Submanifolds of $\mathbb{R}^n$

In this section we want to define manifolds as sufficiently "regular" subsets of $\mathbb{R}^n$. We will see that these can be described in various ways.

2.1 Definition (Regular mappings).

We generalize the notion of regularity of curves from [**86**, 1.2]. A smooth map $f : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, is called REGULAR if the rank of the derivative $f'(x)$ at each point $x \in U$ is as large as possible, i.e. equal to $\min\{n, m\}$.

Note that a map being regular at one point is regular locally around this point, because the rank can not fall locally.

If $m \leq n$, then regularity means that the derivative is surjective at each point.

From linear algebra we know the following relationships for the rank of a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$:

$$\operatorname{rank}(A) := \dim(\operatorname{im}(A)) = \dim(\mathbb{R}^n) - \dim(\ker(A)).$$

Thus for $m \geq n$ regularity means that the derivative is injective at each point.

For the equivalence of the description of "well-behaved" subsets of $\mathbb{R}^n$ to be given in 2.4, we need the following two central results from multidimensional analysis:

2.2 Inverse function theorem.

Let U be open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be smooth, with $f(0) = 0$, and invertible derivative $f'(0)$ at 0. Then f is a local diffeomorphism, i.e. there are open neighborhoods V and V' of 0, such that $f : V \to V'$ is bijective and f^{-1} is smooth.

Without proof, see Real Analysis, e.g. [**81**, 6.2.1] and [**81**, 6.3.15].

2.3 Implicit function theorem.

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be smooth with $f(0, 0) = 0$ and invertible partial derivative $\partial_2 f(0, 0) : \mathbb{R}^m \to \mathbb{R}^m$. Then there is locally a unique solution $y(x)$ of $f(x, y(x)) = 0$

and $x \mapsto y(x)$ is C^∞. More precisely, there is an open 0-neighborhood $U \times V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ such that for $(x, y) \in U \times V$ we have $f(x, y) = 0 \Leftrightarrow y = g(x)$.

Proof. See also [81, 6.2.3] and [81, 6.3.15]. We define $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ with $F(x, y) := (x, f(x, y))$. This mapping is smooth and $F(0, 0) = 0$. Its derivative at $(0, 0)$ is given by the $(n + m) \times (n + m)$-matrix

$$F'(0,0) = \begin{pmatrix} \mathrm{id} & 0 \\ * & \partial_2 f(0,0) \end{pmatrix}$$

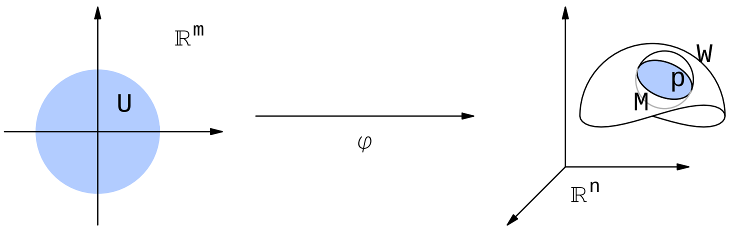
This is invertible, so it follows from the Inverse Function Theorem 2.2 that F^{-1} exists locally and is smooth. Since F is the identity in the first variable, the same holds for F^{-1}. Thus let $(u, g(u, v)) := F^{-1}(u, v)$, then:

$$\begin{aligned} f(x,y) = 0 \Leftrightarrow F(x,y) = (x,0) \Leftrightarrow \\ \Leftrightarrow (x,y) = F^{-1}(x,0) = (x, g(x,0)) \Leftrightarrow y = g(x,0) \qquad \square \end{aligned}$$

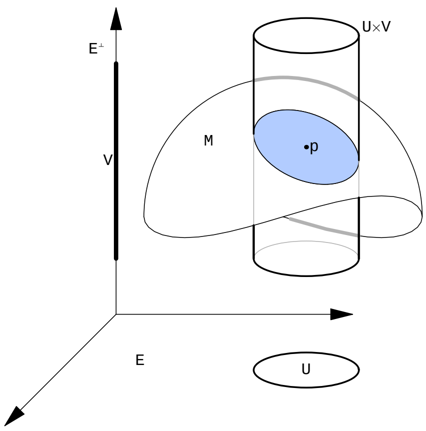
2.4 Proposition (Characterization of submanifolds).

For a subset $M \subseteq \mathbb{R}^n$ with $p \in M$ and $m \leq n$, the following statements are equivalent:

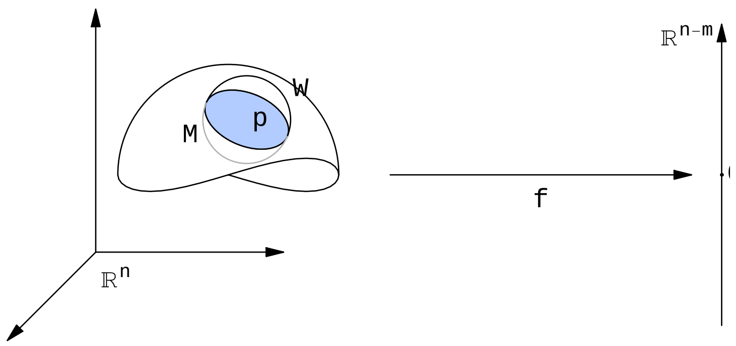
1. *(Local parameterization) There is a smooth and at 0 regular mapping $\varphi : U \to \mathbb{R}^n$, where U is open in $\mathbb{R}^m$ with $0 \in U$ and $\varphi(0) = p$, such that for each open neighborhood $U_1 \subseteq U$ of 0 an open neighborhood W of p exists in $\mathbb{R}^n$ with $\varphi(U_1) = M \cap W$.*



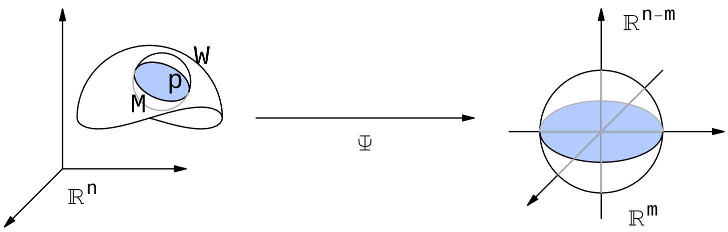
2. *(Local graph) There is a smooth mapping $g : U \to V$, where U is open in an m-dimensional subspace E of $\mathbb{R}^n$ and V is open in in the orthogonal complement $E^\perp$, with $p \in M \cap (U \times V) = \mathrm{graph}(g) := \{(x, g(x)) : x \in U\} \subseteq E \times E^\perp \cong \mathbb{R}^n$.*



3. *(Local equation) There is a smooth and at p regular mapping $f : W \to \mathbb{R}^{n-m}$, where W is open in $\mathbb{R}^n$, with $p \in M \cap W = f^{-1}(0)$.*



4. *(*LOCAL TRIVIALIZATION*) There is a diffeomorphism $\Psi : W \to W'$, where W' is open in $\mathbb{R}^m \times \mathbb{R}^{n-m}$ and W is open in $\mathbb{R}^n$, with $p \in M \cap W = \Psi^{-1}(W' \cap (\mathbb{R}^m \times \{0\}))$.*



Proof. Without loss of generality $p = 0$ since for an affine mapping α the statements hold $p \in M$ iff they hold for M replaced by $\alpha(M)$ and p by $\alpha(p)$ when we compose the claimed regular mappings with α and/or α^{-1}.

($\boxed{1} \Rightarrow \boxed{4}$) Let $\varphi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ be a local parametrization as in $\boxed{1}$. Analogous to [86, 2.3], we want to extend φ to a local diffeomorphism Φ. Let $E \subseteq \mathbb{R}^n$ be the image of $\varphi'(0)$. Due to the regularity of φ, $\dim(E) = m$ and with respect to $E \times E^\perp \cong E \oplus E^\perp = \mathbb{R}^n$ let $\varphi = (\varphi_1, \varphi_2)$, hence $\varphi'(0) = (\varphi_1'(0), \varphi_2'(0))$. Consequently, $\varphi_2'(0) = 0$ and $\varphi_1'(0) : \mathbb{R}^m \to E$ is injective (hence bijective).

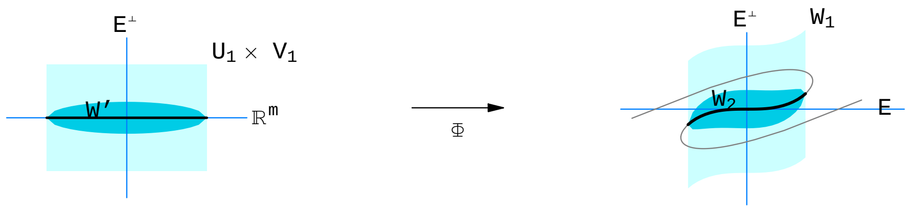
Let $\Phi : \mathbb{R}^m \oplus E^\perp \supseteq U \oplus E^\perp \to E \oplus E^\perp$ be defined by

$$\Phi(u, v) := \varphi(u) + v = (\varphi_1(u), \varphi_2(u) + v).$$

The Jacobi matrix of Φ at $(0, 0)$ has block form:

$$\Phi'(0,0) = \begin{pmatrix} \varphi_1'(0) & 0 \\ \varphi_2'(0) & \mathrm{id} \end{pmatrix}.$$

It is invertible because $\varphi_1'(0) : \mathbb{R}^m \to E$ is bijective! It follows from the Inverse Function Theorem $\boxed{2.2}$ that Φ is a local diffeomorphism, that is, $\exists\, U_1 \subseteq U \subseteq \mathbb{R}^m$ open, $\exists\, V_1 \subseteq E^\perp$ open, and $\exists\, W_1 \subseteq W$ open, so that $\Phi : U_1 \times V_1 \to W_1$ is a diffeomorphism.



In particular, $\varphi(U_1) = \Phi(U_1 \times \{0\}) \subseteq W_1 \subseteq W$. Because of the property 1 of φ, there is an open $W_2 \subseteq \mathbb{R}^n$, and without loss of generality, $W_2 \subseteq W_1$, with $\varphi(U_1) = W_2 \cap M$. Then $U_1 \times \{0\} \subseteq W' := \Phi^{-1}(W_2) \subseteq U_1 \times V_1$, because $\Phi(U_1 \times \{0\}) = \varphi(U_1) = W_2 \cap M$, and furthermore $\Phi : W' \to W_2$ is a diffeomorphism with inverse mapping $\Psi := \Phi^{-1} : W_2 \to W'$. Thus, $\Psi(W_2 \cap M) = U_1 \times \{0\} = W' \cap (\mathbb{R}^m \times \{0\})$ holds.

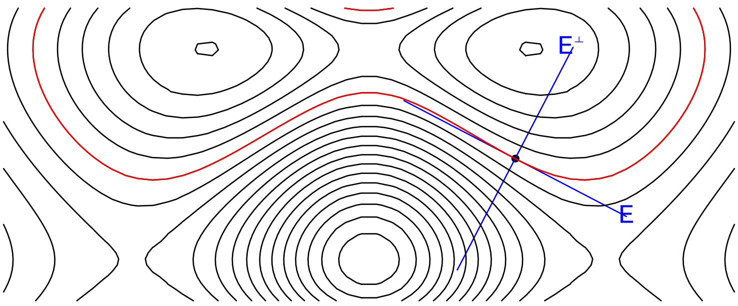
$$\begin{tikzcd}
U \times E^\perp \arrow[r, "\Phi", "0."'] & \mathbb{R}^n \\
U_1 \times V_1 \arrow[u, hook] \arrow[r, tail, two heads, dotted, "1."'] & W_1 \arrow[u, hook] \\
W' := \Phi^{-1}(W_2) \arrow[u, hook] \arrow[r, tail, two heads, dotted, "3."'] & W_2 \arrow[u, hook] \\
W' \cap E = U_1 \arrow[u, hook] \arrow[r, tail, two heads, dotted, "\varphi", "2."'] & W_2 \cap M \arrow[u, hook]
\end{tikzcd}$$

In particular, φ is on U_1 the restriction of the homeomorphism $\Phi : W' \to W_2$, hence $\varphi : U_1 \to M$ is a topological embedding onto the open subset $W_2 \cap M$ of M.

(**4**$\Rightarrow$**3**) Let Ψ be a local trivialization as in 4 and put $f := \mathrm{pr}_2 \circ \Psi$, where $\mathrm{pr}_2 : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ is the projection onto the second factor. Since $f'(z) = \underbrace{\mathrm{pr}_2}_{\text{surj.}} \circ \underbrace{\Psi'(z)}_{\text{bij.}}$ is onto, f is regular. If $z \in W$, then:

$$z \in M \Leftrightarrow \Psi(z) \in \mathbb{R}^m \times \{0\} \Leftrightarrow 0 = (\mathrm{pr}_2 \circ \Psi)(z) = f(z).$$

(**3**$\Rightarrow$**2**) Let $f : W \to \mathbb{R}^{n-m}$ be a local equation as in 3.



We define $E := \ker f'(0)$ and use $\mathbb{R}^n = E \oplus E^\perp$. Because of

$$\dim \underbrace{\ker f'(0)}_{E} + \underbrace{\dim \operatorname{im} f'(0)}_{n-m} = \underbrace{\dim \mathbb{R}^n}_{n},$$

$\dim E = m$ and $\dim E^\perp = n-m$. We are looking for a function $g : E \to E^\perp$, which is implicitly given as solution $g(x) := y$ of $f(x,y) = 0$ (i.e. $(x,y) \in M$). In order to apply the Implicit Function Theorem 2.3 we have to show thatr the second partial derivative of f

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \partial_2 f(0,0) : E^\perp \to \mathbb{R}^{n-m}$$

is bijective: Because of $f'(0)(v_1, v_2) = \partial_1 f(0)(v_1) + \partial_2 f(0)(v_2)$ we have $\partial_1 f(0)(v_1) = f'(0)(v_1, 0) = 0$ for all $v_1 \in E = \ker f'(0)$, thus $f'(0) = \partial_2 f(0) \circ \mathrm{pr}_2 : \mathbb{R}^n \to E^\perp \to \mathbb{R}^{n-m}$ and is surjective by assumption. Hence also $\partial_2 f(0) : E^\perp \to \mathbb{R}^{n-m}$ is surjective and therefore bijective because of $\dim(E^\perp) = n - m$.
By the Implicit Function Theorem 2.3 there exists an open 0-neighborhood $U \times V \subseteq W \subseteq E \times E^\perp$ and a smooth $g : U \to V$, with $g(x) = y \Leftrightarrow f(x,y) = 0$ for each $(x,y) \in U \times V$.

($2 \Rightarrow 1$) Let M be described locally as the graph of $g : E \supseteq U \to V \subseteq E^\perp$. We define the smooth mapping $\varphi : U \to E \times E^\perp \cong \mathbb{R}^n$ by $x \mapsto (x, g(x))$. Remains to show that φ locally describes the set M. For $(x,y) \in U \times V =: W$ we conclude as follows:

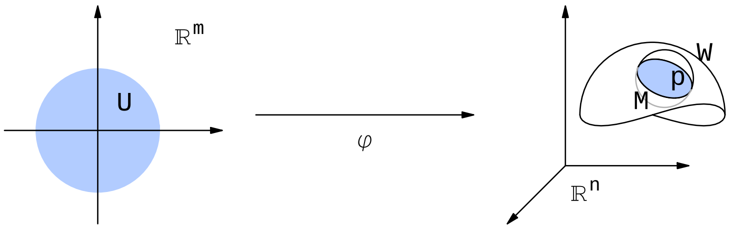
$$(x, y) \in M \Leftrightarrow (x, y) \in \operatorname{graph}(g) \Leftrightarrow y = g(x) \Leftrightarrow (x, y) = (x, g(x)) = \varphi(x).$$

The mapping φ is locally a topological embedding, because $(x, y) \mapsto y$ describes a left inverse. □

Definition (Concrete manifold).

A subset M of $\mathbb{R}^n$ having one of the above equivalent properties for all of its points $p \in M$ is called C^∞-(SUB-)MANIFOLD (OF $\mathbb{R}^n$) of dimension m. Unlike curves, these manifolds do not have self-intersections even for $m = 1$.

A smooth regular mapping $\varphi : \mathbb{R}^m \supseteq U \to M \subseteq \mathbb{R}^n$ with open $U \subseteq \mathbb{R}^m$ and $\varphi(0) = p$, which is a topological embedding onto an open subset of M, is called LOCAL PARAMETERIZATION of M (centered at p). In ($\boxed{1} \Rightarrow \boxed{4}$) we have shown that any φ satisfying $\boxed{1}$ is a local parameterization (on some smaller 0-neighborhood).



The components $u^1, \ldots, u^m$ of the inverse mapping $(u^1, \ldots, u^m) = u = \varphi^{-1} : \varphi(U) \to U$ to a local parameterization φ are called local coordinates of M. Points $p \in M$ can therefore be described locally (after specification of a parameterization φ) by m numbers $u^1(p), \ldots, u^m(p)$.

3. Examples of submanifolds

In this section, we will now give several examples of submanifolds M, providing at the same time precise definitions for the surfaces in $\boxed{1}$.

3.1 The circle.

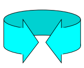
1. Equation: $x^2 + y^2 = R^2$.
   Thus $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^2 + y^2 - R^2$ describes an equation for M that is regular on $W := \mathbb{R}^2 \setminus \{0\}$.
2. Parameterization: $\varphi \mapsto (x, y) := (R \cdot \cos\varphi, R \cdot \sin\varphi)$.
   For all $(x_0, y_0) \in M$ there is a $\varphi_0 \in \mathbb{R}$ (given by $e^{i\varphi_0} = (x_0, y_0)$), s.t. $\varphi \mapsto (x, y)$ is a local parameterization from $U :=]\varphi_0 - \pi, \varphi_0 + \pi[$ to $W \cap M$ with $W := \mathbb{R}^2 \setminus \{(-x_0, -y_0)\}$.
3. Graph: $y = \pm\sqrt{R^2 - x^2}$ or $x = \pm\sqrt{R^2 - y^2}$
   Put $E := \mathbb{R} \times \{0\}$, $U :=]-R, +R[\subset E$, and $V :=]0, +\infty[\subset E^\perp$. Then $M \cap (U \times V) = \{(x, \sqrt{R^2 - x^2}) : x \in U\}$ is a local representation of M as graph of $g : U \to V$.
4. Trivialization: $\Psi^{-1} : (r, \varphi) \mapsto (r \cdot \cos\varphi, r \cdot \sin\varphi)$. Then $\Psi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Psi(M) = \{R\} \times \mathbb{R} \cong \mathbb{R}$. These are just polar coordinates.

3.2 The cylinder.

1. Equation: $x^2 + y^2 + 0 \cdot z = R^2$.
   Note that this is the same equation as that of the circle, but now understood as an equation on $\mathbb{R}^3$.
2. Parameterization: $(\varphi, z) \mapsto (R \cdot \cos\varphi, R \cdot \sin\varphi, z)$. We obtain this parameterization by parametrizing a generator of the cylinder by means of $z \mapsto (R, 0, z)$ and rotating it by the angle φ around the z axis via
$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
   i.e.
$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}.$$
3. Graph: $y = \pm\sqrt{R^2 - x^2}$ or $x = \pm\sqrt{R^2 - y^2}$.
4. Trivialization: $(\varphi, r, z) \leftrightarrow (r \cdot \cos\varphi, r \cdot \sin\varphi, z)$, these are the cylindrical coordinates.

A parameterization $f : \mathbb{R}^m \supseteq U \to M \subseteq \mathbb{R}^n$ is (by definition) LENGTH-PRESERVING if and only if the length of each curve $c : [a, b] \to U \subseteq \mathbb{R}^m$ is equal to that of the image curve $f \circ c : [a, b] \to M \subseteq \mathbb{R}^n$, i.e.
$$\int_a^b |c'(t)|\, dt = \int_a^b |(f \circ c)'(t)|\, dt = \int_a^b |f'(c(t))(c'(t))|\, dt$$
holds. This is exactly fulfilled if $f'(p)$ is an isometry for all $p \in U$, ie
$$|f'(p)(v)| = |v| \text{ for all } v \in \mathbb{R}^m$$
Namely let f be length preserving, $v \in \mathbb{R}^n$ and $c_s : t \mapsto p + t\, s\, v$. Then $c_s : [0, 1] \to U$ for all $s > 0$ close to p and thus
$$s\,|v| = \int_0^1 |c_s'(t)|\, dt = \int_0^1 |f'(c_s(t))(s\, v)|\, dt = s \int_0^1 |f'(c_s(t))(v)|\, dt$$
and, since $c_s \to c_0$ for $s \to 0$ uniformly on $[0, 1]$:
$$|v| = \int_0^1 |f'(c_0(t))(v)|\, dt = \int_0^1 |f'(p)(v)|\, dt = |f'(p)(v)|.$$
The reverse implication is obvious.

The above parameterization $f : (\varphi, z) \mapsto (R\cos\varphi, R\sin\varphi, z)$ is not length preserving for $R \neq 1$, because
$$|f'(\varphi, z)(1, 0)| = |\tfrac{\partial}{\partial\varphi} f(\varphi, z)| = |R(-\sin\varphi, \cos\varphi, 0)| = R \neq |(1, 0)|.$$
However, this can easily be corrected by considering the new parameterization $f : (\varphi, z) \mapsto (R\, e^{i\,\varphi/R}, z)$ with derivative
$$f'(\varphi, z) = \begin{pmatrix} -\sin(\frac{\varphi}{R}) & 0 \\ \cos(\frac{\varphi}{R}) & 0 \\ 0 & 1 \end{pmatrix}$$
The columns now form an orthonormal system, so $f'(\varphi, z)$ is an isometry and thus f is length preserving.



3.3 The cone.

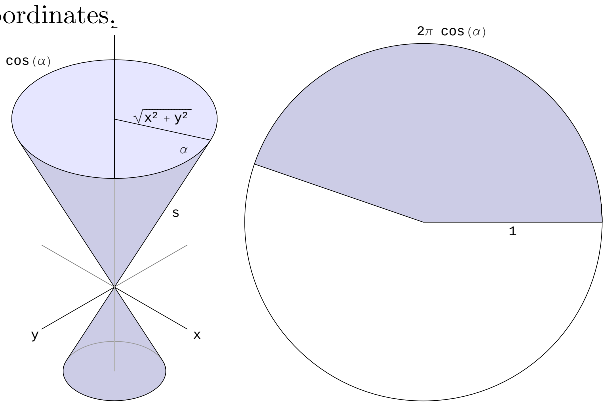
It is formed by rotating a straight line through zero with slope α around the z axis.

1. Equation: $\tan\alpha = z/\sqrt{x^2+y^2}$ or $(x^2+y^2)\tan^2\alpha = z^2$. The former describes the cone, the latter the double cone. The equation is not regular at $(0,0,0)$, so we need to remove the tip, because there the (double) cone is not a manifold.

2. Parameterization: $(\varphi, s) \mapsto (s\cos\alpha\cos\varphi, s\cos\alpha\sin\varphi, s\sin\alpha)$.
   We obtain this parameterization by parameterizing a generator of the cone by arc length as $s \mapsto (s\cos\alpha, 0, s\sin\alpha)$ and rotating that by the angle φ around the z axis via
$$\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix},$$
   i.e.
$$\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}s\cos\alpha\\ 0\\ s\sin\alpha\end{pmatrix} = \begin{pmatrix}s\cos\alpha\cos\varphi\\ s\cos\alpha\sin\varphi\\ s\sin\alpha\end{pmatrix}.$$

3. Graph: $z = \pm\tan\alpha\sqrt{x^2+y^2}$

4. Trivialization: $(\varphi, \alpha, s) \leftrightarrow (s\cos\alpha\cos\varphi, s\cos\alpha\sin\varphi, s\sin\alpha)$, these are the spherical coordinates.



A better parameterization is obtained by unfurling the cone into the plane:
$$(x,y)\mapsto(r,\psi)\mapsto\left(s:=r, \varphi:=\frac{\psi}{\cos\alpha}\right)\mapsto\begin{pmatrix}r\cos\alpha\cos\left(\frac{\psi}{\cos\alpha}\right)\\ r\cos\alpha\sin\left(\frac{\psi}{\cos\alpha}\right)\\ r\sin\alpha\end{pmatrix},$$
where (x,y) are Cartesian coordinates and (ψ, r) are polar coordinates in the plane.

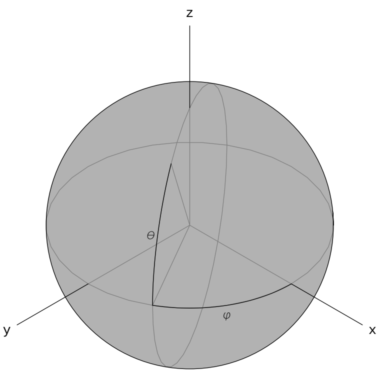
The derivative of this parameterization is the composition of
$$\begin{pmatrix}\cos\alpha\cdot\cos(\frac{\psi}{\cos\alpha}) & -r\cos\alpha\cdot\sin(\frac{\psi}{\cos\alpha})\\ \cos\alpha\cdot\sin(\frac{\psi}{\cos\alpha}) & r\cos\alpha\cdot\cos(\frac{\psi}{\cos\alpha})\\ \sin\alpha & 0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\cos\alpha}\end{pmatrix}\cdot\begin{pmatrix}\cos\psi & -r\sin\psi\\ \sin\psi & r\cos\psi\end{pmatrix}^{-1} =$$
$$= \begin{pmatrix}\cos\alpha\ \cos\psi\ \cos(\frac{\psi}{\cos\alpha}) - \sin\psi\ \sin(\frac{\psi}{\cos\alpha}) & \cos\alpha\ \sin\psi\ \cos(\frac{\psi}{\cos\alpha}) - \cos\psi\ \sin(\frac{\psi}{\cos\alpha})\\ \cos\alpha\ \cos\psi\ \sin(\frac{\psi}{\cos\alpha}) + \sin\psi\ \cos(\frac{\psi}{\cos\alpha}) & \cos\alpha\ \sin\psi\ \sin(\frac{\psi}{\cos\alpha}) + \cos\psi\ \cos(\frac{\psi}{\cos\alpha})\\ \cos\alpha\ \cos\psi & \sin\alpha\ \sin\psi\end{pmatrix},$$
which can be shown to be isometric by a lengthy direct calculation.

3.4 The sphere.

1. Equation: $x^2+y^2+z^2 = R^2$

2. Parameterization: $(\varphi, \vartheta) \mapsto (R\cos\vartheta\cos\varphi, R\cos\vartheta\sin\varphi, R\sin\vartheta)$ with longitudes φ and latitudes ϑ. Again, we obtain this surface by looking at the intersection curve with the x-z plane, which is a (semi-)circle parameterized by $\vartheta \mapsto R(\cos\vartheta, 0, \sin\vartheta)$ and rotating this around the z-axis by some angle φ. So we obtain

$$\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}R\cos\vartheta\\ 0\\ R\sin\vartheta\end{pmatrix} = \begin{pmatrix}R\cos\vartheta\cos\varphi\\ R\cos\vartheta\sin\varphi\\ R\sin\vartheta\end{pmatrix}.$$

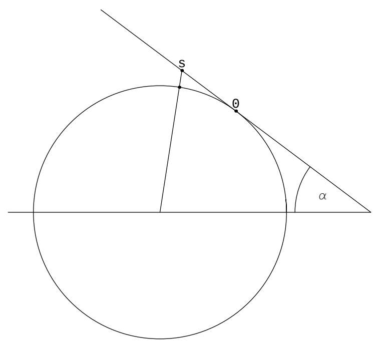


3. Graph: $z = \pm\sqrt{R^2 - x^2 - y^2}$
4. Trivialization: Spherical coordinates.

One can also parametrize a sphere by projecting onto the touching cone with slope α:

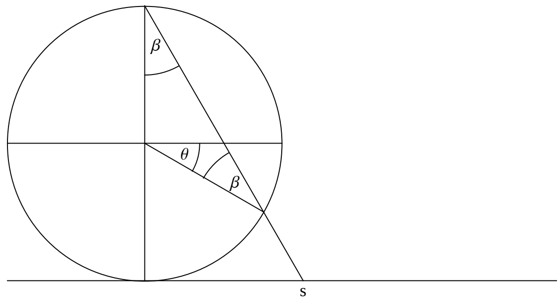
$$(x, y) \mapsto (\varphi, s) \mapsto (\varphi, \vartheta(s)) \mapsto \begin{pmatrix}R\cos\vartheta\cos\varphi\\ R\cos\vartheta\sin\varphi\\ R\sin\vartheta\end{pmatrix},$$

where (φ, s) are the parameters of the above parameterization of the cone and (φ, ϑ) are the parameters of the sphere.



Particular choices for the function ϑ yield the radial projection, or the normal projection to the generators of the cone, see exercise [**86**, 72.42]. In particular, one is interested in angle or area preserving projections. We will show, that a length-preserving parametrization is not possible - one can not form the sphere by furling a sheet of paper.

Especially important is the stereographic projection: There one projects from one point of the sphere (without restricting the generality: the north pole) to the tangential plane in the antipodal point.



Then $2\beta + (\frac{\pi}{2} - \vartheta) = \pi \Rightarrow \beta = \frac{\pi}{4} + \frac{\vartheta}{2}$ and thus

$$\frac{s}{2} = \tan\beta = \tan\left(\frac{\pi}{4} + \frac{\vartheta}{2}\right) = \frac{1 + \tan(\vartheta/2)}{1 - \tan(\vartheta/2)}.$$

This projection is angle preserving and circles are mapped to circles or straight lines, see Exercise [**86**, 72.41].

For seafaring, however, this representation of the sphere is not optimal: There one is particularly interested in the loxodromes, i.e. those curves on the sphere, which intersect the meridians under a fixed angle, because these are the orbits that one travels when keeping constant course with respect to direction north (identifed by the Polar Star or compass). In the stereographic projection, the mappings of the meridians are straight lines through 0, an hence the loxodromes are (logarithmic) spirals. If, on the other hand, we project to the cylinder touching along the equator, then the meridians become parallel straight lines and if we choose the projection angle preserving (the so-called Mercator projection) then the Loxodrome lines, are very easy to draw by plotting the connecting line between start location and destination.

3.5 The n-sphere.

The n-dimensional sphere (or n-sphere for short) is $S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$. The function $f : x \mapsto |x|^2 - 1$ is a regular equation for S^n, because $f'(x)(x) = 2|x|^2 = 2$ for $x \in S^n$. As local coordinates, we use the stereographic projection (but this time to the equatorial plane, giving a factor of $1/2$ with respect to the one previously discussed), i.e. associating to $x \in S^n$ the $y \in \mathbb{R}^n = p^\perp \subset \mathbb{R}^{n+1}$ lying on the line through the choosen pole $p \in S^n$ and x, i.e. $y = p + \lambda(x - p)$ for the $\lambda > 0$ with

$$\begin{aligned}
0 &= \langle p, p + \lambda(x-p)\rangle = |p|^2 - \lambda\langle p, p - x\rangle \\
\Rightarrow \lambda &= \frac{|p|^2}{\langle p, p-x\rangle} = \frac{1}{1 - \langle p, x\rangle} \Rightarrow \\
\Rightarrow y &= \lambda x + (1-\lambda)p = \frac{1}{1 - \langle p, x\rangle}(x - \langle p, x\rangle p).
\end{aligned}$$

Vice versa

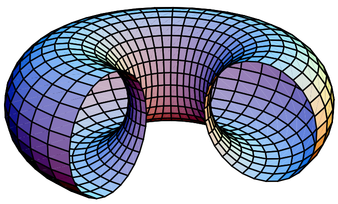
$$\begin{aligned}
& x = p + \mu(y - p) \text{ with } \mu > 0, \text{ such that} \\
& 1 = |x|^2 = \langle p + \mu(y-p), p + \mu(y-p)\rangle \\
& \quad = 1 + 2\langle p, \mu(y-p)\rangle + \mu^2\langle y - p, y - p\rangle \\
\Rightarrow\ & 0 = \mu^2|y-p|^2 + 2\mu\langle p, y - p\rangle = \mu(\mu|y-p|^2 - 2\langle p, p - y\rangle).
\end{aligned}$$

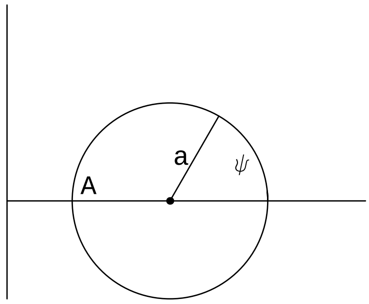
For $\mu := 0$ we get the uninteresting solution $x = p$. The other value is

$$\mu = \frac{2(1 - \langle p, y \rangle)}{|y|^2 - 2\underbrace{\langle y, p \rangle}_{0} + 1} = \frac{2}{|y|^2 + 1} \quad \text{and thus}$$

$$x = \frac{1}{|y|^2 + 1}\Big(2y + (|y|^2 - 1)p\Big)$$

3.6 The torus.





1. Equation: $z^2 + (\sqrt{x^2 + y^2} - A)^2 = a^2$
2. Parameterization:

$$(\varphi, \psi) \mapsto \begin{pmatrix} (A + a\cos\psi)\cos\varphi \\ (A + a\cos\psi)\sin\varphi \\ a\sin\psi \end{pmatrix},$$

with longitudes φ and latitudes ψ. This is not length-preserving.

For the special torus $a^2 := A^2 - 1$ with $A > 1$ we compute the inverse image under the stereographic projection $\mathbb{R}^4 \supset S^3 \to \mathbb{R}^3$ with respect to the point $(0, 0, 0, 1) \in \mathbb{R}^4$ as follows:

$$(x_1, y_1, x_2, y_2) \mapsto \frac{1}{1 - y_2}(x_1, y_1, x_2, 0) \text{ since } z \mapsto \frac{z - \langle z, p \rangle p}{1 - \langle z, p \rangle}.$$

This torus corresponds to the following subset of $\mathbb{R}^4$:

$$\begin{cases} {x_1}^2 + {y_1}^2 + {x_2}^2 + {y_2}^2 = 1 \\ \left(\dfrac{x_2}{1 - y_2}\right)^2 + \left(\dfrac{\sqrt{{x_1}^2 + {y_1}^2}}{1 - y_2} - A\right)^2 = A^2 - 1 \end{cases}$$

Using the first equation, we transform the second one as follows:

$$\begin{aligned} 0 &= \left(\frac{x_2}{1 - y_2}\right)^2 + \left(\frac{\sqrt{{x_1}^2 + {y_1}^2}}{1 - y_2} - A\right)^2 - A^2 + 1 \\ &= \frac{{x_2}^2}{(1 - y_2)^2} + \frac{{x_1}^2 + {y_1}^2}{(1 - y_2)^2} - 2A\frac{\sqrt{{x_1}^2 + {y_1}^2}}{1 - y_2} + 1 \\ &= \frac{1 - {y_2}^2}{(1 - y_2)^2} - 2A\frac{\sqrt{1 - ({x_2}^2 + {y_2}^2)}}{1 - y_2} + 1 \\ \Leftrightarrow 2A\sqrt{1 - ({x_2}^2 + {y_2}^2)} &= 1 + y_2 + (1 - y_2) = 2 \end{aligned}$$

So the torus is described by the following system of equations:

$$\begin{cases} {x_1}^2 + {y_1}^2 + {x_2}^2 + {y_2}^2 = 1 \\ \qquad 1 - ({x_2}^2 + {y_2}^2) = \dfrac{1}{A^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} {x_1}^2 + {y_1}^2 = \dfrac{1}{A^2} \ldots \text{ Circle in } \mathbb{R}^2 \times \{(0,0)\} \\ {x_2}^2 + {y_2}^2 = \dfrac{A^2-1}{A^2} = \dfrac{a^2}{A^2} \ldots \text{ Circle in } \{(0,0)\} \times \mathbb{R}^2 \end{cases}$$

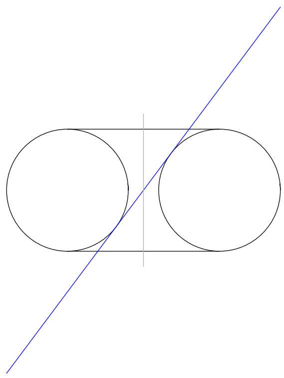
The torus is thus the Cartesian product $S^1 \times S^1$ of two circles standing normal to each other.

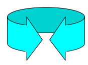
The parameterization

$$(\varphi, \psi) \mapsto \Big(\frac{1}{A}\cos(A\varphi), \frac{1}{A}\sin(A\varphi), \frac{a}{A}\cos(\frac{A\psi}{a}), \frac{a}{A}\sin(\frac{A\psi}{a})\Big)$$

is length-preserving, thus a torus can be generated in $\mathbb{R}^4$ by furling a plane.

Remark: The following special cut through the torus in $\mathbb{R}^3$ results in two intersecting circles:





On the cutting plane $z = \frac{a}{\sqrt{A^2-a^2}}x$ we use the basis with orthonormal vectors $(\frac{\sqrt{A^2-a^2}}{A}, 0, \frac{a}{A})$ and $(0,1,0)$ and we denote the corresponding coordinates with (s, y). Then $x = \frac{\sqrt{A^2-a^2}}{A} \cdot s$ and $z = \frac{a}{A} \cdot s$. If we plug this into the torus equation $z^2 + (\sqrt{x^2+y^2} - A)^2 = a^2$, we get

$$\Big(\frac{a}{A}s\Big)^2 + \left(\sqrt{\frac{A^2-a^2}{A^2}s^2 + y^2} - A\right)^2 = a^2 \;\Leftrightarrow$$

$$\begin{aligned} \Leftrightarrow\; a^2(A^2 - s^2) &= \Big(\sqrt{(A^2-a^2)s^2 + A^2y^2} - A^2\Big)^2 \\ &= (A^2-a^2)s^2 + A^2y^2 + A^4 - 2A^2\sqrt{(A^2-a^2)s^2 + A^2y^2} \\ \Leftrightarrow\; s^2 + y^2 + (A^2 - a^2) &= 2\sqrt{(A^2-a^2)s^2 + A^2y^2} \\ \Leftrightarrow\; \big(s^2 + y^2 + (A^2-a^2)\big)^2 &= 4(A^2-a^2)s^2 + 4A^2y^2 \\ \Leftrightarrow\; \big(A^2 - (s^2 + (y+a)^2)\big) \cdot \big(A^2 - (s^2 + (y-a)^2)\big) &= 0, \end{aligned}$$

and that is the equation of two circles with centers $(0, \pm a)$ in the (s, y) coordinates and radius A.

3.7 The Hopf fibration $S^3 \to S^2$.

It is defined by the following commutative diagram

$$\begin{tikzcd} S^3 \arrow[rr, dotted, "\text{Hopf fibration}"] \arrow[d, hook] & & S^2 \arrow[d, "\text{stereogr.proj.}"] \\ \mathbb{C}^2 \arrow[rr, squiggly] & & \mathbb{C} \\ (z_1, z_2) \arrow[rr, mapsto] & & \frac{z_2}{z_1} \end{tikzcd}$$

Since the inverse to the stereographic projection around $p = (0, 0, 1)$ is the mapping $y \mapsto \frac{2y+(|y|^2-1)p}{|y|^2+1} = \frac{1}{|y|^2+1}(2y, |y|^2 - 1)$, we get the following formula for the Hopf fibration:

$$\begin{aligned}(z_1, z_2) \mapsto & \frac{1}{|\frac{z_2}{z_1}|^2 + 1}\Big(2\frac{z_2}{z_1}, \Big|\frac{z_2}{z_1}\Big|^2 - 1\Big) = \\ &= \frac{z_1\bar{z}_1}{|z_1|^2 + |z_2|^2}\Big(2\frac{z_2}{z_1}, \frac{|z_2|^2 - |z_1|^2}{z_1\bar{z}_1}\Big) \\ &= \underbrace{\frac{1}{|z_1|^2 + |z_2|^2}}_{=1, \text{ because } (z_1,z_2)\in S^3} \Big(2z_2\bar{z}_1, |z_2|^2 - |z_1|^2\Big) \in S^2 \subseteq \mathbb{C} \times \mathbb{R}.\end{aligned}$$

We look at the inverse images in S^3 of a circle of fixed latitude ϑ on S^2, where ϑ:

$$\begin{aligned}(z_1, z_2) \in S^3, \ \Big|\frac{z_2}{z_1}\Big| = r\Big(= \tan\Big(\frac{\pi}{4} + \frac{\vartheta}{2}\Big)\Big) \ \Leftrightarrow \\ \Leftrightarrow \left\{\begin{matrix} |z_2| = r|z_1| \\ (z_1, z_2) \in S^3 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 = r^2|z_1|^2 \\ |z_1|^2 + |z_2|^2 = 1 \end{matrix}\right\} \\ \Leftrightarrow \left\{\begin{matrix} |z_2| = r|z_1| \\ |z_1|^2(1 + r^2) = 1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 = r^2\dfrac{1}{1 + r^2} \\ |z_1|^2 = \dfrac{1}{1 + r^2} \end{matrix}\right\}\end{aligned}$$

This corresponds by 3.6 to a torus in $\mathbb{R}^3$ under the stereographic projection $S^3 \to \mathbb{R}^3$, where $A = \sqrt{r^2 + 1}$ and $a = r$.

We consider the inverse image in S^3 of the South Pole on S^2:

$$(0, 0, -1) \in S^2 \ \hat{=} \ (r = 0) \in \mathbb{R}^2 \ \hat{=} \ (|z_1| = 1, z_2 = 0) \subset S^3,$$

or the North Pole on S^2:

$$(0, 0, +1) \in S^2 \ \hat{=} \ (r = \infty) \subset \mathbb{R}^2 \ \hat{=} \ (z_1 = 0, |z_2| = 1) \subset S^3.$$

We claim in general: The inverse image of each point on S^2 (which is given by $z_0 \in \mathbb{C}$ with $r := |z_0|$ with respect to the stereographic projection $S^2 \to \mathbb{C}$) is a circle in $S^3 \subset \mathbb{R}^4$ obtained by intersecting the sphere $S^3 \subset \mathbb{R}^4$ with the plane $z_2 = z_1 z_0$:

$$\left\{\begin{matrix} (z_1, z_2) \in S^3 \\ \dfrac{z_2}{z_1} = z_0 \in \mathbb{C} \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_2|^2 + |z_1|^2 = 1 \\ z_2 = z_1 z_0 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} |z_1|^2 = \dfrac{1}{1 + r^2} \\ |z_2|^2 = r^2\dfrac{1}{1 + r^2} \\ z_2 = z_1 z_0 \end{matrix}\right\}$$

i.e. z_1 runs through one circle and at the same time z_2 runs through a second circle.

In stereographic coordinates, the first two equations in $\mathbb{R}^3$ correspond to the torus $T : z^2 + (\sqrt{x^2 + y^2} - \sqrt{r^2 + 1})^2 = r^2$. Without restriction of generality, let $r =$

$z_0 \in \mathbb{R}$, otherwise we rotate z_1 by $e^{-i\vartheta}$, which corresponds to a rotation around the z-axes in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$.

$$\text{On the } S^3 : \left\{ \begin{aligned} z_2 &= r z_1 \\ |z_2|^2 &= r^2 \frac{1}{1+r^2} \\ |z_1|^2 &= \frac{1}{1+r^2} \end{aligned} \right\} = \left\{ \begin{aligned} x_2 = r x_1,\ & y_2 = r y_1 \\ |z_2|^2 &= r^2 \frac{1}{1+r^2} \\ |z_1|^2 &= \frac{1}{1+r^2} \end{aligned} \right\}$$

$$\text{Corresponds to } \mathbb{R}^3 : \left\{ \begin{aligned} & z = r x \\ & x^2 + y^2 + z^2 - 1 = 2 r y \\ & z^2 + (\sqrt{x^2+y^2} - \sqrt{r^2+1})^2 = r^2 \end{aligned} \right\}$$

Where we have set $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$ and used the formulas for stereographic projection:

$$x_1 = \frac{2x}{1+|(x,y,z)|^2} \qquad y_1 = \frac{2y}{1+|(x,y,z)|^2}$$
$$x_2 = \frac{2z}{1+|(x,y,z)|^2} \qquad y_2 = \frac{|(x,y,z)|^2 - 1}{1+|(x,y,z)|^2}.$$

So the inverse image of a point is contained in the two intersection circles of the torus with the plane $z = rx$. A more detailed analysis provides that it is exactly the front one with respect to y.

The complement of the filled torus in the S^3 is the interior of another filled torus. These two filled tori are the inverse images of the southern and the northern hemisphere.

3.8 The manifold of linear mappings of fixed rank.

The subspace $L_r(n,m)$ of all $T \in L(n,m)$ of fixed rank r is a submanifold of dimension $r\,(n+m-r)$.

For maximal $r = \min\{n,m\}$ this dimension is $n \cdot m = \dim(L(n,m))$, thus in this case $L_r(n,m)$ is open in $L(n,m)$.

Proof. We describe $L_r(n,m)$ locally as a graph. Let $T_0 \in L_r(n,m)$, that is $\operatorname{rank}(T_0) = \dim \operatorname{im} T_0 = r$. Put $F := \operatorname{im} T_0$ and $E := (\ker T_0)^\perp$. Then $T_0|_E : E \to F$ is injective, and because of $\dim E = n - \dim\ker T_0 = \dim \operatorname{im} T_0 = \dim F = r$ it is even bijective. With respect to the orthogonal decompositions $\mathbb{R}^n = E \oplus E^\perp$ and $\mathbb{R}^m = F \oplus F^\perp$, the mapping T_0 thus has the following form:

$$\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \text{ with } B_0 = 0,\ C_0 = 0,\ D_0 = 0, \text{ and with invertible } A_0.$$

Now let U be the open (because $GL(E) \subseteq L(E,E)$ is open) neighborhood of all matrices $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with invertible A. Then T is in $L_r(\mathbb{R}^n, \mathbb{R}^m)$ if and only if $\dim \operatorname{im} T = r$. We have

$$T \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} Av + Bw \\ Cv + Dw \end{pmatrix}.$$

Thus, $T \begin{pmatrix} v \\ w \end{pmatrix} = 0$ is exactly when $v = -A^{-1}Bw$ and $Cv + Dw = 0$, i.e. $\ker T = \{(-A^{-1}Bw, w) \in E \times E^\perp : CA^{-1}Bw = Dw\}$. Therefore $r = \operatorname{rank} T = \dim \operatorname{im} T =$

$\dim \operatorname{dom} T - \dim \ker T = n - \dim \ker T$ exactly when all $w \in E^\perp$ satisfy the equation $CA^{-1}Bw = Dw$, that is $D = CA^{-1}B$.

The map

$$g : \left\{ \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in L(n,m) : A \text{ invertible} \right\} \to L(E^\perp, F^\perp), \quad \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \mapsto CA^{-1}B$$

is well-defined and smooth on the trace of the open subset U on the linear subspace

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(n,m) : D = 0 \right\},$$

and its graph describes $L_r(n,m)$ in the open set

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(n,m) : A \in GL(E,F) \right\}$$

The dimension of $L_r(\mathbb{R}^n, \mathbb{R}^m)$ is thus $n\,m - (n-r)\,(m-r) = r\,(n+m-r)$. $\square$

3.9 The Graßmann Manifolds $G(r,n)$.

The Graßmann manifold $G(r,n)$ (according to Hermann Graßmann, 1809-1877) of r-planes through 0 in $\mathbb{R}^n$ is a submanifold of $L(n,n)$ of dimension $r\,(n-r)$.

If we choose $r = 1$, we get as special case the projective spaces $\mathbb{P}^{n-1} = G(1,n)$ of the straight lines through 0 in $\mathbb{R}^n$.

Proof. We identify the linear subspaces of $\mathbb{R}^n$ with the orthogonal projections onto them. Thus, $G(r,n)$ is a subset of the manifold $L_r(n,n)$. Let E_0 be a subspace of $\mathbb{R}^n$ of dimension r and P_0 the ortho-projection onto E_0. With respect to the decomposition $\mathbb{R}^n = E_0 \oplus E_0^\perp$ the projection P_0 is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. A neighborhood of P_0 in $L_r(n,n)$ is then given by the matrices $\left(\begin{smallmatrix} A & B \\ C & CA^{-1}B \end{smallmatrix}\right)$ with invertible A. Any linear map P is an ortho-projection if and only if it is idempotent ($P^2 = P$) and self-adjoint ($P = P^t$), or equivalent, if it satisfies the single equation $P^tP = P$. In fact: That P is a projection means $P|_{\operatorname{im} P} = \mathrm{id}$, i.e. $P^2 = P$, and being an orthogonal projection means $\ker(P) = \operatorname{im}(P)^\perp$. From $P^2 = P$ follows $\ker(P) = \operatorname{im}(1-P)$, because $P(1-P) = 0$ and $Px = 0 \Rightarrow x = x - Px = (1-P)x$. Thus, $\ker(P) \perp \operatorname{im}(P)$ is exactly if $0 = \langle (1-P)x, Py \rangle = \langle x, (1-P^t)Py \rangle$ for all x, y, that is $P = P^tP$. Conversely, $P^t = (P^tP)^t = P^tP = P$ follows and thus $P = P^tP = P^2$.

This is the case for $\left(\begin{smallmatrix} A & B \\ C & CA^{-1}B \end{smallmatrix}\right)$ if and only if $A = A^t$ and $B^t = C$ (and then $(CA^{-1}B)^t = B^t(A^t)^{-1}C^t = CA^{-1}B$) and

$$\begin{pmatrix} A^tA + C^tC & A^tB + C^tCA^{-1}B \\ B^tA + B^t(A^t)^{-1}C^tC & B^tB + B^t(A^t)^{-1}C^tCA^{-1}B \end{pmatrix} =$$
$$= \begin{pmatrix} A^t & C^t \\ B^t & B^t(A^t)^{-1}C^t \end{pmatrix} \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} = \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix}$$

or equivalently $A^tA + C^tC = A$ ($\Rightarrow A^t = A$) and thus

$$\begin{aligned} A^tB + C^tCA^{-1}B &= A^tB + (A - A^tA)A^{-1}B = B, \\ B^tA + B^t(A^t)^{-1}C^tC &= B^tA + B^t(A^t)^{-1}(A - A^tA) = B^t(A^t)^{-1}A = C, \\ B^tB + B^t(A^t)^{-1}C^tCA^{-1}B &= B^tB + B^t(A^t)^{-1}(A - A^tA)A^{-1}B \\ &= B^t(A^t)^{-1}B = CA^{-1}B \end{aligned}$$

Together, the equations are $A^tA + C^tC = A$, $B = C^t$ and $D = CA^{-1}B$. These are $r^2 + r\,(n-r) + (n-r)^2$ independent equations, and thus the dimension of $G(r,n)$ should be just $n^2 - (r^2 + n^2 - 2nr + r^2 + nr - r^2) = nr - r^2 = r(n-r)$. These equations describe $G(r,n)$ locally as a graph of $(A,C) \mapsto (B,D) = (C^t, CA^{-1}C^t)$ over the subset $\{(A,C) \in L(E_0, \mathbb{R}^n) : A \in GL(E_0),\ A^tA + C^tC = A\}$

So it remains to show that the equations are regular and for that it is enough(!) to show the regularity of the first equation $A^tA + C^tC - A = 0$. Its derivative in direction (X,Y) is $(X,Y) \mapsto X^tA + A^tX - X + Y^tC + C^tY$. So we have to solve the equation $X^tA + A^tX - X + Y^tC + C^tY = Z$ for $(A,C) = (\mathrm{id}, 0)$, i.e. $X^t = Z$ for (X,Y). Obviously $(Z^t, 0)$ is a solution. □

3.10 Unfurl a 2-fold twisted band.

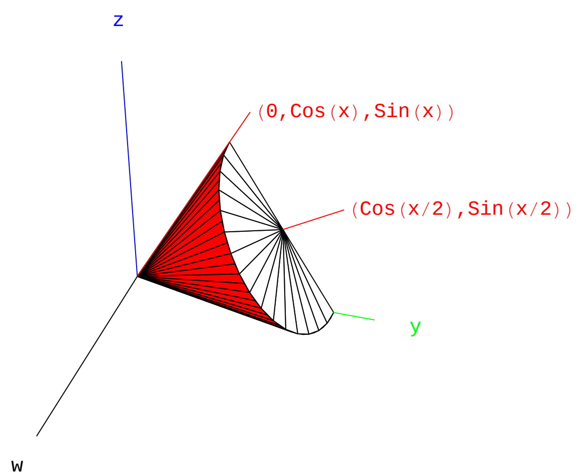
An untwisted piece of band (i.e. rectangle) is parameterized by

$$\varphi_0 : [0, 2\pi] \times [-1, +1] \to \mathbb{R}^3 \subset \mathbb{R}^4, \quad (\vartheta, r) \mapsto (\vartheta, r, 0, 0).$$

A double-twisted band is parameterized by

$$\varphi_\pi : [0, 2\pi] \times [-1, +1] \to \mathbb{R}^3 \subset \mathbb{R}^4, \quad (\vartheta, r) \mapsto (\vartheta, r\cos\vartheta, r\sin\vartheta, 0).$$

We now want to find a diffeotopy $F : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4$ of $\mathbb{R}^4$ (i.e. a smoothly parameterized family $t \mapsto F(t; \_)$ of diffeomorphisms of $\mathbb{R}^m$ with $F(0, \_) = \mathrm{id}$ and $F(\pi, \_)$ the desired diffeomorphism), which converts the non-twisted band into the 2-fold twisted band, i.e. $F(\pi, \varphi_0(\vartheta, r)) = \varphi_\pi(\vartheta, r)$. We refer to the coordinates in $\mathbb{R}^4$ with (x, y, z, w). This diffeotopy $F(t; \_)$ is supposed to leave the hyperplanes normal to the x-axis invariant and act as rotation on them. We denote this rotation in hyperplane $\{x\} \times \mathbb{R}^3 \cong \mathbb{R}^3$ at time t with $R(t, x)$. It should be the rotation by angle t around the axis $\ell := (\cos\frac{x}{2}, \sin\frac{x}{2}, 0)$. We obtain $R(t, x)$ by first rotating ℓ around the w-axis into the y-axis, then turning by the angle $-t$ around the y-axis, and finally rotating the y-axis around the w-axis back into the ℓ.



The matrix representation of $R(t,x)$ with respect to the coordinates (y,z,w) thus looks like this:

$$
\begin{aligned}
&[R(t,x)] = \\
&= \begin{pmatrix} \cos\frac{x}{2} & -\sin\frac{x}{2} & 0 \\ \sin\frac{x}{2} & \cos\frac{x}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}
\begin{pmatrix} \cos\frac{x}{2} & \sin\frac{x}{2} & 0 \\ -\sin\frac{x}{2} & \cos\frac{x}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{x}{2} & -\sin\frac{x}{2} & 0 \\ \sin\frac{x}{2} & \cos\frac{x}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cos\frac{x}{2} & \sin\frac{x}{2} & 0 \\ -\cos t\sin\frac{x}{2} & \cos t\cos\frac{x}{2} & \sin t \\ \sin t\sin\frac{x}{2} & -\sin t\cos\frac{x}{2} & \cos t \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\frac{x}{2} + \cos t\sin^2\frac{x}{2} & (1-\cos t)\cos\frac{x}{2}\sin\frac{x}{2} & -\sin t\sin\frac{x}{2} \\ (1-\cos t)\sin\frac{x}{2}\cos\frac{x}{2} & \sin^2\frac{x}{2} + \cos t\cos^2\frac{x}{2} & \sin t\cos\frac{x}{2} \\ \sin t\sin\frac{x}{2} & -\sin t\cos\frac{x}{2} & \cos t \end{pmatrix}
\end{aligned}
$$

In the boundary points $x = 0$ and $x = 2\pi$,

$$[R(t,0)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$$

and

$$[R(t,2\pi)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

keeps y-axis fixed.

Our wanted diffeotopy is thus

$$F(t;x,y,z,w) := (x, R(t,x)(y,z,w))$$

and the corresponding isotopy

$$
\begin{aligned}
\varphi_t(\vartheta,r) &:= F(t;\varphi_0(\vartheta,r)) = (\vartheta, R(t,\vartheta)(r,0,0)) \\
&= \Big(\vartheta, \tfrac{r}{2}(1+\cos\vartheta+\cos t(1-\cos\vartheta)), \tfrac{r}{2}(1-\cos t)\sin\vartheta, r\sin t\sin\tfrac{\vartheta}{2}\Big)
\end{aligned}
$$

Clearly, $\varphi_t(\vartheta,r) = (\vartheta,r,0,0)$ is for $\vartheta = 0$ and for $\vartheta = 2\pi$. Furthermore, φ_0 and φ_π are the desired boundary values. And by design, all φ_t are embeddings from $[0,2\pi]\times[-1,1]$ into $\mathbb{R}^4$.

4. Examples of Lie groups

Some of the classic examples of manifolds are even Lie groups, i.e. they carry also a smooth group structure. There are lecture courses completely devoted to them, e.g. http://www.mat.univie.ac.at/∼kriegl/Skripten/2010WS.pdf.

4.1 General linear group.

The vector space $L(\mathbb{R}^n,\mathbb{R}^m) = L(n,m) := \{T : \mathbb{R}^n \to \mathbb{R}^m \text{ linear }\}$ is nm-dimensional.

The GENERAL LINEAR GROUP (see also [**86**, 1.2])

$$GL(\mathbb{R}^n) = GL(n) := \{T \in L(n,n) : \det T \neq 0\} \subset L(n,n)$$

is an open (and thus n^2-dimensional) submanifold in $L(n,n)$, because it is given by a continuous strict inequality. With respect to composition, $GL(n)$ is a group.

4.2 Special linear group.

The special linear group is defined by

$$SL(n) := \{T \in L(n,n) : \det(T) = 1\} \subseteq GL(n).$$

So it is given by the equation $\det(T) = 1$, or $f(T) = 0$, where $f : L(n,n) \to \mathbb{R}$ is the function $f(T) := \det(T) - 1$. We assert that this equation is regular, that is, the derivative of the determinant function is surjective. Since the determinant is multilinear in the columns (or even polynomial in the coefficients), its smoothness follows. The derivative at A in direction B is:

$$\begin{aligned}\det{}'(A)(B) &= \tfrac{d}{dt}|_{t=0}\det(A+tB) = \tfrac{d}{dt}|_{t=0}\det(A\cdot(1+tA^{-1}B))\\ &= \tfrac{d}{dt}|_{t=0}\det(tA)\cdot\det(\tfrac{1}{t}+A^{-1}B)\\ &= \tfrac{d}{dt}|_{t=0}t^n\det(A)\cdot\left(\frac{1}{t^n}+\frac{1}{t^{n-1}}\operatorname{trace}(A^{-1}B)+\dots+\det(A^{-1}B)\right)\\ &= \det(A)\operatorname{trace}(A^{-1}B).\end{aligned}$$

This shows the surjectivity of $\det'(A)$ and thus the regularity of det. Without calculating the derivative $\det{}'(A) : L(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ completely, you can proceed as follows:

$$\det{}'(A)(A) = \tfrac{d}{dt}|_{t=0}\underbrace{\det((1+t)A)}_{(1+t)^n\det A} = n(1+t)^{n-1}|_{t=0}\det A = n\det A.$$

Consequently, $\det{}'(A)$ is surjective and $SL(\mathbb{R}^n)$ is a n^2-1 dimensional manifold.

4.3 Orthogonal group.

It is defined by (see also [**86**, 1.2]):

$$O(n) := \{T \in GL(n,n) : T^t \circ T = \mathrm{id}\} = \{T \in GL(n,n) : \langle Tx, Ty\rangle = \langle x,y\rangle\ \forall\, x,y\}.$$

As in Example 4.2, let us now show that the equation $T^t \circ T = \mathrm{id}$ is a regular one. For this purpose we compute the derivative of the quadratic - hence smooth - function $f : GL(n) \to L(n,n)$ given by $f(T) := T^t \circ T = \mathrm{comp}(T^t, T)$:

$$f'(T)\cdot S = \mathrm{comp}(S^t, T) + \mathrm{comp}(T^t, S) = S^t \circ T + T^t \circ S.$$

Since $f(T)$ is obviously symmetric, that is, f has values in the linear subspace $L_{\mathrm{sym}}(n,n) \subseteq L(n,n)$ of the symmetric matrices, we can only hope to have surjectivity for $f'(T) : L(n,n) \to L_{\mathrm{sym}}(n,n)$. The dimension of $L_{\mathrm{sym}}(n,n)$ is obviously $\frac{(n+1)n}{2}$. For an $R \in L_{\mathrm{sym}}(n,n)$ there is an $S \in L(n,n)$ with $R = S^t \circ T + T^t \circ S = (S^t \circ T) + (S^t \circ T)^t$, because $S^t \circ T = \frac{1}{2}R$ has the solution $S = (S^t)^t = (\frac{1}{2}R \circ T^{-1})^t = (T^t)^{-1}\frac{1}{2}R$. Consequently, $f'(T)$ is surjective, and thus $O(n)$ is a submanifold of $L(n,n)$ of dimension $\dim(O(n)) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Note that $\det(T) = \pm 1$ follows from $1 = \det(1) = \det(T^tT) = \det(T)^2$ for $T \in O(n)$. Thus, $O(n) \cong SO(n) \ltimes \mathbb{Z}_2$, where $SO(n) := O(n) \cap SL(n) = O(n) \cap GL_+(n)$ is an open subset of $O(n)$.

More generally, we can consider the Stiefel manifold (due to Eduard Stiefel, 1909-1978)

$$V(k,n) := \{T \in L(k,n) : T^tT = \mathrm{id}\}$$

(see also [**86**, 70.6]). Thus, the elements of $V(k,n)$ are the isometric mappings of $\mathbb{R}^k \to \mathbb{R}^n$, and these can be equivalently described by their values on the standard base in $\mathbb{R}^k$, i.e. by k-tuples of orthonormal vectors in $\mathbb{R}^n$, so-called orthonormal k-frames in $\mathbb{R}^n$.

The function $f : L(k,n) \to L_{\mathrm{sym}}(k,k)$, $T \mapsto T^tT - \mathrm{id}$, is smooth and satisfies $f'(T)(S) = T^tS + S^tT$. So it is regular, because for symmetric R we can solve $f'(T)(S) = R$ with $S := \frac{1}{2}TR$ as before.

4.4 Groups of invariant automorphisms, O_b.

Let us generalize the orthogonal group by considering any bilinear form $b : E \times E \to \mathbb{R}$ on an Euclidean space E. With

$$O_b(E) := \{T \in GL(E) : b(Tx, Ty) = b(x, y)\, \forall\, x, y \in E\}$$

we denote the group of all invertible linear mappings that keep the bilinear form b invariant. The bilinear forms $b : E \times E \to \mathbb{R}$ are in bijective relation to the linear maps $B : E \to E$, by virtue of

$$b(x, y) = \langle Bx, y\rangle = \langle x, B^t y\rangle :$$

since we may consider any bilinear $b : E \times E \to \mathbb{R}$ as mapping $\check{b} : E \to L(E, \mathbb{R}) =: E^*$, which is given by $x \mapsto (y \mapsto b(x, y))$. The scalar product $\langle\_,\_\rangle : E \times E \to \mathbb{R}$ corresponds to a mapping $\iota : E \to E^*$, which is an isomorphism, because $\ker(\iota) = \{x : \langle x, y\rangle = 0 \;\; \forall\; y\} = \{0\}$, and since $\dim(E) = \dim(E^*)$. The composite $B := \iota^{-1} \circ \check{b} : E \to E^* \to E$ is then the sought-after linear mapping, because

$$b(x, y) = \check{b}(x)(y) = (\iota \circ B)(x)(y) = \iota(B(x))(y) = \langle Bx, y\rangle.$$

The equation $b(Tx, Ty) = b(x, y)$ is thus equivalent to $\langle T^tBTx, y\rangle = \langle BTx, Ty\rangle = \langle Bx, y\rangle$, and hence

$$O_b(E) = \{T \in GL(E) : T^tBT = B\}.$$

Thus we have to show that this is a regular equation. For the derivative of the function $f : GL(E) \to L(E)$, which is defined by $f(T) := T^tBT - B$, we obtain $f'(T)(S) = S^tBT + T^tBS$. As with $O(E)$, we can not expect it to be surjective onto $L(E, E)$, but we need a linear subspace $F \subseteq L(E, E)$ in which f has values and on which $f'(T)$ is surjective.

If B is (skew-)symmetric, then the same holds for $f(T)$, so we should use the space $L_\pm(E, E)$ of (skew-)symmetric linear mappings as F. This space has dimension $n(n+1)/2$ (resp. $n(n-1)/2$), where n is the dimension of E. If $U \in F$ and T is the identity then $U = f'(T)(S) = S^tB + BS$ will be solvable in S provided we can find an S with $BS = \frac{1}{2}U$ in S, since then also $S^tB = \pm(BS)^t = \pm\frac{1}{2}U^t = \frac{1}{2}U$. If B is invertible, then $S := \frac{1}{2}B^{-1}U$ is the solution. If $T \in GL(E)$ is arbitrary and B is invertible, then the equation $U = f'(T)(S) = S^tBT + T^tBS$ has the solution $S = \frac{1}{2}B^{-1}(T^{-1})^tU$, because then $T^tBS = \frac{1}{2}U$ and $S^tBT = \pm S^tB^tT = \pm(T^tBS)^t = \pm\frac{1}{2}U^t = \frac{1}{2}U$ follows. Thus, if B is injective, that is, b is not degenerate, or equivalently $x = 0 \Leftarrow \forall\; y : b(x, y) = 0$, then $O_b(E)$ is a submanifold of dimension

$$\dim O_b(E) := \begin{cases} n^2 - n(n+1)/2 = n(n-1)/2 & \text{if } b \text{ is symmetrical} \\ n^2 - n(n-1)/2 = n(n+1)/2 & \text{if } b \text{ is skew-symmetric.} \end{cases}$$

Note that $\det(T) = \pm 1$ for invertible B and $T \in O_b(E)$, because $0 \neq \det(B) = \det(T^tBT) = \det(T)^2 \det(B)$.

4.5 The symmetric case, $O(n,k)$.

In the symmetric case we can find an (orthonormal) base of eigenvectors e_j for B with corresponding eigenvalues $\lambda_j \in \mathbb{R}$ by use of the spectral theorem (i.e. principal axis theorem). Then

$$B(x) = \sum_j \lambda_j \langle x, e_j\rangle e_j$$

and thus

$$b(x,y) = \langle Bx, y\rangle = \sum_j \lambda_j \langle x, e_j\rangle\langle y, e_j\rangle$$

Since we assumed $\ker(B) = \{0\}$, no eigenvalue λ_j is 0, and thus b is represented in the orthogonal basis $f_j := \sqrt{|\lambda_j|}e_j$ as

$$b(x,y) = \sum_{\lambda_j>0} x^j y^j - \sum_{\lambda_j<0} x^j y^j,$$

where $x^j := \langle x, f_j\rangle$ denotes the coordinates of x with respect to the basis (f_j).

One calls such a b also a PSEUDO-EUCLIDEAN PRODUCT. They are of importance for Relativity Theory. Note that there are vectors $x \neq 0$ with norm $b(x,x) = 0$ and also vectors with $b(x,x)$ being negative. Those with vanishing norm are called LIGHT-LIKE, $\sum_{j>k}(x^j)^2 = \sum_{j\le k}(x^j)^2$ (this describes a "cone"), those with positive norm are called SPACE-LIKE and those with negative norm are TIME-LIKE. Consider e.g. the form

$$\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle := x_1y_1 + x_2y_2 - x_3y_3.$$

Then the vectors in the interior of the double cone with the x_3-axis are the time-like ones, those on the outside the space-like and those on the double cones the light-like ones.

The group $O_b(E)$ thus depends, up to isomorphism, only on the signature, defined as the number k of the negative eigenvalues of b, and is therefore also referred to as $O(n,k)$ (and sometimes also as $O(n-k,k)$), where $n = \dim(E)$ is. Note that $O(n,k) = O(n,n-k)$ (replace b with $-b$). The open subgroup $SL(n) \cap O(n,k)$ is denoted $SO(n,k)$. The $O(4,1)$ is also referred to as the LORENTZ GROUP.

4.6 The skew-symmetric case, $Sp(2n)$.

In the skew-symmetric case, we can find a normal form as follows. Let b be a non-degenerate skew-symmetric bilinear form, a so-called SYMPLECTIC FORM. They are important for classical mechanics (see section [**86**, 45]). For a subset $A \subseteq E$, we denote

$$A^\perp := \{x \in E : x \perp y \ \forall\, y \in A\},$$

and call it the ORTHOGONAL COMPLEMENT, where $x \perp y$ stands for $b(x,y) = 0$. Since b is skew-symmetric, $x \perp x$ for all x. Nevertheless, for each linear subspace F we have $\dim E = \dim F + \dim F^\perp$ (in fact: $i^* \circ \check{b} : E \to E^* \to F^*$ is surjective with kernel $F^\perp$, where $i : F \to E$ denotes the inclusion, because $\check{b} : E \to E^*$ is bijective by assumption, and $i^* : E^* \to F^*$ is clearly surjective (choose a left-inverse p to i, then $i^* \circ p^* = \mathrm{id}$) and thus $\dim E = \dim(\ker) + \dim(\mathrm{im}) = \dim(F^\perp) + \dim(F)$). Note that for linear subspaces A and B the equations $A^{\perp\perp} = A$ ($\Leftarrow A \subseteq A^{\perp\perp}$ and dimensional reasons), as well as $(A+B)^\perp = A^\perp \cap B^\perp$ (trivial) and finally $A^\perp + B^\perp = (A^\perp + B^\perp)^{\perp\perp} = (A^{\perp\perp} \cap B^{\perp\perp})^\perp = (A \cap B)^\perp$ hold.

A subset $A \subseteq E$ is called ISOTROPIC if $A \subseteq A^\perp$, that is $b|_{A\times A} = 0$. Let F be maximal among the isotropic subsets. Then $F = F^\perp$ holds (i.e. F is a so-called LAGRANGE SUBSPACE): Otherwise we can add any $y \in F^\perp \setminus F$ to F and get a larger isotropic subset of $F \cup \{y\}$; Because of the bilinearity of b, the orthogonal complement $A^\perp$ is a linear subspace for each subset $A \subseteq E$, and in particular any Lagrangian subspace $F = F^\perp$ is a linear subspace. Consequently $\dim E = \dim F + \dim F^\perp = 2\dim F$, hence the existence of Lagrange subspaces implies that E must be even-dimensional.

We now choose a Lagrange subspace F and a complementary Lagrange subspace F': This is possible because if for an isotropic subspace G with $G \cap F = \{0\}$ still $G + F \subset E$ holds, then $G^\perp + F = G^\perp + F^\perp = (G\cap F)^\perp = \{0\}^\perp = E \supset G + F$

and thus we can choose an $y \in G^\perp \setminus (G+F)$. Therefore $G_1 := \mathbb{R}y + G$ is a larger isotropic subspace with $G_1 \cap F = \{0\}$.
Let $i' : F' \hookrightarrow E$ be the inclusion. Then $i^* \circ \check{b} \circ i' : F' \hookrightarrow E \xrightarrow{\cong} E^* \twoheadrightarrow F^*$ is injective, because the kernel of $i^* \circ \check{b}$ is $F^\perp = F$ and $F \cap F' = \{0\}$, and thus by considering dimension an isomorphism. We claim that the induced isomorphism $E \cong F' \times F \cong F^* \times F$ translates the symplectic form b into the form $(y_1^*, y_1; y_2^*, y_2) \mapsto y_1^*(y_2) - y_2^*(y_1)$: So let $x_j =: y_j' + y_j$ with $y_j \in F$ and $y_j' \in F'$. Since F and F' are isotropic, we have $b(x_1, x_2) = b(y_1', y_2) + b(y_1, y_2') = b(y_1', y_2) - b(y_2', y_1)$. With $y_j^* := (i^* \circ \check{b} \circ i')(y_j')$ we get $b(y_1', y_2) = b(i'y_1', iy_2) = \check{b}(i'y_1')(iy_2) = (i^* \circ \check{b} \circ i')(y_1')(y_2) = y_1^*(y_2)$ and thus is $b(x_1, x_2) = y_1^*(y_2) - y_2^*(y_1)$.

Now we choose a basis $(e_j)_{k<j\le 2k}$ in F (with $2k = n := \dim E$) and take the dual basis $(e^j)_{j>k}$ in F^*. With $(e_j := e'_{k+j})_{j\le k}$ we denote the corresponding basis in F', i.e. $i^* \circ \check{b} \circ i' : e_j \mapsto e^{k+j}$. Then, $(e_j)_{j\le 2k=n}$ is a basis of E, which corresponds to that of $F^* \times F$, and furthermore, $y^*(y) = \sum_{j>k} y_j y^j$, where y_j denote the coordinates of $y^* \in F^*$ with respect to e^j and y^j denote coordinates for $y \in F$ with respect to e_j. So the STANDARD SYMPLECTIC FORM ON $\mathbb{R}^{2k}$ is

$$b(x_1, x_2) = \sum_{j\le k} x_1^j x_2^{j+k} - x_1^{j+k} x_2^j = \langle Jx_1, x_2 \rangle, \text{ with } J = \begin{pmatrix} 0 & -\mathrm{id}_k \\ \mathrm{id}_k & 0 \end{pmatrix}.$$

The corresponding group is denoted $Sp(2k)$, and is called a REAL SYMPLECTIC GROUP. Since $Sp(n)$ does not exist for odd n, $Sp(2k)$ is sometimes referred to as $Sp(k)$ in the literature!

4.7 Reflections.

Now we want to describe those $T \in O_b(E)$ (for symmetric and skew-symmetric b) which have a hyperplane as fixed point set $\{x \in E : Tx = x\}$. Let F be this hyperplane and $0 \ne y \in F^\perp$, that is $F = \{y\}^\perp$. If $y' \notin F$ with $b(y', y) = 1$ (possible because $b(y', y) = 0 \Rightarrow y' \in \{y\}^\perp = F$), then each $x \in E$ can be written as $x = b(x, y)y' + (x - b(x, y)y')$, and $b(x - b(x, y)y', y) = 0$, that is $x - b(x, y)y' \in F$. Any such T must therefore have the following form:

$$T(x) = b(x, y)T(y') + (x - b(x, y)y') = x + b(x, y)(T(y') - y') =: x + b(x, y)y''.$$

That T keeps the form b invariant amounts to

$$\begin{aligned} b(x_1, x_2) &= b(T(x_1), T(x_2)) = b\Big(x_1 + b(x_1, y)y'', x_2 + b(x_2, y)y''\Big) = \\ &= b(x_1, x_2) + b(x_1, y)b(y'', x_2) + b(x_2, y)b(x_1, y'') + b(x_1, y)b(x_2, y)b(y'', y''), \end{aligned}$$

i.e. $b(x_1, y)b(y'', x_2) + b(x_2, y)b(x_1, y'') + b(x_1, y)b(x_2, y)b(y'', y'') = 0$. If we put $x_2 := y'$ and choose $x_1 \perp y$, then $b(x_1, y'') = 0$ follows, so $y'' \in \{y\}^{\perp\perp} = \mathbb{R}\, y$. Let $y'' = \lambda y$ (with $\lambda \ne 0$, since T can not be the identity). Then

$$\begin{aligned} 0 &= \lambda b(x_1, y)b(y, x_2) + \lambda b(x_2, y)b(x_1, y) + b(x_1, y)b(x_2, y)\lambda^2 b(y, y) \\ &= \lambda b(x_1, y)b(x_2, y)(\pm 1 + 1 + \lambda b(y, y)) \end{aligned}$$

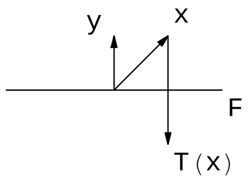
for all x_1 and x_2 if and only if $1 + \lambda b(y, y) = \mp 1$ (choose $x_1 = x_2 := y'$).

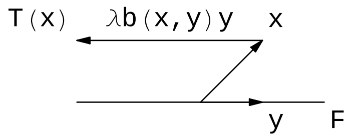
In the symmetric case this is equivalent to $\lambda b(y, y) = -2$ (i.e. $b(y, y) \ne 0$ and $\lambda := -\frac{2}{b(y,y)}$) and in the skew-symmetric one it is always satisfied.

The $T \in O_b(E)$ with a hyperplane $F = \{y\}^\perp$ as fixed point set are therefore precisely those of the form

$$T(x) := \begin{cases} x - 2\frac{b(x,y)}{b(y,y)} y & \text{with } b(y, y) \ne 0 \text{ in the symmetric case,} \\ x + \lambda b(x, y) y & \text{with } 0 \ne \lambda \in \mathbb{R} \text{ in the skew-symmetric case.} \end{cases}$$

These T are also called REFLECTIONS, in analogy to the case where b is an Euclidean metric. A simple calculation shows, that $T^2 = \mathrm{id}$.





In the symplectic case, each reflection is orientation-preserving, because $T(y') = y' + \lambda y$ is on the same side of F as y' and $\left(\begin{smallmatrix} \mathrm{id} & \lambda y \\ 0 & 1 \end{smallmatrix}\right)$ is (because of $y \in F$) the component representation of T with respect to the decomposition $E = F \oplus \mathbb{K}y' \cong F \times \mathbb{K}$. In the symmetric case reflections are orientation-reversing, because $T(y) = y - 2y = -y$ and thus $\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the component representation of T with respect to the decomposition $E = F \oplus \mathbb{K}y \cong F \times \mathbb{K}$.

For $x \neq x'$ it is possible to find a reflection $T : x \mapsto x + \lambda b(x, y)y$ with $Tx = x'$ iff $b(x, x) = b(x', x')$ and $b(x, x') \neq b(x, x)$: In fact, $x' - x = \lambda b(x, y)y$ is valid if and only if $y = \mu(x' - x)$ with $1 = \lambda b(x, y)\mu = \lambda\mu^2(b(x, x') - b(x, x))$. *This reflection T keeps $y^\perp = (x' - x)^\perp$ fixed.* In the symmetric case, the necessary equation $\lambda b(y, y) = \lambda\mu^2 b(x' - x, x' - x) = -2$ follows. Note that for positive definite b, due to the Cauchy Schwarz Inequality, the situation $b(x, x') = b(x, x)$ can not occur. In the symplectic case $b(x, x) = 0 = b(x', x')$ is always fulfilled.

Proposition.

For each (skew)-symmetric non-degenerate bilinear form $b : E \times E \to \mathbb{R}$, the group $O_b(E)$ is generated by the reflections.

It can be shown that in the symmetric case $n = \dim E$ many reflections are sufficient and in the symplectic one $n + 1$ are necessary (see [**35**, Sur les Groups Classique, Hermann, Paris 1967]).

Proof. In the **symmetric case** we choose an orthonormal basis of E (i.e. $b(e_i, e_j) = 0$ for $i \neq j$ and $b(e_i, e_i) = \pm 1$) The images $e'_i := T(e_i)$ from then also an orthonormal basis. We now show by induction that T leaves up to composition with reflections the set $\{e_1, \ldots, e_k\}$ invariant:
In fact, if T keeps $\{e_1, \ldots, e_{k-1}\}$ fixed by induction assumption, and $b(e_k, e'_k) \neq b(e_k, e_k)$, then the reflection S on the orthogonal complement of $e'_k - e_k$ maps e_k to e'_k and leaves $(e'_k - e_k)^\perp \supseteq (e'_k)^\perp \cap (e_k)^\perp \supseteq \{e_1 = e'_1, \ldots, e_{k-1} = e'_{k-1}\}$ fixed, so $S^{-1}T$ keeps even $\{e_1, \ldots, e_k\}$ fixed. On the other hand, if $b(e_k, e'_k) = b(e_k, e_k)$, then we first reflect at the orthogonal complement of e_k (with $b(e_k, e_k) = \pm 1 \neq 0$) and then at that of $e'_k + e_k$ (with $b(e_k + e'_k, e_k + e'_k) = 2(b(e_k, e_k) + b(e_k, e'_k)) = 4b(e_k, e_k) \neq 0$). These reflections leave $(e_k)^\perp \cap (e_k + e'_k)^\perp \supseteq \{e_1, \ldots, e_{k-1}\}$ invariant and their composition maps e_k to $-e_k$ and on to e'_k, so up to them T leaves $\{e_1, \ldots, e_k\}$ invariant.

In the **symplectic case** we prove the statement by induction on $j := n - \dim F$, where $F := \{x : Tx = x\}$. For $j = 0$ we have $T = \mathrm{id}$. So let $j > 0$. For each $y \in E$, we have $b(y, x) = b(Ty, Tx) = b(Ty, x)$ for all $x \in F$, i.e. $Ty - y \in F^\perp$.

If $b(Ty, y) \neq 0$ for some $y \in E$ ($\Rightarrow y \notin F$), then there is a reflection which maps y to Ty and leaves $(Ty - y)^\perp \supseteq F$ fixed. Apart from this reflection T also fixes $F \oplus \mathbb{R}\, y$.
Otherwise, $b(Ty, y) = 0$ for all y. Let first $F \cap F^\perp \neq \{0\}$. Then we choose $0 \neq x \in F \cap F^\perp$ and $y \in E$ with $b(y, x) = 1$ (as in the description of reflections). Then $y \notin F$, since $x \in F^\perp$. Furthermore, $b(Ty, x) = b(Ty, Tx) = b(y, x) = 1$ and therefore reflections exist which map y on $x + y$, respectively Ty to $x + y$

(because $b(y, x+y) = b(y,x) \neq 0$ and $b(Ty, x+y) = b(Ty, x) \neq 0$), and keep $(x+y-y)^\perp \cap (x+y-Ty)^\perp \supseteq F$ fixed. So T fixes $F \oplus \mathbb{R}\, y$ up to these reflections, and we can apply the induction assumption.
If $F = \{0\}$, take $y \neq 0$, extend the isotropic set $\{e_1 := y, e_2 := Ty\}$ to a basis in a Lagrangian subspace, and let $x := e^1 + e^2$ in terms of the dual basis $(e^i)_{i=1}^k$. Then $b(x,y) = 1 = b(x,Ty)$ and we may proceed as just before.
Finally, if $F \neq \{0\}$ and $F \cap F^\perp = \{0\}$, then $E = F \oplus F^\perp$ and b induces on $F^\perp$ a symplectic form, because for $y' \in F^\perp$ with $b(y',y) = 0\,\forall\, y \in F^\perp$ we have $y' \in (F^\perp)^\perp = F$ and thus $y' = 0$. Furthermore, $T \in O_b(E)$ leaves the space $F^\perp$ invariant, because $b(Ty', y) = b(Ty', Ty) = b(y', y) = 0$ for all $y \in F$ and $y' \in F^\perp$. Since $T|_{F^\perp}$ has only 0 as a fixed point, it follows from the previous case that $T|_{F^\perp}$ is a composite of reflections along vectors in $F^\perp$. Such reflections, however, leave $F = F^{\perp\perp}$ fixed and thus T is the composition of these reflections on all E. $\square$

Corollary.

We have $Sp(2k) \subseteq SL(2k)$. $\square$

4.8 Low dimensions.

We will jointly diagonalize the elements of the Abelian among the following groups G, i.e. for each $T \in G$ we will determine the eigenvalues $\lambda_\pm^T$ and associated eigenvectors $e_\pm$ (independent on T). If Λ^T is the diagonal matrix with entries λ_+^T and λ_-^T, and U is the matrix with columns e_+ and e_-, i.e. $U(e_1) = e_+$ and $U(e_2) = e_-$, then $T \cdot U = U \cdot \Lambda^T$, i.e. $U^{-1} \cdot T \cdot U = \Lambda^T$. The conjugation with U thus maps the group G isomorphically to a group of diagonal matrices in $SL_\mathbb{C}(2)$.

$$\boldsymbol{SO(2)} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\} \cong \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{pmatrix} : \lambda \in S^1 \right\} \cong S^1,$$

$$\text{since} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2) \Leftrightarrow \begin{cases} (1) & a^2 + c^2 = 1 \quad (b(e_1, e_1) = 1) \\ (2) & b^2 + d^2 = 1 \quad (b(e_2, e_2) = 1) \\ (3) & ab + cd = 0 \quad (b(e_1, e_2) = 0) \\ (4) & ad - bc = 1 \quad (\det = 1) \end{cases}$$

It follows that

$$\begin{aligned} d \cdot (3) - b \cdot (4): &\quad -b = c(d^2 + b^2) = c, \\ b \cdot (3) + d \cdot (4): &\quad d = a(b^2 + d^2) = a \end{aligned}$$

and thus $a^2 + b^2 = 1$. All this follows more easily from the matrix equation $BT = (T^t)^{-1} B$, with $B = \mathrm{id}$.

The eigenvalues of T are $\lambda_\pm = a \pm i\, b$ with associated eigenvectors $e_\pm = (1, \pm i)$. Thus, the conjugation with $U = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ maps the group $SO(2)$ isomorphically to the diagonal matrices with conjugate complex entries of absolute value 1.

$$\frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} a + i\, b & 0 \\ 0 & a - i\, b \end{pmatrix}$$

$$\boldsymbol{SO(2{,}1)} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 - b^2 = 1 \right\} \cong$$

$$\cong \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} : \lambda \in \mathbb{R} \setminus \{0\} \right\} \cong \mathbb{R} \setminus \{0\},$$

Similar to $SO(2)$, the first equation follows from the matrix equation with $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The first isomorphism is then analogously given by conjugation with the matrix $U = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ of the eigenvectors to the eigenvalues $\lambda_\pm := a \pm b$. Conjugating with U gives

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

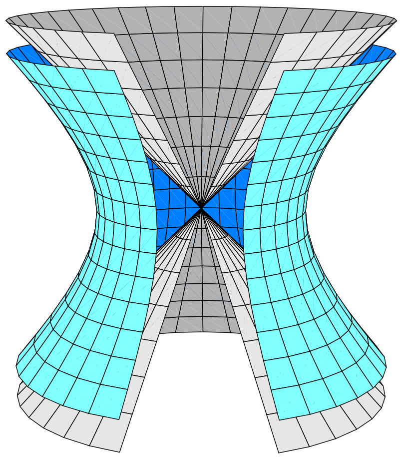
with $(a+b)(a-b) = 1$.

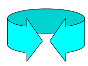
$$\boldsymbol{SL(2)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\} \cong$$
$$\cong \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\},$$

where the isomorphism is given by conjugation with $U := \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$, see [86, 34.5] and [86, 72.62], because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = U^{-1} \cdot \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ \beta_1 - i\beta_2 & \alpha_1 - i\alpha_2 \end{pmatrix} \cdot U = \begin{pmatrix} \alpha_1 + \beta_1 & \alpha_2 - \beta_2 \\ -\alpha_2 - \beta_2 & \alpha_1 - \beta_1 \end{pmatrix}$$
$$\Leftrightarrow \alpha_1 = \frac{a+d}{2},\ \alpha_2 = \frac{b-c}{2},\ \beta_1 = \frac{a-d}{2},\ \beta_2 = -\frac{b+c}{2}$$

Note that the quadric $\{(a,b) \in \mathbb{C}^2 : |a|^2 - |b|^2 = 1\}$ is diffeomorphic to $S^1 \times \mathbb{C}$ because of $(a,b) \mapsto (\frac{a}{|a|}, b) = (\frac{a}{\sqrt{1+|b|^2}}, b)$. However, the induced group structure on $S^1 \times \mathbb{C}$ looks more complicated.





$$\boldsymbol{Sp(2)} = SL(2),$$

$$\text{denn} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2) \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & cb - ad \\ ad - bc & 0 \end{pmatrix}$$
$$\Leftrightarrow ad - bc = 1$$

***SO*(3)** $= PSU(2) = PS^3 = \mathbb{P}^3$, where $PG := G/Z(G)$ for each group G and $Z(G) := \{g \in G : \forall\, h \in G : g \cdot h = h \cdot g\}$ denotes the center of G. Each unit quaternion $q = a + ib + jc + kd$ (see [**86**, 14.16]) acts orthogonally on $\mathbb{R}^4 = \mathbb{H}$ by conjugation and preserves the decomposition $\mathbb{R} \times \mathbb{R}^3$, because $q^{-1} \cdot 1 \cdot q = \frac{\overline{q}}{|q|^2} \cdot q = 1$ and

$$\begin{aligned}|q^{-1} \cdot p \cdot q|^2 &= \overline{(q^{-1} \cdot p \cdot q)} \cdot (q^{-1} \cdot p \cdot q) = \overline{q} \cdot \overline{p} \cdot \overline{\overline{q}} \cdot q^{-1} \cdot p \cdot q \\ &= q^{-1} \cdot |p|^2 \cdot q = |p|^2.\end{aligned}$$

Thus it acts as isometry on $\mathbb{R}^3 \cong \{0\} \times \mathbb{R}^3 \subseteq \mathbb{H}$. The kernel of this group homomorphism $\mathbb{H} \supseteq S^3 \to O(3)$ obviously is $Z(S^3) = Z(\mathbb{H}) \cap S^3 = \{\pm 1\}$. Thus, $S^3 \twoheadrightarrow PS^3 := S^3/Z(S^3)$ is a covering map of groups (see [**86**, 24.19]) and thus PS^3 is a compact connected 3-dimensional Lie group, i.e. openly embeds into $SO(3)$. Since $SO(3)$ is connected (see [**86**, 1.3]), $SO(3) \cong S^3/Z(S^3) = PS^3$ follows.

Geometrically, we saw this also in [**86**, 1.3]: A rotation is defined by the axis of rotation and the angle of rotation, i.e. by a vector $u \in D^3 := \{x \in \mathbb{R}^3 : |x| \le 1\}$ which corresponds to the rotation with the axis $u/|u| \in S^2$ and the rotation angle $\pi|u| \in [-\pi, \pi]/{\sim} \cong S^1$ (Note, that $(x_1, \varphi_1) \neq (x_2, \varphi_2)$ describe the same rotation iff $\varphi_1 = 0 = \varphi_2$ or $(x_1, \varphi_1) = -(x_2, \varphi_2)$). So we get a twofold covering map $S^3 \to S^3/{\sim} = D^3/{\sim} \cong SO(3)$ also from the following diagram

$$\begin{tikzcd}
S^2 \times [-1,1] \arrow[rr, two heads, "\mathrm{id} \times e^{i\pi_}"] \arrow[d] & & S^2 \times S^1 \arrow[d] \\
D^3 \arrow[r, two heads] \arrow[rr, dotted, bend left, two heads] & D^3/{\sim} \arrow[r, dotted, tail, two heads] & SO(3),
\end{tikzcd}$$

where the left vertical mapping is given by $(x, t) \mapsto tx$, the right by $(v, \varphi) \mapsto$ "rotation around v with angle φ" and $\sim$ is the equivalence relation generated by $v \sim -v$ for $v \in S^2$, see also [**86**, 24.40]. However, we do not get the homomorphism property of $S^3 \to SO(3)$ this way.

5. Smooth mappings

In order to relate different manifolds to each other, we also need the notion of smooth mappings between them and that we define now.

5.1 Definition (Smooth mapping).

A mapping $f : M \to N$ between two smooth manifolds $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ is called SMOOTH (C^∞): $\Leftrightarrow$ locally it can be extended to a smooth mapping $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^n$, that is

$$\forall\, p \in M\, \exists\, U \underset{\text{open}}{\subseteq} \mathbb{R}^m\, \exists\, \tilde{f} : U \to \mathbb{R}^n \text{ smooth with } p \in U \text{ and } \tilde{f}|_{M \cap U} = f|_{M \cap U}.$$

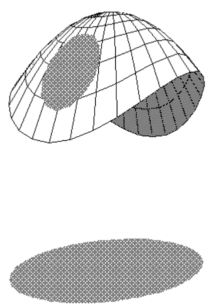
The constant mapping, the identity and the composition of smooth mappings are smooth: Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be smooth and $\tilde{f} : U_1 \to \mathbb{R}^{n_2}$ or $\tilde{g} : U_2 \to \mathbb{R}^{n_3}$ local smooth extensions, then $(g \circ f)^\sim = \tilde{g} \circ \tilde{f} : \tilde{f}^{-1}(U_2) \to \mathbb{R}^{n_3}$ is a local smooth extension of $g \circ f$, so $g \circ f$ is smooth.

5.2 Examples of smooth mappings.

1. For the classical Lie groups G from section 4, the multiplication mult : $G \times G \to G$ is smooth, because for the open subset $GL(E)$ of $L(E,E)$ this is the restriction of the bilinear mapping $(T,S) \mapsto T\,S$, and the other classical Lie groups G are submanifolds in $GL(E)$. The same holds to the inversion inv : $G \to G$, because for $GL(E)$ it is the solution for the implicit equation $\mathrm{mult}(A, \mathrm{inv}(A)) = \mathrm{id}$, to which the Inverse Function Theorem applies. The derivative is given by
$$\mathrm{inv}'(A)(B) = -A^{-1}BA^{-1}.$$
2. Taking the orthogonal complement $\perp$: $G(k,n) \to G(n-k,n)$ is a smooth mapping between Graßmann manifolds (see 3.9) as restriction to $G(k,n) \subseteq L(n,n)$ of the affine mapping $L(n,n) \to L(n,n)$ given by $P \mapsto 1 - P$.
3. The mapping "taking the image" $\mathrm{im} : V(k,n) \to G(k,n)$ is a smooth mapping on the Stiefel manifold (see 4.3), because as mapping $V(k,n) = \{T \in L(k,n) : T^tT = \mathrm{id}\} \to G(k,n) \subset L_k(n,n)$ it is given by $T \mapsto TT^t$: Obviously TT^t is the ortho projection ($(TT^t)^t(TT^t) = T^{tt}T^tTT^t = T\,\mathrm{id}\,T^t = TT^t$) with $\mathrm{im}\,T \supseteq \mathrm{im}\,TT^t \supseteq \mathrm{im}\,TT^tT = \mathrm{im}\,T$.

5.3 Lemma (Charts are diffeomorphisms).

Let $\varphi : U \to M$ be a local parameterization of the manifold M. Then φ is a local diffeomorphism.

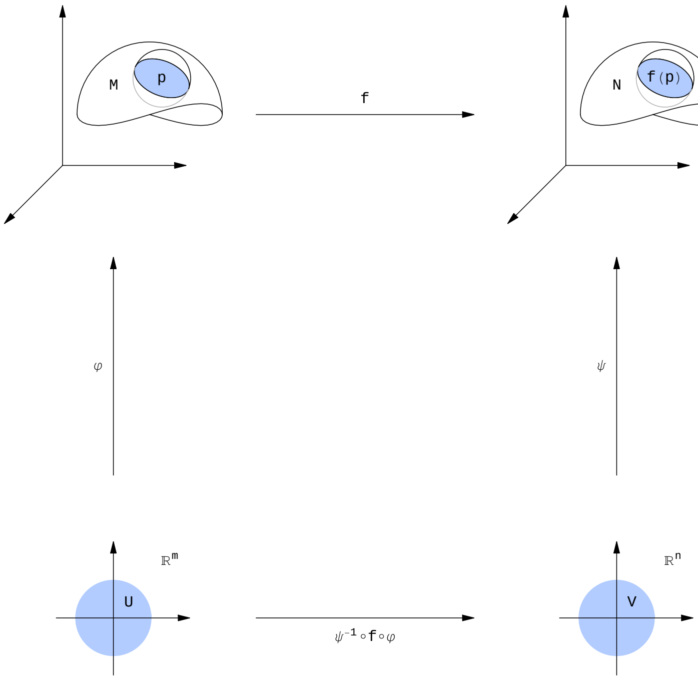


Proof. By definition φ is smooth. In the proof of the implication ($1 \Rightarrow 4$) of theorem 2.4 we have extended φ to a local diffeomorphism $\Phi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^n$. It follows from the bijectivity of $\varphi : U \to M \cap V$ (by assumption) that $\varphi^{-1} : M \cap V \to U$ exists as a map. It is smooth, because locally it can be extended to the smooth mapping Φ^{-1}. □

5.4 Lemma (Smooth mappings).

For a continuous mapping $f : M \to N$ between two manifolds M and N, the following statements are equivalent:

1. *f is smooth.*
2. *For each local parameterization φ of M and each local parameterization ψ of N, the following holds: The mapping $\psi^{-1} \circ f \circ \varphi$ is smooth wherever it is defined.*
3. *For each $p \in M$, a local parameterization φ exists for M centered at p, and a local parameterization ψ exists for N centered at $f(p)$, such that the chart representation $\psi^{-1} \circ f \circ \varphi$ is smooth.*



Proof. (1⇒2) Let $\varphi : U_1 \to V_1 \cap M$ and $\psi : U_2 \to V_2 \cap N$ be local parameterizations. The mapping $\psi^{-1} \circ f \circ \varphi$ is defined exactly for those $x \in U_1$, which satisfy $f(\varphi(x)) \in V_2$. But this is the open set $U_1 \cap (f \circ \varphi)^{-1}(V_2)$. The above mapping is smooth, as it is composed of smooth functions only.

(2⇒3) If the statement holds for all local parameterizations, then also for a specific one.

(3⇒1) We have to show that f is smooth. This is a local property, and locally f can be represented as a composition of smooth mappings as follows:

$$f = \psi \circ \underbrace{(\psi^{-1} \circ f \circ \varphi)}_{\text{smooth by (3)}} \circ \varphi^{-1}. \qquad \square$$

6. Abstract manifolds

Our preliminary definition of a manifold is unsatisfactory: So far, we have been using the properties of the surrounding space in an essential way, which conceptually has nothing to do with the object we want to describe.

In this section we want to get rid of the surrounding Euclidean space, and thus come to the concept of abstract manifolds.

The relevance of this approach is already shown in following examples.

6.1 Examples.

(See also section 1).

1. The Möbius strip topologically results when we identify one pair of opposite edges of a rectangle and supply the resulting object with the induced quotient topology. If we consider a realization of this space in $\mathbb{R}^3$ and cut it along the its middle line, we get a double twisted band. But if we do the same thing topologically, we get in contrast a non-twisted band. However, these two version cannot be transformed continuoulsy into one another in $\mathbb{R}^3$, whereas in $\mathbb{R}^4$ this is possible, as we saw in 3.10.
2. For the "Klein bottle" and the "projective plane" we can not be easily seen in which $\mathbb{R}^n$ they can be embedded. In any case it is not possible in $\mathbb{R}^3$.

We now extend the definition of submanifolds of $\mathbb{R}^n$ to that of abstract manifolds.

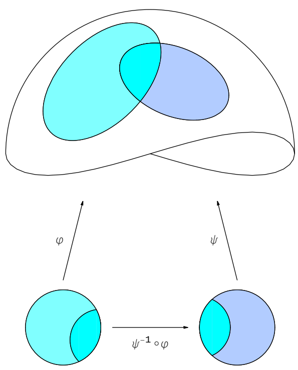
6.2 Definition (Abstract manifold).

Let X be an arbitrary set. A chart (or local parameterization) of X is an injective map $\varphi : \mathbb{R}^m \supseteq U \to X$ defined on an open set $U \subseteq \mathbb{R}^m$.

Two charts φ_1, φ_2 are called C^∞-compatible if the chart change

$$\varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(\varphi_2(U_2)) \to \varphi_2^{-1}(\varphi_1(U_1))$$

is a diffeomorphism of **open** sets. The idea behind this is that every chart φ_1 should be smooth, and by 5.4 $\varphi_2^{-1} \circ \varphi_1$ should be smooth wherever it is defined.



A C^∞-atlas for a set X is a family of C^∞-compatible charts whose images cover X. Two C^∞-atlases are called equivalent if all of their charts are C^∞-compatible to each other, i.e. their union is an C^∞-atlas.

An abstract C^∞-manifold is a set together with an equivalence class of smooth atlases.

6.3 Definition (Topology of a manifold).

On an abstract manifold M one obtains the final topology with respect to the charts by defining:

$$U \subseteq M \text{ is open} :\Leftrightarrow \varphi^{-1}(U) \text{ is open in } \mathbb{R}^m \text{ for each chart of the atlas.}$$

The charts $\varphi : U \to \varphi(U) \subseteq M$ then become homeomorphisms: They are continuous by construction of the topology on M and if $U_1 \subseteq U$ is open, so is $\varphi(U_1) \subseteq M$ because $\psi^{-1}(\varphi(U_1)) = (\varphi^{-1} \circ \psi)^{-1}(U_1)$ is the inverse image under the homeomorphism $\varphi^{-1} \circ \psi$.

Usually it is required that this topology is Hausdorff, i.e. each two disjoint points can be separated by disjoint open neighborhoods, This is done since uniqueness of limits is essential for analysis, and for most (but by no means all, see, e.g., [**86**, 30.15]) manifolds considered in the applications this holds.

The following proposition shows that this definition is really an extension of Definition 2.4.

6.4 Proposition.

Each C^∞ submanifold M of an $\mathbb{R}^n$ is naturally a C^∞ manifold and its topology is the subspace topology.

Proof. An atlas on M is obtained from all local injective parameterizations using 2.4. The chart changes are then smooth by 5.3 and the topology of M is the topology induced from the surrounding $\mathbb{R}^n$ because the parameterizations are local homeomorphisms, see the proof of 2.4. $\square$

6.5 Proposition (Maximal Atlas).

If $\mathcal{A}$ is a C^∞ atlas for M, then

$$\mathcal{A}_{\max} := \{\varphi : \varphi \text{ is a chart for } M \text{ and is compatible with all } \psi \in \mathcal{A}\}$$

is the uniquely determined maximal atlas that includes $\mathcal{A}$.

Proof. We first show that $\mathcal{A}_{\max}$ is a C^∞ atlas: Let φ, $\psi \in \mathcal{A}_{\max}$, then we show that $\varphi^{-1} \circ \psi$ is smooth. So let $x \in \psi^{-1}(\operatorname{im}\varphi)$, i.e. $\psi(x) \in \operatorname{im}\varphi \cap \operatorname{im}\psi$. Since $\mathcal{A}$ is an atlas, the existence of a $\chi \in \mathcal{A}$ with $\psi(x) \in \operatorname{im}\chi$ follows. Thus, $\varphi^{-1}\circ\chi\circ\chi^{-1}\circ\psi = (\chi^{-1}\circ\varphi)^{-1}\circ(\chi^{-1}\circ\psi)$ is defined locally at x. The two bracketed parts are smooth by the definition of $\mathcal{A}_{\max}$ and consequently $\varphi^{-1}\circ\psi$ is also smooth.

Now let $\mathcal{B}$ be a C^∞ atlas that includes $\mathcal{A}$, then we have to show that $\mathcal{B} \subseteq \mathcal{A}_{\max}$. Let $\varphi \in \mathcal{B}$, then φ is compatible with all $\psi \in \mathcal{B}$. Since $\mathcal{B} \supseteq \mathcal{A}$ we have that φ is compatible with all $\psi \in \mathcal{A}$, so by construction $\varphi \in \mathcal{A}_{\max}$. $\square$

6.6 Manifolds via chart changes.

The following considerations show that the chart changes, i.e. a family of local maps $\mathbb{R}^m \to \mathbb{R}^m$, already contain all the information about M. Let $\{g_{\alpha\beta} : \alpha, \beta \in A\}$ be a family of diffeomorphisms of open subsets of finite-dimensional vector spaces, so that $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$ and $g_{\alpha\beta}\circ g_{\beta\gamma} \subseteq g_{\alpha\gamma}$ hold (these are obviously properties of chart changes). Put $U_\alpha := \operatorname{dom} g_{\alpha\alpha}$ and define an equivalence relation on the disjoint union $\bigsqcup_\alpha U_\alpha = \bigcup_\alpha \{\alpha\}\times U_\alpha$ by: $(\alpha, x) \sim (\beta, y) :\Leftrightarrow x = g_{\alpha\beta}(y)$. This is indeed an equivalence relation:

Reflexivity: We have $g_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$, because $g_{\alpha\alpha}^{-1} = g_{\alpha\alpha}$ implies $\operatorname{im} g_{\alpha\alpha} = \operatorname{dom} g_{\alpha\alpha} = U_\alpha$ and $g_{\alpha\alpha}\circ g_{\alpha\alpha} \subseteq g_{\alpha\alpha}$ implies $g_{\alpha\alpha} \subseteq \mathrm{id}$, because $g_{\alpha\alpha}$ is injective being a diffeomorphism. Thus, $(\alpha, x) \sim (\alpha, x)$.

Symmetry: Let $(\alpha, x) \sim (\beta, y)$ be $x = g_{\alpha\beta}(y)$, i.e. $y = g_{\alpha\beta}^{-1}(x) = g_{\beta\alpha}(x)$, i.e. $(\beta, y) \sim (\alpha, x)$.

Transitivity: We have $(\alpha, x) \sim (\beta, y) \sim (\gamma, z)$, i.e. $g_{\alpha\beta}(y) = x$ and $g_{\beta\gamma}(z) = y$. Thus, $g_{\alpha\gamma}(z) = (g_{\alpha\beta}\circ g_{\beta\gamma})(z) = g_{\alpha\beta}(y) = x$, that is $x \sim z$.

Now let $M := \Big(\bigsqcup_{\alpha\in A} U_\alpha\Big)/_\sim$ and let $g_\alpha : U_\alpha \to M$ be defined by $x \mapsto [(\alpha, x)]_\sim$. Then g_α is injective, because $(\alpha, x) \sim (\alpha, y)$ implies $x = g_{\alpha\alpha}(y) = y$.

Moreover, $\mathrm{id}_{U_\alpha} = g_{\alpha\alpha} \supseteq g_{\alpha\beta}\circ g_{\beta\alpha} = g_{\beta\alpha}^{-1}\circ g_{\beta\alpha} = \mathrm{id}_{\operatorname{dom} g_{\beta\alpha}}$ implies $\operatorname{dom}(g_{\beta\alpha}) \subseteq U_\alpha$ and $\operatorname{im} g_{\beta\alpha} = g_{\beta\alpha}(\operatorname{dom}(g_{\beta\alpha})) \subseteq \operatorname{dom}(g_{\alpha\beta}) \subseteq U_\beta$.

Furthermore the chart changes $g_\beta^{-1}\circ g_\alpha$ are given by $y = (g_\beta^{-1}\circ g_\alpha)(x)$ with $x \in U_\alpha$ and $y \in U_\beta \Leftrightarrow g_\beta(y) = g_\alpha(x) \Leftrightarrow (\alpha, x) \sim (\beta, y) \Leftrightarrow x = g_{\alpha\beta}(y) \Leftrightarrow y = g_{\beta\alpha}(x)$ with $x \in \operatorname{dom} g_{\beta\alpha}$ and $y \in \operatorname{im} g_{\beta\alpha}$. Thus, M is a C^∞ manifold with chart changes $g_{\beta\alpha} = g_\beta^{-1}\circ g_\alpha$.

6.7 Definition (Topological manifold).

A topological space M is called a TOPOLOGICAL MANIFOLD $:\Leftrightarrow$ There is a family of homeomorphisms between open subsets of a finite-dimensional vector space and open subsets of M whose images cover M.

Such homeomorphisms are called CHARTS of M.

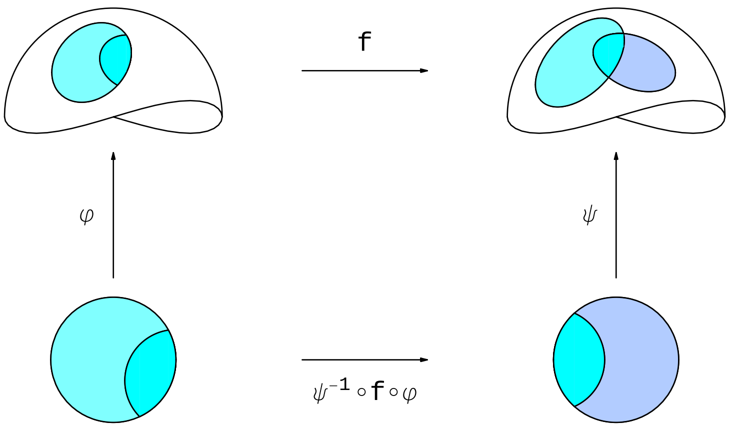
Comments.

1. If M is a topological manifold and $\mathcal{A}$ is the set of all charts for the topological manifold M, then their chart changes are automatically homeomorphisms of open subsets of $\mathbb{R}^m$. So in order to obtain a smooth atlas (and hence recognize M as a smooth manifold) one only needs to find enough of them, such that the corresponding chart changes are differentiable.
2. However, not every topological manifold has a C^∞ atlas. The first example [72] was 10-dimensional. Nowadays 4 is the lowest dimension for which there is an example.

We now want to transfer our differentiability concept for maps between submanifolds to abstract manifolds. Lemma 5.4 suggests the following definition:

6.8 Definition (Smooth mapping).

Let $(M,\mathcal{A})$ and $(N,\mathcal{B})$ be two C^∞ manifolds. A map $f : M \to N$ is called SMOOTH $:\Leftrightarrow$ f is continuous, and for each point $x \in M$, there are charts $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$, so that $x \in \operatorname{im}\varphi$, $f(x) \in \operatorname{im}\psi$ and the CHART REPRESENTATION $\psi^{-1} \circ f \circ \varphi$ of f is smooth. This then also holds to all charts $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$.



In particular, the identity $\mathrm{id} : (M,\mathcal{A}) \to (M,\mathcal{B})$ is a diffeomorphism if and only if the two atlases are equivalent to $\mathcal{A}$ and $\mathcal{B}$.

6.9 Remarks.

1. The continuity of f is assumed in order that the chart representation is defined on an open set.
2. Since the chart change is smooth, it suffices to request for each x the above property for some chart in $\mathcal{A}$ at x and some chart in $\mathcal{B}$ at $f(x)$. The property then follows for all charts.
3. Let us consider $\mathbb{R}$ as a topological manifold. It is very easy to specify two C^∞ structures, namely: $\mathcal{A}_1 := \{\mathrm{id} : \mathbb{R} \to \mathbb{R}\}$, and $\mathcal{A}_2 := \{\varphi(x) = x^3 : \mathbb{R} \to \mathbb{R}\}$. These are incompatible because $\varphi^{-1} \circ \mathrm{id} : x \to \sqrt[3]{x}$ is not smooth (because

$\frac{d}{dx}(\sqrt[3]{x})$ does not exist on 0), so they define two different C^∞ manifold structure on $\mathbb{R}$. However, the two structures are diffeomorphic, so in some sense are the same:

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} & & \cdots \text{ here as manifolds} \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} & & \cdots \text{ here as vector spaces} \end{array}$$

The mapping $f = \sqrt[3]{x}$ is a diffeomorphism: f, f^{-1} are bijective and clearly continuous. Likewise, f is smooth as $(\mathrm{id}^{-1} \circ f \circ \varphi)(x) = f(x^3) = \sqrt[3]{x^3} = x$ is smooth. Similarly f^{-1} is smooth, since $(\varphi^{-1} \circ f^{-1} \circ \mathrm{id})(x) = \varphi^{-1}(x^3) = x$ is smooth.

4. For $\dim M = 4$ and higher it is not true that any two C^∞ atlases of a topological manifold are the same up to a diffeomorphism. For dimension smaller than 4, however, it holds by [**121**]. For example, according to [**115**], S^7 carries at least 15 non-diffeomorphic C^∞ structures; the S^{31} more than $16 \cdot 10^6$. More precisely:

$n = \dim(S^n)$	7	8	9	10	11	12	13	14	15	16	17	18	...
Strukturen auf S^n	28	2	8	6	992	1	3	2	16256	2	16	16	...

For the topological space $\mathbb{R}^n$ with $n \neq 4$ there is exactly one smooth structure. For $n > 4$ this was proved in [**137**]. Quite surprisingly, Kirby proved 1982 that an exotic C^∞ structure exists for $\mathbb{R}^4$. In [**139**] it was shown that there are even uncountable many.

5. The class of C^∞ mappings between manifolds form a category, where a CATEGORY is a class of spaces (objects) and a class of mappings (morphisms), such that for each object the identity is a morphism and the composition of morphisms is one again. It is thus to be shown for three C^∞ manifolds M, N, P and $f : M \to N$ and $g : N \to P$ smooth mappings:
   - $g \circ f : M \to P$ is smooth.
   - $\mathrm{id} : M \to M$ is smooth.

6.10 Lemma (Open submanifold).

Let $(M, \mathcal{A})$ be a C^∞ manifold, and U open in M. Then, U is naturally a C^∞ manifold. An atlas on U is given by the restrictions of charts of M and the topology of this manifold is the trace topology of M.

Proof. The family $\mathcal{A}_U := \{\varphi|_{\varphi^{-1}(U)} : \varphi \in \mathcal{A}\}$ is a C^∞ atlas for U, because the chart changes

$$(\psi|_{\psi^{-1}(U)})^{-1} \circ \varphi|_{\varphi^{-1}(U)} = (\psi^{-1} \circ \varphi)|_{\varphi^{-1}(U)}$$

are C^∞ as restrictions of C^∞ functions. The topology of the manifold U is the trace topology, because a set $W \subseteq U$ is open in the manifold U if and only if $(\varphi|_{\varphi^{-1}(U)})^{-1}(W) = \varphi^{-1}(W) \subseteq \varphi^{-1}(U)$ is open to all charts φ. □

6.11 Remarks.

1. So it makes sense to talk about C^∞ mappings that are defined only on open subsets of a C^∞ manifold.

2. The charts φ of a C^∞ manifold are diffeomorphisms

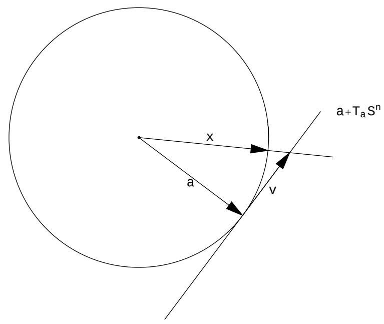
$$\varphi : \mathbb{R}^m \underset{\text{open}}{\supseteq} \operatorname{dom}\varphi \to \operatorname{im}\varphi \underset{\text{open}}{\subseteq} M.$$

In particular, $\mathcal{A}_{\max}$ consists of all those charts φ that are diffeomorphisms onto their images, i.e. $\varphi^{-1} \circ \psi$ is diffeomorphism of open sets for all charts $\psi \in \mathcal{A}$.

6.12 Examples of atlases.

1. $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$
   We consider as charts the radial projections onto the tangential planes

$$\alpha + v = \lambda x \text{ with } |x| = 1 \text{ and } \langle \alpha, v\rangle = 0$$
$$\Rightarrow x \mapsto v := \langle x, \alpha\rangle^{-1} \cdot x - \alpha,$$
$$v \mapsto x := (\alpha + v) \cdot |\alpha + v|^{-1}.$$



   A chart centered at α is

$$\varphi_\alpha : \mathbb{R}^n \cong \alpha^\perp \to \{x \in S^n : \langle x, \alpha\rangle > 0\} \subseteq M$$
$$\varphi_\alpha(v) := (\alpha + v) \cdot |(\alpha + v)|^{-1}$$
$$\varphi_\alpha^{-1}(x) = \langle x, \alpha\rangle^{-1} \cdot x - \alpha.$$

   The family $\{\varphi_\alpha : \alpha \in S^n\}$ forms a C^∞ atlas for S^n. However, already the images of the $\varphi_{\pm e_i}$ $i = 1, \dots, n+1$ cover S^n. Since both φ_α and φ_α^{-1} are smooth on open neighborhoods $\{v : v \neq -\alpha\}$ and $\{x : \langle x, \alpha\rangle > 0\}$ in $\mathbb{R}^{n+1}$, all chart changes are obviously smooth.

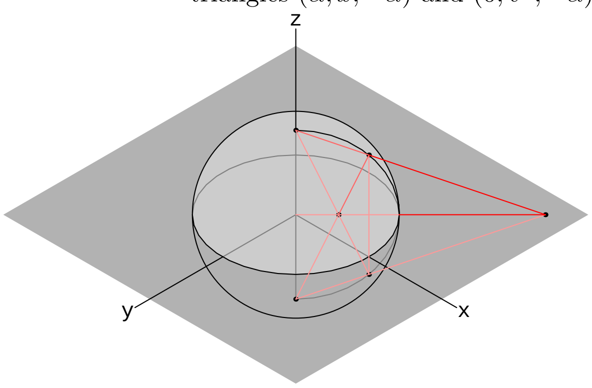
2. The atlas of stereographic projections for S^n has as charts ψ_α with $\alpha \in S^n$:

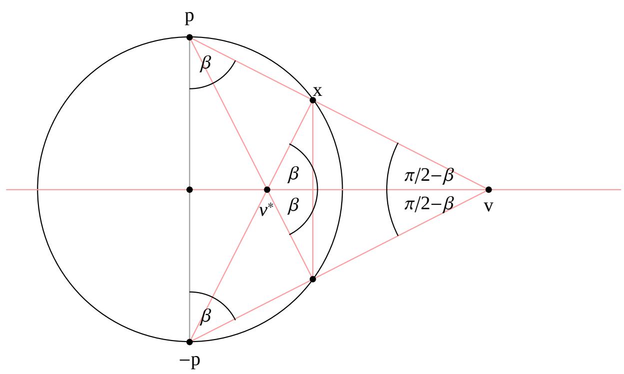
$$\psi_\alpha : \alpha^\perp \to S^n \setminus \{\alpha\}, \quad v \mapsto \alpha + 2(v - \alpha) \cdot (|v|^2 + 1)^{-1}$$

   (see 3.5) with inverse mapping

$$\psi_\alpha^{-1}(x) = (x - \langle x, \alpha\rangle \cdot \alpha) \cdot (1 - \langle x, \alpha\rangle)^{-1}.$$

   The chart ψ_α has $S^n \setminus \{\alpha\}$ as image. For an atlas, it is sufficient to find another chart centered at α, such as $\psi_{-\alpha}$. The chart change for these two particular charts is easily determined by elementary geometric considerations: Let v and v^* be the images of x under ψ_α^{-1} and $\psi_{-\alpha}^{-1}$. The triangles $(\alpha, 0, v)$ and $(\alpha, x, -\alpha)$ have two equal angles, one right angle and one at α, so they are similar. The triangles $(\alpha, x, -\alpha)$ and $(0, v^*, -\alpha)$ are also similar for analogous reasons.





   From the basic proportionality theorem (intercept theorem) we get:

$$\frac{|v|}{1} = \frac{1}{|v^*|} \Rightarrow |v| = |v^*|^{-1} \Rightarrow (\psi_{-\alpha}^{-1} \circ \psi_\alpha)(v) = v^* = v \cdot |v|^{-2}.$$

3. The obvious question whether the two structures on S^n given by 1 and 2 coincide has the following answer: These charts are compatible (that is, produce

the same maximum atlas) because

$$\begin{aligned}\varphi_\alpha^{-1} &: x \mapsto x \cdot \langle x, \alpha\rangle^{-1} - \alpha \\ \psi_\beta &: v \mapsto \beta + 2(v-\beta)\cdot(|v|^2+1)^{-1} \\ \varphi_\alpha^{-1}\circ\psi_\beta &: v \mapsto \frac{\beta + 2(v-\beta)\cdot(|v|^2+1)^{-1}}{\langle \beta + 2(v-\beta)\cdot(|v|^2+1)^{-1}, \alpha\rangle} - \alpha\end{aligned}$$

is an - albeit more complicated - C^∞ diffeomorphism. The compatibility of the charts can also be deduced from the fact that the charts can be considered as local diffeomorphisms of the surrounding space.

4. One-point compactification of $\mathbb{R}^n$:
We define on $\mathbb{R}^n_\infty := \mathbb{R}^n \cup \{\infty\}$ an atlas with two charts χ_0 and χ_∞ given by:

$$\begin{aligned}&\chi_0 : \mathbb{R}^n \to \mathbb{R}^n_\infty \\ &\chi_0(x) = x \\ &\chi_\infty : \mathbb{R}^n \to \mathbb{R}^n_\infty \\ &\chi_\infty(0) = \infty \text{ and } \chi_\infty(x) = x \cdot |x|^{-2} \text{ otherwise.}\end{aligned}$$

The chart changes $\chi_0^{-1}\circ\chi_\infty$ and $\chi_\infty^{-1}\circ\chi_0$ as maps $\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n\setminus\{0\}$ evaluate to $x\mapsto x\cdot|x|^{-2}$. This chart change has already appeared in 2 for the sphere, hence $\mathbb{R}^n_\infty$ is diffeomorphic to S^n. More explicitely, a diffgeomorphism f can be described as follows
Claim: $\mathbb{R}^n_\infty \cong S^n$ via $f(\infty) = e_1$ and $f(x) = \psi_{e_1}(x)$:
It is clear that f is bijective. Remains to show that both f and f^{-1} are smooth. The cases to be examined are:
- $\psi_{e_1}^{-1}\circ f\circ\chi_0 = \chi_0 = \mathrm{id}_{\mathbb{R}^n\setminus\{0\}}$
- $\psi_{e_1}^{-1}\circ f\circ\chi_\infty = \chi_0^{-1}\circ\chi_\infty$
- $\psi_{-e_1}^{-1}\circ f\circ\chi_0 = \chi_\infty^{-1}\circ\chi_0$
- $\psi_{-e_1}^{-1}\circ f\circ\chi_\infty = \mathrm{id}_{\mathbb{R}^n\setminus\{0\}}$

These are all diffeomorphisms, so f is a diffeomorphism.

5. Projective spaces

$$\mathbb{P}^n := \{\ell : \ell \text{ is a straight line through } 0 \text{ in } \mathbb{R}^{n+1}\} = (\mathbb{R}^{n+1}\setminus\{0\})/_\sim$$

where $x\sim y \Leftrightarrow \exists\,\lambda\in\mathbb{R}\setminus\{0\}$, so that $\lambda x = y$. As charts one uses for $0\le i\le n$:

$$\varphi_i : \begin{cases}\mathbb{R}^n \to \mathbb{R}^{n+1} \twoheadrightarrow \mathbb{P}^n \\ (y^1,\dots,y^n) \mapsto \big[(y^1,\dots,y^i,(-1)^i,y^{i+1},\dots,y^n)\big]\end{cases}$$

The sign is chosen so that $\mathbb{P}^n$ will be oriented whenever this is possible, see 27.42.3. Then $\varphi_i : \mathbb{R}^n \to \{x\in\mathbb{R}^{n+1}\setminus\{0\} : x^{i+1}\ne 0\}/_\sim$ is bijective with inverse

$$\varphi_i^{-1} : [(x^0,\dots,x^n)] \mapsto \frac{(-1)^i}{x^i}(x^0,\dots,x^{i-1},x^{i+1},\dots x^n).$$

The chart change is calculated as follows:

$$\begin{aligned}(\varphi_j^{-1}\circ\varphi_i)(y^1,\dots,y^n) &= \varphi_j^{-1}\big[(y^1,\dots,y^i,(-1)^i,y^{i+1},\dots,y^n)\big] \overset{(\text{W.l.o.g. } j>i)}{=\!=\!=} \\ &= \frac{(-1)^j}{y^j}\Big(y^1,\dots,y^i,(-1)^i,y^{i+1},\dots,y^{j-1},y^{j+1},\dots,y^n\Big).\end{aligned}$$

This is a diffeomorphism (on its domain) and additionally orientation-preserving for odd n. So $\mathbb{P}^n$ is a C^∞ manifold. An analogous procedure yields $\mathbb{P}^n_{\mathbb{C}}$ (the space of complex lines in $\mathbb{C}^{n+1}$) with $\dim\mathbb{P}^n_{\mathbb{C}} = 2n$ and also $\mathbb{P}^n_{\mathbb{H}}$ with $\dim\mathbb{P}^n_{\mathbb{H}} = 4n$.

In 3.9 we had given another description of the projective spaces $\mathbb{P}^n$ as Graßmann manifolds $G(1, n+1) \subseteq L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. We had identified straight lines through 0 in $\mathbb{R}^{n+1}$ with the orthogonal projections onto them. We now want to show that this describes diffeomorphic spaces. Let $\bar\varphi_i : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be given by $(y^1, \dots, y^n) \mapsto (y^1, \dots, y^i, (-1)^i, y^{i+1}, \dots, y^n)$. Then $\varphi_i = \pi \circ \bar\varphi_i$, where $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n := \mathbb{R}^{n+1}/_\sim$ denotes the canonical projection $x \mapsto [x]$. For $a, b \in E := \mathbb{R}^{n+1}$, the operator $a \otimes b \in L(E, E)$ is defined by $(a \otimes b)(x) := \langle a, x\rangle b$ (For an explanation of this notation, see 21.3.1). Then $(a, b) \mapsto a \otimes b$ is bilinear and

- $(a \otimes b)^t = b \otimes a$, because $\langle (a \otimes b)x, y\rangle = \langle a, x\rangle \cdot \langle b, y\rangle = \langle x, (b \otimes a)y\rangle$,
- $(a_1 \otimes b_1) \circ (a_2 \otimes b_2) = \langle a_1, b_2\rangle\, a_2 \otimes b_1 : x \mapsto \langle a_1, b_2\rangle \langle a_2, x\rangle b_1$.

The rank 1 linear operators P are exactly those of the form $P = a \otimes b \neq 0$ (in fact, $\operatorname{codim} \ker P = 1 \Rightarrow \ker P = a^\perp$ for some $|a| = 1 \Rightarrow x - \langle a, x\rangle a \in \ker P \Rightarrow P(x) = \langle a, x\rangle\, P(a) = (a \otimes P(a))(x)$) and the ortho-projections among them are those of the form $P = a \otimes a$ with $|a| = 1$: From $P = P^t \circ P = (P(a) \otimes a) \circ (a \otimes P(a)) = |P(a)|^2 a \otimes a$ we get $P(a) = |P(a)|^2\, |a|^2\, a$ and hence $|P(a)| = 1/|a|^3 = 1$, thus $P(a) = a$.
The smooth mapping $a \mapsto \frac{a}{|a|} \otimes \frac{a}{|a|}$ is thus a surjective smooth (since $\otimes$ is bilinear) mapping $f : \mathbb{R}^{n+1} \setminus \{0\} \to G(1, n+1)$ and factorizes to a smooth bijection $\tilde f : \mathbb{P}^n \to G(1, n+1)$. Locally we get a smooth (since $P \mapsto P(a)$ is linear) inverse mapping by sending P near $a \otimes a$ to $\pi(P(a)) \in \mathbb{P}^n$: In fact, for $P = b \otimes b$ with $|b| = 1$ we have $P(a) = \langle b, a\rangle b$ and thus $\tilde f(\pi(P(a))) = \tilde f(\pi(\langle b, a\rangle b)) = \tilde f(\pi(b)) = f(b) = b \otimes b = P$. Conversely, $\pi(f(b)(a)) = \pi\big(\frac{\langle b, a\rangle}{|b|^2} b\big) = \pi(b)$. So $\tilde f$ is the desired diffeomorphism.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\bar\varphi_i"] \arrow[dr, "\varphi_i"'] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[rr, two heads, "x \mapsto \frac{x}{|x|}"] \arrow[d, "\pi"] \arrow[drr, "f"] & & S^n \arrow[d, "x \mapsto x \otimes x"] & \\
& \mathbb{P}^n \arrow[rr, dotted, "\cong", "\tilde f"'] & & G(1, n+1) \arrow[r, hook] & L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})
\end{tikzcd}$$

6.13 Remarks.

Between lowdimensional projective spaces and spheres there are some relationships:

1. The projective line $\mathbb{P}^1 \cong S^1$.

   As charts for S^1 we use $\psi_+ := \psi_{(0,1)}$ and $\psi_- := \psi_{(0,-1)}$, the stereographic projections with respect to the poles $(0, 1)$ and $(0, -1)$ (cf. 6.12). For the chart change we got:

   $$(\psi_{(0,1)}^{-1} \circ \psi_{(0,-1)})(x) = (\psi_{(0,-1)}^{-1} \circ \psi_{(0,1)})(x) = \frac{1}{x} \text{ on } \mathbb{R} \setminus \{0\}$$

   As charts for $\mathbb{P}^1$, we associate to each line through the origin the intersection with the line $y = 1$ (or $x = 1$), see 6.12.5:

   $$\varphi_- : \begin{cases} \mathbb{R} \to \mathbb{P}^1 \setminus [(0, 1)] \\ x \mapsto [(1, x)] \end{cases} \text{ and } \varphi_+ : \begin{cases} \mathbb{R} \to \mathbb{P}^1 \setminus [(1, 0)] \\ x \mapsto [(x, 1)] \end{cases}$$

   The chart φ_- reaches all equivalence classes except $[(0, 1)]$ (which corresponds to the y-axis). This deficiency is corrected by the φ_+ chart. We calculate the inverse mappings:

$$\begin{aligned}
\varphi_-^{-1} &: [(x,y)] = \big[(1,\frac{y}{x})\big] \mapsto \frac{y}{x} \\
\varphi_+^{-1} &: \mathbb{P}^1 \setminus [(1,0)] \to \mathbb{R} \text{ mit} \\
\varphi_+^{-1} &: [(x,y)] \mapsto \frac{x}{y}
\end{aligned}$$

Now for the chart change:

$$(\varphi_+^{-1} \circ \varphi_-)(x) = \varphi_+^{-1}[(1,x)] = \frac{1}{x}, \quad (\varphi_-^{-1} \circ \varphi_+)(x) = \varphi_-^{-1}[(x,1)] = \frac{1}{x},$$

Both are defined on $\mathbb{R} \setminus \{0\}$. Let $f : \mathbb{P}^1 \to S^1$ given by:

$$f := \begin{cases} \psi_- \circ \varphi_-^{-1} \text{ on } \mathbb{P}^1 \setminus [(0,1)] \\ \psi_+ \circ \varphi_+^{-1} \text{ on } \mathbb{P}^1 \setminus [(1,0)]. \end{cases}$$

This mapping is well-defined, as $\psi_-^{-1} \circ \psi_+ = \varphi_-^{-1} \circ \varphi_+$ implies that $\psi_- \circ \varphi_-^{-1} = \psi_+ \circ \varphi_+^{-1}$ on $\mathbb{P}^1 \setminus \{[(1,0)], [(0,1)]\}$. But it is also a diffeomorphism: We only have to show this for the chart representations. On $\mathbb{P}^1 \setminus [(0,1)]$ the chart representation $\psi_-^{-1} \circ f \circ \varphi_- = \psi_-^{-1} \circ \psi_- \circ \varphi_-^{-1} \circ \varphi_- = \mathrm{id}$ is a diffeomorphism because of $f(\operatorname{im} \varphi_-) = \operatorname{im} \psi_-$.

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "f"] & S^1 \\
\mathbb{R} \arrow[u, "\varphi_\pm"] \arrow[r, equal] & \mathbb{R} \arrow[u, "\psi_\pm"']
\end{tikzcd}$$

Analogously for $x \in \mathbb{P}^1 \setminus [(1,0)]$.

The diffeomorphy $\mathbb{P}^1 \cong S^1$ can be seen easier using 6.12.4:

$$\begin{tikzcd}
& \mathbb{R} \arrow[dl, "\varphi_+"'] \arrow[dr, "\chi_\infty"] & \\
\mathbb{P}^1 \arrow[rr, dotted, "\cong", "\varphi_-^{-1}"'] & & \mathbb{R}_\infty \\
& \mathbb{R} \arrow[ul, "\varphi_-"] \arrow[ur, "\chi_0"'] &
\end{tikzcd}$$

2. $\mathbb{P}^1_{\mathbb{C}} \cong S^2$: Geometrically, this can be visualized as follows: $\mathbb{P}^1_{\mathbb{C}}$ is parameterized by the unique intersections of these complex lines through 0 with the complex affine line $g := \{(z,1) : z \in \mathbb{C}\} \cong \mathbb{R}^2$. Only the complex line h parallel to g, that is $h = \{(z,0) : z \in \mathbb{C}\} \in \mathbb{P}^1_{\mathbb{C}}$, is not caught. Those straight lines that are close to h have their intersections far out on g. Thus, the missing straight line h corresponds to the point ∞ in the one-point compacting $\mathbb{R}^2_\infty$ of $\mathbb{R}^2$. But we know that $\mathbb{R}^2_\infty$ and S^2 are diffeomorphic (see Example 6.12).

7. Products and sums of manifolds

The easiest way to make out of manifolds new ones is the formation of products and coproducts (i.e. sums) which we will cover in this section.

7.1 Proposition (Products).

For $i = 1, \dots, n$ let $(M_i, \mathcal{A}_i)$ be a C^∞ manifold. Then, $\prod_{i=1}^n M_i$ is naturally a C^∞ manifold. The atlas on $\prod M_i$ is given by

$$\prod_{i=1}^n \mathcal{A}_i := \{\varphi_1 \times \dots \times \varphi_n : \varphi_i \in \mathcal{A}_i\}.$$

The product $\prod M_i$ has the following universal property: For each C^∞ manifold N and C^∞ mappings $f_i : N \to M_i$, there exists a unique C^∞ mapping $f = (f_1, \dots, f_n)$ with $\mathrm{pr}_i \circ f = f_i$. Where $\mathrm{pr}_i : \prod M_i \to M_i$ is the C^∞ mapping $(x^1, \dots, x^n) \mapsto x^i$. The universal property can also be expressed by the following diagram:

$$\begin{tikzcd} M_i & & \prod_i M_i \arrow[ll, "\mathrm{pr}_i"'] \\ & N \arrow[ul, "f_i"] \arrow[ur, dotted, "\exists\,!f"'] & \end{tikzcd}$$

The topology induced on $\prod_i M_i$ is exactly the product topology.

Proof. Obviously, the topology induced by the atlas $\prod_i \mathcal{A}_i$ is just the product topology, because the product of homeomorphisms φ_i is also a homeomorphism

$$\varphi_1 \times \dots \times \varphi_n : \operatorname{dom} \varphi_1 \times \dots \times \operatorname{dom} \varphi_n \to \operatorname{im} \varphi_1 \times \dots \times \operatorname{im} \varphi_n \subseteq \prod M_i.$$

The chart changes

$$(\psi_1 \times \dots \times \psi_n)^{-1} \circ (\varphi_1 \times \dots \times \varphi_n) = (\psi_1^{-1} \circ \varphi_1) \times \dots \times (\psi_n^{-1} \circ \varphi_n).$$

are products of diffeomorphisms $(\psi_i^{-1} \circ \varphi_i)$ hence are diffeomorphisms, and thus $\prod M_i$ is a C^∞ manifold.

We now claim $\mathrm{pr}_i : \prod M_i \to M_i$ is smooth.

Let $(x^1, \dots, x^n) \in \prod M_i$ and let $\varphi_1 \times \dots \times \varphi_n$ be a chart at this point, i.e. φ_i is chart at x^i. Thus,

$$\varphi_i^{-1} \circ \mathrm{pr}_i \circ (\varphi_1 \times \dots \times \varphi_n) : \mathbb{R}^{m_1 + \dots + m_n} \to \mathbb{R}^{m_i}, \quad (x^1, \dots, x^n) \mapsto x^i$$

is a linear projection, hence smooth.
Let $f_i \in C^\infty(N, M_i)$, then $f : x \mapsto (f_1(x), \dots, f_n(x))$ is the only mapping with $\mathrm{pr}_i \circ f = f_i$ and it is C^∞: If φ is a chart of N then

$$\begin{aligned}(\varphi_1 \times \dots \times \varphi_n)^{-1} \circ f \circ \varphi &= (\varphi_1^{-1} \times \dots \times \varphi_n^{-1}) \circ (f_1 \circ \varphi, \dots, f_n \circ \varphi) \\ &= (\varphi_1^{-1} \circ f_1 \circ \varphi, \dots, \varphi_n^{-1} \circ f_n \circ \varphi).\end{aligned}$$

By assumption the $\varphi_i^{-1} \circ f_i \circ \varphi$ are smooth (because the f_i are smooth), thus also f is smooth. □

7.2 Examples of products.

1. The cylinder is a subset in $\mathbb{R}^3$, namely the Cartesian product of S^1 and an open interval $I \subseteq \mathbb{R}$, hence is a C^∞ manifold.

2. The n-dimensional torus in $\mathbb{R}^{2n}$ is the n-fold Cartesian product of $S^1 \subseteq \mathbb{R}^2$:

$$S^1 \times S^1 \times \dots \times S^1 = \prod_{i=1}^n S^1 = (S^1)^n = T^n.$$

   For $n = 2$ we get the already known “bicycle tube” (see 3.6), but as a subset of $\mathbb{R}^4$ instead of $\mathbb{R}^3$.

7.3 Proposition (Sums).

Let $(M_i, \mathcal{A}_i)$ be C^∞ manifolds. Then the disjoint union $\bigsqcup_i M_i$ is naturally a C^∞ manifold. An atlas $\mathcal{A}$ on $\bigsqcup_i M_i$ is given by $\bigcup_i \mathcal{A}_i$ (here no constraint on the index set is necessary).

In addition, $\bigsqcup M_i$ has the following universal property: For each C^∞ manifold N and for all C^∞ mappings $f_i : M_i \to N$ there is a unique smooth mapping f with

$$f := \bigsqcup_i f_i : \bigsqcup M_i \to N, \text{ so } f|_{M_i} = f_i.$$

This can also be expressed by the following diagram:

$$\begin{tikzcd} M_i \arrow[rr, hook] \arrow[dr, "f_i"'] & & \bigsqcup_i M_i \arrow[dl, dotted, "\exists\, ! f"] \\ & N & \end{tikzcd}$$

Proof. For $\varphi, \psi \in \bigcup \mathcal{A}_i$, either $\varphi^{-1} \circ \psi = \emptyset$ or some i exists with $\varphi, \psi \in \mathcal{A}_i$ and thus $\psi^{-1} \circ \varphi$ is smooth. Open sets in $\bigsqcup M_i$ are unions of open sets in M_i. The universal property is now obvious. □

8. Partitions of unity

To get global constructions from local constructions (such as those treated in Analysis), we need a method to glue them locally. This requires families of "weight" functions, i.e. functions which are non-vanishing only locally, are greater than or equal to 0 and together add up to 1. These are the so-called partitions of unity, which we will discuss in this section.

8.1 Definition (Partition of unity).

Let M be a C^∞ manifold and $\mathcal{U}$ an open covering of M. A SMOOTH PARTITION OF UNITY subordinated to $\mathcal{U}$ is a set $\mathcal{F}$ of smooth functions $M \to \{t \in \mathbb{R} : t \geq 0\}$ with the following properties:

1. The family $\{\operatorname{supp}(f) : f \in \mathcal{F}\}$ is a refinement of $\mathcal{U}$,
   i.e. $\forall\, f \in \mathcal{F}\ \exists\, U_f \in \mathcal{U} : \operatorname{supp}(f) \subseteq U_f$,
   where $\operatorname{supp}(f)$ is the closure of $\{x : f(x) \neq 0\}$.
2. The family $\{\operatorname{supp}(f) : f \in \mathcal{F}\}$ is locally finite,
   i.e. $\forall\, p \in M\ \exists\, U(p)$ so that $\{f \in \mathcal{F} : \operatorname{supp}(f) \cap U(p) \neq 0\}$ is finite.
3. $\sum_{f \in \mathcal{F}} f = 1$.

8.2 Proposition (Partition of unity).

Let $X \subseteq \mathbb{R}^n$ open and $\mathcal{U}$ be an open covering of X. Then there is a C^∞ partition of unity subordinated to $\mathcal{U}$.

Proof. Claim: *X (and indeed every separable metric space) is* LINDELÖF, *i.e. every open covering of X has a countable subcovering.*
So let $\mathcal{U}$ be an open covering of X. Let

$$X_0 := \Big\{(r, x) : 0 < r \in \mathbb{Q}, x \in \mathbb{Q}^n \cap X, \exists\, U \in \mathcal{U} : U_r(x) := \{y : \|y - x\| < r\} \subseteq U\Big\}.$$

Then X_0 is countable and by definition there is a set $U_{r,x} \in \mathcal{U}$ with $U_r(x) \subseteq U_{r,x}$ for each $(r, x) \in X_0$. By the selection principle we can define a function $\Psi : X_0 \to \mathcal{U}$,

$(r,x) \mapsto U_{r,x}$. We claim that the image $\mathcal{U}_0 := \Psi(X_0)$ of Ψ is a countable subcovering of $\mathcal{U}$. Countability is clear. So let $x \in X$ be arbitrary. Since $\mathcal{U}$ is a covering of X, there exists $U \in \mathcal{U}$ with $x \in U$. Since U is open, a $\delta > 0$ exists with $U_\delta(x) \subseteq U$. Let $r \in \mathbb{Q}$ with $0 < 2r < \delta$. Since $\mathbb{Q}^n \cap X$ is dense in X, there is $x_0 \in \mathbb{Q}^n \cap X$ with $d(x_0, x) < r$ and thus $x \in U_r(x_0) \subseteq U_\delta(x) \subseteq U$, i.e. $x \in U_r(x_0) \subseteq U_{r,x_0}$ with $(r, x_0) \in X_0$.

Claim: *There are smooth functions with arbitrarily small support.*
Let us consider the smooth function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(t) := \begin{cases} e^{-\frac{1}{t}} > 0 & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

If we now define a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ for $x_0 \in \mathbb{R}^n$ and $r > 0$ by

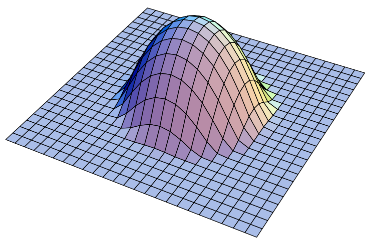
$$\varphi(x) := h(r^2 - \|x - x_0\|^2),$$

then $\varphi(x) \geq 0$ is for all $x \in \mathbb{R}^n$ and

$$0 = \varphi(x) := h(r^2 - \|x - x_0\|^2) \Leftrightarrow$$
$$\Leftrightarrow r^2 - \|x - x_0\|^2 \leq 0 \Leftrightarrow x \notin U_r(x_0),$$

i.e. the support of φ is given by

$$\operatorname{supp} \varphi = \{x : \|x - x_0\| \leq r\}.$$



Claim: *There is a countable locally finite refinement $\{W_n : n \in \mathbb{N}\}$ for $\mathcal{U}$.*
Let $\mathcal{U}$ be the given open covering of X. For each $x \in U \in \mathcal{U}$ we choose an $r > 0$ with $\overline{U_r(x)} \subseteq U$. From the above we know that there is a $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with

$$U_r(x) = \{y : \varphi(y) \neq 0\} =: U_\varphi.$$

These sets are a refinement of $\mathcal{U}$. Since X is Lindelöf, countably many functions exist $\varphi_1, \varphi_2, \dots$ s.t. $\{U_{\varphi_n} : n \in \mathbb{N}\}$ is a covering of X and a refinement of $\mathcal{U}$.

This does not have to be local finite yet, so we define W_n as follows:

$$W_n := \Big\{x : \varphi_n(x) > 0 \wedge \varphi_i(x) < \tfrac{1}{n} \text{ for } 1 \leq i < n\Big\} \subseteq U_{\varphi_n}.$$

It is clear that the W_n are open (given by continuous inequalities) and are subsets of U_{φ_n}.

The W_n form an covering of X, because for each $x \in X$ there is a minimal n_0 with $\varphi_{n_0}(x) > 0$ and thus $x \in W_{n_0}$.

To prove that $\{W_n : n \in \mathbb{N}\}$ is locally finite we define an open neighborhood around x:

$$U(x) := \left\{y : \varphi_{n_0}(y) > \tfrac{1}{2}\varphi_{n_0}(x)\right\}.$$

If $W_k \cap U(x) \neq \emptyset$, then for y chosen in the intersection of these two sets we have:

$$\varphi_i(y) < \tfrac{1}{k} \text{ for all } i < k \quad \text{und} \quad \tfrac{1}{2}\varphi_{n_0}(x) < \varphi_{n_0}(y).$$

If $k > n_0$ is so large that $\frac{1}{k} < \frac{1}{2}\varphi_{n_0}(x)$, then

$$\tfrac{1}{k} < \tfrac{1}{2}\varphi_{n_0}(x) < \varphi_{n_0}(y) < \tfrac{1}{k}$$

yields a contradiction. So there are only finitely many k with $W_k \cap U(x) \neq \emptyset$.

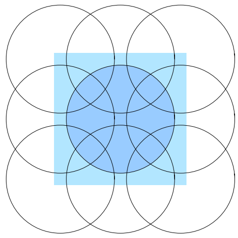
Claim: *There is a partition of unity $\{f_n : n \in \mathbb{N}\}$ with $\{x : f_n(x) \neq 0\} = W_n$.*
For the time being we define smooth function $\psi_n : X \to \{t : 0 \leq t\}$ by

$$\psi_n(x) := h(\varphi_n(x)) \cdot h\Big(\frac{1}{n} - \varphi_1(x)\Big) \cdot \ldots \cdot h\Big(\frac{1}{n} - \varphi_{n-1}(x)\Big).$$

Then

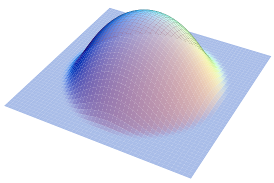
$$\psi_n(x) \neq 0 \Leftrightarrow \Big(\varphi_n(x) > 0\Big) \wedge \Big(\tfrac{1}{n} - \varphi_1(x) > 0\Big) \wedge \ldots \wedge \Big(\tfrac{1}{n} - \varphi_{n-1}(x) > 0\Big) \Leftrightarrow x \in W_n.$$

Since $\{W_n : n\}$ is locally finite, locally only finitely many summands in the sum $\sum_{n=1}^\infty \psi_n$ are not equal to 0, and thus $\psi := \sum_{n=1}^\infty \psi_n \ \in C^\infty(X,\mathbb{R})$. This function ψ vanishes nowhere, because the $\{W_n : n \in \mathbb{N}\}$ form an covering. Now we define $f_n := \frac{\psi_n}{\psi} \ \in C^\infty(X,\mathbb{R})$. Then

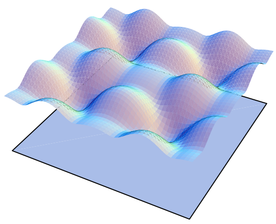


$$\sum f_n = \frac{\sum \psi_n}{\psi} = \frac{\psi}{\psi} = 1$$

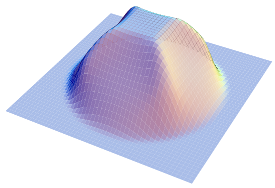
and this proves (3) of 8.1.
(1) and (2) now follow: $\operatorname{supp}(f_n) \subseteq \overline{W_n} \subseteq \overline{U_{\varphi_n}} \subseteq U$ for a $U \in \mathcal{U}$.



ψ_n



ψ



f_n

$\square$

Remarks.

This proof works for Lindelöf spaces X for which the sets $\{x : f(x) \neq 0\}$ with $f \in C^\infty$ form a basis of the topology.

8.3 Corollary (Extending smooth functions).

Let M be a submanifold of $\mathbb{R}^n$. A map $g : M \to \mathbb{R}$ is smooth if and only if there is an open subset $\tilde{M}$ of $\mathbb{R}^n$ that includes M, and a smooth map $\tilde{g} : \tilde{M} \to \mathbb{R}$ that extends g, i.e. $\tilde{g}|_M = g$.

Proof. ($\Leftarrow$) is trivial.

($\Rightarrow$) For each $p \in M$ there is an open neighborhood $U_p \subseteq \mathbb{R}^n$ and a smooth extensionn $\tilde{g}^p : U_p \to \mathbb{R}$. Let $\mathcal{U} := \{U_p : p \in M\}$ and $\tilde{M} := \bigcup \mathcal{U} = \bigcup_{p \in M} U_p$. Then $\tilde{M} \subseteq \mathbb{R}^n$ is open and $M \subseteq \tilde{M}$. By 8.2 there is a partition $\mathcal{F}$ of unity which is subordinated to $\mathcal{U}$, so in particular for each $f \in \mathcal{F}$ there exists a $p(f) \in M$ with $\operatorname{supp}(f) \subseteq U_{p(f)}$. We now define the mapping $\tilde{g}$ as follows:

$$\tilde{g} := \sum_{f \in \mathcal{F}} f \cdot \tilde{g}^{p(f)},$$

where $f \cdot \tilde{g}^{p(f)}$ on $\tilde{M} \setminus \operatorname{supp}(f)$ is extended by 0 (note that a function piecewise smoothly defined on an open covering is itself smooth). In this sum, the individual summands are smooth, but only finitely many are $\neq 0$. But that just means that $\tilde{g} : \tilde{M} \to \mathbb{R}$ is also smooth. To show the last equation, we restrict $\tilde{g}$ to M and calculate for a $x \in M$:

$$\tilde{g}(x) = \sum_{f \in \mathcal{F}} f(x) \cdot \underbrace{\tilde{g}^{p(f)}(x)}_{g(x)} = \underbrace{\Big(\sum_{f \in \mathcal{F}} f(x)\Big)}_{1} \cdot g(x) = g(x). \qquad \square$$

8.4 Corollary (Partition of unity for manifolds).

Each submanifold of $\mathbb{R}^n$ has a subordinate C^∞ partition of unity for every open covering.

Proof. Let $\mathcal{U}$ be an open covering of M. Without restricting generality, the $U \in \mathcal{U}$ can be chosen so small that they are images of parametrizations and thus there are open subsets $\tilde{U}$ of $\mathbb{R}^n$, which trivialize M locally. In particular they satisfy $\tilde{U} \cap M = U$ (and $\tilde{U} \cap M$ is closed in $\tilde{U}$). Then $X := \bigcup_{U \in \mathcal{U}} \tilde{U} \subseteq \mathbb{R}^n$ is open (and M is closed in X: Namely given $x \in X \setminus M$, then $\exists\ U \in \mathcal{U} : x \in \tilde{U} \setminus M$ and thus $\exists\ U(x) \subseteq \tilde{U} : U(x) \cap M = \emptyset$).

By 8.2, a partition $\mathcal{F}$ of unity exists on X, which is subordinated to $\tilde{\mathcal{U}} := \{\tilde{U} : U \in \mathcal{U}\}$. So $\{f|_M : f \in \mathcal{F}\}$ is a partition of unity, which is subordinated to $\mathcal{U}$. □

8.5 Proposition.

Each closed set of $A \subseteq \mathbb{R}^n$ is the zero set of a C^∞ function.

Compare this with Theorem 2.4 on zero sets of regular mappings.

Proof. Let $A \subseteq \mathbb{R}^n$ be closed and $x \in \mathbb{R}^n \setminus A$, then there is a smooth $f_x \geq 0$ with $x \in \operatorname{supp} f_x \subseteq \mathbb{R}^n \setminus A$, compact. Let $\mathcal{U}$ be an open covering of $\mathbb{R}^n \setminus A$ with sets of the form $U_x = \{y : f_x(y) > 0\}$, where $x \in \mathbb{R}^n \setminus A$. Since $\mathbb{R}^n \setminus A$ is Lindelöf, $\mathcal{U}$ has a countable subcovering. Let f_1, f_2, … be the corresponding functions. Without loss of generality

$$\left| \frac{\partial f_k^{t_1+\ldots+t_n}}{\partial x_1^{t_1} \ldots \partial x_n^{t_n}}(x) \right| \leq \frac{1}{2^k} \text{ for } x \in \mathbb{R}^n \text{ and } t_1 + \cdots + t_n \leq k.$$

This can be achieved by multiplying f_k by a sufficiently small number. The series $\sum_{k=0}^\infty f_k$ then converges uniformly in all (partial) derivatives, thus defines a smooth function $f \geq 0$ with $f(x) = 0 \Leftrightarrow f_k(x) = 0$ for all $k \Leftrightarrow x \notin U_{f_k}$ for all $k \in \mathbb{N} \Leftrightarrow x \in A$. □

9. Topological properties of manifolds

9.1 Lemma (Topology on Hausdorff manifolds).

Let M be a Hausdorff C^∞ manifold, then:

1. *M is locally compact (i.e. $\forall\ x \in M\ \exists\ U_x$ with compact closure $\overline{U_x}$; in other words there are relatively compact neighborhoods).*
2. *The C^∞ functions $M \to \mathbb{R}$ separate points. They even separate points from closed sets (that is, for $x \notin A$, where A is closed, a smooth $f : M \to \mathbb{R}$ exists, such that $f(x) = 1$ and $f(y) = 0$ for all $y \in A$). In particular, M is completely regular.*

Proof. 1 Let a $x \in M$ and φ a chart at x with open $\operatorname{dom} \varphi \subseteq \mathbb{R}^m$. Without restriction of generality $\varphi(0) = x$ holds. Let $B_x \subseteq \mathbb{R}^m$ be a ball around 0, with $\overline{B_x} \subseteq \operatorname{dom} \varphi$ compact. Then $\varphi(B_x)$ is an open neighborhood of x, and $\varphi(\overline{B_x})$ is compact in M as φ is continuous and hence closed since M is Hausdorff. So $\overline{\varphi(B_x)} = \varphi(\overline{B_x})$ and hence is compact.

2 When $x \notin A$ and A is closed, then there exists a relatively compact neighborhood W_x of x whose (compact) closure is completely contained in a chart φ centered at x.

Thus $\varphi^{-1}(\overline{W_x})$ is compact and $\varphi^{-1}(W_x)$ is an open neighborhood of $0 = \varphi^{-1}(x) \notin \varphi^{-1}(A)$. Hence there is an $r > 0$, such that

$$B_r := \{y \in \mathbb{R}^m : |y| \le r\} \subseteq \varphi^{-1}(W_x) \text{ and } B_r \cap \varphi^{-1}(A) = \emptyset, \text{ hence } \varphi(B_r) \subseteq M \setminus A.$$

From theorem 8.5, we know that there is a smooth mapping $f : \mathbb{R}^m \to \mathbb{R}$, with

$$f(0) = 1 \text{ and } \operatorname{supp} f \subseteq B_r.$$

Let's define $g : M \to \mathbb{R}$ by

$$g(z) := \begin{cases} f(\varphi^{-1}(z)) & \text{for } z \in \operatorname{im}\varphi \\ 0 & \text{for } z \in M \setminus \varphi(B_r). \end{cases}$$

This definition makes sense and gives a smooth mapping, because $f(\varphi^{-1}(z)) = 0$ for all $z \in \operatorname{im}\varphi \setminus \varphi(B_r)$. Moreover, $g(x) = f(0) = 1$ and $g|_A = 0$, because $A \subseteq M \setminus \varphi(B_r)$. This proves the claim. □

9.2 Definition (Paracompactness).

A topological space X is said to be PARACOMPACT if there is a local finite refinement $\mathcal{V}$ for each open covering $\mathcal{U}$ of X, i.e. $\mathcal{V}$ is an open covering of X satisfying:

1. For all $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subseteq U$ ("refinement").
2. For all $x \in X$ there is an U_x, so that $U_x \cap V \neq \emptyset$ at most for finally many $V \in \mathcal{V}$ ("local finiteness").

9.3 Theorem (Paracompact manifolds).

For Hausdorff C^∞ manifolds M the following statements are equivalent:

1. *M has C^∞ partitions of unity.*
2. *M is* METRIZABLE, *i.e. there is a metric that generates the topology.*
3. *M is paracompact, that is, for each open covering there is a local finite refinement that still covers M.*
4. *Each connected component is σ-*COMPACT, *meaning that it is the union of countably many compact subsets.*
5. *Each connected component is* LINDELÖF, *which means that there is a countable subcovering for each open covering.*

Remark (Other topological properties used in the literature).

Not all (continuous) Hausdorff C^∞ manifolds possess the above properties, e.g. the following "long ray" shows: Let Ω be the set of countable ordinals (that is, the smallest uncountable ordinal),

$$\Omega = \Big\{0, 1, 2, \dots, \omega, \omega+1, \dots, 2\omega, \dots, \omega^2, \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots\dots\Big\}.$$

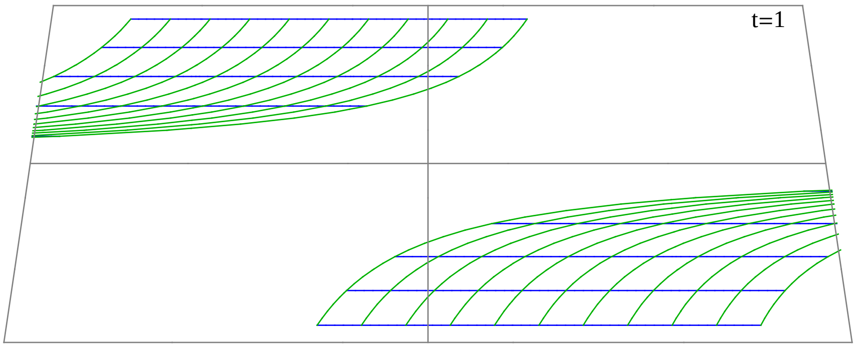
We consider $\Omega \times [0,1) \setminus \{(0,0)\}$, provided with the lexicographic order, that is, $((\alpha,t) \le (\beta,s)) \Leftrightarrow (\alpha < \beta \text{ or } (\alpha = \beta \text{ and } t \le s))$. This "ray" can be made into a C^∞ manifold with the ordering topology, which is indeed Hausdorff, but not paracompact, see [**136**, Vol.I, Appendix A].

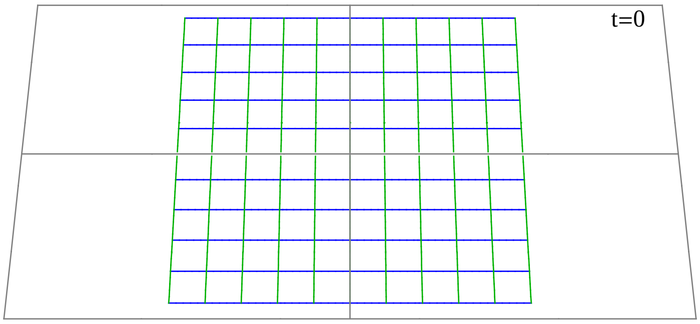
For metric spaces the properties Lindelöf, separabel and the 2nd countability axiom (that is, existence of a countable basis of the topology) are known to be equivalent (see [**79**, 3.3.1]).

Often, for manifolds, only **separability** is assumed (e.g., [5] and [113]), but, by modifying the Prüfer surface, it was shown in [23] that there are non-metrizable separable analytical surfaces:
The modified surface S of Prüfer is the quotient of $\bigsqcup_{\mathbb{R}} \mathbb{R}^2/_\sim$, where

$$(x, y; t) \sim (x', y'; t') :\Leftrightarrow y = y' \text{ and } \begin{cases} x = x' & \text{if } t = t' \\ xy + t = x'y' + t' & \text{andernfalls} \end{cases}$$

This is a separable ($\mathbb{Q}^2$ is dense) Hausdorff analytical surface, which does not satisfy the 2nd countability axiom, because $\bigsqcup_{\mathbb{R}}\{(0,0)\}$ is uncountable and discrete.





On the other hand, sometimes (for example, [147], [74], and [19]), even the second **second countability axiom** is presupposed for manifolds. However, this implies that there are only countably many connected components and thus, e.g. Theorem $\boxed{18.9}$ fails, as the foliation of the torus with irrational slope shows.

Proof of theorem $\boxed{9.3}$.

($\boxed{1}\Rightarrow\boxed{2}$) Using C^∞ partitions of 1, we can glue local Riemann metrics into a global Riemann metric. By [89, 32.3] this provides a topology generating metric d on the connected components of M. Then

$$\tilde{d}(x, y) := \begin{cases} \frac{d(x,y)}{1+d(x,y)} & \text{for } x \text{ and } y \text{ in the same connected component} \\ 1 & \text{for } x \text{ and } y \text{ in different connected components} \end{cases}$$

defines a topology generating metric on all M.

($\boxed{2}\Rightarrow\boxed{3}$) Let $\mathcal{W}$ be an open covering. Using the axiom of choice (see [79, 1.3.9]) we provide $\mathcal{W}$ with a well ordering $\prec$. For $W \in \mathcal{W}$ and $n \in \mathbb{N}$, put $W_n := \bigcup_{x \in M_{W,n}} U_{1/2^n}(x)$ where $U_r(x)$ is the open ball around x with radius r and

$$M_{W,n} := \left\{ x \in X : \begin{matrix} (i) & \forall\, V \prec W : x \notin V \\ (ii) & \forall\, j < n \,\forall\, V \in \mathcal{W} : x \notin V_j \\ (iii) & U_{3/2^n}(x) \subseteq W \end{matrix} \right\}$$

Then $\{W_n : W \in \mathcal{W}, n \in \mathbb{N}\}$ is a local-finite refinement of $\mathcal{W}$:
Refinement: $W_n \subseteq W$, because $U_{1/2^n}(x) \subseteq U_{3/2^n}(x) \subseteq W$ for $x \in M_{W,n}$.
Covering: Let $x \in X$, $V := \min\{W \in \mathcal{W} : x \in W\}$, $\exists\, n \in \mathbb{N} : U_{3/2^n}(x) \subseteq V \Rightarrow$ either (ii) and thus $x \in M_{V,n} \subseteq V_n$ or $x \in W_j$ for a $j < n$ and $W \in \mathcal{W}$.
Local-finite: Let $x \in X$ and $V := \min\{W \in \mathcal{W} : \exists\, n : x \in W_n\}$ then $\exists\, n : x \in V_n$ and thus $\exists\, j : U_{2/2^j}(x) \subseteq V_n$.
Claim: $i \geq n + j \Rightarrow \forall\, W \in \mathcal{W} : U_{1/2^{n+j}}(x) \cap W_i = \emptyset$.
Because of $i > n$, $\forall\, y \in M_{W,i} : y \notin V_n$ is by (ii) and thus $d(y, x) \geq 2/2^j$ because of $U_{2/2^j}(x) \subseteq V_n$. From $n + j \geq j$ and $i \geq j$ we conclude that

$$U_{1/2^{n+j}}(x) \cap U_{1/2^i}(y) \subseteq U_{1/2^j}(x) \cap U_{1/2^j}(y) = \emptyset,\ U_{1/2^{n+j}}(x) \cap W_i = \emptyset.$$

Claim: $i < n + j \Rightarrow U_{1/2^{n+j}}(x) \cap W_i \neq \emptyset$ for at most one $W \in \mathcal{W}$.
Let $p \in W_i$ and $p' \in W'_i$ for $W, W' \in \mathcal{W}$, without restriction of generality $W \prec W'$; i.e. $\exists\, y \in M_{W,i} : p \in U_{1/2^i}(y)$ and thus $U_{3/2^i}(y) \subseteq W$ by (iii) and $\exists\, y' \in M_{W',i} :$ $p' \in U_{1/2^i}(y')$ and thus $y' \notin W$ by (i). $\Rightarrow d(y, y') \geq 3/2^i \Rightarrow d(p, p') \geq d(y', y) - d(p, y) - d(p', y') > 1/2^i \geq 2/2^{n+j}$ in contradiction toh $p, p' \in U_{1/2^{n+j}}(x)$.

As a result, $U_{1/2^{n+j}}(x)$ is met only by a finite number of W_j's.

(3 ⇒ 4) Let M_0 be a connected component of M. There is a covering with relatively compact sets (see Lemma 9.1). This can, since M_0 is paracompact, be assumed to be local-finite. If $\mathcal{U}$ is such an covering, then:

$$\{U \in \mathcal{U} : U \cap W \neq \emptyset\} \text{ is finitefor every } W \in \mathcal{U},$$

because there is an V_x for every $x \in \overline{W}$, so that V_x o meets only finitely $U \in \mathcal{U}$. Since $\overline{W}$ is compact, there is a finite subcovering $\{V_{x_1}, \dots, V_{x_n}\}$ of $\overline{W}$. Let $U \in \mathcal{U}$ with $U \cap W \neq \emptyset$. Thus, there is an i with$U \cap V_{x_i} \neq \emptyset$. For the finitely many i this case only occurs for finitely many $U \in \mathcal{U}$, so

$$\{U \in \mathcal{U} : U \cap W \neq \emptyset\}$$

is finite.

We now choose a $W_1 \in \mathcal{U}$. Let W_2 be the union of those finitely many $U \in \mathcal{U}$, whose intersection with $W_1 \neq \emptyset$.
Now, let inductively W_n be the union of $U \in \mathcal{U}$ whose intersection with $W_{n-1} \neq \emptyset$. Every W_i is the union of finitely many relatively compact sets, thus is relatively compact itself. If $W := \bigcup_n W_n$, then W is open. We want to show that $W = M_0$. For that it suffices to show that $M_0 \setminus W$ is open. So let $x \notin W$, then there is a $U \in \mathcal{U}$ with $x \in U$. Clearly $U \cap W = \emptyset$ holds, otherwise there would be a n with $U \cap W_n \neq \emptyset$, sthus $x \in U \subseteq W_{n+1} \subseteq W$. This is a contradiction.

Hence $M_0 = W \cup (M_0 \setminus W)$, and W and $M_0 \setminus W$ are both open. Since M_0 is open, W or $M_0 \setminus W$ must be empty. But $W \neq \emptyset$, so $M_0 \setminus W = \emptyset$, and so $M_0 = W$. The equation $M_0 = \bigcup_n \overline{W_n}$ shows the σ-compactness of M_0.

(4 ⇒ 5) Let X be a connected component, i.e. $X = \bigcup_{n \in \mathbb{N}} K_n$ with compact K_n. Each open covering $\mathcal{U}$ of X thus has a finite subcovering $\mathcal{U}_n$ of K_n. And so $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is a countable subcovering of X, i.e. X is Lindelöf.

(5 ⇒ 1) In 8.2 a proof for the existence of C^∞ partitions of unity was given using as prerequisite Lindelöf and the existence of C^∞ functions with arbitrarily small carriers only. Also the latter assumption is satisfied here because of 9.1.2. □

This theorem is actually a proposition about locally compact Hausdorff spaces (replacing C^∞ partitions with continuous partitions in 1). However, the proof of (1 ⇒ 2) can not be done as above, but (5 ⇒ 2) follows directly from the Metrizability Theorem [**79**, 3.3.10] of Nagata and Smirnov and (1 ⇒ 3) holds obviously: Namely let $\mathcal{U}$ be an open covering and $\mathcal{F}$ an associated partition of unity. Then for each $x \in M$ there is an neighborhood U_x so that $I := \{f \in \mathcal{F} : \operatorname{supp} f \cap U_x \neq \emptyset\}$ is finite (this corresponds to the 2nd condition for a partition of unity). Thus, $(\{x : f(x) > 0\})_{f\in\mathcal{F}}$ is a local finite refinement of $\mathcal{U}$.

To conclude this excursion into topology, we make some remarks on dimension theory (for more detailed explanations see [**38**]):

9.4 Definition (Covering dimension).

Let X a paracompact Hausdorff space. The COVERING DIMENSION of X is said to be at most n ($\text{cov-dim}\, X \le n$ for short) if there is an open refinement of order $n+1$ for each open covering of X ($\mathcal{U}$ is said to be of ORDER $n+1$ if the intersection of any $n+2$ different sets from $\mathcal{U}$ is always empty). By definition, $\text{cov-dim}\, X = n \Leftrightarrow \text{cov-dim}\, X \le n$ but not $\text{cov-dim} \le n-1$.

9.5 Proposition (Properties of the covering dimension).

The following holds:

1. $\text{cov-dim}\,[0,1]^n = n$.
2. *If A is closed in X, then*
$$\text{cov-dim}\, A \le \text{cov-dim}\, X.$$
3. *For any locally finite closed covering $\mathcal{A}$ of X:*
$$\text{cov-dim}\, X \le \sup\{\text{cov-dim}\, A : A \in \mathcal{A}\}.$$

Without proof, see [**38**, S.295,268,278]

9.6 Corollary.

Each m-dimensional paracompact Hausdorff manifold M has $\text{cov-dim}\, M = m$.

Proof. M has an open covering by sets $\varphi((0,1)^m)$, where φ are charts for M which are defined on neighborhoods of $[0,1]^m$. Since M is paracompact, there is a locally finite refinement $\mathcal{U}$. If $\mathcal{U}^- := \{\overline{V} : V \in \mathcal{U}\}$, then $\mathcal{U}^-$ is a locally finite closed covering. $\varphi^{-1}(\overline{V}) \subseteq [0,1]^m$. Since φ is homeomorphism and thus preserves cov-dim , we get by 9.5:

$$\text{cov-dim}\, \overline{V} = \text{cov-dim}\, \varphi^{-1}(\overline{V}) \overset{(2)}{\le} \text{cov-dim}\,[0,1]^m \overset{(1)}{=\!=} m,$$
$$\text{cov-dim}\, M \overset{(3)}{\le} \sup\{\text{cov-dim}\, \overline{V} : \overline{V} \in \mathcal{V}\},$$

thus $\text{cov-dim}\, M \le m$. Conversely, the following holds: If $\varphi : [0,1]^m \to M$ is a chart, then $\varphi([0,1]^m)$ is closed in M, so by 9.5:

$$\text{cov-dim}\, M \overset{(2)}{\ge} \text{cov-dim}\, \varphi([0,1]^m) = \text{cov-dim}\,[0,1]^m \overset{(1)}{=\!=} m.$$

Together, the claim follows: $\text{cov-dim}\, M = m$. □

9.7 Corollary.

Let M be a paracompact and connected Hausdorff manifold. If $\mathcal{O}$ is an open covering of M, then $p \le \dim(M) + 1$ exists and a refinement of $\mathcal{O}$ of the form:

$$\mathcal{V} = \{V_i^n : i \le p, n \in \mathbb{N}\},$$

such that $V_i^n \cap V_i^m = \emptyset\ \forall\, n \neq m$.

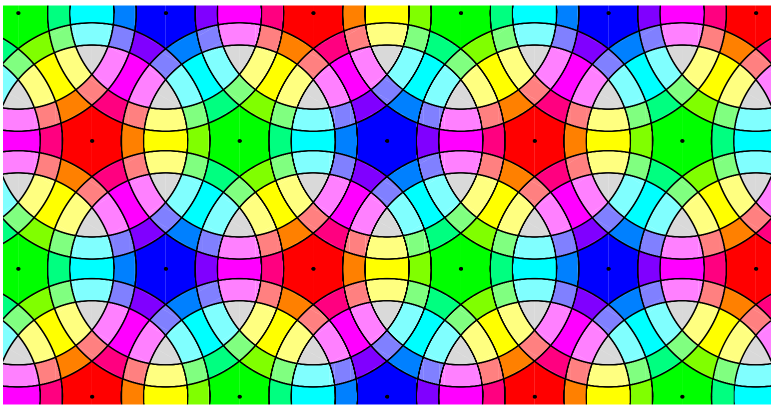
Proof. By 9.6, the covering dimension of M is equal to $\dim M$, so there exists a refinement $\mathcal{O}'$ of order $p \le \text{cov-dim}\, M + 1$ for the open covering $\mathcal{O}$. Since M is paracompact, there is a locally finite refinement $\mathcal{O}''$, and since M is by 9.3 Lindelöf, we can assume that this covering $\mathcal{O}''$ is countable. Thus, without restricting the generality, $\mathcal{O}$ is a countable locally finite covering of order p.

We now show by induction on p that any such covering has a refinement of the desired form.
For this we shrink the sets in $\mathcal{O}$ to obtain a smaller covering $\mathcal{U}$. That is to say, we construct $U \in \mathcal{U}$ with $\bar{U} \subseteq O$ for each $O \in \mathcal{O}$ in such a way that the U still form an covering.
This can be done inductively: Let $\mathcal{O} := \{O_n : n \in \mathbb{N}\}$. Between $M \setminus \bigcup_{n\ge 2} O_n$ and O_1 (the former is closed, the later is open) we squeeze U_1 and $\bar{U}_1$ and get a covering $\{U_1\} \cup \{O_n : n > 1\}$ (This can be done recursively because by 8.5 a C^∞ function f exists with support in $\bar{O}_1$ which is identical 1 on $M \setminus \bigcup_{n\ge 1} O_n$. Now we may put $U_1 := \{x : f(x) > 1/2\}$). In the second step, we find similarly a U_2 between $M \setminus U_1 \cup \bigcup_{n>2} O_n$ and O_2; and so on.

Now let's look at the two families:

$$\begin{aligned}\mathcal{V}_p &:= \text{ the set of intersections of each } p \text{ of the } O \in \mathcal{O},\\ \mathcal{A}_p &:= \text{ the set of intersections of each } p \text{ of the } \bar{U} \text{ for } U \in \mathcal{U}\end{aligned}$$

and denote their union with $V_p := \bigcup \mathcal{V}_p$ and $A_p := \bigcup \mathcal{A}_p$.
In the following image, the large disks are the sets in $\mathcal{O}$, the small disks are those in $\mathcal{U}$, the dark (red/green/blue) "hexagons "are those points which lie in exactly one $O \in \mathcal{O}$, the points in the the next brighter stripes are in exactly 2 of the O's, the larger "triangles" are in exactly 3 of the $O's$ (thus being the elements of $\mathcal{V}_p$) and the white little "triangles" are the elements of $\mathcal{A}_p$.



The family $\mathcal{V}_p$ consists of open disjoint sets, because if we assume that two different members of $\mathcal{V}_p$ have non-empty intersections, then at least $p+1$ of the $O \in \mathcal{O}$ would have a nonempty intersection, and that is a contradiction. As a result, $V_p \subseteq M$ is open.

The family $\mathcal{A}_p$ consists of closed sets and is locally finite, since the corresponding elements of $\mathcal{V}_p$ are disjoint. Thus, A_p is itself closed as a locally finite union of closed sets and $A_p \subseteq V_p$ holds.

We now claim that $\mathcal{U}$ is a countable local finite covering of $M \setminus A_p$ of order less than p.
Assuming there are p sets in $\mathcal{U}$ whose intersection - restricted to $M \setminus A_p$ - is not

empty, then this intersection is also included in the intersection of the unrestricted closures and is therefore contained in A_p by construction. This is a contradiction.

Thus, by induction hypothesis, there exists a refinement of the form $\{V_i^n : i < p,\ n \in \mathbb{N}\}$ of $\mathcal{U}$, which covers $M \setminus A_p$ and thus $V_i^n \cap V_i^m = \emptyset\, \forall\, i < p\, \forall\, n \neq m$.

Together with the disjoint family $\mathcal{V}_p =: \{V_p^n : n \in \mathbb{N}\}$, these sets form the desired refinement of $\mathcal{O}$. □

9.8 Corollary (Finit Atlas).

Each connected, paracompact, smooth Hausdorff manifold of dimension m has an atlas with at most $m+1$ charts.

Proof. Let $\mathcal{O}$ be an open covering of such a manifold M by images $\varphi(U)$ of charts $\varphi : U \to M$ with open U in $\mathbb{R}^m$. Without restricting generality, $\mathcal{O}$ is countable (since M is Lindelöf), i.e.

$$\mathcal{O} = \{\varphi_i(U_i) : i \in \mathbb{N}\},$$

where we can assume the U_i to be disjoint. By 9.7 there is $p \leq \dim M + 1$ and a refinement of the form:

$$\{O_i^n : i \leq p,\ n \in \mathbb{N}\}$$

with $O_i^n \cap O_i^m = \emptyset$ for all $n \neq m$. To O_i^n there is a diffeomorphic $U_i^n \subseteq \mathbb{R}^m$, by means of some chart φ_i^n. We define now

$$\varphi_i : \begin{cases} \bigcup_n U_i^n \to \bigcup_n O_i^n \\ x \mapsto \varphi_i^n(x) \in O_i^n \text{ for } x \in U_i^n. \end{cases}$$

Thus, the φ_i are diffeomorphisms whose images cover M, i.e. $\{\varphi_i : 1 \leq i \leq p\}$ is a C^∞ atlas. □

As a simple conclusion we will show in 11.11 that every such manifold can be realized as a submanifold of a $\mathbb{R}^n$ up to a diffeomorphism.

III. Tangent space

In this chapter, we will transfer the notion of derivative to mappings between manifolds. Since derivatives carry directional vectors to one another, they do not act between the manifolds but between their tangent spaces, which we will introduce now. As an application we will then discuss some simple infinitesimal properties such as immersivity and submersivity of mappings. Under additional local and/or global properties we get embeddings, fiber bundles and as a special case covering maps.

10. Tangent space and derivatives

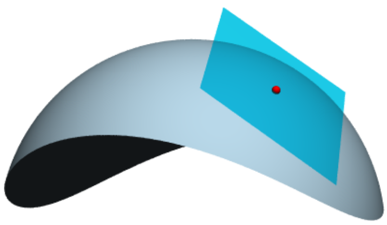
The derivative $f'(x)$ of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at x is defined as the linear approximation to the f function shifted to 0. This can not be readily transferred to manifolds, because in order to speak about linear mappings $f'(x)$ they must be between vector spaces (and not like f between manifolds). Thus, first of all, we need a linear approximation to a manifold M at point $x \in M$. This should then become the domain or range of the linear approximation of f at x.

10.1 Proposition (Description of the tangent space).

Let M be a submanifold of $\mathbb{R}^n$ and $p \in M$. Then the following subsets of $\mathbb{R}^n$ are identical:

1. $\operatorname{im} \varphi'(0)$*, where φ is a local parameterization of M with $\varphi(0) = p$.*
2. $\{c'(0) : c : I \to M$ *smooth,* $c(0) = p$*, I an open interval with* $0 \in I\}$.
3. $\ker f'(p)$*, where f is a regular equation describing M locally at p.*
4. $\operatorname{graph} g'(\bar{p})$ *where M is locally described at p as a graph of function g with* $p = (\bar{p}, g(\bar{p}))$.

In particular, this subset is a linear subspace due to $\boxed{1}$ or $\boxed{3}$ and because of $\boxed{2}$ it is independent on the choosen parameterization, the equation, and the mapping describing it as graph.



Proof. ($\boxed{1} \subseteq \boxed{2}$) Let $\varphi'(0)(v) \in \operatorname{im}(\varphi'(0))$ with $v \in \mathbb{R}^n$. If we define a smooth locally in M lying curve c by

$$c(t) := \varphi(0 + tv)$$

then we get $c'(0) = \varphi'(0)(v)$.

($\boxed{2} \subseteq \boxed{3}$) We now consider $c'(0)$ for a curve $c \in C^\infty(I, M)$ with $c(0) = p$ and a locally regular equation f for M,

$$f'(\underbrace{p}_{c(0)})(c'(0)) = (\underbrace{f \circ c}_{0})'(0) = 0.$$

i.e. $c'(0) \in \ker f'(p)$

($\boxed{3} \subseteq \boxed{1}$) As we have already shown $\boxed{1} \subseteq \boxed{2} \subseteq \boxed{3}$, it suffices to show that the subspaces in $\boxed{1}$ and $\boxed{3}$ have the same dimension:

$$\begin{aligned} \dim \operatorname{im} \varphi'(0) &= \dim \mathbb{R}^m = m \\ \dim \ker f'(p) &= n - \dim \underbrace{\operatorname{im} f'(p)}_{\mathbb{R}^{n-m}} = m. \end{aligned}$$

($\boxed{1} = \boxed{4}$) A parameterization of M is given by $\varphi(u) := (u, g(u))$.

$$\operatorname{im} \varphi'(\bar{p}) = \operatorname{im}(\mathrm{id}, g'(\bar{p})) = \{(v, g'(\bar{p})(v)) : v \in \mathbb{R}^m\} = \operatorname{graph} g'(\bar{p}). \qquad \square$$

10.2 Definition (Tangent space and tangent mapping).

The subspace of $\mathbb{R}^n$ described in $\boxed{10.1}$ is called the TANGENT SPACE for M at the point p and is denoted by T_pM. Its elements are called TANGENT VECTORS.

For smooth $f : M \to N$ the TANGENT MAPPING of f at the point $p \in M$ is defined by

$$T_pf : \begin{cases} T_pM \to T_{f(p)}N \\ c'(0) \mapsto (f \circ c)'(0) \text{ für } c \in C^\infty(I, M) \text{ mit } c(0) = p \end{cases}$$

This definition makes sense, i.e. does not depend on the choice of c, but only on $c'(0)$: Let c_1 and c_2 be two such curves with $c_1'(0) = c_2'(0)$. For $f : \mathbb{R}^m \supseteq M \to N \subseteq \mathbb{R}^n$ there is an open neighboorhood $U(p) \subseteq \mathbb{R}^m$ of p and a smooth mappping

$$\tilde{f} : \mathbb{R}^m \supseteq U(p) \to \mathbb{R}^n \text{ with } \tilde{f}|_M = f|_{U(p)},$$

hence

$$(f \circ c_1)'(0) = (\tilde{f} \circ c_1)'(0) = \tilde{f}'(p)(c_1'(0)) = \tilde{f}'(p)(c_2'(0)) \overset{\text{analog}}{=\!=\!=} (f \circ c_2)'(0).$$

The tangent mapping T_pf is linear, because $(T_pf)(c'(0)) = (f \circ c)'(0) = \tilde{f}'(p)(c'(0))$, hence

$$T_pf = \tilde{f}'(p)|_{T_pM}, \text{ where } \tilde{f} \text{ is a local extension of } f.$$

10.3 Example (Quadrics).

Let $f : E \to \mathbb{R}$ be a quadratic (i.e. $f(tx) = t^2 f(x)$ smooth form and $c \neq 0$. Then the quadric $M := f^{-1}(c) = \{x \in E : f(x) = c\}$ is a submanifold of E, because differentiating the homogeneity equation yields $f'(tx)(tv) = t^2 f'(x)(v)$, i.e. $f'(tx)(v) = t f'(x)(v)$, and furthermore $f''(tx)(tw, v) = t f''(x)(w, v)$, i.e. $f''(x) = f''(tx) = f''(0)$ for $t \to 0$. Thus, according to Taylor's theorem, $f(x) = b(x, x)$, where $b := \frac{1}{2} f''(0)$ is a symmetric quadratic form.

The derivative of f is $f'(x)(v) = b(x, v) + b(v, x) = 2b(x, v)$ and thus surjective with respect to v for each $x \in M$, because $2b(x, x) = f(x) = c \neq 0$. The tangent space of M at x is

$$T_xM = \{v \in E : b(x, v) = 0\} =: x^\perp.$$

A first example of a quadric is the sphere $S^n = f^{-1}(1)$, where $f(x) := |x|^2$.

The special linear group $SL(E) := \{T \in L(E) : \det(T) = 1\}$ (see 4.2) has as a tangent space at $\mathrm{id} \in SL(E)$ the subspace $\{T \in L(E) : 0 = \det'(\mathrm{id})(T) = \mathrm{spur}(T)\}$ of the trace-free linear mappings.

The orthogonal group $O(E) := \{T \in L(E) : T^t T = \mathrm{id}\}$ (see 4.3) has as tangent space at id the subspace $\{T \in L(E) : 0 = f'(\mathrm{id})(T) = T^t + T\}$ of the skew-symmetric (that is, anti-self-adjoint) linear mappings, where f is the quadratic mapping $f : T \mapsto T^t T$.

More generally, for a bilinear non-degenerate form $b : E \times E \to \mathbb{R}$, the tangent space at id of the group $O_b(E) := \{T \in L(E) : b(Tx, Ty) = b(x, y)\, \forall\, x, y \in E\} = \{T \in L(E) : T^t B T = B\}$ (with $b(x, y) = \langle Bx, y\rangle$) treated in 4.4 is the subspace $\{T \in L(E) : T^t B + BT = 0\}$ of those linear mappings which are skew-symmetric with respect to B.

For the groups G treated in 4.4-4.8, we obtain the following descriptions of the tangent space at $\mathrm{id} \in G$ (using $\det'(A)(B) = \det(A)\,\mathrm{trace}(A^{-1}B)$ from 4.2) in a corresponding manner, from which we can easily read off the dimension of G.

G	$T_{\mathrm{id}}G$	$\dim_{\mathbb{R}}$
$GL(n)$	$L(n)$	n^2
$GL_{\mathbb{C}}(n)$	$L_{\mathbb{C}}(n)$	$2n^2$
$GL_{\mathbb{H}}(n)$	$L_{\mathbb{H}}(n)$	$4n^2$
$SL(n)$	$\{T \in L(n) : \mathrm{trace}_{\mathbb{R}}(T) = 0\}$	$n^2 - 1$
$SL_{\mathbb{C}}(n)$	$\{T \in L_{\mathbb{C}}(n) : \mathrm{trace}_{\mathbb{C}}(T) = 0\}$	$2(n^2 - 1)$
$SL_{\mathbb{H}}(n)$	$\{T \in L_{\mathbb{H}}(n) : \mathrm{trace}_{\mathbb{R}}(T) = 0\}$	$4n^2 - 1$
$O(n), SO(n)$	$\{T \in L(n) : T^t + T = 0\}$	$n(n-1)/2$
$O(n,k), SO(n,k)$	$\{T \in L(n) : T^t I_k + I_k T = 0\}$	$n(n-1)/2$
$O_{\mathbb{C}}(n), SO_{\mathbb{C}}(n)$	$\{T \in L_{\mathbb{C}}(n) : T^t + T = 0\}$	$n(n - 1)$
$U(n)$	$\{T \in L_{\mathbb{C}}(n) : T^* + T = 0\}$	n^2
$U(n,k)$	$\{T \in L_{\mathbb{C}}(n) : T^* I_k + I_k T = 0\}$	n^2
$SU(n)$	$\{T \in L_{\mathbb{C}}(n) : T^* + T = 0, \mathrm{trace}_{\mathbb{C}}(T) = 0\}$	$n^2 - 1$
$SU(n,k)$	$\{T \in L_{\mathbb{C}}(n) : T^* I_k + I_k T = 0, \mathrm{trace}_{\mathbb{C}}(T) = 0\}$	$n^2 - 1$
$Q(n)$	$\{T \in L_{\mathbb{H}}(n) : T^* + T = 0\}$	$n(2n + 1)$
$Q(n,k)$	$\{T \in L_{\mathbb{H}}(n) : T^* I_k + I_k T = 0\}$	$n(2n + 1)$
$Q_-(n)$	$\{T \in L_{\mathbb{H}}(n) : T^* i + iT = 0\}$	$n(2n - 1)$
$Sp(2n)$	$\{T \in L(2n) : T^t J + JT = 0\}$	$n(2n + 1)$
$Sp_{\mathbb{C}}(2n)$	$\{T \in L_{\mathbb{C}}(2n) : T^t J + JT = 0\}$	$2n(2n+1)$

In detail, this means e.g. for $O(n, k)$, that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in T_{\mathrm{id}} O(n,k) \subseteq L(\mathbb{R}^k \times \mathbb{R}^{n-k}) \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0$$

$$\Leftrightarrow A^t + A = 0, \quad B^t = C, \quad D^t + D = 0$$

and for the $Sp(2n)$, that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in T_{\mathrm{id}} Sp(2n) \subseteq L(\mathbb{R}^n \times \mathbb{R}^n) \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0$$

$$\Leftrightarrow C^t = C, \quad -A^t = D, \quad B^t = B.$$

10.4 Lemma.

Chain Rule: *For manifolds M, N, P and smooth mappings f, g with $M \xrightarrow{f} N \xrightarrow{g} P$, we have*

$$T_p(g \circ f) = T_{f(p)}g \circ T_p f.$$

For the identity $\mathrm{id} : M \to M$ *we have*

$$T_p(\mathrm{id}_M) = \mathrm{id}_{T_pM} : T_pM \to T_pM.$$

product rule: *If $f, g : M \to \mathbb{R}$ are smooth, then*

$$T_p(f \cdot g) = f(p) \cdot T_pg + g(p) \cdot T_pf.$$

Theorem on inverse functions: *A smooth mapping f is a local diffeomorphism around p if and only if T_pf is an isomorphism.*

Proof. By extending all occurring functions smoothly to neighborhoods in the surrounding vector spaces, the chain and product rules follow from the classical versions, see [**81**, 6.1.9] and [**81**, 6.1.13]. Furthermore, the theorem about inverse functions is only local in nature and therefore also a consequence of the classical theorem 2.2. □

Unfortunately, we can not directly use the descriptions of the tangent space given in 10.1 for abstract manifolds, since we have used the surrounding vector space in an essential way. So we need other (more abstract) descriptions. For this we pay attention to the action of $v \in T_pM$ on $f \in C^\infty(M, \mathbb{R})$ through $f \mapsto T_pf \cdot v$ and give the following

10.5 Definition (Derivation).

A mapping $\partial : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ is called DERIVATION over $p \in M$ if it is linear and fulfills the product rule, that is for $f, g \in C^\infty(M, \mathbb{R})$ and $\alpha \in \mathbb{R}$ we assume:

1. $\partial(f + g) = \partial f + \partial g$
2. $\partial(\alpha f) = \alpha \cdot \partial f$
3. $\partial(f \cdot g) = \partial f \cdot g(p) + f(p) \cdot \partial g$

With $\mathrm{Der}_p(C^\infty(M, \mathbb{R}), \mathbb{R})$ we denote the set of all derivations over $p \in M$. With respect to the pointwise operations, this is a vector space.

10.6 Theorem (Tangent vectors as derivatives).

The mapping

$$\begin{aligned} T_pM \times C^\infty(M, \mathbb{R}) &\to \mathbb{R} \\ (v, f) &\mapsto (T_pf)(v) \end{aligned}$$

induces a linear isomorphism

$$\Phi_p : \begin{cases} T_pM \to \mathrm{Der}_p(C^\infty(M, \mathbb{R}), \mathbb{R}) \\ \quad v \mapsto \partial_v\Big(: f \mapsto (T_pf)(v)\Big) \end{cases}$$

For each smooth $f : M \to N$ the tangent mapping T_pf of f corresponds via Φ_p to the following mapping for derivations:

$$\begin{tikzcd}
T_pM \arrow[r, "\Phi_p"] \arrow[d, "T_pf"'] & \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R}) \ni \arrow[d, "(f^*)^*"] & \partial \arrow[d, mapsto] \\
T_{f(p)}N \arrow[r, "\Phi_{f(p)}"] & \mathrm{Der}_{f(p)}(C^\infty(N,\mathbb{R}),\mathbb{R}) \ni & (g \mapsto \partial(g \circ f)) = \partial \circ f^*
\end{tikzcd}$$

Proof. Well-definedness: The mapping $T_pM \times C^\infty(M,\mathbb{R}) \to \mathbb{R}$, $(v,f) \mapsto (T_pf)(v)$ is clearly bilinear, so it induces a linear map $T_pM \to L(C^\infty(M,\mathbb{R}),\mathbb{R})$ by $v \mapsto (f \mapsto (T_pf)(v))$. This mapping has values in the space of derivations over p, because let $f, g : M \to \mathbb{R}$ be two smooth functions and let $v \in T_pM$ then by the product rule 10.4 the following holds::

$$\partial_v(f \cdot g) = T_p(f \cdot g)(v) = f(p) \cdot (T_pg)(v) + g(p) \cdot (T_pf)(v).$$

Commutativity of the diagram: Let $f : M \to N$ be smooth and $p \in M$. Then the above diagram commutes since for $v \in T_pM$ and $g \in C^\infty(N,\mathbb{R})$ is $(\Phi_{f(p)} \circ T_pf)(v)(g) = (T_{f(p)}g)((T_pf)(v)) = (T_p(g \circ f))(v) = \Phi_p(v)(g \circ f)$, because $\partial := \Phi_p(v)$ acts on $h \in C^\infty(M,\mathbb{R})$ by $\partial(h) = (T_ph)(v)$.

Locality of derivations: Each derivation ∂ of $C^\infty(M,\mathbb{R})$ over $p \in M$ is a local operator, that is, the value $\partial(f)$ depends only on $f \in C^\infty(M,\mathbb{R})$ near p:
So let $f_1, f_2 \in C^\infty(M,\mathbb{R})$ with $f_1 = f_2$ near p. Let $f := f_1 - f_2$ and $g \in C^\infty(M,\mathbb{R})$ be choosen such that $g(p) = 1$ and that the carrier of g is included in the set of x with $f(x) = 0$. Then:

$$0 = \partial(0) = \partial(g \cdot f) = \underbrace{g(p)}_{1} \cdot \partial(f) + \underbrace{f(p)}_{0} \cdot \partial(g) = \partial(f).$$

In particular, $\partial(f) = 0$ for all constant functions f, because $\partial(1) = \partial(1 \cdot 1) = 1 \cdot \partial(1) + \partial(1) \cdot 1$, thus $\partial(1) = 0$.

Bijectivity for open submanifolds: First we want to prove the bijectivity of Φ for the special case $0 = p \in M$ with $M \subseteq \mathbb{R}^m$ open. If $(e_i)_{i=1}^m$ is the standard basis in $\mathbb{R}^m$, then each vector $v \in T_pM = \mathbb{R}^m$ can be developed in the basis as $v = \sum_i v^i e_i$. Let us consider

$$\Phi : T_pM \ni v \mapsto \partial_v \in \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R})$$
$$\text{with } \partial_v(f) := (T_pf)(v) = f'(p)(v) = \sum_{i=1}^m (\partial_i f)(p) \cdot v^i,$$

where $\partial_i f$ is the i-th partial derivative of f, i.e.

$$(\partial_i f)(p) = \tfrac{\partial}{\partial t}\big|_{t=0} f(p + te^i) = f'(p)(e^i).$$

The derivation ∂_v is nothing else than "taking the directional derivative d_v in direction v at p" and Φ is injective, because the components of v can be reconstructed uniquely from ∂_v by

$$\partial_v(\mathrm{pr}_j) = \sum_{i=1}^m \underbrace{(\partial_i \,\mathrm{pr}_j)(p)}_{\delta_{i,j}} \cdot v^i = v^j.$$

Moreover, Φ is also surjective, because for $\partial \in \mathrm{Der}_0(C^\infty(U,\mathbb{R}),\mathbb{R})$, $f \in C^\infty(U,\mathbb{R})$ and x near 0, the following holds:

$$f(x)-f(0)=\int_0^1 f'(tx)(x)dt=\int_0^1 \sum_i (\partial_i f)(tx)x^i dt=\sum_{i=1}^m x^i \underbrace{\int_0^1 (\partial_i f)(tx)dt}_{=:h_i(x)}$$

and furthermore, because ∂ is a local operator,

$$\begin{aligned}\partial(f)&=\partial(f(0))+\partial\left(\sum_{i=1}^m \mathrm{pr}^i \cdot h_i\right)=0+\sum_{i=1}^m\Big(\underbrace{\partial(\mathrm{pr}^i)}_{=:v^i}\ \underbrace{h_i(0)}_{(\partial_i f)(0)}+\underbrace{\mathrm{pr}^i(0)}_{=0}\cdot\partial(h_i)\Big)\\&=\sum_{i=1}^m v^i\cdot(\partial_i f)(0).\end{aligned}$$

So $\partial(f)=\partial_v(f)=\Phi(v)(f)$ for all f.

Bijectivity in general: Now let M be a submanifold of $\mathbb{R}^n$ and φ a local parameterization of M centered at some point $p \in M$. The following diagram shows that Φ_p is an isomorphism:

$$\begin{tikzcd}
\mathbb{R}^m = T_0 U \arrow[rr, "T_0\varphi", "\cong"'] \arrow[d, "\Phi_0"', "\cong"] & & T_p M \hookrightarrow \mathbb{R}^n \arrow[d, "\Phi_p"'] \\
\mathrm{Der}_0(C^\infty(U,\mathbb{R}),\mathbb{R}) \arrow[r, "(\varphi^*)^*", "\cong"'] & \mathrm{Der}_p(C^\infty(\varphi(U),\mathbb{R}),\mathbb{R}) \arrow[r, "(\mathrm{incl}^*)^*", "\cong"'] & \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R})
\end{tikzcd}$$

Where $T_0\varphi$ is an isomorphism by 5.3 and 10.4; Φ_0 is one because of the previous case; $(\varphi^*)^*:\partial\mapsto(f\mapsto\partial(f\circ\varphi))$ is one because $\varphi:U\to\varphi(U)$ is a diffeomorphism; and finally $(\mathrm{incl}^*)^*$ is one, since derivations are local operators; So also Φ_p is an isomorphism, and thus the theorem proved. □

We can now use the theorem 10.6 to define the tangent space of abstract manifolds as follows:

10.7 Definition (Tangent space of abstract manifolds).

The TANGENT SPACE at p of an abstract manifold M is the vector space

$$T_pM := \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R}).$$

Note that for submanifolds $M \subseteq \mathbb{R}^n$, and in particular for open subsets, we have replaced the tangent space $T_pM \subseteq \mathbb{R}^n$ defined in 10.2 with a nonidentical but canonically isomorphic vector space $T_pM \subseteq L(C^\infty(M,\mathbb{R}),\mathbb{R})$.

For $f \in C^\infty(M,N)$ and $p\in M$, the linear map $T_pf=(f^*)^*:T_pM\to T_{f(p)}N$ defined by

$$\partial\mapsto\Big((T_pf)(\partial):g\mapsto\partial(g\circ f)\Big)\text{ for }\partial\in T_pM\text{ and }g\in C^\infty(N,\mathbb{R})$$

is called the TANGENT MAP of f at p.

10.8 Basis for the tangent space.

If, as for mappings between $\mathbb{R}^m$'s, we want to describe the derivative of a mapping between manifolds as a matrix (the Jacobi matrix) we need a basis. But even if we have chosen a basis in the surrounding vector space, we still do not get a distinguished basis of the tangent space (think, for example, of the sphere S^2). But we can proceed as follows. Let M be a manifold, $p \in M$ and φ a chart of M centered at p. Then $T_0\varphi : T_0\mathbb{R}^m \to T_pM$ is a linear isomorphism by 6.11.2

and 10.4 if M is a submanifold of $\mathbb{R}^n$, but also in the general case of an abstract manifold, since 10.4 is quite easy to show for them. The standard basis $(e_i)_{i=1}^m$ of $\mathbb{R}^m$ is mapped by the isomorphism $\Phi : \mathbb{R}^m \cong \mathrm{Der}_0(C^\infty(U,\mathbb{R}),\mathbb{R})$ of 10.6 to the basis of the partial derivatives $(\partial_i|_0)_{i=1}^m$ in T_0U. The isomorphism $T_0\varphi$ further maps this basis to a basis $(\partial_i^\varphi|_p)_{i=1}^m$ in T_pM, which is defined on $f \in C^\infty(M,\mathbb{R})$ by

$$\partial_i^\varphi|_p(f) := (T_0\varphi)(\partial_i|_0)(f) = (\varphi^*)^*(\partial_i|_0)(f) = \partial_i|_0(\varphi^*(f)) = \partial_i(f\circ\varphi)(0),$$

i.e. by taking the partial derivative of the chart representation $f\circ\varphi$ of f in direction e_i at $0 = \varphi^{-1}(p)$. In summary:

$$\begin{array}{ccccc} \mathbb{R}^m & \xrightarrow[\Phi]{\cong} & T_0U & \xrightarrow[T_0\varphi]{\cong} & T_pM = \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R}) \\ e_i & \longmapsto & \partial_i|_0 & \longmapsto & \Big(\partial_i^\varphi|_p : f \mapsto \partial_i(f\circ\varphi)(\varphi^{-1}(p))\Big) \end{array}$$

In the case of a submanifold $M \subseteq \mathbb{R}^n$, $\partial_i^\varphi|_{\varphi(0)} := (T_0\varphi)(\partial_i|_0) \hat{=} (\partial_i\varphi)(0) := \varphi'(0)(e_i)$ (via the embedding $T_pM \hookrightarrow \mathbb{R}^n$) is just the i-th partial derivative of the parameterization $\varphi : U \to M \subseteq \mathbb{R}^n$.

If $(u^1,\dots,u^m) := \varphi^{-1} : M \supseteq \varphi(U) \to U \subseteq \mathbb{R}^m$ are the local coordinates associated to φ, we also write

$$\frac{\partial}{\partial u^i}\Big|_p := \partial_i^\varphi|_p$$

instead of $\partial_i^\varphi|_p \in T_pM = \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R})$. This (uncommon) notation ∂_i^φ expresses better that this derivation depends on the chart φ and not, as one might erroneously deduce from the notation $\frac{\partial}{\partial u^i}$, only on the ith component u^i of the inverse function $\varphi^{-1} = (u^1,\dots,u^m)$ (see 10.10). The name $\frac{\partial}{\partial u^i}$, however, is the more common and does not cause any problems if it is interpreted only as $\left(\frac{\partial}{\partial u}\right)_i$ and not as $\frac{\partial}{\partial (u^i)}$.

If φ is not centered at p then, more generally,

$$\frac{\partial}{\partial u^i}\Big|_p(f) = \partial_i^\varphi|_p(f) := \partial_i(f\circ\varphi)(\varphi^{-1}(p)) \text{ for } f \in C^\infty(M,\mathbb{R})$$

and in particular

$$\frac{\partial}{\partial u^i}\Big|_p(u^j) = \partial_i(u^j\circ\varphi)(\varphi^{-1}(p)) = \partial_i(\mathrm{pr}^j)(\varphi^{-1}(p)) = \delta_i^j,$$

because of the local nature of $\frac{\partial}{\partial u^i}\big|_p$ the previous formula also holds for $f := u^i \in C^\infty(\varphi(U),\mathbb{R})$.

10.9 Transformation behavior of tangent vectors.

For $g \in C^\infty(M,N)$ and $p \in M$, let $\varphi^{-1} = u = (u^1,\dots,u^m)$ be local coordinates of M at p and $\psi^{-1} = v = (v^1,\dots,v^n)$ local coordinates of N at $g(p)$. We know that $T_pg : T_pM \to T_{g(p)}N$ is linear by 10.5 and $(\frac{\partial}{\partial u^i}\big|_p)$ is a basis of T_pM and $(\frac{\partial}{\partial v^i}\big|_{f(p)})$ is one of $T_{f(p)}N$ by 10.8. What is the matrix representation $[T_pg]$ of T_pg with respect to these bases?

Let $\bar{g} := \psi^{-1} \circ g \circ \varphi$ be the chart representation of g. Since, according to the definition of $T_p g$ and because of 10.4 and 10.6, the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\bar{g}"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"'] \\ M \arrow[r, "g"] & N \end{tikzcd} \implies \begin{tikzcd} \mathbb{R}^m \arrow[rr, "(\bar{g})'(0)"] \arrow[d, "\Phi"', "\cong"] & & \mathbb{R}^n \arrow[d, "\Phi"', "\cong"] \\ \mathrm{Der}_0(C^\infty(U,\mathbb{R}),\mathbb{R}) \arrow[r, "T_0\bar{g}"] \arrow[d, "T_0\varphi"', "\cong"] \arrow[ur, phantom, "\boxed{10.6}"] & & \mathrm{Der}_0(C^\infty(V,\mathbb{R}),\mathbb{R}) \arrow[d, "T_0\psi"', "\cong"] \\ \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R}) \arrow[rr, "T_p g"] \arrow[ur, phantom, "\boxed{10.4}"] & & \mathrm{Der}_{g(p)}(C^\infty(N,\mathbb{R}),\mathbb{R}) \end{tikzcd}$$

the corresponding basis is mapped as follows:

$$\begin{tikzcd} e_i \arrow[r, mapsto, "(\bar{g})'(0)"] \arrow[d, mapsto, "\Phi"', "\cong"] & \bar{g}'(0)(e_i) \arrow[r, equal] & \sum_j \partial_i \bar{g}^j(0) \cdot e_j \arrow[d, mapsto, "\Phi"', "\cong"] \\ \partial_i \arrow[r, mapsto, "T_0\bar{g}"] \arrow[d, mapsto, "T_0\varphi"', "\cong"] & (T_0\bar{g})(\partial_i) \arrow[r, equal] & \sum_j \partial_i \bar{g}^j(0) \cdot \partial_j \arrow[d, mapsto, "T_0\psi"', "\cong"] \\ \partial_i^\varphi \arrow[r, mapsto, "T_p g"] & (T_p g)(\partial_i^\varphi) \arrow[r, equal] & \sum_j \partial_i \bar{g}^j(0) \cdot \partial_j^\psi \end{tikzcd}$$

So for the components ξ^i of tangent vectors $\xi = \sum_i \xi^i \cdot \partial_i^\varphi \in T_p M$ we get the following:

$$\begin{aligned} (T_p g)(\xi) &= (T_p g)\Big(\sum_i \xi^i \cdot \partial_i^\varphi\Big) = \sum_i \xi^i \cdot (T_p g)(\partial_i^\varphi) = \sum_i \xi^i \cdot \sum_j \partial_i \bar{g}^j(0) \cdot \partial_j^\psi \\ &= \sum_j \Big(\sum_i \xi^i \cdot \partial_i \bar{g}^j(0)\Big) \cdot \partial_j^\psi \end{aligned}$$

The components η^j of $\eta = \sum_j \eta^j \cdot \partial_j^\psi := (T_p g)(\xi) \in T_{g(p)} N$ are therefore given by

$$\eta^j = \sum_i \xi^i \cdot \partial_i \bar{g}^j(0),$$

or in matrix notation

$$\begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} = \begin{pmatrix} \partial_1 \bar{g}^1(0) & \dots & \partial_m \bar{g}^1(0) \\ \vdots & \ddots & \vdots \\ \partial_1 \bar{g}^n(0) & \dots & \partial_m \bar{g}^n(0) \end{pmatrix} \cdot \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^m \end{pmatrix},$$

i.e. is just multiplication with the Jacobi matrix of the coordinate representation $\bar{g} = \psi^{-1} \circ g \circ \varphi$ of g.

In particular, if we choose $g = \mathrm{id}_M$ and two charts φ and ψ centered at $p \in M$, then $\bar{g}$ is the chart change $\psi^{-1} \circ \varphi$ from coordinates $(u^1, \dots, u^m) := \varphi^{-1}$ to coordinates $(v^1, \dots, v^m) = \psi^{-1}$. If we consider the above formula $\partial_i^\varphi = (T_p\,\mathrm{id})(\partial_i^\varphi) = (T_p g)(\partial_i^\varphi) = \sum_j \partial_i \bar{g}^j(0) \cdot \partial_j^\psi$ to be formally a multiplication of matrices, then

$$\begin{pmatrix} \partial_1^\varphi \\ \vdots \\ \partial_m^\varphi \end{pmatrix} = \begin{pmatrix} \partial_1 \bar{g}^1(0) & \dots & \partial_1 \bar{g}^m(0) \\ \vdots & \ddots & \vdots \\ \partial_m \bar{g}^1(0) & \dots & \partial_m \bar{g}^m(0) \end{pmatrix} \cdot \begin{pmatrix} \partial_1^\psi \\ \vdots \\ \partial_m^\psi \end{pmatrix}$$

Thus we get the basis (∂_i^φ) formally from the basis (∂_j^ψ) by multiplying it with the transposed Jacobian matrix of the inverse chart change $\psi^{-1} \circ \varphi = (\varphi^{-1} \circ \psi)^{-1}$ from φ to ψ.

If we now use the notation $\frac{\partial}{\partial u^i} := \partial_i^\varphi$, $\frac{\partial}{\partial v^j} := \partial_j^\psi$ and $\frac{\partial f}{\partial u^i} := \frac{\partial}{\partial u^i} f$ and note that $\partial_i^\varphi(f)(p) = (\partial_i^\varphi)|_p(f) = \partial_i(f \circ \varphi)(\varphi^{-1}(p))$ and thus

$$\begin{aligned}\partial_i \bar{g}^j(0) &= \partial_i((\psi^{-1} \circ g \circ \varphi)^j)(\varphi^{-1}(p)) = \partial_i(v^j \circ g \circ \varphi)(\varphi^{-1}(p)) \\ &= \partial_i^\varphi(v^j \circ g)(p) = \tfrac{\partial}{\partial u^i}(v^j \circ g)(p),\end{aligned}$$

then the above formula for $g = \mathrm{id}$ states that

$$\frac{\partial}{\partial u^i} = \partial_i^\varphi = (T_p \,\mathrm{id})\,(\partial_i^\varphi) = \sum_j \partial_i \bar{g}^j(0) \cdot \partial_j^\psi = \sum_j \frac{\partial v^j}{\partial u^i} \cdot \frac{\partial}{\partial v^j}$$

(note the memo-technical advantage of this notation) or in (formal) matrix notation:

$$\begin{pmatrix} \frac{\partial}{\partial v^1} \\ \vdots \\ \frac{\partial}{\partial v^m} \end{pmatrix} = \begin{pmatrix} \frac{\partial u^1}{\partial v^1} & \dots & \frac{\partial u^m}{\partial v^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^1}{\partial v^m} & \dots & \frac{\partial u^m}{\partial v^m} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u^1} \\ \vdots \\ \frac{\partial}{\partial u^m} \end{pmatrix}$$

and

$$\begin{pmatrix} \eta^1 \\ \vdots \\ \eta^m \end{pmatrix} = \begin{pmatrix} \frac{\partial v^1}{\partial u^1} & \dots & \frac{\partial v^1}{\partial u^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial v^m}{\partial u^1} & \dots & \frac{\partial v^m}{\partial u^m} \end{pmatrix} \cdot \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^m \end{pmatrix}.$$

10.10 Example.

Be $M = \mathbb{R}^3$. We choose 3 different coordinate systems:

1. Cartesian coordinates (x, y, z) with associated basis vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.
2. Cylinder coordinates (r, φ, z) with associated basis vectors $\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}$.
3. Spherical coordinates (R, φ, ϑ) with associated basis vectors $\frac{\partial}{\partial R}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \vartheta}$.

For the Jacobi matrices of the chart change (2) $\to$ (1) we get: $x = r \cdot \cos\varphi, y = r \cdot \sin\varphi, z = z$

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r \cdot \sin\varphi & 0 \\ \sin\varphi & r \cdot \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

For (3) $\to$ (2) we have: $r = R \cdot \cos\vartheta$, $z = R \cdot \sin\vartheta$

$$\begin{pmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \varphi} & \frac{\partial r}{\partial \vartheta} \\ \frac{\partial \varphi}{\partial R} & \frac{\partial \varphi}{\partial \varphi} & \frac{\partial \varphi}{\partial \vartheta} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & 0 & -R \cdot \sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & R \cdot \cos\vartheta \end{pmatrix};$$

And finally for (1) $\to$ (3):
$R = \sqrt{x^2 + y^2 + z^2}$, $\varphi = \arctan(y/x)$, $\vartheta = \arctan(z/\sqrt{x^2 + y^2})$. The calculation of the Jacobi matrix of this chart change is left to the reader as an exercise.

If we use new coordinates $\bar{x} := x$, $\bar{y} := x + y$ on $\mathbb{R}^2$ instead of the Cartesian coordinates x, y, then the respective first coordinates coincide, but not the corresponding derivations

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial \bar{x}}{\partial x}}_{=1} \cdot \frac{\partial}{\partial \bar{x}} + \underbrace{\frac{\partial \bar{y}}{\partial x}}_{=1} \cdot \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \neq \frac{\partial}{\partial \bar{x}}.$$

So $\frac{\partial}{\partial u^i}$ depends not only on u^i but on $u = (u^1, \dots, u^m)$!

There is also the possibility of describing the tangent space of an abstract manifold more geometrically.

10.11 Lemma (Tangent vectors via curves).

Let $C^\infty_x(\mathbb{R},M) := \{c \in C^\infty(\mathbb{R},M) : c(0)=x\}$ be the set of smooth curves through $x \in M$. For such a smooth curve c and a smooth function $f : M \to \mathbb{R}$ let $\partial_c(f) := (f \circ c)'(0)$. Then $c \mapsto \partial_c$ defines a surjective mapping

$$\partial : C^\infty_x(\mathbb{R},M) \twoheadrightarrow \mathrm{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R}).$$

So we can identify T_xM with $C^\infty_x(\mathbb{R},M)/\sim$, where $\sim$ is the following equivalence relation on $C^\infty_x(\mathbb{R},M)$:

$$c_1 \sim c_2 \quad :\Leftrightarrow \quad \forall\, f \in C^\infty(M,\mathbb{R}) : (f\circ c_1)'(0) = (f \circ c_2)'(0).$$

The tangent mapping of a smooth function $g : M \to N$ looks like this in this description:

$$(T_xg)(\partial_c) = \partial_{g\circ c}.$$

This corresponds to the description of T_xM for submanifolds of $\mathbb{R}^n$ in 10.1.2. However, it has the disadvantage of being unable to recognize the vector space structure of T_xM and the linearity of T_xg.

Proof. The following calculation shows that ∂_c is a derivation via x:

$$\begin{aligned}
\partial_c(f+g) &= \Big((f+g)\circ c\Big)'(0) = \Big((f\circ c)+(g\circ c)\Big)'(0)\\
&= (f\circ c)'(0) + (g\circ c)'(0) = \partial_c f + \partial_c g\\
\partial_c(\lambda f) &= \Big((\lambda f)\circ c\Big)'(0) = \Big(\lambda(f\circ c)\Big)'(0)\\
&= \lambda (f\circ c)'(0) = \lambda\cdot \partial_c f\\
\partial_c(f\cdot g) &= \Big((f\cdot g)\circ c\Big)'(0) = \Big((f\circ c)\cdot(g\circ c)\Big)'(0)\\
&= (f\circ c)'(0)\cdot(g\circ c)(0) + (f\circ c)(0)\cdot(g\circ c)'(0)\\
&= (\partial_c f)\cdot g(x) + (\partial_c g)\cdot f(x)
\end{aligned}$$

In order to show that assignment $c \mapsto \partial_c$ is surjective, we choose local coordinates $\varphi^{-1} = (u^1,\dots,u^m)$ centered at $x \in M$. Each element of $T_xM = \mathrm{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R})$ then has the form $\sum_{i=1}^m \xi^i \frac{\partial}{\partial u^i}|_x$. We now define a (local) curve $c : \mathbb{R} \to M$ by $c(t) := \varphi(t\xi^1,\dots,t\xi^m)$, i.e. $u^i(c(t)) := t\xi^i$ for $i = 1,\dots,m$, then for $f \in C^\infty(M,\mathbb{R})$:

$$\begin{aligned}
(f\circ c)'(0) &= \Big((f\circ\varphi)\circ(\varphi^{-1}\circ c)\Big)'(0) = (f\circ\varphi)'(0)\Big((\varphi^{-1}\circ c)'(0)\Big)\\
&= (f\circ\varphi)'(0)(\xi^1,\dots,\xi^m) = \sum_{i=1}^m \partial_i(f\circ\varphi)(0)\xi^i\\
&= \sum_{i=1}^m \tfrac{\partial}{\partial u^i}|_x(f)\xi^i = \sum_{i=1}^m \xi^i \tfrac{\partial}{\partial u^i}|_x(f).
\end{aligned}$$

Thus, ∂_c is the given element of T_xM, with the single flaw that c is only locally defined. However, since the above calculation depends only on the appearance of c near 0, we can reparameterize c so that nothing changes near 0, but c remains entirely in $\mathrm{dom}(\varphi)$.

The fact that T_xg has the given form is immediately apparent:

$$\Big((T_xg)(\partial_c)\Big)(f) = \partial_c(f\circ g) = \Big((f\circ g)\circ c\Big)'(0) = \Big(f\circ(g\circ c)\Big)'(0) = \partial_{g\circ c}(f). \qquad \square$$

Especially among physicists the following description of the tangent space is common:

10.12 Lemma (Tangent vectors via coordinates).

Assume that for each local parameterization φ of M centered around x, there are coordinates $(\xi^i_\varphi)_{i=1}^m$ of a vector $\xi_\varphi \in \mathbb{R}^m$ specified so that they transform correctly, i.e. $\xi_{\varphi_2} = (\varphi_2^{-1} \circ \varphi_1)'(0) \cdot \xi_{\varphi_1}$ for any two charts φ_1 and φ_2 with chart change mapping $\varphi_2^{-1} \circ \varphi_1$, or in coordinates $\xi^i_{\varphi_2} = \sum_{j=1}^m \partial_j(\varphi_2^{-1} \circ \varphi_1)^i(0) \cdot \xi^j_{\varphi_1}$. Such a coordinate scheme corresponds to a unique tangent vector in T_xM and vice versa.

If $g : M \to N$ is a smooth function, T_xg maps such a scheme $\xi_\varphi \in \mathbb{R}^m$ to the scheme $\eta_\psi \in \mathbb{R}^n$, with $\eta_\psi = (\psi^{-1} \circ g \circ \varphi)'(0) \cdot \xi_\varphi$.

Proof. Let $\xi_\varphi \in \mathbb{R}^m$ be given for a local parameterization φ and let $(u^1, \dots, u^m) := \varphi^{-1}$ be the associated local coordinates. Then we define a derivation $\partial_\xi \in T_xM$ by $\partial_\xi := \sum_{i=1}^m \xi^i_\varphi \frac{\partial}{\partial u^i}$. This definition makes sense, that is, is independent of the choice of the chart φ, because the ξ_φ transform in the same way as the coefficients of a derivation with respect to the basis $(\frac{\partial}{\partial u^i})$.

Conversely, the coefficients ξ^i_φ of a derivation $\partial \in T_xM$ with respect to the basis $(\frac{\partial}{\partial u^i})$ belonging to $\varphi = (u^1, \dots, u^m)^{-1}$, form exactly one correctly transforming coordinate scheme.

The fact that T_xg maps these schemes in the manner indicated follows immediately from the coordinate representation of T_xg with respect to bases $(\frac{\partial}{\partial u^i})$ and $(\frac{\partial}{\partial v^j})$ of T_xM and $T_{g(x)}N$. $\square$

11. Immersions

In the remainder of this chapter, we will use the tangent mapping to study specific properties of smooth mappings. In particular, we are interested in the correct concept of "subobjects"and "quotient objects" of manifolds.

11.1 Definition (Immersions and submersions).

Let $f \in C^\infty(M, N)$, where M, N are manifolds. Then

f is IMMERSIVE $:\Leftrightarrow T_xf$ is injective $\forall\, x \in M$;
f is SUBMERSIVE $:\Leftrightarrow T_xf$ is surjective $\forall\, x \in M$;
f is REGULAR $:\Leftrightarrow \operatorname{rang}(T_xf)$ is maximal $(= \min\{\dim T_xM, \dim T_{f(x)}N\})$ $\forall\, x$.

Note that a mapping is immersive if and only if it is regular and $\dim M \le \dim N$ holds. Likewise, it is submersive if and only if it is regular and $\dim M \ge \dim N$ holds.

11.2 Rank-Theorem.

Let $f \in C^\infty(M, N)$ and $r \in \mathbb{N}$. Then $\operatorname{rank}(T_xf) = r \quad \forall\, x \in M$ if and only if there is a chart φ centered at x for each $x \in M$ and a chart ψ centered at $f(x)$, s.t. the locally defined chart representation

$$\psi^{-1} \circ f \circ \varphi : \mathbb{R}^r \times \mathbb{R}^{m-r} \to \mathbb{R}^r \times \mathbb{R}^{n-r}$$

has the form $(x, y) \mapsto (x, 0)$.

Note that, w.l.o.g. we may assume (by restricting φ to $\varphi^{-1}(f^{-1}(\operatorname{im}\psi))$) that $f(\operatorname{im}\varphi) \subseteq \operatorname{im}\psi$ and thus the following diagram commutes:

$$\begin{tikzcd}
M \supseteq \operatorname{im}\varphi \arrow[r, "f|_{\operatorname{im}\varphi}"] & \operatorname{im}\psi \subseteq N \\
\mathbb{R}^m \supseteq \operatorname{dom}\varphi \arrow[u, "\varphi", "\cong"'] \arrow[r, "\psi^{-1}\circ f\circ\varphi"'] & \operatorname{dom}\psi \subseteq \mathbb{R}^n \arrow[u, "\cong", "\psi"']
\end{tikzcd}$$

By further shrinking $\operatorname{dom}\varphi$, we obtain the form $\operatorname{dom}\varphi = W_1 \times W_2 \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r}$ and $\operatorname{dom}\psi \cap \mathbb{R}^r = W_1$ (or even the form $\operatorname{dom}\psi = W_1 \times W_3 \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r}$ using a compactness argument on W_1).

Proof. ($\Leftarrow$) We have $\operatorname{rank} T_x f = \operatorname{rank} T_0(\psi^{-1}\circ f\circ\varphi) = \operatorname{rank}((x,y)\mapsto(x,0)) = r$.

($\Rightarrow$) Without loss of generality, let $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $x = 0$, and $f(x) = 0$. The idea of the proof is that f looks locally roughly like the derivative $f'(0)$, and being a linear map of rank r this is up to change of basis of the form $(x,y)\mapsto(x,0)$: Namely, let $F_1 := \operatorname{Bild}(f'(0))$, $F_2 := F_1^\perp$, $E_2 := \operatorname{Ker}(f'(0))$, and $E_1 := E_2^\perp$. Then $r = \operatorname{rank}(f'(0)) = \dim(F_1)$, and furthermore $\dim(E_1) := m - \dim(E_2) = \dim(F_1)$ and the component representation of $f'(0) : E_1 \oplus E_2 \to F_1 \oplus F_2$ has the following form:

$$f'(0) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

with invertible $A \in L(E_1, F_1)$ and, if we write $f = (f_1, f_2)$, then $A = \partial_1 f_1(0,0)$.

We now try to use local diffeomorphisms to bring the map f to the desired shape. For this we first consider a slightly modified variant of f, namely the smooth map $\varphi^{-1} : E_1 \oplus E_2 \to F_1 \oplus E_2$ given by $\varphi^{-1} : (x_1,x_2) \mapsto (f_1(x_1,x_2), x_2)$ (we will justify the notation as inverse). The Jacobi matrix of φ^{-1} in 0 looks like this:

$$(\varphi^{-1})'(0) = \begin{pmatrix} \partial_1 f_1(0,0) & \partial_2 f_1(0,0) \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

So $(\varphi^{-1})'(0)$ is invertible and because of the Inverse Function Theorem 2.2, φ^{-1} is a local diffeomorphism. If φ is the local diffeomorphism inverse to φ^{-1} and $g := f\circ\varphi$, then $g = (g_1, g_2)$ has the following form

$$g(y_1,y_2) = (y_1, g_2(y_1,0)),$$

because

$$\begin{aligned} x = (x_1,x_2) &:= \varphi(y_1,y_2) \Rightarrow \\ y = (y_1,y_2) &= \varphi^{-1}(x_1,x_2) = (f_1(x_1,x_2),x_2) \Rightarrow \\ y_1 &= f_1(x_1,x_2) = f_1(\varphi(y_1,y_2)) = g_1(y_1,y_2). \end{aligned}$$

Furthermore, $\operatorname{Rang} g'(y) = \operatorname{Rang} f'(\varphi(y)) = r$ holds, since φ is a local diffeomorphism. So in the component representation of $g'(y)$

$$g'(y) = \begin{pmatrix} \mathrm{id} & 0 \\ \partial_1 g_2(y) & \partial_2 g_2(y) \end{pmatrix}$$

the bottom right corner has to be $\partial_2 g_2(y) = 0$ and thus $g_2(y_1,y_2) = g_2(y_1,0)$.

In order to make the second component of g zero, we use $\psi^{-1} : F_1 \oplus F_2 \to F_1 \oplus F_2$ (the notation as inverse will be also justified) defined by

$$\psi^{-1}(y_1,y_2) := (y_1, y_2 - g_2(y_1,0)).$$

The component representation of $(\psi^{-1})'(x)$ is given by

$$(\psi^{-1})'(y_1,y_2) = \begin{pmatrix} \mathrm{id} & 0 \\ -\partial_1 g_2(y_1,0) & \mathrm{id} \end{pmatrix}$$

and thus ψ^{-1} is a local diffeomorphism, i.e. is really the inverse of some ψ. Furthermore

$$\begin{aligned}(\psi^{-1}\circ f\circ\varphi)(y_1,y_2) &= \psi^{-1}(y_1, g_2(y_1,0)) \\ &= (y_1, g_2(y_1,0) - g_2(y_1,0)) = (y_1,0). \qquad \square\end{aligned}$$

11.3 Corollary (Characterization of diffeomorphisms).

For smooth mappings f the following holds:

$$f \text{ is diffeomorphism } \Leftrightarrow f \text{ and } T_x f \text{ are bijective for all } x\in M.$$

Proof. ($\Rightarrow$) The mapping f is clearly bijective. That $T_x f$ is also bijective has already been shown in 10.4.

($\Leftarrow$) The mapping $g := f^{-1}$ is well-defined and continuous since f is open as local diffeomorphism. By the Rank-Theorem 11.2, there are charts φ at x and ψ at $f(x)$, such that $f(\mathrm{im}(\varphi))\subseteq \mathrm{im}(\psi)$ and $\psi^{-1}\circ f\circ\varphi = \mathrm{id}$.

$$\begin{tikzcd}
 & \operatorname{im}\varphi \arrow[r, "f"] & \operatorname{im}\psi & \\
\mathbb{R}^m & \operatorname{dom}\varphi \arrow[l, hook'] \arrow[u, "\varphi"', "\cong"] \arrow[r, "\mathrm{id}"] & \operatorname{dom}\psi \arrow[u, "\psi", "\cong"'] \arrow[r, hook] & \mathbb{R}^m
\end{tikzcd}$$

Without restriction of generality we can therefore assume $\operatorname{dom}\psi = \operatorname{dom}\varphi$. Thus $z\mapsto f^{-1}(z) = (\varphi\circ\psi^{-1})(z)$ is smooth on $\operatorname{im}\psi$ and hence f is a diffeomorphism since ψ^{-1} restricted to $\operatorname{im}\psi$ is one. $\square$

11.4 Characterization of Immersions.

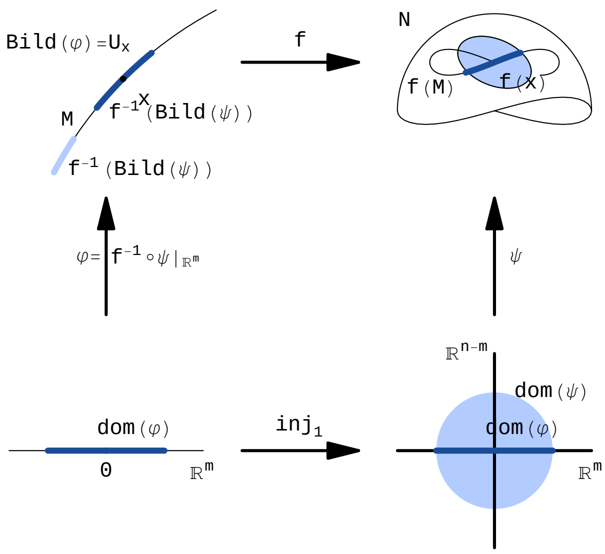
We now want to try to find out which subsets M of manifolds N can be made into manifolds such that the inclusion $f := \mathrm{incl} : M\to N$ is smooth and that the tangent spaces $T_x M$ of M are mapped by $T_x f$ bijectively onto subspaces of $T_{f(x)}N$, i.e. f is an immersion. For this we have to try to express the property that f is an immersion by using charts from N.

By the Rank-Theorem 11.2 immersions look like inclusions $\mathrm{incl}:\mathbb{R}^m\hookrightarrow\mathbb{R}^n$ with respect to suitable charts φ centered at $x\in M$ and ψ centered at $f(x)\in N$ with $f(\operatorname{im}\varphi)\subseteq\operatorname{im}\psi$ and $\operatorname{dom}\psi\cap\mathbb{R}^m = \operatorname{dom}\varphi$. So $f|_{\operatorname{im}\varphi}$ is bijective from $\operatorname{im}\varphi$ to its image

$$\operatorname{im}(f|_{\operatorname{im}\varphi}) = f(\operatorname{im}\varphi) = \psi(\mathrm{incl}(\operatorname{dom}\varphi)) = \psi(\operatorname{dom}\psi\cap\mathbb{R}^m) = \psi(\mathbb{R}^m),$$

and thus

$$\varphi = f|_{\operatorname{im}\varphi}^{-1}\circ f|_{\operatorname{im}\varphi}\circ\varphi = f|_{\operatorname{im}\varphi}^{-1}\circ\psi\circ\mathrm{incl} = f|_{\operatorname{im}\varphi}^{-1}\circ\psi|_{\mathbb{R}^m}.$$



Thus by appropriate choices of neighborhoods $U_x := \operatorname{im}\varphi$ of $x \in M$ and charts ψ of N centered at $f(x)$ we obtain an atlas $\varphi := f|_{U_x}^{-1} \circ \psi|_{\mathbb{R}^m}$ for M and the chart representaion of f then looks like the inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$. This shows the direction ($\boxed{1} \Rightarrow \boxed{2}$) of the following

Proposition. Characterization of immersions.

For $f \in C^\infty(M,N)$ the following statements are equivalent:

1. *f is immersive;*
2. *$\forall\, x \in M\ \exists\, U_x$ open neighborhood of x in M and a chart ψ centered at $f(x)$ in N, such that $f|_{U_x}^{-1} \circ \psi|_{\mathbb{R}^m} : \operatorname{dom}\psi \cap \mathbb{R}^m \to U_x$ is a well-defined diffeomorphism (and thus a chart for M);*
3. *f has locally a left-side inverse, that is $\forall\, x \in M\ \exists\, U_x$ open neighborhood of x in M and $\exists\, h : N \supseteq V_{f(x)} \to M$ smooth with $V_{f(x)} \supseteq f(U_x)$ open and $h \circ f = \mathrm{id}_{U_x}$.*

Proof. We just showed ($\boxed{1} \Rightarrow \boxed{2}$).

($\boxed{2} \Rightarrow \boxed{3}$) Let $\varphi := f|_{U_x}^{-1} \circ \psi|_{\mathbb{R}^m} : \operatorname{dom}\psi \cap \mathbb{R}^m \to U_x$ be the diffeomorphism with U_x and ψ as in $\boxed{2}$. By shrinking the chart ψ we can achieve that $\operatorname{dom}\psi$ is of the form $W_1 \times W_2 \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$. Now we put $V_{f(x)} := \operatorname{im}\psi$ and $h := \varphi \circ \mathrm{pr}_1 \circ \psi^{-1}$, where $\mathrm{pr}_1 : \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \supseteq W_1 \times W_2 \to W_1 \subseteq \mathbb{R}^m$ denotes the canonical projection onto the first factor. Then $h : V_{f(x)} \to U_x$ is smooth and

$$h \circ f \circ \varphi = \varphi \circ \mathrm{pr}_1 \circ \psi^{-1} \circ f \circ \varphi = \varphi \circ \mathrm{pr}_1 \circ \mathrm{incl}_1 = \varphi = \mathrm{id} \circ \varphi,$$

so $h \circ f = \mathrm{id}$ on $\operatorname{im}\varphi = U_x$.

($\boxed{3} \Rightarrow \boxed{1}$) Because of $h \circ f = \mathrm{id}$ locally at x, the identity $\mathrm{id} = T_x\,\mathrm{id} = T_{f(x)}h \circ T_x f$ holds, so $T_x f$ is injective and thus f is an immersion. □

11.5 Corollary.

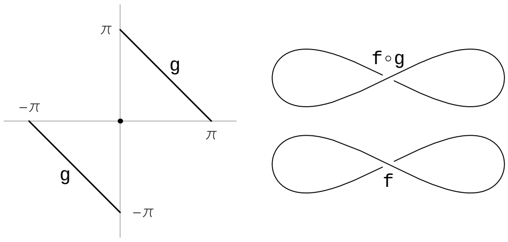
Let $f \in C^\infty(M,N)$ be an immersion and $g : P \to M$ be a continuous mapping with $f \circ g \in C^\infty(P,N)$. Then g is smooth.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ P \arrow[u, "C\ni g"] \arrow[ur, "f\circ g\in C^\infty"'] & \end{tikzcd}$$

Proof. Let $z \in P$ and $x := g(z)$. There are U_x and $h : f(U_x) \subseteq V_{f(x)} \to M$ as in 11.4.3. Since g is continuous, $g^{-1}(U_x)$ is an open neighborhood of z on which $g = (h \circ f) \circ g = h \circ (f \circ g)$ is smooth. $\square$

11.6 Remarks.

1. For 11.5, the continuity of g is essential: Let $g :]-\pi, \pi[\to]-\pi, \pi[$ be defined by
$$g : t \mapsto \begin{cases} \pi - t & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -\pi - t & \text{for } t < 0 \end{cases}$$
and the immersion $f : (-\pi, \pi) \to \mathbb{R}^2$ be defined by $f(t) := (\sin t, -\sin 2t)$.



Then $f \circ g$ is smooth, but g is not continuous, ergo also not smooth.

2. A manifold M, whose underlying set is a subset of a manifold N, is called an IMMERSIVE SUBMANIFOLD, if the inclusion incl : $M \to N$ is an immersion. An immersive submanifold is generally not a submanifold in the sense of 2.4, or more generally of 11.10: The mapping $f : (-pi, pi) \to \mathbb{R}^2$ from 1 is an injective immersion, but the immersive submanifold $\operatorname{im}(f) \cong (\pi, \pi)$ is not a submanifold of $\mathbb{R}^2$.

3. The manifold structure of an immersive submanifold is generally not determined by that of N as 1 shows: f and $f \circ g$ induce two different manifold structures on $M = \operatorname{im}(f)$.

11.7 Definition (Initial and final mappings).

Let $f \in C^\infty(M, N)$. The map f is called INITIAL MAPPING :⇔ for each mapping $g : P \to M$ with the property that $f \circ g$ is smooth, g itself is smooth.

The mapping f is called FINAL MAPPING :⇔ for each $g : N \to P$ with the property that $g \circ f$ is smooth, g itself is smooth.

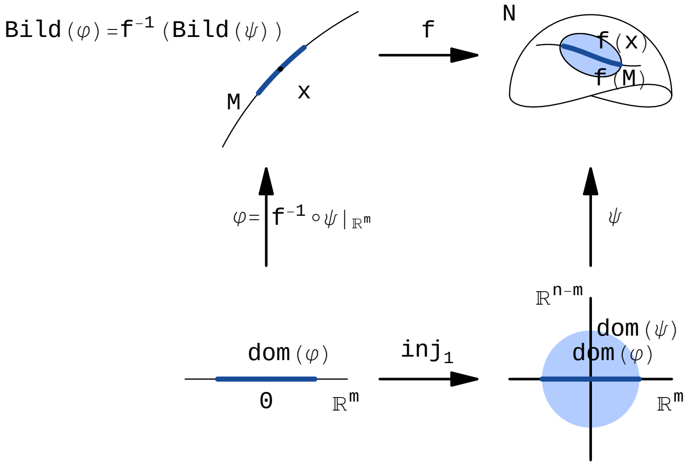
11.8 Definition (Embedding).

It is $f : M \to N$ smooth, then f is called EMBEDDING :⇔ f is an injective immersion and $f : M \to f(M)$ is a homeomorphism onto its image $f(M)$ supplied with the trace topology of N.

11.9 Proposition (Characterization of embeddings).

For f in $C^\infty(M, N)$, the following statements are equivalent:

1. *f is an embedding;*
2. *For each $x \in M$, there is a chart ψ of N centered at $f(x)$ so that $f^{-1} \circ \psi|_{\mathbb{R}^m}$: $\operatorname{dom} \psi \cap \mathbb{R}^m \to f^{-1}(\operatorname{im} \psi)$ is a well-defined diffeomorphism (and thus a chart);*
3. *f has local left inverses in the following sense: $\forall\, x \in M\ \exists\, h : V_{f(x)} \to M$ smooth on an open neighborhood $V_{f(x)}$ from $f(x)$ in N with $h \circ f = \mathrm{id}$ on $f^{-1}(V_{f(x)})$.*



Note that the difference to the formulation of immersions in 11.4 is only that the image of the constructed charts φ is now all of $f^{-1}(\operatorname{im} \psi)$ and not just an open neighborhood U_x of x, i.e. $\operatorname{im} \psi \cap \operatorname{im} f$ may consist only of a part which looks like $\mathbb{R}^m \subseteq \mathbb{R}^n$.

Proof.

(1 $\Rightarrow$ 2) Let f be an embedding. Since f is an immersion, there exists, for $x \in M$ by 11.4, an open $U_x \subseteq M$ and a chart ψ centered at $f(x)$, s.t.

$$f^{-1} \circ \psi|_{\mathbb{R}^m} : \operatorname{dom} \psi \cap \mathbb{R}^m \to U_x$$

is a well-defined diffeomorphism. We want to achieve $U_x = f^{-1}(\operatorname{im} \psi)$ by resizing $\operatorname{dom} \psi$. Since f is a homeomorphism onto its image, there exists an open $W \subseteq \operatorname{im} \psi$ with $W \cap \operatorname{im} f = f(U_x)$. Without loss of generality $\operatorname{im} \psi = W$, hence $U_x = f^{-1}(\operatorname{im} \psi)$, because

$$U_x \overset{f \text{ inj.}}{=\!=} (f^{-1} \circ f)(U_x) = f^{-1}(W \cap \operatorname{im} f) = f^{-1}(\operatorname{im} \psi \cap \operatorname{im} f) = f^{-1}(\operatorname{im} \psi).$$

(2 $\Rightarrow$ 3) The same definition of h as in the corresponding proof of 11.4 now provides a left inverse on $U_x = f^{-1}(\operatorname{im} \psi)$.

(3 $\Rightarrow$ 1) by 11.4 f is immersive.

Furthermore, f is injective, otherwise there are $x_1 \neq x_2$ with $y := f(x_1) = f(x_2)$ and a local left-inverse $h : V_y \to M$ as in 3. Then $x_i \in f^{-1}(V_y)$ and thus $x_i = (h \circ f)(x_i) = h(y)$ is independent of i, a contradiction.

Finally, f is a homeomorphism onto its image: Let $(x_i)_i$ be a net in M for which $f(x_i)$ converges to an $f(x_\infty)$. Let $h : V \to M$ be a local left inverse as in 3 with an open neighborhood V of $f(x_\infty)$. Then $f(x_i)$ is finally in V and thus x_i finally in $f^{-1}(V)$ and hence converges $x_i = (h \circ f)(x_i) = h(f(x_i)) \to h(f(x_\infty)) = x_\infty$. □

11.10 Definition (Submanifold).

A subset M of a manifold N, which itself is a manifold and for which the inclusion $\mathrm{incl} : M \hookrightarrow N$ is an embedding (hence possesses the equivalent properties of 11.9) is called (regular) submanifold of N.

Any subset $M \subseteq N$ having for each point $x \in M$ a chart ψ of N centered at x for which $M \cap \operatorname{im}\psi = \psi(\mathbb{R}^m)$ holds, is itself a manifold of dimension m with the smooth atlas formed by these restrictions $\psi|_{\mathbb{R}^m}$ and, furthermore, the inclusion $\mathrm{incl} : M \hookrightarrow N$ is by construction and by 11.9.2 an embedding. Thus M is a regular submanifold of N.

This shows that the definition for regular submanifolds of $N = \mathbb{R}^n$ coincide with that given in 2.4 for submanifolds of $\mathbb{R}^n$, because charts ψ of $N = \mathbb{R}^n$ as in 11.9.2 (i.e. with $M \cap \operatorname{im}\psi = \psi(\mathbb{R}^m)$) are just local trivializations in the sense of 2.4.4.

The image $f(M)$ of each embedding $f : M \to N$ is obviously a regular submanifold of N and the embedding induces a diffeomorphism $f : M \to f(M)$:

By 11.9.2, $f(M) \cap \operatorname{im}\psi = \psi(\mathbb{R}^m)$, i.e. $f(M)$ is a regular submanifold with $\psi|_{\mathbb{R}^m}$ as charts and f^{-1} is (locally) smooth. So up to diffeomorphisms, embeddings are nothing else but the inclusion of regular submanifolds.

$$\begin{tikzcd}
M \arrow[r, tail, two heads, "f"] & f(M) \arrow[r, hook] & N \\
f^{-1}(\operatorname{im}\psi) \arrow[u, hook] \arrow[r, tail, two heads, "f"] & \psi(\mathbb{R}^m) \arrow[u, hook] \arrow[r, hook] & \operatorname{im}\psi \arrow[u, hook] \\
& \operatorname{dom}\psi \cap \mathbb{R}^m \arrow[ul, dotted, "\cong", "f^{-1}\circ\psi|_{\mathbb{R}^m}"'] \arrow[u, "\cong"', "\psi|_{\mathbb{R}^m}"] \arrow[r, hook] & \operatorname{dom}\psi \arrow[u, "\cong"', "\psi"]
\end{tikzcd}$$

11.11 Whitney Embedding Theorem.

Let M be a connected σ-compact (and thus paracompact) C^∞ manifold of dimension m. Then M can be embedded into some finite-dimensional vector space. Thus each "abstract" manifold can be realized as a submanifold of some $\mathbb{R}^n$.

Proof. Let $\{\psi_i : 0 \le i \le m\}$ be a finite atlas on 9.8 (for an elementary proof of 11.11 without using dimension theory, see for example [19, S.73]). Furthermore, let f_i be a partition of unity subordinated to $\{\operatorname{im}\psi_i\}$ and let $f : M \to \prod_{i=0}^m (\mathbb{R}\times\mathbb{R}^m)$ be the smooth mapping

$$x \mapsto \big(f_i(x), f_i(x)\psi_i^{-1}(x)\big)_{i=0}^m.$$

In order to apply 11.9, we show the existence of local left-inverses $g_i : (\mathbb{R} \times \mathbb{R}^m)^{m+1} \supseteq V_i \to M$ for an open covering $\{V_i : 0 \le i \le m\}$ of $f(M)$.
Let

$$\begin{aligned} V_i :&= \left\{(t,y) : t_i > 0, \frac{1}{t_i} y_i \in \operatorname{dom}\psi_i\right\}, \\ g_i :&V_i \to M, \quad (t,y) \mapsto \psi_i(t_i^{-1}\cdot y_i) \end{aligned}$$

Then $g_i \circ f = \mathrm{id}$ on $f^{-1}(V_i)$, because

$$f^{-1}(V_i) \ni x \mapsto (g_i \circ f)(x) = \psi_i\left(\frac{f_i(x)\psi_i^{-1}(x)}{f_i(x)}\right) = \psi_i(\psi_i^{-1}(x)) = x. \quad \square$$

11.12 Remarks.

1. By the proof of 11.11, m-dimensional manifolds can be embedded into $\mathbb{R}^{(m+1)^2}$. But this also works for lower dimensional $\mathbb{R}^n$'s. Namely, M can be embedded in $\mathbb{R}^n$, where
   (i) for $n = 2m+1$, the proof is relatively simple, see [65, S.55];
   (ii) for $n = 2m$ this is due to [153].
   Conjecture: The minimum $n = 2m - \alpha(m) + 1$, where $\alpha(m)$ is the number of ones in the binary number development of m.

   A related question is that about the minimum n for the existence of an immersion $M \to \mathbb{R}^n$?
   (i) For $n = 2m$, the proof is relatively simple, see [65, S.24].
   (ii) For $n = 2m-1$, it is due to [153].
   The conjecture that the minimum is $n = 2m - \alpha(m)$ was finally be proved for compact manifolds in [30] and in general in [20].

2. The Rank-Theorem provides us in a simple way with more regular submanifolds:
   Let $f \in C^\infty(M,N)$ with $\mathrm{rank}(T_x f) = r\ \forall\, x \in M$. Then $f^{-1}(y)$ is a regular submanifold of M for each y.

   Proof. This is a local property, so without loss of generality we may assume that $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ are open. Then it follows from 11.2 that f looks locally like $(x,y) \mapsto (x,0)$, and thus the $f^{-1}(0)$ preimage looks like $\{0\} \times \mathbb{R}^{m-r}$. $\square$

11.13 Corollary (Retracts are manifolds).

Let $f \in C^\infty(M,M)$ with $f \circ f = f$. Then $A := f(M)$ is regular submanifold, i.e. smooth retracts of manifolds are again manifolds.

Proof. Note that $x \in A := f(M)$ holds if and only if $f(x) = x$ holds: In fact $x = f(y) \Rightarrow f(x) = f(f(y)) = f(y) = x$, and vice versa $x = f(x) \in f(M)$.

Let $x_0 \in A$ and $\varphi : \mathbb{R}^m \supseteq U \to \varphi(U) \subseteq M$ be a chart centered at x_0. For all y in the neighborhood $V := f^{-1}(\varphi(U)) \cap \varphi(U)$ of x_0 we have:

$$\begin{tikzcd}
M \arrow[rr, "f"] & & M \\
\varphi(U) \arrow[u, hook] & V \arrow[l, hook', "\varphi"'] \arrow[r, "f|_V"] & \varphi(U) \arrow[u, hook] \\
U \arrow[u, "\varphi", "\cong"'] & \varphi^{-1}(V) \arrow[l, hook', "\varphi"'] \arrow[u, "\varphi", "\cong"'] \arrow[r, "\bar f"] & U \arrow[u, "\varphi", "\cong"']
\end{tikzcd}$$

$$\begin{aligned}
y \in f(M) &\Leftrightarrow f(y) = y \\
&\Leftrightarrow (\varphi^{-1} \circ f \circ \varphi)(\varphi^{-1}(y)) = (\varphi^{-1} \circ f)(y) = \varphi^{-1}(y) \\
&\Leftrightarrow (\mathrm{id} - \bar f)(\varphi^{-1}(y)) = 0,
\end{aligned}$$

where $\bar f := \varphi^{-1} \circ f \circ \varphi : U \supseteq \varphi^{-1}(V) \to U \subseteq \mathbb{R}^m$ is the chart representation of f. For $z \in \bar f^{-1}(\varphi^{-1}(V))$ we have:

$$\begin{aligned}
(\bar f \circ \bar f)(z) = (\varphi^{-1} \circ f \circ \varphi \circ \varphi^{-1} \circ f \circ \varphi)(z) &= (\varphi^{-1} \circ f^2 \circ \varphi)(z) = \\
&= (\varphi^{-1} \circ f \circ \varphi)(z) = \bar f(z),
\end{aligned}$$

i.e., without loss of generality, $0 \in U \subseteq \mathbb{R}^m$ is open and $f : U \to \mathbb{R}^m$ satisfies $f(0) = 0$ and $f \circ f = f$. We have to show that $\mathrm{id} - f = 0$ is a regular equation

locally at 0:
Obviously, $\mathrm{rang}(T_z(\mathrm{id}-f)) \geq \mathrm{rang}(T_0(\mathrm{id}-f)) =: r$ for all z close to 0. Conversely, from $f\circ(\mathrm{id}-f)=0$ it follows that $T_{(\mathrm{id}-f)(z)}f\circ(\mathrm{id}-T_zf)=0$ and thus $\mathrm{im}(\mathrm{id}-T_zf)\subseteq \mathrm{Ker}(T_{(\mathrm{id}-f)(z)}f)$. Thus locally

$$\mathrm{rang(T_z(id\text{-}f))} \leq \dim\mathrm{Ker}(T_{(\mathrm{id}-f)(z)}f) = m - \dim\mathrm{im}(T_{(\mathrm{id}-f)(z)}f) \leq \\ \leq m - \dim\mathrm{im}(T_0 f) = \dim\mathrm{im}(\mathrm{id}-T_0 f) = r,$$

where we used the obvious equation $T_0\mathbb{R}^m = \mathrm{im}(T_0f)\oplus\mathrm{im}(\mathrm{id}-T_0f)$ for the linear projection T_0f.

Now use 11.12.2. □

11.14 Remark.

Conversely, it can be shown that each submanifold M of a manifold N is the retract of an open set in N, see [**86**, 62.9] or [**65**, S.110]. Together with the embedding theorem 11.11, this implies that connected σ-compact manifolds are - up to diffeomorphisms - precisely the retracts of open subsets of finite-dimensional vector spaces.

For $f\in C^\infty(M,N)$, the graph of f is defined as

$$\mathrm{graph}(f) := \{(x,f(x)) : x\in M\}\subseteq M\times N.$$

It is a regular submanifold. The proof remains as an exercise. Note: $\mathrm{graph}(f)\cong M$.

11.15 Sard's Theorem.

The set of critical values of any smooth mapping between σ-compact manifolds has Lebesgue measure 0.

Definition.

Here we call a point $x\in M$ CRITICAL for a mapping $f: M\to N$, if $T_xf: T_xM\to T_{f(x)}N$ does not have maximal rank $\min\{\dim M,\dim N\}$, i.e. f is not regular at x. A point $y\in N$ is called a CRITICAL VALUE of f if a critical point $x\in f^{-1}(y)$ exists. Sometimes one only asks for critical points that T_xf is not surjective, i.e. $\mathrm{rank}(T_xf)<\dim N$. At least for the Theorem of Sard it makes however no big difference, because only in the case of $\dim M<\dim N$ there are then more critical values (namely all in the image), however, these also form a Lebesgue zero set according to the corollary in 11.16.

A set $N\subseteq\mathbb{R}^n$ is called a LEBESGUE ZERO SET if for each $\varepsilon>0$ a sequence of cubes (or cubes or spheres) $(Q_k)_{k\in\mathbb{N}}$ exists with $N\subseteq\bigcup_{k\in\mathbb{N}}Q_k$ and $\sum_{k\in\mathbb{N}}|Q_k|<\varepsilon$, where we write $|Q|$ for the volume of Q.
A subset $N\subseteq M$ of a smooth manifold M is called a LEBESGUE ZERO SET if the inverse image under each chart is a Lebesgue zero set.

The Theorem of Sard is also valid, if $f\in C^r(M,N)$ with $r>\min\{0,\dim M-\dim N\}$. However, in [**151**] an C^1 mapping $f:\mathbb{R}^2\to\mathbb{R}$ was constructed which is critical but not constant on an arc I. Thus, the graph of f is an surface $S\subseteq\mathbb{R}^3$ on which there is an arc $f(I)$, so that the tangent plane at S is horizontal in each point, but nevertheless $f(I)$ does not have constant height. See also [**19**, S.58] and [**65**, p.68].

11.16 Lemma.

Let $U\subseteq\mathbb{R}^m$ be open and $N\subseteq U$ a Lebesgue zero set. Furthermore, let $f:U\to\mathbb{R}^m$ be a C^1 map. Then $f(N)$ is also a Lebesgue zero set.

Proof. Since U is the union of countable many compact convex sets (e.g. the spheres contained in U with rational coordinates of their center and rational radius) and because the countable union of Lebesgue zero sets is again a Lebesgue zero set, we may assume that N is contained in such a compact convex set $K \subseteq U$.

Because $f : U \to \mathbb{R}^m$ is C^1, $\kappa := \sup\{\|f'(x)\| : x \in K\} < \infty$. Let $Q \subseteq K$ be a cube with side length a. Then by the fundamental theorem of calculus

$$|f(x_1) - f(x_0)| = \Big|\int_0^1 f'(x_0 + t(x_1 - x_0))\,(x_1 - x_0)\,dt\Big| \le \kappa \cdot |x_1 - x_0| \le \kappa\, a\, \sqrt{m}.$$

for all $x_1, x_0 \in Q$. So $f(Q)$ is contained in a cube with side length $2\,\kappa\, a\,\sqrt{m}$ and volume $(2\,\kappa\, a\,\sqrt{m})^m = (2\,\kappa\sqrt{m})^m\,|Q|$. The image of a countable covering with cubes (which we may assume to be contained in K) of total volume smaller than $\delta := \varepsilon/(2\,\kappa\sqrt{m})^m > 0$ is thus contained in a covering with cubes of total volume smaller than $(2\,\kappa\sqrt{m})^m \cdot \delta = \varepsilon$. □

Thus, a subset $N \subseteq M$ of a manifold is a Lebesgue zero set if and only if the inverse images under the charts of a fixed atlas are Lebesgue zero sets.

Corollary.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 and $n < m$, then $f(\mathbb{R}^n)$ is a Lebesgue zero set.

Proof. Apply the lemma in 11.16 to $\tilde f := f \circ \mathrm{pr} : \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \to \mathbb{R}^m$ and the Lebesque zero set $N := \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^m$. □

We need also the

11.17 Theorem of Fubini.

Let $N \subseteq \mathbb{R}^n$ be compact and $N \cap (\{t\} \times \mathbb{R}^{n-1})$ a Lebesgue zero set in $\{t\} \times \mathbb{R}^{n-1}$ for all $t \in \mathbb{R}$. Then N is a Lebesgue zero set in $\mathbb{R}^n$.

For a proof, see [**19**, S.59] or [**82**, 7.6.9].

Proof of the Theorem of Sard 11.15. Note that if there is a neighborhood U_x for each point x in a set $X \subseteq \mathbb{R}^m$, s.t. $f(U_x \cap X)$ is a Lebesgue zero set, then $f(X) = \bigcup_{x\in X} f(U_x \cap X)$ is also a Lebesgue zero set, because countably many of the U_x already cover X (which is Lindelöff by the proof of 8.2).

Hence it suffices to consider case $f : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$. Let D be the set of critical points. We make induction on m. For $m = 0$, this is trivial.

In the induction step we want to apply 11.17, however the set of critical values is not compact, but the critical points are a countable union of compact sets, because the set of points x, where the determinant of a fixed $r \times r$ submatrix of $f'(x)$ vanishes, is closed, i.e. is the countable union of their intersections with the compact balls $B_n(x)$ for $n \in \mathbb{N}$, and the set of critical values thus is a countable union of the compact images of all these compact sets (and hence 11.17 applies).

Let

$$D_k := \Big\{x \in U : \frac{\partial^\alpha}{\partial x^\alpha} f(x) = 0 \text{ for all } |\alpha| \le k\Big\}.$$

The D_k are closed and $D \supseteq D_1 \supseteq D_2 \supseteq \dots$.

$f(D \setminus D_1)$ is a Lebesgue zero set:
Let $x \in D \setminus D_1$. Without loss of generality $\frac{\partial}{\partial x^1} f_1(x) \ne 0$. Then $h : U \to \mathbb{R}^m$,

$(x^1,\dots,x^m)\mapsto(f_1(x),x^2,\dots,x^m)$ is a local diffeomorphism and $g:=f\circ h^{-1}$ has the form

$$g:(f_1(x),x^2,\dots)\mapsto(x^1,\dots,x^n)\mapsto(f_1(x),\dots,f_n(x))$$
$$g:(t,x^2,\dots,x^m)\mapsto(t,g^2(t,x),\dots,g^n(t,x)).$$

The hyperplane $\{t\}\times\mathbb{R}^{m-1}\cong\mathbb{R}^{m-1}$ is mapped into the hyperplane $\{t\}\times\mathbb{R}^{n-1}$ by g, and the restriction $g_t(x):=(g^2(t,x),\dots,g^n(t,x))$ of g to it has x as a critical point if and only if (t,x) is a critical point of g. By induction, the critical values of g_t are a Lebesgue zero set, and according to Fubini's theorem 11.17, also those of g, but these are the ones of $f=g\circ h$ because h is a local diffeomorphism.

$f(D_k\setminus D_{k+1})$ is a Lebesgue zero set:
Let $x\in D_k\setminus D_{k+1}$. W.l.o.g. $\frac{\partial^{k+1}f_1}{\partial x^1\partial x^{m_1}\dots\partial x^{m_k}}(x)\neq 0$. Put $w:=\frac{\partial^k f_1}{\partial x^{m_1}\dots\partial x^{m_k}}$. Then $w|_{D_k}=0$ and $\frac{\partial w}{\partial x_1}(x)\neq 0$. Let $h(x):=(w(x),x_2,\dots,x_m)$. Then $h:U\to\mathbb{R}^m$ is a local diffeomorphism (say between U_x and $h(U_x)$) with $h(D_k)\subseteq\{0\}\times\mathbb{R}^{m-1}\subseteq\mathbb{R}^m$. We consider the mapping $g:=f\circ h^{-1}:h(U_x)\to\mathbb{R}^n$ and its restriction $g_0:=g|_{\{0\}\times\mathbb{R}^{m-1}}$. Since all partial derivatives of f of order $\le k$ vanish on D_k and thus, in particular, those of order 1 of $g_0=f\circ h^{-1}|_{\{0\}\times\mathbb{R}^{m-1}}$ vanish on $h(U_x\cap D_k)$, we obtain that $h(U_x\cap D_k)$ is contained in the set of critical points of g_0. By induction hypothesis $f(U_x\cap D_k)=g_0(h(U_x\cap D_k))$ is a Lebesgue zero set and hence $f(D^k\setminus D^{k+1})$ is also one.

$f(D_k)$ is a Lebesgue zero set for each $k>\frac{m}{n}-1$:
Let Q be a cube with side-length a. By the Taylor formula we get

$$\begin{aligned}|f(x+h)-f(x)|&=\Big|\int_0^1\frac{(1-t)^k}{k!}f^{(k+1)}(x+th)(h,\dots,h)\,dt\Big|\\&\le\underbrace{\sup\Big\{\|f^{(k+1)}(x)\|:x\in Q\Big\}\int_0^1\frac{(1-t)^k}{k!}\,dt}_{=:\tau}\,|h|^{k+1}\le\tau\,|h|^{k+1}\end{aligned}$$

for all $x\in D_k\cap Q$. We decompose Q into N^m cubes with side-length $\frac{a}{N}$. Let Q' be such a smaller cube containing a point $x\in D_k$. Then each point in Q' is of the form $x+h$ with $|h|\le\frac{a}{N}$ and thus $f(Q')$ is contained in a cube of edge length $2\tau\left(\frac{a}{N}\right)^{k+1}$. All these cubes together have a total volume of at most $N^m\,\frac{(2\,\tau\,a^{k+1})^n}{N^{(k+1)n}}$ and for $(k+1)n>m$ this term converges to zero for $N\to\infty$. □

11.18 Retraction Theorem.

There is no continuous retraction $\mathbb{D}^n:=\{x\in\mathbb{R}^n:|x|\le 1\}\to S^{n-1}$.

A RETRACTION f to a subset $Y\subseteq X$ is a mapping $f:X\to Y$ which fulfills $f|_Y=\mathrm{id}$, i.e. a left inverse to the inclusion $Y\hookrightarrow X$. More intuitively, a DEFORMATION RETRACT from Y to X is defined to be a continuous mapping $F:[0,1]\times X\to X$ with the following properties:

- $\forall\,t\in[0,1]\ \forall\,y\in Y$: $F(t,y)=y$.
- $\forall\,x\in X$: $F(0,x)=x$.
- $\forall\,x\in X$: $F(1,x)\in Y$.

If put $F_t(x):=F(t,x)$ with $F_t:X\to X$, then $F_t|_Y=\mathrm{id}_Y$, $F_0=\mathrm{id}_X$, and $F_1:X\to Y$ is a retraction.

Conversely, we can extend any retraction $f:\mathbb{D}^n\to S^{n-1}\subseteq\mathbb{D}^n$ to a deformation retract $F(t,x):=(1-t)\,x+t\,f(x)$.

Proof. Suppose there were a retraction f.
We first want to show that w.l.o.g. f is C^∞:
Using f we get a retraction $f_1 : \mathbb{R}^n \to S^{n-1}$ which is C^∞ on a neighborhood of S^{n-1}, e.g.

$$f_1(x) := \begin{cases} f(x/|x|) = x/|x| & \text{for } 1/2 \le |x| \\ f(2x) & \text{for } |x| \le 1/2. \end{cases}$$

By Exercise [98, EX5] (or the Stone-Weierstrass Theorem) there exists a smooth function $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ with $\|f_2 - f_1\|_\infty < 1$. Now let

$$h \in C^\infty(\mathbb{R}^m, [0,1]) \text{ with } h(x) = \begin{cases} 1 & \text{for } |x| \le \frac{1}{2} \\ 0 & \text{for } |x| \ge 1 \end{cases}$$

and $f_3(x) := (1 - h(x))\, f_1(x) + h(x)\, f_2(x)$. Then

$$|f_3(x) - f_1(x)| = h(x) \cdot |f_2(x) - f_1(x)| \le |f_2(x) - f_1(x)| < 1 = |f_1(x)| \;\Rightarrow$$
$$\Rightarrow\; f_3(x) \ne 0 \text{ for all } x \text{ and } f_3(x) = f_1(x) = x/|x| \text{ for } |x| \ge 1.$$

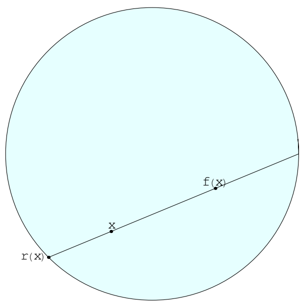
Finally, $f_4(x) := f_3(x)/|f_3(x)|$ is the sought-after C^∞-retraction. We call this again f. By the Theorem of Sard 11.15, there exists a regular value $y \in S^{n-1}$ of f, and thus $M := f^{-1}(y)$ is a 1-dimensional submanifold of $\mathbb{R}^n$ (which intersects S^{n-1} radially) and $y \in M \cap S^{n-1}$. Let $z \in M$ be another intersection point of the connected component of y in M with S^{n-1} (It exists, since the connected component of y is unbounded, hence homeomorphic to $\mathbb{R}$ (see [82, 7.6.12]) and thus must leave $\mathbb{D}^n$ again because $f^{-1}(y) \cap \mathbb{D}^n$ is compact). Then $f(z) = z \ne y$ gives a contradiction to $z \in f^{-1}(y)$. □

11.19 Brouwer's fixed point theorem.

Every continuous $f : \mathbb{D}^n \to \mathbb{D}^n$ has at least one fixed point.

Proof.

Suppose $f : \mathbb{D}^n \to \mathbb{D}^n$ has no fixed point. Then a continuous retraction $r : \mathbb{D}^n \to S^{n-1}$ can be defined by mapping $x \in \mathbb{D}^n$ to the intersection point of the straight line from $f(x)$ to x with S^{n-1} which is closer to x. A contraction to 11.18. □



Explicitly, r is given by:

$$r(x) := x - \lambda(f(x) - x), \text{ where } \lambda \ge 0 \text{ and}$$
$$0 = |r(x)|^2 - 1 = \lambda^2 |f(x) - x|^2 - 2\lambda\langle x | f(x) - x\rangle + |x|^2 - 1,$$
$$\text{that is } \lambda = \frac{\langle x|f(x) - x\rangle + \sqrt{\langle x|f(x) - x\rangle^2 + |f(x) - x|^2\,(1 - |x|^2)}}{|f(x) - x|^2}.$$

12. Submersions

12.1 Proposition (Characterization of submersions).

Let $f \in C^\infty(M,N)$, then

$$f \text{ is a submersion} \Leftrightarrow f \text{ has local sections.}$$

(I.e., $\forall\ x \in M\ \exists\, U_{f(x)} \subseteq N\ \exists\, g^x \in C^\infty(U_{f(x)}, M) : g^x(f(x)) = x$ and $f \circ g^x = \mathrm{id}$ on $U_{f(x)}$. So locally there exists a right inverse.)

Proof. ($\Rightarrow$) By the Rank-Theorem 11.2, charts φ exist around x and ψ around $f(x)$, so that the following diagram commutes:

$$\begin{tikzcd} & \operatorname{im}\varphi \arrow[r, "f"] & \operatorname{im}\psi \\ \mathbb{R}^n \times \mathbb{R}^{m-n} \cong \mathbb{R}^m \supseteq & \operatorname{dom}\varphi \arrow[u, "\varphi"] \arrow[r, "\mathrm{pr}_1"] & \operatorname{dom}\psi \subseteq \mathbb{R}^n \arrow[u, "\psi"'] \end{tikzcd}$$

Here $\mathrm{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ denotes the natural projection. If we now put $U_{f(x)} := \operatorname{im}(\psi)$ and $g^x := \varphi \circ \mathrm{incl}_1 \circ \psi^{-1}$, then g^x is smooth with $g^x(f(x)) = x$ and

$$f \circ g^x = f \circ \varphi \circ \mathrm{incl}_1 \circ \psi^{-1} = \psi \circ \mathrm{pr}_1 \circ \mathrm{incl}_1 \circ \psi^{-1} = \mathrm{id}_{U_{f(x)}}.$$

($\Leftarrow$) Let $U_{f(x)}$ and g^x be as assumed, then

$$T_{f(x)}\,\mathrm{id} = \mathrm{id}_{T_{f(x)}N} = T_{f(x)}(f \circ g^x) = T_x f \circ T_{f(x)} g^x \Rightarrow T_x f \text{ is surjective.} \qquad \square$$

12.2 Corollary (Submersions are open and final).

Each submersion $f : M \to N$ is an open mapping. Each surjective submersion f is also final.

Proof. *f is open*: Let $U \subseteq M$ be open and $y \in f(U)$, i.e. $y := f(x)$ for some $x \in U$. By 12.1 there exists a y-neighborhood U_y and a smooth $g^x : U_y \to M$ with $g^x(y) = x$ and $f \circ g^x = \mathrm{id}_{U_y}$. Without loss of generality, $U_y \subseteq (g^x)^{-1}(U)$. Hence $U_y = (f \circ g^x)(U_y) \subseteq f(U) \Rightarrow f(U)$ is open.

f is final: Let $g : N \to P$ be such that $g \circ f$ is smooth. Since f is surjective, there is an $x \in M$ with $f(x) = y$ for each $y \in N$. As before, by 12.1 $\exists\, g^x \in C^\infty(U_y, M)$. Hence

$$g|_{U_y} = g \circ (f \circ g^x) = (g \circ f) \circ g^x \text{ is smooth,}$$

so g is smooth everywhere. $\square$

13. Fiber bundles

By the Rank-Theorem 11.2, for submersive maps $f : P \to M$ there are charts $\varphi : \mathbb{R}^n \times \mathbb{R}^{m-n} \supseteq W_1 \times W_2 \to P$ and $\psi : W_1 \to M$ so that the chart representation of f is given by $\mathrm{pr}_1 : W_1 \times W_2 \to W_1$. The composition $\Psi := \varphi \circ (\psi^{-1} \times \mathrm{id}) : \operatorname{im}\psi \times W_2 \to W_1 \times W_2 \to \operatorname{im}\varphi$ is then a diffeomorphism such that $f \circ \Psi = \mathrm{pr}_1$:

$$\begin{tikzcd}
P \arrow[d, "f"] & \operatorname{im}\varphi \arrow[l, hook'] \arrow[d, "f|_{\operatorname{im}\varphi}"'] & W_1 \times W_2 \arrow[l, "\cong", "\varphi"'] \arrow[d, "\mathrm{pr}_1"] & \operatorname{im}\psi \times W_2 \arrow[l, "\cong", "\psi^{-1}\times\mathrm{id}"'] \arrow[d, "\mathrm{pr}_1"] \arrow[ll, bend right, "\Psi"'] \\
M & \operatorname{im}\psi \arrow[l, hook'] & W_1 \arrow[l, "\cong", "\psi"'] & \operatorname{im}\psi \arrow[l, "\cong", "\psi^{-1}"'] \arrow[ll, bend left, "\mathrm{id}"]
\end{tikzcd}$$

We will now get to know an even stronger property of mappings f:

13.1 Definition (Fiber bundle).

A smooth map $p : P \to M$ is called fiber bundle $:\Leftrightarrow$ p is locally trivial, i.e. $\forall\ y \in M$ exists an open neighborhood $U \subseteq M$ and a trivialization Ψ of p over U, that is a diffeomorphism $\Psi : U \times F \to p^{-1}(U)$ for some manifold F, such that the following diagram commutes:

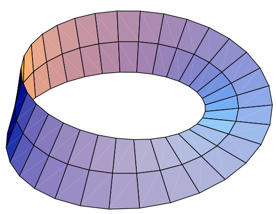
$$\begin{tikzcd}
P \arrow[dr, "p"'] & p^{-1}(U) \arrow[l, hook'] \arrow[dr, "p|_{p^{-1}(U)}"] & & U \times F \arrow[ll, "\cong", "\Psi"'] \arrow[dl, "\mathrm{pr}_1"] \\
& M & U \arrow[l, hook'] &
\end{tikzcd}$$

The manifold F is called typical fiber (On connected components of M, all fibers are diffeomorphic).

A covering map is a fiber bundle p with discrete typical fiber F. This is the smooth version of the definition we used in [**86**, 3.7] and in [**91**, 6.1].

13.2 Examples of fiber bundles.

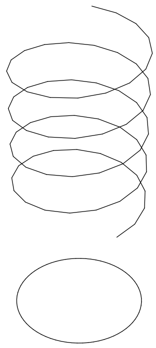
1. For two manifolds M and F, the canonical projection $\mathrm{pr}_1 : M \times F \to M$, $(x, y) \mapsto x$ is a fiber bundle with typical F fiber. Such fiber bundles are called globally trivial (or just trivial, for short).
2. The projection Möb $\to S^1$ of the Möbius strip to its centerline is a fiber bundle with typical fiber $(-1, 1) \cong \mathbb{R}$ and which is not globally trivial (i.e. not diffeomorphic to the cylinder). The restriction of this projection to the boundary of the (closed) Möbiusstrip is up to diffeomorphisms $S^1 \to S^1$, $z \mapsto z^2$, which is a 2-fold covering map but obviously not trivial, since the domain S^1 is connected, hence not diffeomorphic to $S^1 \times \{-1, +1\}$.



3. The Hopf fibration: $S^3 \to S^2$ is a fiber bundle with typical fiber S^1, see $\boxed{3.7}$.

Examples of covering maps are:

4. The map $\mathbb{R} \to S^1$ given by $\varphi \mapsto (\cos\varphi, \sin\varphi)$ is a countable covering map.



5. The following mapping $\mathbb{R} \times (-1, 1) \to \text{Möb}$
$$\begin{pmatrix} \varphi \\ t \end{pmatrix} \mapsto \begin{pmatrix} (1 + t\cos\varphi)\cos(2\varphi) \\ (1 + t\cos\varphi)\sin(2\varphi) \\ t\sin\varphi \end{pmatrix}$$
is a countable covering map. Its factors over the mapping
$$\mathbb{R} \times (-1, 1) \to S^1 \times (-1, 1), \quad (\varphi, t) \mapsto (\cos\varphi, \sin\varphi, t)$$
to a two-fold covering map of the Möbius strip by the cylinder $S^1 \times (-1, 1)$:
$$\begin{tikzcd} & \mathbb{R} \times (-1,1) \arrow[dl] \arrow[dr] & \\ S^1 \times (-1,1) \arrow[rr, dotted] & & \text{Möb} \end{tikzcd}$$
$$(x, y, t) \mapsto \Big((1 + tx)(x^2 - y^2), (1 + tx)2xy, ty\Big).$$

6. $S^n \to \mathbb{P}^n$, $x \mapsto \mathbb{R} \cdot x$ is a two-fold covering map, see Exercise [**86**, 72.53].

7. $S^3 \to SO(3)$ and $S^3 \times S^3 \to SO(4)$ are two-fold covering maps, see 4.8 or Exercise [**86**, 72.66] and [**86**, 72.67].

IV. Vector fields

Ordinary differential equations are described on manifolds by vector fields. In order to be able to speak of the smoothness of these, we need the tangent bundle as a manifold, or better as a vector bundle, and we provide these two things in the first two sections. The next two are devoted to the differential equations and their solutions, the local flows. It then treats the Lie bracket as an obstruction to the commutativity of the local flows of two vector fields. Finally, we generalize flows to integral manifolds of subbundles and prove the Theorem of Frobenius on their existence.

14. Tangent bundle

14.1 Motivation.

We want to treat ordinary differential equations of 1st order on manifolds. For this we first consider the classical case: If a differential equation $x'(t) = f(x(t))$ is given, where $f : U \to \mathbb{R}^n$ with open $U \subseteq \mathbb{R}^n$, then there exists a locally defined differentiable curve $x : (a, b) \to U$ being a solution with initial condition $x(0) = x_0$.

Our aim is to replace U by a manifold M. As solution curve $x : (a, b) \to M$ we should get a differentiable curve in the manifold. Its derivative $x'(t)$ at t is a tangent vector in $T_{x(t)}M$. The function f constituting the differential equation must therefore map points $x \in M$ to tangent vectors at these points:

$$f : M \ni p \mapsto f(p) \in T_pM, \quad \text{i.e.} \quad f : M \to \bigsqcup_{p \in M} T_pM,$$

where $\bigsqcup_{p\in M} T_pM$ denotes the disjoint union of all T_pM with $p \in M$.

14.2 Definition (Tangent bundle).

If M is a manifold, then the tangent space of M is defined by:

$$TM := \bigsqcup_{p\in M} T_pM := \bigcup_{p\in M} \{p\} \times T_pM.$$

On TM, $\pi_M : \{p\} \times T_pM \ni (p, v) \mapsto p \in M$ defines the so-called foot-point mapping. Each smooth $f : M \to N$ induces a mapping $Tf : TM \to TN$, the so-called tangent map of f, defined by $(Tf)(p, v) := (f(p), T_pf(v))$ using the tangent maps $T_pf : T_pM \to T_{f(p)}N$ of f at p. Thus

$$Tf|_{T_pM} = T_pf : T_pM \cong \{p\} \times T_pM \xrightarrow{Tf} \{f(p)\} \times T_{f(p)}N \cong T_{f(p)}N$$

is linear on the fibers $\pi^{-1}(p) = \{p\} \times T_pM \cong T_pM$ of π.

If $f : M \to N$ and $g : N \to P$ are smooth, then the chain rule from 10.4 takes the very simple form

$$T(g \circ f) = Tg \circ Tf,$$

as the following calculation shows:

$$(T(g\circ f))(x,v) = \Big((g\circ f)(x), T_x(g\circ f)(v)\Big)$$
$$\overset{\boxed{10.4}}{=\!=} \Big(g(f(x)), T_{f(x)}g\big((T_x f)(v)\big)\Big)$$
$$(Tg\circ Tf)(x,v) = Tg(Tf(x,v)) = Tg(f(x), T_x f(v))$$
$$\overset{\boxed{10.4}}{=\!=} \Big(g(f(x)), T_{f(x)}g\big((T_x f)(v)\big)\Big)$$

Furthermore, $T\,\mathrm{id}_M = \mathrm{id}_{TM}$ and $(Tf)^{-1} = T(f^{-1})$ for diffeomorphisms f.

14.3 Remarks.

In order to request reasonable properties (in particular differentiability) of $f : M \to TM$, we need a smooth manifold structure on $TM = \bigsqcup_{x\in M} T_x M$.

For the moment, let $M = U \subseteq \mathbb{R}^m$ be open. Then $T_pM = \mathbb{R}^m$ (more precisely, $T_pM \cong \mathbb{R}^m$) and thus

$$TM = \bigcup_{p\in M} \{p\}\times\mathbb{R}^m = M\times\mathbb{R}^m.$$

For a smooth mapping $f : \mathbb{R}^m \supseteq U \to V \subseteq \mathbb{R}^n$ the tangent mapping is $Tf : U\times\mathbb{R}^m \to V\times\mathbb{R}^n$ given by

$$(Tf)(x,v) = (f(x), f'(x)(v)).$$

Let next M be a submanifold of $\mathbb{R}^n$ and let $\varphi : \mathbb{R}^m \supseteq U \to W\cap M$ be a local parameterization. Then

$$TM = \bigcup_{p\in M} \{p\}\times T_pM \subseteq M\times\mathbb{R}^n \subseteq \mathbb{R}^n\times\mathbb{R}^n = \mathbb{R}^{2n}.$$

and $T\varphi : \mathbb{R}^{2m} \supseteq U\times\mathbb{R}^m = TU \to TM$, $(x,v)\mapsto(\varphi(x),\varphi'(x)(v))$ is a local parameterization of TM: It is defined on the open subset TU of $\mathbb{R}^{2m}$ and there clearly C^∞. Furthermore, its derivative at a point $(x,v)\in TU = U\times\mathbb{R}^m$ in direction $(w,h)\in\mathbb{R}^m\times\mathbb{R}^m$ is given by

$$(T\varphi)'(x,v)(w,h) = (\varphi'(x)(w)+0, \varphi''(x)(w,v)+\varphi'(x)(h)).$$

The Jacobi matrix of $T\varphi$ at (x,v) thus is:

$$\begin{pmatrix}\varphi'(x) & 0\\ \varphi''(x)(\_,v) & \varphi'(x)\end{pmatrix}.$$

Since φ is regular, $\varphi'(x)$ is injective and thus the same is true for the Jacobi matrix of $T\varphi$, i.e. $T\varphi$ is regular.

Let $f : M\to N$ smooth and $\varphi : \mathbb{R}^m \to M$ and $\psi : \mathbb{R}^n\to N$ be local parameterizations and, by what we have just shown, $T\varphi$ and $T\psi$ are local parameterizations of TM and TN. The local representation of Tf with respect to these parametrizations is:

$$(T\psi)^{-1}\circ Tf\circ T\varphi = T(\psi^{-1})\circ Tf\circ T\varphi = T(\psi^{-1}\circ f\circ\varphi).$$

Since the local representation $\psi^{-1}\circ f\circ\varphi$ of f is smooth, the same holds for Tf.

Finally, if M is an abstract manifold then we should be able to define a smooth atlas $\{T\varphi : TU = U\times\mathbb{R}^m \to TM\}$ of TM using the charts $\varphi : \mathbb{R}^m \supseteq U \to M$ of M. In fact, the same calculation as just before but for $f = \mathrm{id}_M$ shows that the chart change $(T\psi)^{-1}\circ T\varphi = T(\psi^{-1}\circ\varphi)$ is smooth.

14.4 Lemma (Tangent bundle as fiber bundle).

For each manifold M, the tangent bundle $TM \xrightarrow{\pi} M$ is a fiber bundle.

Proof. We need to find local trivializations of $TM \xrightarrow{\pi} M$. Let $\varphi : U \to M$ a chart for M. Then $\varphi : U \to \varphi(U) \subseteq M$ is a diffeomorphism to an open subset of M, and $T\varphi : U \times \mathbb{R}^m = TU \to TM$ is a chart for TM by 14.3. The image of $T\varphi$ is

$$\begin{aligned}\operatorname{im}(T\varphi) &= \{(x,v) \in TM : x \in \operatorname{im}\varphi =: V,\ v \in T_xM\}\\ &= \{(x,v) \in TM : x \in V\} = \pi_M^{-1}(V).\end{aligned}$$

A trivialization $\Psi := T\varphi \circ (\varphi^{-1} \times \mathbb{R}^m)$ of π over V is now given by the following diagram:

$$\begin{tikzcd}
TM \arrow[d, "\pi"'] & \pi^{-1}(V) \arrow[l, hook'] \arrow[d, "\pi"'] & TU \arrow[l, "\cong", "T\varphi"'] \arrow[d, "\pi"'] \arrow[r, leftrightarrow, "\cong"'] & U \times \mathbb{R}^m \arrow[d, "\mathrm{pr}_1"'] & V \times \mathbb{R}^m \arrow[l, "\cong", "\varphi^{-1}\times\mathbb{R}^m"'] \arrow[d, "\mathrm{pr}_1"'] \arrow[llll, bend right, "\Psi"'] \\
M & V \arrow[l, hook'] & U \arrow[l, "\varphi", "\cong"'] \arrow[r, equal] & U & V \arrow[l, "\varphi^{-1}", "\cong"'] \arrow[llll, bend left, "\mathrm{id}"]
\end{tikzcd}$$

□

Remark.

We have an additional structure on TM because the fibers $T_xM = \pi^{-1}(x)$ are vector spaces and $T_0\varphi : \mathbb{R}^m = T_0\mathbb{R}^m \to T_xM$ is linear.

14.5 Definition (Vector bundle).

A fiber bundle $p : E \to M$ is called a VECTOR BUNDLE (VB, for short) if all the fibers $p^{-1}(x) =: E_x$ are vector spaces and for each $x_0 \in M$ there is an open neighborhood $U \subseteq M$ and a local trivialization Ψ,

$$\begin{tikzcd}
p^{-1}(U) \arrow[dr, "p"'] & & U \times \mathbb{R}^k \arrow[ll, "\Psi", "\cong"'] \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

which is fiber-linear, i.e. $\Psi_x := \Psi(x,.) : \mathbb{R}^k \to E_x$ is linear for each $x \in U$. Such a local trivialization is called vector bundle chart.

A vector bundle $E \to M$ can be conceived as a family $\{E_x : x \in M\}$ of vector spaces, which is parameterized in a certain sense smoothly by M.

14.6 Proposition (The tangent bundle as vector bundle).

The tangent bundle $TM \to M$ of each manifold M is a vector bundle.

Proof. Let $\varphi : \mathbb{R}^m \supseteq U \xrightarrow{\cong} V \subseteq M$ be a local parameterization of M. Then, by 14.4, we get a local trivialization Ψ for TM as the top arrow in the following commutative diagram:

$$\begin{tikzcd}
TM|_V \arrow[d, "\pi"'] & U \times \mathbb{R}^m \arrow[l, "\cong", "T\varphi"'] \arrow[r, "\cong", "\varphi\times\mathrm{id}"'] \arrow[d, "\mathrm{pr}_1"] & V \times \mathbb{R}^m \arrow[d, "\mathrm{pr}_1"] \arrow[ll, bend right, "\Psi"'] \\
V & U \arrow[l, "\varphi", "\cong"'] \arrow[r, "\varphi", "\cong"'] & V \arrow[ll, bend left, "\mathrm{id}"]
\end{tikzcd}$$

Remains to show that $v \mapsto \Psi(x,v)$ of $\mathbb{R}^m \to \{x\} \times T_xM \cong T_xM$ is linear. However, this mapping

$$v \mapsto (x,v) \overset{\varphi^{-1}\times\mathrm{id}}{\longmapsto} (\varphi^{-1}(x),v) \overset{T\varphi}{\longmapsto} (\underbrace{\varphi(\varphi^{-1}(x))}_{=x}, T_{\varphi^{-1}(x)}\varphi\cdot v) \mapsto T_{\varphi^{-1}(x)}\varphi\cdot v$$

is $T_{\varphi^{-1}(x)}\varphi$ and thus clearly linear. □

14.7 Remarks.

1. For vector bundle charts $\psi_U : U \times \mathbb{R}^k \to p^{-1}(U)$ and $\psi_V : V \times \mathbb{R}^k \to p^{-1}(V)$ the vector bundle chart change
$$\psi_V^{-1} \circ \psi_U : (U \cap V) \times \mathbb{R}^k \to p^{-1}(U \cap V) \to (U \cap V) \times \mathbb{R}^k$$
is of the form
$$(x,v) \mapsto \Big(\underbrace{(\mathrm{pr}_1 \circ \psi_V^{-1} \circ \psi_U)(x,v)}_{=x}, \underbrace{(\mathrm{pr}_2 \circ \psi_V^{-1} \circ \psi_U)(x,v)}_{=:\psi_{VU}(x)\cdot v}\Big).$$
The essential component $(\mathrm{pr}_2 \circ \psi_V^{-1} \circ \psi_U) : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ is described by $\psi_{UV} := (\mathrm{pr}_2 \circ \psi_V^{-1} \circ \psi_U)^\vee : U \cap V \to L(k,k)$ (note that $\psi_V^{-1} \circ \psi_U$ is fiber-linear). This mapping ψ_{VU} is called TRANSITION FUNCTION. It has values in $GL(k) \subseteq L(k,k)$ because the inverse to $\psi_{VU}(x)$ is $\psi_{UV}(x)$.

2. In the case of the tangent bundle $TM \to M$, we obtain the transition functions as follows:
$$\begin{aligned}
\psi_i(x,v) &:= (x, T_{\varphi_i{}^{-1}(x)}\varphi_i \cdot v) \quad \Rightarrow \\
\psi_i{}^{-1}(x,w) &:= (x, (T_{\varphi_i{}^{-1}(x)}\varphi_i)^{-1} \cdot w) \quad \Rightarrow \\
(x, \psi_{i,j}(x)(v)) &:= (\psi_i{}^{-1} \circ \psi_j)(x,v) = \psi_i{}^{-1}\left(x, T_{\varphi_j{}^{-1}(x)}\varphi_j \cdot v\right) \\
&= \left(x, (T_{\varphi_i{}^{-1}(x)}\varphi_i)^{-1} \cdot T_{\varphi_j{}^{-1}(x)}\varphi_j \cdot v\right) \\
&= \left(x, T_{\varphi_j{}^{-1}(x)}(\varphi_i{}^{-1} \circ \varphi_j) \cdot v\right) \quad \Rightarrow \\
\psi_{i,j}(x) &= T_{\varphi_j{}^{-1}(x)}(\varphi_i{}^{-1} \circ \varphi_j) = (\varphi_i{}^{-1} \circ \varphi_j)'(\varphi_j{}^{-1}(x)) \quad \Rightarrow \\
\psi_{i,j} &= (\varphi_i{}^{-1} \circ \varphi_j)' \circ \varphi_j{}^{-1}.
\end{aligned}$$
So these transition functions are essentially the derivatives of the chart changes $\varphi_i{}^{-1} \circ \varphi_j$ for M.

3. The transition functions of vector bundles satisfy the cozykel equations:
$$\psi_{U_3U_2}(x) \circ \psi_{U_2U_1}(x) = \psi_{U_3U_1}(x) \text{ for all } x \in U_1 \cap U_2 \cap U_3$$
$$\psi_{UU}(x) = \mathrm{id}_{\mathbb{R}^n} \text{ for all } x \in U$$

4. By construction, the mapping $\hat\psi_{UV} : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$, $\hat\psi_{VU} : (x,v) \mapsto \psi_{VU}(x) \cdot v$, is smooth. We now claim that its smoothness is equivalent to $\psi_{VU} : U \cap V \to GL(k) \subseteq L(k,k)$ being smooth. To prove this, we use the smooth (bilinear) evaluation map $\mathrm{ev} : L(k,k) \times \mathbb{R}^k \to \mathbb{R}^k$, $(A,v) \mapsto A \cdot v$.
($\Leftarrow$) is valid since
$$\hat\psi_{VU} : (U \cap V) \times \mathbb{R}^k \xrightarrow{\psi_{VU}\times\mathbb{R}^k} L(k,k) \times \mathbb{R}^k \xrightarrow{\mathrm{ev}} \mathbb{R}^k$$
($\Rightarrow$) We have that $\psi_{VU} : U \cap V \to L(k,k)$ is C^∞, provided $ev_y \circ \psi_{VU}$ is smooth $\forall\, y \in \mathbb{R}^k$, where $\mathrm{ev}_y : L(k,k) \to \mathbb{R}^k$ is the mapping $T \mapsto T(y)$: This is the case, since
$$(ev_y \circ \psi_{VU})(x) = \psi_{VU}(x) \cdot y = \hat\psi_{VU}(x,y).$$

5. Now let M be a manifold and $p : E \to M$ a map defined on a set E such that a family of fiber-preserving (i.e., $p \circ \psi_U = \mathrm{pr}_1$) bijective mappings $\psi_U : U \times \mathbb{R}^k \to p^{-1}(U)$ exists, where the corresponding U form an open covering of M and the associated transition functions $\psi_{VU} : U \cap V \to GL(k)$ are well-defined and smooth.
Then we can supply E with a unique manifold structure, such that $p : E \to M$ becomes a vector bundle with vector bundle charts ψ_U.
As parameterizations of E we can use $\psi_U \circ (\varphi \times \mathbb{R}^k)$, where the ψ_U are the given fiber-preserving mappings and φ are the charts of M.
$$\begin{tikzcd} E \arrow[d, "p"'] & p^{-1}(U) \arrow[l, hook'] \arrow[d, "p"] & U \times \mathbb{R}^k \arrow[l, "\psi_U"] \arrow[d, "\mathrm{pr}_1"] & W \times \mathbb{R}^k \arrow[l, "\varphi\times\mathbb{R}^k"] \arrow[r, hook] \arrow[d, "\mathrm{pr}_1"] & \mathbb{R}^m \times \mathbb{R}^k \arrow[d, "\mathrm{pr}_1"] \\ M & U \arrow[l, hook'] \arrow[r, equal] & U & W \arrow[l, "\varphi"] \arrow[r, hook] & \mathbb{R}^m \end{tikzcd}$$
The chart change mapping of E are then
$$(\psi_V \circ (\varphi_2 \times \mathbb{R}^k))^{-1} \circ (\psi_U \circ (\varphi_1 \times \mathbb{R}^k)) = (\varphi_2^{-1} \circ \varphi_1, \hat{\psi}_{VU} \circ (\varphi_1 \times \mathbb{R}^k)).$$
By construction, the ψ_U are fiber bundle charts and we can turn the fibers E_x into vector spaces by making the ψ_U fiber-linear.

6. From 6.7 we know that any manifold can be recovered from their chart changes. For transition functions of a VB, we have a similar situation: Let $\mathcal{U}$ be an open covering of M. A coherent family of transition functions, i.e. a family of smooth functions $\psi_{VU} : U \cap V \to GL(k)$ for $U, V \in \mathcal{U}$ that satisfies the cozykel equations (3) defines a vector bundle being unique to isomorphisms.
In order to prove that, we define $E_x := \{(U, w) : x \in U \in \mathcal{U},\ w \in \mathbb{R}^k\}/\sim$, where
$$(U, w) \sim (V, w') \Leftrightarrow w' = \psi_{VU}(x) \cdot w.$$
Then E_x is a vector space with $\psi_U(x) : w \mapsto [(U, w)]$ being a vector space isomorphism $\mathbb{R}^k \to E_x$. The disjoint union
$$E := \bigsqcup_{x \in M} E_x := \bigcup_{x \in M} (\{x\} \times E_x)$$
is a vector bundle over M with the foot-point map $p : E \ni (x, v) \mapsto x \in M$, because $E|_U := p^{-1}(U) = \bigsqcup_{x \in U} E_x \cong U \times \mathbb{R}^k$ via the trivialization ψ_U defined by $\psi_U(x, w) := (x, [(U, w)])$. For the chart changing we have:
$$\begin{aligned} &(\psi_V^{-1} \circ \psi_U)(x, w) = (x, w') \Leftrightarrow \\ &\qquad \Leftrightarrow (x, [(V, w')]) = \psi_V(x, w') = \psi_U(x, w) = (x, [(U, w)]), \end{aligned}$$
hence $w' = \psi_{VU}(x) \cdot w$.

14.8 Definition (Vector bundle homomorphisms).

If $p : V \to M$ and $q : W \to N$ are vector bundles, then a smooth function $\bar{f}$ is called a VECTOR BUNDLE HOMOMORPHISM over a smooth function $f : M \to N$, if the following diagram commutes and $\bar{f}_x : V_x \to W_{f(x)}$ is linear $\forall\, x \in M$.
$$\begin{tikzcd} V \arrow[r, "\bar{f}"] \arrow[d, "p"'] & W \arrow[d, "q"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

14.9 Definition (Vector subbundle).

Let $p : E \to M$, $q : F \to M$ be two vector bundles, so that F_x is linear subspace of $E_x\ \ \forall\, x \in M$. Then, $q : F \to M$ is called a VECTOR SUBBUNDLE of $p : E \to M$ if there is a VB atlas $\{\psi_U : U \times \mathbb{R}^k \to p^{-1}(U)\}$ for E, which maps $U \times \mathbb{R}^l \subseteq U \times \mathbb{R}^k$ onto $F|_U$ for some $l \le k$, i.e. $\psi_U : U \times \mathbb{R}^k \cong E|_U = p^{-1}(U)$ and $\psi|_{(U\times\mathbb{R}^l)} : U \times \mathbb{R}^l \cong F|_U = q^{-1}(U)$.

$$
\begin{tikzcd}
& F \arrow[dl, hook'] \arrow[drr, "q"] & & q^{-1}(U) \arrow[ll, hook'] \arrow[dl, hook'] \arrow[drr] & & U\times\mathbb{R}^l \arrow[ll, two heads, tail, "\psi_U|_{U\times\mathbb{R}^l}", "\cong"'] \arrow[dl, hook'] \arrow[dd, "\mathrm{pr}_1"] \\
E \arrow[ddr, "p"'] & & p^{-1}(U) \arrow[ll, hook'] \arrow[dr] \arrow[ddr] & & U\times\mathbb{R}^k \arrow[ll, two heads, tail, "\psi_U", "\cong"'] \arrow[ddl] & \\
& & & M \arrow[dll, equal] & & U \arrow[ll, hook', "\mathrm{pr}_1"'] \arrow[dll, equal] \\
& M & & U \arrow[ll, hook'] & &
\end{tikzcd}
$$

This means that $\psi_U(x)$ maps the “constant” subspace $\mathbb{R}^l$ precisely onto F_x.

15. Vector fields

15.1 Definition (Sections of bundles).

A SECTION σ of a vector bundle (or fiber bundle) $E \xrightarrow{p} M$ is a mapping $\sigma : M \to E$ that satisfies $p \circ \sigma = \mathrm{id}_M$. The sections of the tangent bundle $TM \to M$ are called VECTOR FIELDS (VF, for short) on the manifold M.

The space of all smooth sections is denoted

$$C^\infty(M \xleftarrow{p} E) := \{\sigma \in C^\infty(M, E) : p \circ \sigma = \mathrm{id}\}$$

and also $\Gamma(E \xrightarrow{p} M)$ or $\Gamma(E)$ for short, if the base space M and p is clear.

The set of all smooth vector fields on M is also denoted

$$\mathfrak{X}(M) := C^\infty(M \xleftarrow{\pi_M} TM).$$

Sections can be added pointwise and they can be multiplied pointwise by real-valued functions f on M. Thus, $C^\infty(M \xleftarrow{p} E)$ is a vector space and even a module over $C^\infty(M, \mathbb{R})$, that is a “vector space” over the ring $C^\infty(M, \mathbb{R})$ (instead of over a field), i.e.:

$$
\begin{aligned}
(f + g)\xi &= f\xi + g\xi, & f(\xi + \eta) &= f\xi + f\eta, \\
(f \cdot g)\xi &= f(g \cdot \xi), & 1 \cdot \xi &= \xi
\end{aligned}
$$

We want to do calculations with vector fields or more generally with sections of vector bundles. For this we need local representations.

15.2 Local description of sections.

Locally, a section s is given by a map $\bar{s} = (s^1, \dots, s^k) : M \to \mathbb{R}^k$ of the basis M into the typical fiber $\mathbb{R}^k$.

$$s(x) \longleftarrow\!\shortmid\; \psi_U^{-1}(s(x)) = (x, \bar{s}(x))$$

$$\begin{tikzcd}
E|_U \arrow[dr, "p"] & & U \times \mathbb{R}^k \arrow[ll, "\psi_U"', "\cong"] \arrow[dl, "\mathrm{pr}_1"'] \\
& U & \\
& U \arrow[u, "\mathrm{id}"] \arrow[uul, bend left, "s"] \arrow[uur, bend right, "(\mathrm{id},\bar{s})"'] &
\end{tikzcd}$$

In particular, for the tangent bundle we get: The vector fields ξ correspond locally to maps $(\xi^i_\varphi)_{i=1}^m : M \to \mathbb{R}^m$ whose form depends on the choice of the chart φ:

$$\begin{tikzcd}
TM|_{\mathrm{im}\,\varphi} & & \mathrm{im}\,\varphi \times \mathbb{R}^m \arrow[ll, "T\varphi\circ(\varphi^{-1}\times\mathbb{R}^m)"', "\cong"] \\
& \mathrm{im}(\varphi) \arrow[ul, bend left, "\xi|_{\mathrm{im}\,\varphi}"] \arrow[ur, bend right, "(\mathrm{id},(\xi^i_\varphi)_{i=1}^m)"'] &
\end{tikzcd}$$

We have seen in 10.8 that for local coordinates $(u^1, \dots, u^m) = \varphi^{-1}$ on M the derivations $\big(\partial^\varphi_1|_p = \frac{\partial}{\partial u^1}|_p, \dots, \partial^\varphi_m|_p = \frac{\partial}{\partial u^m}|_p\big)$ form a basis of T_pM for each p in the domain of definition of the chart φ and the isomorphism $T\varphi \circ (\varphi^{-1} \times \mathbb{R}^m)$ maps the standard basis (x, e_i) to $\frac{\partial}{\partial u^i}|_x$. Each vector field ξ can thus be written on U as $\xi = \sum_{i=1}^m \xi^i_\varphi \partial^\varphi_i$, where ∂^φ_i are the vector fields $p \mapsto \partial^\varphi_i|_p = \frac{\partial}{\partial u^i}|_p$. The subscript φ of the components ξ^i_φ of ξ with respect to the basis ∂^φ_i indicates the dependence of these components on the basis, which in turn depends on φ. In most cases, however, we will omit this index as is commonly done. We can calculate the components ξ^i by applying $\xi = \sum_i \xi^i \frac{\partial}{\partial u^i}$ to the local coordinate function u^j: $\xi(u^j) = \big(\sum_i \xi^i \frac{\partial}{\partial u^i}\big)(u^j) = \sum_i \xi^i \delta^j_i = \xi_j$. So $\xi = \sum_i \xi(u^i) \frac{\partial}{\partial u^i}$.

15.3 Corollary.

A vector field ξ is smooth if and only if all components ξ^i_φ are smooth.

Proof. This follows immediately from the fact that the local sections $\frac{\partial}{\partial u^i}$ are smooth, which in turn follows from the diagram

$$\begin{tikzcd}
TM|U \arrow[d, "\pi|_U"'] & V \times \mathbb{R}^m \arrow[l, "T\varphi"'] \arrow[r, "\varphi\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_1"] & U \times \mathbb{R}^m \arrow[d, "\mathrm{pr}_1"] \\
U & V \arrow[l, "\varphi"'] \arrow[r, "\varphi"] & U
\end{tikzcd}$$

because the section $\frac{\partial}{\partial u^i}$ on the far left corresponds to the constant section $x \mapsto (x, e^i)$ on the right. $\square$

15.4 Examples of globally (none-)trivial vector bundles.

1. The tangent bundle of $S^n \subseteq \mathbb{R}^{n+1}$ as a sub bundle of $T\mathbb{R}^{n+1}|_{S^n} = S^n \times \mathbb{R}^{n+1}$ is given by $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : \langle x, v\rangle = 0\}$. In particular, $TS^1 = \{(x, y, u, v) : x^2 + y^2 = 1,\ xu + yv = 0\}$, i.e. $TS^1 \cong S^1 \times \mathbb{R}$ using $(x, y, t) \mapsto$

$(x, y, -ty, tx)$. Thus, the tangent bundle of the S^1 is trivial, and indeed it is the cylinder.

2. The projection of the Möbius strip onto its center line S^1 is a VB whose fiber is $(-1, 1) \cong \mathbb{R}$.
   However, this VB is not trivial, otherwise we would have a global trivialization ψ:

$$\begin{tikzcd} S^1 \times \mathbb{R} \arrow[rr, "\psi", "\cong"'] \arrow[dr] & & \text{Möb} \arrow[dl] \\ & S^1 & \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[rr, "\psi(\_,1)"] \arrow[dr] & & \text{Möb} \arrow[dl] \\ & S^1 & \end{tikzcd}$$

   with $\psi(S^1, 1) \cap S^1 := \psi(S^1, 1) \cap \psi(S^1, 0) = \emptyset$. But there is no such mapping.

3. The tangent bundle TS^2 of the sphere. In order to answer the question of whether it is also trivial, let us assume that there is a trivialization of $\psi : S^2 \times \mathbb{R}^2 \to TS^2$. With $\psi(\_, e_1)$, one would have a continuous mapping which maps each $x \in S^2$ to a nonvanishing tangent vector, but such a map does not exist (by the Hairy Ball Theorem 29.11).

4. Since S^3 carries a smooth group structure, $TS^3 \cong S^3 \times \mathbb{R}^3$ is a trivial vector bundle via the tangent map of the left multiplication, see 15.9.3.

15.5 Definition (Linear independent vector fields).

A family of vector fields $\{\xi_1, \dots, \xi_k\}$ on M is said to be LINEARLY INDEPENDENT (everywhere) if $\{\xi_i|_p : 1 \le i \le k\}$ is linearly independent in T_pM for each $p \in M$.

15.6 Remark (Parallelizable manifolds).

A manifold M is called PARALLELIZABLE if its tangent bundle is trivial. This is the case iff it has $m := \dim M$ linearly independent vector fields everywhere: In fact, if TM is trivial, that is

$$\begin{tikzcd} TM \arrow[dr, "\pi_M"'] & & M \times \mathbb{R}^m \arrow[ll, dotted, "\cong"', "\psi"] \arrow[dl, "\mathrm{pr}_1"] \\ & M & \end{tikzcd}$$

then the $\xi_i : x \mapsto \psi(x, e_i)$ for $1 \le i \le m$ are linear independent vector fields. Conversely, $\psi(x, (v^i)_{i=1}^m) := \sum_i v^i \xi_i(x)$ defines a trivialization of TM if $\{\xi_i\}_{i=1}^m$ is linearly independent.

For example, S^1 has a linearly independent vector field because its tangent bundle is trivial. The following theorem provides information on how many linearly independent vector fields exist on the higher-dimensional spheres ("how trivial their tangent bundle is").

15.7 Theorem (Linear independent vector fields on the spheres).

On S^n, m linearly independent vector fields can be choosen if and only if $n + 1 = 2^{4a+b} \cdot c$ with $a \in \mathbb{N}_0$, $b \in \{0, 1, 2, 3\}$, odd c, and $m + 1 \le 8 \cdot a + 2^b$.

Without proof. The result was obtained by [36], [68] and [1].

The number of linearly independent vector fields on the spheres is related to the structure of certain algebras:

15.8 Proposition.

Let $b : \mathbb{R}^{k+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilinear mapping, such that

1. *$b(v,x) = 0$ implies $v = 0$ or $x = 0$ (zero diviser free),*
2. *$\exists\ v_0 \in \mathbb{R}^{k+1}$, such that $b(v_0, x) = x\ \forall\ x \in \mathbb{R}^{n+1}$ (left unit).*

Then k linearly independent vector fields exist on the S^n.

Proof. If $v \in \mathbb{R}^{k+1}$, then the mapping $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, given by $x \mapsto b(v,x)$, is linear. Using the radial projection $\rho : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ and the canonical inclusion $\mathrm{incl} : S^n \to \mathbb{R}^{n+1}$, a smooth vector field $\xi_v : S^n \to TS^n$ can be defined as follows: $\xi_v = T\rho \circ b(v,.) \circ \mathrm{incl}$. If $\{v_0, v_1, \dots v_k\}$ are linearly independent in $\mathbb{R}^{k+1}$, then $\{\xi_{v_1}, \dots \xi_{v_k}\}$ are linearly independent everywhere: Let

$$0 = \sum_{i=1}^k \lambda_i \xi_{v_i}|_x = \sum_{i=1}^k \lambda_i T_x\rho(b(v_i,x)) = T_x\rho\Big(\sum_{i=1}^k \lambda_i b(v_i,x)\Big)$$

The kernel of $T_x\rho$ is the line created by x in $\mathbb{R}^{n+1}$ $\Rightarrow$

$$\Rightarrow \sum_{i=1}^k \lambda_i b(v_i,x) = -\lambda_0 x = -\lambda_0 b(v_0,x) \text{ for some } \lambda_0 \in \mathbb{R}$$

$$\Rightarrow b\Big(\sum_{i=0}^k \lambda_i v_i, x\Big) = 0 \xRightarrow{x \neq 0} \sum_{i=0}^k \lambda_i v_i = 0 \xRightarrow{v_i \text{ l.u.}} \lambda_i = 0\ \forall\ i. \qquad \square$$

15.9 Corollary.

1. *The spheres S^1, S^3, and S^7 are parallelizable:*
   As bilinear functions $b : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, which fulfill the properties (i), (ii) of 15.8, the following $\mathbb{R}$-algebra multiplications can be used:

   $$n = 1 : \quad \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$
   $$n = 3 : \quad \mathbb{H} \times \mathbb{H} \to \mathbb{H}$$
   $$n = 7 : \quad \mathbb{O} \times \mathbb{O} \to \mathbb{O}\,, \text{ where } \mathbb{O} \cong \mathbb{R}^8 \text{ are the Cayley numbers (also called octaves or octonions).}$$

   By 15.7 these are the only parallelizable spheres, because $2^{4a+b} \cdot c = 8a + 2^b$ implies $c = 1$ and furtheron $a = 0$, i.e. $n = 2^b - 1$ for $b \in \{0,1,2,3\}$.
2. *If n is odd then S^n has a non-vanishing vector field. For $b : \mathbb{R}^2 \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, where $n+1 = 2k$ is for $k \in \mathbb{N}$, scalar multiplication with complex numbers can be used:*

   $$\mathbb{C} \times \mathbb{C}^k \to \mathbb{C}^k, \qquad (\lambda; \lambda^1, \dots, \lambda^k) \mapsto (\lambda \cdot \lambda^1, ., \lambda \cdot \lambda^k)$$
3. *If G is a Lie group with neutral element $e \in G$, then $TG \cong G \times T_eG$ via the isomorphism given by*

   $$\xi \mapsto (\pi(\xi), TL_{\pi(\xi)^{-1}} \cdot \xi) = (\pi(\xi), T\mu(0_{\pi(xs)^{-1}}, \xi)),$$

   for details see [86, 67.2].

16. Ordinary differential equations of first order

16.1 Definition (Integral curve).

Let $\xi \in \mathfrak{X}(M)$ and I be an open interval with $0 \in I$. Then $c : I \to M$ is called INTEGRAL CURVE (or solution curve) of the vector field ξ through p :⇔

$$c(0) = p \quad \text{ and } \quad c'(t) = \xi_{c(t)} \text{ for } t \in I.$$

We will use the following classical existence and uniqueness theorem for the solution of ordinary differential equations in vector spaces.

16.2 Theorem on ordinary differential equations.

Let E be a Euclidean vector space (or more generally a Banach space) and let $f : \mathbb{R} \times E \to E$ be a smooth function. Then an open interval $I \subseteq \mathbb{R}$ exists around 0 and an open ball U around 0 in E, such that for all $x \in U$ there is a unique solution $c_x : I \to E$ of the ordinary differential equation

$$c_x'(t) = f(t, c_x(t)) \text{ with } c_x(0) = x.$$

Furthermore, $(t, x) \mapsto c_x(t)$ is smooth as mapping $I \times U \to E$.

Without proof. See, e.g. [**81**, 6.2.15] or [**35**, 10.8.1 und 10.8.2].

Now we can prove the following global version for manifolds.

16.3 Theorem about ordinary differential equations on manifolds.

If $\xi \in \mathfrak{X}(M)$, then:

1. *For each $p \in M$, there exists a uniquely determined maximum integral curve $c_p : (t_-^p, t_+^p) \to M$ to ξ through p (i.e. any other integral curve is a restriction of c_p).*
2. *If $t_+^p < \infty$, then $\lim_{t \nearrow t_+^p} c(t) = \infty$ follows, i.e. for any compact set of $K \subseteq M$, $c(t)$ is not in K for all t sufficiently close to t_+^p.*
3. *The set $U = \{(t, p);\ t_-^p < t < t_+^p\} \subseteq \mathbb{R} \times M$ is an open neighborhood of $\{0\} \times M$. The map $\mathrm{Fl}^\xi : U \to M$, defined by $\mathrm{Fl}^\xi(t, p) := c_p(t)$, is C^∞ and is called the local flow of the vector field. If $q := \mathrm{Fl}^\xi(s, p)$ exists, then $\mathrm{Fl}^\xi(t+s, p)$ exists if and only if $\mathrm{Fl}^\xi(t, q)$ exists, and the two coincide. This equation is also called "one-parameter subgroup property", because for globally defined Fl^ξ it says: $(\mathrm{Fl}^\xi)^\vee : \mathbb{R} \to \mathrm{Diff}(M)$ is a group homomorphism.*

Proof. ($\boxed{1}$) **Local existence and uniqueness:** Without loss of generality, $U \subseteq \mathbb{R}^m$ is open and $\xi : U \to \mathbb{R}^m$ is smooth. We are searching for a c with $c'(t) = \xi_{c(t)}$ and $c(0) = x$. This is an ordinary differential equation whose local solutions exists by $\boxed{16.2}$ and are unique because ξ is locally Lipschitz. It is C^∞, since ξ is smooth.

Global existence and uniqueness: Let c_1, c_2 be two integral curves. The set $\{t \geq 0 : c_1(t) = c_2(t)\}$ is a closed subset of $\operatorname{dom} c_1 \cap \operatorname{dom} c_2$. Suppose it is not the whole set, then there is a t in the difference. Put $t_0 := \inf\{0 < t \in \operatorname{dom} c_1 \cap \operatorname{dom} c_2 : c_1(t) \neq c_2(t)\}$. Clearly $c_1(t_0) = c_2(t_0)$. Now, however, $t \mapsto c_1(t_0 + t)$ and $t \mapsto c_2(t_0 + t)$ are integral curves through $c_1(t_0) = c_2(t_0)$ and thus coincide locally. This is a contradiction to the property of the infimum. Thus, $c_p := \bigcup\{c : c \text{ is integral curve through } p\}$ is the well-defined uniquely determined maximal integral curve through p. We put $(t_-^p, t_+^p) := \operatorname{dom} c_p$.

($\boxed{3}$) Because of $\boxed{1}$, $\{0\} \times M \subset U$ and $\mathrm{Fl}^\xi(0, p) = c_p(0) = p$.
One-parameter subgroup property: Let $q := \mathrm{Fl}(s, p)$ exist, i.e. $t_-^p < s < t_+^p$, since the maximal integral curve $r \mapsto \mathrm{Fl}(r, p)$ with initial value p is defined for $t_-^p < r < t_+^p$. The maximal integral curve $t \mapsto \mathrm{Fl}(t, \mathrm{Fl}(s, p))$ with initial value q is defined for $t_-^q < t < t_+^q$. For t with $t_-^p < t + s < t_+^p$ also $t \mapsto \mathrm{Fl}(t + s, p)$ is a solution with initial value $q = \mathrm{Fl}(s, p)$. So because of the maximality and uniqueness of $t \mapsto \mathrm{Fl}(t, q)$ we get equality and $t_-^q \leq t_-^p - s < -s < t_+^p - s \leq t_+^q$. In particular, $\mathrm{Fl}(-s, q)$ exists and agrees with $\mathrm{Fl}(-s + s, p) = p$. For symmetry reasons, it follows

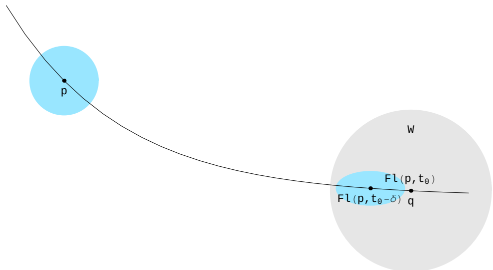
that $t^p_- \le t^q_- + s$ and $t^q_+ + s \le t^p_-$. Together this results in $t^p_\pm = t^q_\pm + s$ and thus $\mathrm{Fl}(t+s,p)$ exists exactly when $\mathrm{Fl}(t,q)$ exists and the two coincide.

We now show that $U \subseteq \mathbb{R} \times M$ is open and Fl is C^∞ on it: For $p \in M$ let

$$I := \Big\{t' \in \big[0, t^p_+\big) : \text{ Fl is locally around } [0,t'] \times \{p\} \text{ defined and smooth}\Big\}.$$

We indirectly show that $I = [0, t^+_p)$ (and analogous for t^p_-):
Suppose $I \subset \big[0, t^p_+\big)$. Let $t_0 := \inf(\big[0, t^p_+\big) \setminus I)$ and $q := \mathrm{Fl}(t_0, p)$. For $p \in M$ there is by 16.2 an open neighborhood of $(0,p) \in \mathbb{R} \times M$ on which the flow Fl is defined and smooth, thus $t_0 > 0$.

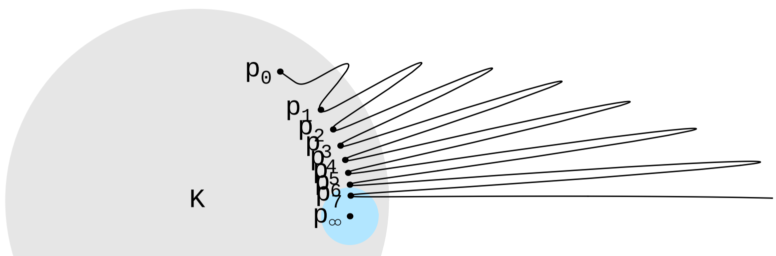
Furthermore, Fl is smooth on a neighborhood $(-\varepsilon, \varepsilon) \times W$ of $(0,q)$, and because of the continuity of $t \mapsto \mathrm{Fl}(t,p)$ at t_0, a $0 < \delta < \varepsilon$ exists such that $\mathrm{Fl}(t_0 - \delta, p)$ is contained in W. By construction of t_0 the flow Fl is smooth on a neighborhood of $[0, t_0 - \delta] \times \{p\}$. Thus, $x \mapsto \mathrm{Fl}(t_0 - \delta, x)$ maps a neighborhood of p smoothly into W, and hence the composition $(s,x) \mapsto \mathrm{Fl}(s, \mathrm{Fl}(t_0 - \delta, x))$ is smooth on a neighborhood of $[0,\delta] \times \{p\}$. Because of the one-parameter subgroup property, $\mathrm{Fl}(s, \mathrm{Fl}(t_0 - \delta, x)) = \mathrm{Fl}(s + t_0 - \delta, x)$, i.e. Fl is smooth locally around $[t_0 - \delta, t_0] \times \{p\}$. Overall, Fl is smooth on a neighborhood of $([0, t_0 - \delta] \cup [t_0 - \delta, t_0]) \times \{p\} = [0, t_0] \times \{p\}$, and thus containing a neighborhood of t_0 in I, a contradiction to the assumption.



(2) Let $K \subseteq M$ compact. Suppose $t_n \to t^p_+ < \infty$ exist, such that $p_n := c_p(t_n) \in K$ for all n. Without loss of generality $p_n \to p_\infty \in K$ holds (because K is compact). By 3, a $\delta > 0$ exists such that the flow $\mathrm{Fl}(t,q)$ is well-defined for $|t| \le \delta$ and q near p_∞. For sufficiently large n let p_n be such values for q, i.e. $\mathrm{Fl}(t, p_n)$ is well-defined for $|t| < \delta$. On the other hand:

$$\mathrm{Fl}(t, p_n) = \mathrm{Fl}(t, c_p(t_n)) = \mathrm{Fl}(t, \mathrm{Fl}(t_n, p)) = \mathrm{Fl}(t + t_n, p) = c_p(t + t_n).$$

So $c_p(s)$ is well-defined not only for $0 \le s < t^p_+$ but also for $s = t + t_n$ with $|t| < \delta$ and n sufficiently large, that is $t_n - \delta < s < t_n + \delta$. Let n be so large that $t_n > t^p_+ - \delta$. Then c_p is well-defined on $[0, t^p_+) \cup (t_n - \delta, t_n + \delta) \supseteq [0, t^p_+]$, a contradiction to the assumption that the solution curves are defined only up to t^p_+.



□

16.4 Example (Exponential mapping).

For $T \in L(n,n)$, matrix multiplication $T_* : S \mapsto T \circ S$ with T from left defines a vector field on $L(n,n)$. We are searching for the solution curve $c : \mathbb{R} \to L(n,n)$, which msatisfies $c'(t) = T_*(c(t)) := T \circ c(t)$ with given $c(0) = S \in L(n,n)$. Define

$$\exp(T) := \sum_{k=0}^{\infty} \frac{1}{k!} T^k$$

and show that the series is absolutely convergent. The solution of the above differential equation with initial value S is then $c(t) = \exp(tT) \circ S$, and the global flow is $\mathrm{Fl}(t,S) = \exp(tT) \circ S$, see Exercise [**86**, 72.50].

16.5 Definition (Complete vector fields).

A vector field $\xi \in \mathfrak{X}(M)$ is called complete if Fl^ξ is defined globally (i.e. on $\mathbb{R} \times M$).

16.6 Remarks.

1. From 16.3.2 follows directly:
   If M is compact, then each vector field is complete.

2. If M has a non-compact connected component, then there are incomplete vector fields, for example: $M := \mathbb{R}$, $\xi(x) := 1 + x^2$, i.e. $c'(t) = 1 + c(t)^2$. For the initial value $c(0) = 0$, the solution $c(t) = \tan(t)$ is then defined only for $t \in (-\pi/2, \pi/2)$.

3. Let $M := \mathbb{R}^2$, $\xi(x,y) := y \frac{\partial}{\partial x}$ and $\eta(x,y) := (x^2/2) \frac{\partial}{\partial y}$. We claim that ξ and η are complete:

$$\begin{aligned}
&\mathrm{Fl}^\xi(t;x,y) = (x+ty, y),\\
&\mathrm{Fl}^\eta(t;x,y) = (x, y + tx^2/2), \quad \text{because}\\
&\frac{d}{dt}\mathrm{Fl}^\xi(t;x,y) = (y,0) = y \cdot \tfrac{\partial}{\partial x} + 0 \cdot \tfrac{\partial}{\partial y} = \xi(Fl(t;x,y)) \text{ and analog for } \eta.
\end{aligned}$$

   But $\xi + \eta$ is not complete: Let $c(t) = (x(t), y(t))$ be a solution curve of $(\xi + \eta)|_{(x,y)} = y \frac{\partial}{\partial x} + \frac{x^2}{2} \frac{\partial}{\partial y}$. Then $x'(t) = y(t)$ and $y'(t) = x(t)^2/2$, hence $\frac{d}{dt} x'(t)^2 = 2x'(t)x''(t) = 2x'(t)y'(t) = x'(t)x(t)^2 = \frac{d}{dt} x(t)^3/3 \Rightarrow x'(t)^2 = x(t)^3/3 + C$. Solving the differential equation by separation of variables for the initial value $y_0^2 = x_0^3/3$ with $x_0 > 0$ gives $C = 0$ and hence $x(t) = \mathrm{pr}_1(\mathrm{Fl}^{\xi+\eta}(t;x_0,y_0)) = \frac{12x_0}{12 - 4\sqrt{3x_0}t + x_0 t^2}$ is not globally defined.

4. Let $\mathrm{Fl}^\xi_t(p) := \mathrm{Fl}^\xi(t,p)$. Because for small t the flow Fl^ξ_t exists in an open neighborhood of p and Fl^ξ_{-t} exists in an open neighborhood of $\mathrm{Fl}^\xi_t(p)$, we get by the 1-parameter subgroup property that locally $(\mathrm{Fl}^\xi_t)^{-1} = \mathrm{Fl}^\xi_{-t}$ for all small t. So the flow Fl^ξ_t is a local diffeomorphism.

17. Lie bracket

In 10.6 we saw that we identified T_pM with $\mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R})$. Namely, for local coordinates $(u^1,\dots,u^m)$, the action of a tangent vector $v = \sum_i v^i \frac{\partial}{\partial u^i}\big|_p \in T_pM$ on

$f \in C^\infty(M, \mathbb{R})$ was given by:

$$v(f) = \Big(\sum_i v^i \tfrac{\partial}{\partial u^i}|_p\Big)(f) = \sum_i v^i \cdot \tfrac{\partial}{\partial u^i}|_p(f) \text{ and in particular}$$
$$v(u^j) = \sum_i v^i \cdot \tfrac{\partial}{\partial u^i}|_p(u^j) = v^j, \text{ because}$$
$$\tfrac{\partial}{\partial u^i}|_p(u^j) = \partial_i(u^j \circ \varphi)(\varphi^{-1}(p)) = \partial_i(\mathrm{pr}_j)(\varphi^{-1}(p)) = \delta_i^j.$$

We now want to see what happens when we vary the point p, hence we consider the following mappings:

17.1 Proposition (Vector fields as derivations).

There is a bilinear mapping

$$\mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}),$$
$$(\xi, f) \mapsto \xi \cdot f = \xi(f) \Big(: p \mapsto \xi_p(f) \in \mathbb{R}\Big).$$

This bilinear mapping induces an $\mathbb{R}$ linear isomorphism of $\mathfrak{X}(M)$

$$\mathrm{Der}(C^\infty(M, \mathbb{R})) := \Big\{\partial \in L(C^\infty(M, \mathbb{R})) : \partial(f \cdot g) = \partial(f) \cdot g + f \cdot \partial(g)\Big\}.$$

In addition: $(f \cdot \xi) \cdot g = f \cdot (\xi \cdot g)$, i.e. this isomorphism is even $C^\infty(M, \mathbb{R})$-linear, where $\mathrm{Der}(C^\infty(M, \mathbb{R}))$ is made into a module over the commutative algebra $C^\infty(M, \mathbb{R})$ by $(f \cdot \partial)(g) := f \cdot \partial(g)$. Note that $\xi \cdot f \in C^\infty(M, \mathbb{R})$ whereas $f \cdot \xi$ is the pointwise product in $\mathfrak{X}(M)$.

Proof. We define:

$$\xi(f)(x) = (\xi(f))(x) := \xi(x)(f) = (T_x f)(\xi(x)) = (\mathrm{pr}_2 \circ Tf \circ \xi)(x).$$

So $\xi(f) = \mathrm{pr}_2 \circ Tf \circ \xi \in C^\infty(M, \mathbb{R})$.

The assignment $(\xi, f) \mapsto \xi(f)$ is linear in ξ because $T_x f$ is linear. It is linear in f since $\xi(x) \in \mathrm{Der}_x$ is linear.

The induced mapping $f \mapsto \xi(f)$ is a derivation because

$$\begin{aligned}
\xi(fg)(x) &= \xi(x)(fg) = \xi(x)(f) \cdot g(x) + f(x) \cdot \xi(x)(g) \\
&= \xi(f)(x) \cdot g(x) + f(x) \cdot \xi(g)(x) \\
&= \big(\xi(f) \cdot g\big)(x) + \big(f \cdot \xi(g)\big)(x) \\
&= \big(\xi(f)\, g + f\, \xi(g)\big)(x).
\end{aligned}$$

The induced mapping $\mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M, \mathbb{R}))$ is surjective:
Let $\partial \in \mathrm{Der}(C^\infty(M, \mathbb{R}))$ be given. We are looking for a vector field $\xi \in \mathfrak{X}(M)$, which fulfills $\xi(x)(f) = \partial(f)(x)$. So

$$\xi(x) := \mathrm{ev}_x \circ \partial \in \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R}) = T_x M.$$

Remains to show that ξ is smooth. Note that ∂ is a local operator, i.e. $f|_U = 0 \Rightarrow \partial f|_U$ for each open subset $U \subseteq M$: For $x \in U$ choose $\rho \in C^\infty(M, \mathbb{R})$ with $\rho(x) = 1$ and $\operatorname{supp} \rho \subseteq U$. Then $0 = \partial(0)(x) = \partial(\rho f)(x) = \partial(\rho)(x) \cdot 0 + 1 \cdot \partial(f)(x) = \partial(f)(x)$. Let now $(u^1, \dots, u^m)$ be local coordinates. Then $\xi(x) = \sum_i \xi(x)^i \frac{\partial}{\partial u^i}|_x$, and the components

$$\xi^i(x) = \xi(x)^i = (\mathrm{ev}_x \circ \partial)^i = (\mathrm{ev}_x \circ \partial)(u^i) = \partial(u^i)(x)$$

are smooth in x. Thus $\xi \in \mathfrak{X}(M)$. That the two mappings $\xi \leftrightarrow \partial$ are inverse to each other, is clear, because

$$\begin{aligned}\xi^i(x) &= (\mathrm{ev}_x \circ \partial)(u^i) = \partial(u^i)(x) = \xi_x(u^i) = \xi^i(x)\\ \partial(f)(x) &= \xi(x)(f) = (\mathrm{ev}_x \circ \partial)(f) = \partial(f)(x).\end{aligned}$$

Finally, we show the $C^\infty(M,\mathbb{R})$ linearity:

$$\begin{aligned}(f\cdot\xi)(g)|_p &= (f\cdot\xi)_p \cdot g = (f_p \cdot \xi_p)\cdot g = f(p)\cdot(\xi_p\cdot g)\\ &= f(p)\cdot(\xi\cdot g)_p = f(p)\cdot \xi(g)(p) = (f\cdot\xi(g))|_p. \qquad \square\end{aligned}$$

17.2 Corollary (Space of vector fields as Lie algebra).

The assignment:

$$\begin{aligned}&\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M),\\ &\qquad (\xi,\eta)\mapsto[\xi,\eta]\big(: f\mapsto \xi(\eta(f))-\eta(\xi(f))\big),\end{aligned}$$

defines a bilinear mapping that turns $\mathfrak{X}(M)$ into a Lie algebra, i.e. the following equations hold:

1. *skew-symmetry:* $[\xi,\eta]+[\eta,\xi]=0$;
2. *"Jacobi Identity":* $[\xi,[\eta,\chi]]+[\eta,[\chi,\xi]]+[\chi,[\xi,\eta]]=0$;
3. *Additionally we have:* $[f\xi,g\eta]=fg\cdot[\xi,\eta]+f\xi(g)\cdot\eta-g\eta(f)\cdot\xi$.

Proof. We prove this for the space $\mathrm{Der}(A)$ of the derivations of an arbitrary associative algebra A instead of $C^\infty(M,\mathbb{R})$. For this we define the Lie bracket of $\xi,\eta\in\mathrm{Der}(A)$ by $[\xi,\eta]:=\xi\circ\eta-\eta\circ\xi$.

Then $[\xi,\eta]\in\mathrm{Der}(A)$ holds, because obviously $[\xi,\eta]$ is linear and for $f,g\in A$:

$$\begin{aligned}[\xi,\eta](f\cdot g) &= \xi(\eta(f\cdot g))-\eta(\xi(f\cdot g))\\ &= \xi(f\cdot\eta(g))+\xi(\eta(f)\cdot g)-\eta(f\cdot\xi(g))-\eta(\xi(f)\cdot g)\\ &= f\cdot\xi(\eta(g))+\xi(f)\eta(g)+\eta(f)\xi(g)+\xi(\eta(f))\cdot g\\ &\qquad - f\cdot\eta(\xi(g))-\eta(f)\xi(g)-\xi(f)\eta(g)-\eta(\xi(f))\cdot g\\ &= f\cdot[\xi,\eta](g)+[\xi,\eta](f)\cdot g.\end{aligned}$$

The mapping $(\xi,\eta)\mapsto[\xi,\eta]$ is bilinear because the composition in $L(A,A)$ is bilinear and the subtraction in $L(A,A)$ is linear.

It is skew-symmetric because

$$[\xi,\eta]=\xi\circ\eta-\eta\circ\xi=-(\eta\circ\xi-\xi\circ\eta)=-[\eta,\xi]$$

and it satisfies the Jacobi equation because

$$\begin{aligned}&[\xi,[\eta,\chi]]+[\eta,[\chi,\xi]]+[\chi,[\xi,\eta]]\\ &\quad=[\xi,\eta\circ\chi-\chi\circ\eta]+[\eta,\chi\circ\xi-\xi\circ\chi]+[\chi,\xi\circ\eta-\eta\circ\xi]\\ &\quad=\xi\circ(\eta\circ\chi-\chi\circ\eta)-(\eta\circ\chi-\chi\circ\eta)\circ\xi\\ &\qquad+\eta\circ(\chi\circ\xi-\xi\circ\chi)-(\chi\circ\xi-\xi\circ\chi)\circ\eta\\ &\qquad+\chi\circ(\xi\circ\eta-\eta\circ\xi)-(\xi\circ\eta-\eta\circ\xi)\circ\chi\\ &\quad=0\end{aligned}$$

Finally we show point (3):

$$
\begin{aligned}
\big([f\cdot\xi, g\cdot\eta]\big)(h) &= \Big((f\cdot\xi)\circ(g\cdot\eta) - (g\cdot\eta)\circ(f\cdot\xi)\Big)(h) = \\
&= (f\cdot\xi)\big((g\cdot\eta)(h)\big) - (g\cdot\eta)\big((f\cdot\xi)(h)\big) \\
&= (f\cdot\xi)\big(g\cdot\eta(h)\big) - (g\cdot\eta)\big(f\cdot\xi(h)\big) \\
&= f\cdot\xi(g)\cdot\eta(h) + f\cdot g\cdot\xi\big(\eta(h)\big) \\
&\qquad - g\cdot\eta(f)\cdot\xi(h) - g\cdot f\cdot\eta\big(\xi(h)\big) \\
&= \Big(f\cdot g\cdot[\xi,\eta] + f\cdot\xi(g)\cdot\eta - g\cdot\eta(f)\cdot\xi\Big)(h). \qquad \square
\end{aligned}
$$

Remark.

The VB chart representation of $[\xi,\eta]$ looks as follows:

$$
\begin{aligned}
[\xi,\eta] &= \Big[\sum_i \xi^i\frac{\partial}{\partial u^i}, \sum_k \eta^k\frac{\partial}{\partial u^k}\Big] \overset{\text{bilinearity}}{=\!=\!=} \sum_{i,k}\Big[\xi^i\frac{\partial}{\partial u^i}, \eta^k\frac{\partial}{\partial u^k}\Big] \\
&\overset{\boxed{3}}{=\!=} \sum_{ik}\Big(\xi^i\eta^k\Big[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}\Big] + \xi^i\Big(\frac{\partial}{\partial u^i}\cdot\eta^k\Big)\frac{\partial}{\partial u^k} - \eta^k\Big(\frac{\partial}{\partial u^k}\cdot\xi^i\Big)\frac{\partial}{\partial u^i}\Big) \\
&= \sum_k\sum_i\Big(\xi^i\frac{\partial}{\partial u^i}\eta^k - \eta^i\frac{\partial}{\partial u^i}\xi^k\Big)\frac{\partial}{\partial u^k}, \quad \text{since } \Big[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}\Big] = 0,
\end{aligned}
$$

in fact:

$$
\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^k}f = \frac{\partial}{\partial u^i}\big(\partial_k(f\circ\varphi)\circ\varphi^{-1}\big) = \partial_i\Big(\partial_k(f\circ\varphi)\circ\varphi^{-1}\circ\varphi\Big)\circ\varphi^{-1} = \partial_i\partial_k(f\circ\varphi)\circ\varphi^{-1}.
$$

So the coefficient of $[\xi,\eta]$ with respect to $\frac{\partial}{\partial u^k}$ is

$$
[\xi,\eta]^k = \sum_i\left(\xi^i\frac{\partial\eta^k}{\partial u^i} - \eta^i\frac{\partial\xi^k}{\partial u^i}\right).
$$

Conversely, this local formula can be used to define the Lie bracket. But we have to check compatibility with chart changes. This is done as follows:

$$
\begin{aligned}
&\sum_{\bar\imath,\bar\jmath}\Big(\xi^{\bar\imath}\frac{\partial\bar\eta^{\bar\jmath}}{\partial\bar u^{\bar\imath}} - \bar\eta^{\bar\imath}\frac{\partial\bar\xi^{\bar\jmath}}{\partial\bar u^{\bar\imath}}\Big)\frac{\partial}{\partial\bar u^{\bar\jmath}} \\
&\quad= \sum_{\bar\imath,\bar\jmath}\left(\sum_i\xi^i\frac{\partial\bar u^{\bar\imath}}{\partial u^i}\frac{\partial}{\partial\bar u^{\bar\imath}}\Big(\sum_j\frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\eta^j\Big) - \sum_i\eta^i\frac{\partial\bar u^{\bar\imath}}{\partial u^i}\frac{\partial}{\partial\bar u^{\bar\imath}}\Big(\sum_j\frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\xi^j\Big)\right)\frac{\partial}{\partial\bar u^{\bar\jmath}} \\
&\quad= \sum_{\bar\jmath}\left(\sum_i\xi^i\frac{\partial}{\partial u^i}\Big(\sum_j\frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\eta^j\Big) - \sum_i\eta^i\frac{\partial}{\partial u^i}\Big(\sum_j\frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\xi^j\Big)\right)\frac{\partial}{\partial\bar u^{\bar\jmath}} \\
&\quad= \sum_{\bar\jmath}\left(\sum_i\xi^i\sum_j\Big(\frac{\partial^2\bar u^{\bar\jmath}}{\partial u^i\partial u^j}\eta^j + \frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\frac{\partial}{\partial u^i}\eta^j\Big)\right. \\
&\qquad\left. - \sum_i\eta^i\sum_j\Big(\frac{\partial^2\bar u^{\bar\jmath}}{\partial u^i\partial u^j}\xi^j + \frac{\partial\bar u^{\bar\jmath}}{\partial u^j}\frac{\partial}{\partial u^i}\xi^j\Big)\right)\frac{\partial}{\partial\bar u^{\bar\jmath}} \\
&\quad= \sum_{i,j}\Big(\xi^i\frac{\partial\eta^j}{\partial u^i} - \eta^i\frac{\partial\xi^j}{\partial u^i}\Big)\frac{\partial}{\partial u^j}
\end{aligned}
$$

For open $M \subseteq \mathbb{R}^n$, the following holds:

$$[\xi,\eta]_x^k = \sum_i (\xi_x^i \partial_i \eta^k|_x - \eta_x^i \partial_i \xi^k|_x) = (\eta^k)'(x)(\xi_x) - (\xi^k)'(x)(\eta_x)$$
$$\text{that is } [\xi,\eta](x) = \eta'(x)\cdot\xi_x - \xi'(x)\cdot\eta_x.$$

Example.

The Lie-bracket $[\xi,\eta]$ of the two complete vector fields ξ and η from 16.6.3 is not complete:

$$[\xi,\eta] = [y\tfrac{\partial}{\partial x}, \tfrac{x^2}{2}\tfrac{\partial}{\partial y}] = y\tfrac{x^2}{2}[\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y}] + y\tfrac{\partial}{\partial x}(\tfrac{x^2}{2})\tfrac{\partial}{\partial y} - \tfrac{x^2}{2}\tfrac{\partial}{\partial y}(y)\tfrac{\partial}{\partial x} = yx\tfrac{\partial}{\partial y} - \tfrac{x^2}{2}\tfrac{\partial}{\partial x}$$

$$c(t) = \begin{pmatrix} c_1(t) \\ c_2(t)\end{pmatrix} \text{ is a solution curve } \Leftrightarrow \begin{cases} \dfrac{d}{dt}c_1(t) = -c_1^2(t)/2 \\ \dfrac{d}{dt}c_2(t) = c_1(t)\cdot c_2(t)\end{cases}$$

Thus $c_1(t) = 2(t+A)^{-1}$ and $c_2(t) = (t+A)^2\cdot B$. The initial condition $c(0) = (x,y)$ results in $A = \frac{2}{x}$ and $B = \frac{x^2 y}{4}$. Thus,

$$c_{(x,y)}(t) = \mathrm{Fl}_t^{[\xi,\eta]}(x,y) = \Big(\tfrac{2x}{2+tx}, (t+\tfrac{2}{x})^2\,\tfrac{x^2y}{4}\Big) = \Big(\tfrac{x}{1+tx/2}, (1+tx/2)^2 y\Big).$$

For $t = -\frac{2}{x}$, the flow is undefined, i.e. $[\xi,\eta]$ is not complete.

17.3 Definition (Relatedness of vector fields).

Let $f: M \to N$ be smooth.
A vector field $\xi \in \mathfrak{X}(M)$ is called f-related to a vector field $\eta \in \mathfrak{X}(N)$:⇔ $Tf \circ \xi = \eta \circ f$.

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN & \\ M \arrow[u, "\xi"] \arrow[r, "f"] & N \arrow[u, "\eta"'] \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

The vector field ξ is f-related to η if and only if $\xi(g\circ f) = \eta(g)\circ f$ for all smooth $g: N \to \mathbb{R}$.

($\Rightarrow$) $\xi(g\circ f)(p) = \xi_p(g\circ f) = (T_pf\cdot\xi_p)\,g = \eta_{f(p)}g = \eta(g)(f(p)) = (\eta(g)\circ f)(p)$

($\Leftarrow$) $(Tf\circ\xi)_p g = (Tf\cdot\xi_p)g = \xi_p(g\circ f) = \xi(g\circ f)(p) = (\eta(g)\circ f)(p) = \eta(g)(f(p)) = \eta_{f(p)}(g)$.

17.4 Remark (Push-forward of vector fields).

For general f it is not possible to find a vector field f to which a given vector field is related to. However, if f is a diffeomorphism, then $f_*\xi := Tf\circ\xi\circ f^{-1}$ is a vector field on N for each vector field ξ on M.

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "\xi"] \arrow[r, "f"] & N \arrow[u, "f_*\xi"'] \end{tikzcd}$$

The vector field ξ is f-related to $f_*\xi$ by construction. Conversely, one has the following statement:

17.5 Lemma (Pull-back of vector fields).

Let $f: M \to N$ be an immersion, $\eta \in \mathfrak{X}(N)$ and $\eta_{f(p)} \in \mathrm{im}(T_pf)$ for all $p \in M$, then $\exists\,!\,\xi(=: f^*\eta) \in \mathfrak{X}(M)$, s.t. ξ is f-related to η.

Proof. Since T_pf is injective, to each $\eta_{f(p)}$ in $\mathrm{im}(T_pf)$ there is a unique preimage $\xi_p \in T_pM$. It remains to show that this vector field $\xi: M \to TM$ is smooth. Since f is an immersion, according to 11.4, charts φ and ψ exist centered at p and $f(p)$, respectively, so that $\psi^{-1}\circ f\circ\varphi = \mathrm{incl}_{\mathbb{R}^m\to\mathbb{R}^n}$. Let $\xi = \sum \xi^i_\varphi \partial^\varphi_i$. It suffices to show that the $\xi^i_\varphi : M \to \mathbb{R}$ are smooth. Since

$$\begin{aligned}(\xi^i_\varphi)_p &= \xi_p(\mathrm{pr}_i\circ\varphi^{-1}) = \xi_p(\mathrm{pr}_i\circ\psi^{-1}\circ f) = (Tf\circ\xi_p)(\mathrm{pr}_i\circ\psi^{-1})\\ &= \eta_{f(p)}(\mathrm{pr}_i\circ\psi^{-1}) = (\eta^i_\psi)_{f(p)} = (\eta^i_\psi\circ f)_p \text{ is smooth in } p\,,\end{aligned}$$

it follows that ξ_p is smooth locally around p. The f-relatedness follows directly from the construction of ξ. $\square$

17.6 Remark.

We have shown that vector fields can be transported using diffeomorphisms f:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "\xi"] \arrow[r, "f"] & N \arrow[u, "f_*\xi"'] \end{tikzcd} \qquad \begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "f^*\eta"] \arrow[r, "f"] & N \arrow[u, "\eta"'] \end{tikzcd}$$

Here, $f^*\eta := Tf^{-1}\circ\eta\circ f$ by 17.5 and $f_*\xi := Tf\circ\xi\circ f^{-1}$ by 17.4. Then the following holds:

$$f_*(f^*\eta) = Tf\circ(f^*\eta)\circ f^{-1} = Tf\circ Tf^{-1}\circ\eta\circ f\circ f^{-1} = \eta$$

and analog $f^*(f_*\xi) = \xi$, i.e. $f^* : \mathfrak{X}(N) \to \mathcal{X}(M)$ and $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ are inverse to each other for diffeomorphisms f.

17.7 Proposition.

Let vector fields ξ_i be f-related to η_i for $i = 1, 2$. Then:

1. *$\xi_1 + \xi_2$ is f-related to $\eta_1 + \eta_2$.*
2. *$[\xi_1, \xi_2]$ is f-related to $[\eta_1, \eta_2]$.*
3. *$(g\circ f)\cdot\xi$ is f-related to $g\cdot\eta$, where $g : N \to \mathbb{R}$ is smooth.*

Proof.
1 follows from the linearity of T_pf.
3 follows analogously, because of

$$\begin{aligned}\Big(Tf\circ\big((g\circ f)\cdot\xi\big)\Big)(p) &= Tf\Big(g\big(f(p)\big)\cdot\xi_p\Big) = g\big(f(p)\big)\cdot(T_pf)\,\xi_p\\ &= g\big(f(p)\big)\cdot\eta_{f(p)} = \Big((g\cdot\eta)\circ f\Big)(p).\end{aligned}$$

2 follows because

$$\begin{aligned}[\xi_1,\xi_2](g\circ f) &= \xi_1\big(\xi_2(g\circ f)\big) - \xi_2\big(\xi_1(g\circ f)\big)\\ &\overset{17.3}{=\!=} \xi_1\big(\eta_2(g)\circ f\big) - \xi_2\big(\eta_1(g)\circ f\big)\\ &\overset{17.3}{=\!=} \big(\eta_1(\eta_2(g))\big)\circ f - \big(\eta_2(\eta_1(g))\big)\circ f = \big([\eta_1,\eta_2]\,g\big)\circ f.\end{aligned}$$ $\square$

17.8 Lemma.

Let $f \in C^\infty(M,N)$, $\xi \in \mathfrak{X}(M)$, and $\eta \in \mathfrak{X}(N)$.
Then ξ is f-related to η $\Leftrightarrow$ $f\circ\mathrm{Fl}^\xi = \mathrm{Fl}^\eta\circ(\mathrm{id}\times f)$ locally at $\{0\}\times M$.

Proof.

(⇐) We have

$$\tfrac{d}{dt}|_{t=0} f\big(\mathrm{Fl}^\xi(t,p)\big) = Tf\big(\mathrm{Fl}^\xi(\_,p)'(0)\big) = Tf(\xi_p) \text{ and}$$
$$\tfrac{d}{dt}|_{t=0} \mathrm{Fl}^\eta\big(t, f(p)\big) = \eta\Big(\mathrm{Fl}^\eta\big(0, f(p)\big)\Big) = \eta\big(f(p)\big).$$

(⇒) The curve $\mathrm{Fl}^\eta(\_, f(p))$ is the unique integral curve to η with start value $f(p)$. On the other hand, $f \circ \mathrm{Fl}^\xi(\_,p)$ has value $f(p)$ at $t=0$ and by differentiating we obtain:

$$\big(f \circ \mathrm{Fl}^\xi(\_,p)\big)'(t) = Tf\Big(\big(\mathrm{Fl}^\xi(\_,p)\big)'(t)\Big) = (Tf \cdot \xi)|_{\mathrm{Fl}^\xi(t,p)} = \eta|_{f(\mathrm{Fl}^\xi(t,p))}.$$

The equality of the two terms now follows from the uniqueness of the integral curves of η. □

17.9 Definition (Lie derivative).

1. For $\xi \in \mathfrak{X}(M)$ the Lie derivative $\mathcal{L}_\xi : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ in direction ξ on functions $f \in C^\infty(M,\mathbb{R})$ is defined by
$$f \mapsto \Big(p \mapsto \tfrac{d}{dt}|_{t=0} (\mathrm{Fl}^\xi_t)^* f(p) = \tfrac{d}{dt}|_{t=0} \big(f \circ \mathrm{Fl}^\xi\big)(t,p)\Big).$$
2. For $\xi \in \mathfrak{X}(M)$ the Lie derivative $\mathcal{L}_\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ in direction ξ on vector fields $\eta \in \mathfrak{X}(M)$ is defined by
$$\eta \mapsto \Big(p \mapsto \tfrac{d}{dt}|_{t=0} (\mathrm{Fl}^\xi_t)^* \eta(p) = \tfrac{d}{dt}|_{t=0} \big(T\,\mathrm{Fl}^\xi_{-t} \circ \eta \circ \mathrm{Fl}^\xi_t\big)(p)\Big).$$
Note that $T\,\mathrm{Fl}^\xi_{-t} \circ \eta \circ \mathrm{Fl}^\xi_t : M \to TM$ is locally a section for all t near 0, and thus $t \mapsto (T\,\mathrm{Fl}^\xi_{-t} \circ \eta \circ \mathrm{Fl}^\xi_t)(p)$ is a locally defined curve in the vector space T_pM for each p (whereas $t \mapsto (T\,\mathrm{Fl}^\xi_{-t} \circ \eta \circ \mathrm{Fl}^\xi_t)$ is not a well-defined curve in $\mathfrak{X}(M)$) and thus the derivative $\frac{d}{dt}|_{t=0} (T\,\mathrm{Fl}^\xi_{-t} \circ \eta \circ \mathrm{Fl}^\xi_t)(p)$ is also in T_pM.

The following theorem shows that we already know the Lie derivative of functions and of vector fields.

17.10 Proposition (Descriptions of the Lie derivative).

1. *For $\xi \in \mathfrak{X}(M)$ and $f \in C^\infty(M,\mathbb{R})$:*
$$\mathcal{L}_\xi f = \xi f.$$
2. *For $\xi, \eta \in \mathfrak{X}(M)$:*
$$\mathcal{L}_\xi(\eta) = [\xi, \eta].$$

Proof. $\boxed{1}$ Since $\xi_p = c_p'(0)$ with $c_p := \mathrm{Fl}^\xi(\_,p)$ we get

$$(\mathcal{L}_\xi f)_p = \tfrac{d}{dt}|_{t=0} (\mathrm{Fl}^\xi_t)^* f(p) = (f \circ \mathrm{Fl}^\xi(\_,p))'(0) = (f \circ c_p)'(0) = \xi_p(f) = (\xi f)_p.$$

$\boxed{2}$ Let $\alpha : \mathbb{R}^2 \to \mathbb{R}$ be locally defined by

$$\alpha(t,s) := (\eta|_{\mathrm{Fl}^\xi(t,p)})(f \circ \mathrm{Fl}^\xi_s) = T\,\mathrm{Fl}^\xi_s(\eta_{\mathrm{Fl}^\xi(t,p)})(f) = (T\,\mathrm{Fl}^\xi_s \circ \eta \circ \mathrm{Fl}^\xi_t)_p(f).$$

Then

$$\begin{aligned}
&\alpha(t,0) = \eta_{\mathrm{Fl}^\xi(t,p)}\, f = (\eta f)(\mathrm{Fl}^\xi(t,p)) \\
&\alpha(0,s) = \eta_p(f \circ \mathrm{Fl}^\xi_s) \\
\Rightarrow\quad &\partial_1 \alpha|_{(0,0)} = \tfrac{d}{dt}|_{t=0} (\eta f)(\mathrm{Fl}^\xi(t,p)) = \xi_p(\eta f) \\
&\partial_2 \alpha|_{(0,0)} = \tfrac{d}{dt}|_{t=0} \Big(\eta_p\big(f \circ \mathrm{Fl}^\xi(t,.)\big)\Big) = \eta_p\Big(\tfrac{d}{dt}|_{t=0}\big(f \circ \mathrm{Fl}^\xi(t,.)\big)\Big) = \eta_p(\xi f),
\end{aligned}$$

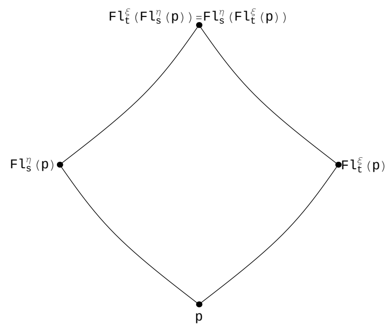
because η_p is linear. Thus

$$\tfrac{d}{dt}|_{t=0}\alpha(t,-t) = \partial_1\alpha(0,0) - \partial_2\alpha(0,0) = \xi_p(\eta f) - \eta_p(\xi f) = [\xi,\eta]_p f$$
$$\text{and } \tfrac{d}{dt}|_{t=0}\alpha(t,-t) = \tfrac{d}{dt}|_{t=0}(T\,\mathrm{Fl}^\xi_{-t}\circ\eta\circ\mathrm{Fl}^\xi_t)_p f = \mathcal{L}_\xi(\eta)_p f. \quad \square$$

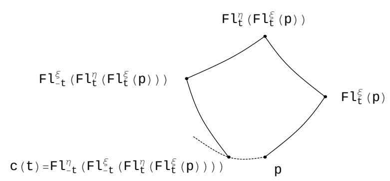
17.11 Proposition.

The Lie bracket is an obstruction to the commutativity of the flows. More precisely, this means:

1. *We have $[\xi,\eta]=0 \Leftrightarrow \mathrm{Fl}^\xi_t\circ\mathrm{Fl}^\eta_s = \mathrm{Fl}^\eta_s\circ\mathrm{Fl}^\xi_t$ (These mappings are defined locally for small t and s).*



2. *Let $c:\mathbb{R}\to M$ be defined locally by $c(t) := (\mathrm{Fl}^\eta_{-t}\circ\mathrm{Fl}^\xi_{-t}\circ\mathrm{Fl}^\eta_t\circ\mathrm{Fl}^\xi_t)_p$. Then: $c(0)=p$, $c'(0)=0$, $c''(0)\in T_pM$ is well defined and $c''(0)=2[\xi,\eta]_p$.*



Proof.
($\boxed{1}$) ($\Leftarrow$) We have

$$\mathrm{Fl}^\xi_t(\mathrm{Fl}^\eta(s,p)) = (\mathrm{Fl}^\xi_t\circ\mathrm{Fl}^\eta_s)(p) = (\mathrm{Fl}^\eta_s\circ\mathrm{Fl}^\xi_t)(p) = \mathrm{Fl}^\eta(s,\mathrm{Fl}^\xi_t(p)),$$

that is $\mathrm{Fl}^\xi_t\circ\mathrm{Fl}^\eta = \mathrm{Fl}^\eta\circ(1\times\mathrm{Fl}^\xi_t)$

$\xRightarrow{\boxed{17.8}}$ η is Fl^ξ_t-related to η, i.e. $T\,\mathrm{Fl}^\xi_t\circ\eta = \eta\circ\mathrm{Fl}^\xi_t$.

$\Rightarrow$ $\eta = T\,\mathrm{Fl}^\xi_{-t}\circ\eta\circ\mathrm{Fl}^\xi_t$ because $(\mathrm{Fl}^\xi_t)^{-1} = \mathrm{Fl}^\xi_{-t}$ is a local diffeomorphism

$\Rightarrow$ $0 = \frac{d}{dt}\Big|_{t=0}\eta = \frac{d}{dt}\Big|_{t=0} T\,\mathrm{Fl}^\xi_{-t}\circ\eta\circ\mathrm{Fl}^\xi_t \overset{\boxed{17.10.2}}{=\!=\!=} [\xi,\eta]$.

($\Rightarrow$) From $[\xi,\eta]=0$ we get:

$$\begin{aligned}
\tfrac{d}{dt}(T\operatorname{Fl}^\xi_{-t}\circ\eta\circ\operatorname{Fl}^\xi_t)(p) &= \tfrac{d}{ds}\big|_{s=0}(T\operatorname{Fl}^\xi_{-(t+s)}\circ\eta\circ\operatorname{Fl}^\xi_{t+s})(p)\\
&= \tfrac{d}{ds}\big|_{s=0}\Big(T\operatorname{Fl}^\xi_{-t}\circ T\operatorname{Fl}^\xi_{-s}\circ\eta\circ\operatorname{Fl}^\xi_s\circ\operatorname{Fl}^\xi_t\Big)(p)\\
&= T\operatorname{Fl}^\xi_{-t}\Big(\tfrac{d}{ds}\big|_{s=0}(T\operatorname{Fl}^\xi_{-s}\circ\eta\circ\operatorname{Fl}^\xi_s)\,(\operatorname{Fl}^\xi_t(p))\Big)\\
&\overset{\boxed{17.10.2}}{=\!=\!=} (T\operatorname{Fl}^\xi_{-t}\circ[\xi,\eta]\circ\operatorname{Fl}^\xi_t)(p)=0
\end{aligned}$$

So $T\operatorname{Fl}^\xi_{-t}\circ\eta\circ\operatorname{Fl}^\xi_t = T\operatorname{Fl}^\xi_0\circ\eta\circ\operatorname{Fl}^\xi_0=\eta$ is constant in t, that is $\eta\circ\operatorname{Fl}^\xi_t = T\operatorname{Fl}^\xi_t\circ\eta$. Thus, η is Fl^ξ_t-related to η. By $\boxed{17.8}$ we finally obtain $\operatorname{Fl}^\eta_s\circ\operatorname{Fl}^\xi_t=\operatorname{Fl}^\xi_t\circ\operatorname{Fl}^\eta_s$.

($\boxed{2}$) Let $c:\mathbb{R}\to M$ be locally defined and C^∞. Then $c':\mathbb{R}\to TM$ is the canonical lift of c. The curve $c'':\mathbb{R}\to T(TM)$ can also be understood as a lift of c.

$$\left.\begin{aligned} c&=\pi_M\circ c'\\ c'&=\pi_{TM}\circ c''\end{aligned}\right\}\Rightarrow\ c=\pi_M\circ\pi_{TM}\circ c''$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "c''"] \arrow[d, equal] & T(TM) \arrow[d, "\pi_{TM}"] \\
\mathbb{R} \arrow[r, "c'"'] \arrow[d, equal] & TM \arrow[d, "\pi_M"] \\
\mathbb{R} \arrow[r, "c"'] & M
\end{tikzcd}$$

If $c'(0)=0$, then $c''(0)$ can also be understood as the derivation $f\mapsto c''(0)f:=(f\circ c)''(0)$: This is linear and

$$\begin{aligned}
c''(0)(fg) &= \big((fg)\circ c\big)''(0)=\big((f\circ c)\,(g\circ c)\big)''(0)\\
&= (f\circ c)''(0)\,(g\circ c)(0)+2(f\circ c)'(0)\,(g\circ c)'(0)+(f\circ c)(0)\,(g\circ c)''(0)\\
&= \big(c''(0)f\big)\,g\big(c(0)\big)+f\big(c(0)\big)\,\big(c''(0)g\big).
\end{aligned}$$

So $c''(0)$ acts as a derivation over $c(0)=p$, that is $c''(0)\in T_pM$.

$$\begin{aligned}
\text{Let }\alpha_0(t,s)&:=(\operatorname{Fl}^\eta_t\circ\operatorname{Fl}^\xi_s)(p)\\
\alpha_1(t,s)&:=(\operatorname{Fl}^\xi_{-t}\circ\operatorname{Fl}^\eta_s\circ\operatorname{Fl}^\xi_s)(p)\\
\alpha_2(t,s)&:=(\operatorname{Fl}^\eta_{-t}\circ\operatorname{Fl}^\xi_{-s}\circ\operatorname{Fl}^\eta_s\circ\operatorname{Fl}^\xi_s)(p).
\end{aligned}$$

$$\begin{aligned}
\text{Then }c(t)&=\alpha_2(t,t)\\
\alpha_2(0,s)&=\alpha_1(s,s)\\
\alpha_1(0,s)&=\alpha_0(s,s).
\end{aligned}$$

If $f\in C^\infty(M,\mathbb{R})$, then:

$$\begin{aligned}
\partial_1(f\circ\alpha_0)&=(\eta f)\circ\alpha_0\\
\partial_1(f\circ\alpha_1)&=-(\xi f)\circ\alpha_1\\
\partial_1(f\circ\alpha_2)&=-(\eta f)\circ\alpha_2\\
\partial_2(f\circ\alpha_0)(0,s)&=(\xi f)(\alpha_0(0,s))\\
\partial_2(f\circ\alpha_1)(0,s)&=\partial_1(f\circ\alpha_0)(s,s)+\partial_2(f\circ\alpha_0)(s,s)\\
\partial_2(f\circ\alpha_2)(0,s)&=\partial_1(f\circ\alpha_1)(s,s)+\partial_2(f\circ\alpha_1)(s,s)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow c'(0)f &= (f\circ c)'(0) = \tfrac{d}{dt}|_{t=0}(f\circ\alpha_2)(t,t)\\
&= \partial_1(f\circ\alpha_2)(0,0) + \partial_2(f\circ\alpha_2)(0,0)\\
&= -(\eta f)_p + \partial_1(f\circ\alpha_1)(0,0) + \partial_2(f\circ\alpha_1)(0,0)\\
&= -(\eta f)_p - (\xi f)_p + \partial_1(f\circ\alpha_0)(0,0) + \partial_2(f\circ\alpha_0)(0,0) = 0\\
c''(0)f &:= (f\circ c)''(0) = (\tfrac{d}{dt})^2|_{t=0}(f\circ\alpha_2)(t,t)\\
&= \partial_1^2(f\circ\alpha_2)(0,0) + 2\partial_2\partial_1(f\circ\alpha_2)(0,0) + \partial_2^2(f\circ\alpha_2)(0,0)
\end{aligned}$$

$$\begin{aligned}
\partial_1^2(f\circ\alpha_2)(0,0) &= \partial_1(-(\eta f)\circ\alpha_2)(0,0) = (-\eta(-\eta f))\alpha_2(0,0) = (\eta(\eta f))_p\\
\partial_2\partial_1(f\circ\alpha_2)(0,0) &= \partial_2((-\eta f)\circ\alpha_2)(0,0)\\
&= \partial_1((-\eta f)\circ\alpha_1)(0,0) + \partial_2((-\eta f)\circ\alpha_1)(0,0)\\
&= (\xi\eta f)_p + \partial_1(-\eta f\circ\alpha_0)(0,0) + \partial_2(-\eta f\circ\alpha_0)(0,0)\\
&= (\xi\eta f)_p - (\eta\eta f)_p - (\xi\eta f)_p = -(\eta\eta f)_p
\end{aligned}$$

$$\begin{aligned}
\partial_2^2(f\circ\alpha_2)(0,0) &= \partial_1^2(f\circ\alpha_1)(0,0) + 2\partial_1\partial_2(f\circ\alpha_1)(0,0) + \partial_2^2(f\circ\alpha_1)(0,0)\\
\partial_1^2(f\circ\alpha_1)(0,0) &= (\xi\xi f)_p\\
\partial_2\partial_1(f\circ\alpha_1)(0,0) &= \partial_2((-\xi f)\circ\alpha_1)(0,0)\\
&= \partial_1(-\xi f\circ\alpha_0)(0,0) + \partial_2(-\xi f\circ\alpha_0)(0,0)\\
&= -(\eta\xi f)_p - (\xi\xi f)_p\\
\partial_2^2(f\circ\alpha_1)(0,0) &= \partial_1^2(f\circ\alpha_0)(0,0) + 2\partial_2\partial_1(f\circ\alpha_0)(0,0) + \partial_2^2(f\circ\alpha_0)(0,0)\\
&= (\eta\eta f)_p + 2(\xi\eta f)_p + (\xi\xi f)_p
\end{aligned}$$

By collecting the results we finally obtain:

$$\begin{aligned}
c''(0)f &= \eta\eta f - 2\eta\eta f + \xi\xi f - 2\eta\xi f - 2\xi\xi f + \eta\eta f + 2\xi\eta f + \xi\xi f\\
&= 2(\xi\eta f - \eta\xi f) = 2[\xi,\eta]f. \qquad \square
\end{aligned}$$

17.12 Proposition (Commutating flows are coming from charts).

Let $\{\xi_i\}_{i=1}^k$ be linear independent vector fields on M with $[\xi_i,\xi_j]=0\ \forall\ i,j$. Then there is a chart φ, so that locally $\xi_i = \partial_i^\varphi$ for $i=1\dots k$.

Proof. Without loss of generality, $M\subseteq\mathbb{R}^n$ is open, $p=0$ and $\xi_i(0)=e_i$ for $i=1,\dots k$. Let

$$\begin{aligned}
\varphi(t_1,\dots,t_n) &:= \mathrm{Fl}^{\xi_1}\Big(t_1, \mathrm{Fl}^{\xi_2}(t_2,\dots \mathrm{Fl}^{\xi_k}(t_k;0,\dots 0,t_{k+1},\dots t_n)\dots)\Big)\\
&= \Big(\mathrm{Fl}^{\xi_1}_{t_1}\circ\dots\circ\mathrm{Fl}^{\xi_k}_{t_k}\Big)(0,\dots,0,t_{k+1},\dots t_n).
\end{aligned}$$

Then $\varphi(0) = p$ and φ is a local diffeomorphism because the partial derivatives for $i \le k$ have the following form:

$$\begin{aligned}
\partial_i \varphi(t_1, \dots, t_n) &= \tfrac{\partial}{\partial t_i}\Big(\mathrm{Fl}^{\xi_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{\xi_i}_{t_i} \circ \dots \circ \mathrm{Fl}^{\xi_k}_{t_k}\Big)(0, \dots, 0, t_{k+1}, \dots t_n) \\
&= \tfrac{\partial}{\partial t_i}\Big(\mathrm{Fl}^{\xi_i}_{t_i} \circ \mathrm{Fl}^{\xi_1}_{t_1} \circ \dots \circ \overbracket{\mathrm{Fl}^{\xi_i}_{t_i}} \circ \dots \circ \mathrm{Fl}^{\xi_k}_{t_k}\Big)(0, \dots, 0, t_{k+1}, \dots t_n) \\
&= \xi_i\Big(\Big(\mathrm{Fl}^{\xi_i}_{t_i} \circ \mathrm{Fl}^{\xi_1}_{t_1} \dots \circ \overbracket{\mathrm{Fl}^{\xi_i}_{t_i}} \circ \dots \circ \mathrm{Fl}^{\xi_k}_{t_k}\Big)(0, \dots, 0, t_{k+1}, \dots t_n)\Big) \\
&= \xi_i\Big(\Big(\mathrm{Fl}^{\xi_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{\xi_k}_{t_k}\Big)(0, \dots, 0, t_{k+1}, \dots t_n)\Big) \\
&= \xi_i\Big(\varphi(t_1, \dots, t_n)\Big),
\end{aligned}$$

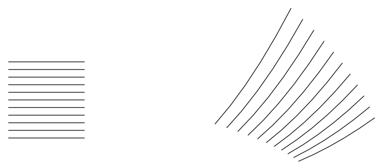
where $\overbracket{\dots}$ means that the corresponding term is to be omitted.
So $\xi_i = \partial_i^\varphi$ is for $i \le k$. For $i > k$ and $t_1 = \dots = t_k = 0$ the following holds:

$$\partial_i|_{t_i=0}\varphi(0, \dots, 0, t_i, 0, \dots, 0) = (\tfrac{\partial}{\partial t_i})(0, \dots 0, t_i, 0, \dots 0) = e_i.$$

Thus, $\varphi'(0) = \mathrm{id}_{\mathbb{R}^n}$, because $\xi_i(0) = e_i$, and $\partial_i^\varphi(q) = \partial_i(\varphi)(\varphi^{-1}q) = \xi_i(q)$ für $i \le k$. □

17.13 Remarks.

1. The reverse holds as well: If φ is a chart then the Lie brackets of the basis vector fields ∂_i^φ vanish and thus their flows commute pairwise.
2. Let $\xi \in \mathfrak{X}(M)$ with $\xi_p \ne 0$. Then there is a chart φ with $\xi = \partial_1^\varphi$ for $k = 1$ by 17.12. Since ∂_1 is obviously φ-related to ∂_1^φ, we have $\varphi(\mathrm{Fl}^{\partial_1}(t, x)) = \mathrm{Fl}^\xi(t, \varphi(x))$ by 17.8 and thus
$$\mathrm{Fl}^\xi(t, p) = \varphi(\mathrm{Fl}^{\partial_1}(t, \varphi^{-1}(p))) = \varphi(\varphi^{-1}(p) + te_1).$$
The flow of each non-stationary vector field is thus given up to diffeomorphisms φ by the translation $x \mapsto x + t\, e_1$ with constant velocity vector e_1.



3. Let $\xi_p = 0$ (i.e. p is a zero of the vector field) and thus $\mathrm{Fl}^\xi(t, p) = p$, i.e. p be a fixed point (stationary point) of the local flow. Without loss of generality, $U \subseteq \mathbb{R}^m$ is open and $\xi : U \to \mathbb{R}^m$ with $\xi(0) = 0$. Then $\xi'(0) : \mathbb{R}^m \to \mathbb{R}^m$ is linear, and the Eigenvalues of $\xi'(0)$ generically determine the local behavior of the flow (see books on dynamical systems).

17.14 Proposition.

Let $M \subseteq \mathbb{R}^3$ be a surface and X_1, X_2 pointwise linear independent vector fields on M. Then there is a local parameterization φ of M with $\partial_i\varphi(u)$ parallel to $X_i(\varphi(u))$ for $i \in \{1, 2\}$.

For hypersurfaces in $\mathbb{R}^n$ with $n > 3$ the analogue proposition is wrong!

Direct proof. Let ψ be a local parameterization of M and $Y_i := \psi^{-1}(X_i)$ the local vector fields on $\mathbb{R}^2$ with $T_v\psi \cdot Y_i(v) = X_i(\psi(v))$. We are looking for a local diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, $(v^1, v^2) \mapsto (u^1, u^2)$ with $\varphi := \psi \circ h^{-1}$ as desired, i.e. $\partial_i\varphi(u) = T_{h^{-1}(u)}\psi \cdot (h^{-1})'(u) \cdot e_i$ parallel to $X_i(\varphi(u)) = X_i(\psi(h^{-1}(u))) = T_{h^{-1}(u)}\psi \cdot Y_i(h^{-1}(u))$. This means

$$0 = (u^j)'(v) \cdot Y_i(v) = \sum_{k=1}^{2} \partial_k u^j(v) \cdot Y_i^k(v) \text{ for } j \neq i,$$

so $h'(v) \cdot Y_i(v)$ is proportional to e_i, because $(h^{-1})'(u) \cdot e_i = h'(h^{-1}(u))^{-1} \cdot e_i$ is parallel to $Y_i(h^{-1}(u))$. The above partial differential equations of the form

$$\partial_1 u(v) \cdot Y^1(v) + \partial_2 u(v) \cdot Y^2(v) = 0$$

are solvable because if $t \mapsto v(t)$ is an integral curve of vector field Y, then

$$\begin{aligned}\frac{d}{dt} u(v(t)) &= \partial_1 u(v(t)) \cdot (v^1)'(t) + \partial_2 u(v) \cdot (v^2)'(t)\\ &= \partial_1 u(v(t)) \cdot Y^1(v(t)) + \partial_2 u(v(t)) \cdot Y^2(v(t)) = 0\end{aligned}$$

for each solution u of the partial differential equation, i.e. $u \circ v$ constant. Hence $u(\mathrm{Fl}^Y(t, v)) = u(v)$. Thus, if we specify u on a curve normal to Y, then u is locally defined and satisfies this partial differential equation. □

Proof by means of commutating vector fields. Compare this to 17.12. Let X_1, X_2 pointwise linearly independent. Then local functions exist with $a_i > 0$

$$\begin{aligned}0 = [a_1X_1, a_2X_2] &= a_1a_2[X_1, X_2] + a_1X_1(a_2)X_2 - a_2X_2(a_1)X_1\\ &= a_1a_2\Big([X_1, X_2] + \frac{X_1(a_2)}{a_2}X_2 - \frac{X_2(a_1)}{a_1}X_1\Big)\end{aligned}$$

and thus by 17.12 for $k = 2$ a chart φ with $\partial_i\varphi = a_iX_i$ for $i = 1, 2$: We have $[X_1, X_2] = b_1X_1 + b_2X_2$ with smooth coefficients functions b_1 and b_2 and therefore we have to solve only the partial differential equation of first order $\frac{X_1(a_2)}{a_2} = b_2$ and analogously $\frac{X_2(a_1)}{a_1} = -b_1$, which is obviously possible, since by 17.12 for $k = 1$ we find a chart φ with $X_1 = \partial_1^\varphi$ and then $\frac{X_1(a_2)}{a_2} = b_2$ is an ordinary differential equation with additional parameter. □

18. Integral manifolds

18.1 Remark.

We have seen in 16.6 that integral curves of vector fields are not always globally defined. Intuitively speaking, they are not defined for all $t \in \mathbb{R}$ because they escape to "infinity" in finite time. Thus the solution curves are "too fast", i.e. their velocity vectors are too large. But we could make the flow global by reducing its speed.

More abstractly this means:

i. Instead of vector fields we consider one-dimensional subspaces $E_p \subseteq T_pM \ \forall\ p \in M$, i.e. vector subbundles.

ii. Instead of solution curves, we consider integral manifolds, i.e. 1-dimensional submanifolds N of M, for which $T_pN = E_p$ holds. We can also consider these concepts in the higher-dimensional case:

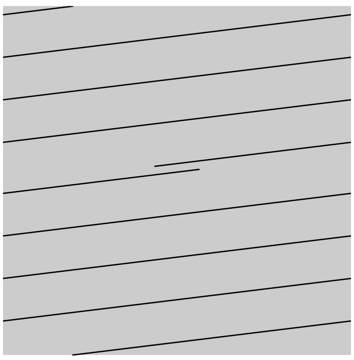
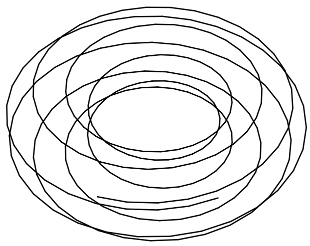
18.2 Definition (Integral manifold).

Let E be a vector subbundle of $\pi : TM \to M$ (referred to as DISTRIBUTION in the (older) literature). Then we understand by an INTEGRAL MANIFOLD N of E a connected manifold structure on a subset $N \subseteq M$ such that the inclusion $\mathrm{incl} : N \to M$ is an immersion and $T_p \,\mathrm{incl} : T_pN \to E_p$ is a bijection for each $p \in N$.

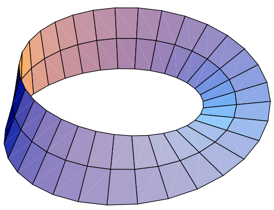
18.3 Examples.

1. For one-dimensional vector subbundles, which are spanned locally by a vector field, integral curves always exist for this vector field, and thus also integral manifolds of the bundle.

   For example: If a "constant" vector field at the torus has an irrational slope, then each of its integral manifolds lies dense in the torus.

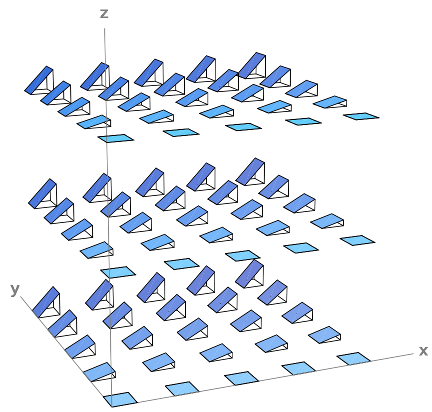
2. Note however, that the vector subbundles E is generally not spanned globally by a VF. An example is the subbundle E of the tangent bundle of Möbius strip M consisting of all velocity vectors of curves in the fibers of $M \to S^1$.



3. In the multidimensional case, it is generally not true that each vector subbundle has integral manifolds. Consider the following example:
   $M = \mathbb{R}^3$ with $E_{xyz} = \langle\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\}\rangle = \{(\lambda, \mu, \lambda y) : \lambda, \mu \in \mathbb{R}\} \subseteq T_{(x,y,z)}\mathbb{R}^3$.

Suppose there exists an integral manifold N through $(0,0,0)$.

Since $E_{(0,y,z)} \cap (\{0\} \times \mathbb{R}^2) = \mathbb{R} \cdot e_2$ the intersection $N \cap (\{0\} \times \mathbb{R}^2)$ is locally the y-axes $\mathbb{R} \cdot e_2$.
Since $E_{(x,y_0,z)} \cap (\mathbb{R} \times \{0\} \times \mathbb{R}) = \mathbb{R} \cdot (1,0,y_0)$ the intersection $N \cap (\mathbb{R} \times \{y_0\} \times \mathbb{R})$ for fixed y_0 is locally near $(0,y_0,0)$ the line $(0,y_0,0) + \mathbb{R} \cdot (1,0,y_0) = \{(x,y_0,xy_0) : x \in \mathbb{R}\}$. Thus N locally at 0 contains $\{(x,y,x\,y) : x,y \in \mathbb{R}\}$ and hence its tangent space $T_{(x,y,x\,y)}N$ contains $(0,1,x)$ which lies in $E_{(x,y,x\,y)} = \{(\lambda,\mu,\lambda\,y) : \lambda,\mu \in \mathbb{R}\}$ **only for** $x = 0$. Therefore an integral manifold through 0 does not exist.



18.4 Remark. Finding necessary conditions for integrability.

Suppose E is a subbundle of TM that has an integral manifold through each point. Let $p \in M$ and let N be such an integral manifold through p. Furthermore, let ξ and η be vector fields on M with $\xi_x, \eta_x \in E_x$ for all x. Because of lemma 17.5, vector fields $\bar\xi$ and $\bar\eta$ exist on N, such that $\bar\xi, \bar\eta$ are related to ξ, η with respect to incl. Then $[\bar\xi, \bar\eta]$ is a vector field on N which is incl-related to $[\xi, \eta]$ by 17.7.2. Thus we get $[\xi,\eta]_p = T\,\mathrm{incl}\,[\bar\xi,\bar\eta]_p \in E_p$.

18.5 Definition (Integrable subbundle).

A vector subbundle E of TM is called INTEGRABLE $:\Leftrightarrow$ for every two smooth vector fields ξ, η on M: $\xi_p, \eta_p \in E_p\ \forall\, p \Rightarrow [\xi,\eta]_p \in E_p\, \forall\, p$.

Exercise: Show that the subset of 18.3.3 is not integrable.
Hint: Consider the two generating vector fields.

18.6 Local Integrability Theorem of Frobenius.

Let E be a vector subbundle of π: $TM \to M$. Then E is integrable if and only if for each $p \in M$ there exists an integral manifold through p (moreover, there is a chart φ centered at p, such that $\varphi(\mathbb{R}^k \times \{a\})$ is an integral manifold for each a).

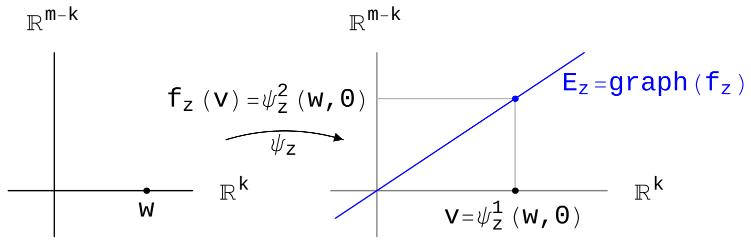
The images $\varphi(\mathbb{R}^k \times \{a\})$ are called PLAQUES.

Proof. $(\Leftarrow)$ We already showed that in 18.4.

$(\Rightarrow)$ Without loss of generality, $M \subseteq \mathbb{R}^m$ is open and $\psi : M \times \mathbb{R}^m \to M \times \mathbb{R}^m$ is a VB chart trivializing $E \subseteq M \times \mathbb{R}^m$, that is $\psi_z := \psi(z,.) : \mathbb{R}^k \times \{0\} \to E_z$ is an isomorphism for each $z \in M$. By applying a rotation to M and hence to T_0M we may assume that $E_0 = \mathbb{R}^k \times \{0\}$ and by composing ψ with $\mathrm{id} \times \psi_0^{-1}$ we may further assume that $\psi_0 = \mathrm{id}$, in particular, $\mathrm{pr}_k \circ \psi_0 \circ \mathrm{incl}_k = \mathrm{id} \in GL(k)$. Thus, $\mathrm{pr}_k \circ \psi_z \circ \mathrm{incl}_k \in GL(k)$ for all z close to 0.

We now want to represent each of the subspaces E_z as a graph of a linear mapping $f_z : \mathbb{R}^k \to \mathbb{R}^{m-k}$. Because of $\mathrm{graph}(f_z) := \{(v, f(z)v) : v \in \mathbb{R}^k\}$ and $E_z = \{(\psi_z^1(w,0), \psi_z^2(w,0)) : w \in \mathbb{R}^k\}$ we need $f_z(v) = \psi_z^2(w,0)$ with $\psi_z^1(w,0) = v$ for a (uniquely determined) $v \in \mathbb{R}^k$, i.e. $f : M \to L(k, m-k)$ must be given by:

$$f_z := f(z) = \psi_z^2 \circ (\psi_z^1|_{\mathbb{R}^k})^{-1} = \mathrm{pr}_{m-k} \circ \psi_z \circ \mathrm{incl}_k \circ (\mathrm{pr}_k \circ \psi_z \circ \mathrm{incl}_k)^{-1},$$



What tells us integrability in this context?
For $\xi \in \mathfrak{X}(M)$ we have: $\xi_p \in E_p \Leftrightarrow \xi_p \in \operatorname{graph} f(p) \Leftrightarrow \xi_p = (\xi_1|_p, \xi_2|_p)$, with $f(p)(\xi_1|_p) = (\xi_2|_p)$. Let $\xi, \eta : \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^{m-k} \times \mathbb{R}^k$ with ξ_p, $\eta_p \in E_p$. By assumption $[\xi,\eta]_p \in E_p$, thus

$$[\xi,\eta](p) = \big([\xi,\eta]_1(p), f(p)([\xi,\eta]_1(p))\big) \text{ and on the other hand}$$

$$\begin{aligned}
[\xi,\eta](p) &= \eta'(p)\big(\xi(p)\big) - \xi'(p)\big(\eta(p)\big)\\
&= \Big(\eta_1'(p)\big(\xi(p)\big) - \xi_1'(p)\big(\eta(p)\big), \eta_2'(p)\big(\xi(p)\big) - \xi_2'(p)\big(\eta(p)\big)\Big)\\
&= \bigg(\eta_1'(p)\big(\xi(p)\big) - \xi_1'(p)\big(\eta(p)\big), f'(p)\big(\xi(p)\big)\big(\eta_1(p)\big) + f(p)\Big(\eta_1'(p)\big(\xi(p)\big)\Big)\\
&\qquad - f'(p)\big(\eta(p)\big)\big(\xi_1(p)\big) - f(p)\Big(\xi_1'(p)\eta(p)\Big)\bigg)\\
&= \Big([\xi,\eta]_1(p), f(p)([\xi,\eta]_1(p)) + f'(p)\big(\xi(p)\big)\big(\eta_1(p)\big) - f'(p)\big(\eta(p)\big)\big(\xi_1(p)\big)\Big)
\end{aligned}$$

For $v_1 := \xi_1(p)$ and $v_2 := \eta_1(p)$ with $v_1, v_2 \in \mathbb{R}^k$ we get:

$$f'(p)(v_1, f(p)v_1)v_2 = f'(p)(v_2, f(p)v_2)v_1.$$

We want to find a $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\varphi(\mathbb{R}^k \times \{a\})$ is an integral manifold for all a, i.e. $(\partial_1\varphi)(z) : \mathbb{R}^k \to E_{\varphi(z)}$ should be an isomorphism. Without loss of generality (as will be seen), we further restrict the appearance of φ by the following condition:

$$\varphi(0,y) = (0,y), \quad (\partial_1\varphi)(z)\cdot v = (v, f(\varphi(z))v).$$

If $\varphi(x,y) =: (\varphi_1(x,y), \varphi_2(x,y))$, then

$$\begin{aligned}
&(\partial_1\varphi_1(z)\cdot v, \partial_1\varphi_2(z)\cdot v) = \partial_1\varphi(z)\cdot v = (v, f(\varphi(z))\cdot v)\\
&\Rightarrow\ \varphi_1(x,y) = \varphi_1(0,y) + x = x \ \Rightarrow\ \varphi(x,y) = (x, \varphi_2(x,y)) =: (x, g_y(x)),\\
&\quad \text{where } g_y(0) = y \text{ and } g_y'(x) = f(x, g_y(x)) \text{ has to hold.}
\end{aligned}$$

Now everything follows from the following proposition: □

18.7 Theorem of Frobenius for total differential equations.

Let $f : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n \to L(k,n)$ be locally C^∞. Then for each $(x_0,y_0) \in \mathbb{R}^m$ there is a local C^∞ mapping $g_{x_0,y_0} : \mathbb{R}^k \to \mathbb{R}^n$ with $g'_{x_0,y_0}(x)v = f(x, g_{x_0,y_0}(x))v$ and $g_{x_0,y_0}(x_0) = y_0$ if and only if $f'(z)(v_1, f(z)v_1)v_2$ is symmetric in v_1, v_2.
Further, the mapping $(x_0, y_0, x) \mapsto g_{x_0,y_0}(x)$ is C^∞.

Remark.

Let $\{e_1, \dots, e_k\}$ be a basis for $\mathbb{R}^k$ and put $f_i(z) := f(z)e_i$. Then $f(z)v = \sum_{i=1}^k f_i(z)v^i$ and $\partial_i g(x) = f_i(x, g(x))$ with $1 \le i \le k$ is a system of partial differential equations. We will prove 18.7 in a basis-free way. (A proof in coordinates can be found, for example, in [**136**, Vol.I, S.254].)

Proof of 18.7. See [**81**, 6.5.1]. Let $E = \mathbb{R}^k$ and $F = \mathbb{R}^{m-k}$.

($\Rightarrow$) For $z_0 = (x_0, y_0) \in E \times F$ let g be a local solution of the above differential equation with initial condition $g(x_0) = y_0$. Then $g' = f \circ (\mathrm{id}, g)$ and by the chain rule

$$g''(x_0)(v_1, v_2) = (g')'(x_0)(v_1)(v_2) = \mathrm{ev}_{v_2}\Big((g')'(x_0)(v_1)\Big) = \mathrm{ev}_{v_2}\Big((f \circ (\mathrm{id}, g))'(x_0)(v_1)\Big)$$
$$= \mathrm{ev}_{v_2}\Big(f'\big(x_0, g(x_0)\big)\big(v_1, g'(x_0)(v_1)\big)\Big) = f'(z_0)\Big(v_1, f(z_0)(v_1)\Big)(v_2)$$

Since $g''(x_0)$ is symmetric by the Theorem of Schwarz, thus the same hold for the right side.

($\Leftarrow$) Let $(x_0, y_0) \in E \times F$. We try to reduce the total differential equation to an ordinary one by first examining what happens at x_0 in the direction of $v \in E$.
For the moment we assume that a local solution g of the total differential equation with initial value $g(x_0) = y_0$ exists and put $\bar g(t, v) := g(x_0 + tv)$. Then

$$\frac{\partial}{\partial t}\bar g(t, v) = g'(x_0 + tv) \cdot v = f(x_0 + tv, g(x_0 + tv)) \cdot v = f(x_0 + tv, \bar g(t, v)) \cdot v,$$
$$\bar g(0, v) = g(x_0) = y_0.$$

This is an ordinary differential equation for $\bar g$ which thus locally (i.e. for $|t| \le \varepsilon$, $\|v\| \le \varepsilon$ with a certain $\varepsilon > 0$) has a unique solution $\bar g$, which depends smoothly on (t, v, x_0, y_0). From this we should get a solution g of the total differential equation by taking $g(x) := \bar g(t, v)$ with $tv := x - x_0$. Obviously $t = 1$ would be nice, but the solution $\bar g$ need not exist till then. Thus we choose $t := \varepsilon$ and hence $v := \frac{x - x_0}{\varepsilon}$ and $g(x) := \bar g(\varepsilon, \frac{x-x_0}{\varepsilon})$ for $\|x - x_0\| \le \varepsilon^2$. Then we have to calculate $g'(x)(w)$ and in particular $\partial_2 \bar g$. The idea is that

$$\partial_2 \bar g(t, v)(w) = \frac{\partial}{\partial s}\Big|_{s=0} \bar g(t, v + sw) = \frac{\partial}{\partial s}\Big|_{s=0} g(x_0 + t(v + sw))$$
$$= g'(x_0 + tv)(tw) = f(x_0 + tv, g(x_0 + tv))(tw)$$
$$= f(x_0 + tv, \bar g(t, v))(tw)$$

should be valid. Thus we define $k : \mathbb{R} \to F$ by

$$k(t) := \partial_2 \bar g(t, v)(w) - f(x_0 + tv, \bar g(t, v))(tw).$$

Then we get $k(0) = \partial_2 \bar g(0, v)(w) - f(x_0 + 0v, \bar g(0, v))(0w) = 0$ and, after applying the chain rule, - where we omit for the sake of clarity the argument (t, v) of $\bar g$ and of its derivatives as well as the argument $(x_0 + tv, \bar g(t, v))$ of f and its derivatives -

$$\frac{d}{dt}k(t) = \frac{\partial}{\partial t}\Big(\partial_2 \bar g(t, v)(w) - f(x_0 + tv, \bar g(t, v))(tw)\Big)$$
$$= \partial_2 \underbrace{\frac{\partial}{\partial t}\bar g(t, v)}_{f(x_0 + tv, \bar g(t, v)) \cdot v}(w) - \Big(\partial_1 f \cdot v \cdot tw + \partial_2 f \cdot \underbrace{\frac{\partial}{\partial t}\bar g}_{f \cdot v} \cdot tw + f \cdot w\Big)$$
$$= \Big(\partial_1 f \cdot tw \cdot v + \partial_2 f \cdot (\partial_2 \bar g \cdot w) \cdot v + f \cdot w\Big) - \Big(f' \cdot (v, f \cdot v) \cdot tw + f \cdot w\Big)$$
$$\xlongequal{\text{Int.Cond.}} \partial_1 f \cdot tw \cdot v + \partial_2 f \cdot (\partial_2 \bar g \cdot w) \cdot v - f' \cdot (tw, f \cdot tw) \cdot v$$
$$= \partial_2 f \cdot (\partial_2 \bar g \cdot w - f \cdot tw) \cdot v = \partial_2 f \cdot k(t) \cdot v.$$

Since this is a linear differential equation (with non-constant coefficients) and $k(0) = 0$ we conclude $k = 0$. Thus, for $g(x) := \bar g(t, v)$ with $t := \varepsilon$ and $v := \frac{x-x_0}{\varepsilon}$ we have

$$g'(x)(w) = \partial_2 \bar g(\varepsilon, \tfrac{x-x_0}{\varepsilon})(\tfrac{1}{\varepsilon}w) = \partial_2 \bar g(t, v)(\tfrac{1}{\varepsilon}w) = f(x_0 + tv, \bar g(t, v))\,(t\tfrac{1}{\varepsilon}w)$$
$$= f\Big(x, \bar g(\varepsilon, \tfrac{x-x_0}{\varepsilon})\Big)(w) = f(x, g(x))(w). \quad \square$$

18.8 Special cases.

In particular we get (if $f : \mathbb{R}^m \times \mathbb{R}^n \to L(m, n)$ only depends on one factor):

1. For $f : \mathbb{R}^m \to L(m, n)$: $f'(x) \cdot v_1 \cdot v_2 = f'(x) \cdot v_2 \cdot v_1 \Leftrightarrow$ there is a local $g : \mathbb{R}^m \to \mathbb{R}^n$ with $g(0) = 0$ and $g'(x)v = f(x)v$, that is $g' = f$.
2. For $f : \mathbb{R}^n \to L(m, n)$: $f'(y)(f(y)v_1)v_2 = f'(y)(f(y)v_2)v_1 \Leftrightarrow$ there is a local $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ with $g_y(0) = y$ and $g'_y(x) = f(g_y(x))$.

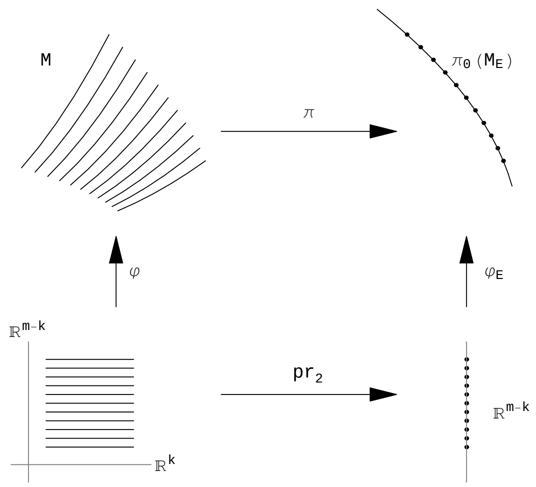
18.9 Integrability Theorem of Frobenius, global version.

If E is an integrable subbundle of TM, then:

1. *There is a manifold structure M_E on M such that the inclusion* $\mathrm{incl} : M_E \to M$ *is an immersion with $T\,\mathrm{incl}(TM_E) = E$, i.e. $T\,\mathrm{incl} : TM_E \to E \subseteq TM$ is bijective.*
2. *Let $f : N \to M$ be smooth with $Tf(TN) \subseteq E$. Then $f : N \to M_E$ is smooth.*
3. *Each connected component of M_E is an initial submanifold of M, and is paracompact if M is it.*
4. *If N is a connected integral manifold, then N is an open submanifold of a connected component of M_E (hence the later ones are called* MAXIMAL INTEGRAL MANIFOLDS*).*

In this situation, one speaks of the FOLIATION M_E induced by E on M. The maximal integral manifolds are called LEAVES OF THE FOLIATION (**Attention:** This is something different than the leaves of a covering).

Proof. By 18.6, there are charts φ, such that $\varphi\,(\mathbb{R}^k \times \{a\})$ is an integral manifold for each a, i.e. $T_z\varphi(\mathbb{R}^k \times \{0\}) = E_{\varphi(z)}$ for all $z \in \operatorname{dom}\varphi$.



Let $f : N \to M$ be smooth with $\mathrm{im}(Tf) \subseteq E$ *and $f(p) \in \operatorname{im}\varphi$, then f is locally in some plaque $\varphi(\mathbb{R}^k \times \{a\})$*: In fact, for $\bar f := \varphi^{-1} \circ f$ we have

$$\left.\begin{aligned} \mathrm{im}(T_x\bar f) = T_{f(x)}\varphi^{-1}(\mathrm{im}\, T_x f) \subseteq T_{f(x)}\varphi^{-1}(E_{f(x)}) = \mathbb{R}^k \times \{0\} \\ \exists\, a : \ \bar f(p) \in \mathbb{R}^k \times \{a\} \end{aligned}\right\} \Rightarrow \mathrm{im}\,\bar f \subseteq \mathbb{R}^k \times \{a\}.$$

(1) *The family*

$$\left\{\varphi|_{(\mathbb{R}^k\times\{a\})},\ \varphi \text{ is trivializing for } E \text{ as in } 18.6,\ a \in \mathbb{R}^{m-k}\right\}$$

is an atlas on the set M:
For this we have to show that the chart changes are smooth maps defined on open sets:

Consider φ_1, φ_2; a_1, a_2 and $p \in \varphi_1(\mathbb{R}^k \times \{a_1\}) \cap \varphi_2(\mathbb{R}^k \times \{a_2\})$. Since $\varphi_1|_{(\mathbb{R}^k \times \{a_1\})} : \mathbb{R}^k \times \{a_1\} \to M$ is an integral manifold, the image $\operatorname{im}(\varphi_1|_{(\mathbb{R}^k \times \{a_1\})})$ is contained locally in $\varphi_2(\mathbb{R}^k \times \{a_2\})$. So

$$\Big(\varphi_2|_{(\mathbb{R}^k \times \{a_2\})}\Big)^{-1} \circ \Big(\varphi_1|_{(\mathbb{R}^k \times \{a_1\})}\Big)$$

is well-defined locally and smooth as a restriction of $\varphi_2^{-1} \circ \varphi_1$.

We denote the so obtained manifold by M_E. *The inclusion $M_E \hookrightarrow M$ is an immersion* because the inclusion $\mathbb{R}^k \times \{a\} \hookrightarrow \mathbb{R}^m$ is its chart representation. Moreover, $T\,\mathrm{incl}(M_E) = E$ since $T_x\varphi(\mathbb{R}^k \times \{0\}) = E_{\varphi(x)}$ for all $x \in \operatorname{dom}\varphi$.

($\boxed{2}$) Let $f : N \to M$ be smooth and $\operatorname{im}(Tf) \subseteq E$. Then f lies locally in a plaque $\varphi(\mathbb{R}^k \times \{a\})$ and thus $\big(\varphi|_{(\mathbb{R}^k \times \{a\})}\big)^{-1} \circ f$ is locally well-defined and smooth, i.e. $f : N \to M_E$ is smooth.

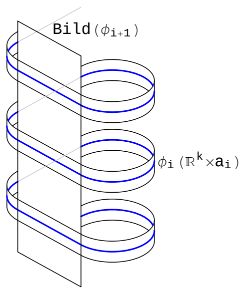
($\boxed{3}$) *With M also M_E is paracompact*: Without restricting generality, M is connected and let C be a connected component of M_E. By $\boxed{9.3}$ it suffices to show that *C is covered by countable many chart images* $\varphi(\mathbb{R}^k \times \{a\})$, hence is σ-compact.

Let $\mathcal{A}$ be a countable family of E-trivializing charts which covers M and let $p_0 \in C$ be fixed and $p \in C$ arbitrary. Since a curve exists in C connecting p_0 and p, there are finitely many charts $\varphi_1, \dots, \varphi_n \in \mathcal{A}$ and some $a_1, \dots, a_n$, such that:
$p_0 \in \varphi_1(\mathbb{R}^k \times \{a_1\})$, $p \in \varphi_n(\mathbb{R}^k \times \{a_n\})$ and $\varphi_i(\mathbb{R}^k \times \{a_i\}) \cap \varphi_{i+1}(\mathbb{R}^k \times \{a_{i+1}\}) \neq \emptyset$.

For given φ_i, φ_{i+1}, a_i, there are at most countably many a_{i+1} with

$$\varphi_i(\mathbb{R}^k \times \{a_i\}) \cap \varphi_{i+1}(\mathbb{R}^k \times \{a_{i+1}\}) \neq \emptyset,$$

because otherwise there would be a covering of $\varphi_i(\mathbb{R}^k \times \{a_i\}) \cap \operatorname{im}\varphi_{i+1}$ by uncountable many disjoint and (in the topology induced by $(\varphi_i|_{\mathbb{R}^k \times \{a_i\}})^{-1}(\operatorname{im}\varphi_{i+1}) \subseteq \mathbb{R}^k)$ open sets $\varphi_{i+1}(\mathbb{R}^k \times \{a\})$, which would give a contradiction to the Lindelöf property.



Thus there are only countably many finite sequences $(\varphi_i, a_i)_i$ that satisfy the condition $\varphi_i(\mathbb{R}^k \times \{a_i\}) \cap \varphi_{i+1}(\mathbb{R}^k \times \{a_{i+1}\}) \neq \emptyset$. Each $p \in C$ is reached by an appropriate sequence. So C is covered by countably many chart images $\varphi(\mathbb{R}^k \times \{a\})$.

The connected component C is an initial submanifold: Let $f : N \to C \subseteq M$ be smooth. Locally f has values in $\operatorname{im}\varphi$ and globally also in C. However, since C (as a countable union of plaques) meets at most countably many plaques of φ, f lies locally in one plaque (different plaques are not connected in $\operatorname{im}\varphi$ with each other). Thus $\operatorname{im}(Tf) \subseteq T(\varphi(\mathbb{R}^k \times \{a\})) = E|_{\varphi(\mathbb{R}^k \times \{a\})}$, hence $f : N \to M_E$ is smooth by $\boxed{2}$.

($\boxed{4}$) Let $N \to M$ be a connected integral manifold, then $\mathrm{incl} : N \to M_E$ is smooth by $\boxed{2}$. Furthermore, $\mathrm{incl} : N \to M_E$ is injective and immersive (since incl : $N \to M$ is so) and submersive (since $T\,\mathrm{incl} : T_pN \to E_p$ is bijective), hence a local diffeomorphism. Thus, incl : $N \hookrightarrow M_E$ is a diffeomorphism onto an open subset of M_E. □

18.10 Proposition.

Let $f : M \to N$ be smooth and $x \mapsto T_xf$ have constant rank r. Then, $\ker(Tf) := \bigsqcup_{x\in M} \ker(T_xf)$ is an integrable vector subbundle of TM, and the connected components of the level sets $f^{-1}(q)$ are the maximal integral manifolds for $\ker(Tf)$.

18.11 Definition (Riemannian manifold).

A RIEMANNIAN METRIC on a manifold M is a function g which associates to each point $x \in M$ a positive definite symmetric bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$ such that for any vector fields $\xi, \eta \in \mathfrak{X}(M)$ the mapping $x \mapsto g_x(\xi_x, \eta_x)$ from M to $\mathbb{R}$ is smooth.

A RIEMANNIAN MANIFOLD is a manifold M together with a specified Riemannian metric g.

If the metric is specified only up to multiples with smooth positive functions, then one speaks of a CONFORMAL MANIFOLD.

Substituting the condition of positive definiteness by that of non-degeneracy, that is, $v \mapsto \langle v, \cdot\rangle$ and $v \mapsto \langle \cdot, v\rangle$ are injective as mappings $\mathbb{R}^m \to (\mathbb{R}^m)^*$, one obtains the notion of a PSEUDO-RIEMANNIAN METRIC and the corresponding manifolds are called PSEUDO-RIEMANNIAN MANIFOLDS. If the signature is -1, then one speaks of a LORENTZIAN MANIFOLD.

If one considers complex manifolds and replaces the condition "bilinear form" by "Hermitian form", one speaks of HERMITIAN MANIFOLDS. The real part of the Hermitian form is a Riemannian metric.

18.12 Definition (Length and distance).

Let (M, g) be a Riemannian manifold, then we can define the LENGTH OF TANGENT VECTORS $\xi_x \in T_xM$ as $|\xi_x| := \sqrt{g_x(\xi_x, \xi_x)}$.

If $c : [0, 1] \to M$ is a smooth curve in M, let the LENGTH of c is defined by

$$L(c) := \int_0^1 \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt.$$

As one easily convinces oneself, we also have a metric $d_g : M \times M \to \mathbb{R}^+$ in the sense of topology for connected Riemannian manifolds (M, g):

$$d_g(p, q) := \inf\Big\{L(c) : c \in C^\infty(\mathbb{R}, M);\ c(0) = p,\ c(1) = q\Big\}.$$

For each smooth immersive $f : N \to M$ the mapping $(v, w) \mapsto g_{f(x)}(T_xf \cdot v, T_xf \cdot w)$ for $v, w \in T_xN$ defines a Riemannian metric f^*g on N and it satisfies:

$$\begin{aligned}
L_{f^*g}(c) &= L_g(f \circ c) \text{ and thus}\\
d_g(f(x), f(y)) &= \inf\big\{L_g(c) : c \text{ connects } f(x) \text{ to } f(y)\big\}\\
&\le \inf\big\{L_g(f \circ c) : c \text{ connects } x \text{ to } y\big\}\\
&= \inf\big\{L_{f^*g}(c) : c \text{ connects } x \text{ to } y\big\}\\
&= d_{f^*g}(x, y), \text{ hence}\\
f\Big(\big\{x : d_{f^*g}(x, x_0) < r\big\}\Big) &\subseteq \big\{y : d_g(y, f(x_0)) < r\big\}
\end{aligned}$$

We now show that *the metric d_g generates the topology*:

To see that the identity from the manifold M to the metric space (M, d_g) is continuous, we use that for the chart representation $\varphi^* g$ with respect to a chart $\varphi : \mathbb{R}^m \supseteq U \to \varphi(U) \subseteq M$ and for all x in a compact subset of U we have the inequalities

$$M_1^2 \cdot |v|^2 \leq (\varphi^* g)_x(v, v) \leq M_2^2 \cdot |v|^2$$

with some constants $M_1, M_2 > 0$, hence

$$\varphi\Big(\big\{x : |x - x_0| \leq \frac{\varepsilon}{M_2}\big\}\Big) \subseteq \varphi\Big(\big\{x : d_{\varphi^* g}(x, x_0) \leq \varepsilon\big\}\Big) \subseteq \big\{y : d_g(y, \varphi(x_0)) \leq \varepsilon\big\}$$

and $d_g : M \times M \to \mathbb{R}$ is continuous.
In fact: On the one hand $\{(\varphi^* g)_x(w, w) : |w| \leq 1, x \text{ in a compact set}\}$ is compact, so bounded by some M_2^2 and thus $(\varphi^* g)_x(v, v) = |v|^2\, (\varphi^* g)_x(w, w) \leq M_2^2\, |v|^2$ with $v =: |v|\, w$. On the other hand,

$$M_1^2 := \inf\Big\{(\varphi^* g)_x(v, v)/|v|^2 : v \neq 0, x \text{ in a compact set}\Big\} > 0,$$

otherwise, x_n and $v_n \neq 0$ exist with

$$(\varphi^* g)_{x_n}(w_n, w_n) = (\varphi^* g)_{x_n}(v_n, v_n)/|v_n|^2 \to 0 \text{ for } w_n := v_n/|v_n|.$$

and for accumulation points x_∞ of x_n and w_∞ of w_n we have $|w_\infty| = 1$ but $(\varphi^* g)_{x_\infty}(w_\infty, w_\infty) = 0$.

Conversely, let $\varphi : U \to M$ be a chart centered at y_0 and V a relatively compact open neighborhood of 0 with $\bar{V} \subseteq U$. According to the above, there is an $M_1 > 0$ with $M_1^2 \cdot |v|^2 \leq (\varphi^* g)_x(v, v)$ for all $x \in \bar{V}$. Let $\varepsilon > 0$ with $\{x : M_1 |x| \leq \varepsilon\} \subseteq V$ and $\bar{\varphi} := \varphi|_V$. Then $d_{\bar{\varphi}^* g}(x, 0) \geq M_1 |x|$ for all $x \in \bar{V}$, because

$$L_{\bar{\varphi}^* g}(c) = \int_0^1 \sqrt{(\varphi^* g)_{c(t)}(c'(t), c'(t))}\, dt \geq M_1 \int_0^1 |c'(t)|\, dt \geq M_1\, |c(1) - c(0)|$$

for each smooth curve $c : [0, 1] \to V$. Finally,

$$\begin{aligned} \big\{y : d_g(y, y_0) < \varepsilon\big\} &\subseteq \bar{\varphi}\big(\{x \in V : d_{\bar{\varphi}^* g}(x, 0) < \varepsilon\}\big) \\ &\subseteq \bar{\varphi}\big(\{x \in V : M_1 |x| < \varepsilon\}\big) \subseteq \varphi(U). \end{aligned}$$

Otherwise, there is a smooth curve $c : [0, 1] \to M$ with $c(0) = y_0$ and $L_g(c) < \varepsilon$, but $y = c(1)$ is not in the open set $\bar{\varphi}(\{x \in V : d_{\bar{\varphi}^* g}(x, 0) < \varepsilon\})$. Now choose t_∞ minimal with $c(t_\infty) \notin \bar{\varphi}(\{x \in V : d_{\bar{\varphi}^* g}(x, 0) < \varepsilon\})$. Then $c(t) \in \bar{\varphi}(\{x \in V : d_{\bar{\varphi}^* g}(x, 0) < \varepsilon\})$ for all $t < t_\infty$ and has a accumulation point x_∞ in the compact set $\{x \in \bar{V} : d_{\bar{\varphi}^* g}(x, 0) \leq \varepsilon\} \subseteq \bar{V} \subseteq U$ for $t \to t_\infty$. Thus, $c(t_\infty) = \bar{\varphi}(x_\infty) \in \varphi(U)$ and hence $\varphi^{-1} \circ c|_{[0, t_\infty]}$ is a well-defined smooth curve in V with

$$L_{\bar{\varphi}^* g}(\bar{\varphi}^{-1} \circ c|_{[0, t_\infty]}) = L_g(\bar{\varphi} \circ \bar{\varphi}^{-1} \circ c|_{[0, t_\infty]}) = L_g(c|_{[0, t_\infty]}) < \varepsilon,$$

i.e. $d_{\bar{\varphi}^* g}((\bar{\varphi}^{-1} \circ c)(t_\infty), 0) < \varepsilon$, a contradiction.

It is interesting to actually find a shortest connection between two points. This is a variation problem which we will address in paragraph [**86**, 57].

VI. Differential Forms

In this chapter, we start with 1-forms and the necessary cotangent spaces. Then we generalize these forms to differential forms of higher degree (in short: n forms). After a motivational section, we put together the necessary multilinear algebra and glue the tensor spaces constructed from the tangential and cotangent spaces to form tensor bundles. As sections of the bundles of alternating tensors we obtain the differential forms. We treat the most important operations on them: The outer derivative, the Lie derivative, and the insertion homomorphism. In particular, we will take a closer look at this for Riemannian manifolds. As an application we introduce the De Rham cohomology.

19. Constructions and 1-forms

19.1 Motivation.

For path integrals in $\mathbb{R}^m$, the notion of 1-form is important because these are the objects that can be integrated along curves (see [**86**, 3.10] or [**81**, 6.5.6]). We now want to extend this concept to manifolds. Recall that a 1-form ω on an open subset $M \subseteq \mathbb{R}^m$ is a map $\omega : M \to L(\mathbb{R}^m, \mathbb{R})$. The path integral of ω along a curve $c : \mathbb{R} \to M$ is then defined as the usual Riemann integral of $t \mapsto \omega(c(t))(c'(t))$. On a general manifold M the velocity vector $c'(t) \in T_{c(t)}M$ and thus $\omega(x)$ should be in $L(T_xM, \mathbb{R}) = (T_xM)^*$ for each $x \in M$.

19.2 Definition (1-forms).

By a 1-FORM on a manifold M we understand a mapping ω which associates to each point $x \in M$ a linear functional $\omega(x) \in (T_xM)^*$.

Let $f : M \to \mathbb{R}$ be a smooth function. Then we have a 1-form, the TOTAL DIFFERENTIAL df of f, given by $df(x)(v) := v(f) \in \mathbb{R}$ for all $v \in T_xM = \operatorname{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R})$.

We now want to describe 1-forms in local coordinates. For this we need coordinates in $(T_xM)^*$. If E is a m-dimensional vector space and $(g_i)_{i=1}^m$ is a basis in E, one obtains a basis $(g^i)_{i=1}^m$ for E^*, the so-called DUAL BASIS, by specifying the functionals g^i on the basis $(g_j)_{j=1}^m$ by $g^i(g_j) := \delta^i_j$, where δ^i_j is the Kronecker delta symbol, i.e. $\delta^i_i := 1$ and $\delta^i_j := 0$ for $i \neq j$.

Now let $(u^1, \dots, u^m)$ be local coordinates on M. Then $(\frac{\partial}{\partial u^i}|_x)_{i=1}^m$ is a basis of T_xM. If we now calculate the total differential du^i of the i-th coordinate functions u^i, we obtain:

$$du^i|_x\big(\tfrac{\partial}{\partial u^j}|_x\big) = \tfrac{\partial}{\partial u^j}|_x(u^i) = \partial_j(u^i \circ \varphi)(\varphi^{-1}(x)) = \partial_j(\operatorname{pr}_i)(\varphi^{-1}(x)) = \delta^i_j.$$

So $(du^i|_x)_{i=1}^m$ is precisely the dual basis to the basis $(\frac{\partial}{\partial u^i}|_x)_{i=1}^m$ of T_xM and $du^i|_x(\xi_x) = \xi^i_x$ for $\xi_x = \sum_i \xi^i_x \frac{\partial}{\partial u^i} \in T_xM$. For the total differential df of a function $f : M \to \mathbb{R}$

we get

$$df = \sum_i \frac{\partial f}{\partial u^i} \cdot du^i,$$

because $df(x)(\xi_x) = \xi_x(f) = \Big(\sum_i \xi_x^i \frac{\partial}{\partial u^i}\Big)(f) = \sum_i du^i(\xi_x)\,\frac{\partial f}{\partial u^i} = \Big(\sum_i \frac{\partial f}{\partial u^i} \cdot du^i\Big)(\xi_x)$.

19.3 Transformation behavior of vectors.

(Compare this with [**86**, 1.1] and 10.9) In the following, let E be a finite-dimensional vector space, $\mathcal{G} := (g_i)_{i=1}^m$ be a basis in E, and x^i the components (coordinates) of a point x in E with respect to $\mathcal{G}$, i.e. $x = \sum_{i=1}^m x^i g_i$. Let $\bar{\mathcal{G}} := (\bar g_j)_j$ be a second basis and $\bar x^j$ the coordinates of x with respect to $\bar{\mathcal{G}}$. Let A be the isomorphism of E, which maps the g_i to $\bar g_i$. If one represents the vectors $\bar g_j$ with respect to the basis $\mathcal{G}$, i.e. $\bar g_j = \sum_{i=1}^m a_j^i g_i$, then $[A] := (a_j^i)_{i,j}$ (where the upper index i numbers the rows and the lower j the columns of the matrix) is

- the matrix representation $[A]_{\mathcal{G},\mathcal{G}}$ of A with respect to the basis $\mathcal{G}$ for $\operatorname{dom}(A) = E$ and for $\operatorname{im}(A) = E$,
- and the matrix representation $[\mathrm{id}]_{\bar{\mathcal{G}},\mathcal{G}}$ of the identity with respect to the basis $\bar{\mathcal{G}}$ for $\operatorname{dom}(\mathrm{id}) = E$ and the basis $\mathcal{G}$ for $\operatorname{im}(\mathrm{id}) = E$,
- and also the matrix representation $[A]_{\bar{\mathcal{G}},\bar{\mathcal{G}}}$ with respect to the basis $\bar{\mathcal{G}}$ for $\operatorname{dom}(A) = E$ and for $\operatorname{im}(A) = E$.

In fact, the first two representations follow from [**86**, 1.1], according to which the j-th column of the matrix representation of a linear mapping are the coefficients of the image of the j-th basis vector with respect to the basis in the range space. On the other hand, $[A]_{\mathcal{G},\bar{\mathcal{G}}} = 1$, because $A(g_i) = \bar g_i$ hence $[A(g_i)]_{\bar{\mathcal{G}}} = (\delta_i^j)_j$, and thus $[A]_{\bar{\mathcal{G}},\bar{\mathcal{G}}} = [A \circ \mathrm{id}]_{\bar{\mathcal{G}},\bar{\mathcal{G}}} = [A]_{\mathcal{G},\bar{\mathcal{G}}} \cdot [\mathrm{id}]_{\bar{\mathcal{G}},\mathcal{G}} = 1 \cdot [A]_{\mathcal{G},\mathcal{G}} = [A]$.

Summarized: $[A] = [A]_{\mathcal{G},\mathcal{G}} = [A]_{\bar{\mathcal{G}},\bar{\mathcal{G}}} = [\mathrm{id}]_{\bar{\mathcal{G}},\mathcal{G}}$ and $[A]_{\mathcal{G},\bar{\mathcal{G}}} = 1$.

For the transformation behavior of the components, we thus obtain:

$$[x]_{\mathcal{G}} = 1 \cdot [x]_{\mathcal{G}} = [A]_{\mathcal{G},\bar{\mathcal{G}}} \cdot [x]_{\mathcal{G}} = [A(x)]_{\bar{\mathcal{G}}} = [A]_{\bar{\mathcal{G}},\bar{\mathcal{G}}} \cdot [x]_{\bar{\mathcal{G}}} = [A] \cdot [x]_{\bar{\mathcal{G}}}$$

Conversely, $A^{-1} : E \to E$ is given by $A^{-1} : \bar g_j \mapsto g_j$ with matrix representation $[A^{-1}] = [A]^{-1} =: (b_j^i)_{i,j}$.

Let $E^* := L(E, \mathbb{R})$ denote the dual space to E and let $\mathcal{G}^* := (g^i)_{i=1}^m$ be the dual basis to $\mathcal{G} = (g_i)_{i=1}^m$ defined by $g^i(g_j) := \delta_j^i$. Each vector $x^* \in E^*$ can then be written in the form $x^* = \sum_{i=1}^m x_i g^i$, with coefficients $x_i = x^*(g_i) \in \mathbb{R}$.

How do these coordinates transform now?

The matrix representation $[T^*]_{\bar{\mathcal{G}}^*,\mathcal{G}^*}$ of the adjoint of a linear mapping T is the transpose of the matrix representation $(t_k^i)_{k,i} := [T]_{\mathcal{G},\bar{\mathcal{G}}}$ of T, i.e. $[T^*]_{\bar{\mathcal{G}}^*,\mathcal{G}^*} = [T]_{\mathcal{G},\bar{\mathcal{G}}}^t$, because

$$T^*(\bar g^i)(g_j) := \bar g^i(T(g_j)) = \bar g^i\Big(\sum_k t_j^k \bar g_k\Big) = \sum_k t_j^k \bar g^i(\bar g_k) = t_j^i = \sum_k t_k^i g^k(g_j).$$

By applying this to the basis-transformation map $A : g_i \mapsto \bar g_i$, we get

$$[A^*]_{\bar{\mathcal{G}}^*,\mathcal{G}^*} = [A]_{\mathcal{G},\bar{\mathcal{G}}}^t = 1^t = 1, \text{ i.e. } A^*(\bar g^j) = g^j$$

and furthermore $g^j = A^*(\bar g^j) = \sum_{i=1}^m a_i^j \bar g^i$, because $[A^*]_{\bar{\mathcal{G}}^*,\bar{\mathcal{G}}^*} = [A]^t_{\bar{\mathcal{G}},\mathcal{G}} = [A]^t$. And thus the transformation behavior for the coordinates of dual vectors $x^* \in E^*$ is:

$$\sum_i \bar x_i \bar g^i = x^* = \sum_j x_j g^j = \sum_{i,j} x_j a_i^j \bar g^i \;\Rightarrow\; \bar x_i = \sum_j a_i^j x_j.$$

Comparing the transformation formulas, we conclude that the components x_i of the dual vectors $x^* \in E^*$ transform like the basis vectors g_i of the original space:

$$\bar x_j = \sum a_j^i x_i, \quad \bar g_j = \sum a_j^i g_i \;; \quad x_i = \sum b_i^j \bar x_j, \quad g_i = \sum b_i^j \bar g_j.$$

On the other hand, the components x^i of a vector $x \in E$ transform like the vectors of the dual basis g^i:

$$\bar x^j = \sum b_i^j x^i, \quad \bar g^i = \sum b_j^i g^i \;; \quad x^i = \sum a_j^i \bar x^j, \quad g^i = \sum a_j^i \bar g^j.$$

This fact also motivates the use of “upper” and “lower” indices: The component vectors of dual vectors transform like the basis in the original space (they TRANSFORM COVARIANTLY), the dual basis and the component vectors in the original space TRANSFORM CONTRAVARIANTLY.

However, compare that with the following

19.4 Definition (Co/Contra-variant functor).

By a FUNCTOR $\mathcal{F}$ on a category is meant an assignment which associates to each space M another space $\mathcal{F}(M)$ and associates to each morphism $f : M \to N$ a corresponding morphism $\mathcal{F}(f)$ between $\mathcal{F}(M)$ and $\mathcal{F}(N)$, such that $\mathcal{F}(\mathrm{id}_M) = \mathrm{id}_{\mathcal{F}(M)}$ and $\mathcal{F}$ applied to the composition of two morphisms is the composition of the associated morphisms.

One calls a functor $\mathcal{F}$ COVARIANT if $\mathcal{F}(f)$ runs in the same direction as f, that is $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(N)$ for $f : M \to N$. It is called CONTRAVARIANT if $\mathcal{F}(f)$ runs in the opposite direction, that is $\mathcal{F}(f) : \mathcal{F}(N) \to \mathcal{F}(M)$ for $f : M \to N$. In particular, the dual-space functor $(f : E \to F) \mapsto (f^* : F^* \to E^*)$ is **contravariant**.

19.5 Transformation behavior of 1-forms.

Let $\varphi^{-1} = (u^1, \dots, u^m)$ and $\psi^{-1} = (v^1, \dots, v^m)$ be charts of a manifold M, and let $\partial_i^\varphi = \frac{\partial}{\partial u^i}$ and $\partial_j^\psi = \frac{\partial}{\partial v^j}$ be the (local) basis vector fields of the tangent bundle. These are related according to 10.9 as follows:

$$\partial_j^\psi|_x = \sum_{i=1}^m \partial_j^\psi|_x(\varphi^{-1})^i\, \partial_i^\varphi|_x \text{ or more classically } \frac{\partial}{\partial v^j} = \sum_{i=1}^m \frac{\partial u^i}{\partial v^j}\frac{\partial}{\partial u^i}$$

$$\partial_j^\varphi|_x = \sum_{i=1}^m \partial_j^\varphi|_x(\psi^{-1})^i\, \partial_i^\psi|_x \text{ or more classically } \frac{\partial}{\partial u^j} = \sum_{i=1}^m \frac{\partial v^i}{\partial u^j}\frac{\partial}{\partial v^i}$$

If $a_j^i := \frac{\partial u^i}{\partial v^j}$ and $b_i^j := \frac{\partial v^j}{\partial u^i}$ are the coefficients of the Jakobi-matrix of the chart changes and the vector field ξ has the representations $\xi = \sum \xi^i \frac{\partial}{\partial u^i} = \sum \eta^j \frac{\partial}{\partial v^j}$, then

$$\xi = \sum_j \eta^j \sum_i \frac{\partial u^i}{\partial v^j}\frac{\partial}{\partial u^i} = \sum_i \left(\sum_j \eta^j \left(\frac{\partial u^i}{\partial v^j}\right)\right)\frac{\partial}{\partial u^i}$$

$$\Rightarrow \quad \xi^i = \sum_j \frac{\partial u^i}{\partial v^j}\eta^j = \sum_j a_j^i \eta^j$$

and analogously $\eta^j = \sum_j b_j^i \xi^j$ holds.

For cotangent vectors we get the following transformation formulas because of 19.3:

$$du^i = \sum_j \frac{\partial u^i}{\partial v^j} dv^j = \sum_j a^i_j dv^j$$
$$dv^j = \sum_i \frac{\partial v^j}{\partial u^i} du^i = \sum_i b^j_i du^i.$$

Therefore, the components of the cotangent vectors transform covariantly, so sections in the cotangent bundle (i.e. 1-forms) are also called COVARIANT VECTOR FIELDS.

19.6 Construction of the dual bundle.

In order to be able to talk about the smoothness of 1-forms, we have to turn the disjoint union $T^*M := (TM)^* := \bigsqcup_{x\in M}(T_xM)^*$ into a smooth manifold, or better, a vector bundle. More generally, for an arbitrary vector bundle $E \xrightarrow{p} M$, we want to make the disjoint union $E^* := \bigsqcup_{x\in M}(E_x)^*$ into a vector bundle. Let trivializations $\varphi : U \times \mathbb{R}^k \xrightarrow{\cong} E|_U$ of E over open sets $U \subseteq M$ be given. We need to construct trivializations

$$\varphi^* : \bigsqcup_{x\in U}(\mathbb{R}^k)^* = U \times (\mathbb{R}^k)^* \xrightarrow{\cong} E^*|_U = \bigsqcup_{x\in U}(E_x)^*$$

Fiber-wise, we may define φ^* as $(\varphi^*)_x := ((\varphi_x)^*)^{-1} = ((\varphi_x)^{-1})^* : (\mathbb{R}^k)^* \to (E_x)^*$, where $(\varphi_x)^* : (E_x)^* \to (\mathbb{R}^k)^*$ denotes the adjoint mapping to the isomorphism $\varphi_x : \mathbb{R}^k \to E_x$.

Let $\psi : U \cap V \to GL(\mathbb{R}^k)$ be the transition function for two vector bundle charts of E. The transition functions ψ^* belonging to the trivializations φ^* are then given by

$$\psi^*(x) := (\psi(x)^*)^{-1} \in GL((\mathbb{R}^k)^*) \cong GL(\mathbb{R}^k),$$

where $\psi(x)^*$ denotes the adjoint mapping to the linear isomorphism $\psi(x) : \mathbb{R}^k \to \mathbb{R}^k$. Since $A \mapsto A^*$, $L(\mathbb{R}^k, \mathbb{R}^l) \to L((\mathbb{R}^l)^*, (\mathbb{R}^k)^*)$, is linear, inversion $A \mapsto A^{-1}$ of $GL(\mathbb{R}^k) \to GL(\mathbb{R}^k)$ is smooth, and $\psi : U \cap V \to GL(\mathbb{R}^k)$ is also smooth as a transition function of vector bundle E, the same holds for the composition ψ^*

$$\begin{tikzcd}
 & & GL(\mathbb{R}^k) \arrow[dr, "\mathrm{inv}"] & \\
U \cap V \arrow[r, "\psi"] & GL(\mathbb{R}^k) \arrow[ur, "(\text{-})^*"] \arrow[dr, "\mathrm{inv}"'] & & GL(\mathbb{R}^k) \\
 & & GL(\mathbb{R}^k) \arrow[ur, "(\text{-})^*"'] &
\end{tikzcd}$$

Thus, the ψ^* form a cocycle of transition functions for a smooth vector bundle $E^* \to M$ and the φ^* are the associated vector bundle charts. This vector bundle $E^* \to M$ is called the DUAL BUNDLE of $E \to M$.

In the special case, where $E \to M$ is the tangent bundle $TM \to M$, the dual bundle $T^*M := (TM)^* \to M$ is called COTANGENT BUNDLE of M.

The space $C^\infty(M \leftarrow T^*M)$ of the smooth sections of the cotangent bundle (i.e. 1-forms) is denoted $\Omega^1(M)$.

19.7 Smooth 1-Forms.

How to check whether a 1-form ω is smooth? Well, this is the case locally around a point $x \in M$ if and only if its representation with respect to a trivialization $T^*M|_U \cong U \times \mathbb{R}^m$ with $x \in U \subseteq M$ is smooth. By 19.6, the trivializations of T^*M are obtained by dualizing those of TM. To a chart $\varphi : \mathbb{R}^m \supseteq U \to \varphi(U) \subseteq M$ with associated local coordinates $(u^1, \dots, u^m) = \varphi^{-1}$, the corresponding local trivialization of $TM \to M$ in 14.4 was given by

$$TM \supseteq T(\varphi(U)) \xleftarrow{T\varphi} TU \cong U \times \mathbb{R}^m \xleftarrow{\varphi^{-1}\times\mathbb{R}^m} \varphi(U) \times \mathbb{R}^m.$$

The standard basis (e_i) in $\{x\} \times \mathbb{R}^m$ corresponds to the basis $(\frac{\partial}{\partial u^i}\big|_x) \in T_xM$ of the directional derivatives. The dual mapping to $T_{\varphi^{-1}(x)}\varphi : \mathbb{R}^m \to T_xM$ thus maps the dual basis (du^i) of $(T_xM)^*$ to the dual basis (e^i) of $(\mathbb{R}^m)^* \cong \mathbb{R}^m$. The local trivialization of T^*M hence maps e^i to du^i, and a 1-form ω is smooth if and only if all of its local coordinates (coefficients) ω_i - given by $\omega = \sum_i \omega_i \, du^i$ - are smooth.

19.8 Lemma (Sections of the dual bundle).

If $p : E \to M$ is a vector bundle, then we have the following descriptions for the smooth sections of the dual bundle $E^* := \bigsqcup_x (E_x)^* \to M$:

$$\begin{aligned} C^\infty(M\leftarrow E^*) &:= \Gamma(E^* \to M) := \{\sigma \in C^\infty(M, E^*) : \forall\, x : \sigma(x) \in E^*_x\} \\ &\cong \{s \in C^\infty(E, \mathbb{R}) : \forall\, x : s|_{E_x} \in L(E_x, \mathbb{R})\} \\ &\cong \textit{The space of vector bundle homomorphisms } E \to M \times \mathbb{R} \textit{ over } \mathrm{id}_M. \end{aligned}$$

Proof. We only need to show that the sections $\sigma \in C^\infty(M\leftarrow E^*)$ correspond exactly to the fiber-wise linear smooth mappings $s : E \to \mathbb{R}$.

If we define $\sigma \leftrightarrow s$ by $\sigma(x) = s|_{E_x} =: s_x$, then this gives a correspondance between the mappings σ with graph $\{(x, \sigma(x)) : x \in M\} \subseteq M \times E^*$ and $s = \bigsqcup_{x\in M} s_x$. Remains to show that σ is smooth if and only if s is. This is a local property. Let $\varphi : U \times \mathbb{R}^k \to E|_U$ be a vector bundle chart of $p : E \to M$ and $\varphi^* : U \times (\mathbb{R}^k)^* \to E^*|_U$ its associated chart of $E^* \to M$. Locally, σ is given by $\bar\sigma : U \to (\mathbb{R}^k)^* = L(\mathbb{R}^k, \mathbb{R})$ with $\sigma(x) = (\varphi^*)_x(\bar\sigma(x)) = \bar\sigma(x) \circ (\varphi_x)^{-1}$ and s by $\bar s := s \circ \varphi : U \times \mathbb{R}^k \to \mathbb{R}$. So $\bar s(x, v) = s_x(\varphi_x(v)) = \sigma(x)(\varphi_x(v)) = \bar\sigma(x)(v)$. If σ (and therefore also $\bar\sigma$) smooth, then $\bar s : (x, v) \mapsto \bar s(x, v) = \bar\sigma(x)(v) = (\mathrm{eval} \circ (\bar\sigma \times \mathrm{id}_{\mathbb{R}^k}))(x, v)$ is also smooth.
Conversely: Let s be smooth, then also $\bar s$ is smooth and thus also $\mathrm{eval}_v \circ \bar\sigma = \bar s(., v)$ for each v. Hence $\bar\sigma : U \to L(\mathbb{R}^k, \mathbb{R})$ is smooth, and so also σ. □

19.9 Remark.

Next we want an algebraic description of smooth 1-forms, similarly as that for smooth vector fields in 17.1. We can apply a 1-form ω to a vector field ξ pointwise (since $\omega_x \in (T_xM)^* = L(T_xM, \mathbb{R})$ and $\xi_x \in T_xM$) and get a function $\omega(\xi) : M \to \mathbb{R}$ with $x \mapsto \omega_x(\xi_x)$. In local coordinates this looks as follows:

$$\omega = \sum_i \omega_i \, du^i \; ; \quad \xi = \sum_i \xi^i \tfrac{\partial}{\partial u^i}$$
$$\omega(\xi) = \Big(\sum_i \omega_i \, du^i\Big)\Big(\sum_j \xi^j \tfrac{\partial}{\partial u^j}\Big) = \sum_{i,j} \omega_i \xi^j \, du^i(\tfrac{\partial}{\partial u^j}) = \sum_i \omega_i \xi^i.$$

So the resulting function $\omega(\xi)$ is smooth if ω and ξ are smooth. And clearly, the mapping $(\omega, \xi) \mapsto \omega(\xi)$ is bilinear as a map of $\Omega^1(M) \times \mathfrak{X}(M) \to C^\infty(M, \mathbb{R})$.

19.10 Lemma (Space of 1-forms as $C^\infty(M, \mathbb{R})$-linear mappings).

The bilinear map $\Omega^1(M)\times\mathfrak{X}(M)\to C^\infty(M,\mathbb{R})$ induces a $C^\infty(M,\mathbb{R})$-linear isomorphism

$$\Omega^1(M)\cong \operatorname{Hom}_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M),C^\infty(M,\mathbb{R})),$$

where the space on the right-hand side consists of all $C^\infty(M,\mathbb{R})$-linear maps (i.e. $C^\infty(M,\mathbb{R})$-module homomorphisms) from $\mathfrak{X}(M)$ to $C^\infty(M,\mathbb{R})$.

Proof. Clearly, this bilinear mapping induces a linear mapping from $\Omega^1(M)$ into the space $L\big(\mathfrak{X}(M),C^\infty(M,\mathbb{R})\big)$ of the linear mappings.

Each $\omega\in\Omega^1(M)$ also acts $C^\infty(M,\mathbb{R})$-linearly on $\xi\in\mathfrak{X}(M)$, because

$$\omega(f\cdot\xi)|_x=\omega_x((f\cdot\xi)_x)=\omega_x(f(x)\cdot\xi_x)=f(x)\cdot\omega_x(\xi_x)=(f\cdot\omega(\xi))_x$$

Furthermore, $\Omega^1(M)\to\operatorname{Hom}_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M),C^\infty(M,\mathbb{R}))$ is actually $C^\infty(M,\mathbb{R})$-linear, because $(f\cdot\omega)(\xi)|_x=(f\cdot\omega)_x(\xi_x)=(f(x)\omega_x)(\xi_x)=f(x)\cdot\omega_x(\xi_x)=f(x)\cdot\omega(\xi)|_x=(f\cdot\omega(\xi))_x$.

Conversely, let $\omega\in\operatorname{Hom}_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M),C^\infty(M,\mathbb{R}))$ be given.

Then ω acts locally, that is, $\xi=0$ on $U\subseteq M$ implies $\omega(\xi)=0$ on U: For $x\in U$ we choose $f\in C^\infty(M,\mathbb{R})$ with $f(x)=1$ and $\operatorname{supp}(f)\subseteq U$. Then $f\cdot\xi=0$ and thus

$$0=\omega(0)=\omega(f\cdot\xi)=f\cdot\omega(\xi)\quad\Rightarrow\quad 0=f(x)\cdot\omega(\xi)(x)=\omega(\xi)(x).$$

Moreover, ω acts even point-wise, i.e. $\xi(x)=0$ implies $\omega(\xi)(x)=0$, because

$$\omega(\xi)(x)=\omega\Big(\sum_i\xi^i\tfrac{\partial}{\partial u^i}\Big)(x)=\sum_i\underbrace{\xi^i(x)}_{=0}\cdot\omega(\tfrac{\partial}{\partial u^i})(x)=0.$$

Thus we may define a 1-form ω by $\omega(x)(\xi_x):=\omega(\xi)(x)$, where $\xi\in\mathfrak{X}(M)$ is arbitrarily chosen such that $\xi(x)=\xi_x$. The 1-form ω is smooth, because locally

$$\omega=\sum_i\omega_i\,du^i\text{ with }\omega_i=\omega(\tfrac{\partial}{\partial u^i}).$$

That these two assignments are inverse to each other is obvious. □

Note that this proof can be generalized directly to one for

$$C^\infty(M\leftarrow E^*)\cong\operatorname{Hom}_{C^\infty(M,\mathbb{R})}(C^\infty(M\leftarrow E),C^\infty(M,\mathbb{R})).$$

19.11 Lemma (Pull-back of sections of dual bundles).

Let $p:E\to M$ and $q:F\to N$ be vector bundles and $f:E\to F$ a vector bundle homomorphism with base map f_0. Then $f^*:C^\infty(N\leftarrow F^*)\to C^\infty(M\leftarrow E^*)$ is well-defined by

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ M \arrow[r, "f_0"] & N \end{tikzcd}$$

$$f^*(s)_x\cdot v_x:=s_{f_0(x)}\cdot f(v_x)\ \textit{for } s\in C^\infty(N\leftarrow F^*),\ x\in M,\ \textit{and } v_x\in E_x.$$

Compare this with 17.5 and 17.4. If $p:E=M\times\mathbb{R}\to M$ and $q:F=N\times\mathbb{R}\to N$ are trivial bundles with $f(x,t)=(f_0(x),t)$, the just defined pullback f^* generalizes that for functions $g\in C^\infty(N,\mathbb{R})$, because then the isomorphism $C^\infty(N,\mathbb{R})\cong C^\infty(N\leftarrow F^*)$ is given by $g\mapsto\big(s:N\ni y\mapsto(F_y\ni v\mapsto g(y)\cdot v\in F_y^*)\big)$.

Proof. We have to show that $f^*(s)$ is smooth. Using local trivializations reduces the problem to trivial bundles. So let us consider bundles $p:U\times\mathbb{R}^k\to U$, $q:V\times\mathbb{R}^i\to V$, and mappings $f:U\to L(\mathbb{R}^k,\mathbb{R}^i)$, $f_0:U\to V$, and $s:V\to(\mathbb{R}^i)^*$. Then

$$f^*(s)_x\cdot v_x:=s_{f_0(x)}\cdot f(v_x)=(s\circ f_0)(x)\cdot f_x(v_x)=\operatorname{comp}((s\circ f_0)(x),f_x)\cdot v_x,$$

i.e.

$$\begin{tikzcd} U \arrow[rr, "f^*(s)"] \arrow[dr, "(s\circ f_0, f)"'] & & (\mathbb{R}^k)^* \\ & (\mathbb{R}^i)^* \times L(\mathbb{R}^k, \mathbb{R}^i) \arrow[ur, "\mathrm{comp}"'] & \end{tikzcd}$$

commutes and $f^*(s)$ is smooth as a composite of two C^∞ functions. □

19.12 Pull-back of 1-forms in local coordinates.

Let $f : M \to N$ be smooth and $\omega \in \Omega^1(N)$. Then $f^*\omega \in \Omega^1(M)$ is defined by

$$(f^*\omega)_x(\xi) := \big((Tf)^*\omega\big)(x)(\xi) \xlongequal{\boxed{19.11}} \omega_{f(x)}(T_xf\cdot\xi) \text{ for } x \in M \text{ and } \xi \in T_xM.$$

In particular, we have

$$\begin{aligned} f^*(dg)_p(\xi_p) &= (dg)_{f(p)}(T_pf\cdot\xi_p) = \mathrm{pr}_2\cdot T_{f(p)}g\cdot T_pf\cdot\xi_p \\ &= \mathrm{pr}_2\cdot T_p(g\circ f)\cdot\xi_p = d(g\circ f)_p\cdot\xi_p, \\ \text{i.e.}\quad f^*(dg) &= d(g\circ f) = d(f^*g) \text{ for } g\in C^\infty(N,\mathbb{R}), \\ \text{or even shorter: }& f^*\circ d = d\circ f^*. \end{aligned}$$

We want to express $f^*\omega$ in local coordinates. Let $(u^1,\dots,u^i)$ be local coordinates around $x \in M$ and $(v^1,\dots,v^j)$ local coordinates around $y := f(x) \in N$. Furthermore, let $\omega = \sum_j \omega_j\, dv^j$ be the coordinate representation of ω at y and $f^*\omega = \sum_i \eta_i\, du^i$ that of $f^*\omega$ at x. If we apply $f^*\omega$ to $\xi := \frac{\partial}{\partial u^i}\big|_x$ then

$$\omega_{f(x)}\Big(T_xf\cdot\frac{\partial}{\partial u^i}\Big|_x\Big) = (f^*\omega)_x\Big(\frac{\partial}{\partial u^i}\Big|_x\Big) = \Big(\sum_k \eta_k\, du^k\Big)\Big(\frac{\partial}{\partial u^i}\Big|_x\Big) \xlongequal{\boxed{19.9}} \eta_i$$

$$\omega_{f(x)}\Big(T_xf\cdot\frac{\partial}{\partial u^i}\Big|_x\Big) \xlongequal{\boxed{10.9}} \Big(\sum_j \omega_j\, dv^j\Big)_y\Big(\sum_l \frac{\partial f^l}{\partial u^i}\Big|_x \frac{\partial}{\partial v^l}\Big|_y\Big) \xlongequal{\boxed{19.9}} \sum_j \omega_j(y)\,\frac{\partial f^j}{\partial u^i}\Big|_x$$

So

$$f^*\Big(\sum_j \omega_j\, dv^j\Big) = \sum_i\Big(\sum_j (\omega_j\circ f)\,\frac{\partial f^j}{\partial u^i}\Big)\, du^i.$$

Note that the path integral of [86, 3.10] of a 1-form $\omega \in \Omega^1(U)$ on an open set $U \subseteq \mathbb{R}^m$ along a smooth curve $c : I \to U$ is therefore just given by

$$\int_c \omega = \int_c \sum_i \omega_i(x)\, dx^i := \int_0^1 \sum_i \omega_i(c(t))\,\frac{dc^i}{dt}\, dt = \int_0^1 c^*(\omega).$$

Hence for an abstract manifold M, we can also define the PATH INTEGRAL $\int_c \omega$ of a 1-form $\omega \in \Omega^1(M)$ along a curve $c : I \to M$ by

$$\int_c \omega := \int_0^1 c^*(\omega).$$

We will generalize this definition further in section $\boxed{28}$.

20. Motivation for forms of higher order

20.1 The Riemannian metric as a tensor field.

In 18.11 we defined Riemannian metrics as mappings which associate a bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$ to each $x \in M$ in such a way that $x \mapsto g_x(\xi_x, \eta_x)$, $M \to \mathbb{R}$ is smooth for every two smooth vector fields $\xi, \eta \in \mathfrak{X}(M)$. If we write the two vector fields by means of local coordinates $(u^1, \dots, u^m)$ as $\xi = \sum_i \xi^i \frac{\partial}{\partial u^i}$ and $\eta = \sum_i \eta^i \frac{\partial}{\partial u^i}$, we obtain

$$g_x(\xi_x, \eta_x) = \sum_{i,j} \xi_x^i \eta_x^j \, g_x\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \sum_{i,j} du^i(\xi)|_x \, du^j(\eta)|_x \, g_{i,j}(x),$$

where we put $g_{i,j}(x) := g_x\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$. Note that $(\xi_x, \eta_x) \mapsto du^i(\xi)|_x \cdot du^j(\eta)|_x$ is a bilinear mapping $T_xM \times T_xM \to \mathbb{R}$, which we denote $du^i|_x \otimes du^j|_x$. So locally we have

$$g = \sum_{i,j} g_{i,j} \, du^i \otimes du^j$$

20.2 Hessian form.

If a function $f : M \to \mathbb{R}$ has a local extremum at $x \in M$, then $T_xf : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}$ is the zero mapping. In order to reverse this implication, we need the 2nd derivative: Let M be an open subset of $\mathbb{R}^m$, (or M a submanifold that we replace with an open neighborhood and f an extension to it).

$$\begin{gathered}
M \times \mathbb{R}^m = TM \xrightarrow{Tf} T\mathbb{R} = \mathbb{R} \times \mathbb{R} \\
Tf(x,v) = (f(x), f'(x)(v)) \\
M \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m = T^2M := T(TM) \xrightarrow{T^2f} T^2\mathbb{R} = \mathbb{R}^4 \\
T^2f(x,v;y,w) = \Big(f(x), f'(x)(v), f'(x)(y), f''(x)(v,y) + f'(x)(w)\Big)
\end{gathered}$$

For submanifolds $M \subseteq \mathbb{R}^m$ we have $f''(x)(v,y) = \mathrm{pr}_4(T^2f(x,v;y,0))$ provided $(x,v;y,0)$ is in the second tangent space T^2M.

20.3 Example.

Second derivative of functions on the circle:

$$\begin{aligned}
S^1 &= \{x \in \mathbb{R}^2 : |x| = 1\} \\
TS^1 &= \{(x,v) \in (\mathbb{R}^2)^2 : |x| = 1, \langle x, v\rangle = 0\} \\
T^2S^1 &= \{(x,v;y,w) \in (\mathbb{R}^2)^4 : |x| = 1, \langle x, v\rangle = \langle x, y\rangle = 0, \\
&\qquad\qquad \langle y, v\rangle + \langle x, w\rangle = 0\}
\end{aligned}$$

Thus, $(x,v;y,0) \in T^2S^1$ if and only if $|x| = 1$, $v \perp x$, $y \perp x$, and $v \perp y$, thus only if $v = 0$ or $y = 0$. Hence $f''(x) : T_xM \times T_xM \to \mathbb{R}$ can not be meaningfully defined on a general manifold.

If $T_xf = 0$ however, then this is nevertheless possible. Let $\xi_x, \eta_x \in T_xM$ and define $f''(x)(\xi_x, \eta_x) := \eta_x(\xi(f))$, where ξ is a vector field with $\xi(x) = \xi_x$. Let us express ξ_x and η_x in local coordinates, i.e. $\xi_x = \sum_i \xi^i \frac{\partial}{\partial u^i}$, respectively $\eta_x = \sum_i \eta^i \frac{\partial}{\partial u^i}$. Then

$$\xi(f) = \sum_i \xi^i \frac{\partial f}{\partial u^i}$$

$$\begin{aligned}\eta_x(\xi(f)) &= \Big(\sum_j \eta^j \frac{\partial}{\partial u^j}\Big)\Big(\sum_i \xi^i \cdot \frac{\partial f}{\partial u^i}\Big)\Big|_x \\ &= \sum_j \eta^j \sum_i \frac{\partial}{\partial u^j}\Big(\xi^i \cdot \frac{\partial}{\partial u^i} f\Big)\Big|_x = \sum_j \eta^j \sum_i \Big(\frac{\partial \xi^i}{\partial u^j}\frac{\partial f}{\partial u^i} + \xi^i \frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i} f\Big)_x \\ &= \sum_{i,j} \xi^i \eta^j \frac{\partial^2 f}{\partial u^j \partial u^i}(x), \text{ because } \frac{\partial f}{\partial u^i}\Big|_x = 0.\end{aligned}$$

Thus we have shown that the above definition is independent of the extension ξ and yields the usual second derivative in local coordinates, provided $f'(x) = 0$.

Therefore $f''(x) : T_xM \times T_xM \to \mathbb{R}$ is given under this condition by

$$\begin{aligned}f''(x)(\xi,\eta) &= \sum_{i,j} \xi^i \eta^j \frac{\partial^2 f}{\partial u^j \partial u^i}\Big|_x = \sum_{i,j} du^i(\xi) du^j(\eta) \frac{\partial^2 f}{\partial u^j \partial u^i}\Big|_x \\ &= \Big(\sum_{i,j} \frac{\partial^2 f}{\partial u^j \partial u^i} du^i \otimes du^j\Big)(x)(\xi,\eta).\end{aligned}$$

For short, $f''(x) = \sum_{i,j} \frac{\partial^2 f}{\partial u^j \partial u^i}(x)\, du^i|_x \otimes du^j|_x$. How does this expression transform when changing from coordinates u^i to new coordinates v^j? We have $dv^i = \sum_j \frac{\partial v^i}{\partial u^j} du^j$ and $\frac{\partial}{\partial u^j} = \sum_k \frac{\partial v^k}{\partial u^j}\frac{\partial}{\partial v^k}$. So $\frac{\partial}{\partial u^j}(f) = \sum_k \frac{\partial v^k}{\partial u^j}\frac{\partial f}{\partial v^k}$ and

$$\begin{aligned}\frac{\partial^2}{\partial u^i \partial u^j}(f) &= \frac{\partial}{\partial u^i}\Big(\frac{\partial}{\partial u^j}(f)\Big) = \frac{\partial}{\partial u^i}\Big(\sum_k \frac{\partial v^k}{\partial u^j}\frac{\partial f}{\partial v^k}\Big) \\ &= \sum_k \Big(\frac{\partial f}{\partial v^k} \cdot \frac{\partial^2 v^k}{\partial u^i \partial u^j} + \frac{\partial v^k}{\partial u^j} \cdot \Big(\sum_l \frac{\partial v^l}{\partial u^i}\frac{\partial}{\partial v^l}\Big)\frac{\partial f}{\partial v^k}\Big) \\ &= \sum_k \frac{\partial^2 v^k}{\partial u^i \partial u^j} \cdot \frac{\partial f}{\partial v^k} + \sum_{k,l} \frac{\partial v^l}{\partial u^i} \cdot \frac{\partial v^k}{\partial u^j} \cdot \frac{\partial^2 f}{\partial v^l \partial v^k}.\end{aligned}$$

Thus,

$$\sum_{i,j} \frac{\partial^2 f}{\partial u^i \partial u^j} du^i \otimes du^j = \sum_{l,k} \frac{\partial^2 f}{\partial v^l \partial v^k} dv^l \otimes dv^k + \sum_{i,j}\Big(\sum_k \frac{\partial^2 v^k}{\partial u^i \partial u^j}\Big)\frac{\partial f}{\partial v^k} du^i \otimes du^j,$$

and the second summand disappears at x, since we assumed $\frac{\partial f}{\partial v^k}|_x = 0$.

20.4 Exact 1-forms.

For a smooth function $f : M \to \mathbb{R}$, with $M \subseteq \mathbb{R}^m$ open, $f' : M \to L(\mathbb{R}^m, \mathbb{R})$ is smooth. Of course, one is interested whether the converse holds, i.e. under which conditions on a 1-form $\omega : M \to L(\mathbb{R}^m, \mathbb{R})$ does there exist a function $f : M \to \mathbb{R}$ with $\omega = f'$. Such an ω is called an EXACT 1-FORM. The special case 18.8.1 of the Frobenius Theorem 18.7 provides an integrability condition (see also [81, 6.5.2]) for this:

$$\text{Such an } f \text{ exists locally} \Leftrightarrow \underbrace{\omega'(x)(v_1)\cdot v_2 - \omega'(x)(v_2)\cdot v_1}_{=:2d\omega(x)(v_1,v_2)} = 0\ \forall\, v_1, v_2 \in \mathbb{R}^m.$$

The just defined $d\omega : M \to L(\mathbb{R}^m, \mathbb{R}^m; \mathbb{R})$ is for fixed $x \in M$ alternating (=skew-symmetric) and bilinear. So, if we denote the space of all bilinear alternating functions $E \times E \to F$ by $L^2_{\text{alt}}(E, F)$, then $d\omega : M \to L^2_{\text{alt}}(\mathbb{R}^m, \mathbb{R})$ and one calls $d\omega$ a 2-FORM.

In general, a mapping $\omega : M \to L^k_{\text{alt}}(\mathbb{R}^m, \mathbb{R})$ is called k-FORM, where $L^k_{\text{alt}}(E, F)$ denotes the space of the alternating k-linear functions $E \times \ldots \times E \to F$. If $M = \mathbb{R}^m$, then the condition $d\omega = 0$ is sufficient to garantee a globally defined $f : M \to \mathbb{R}$ with $\omega = f'$. If $M \subseteq \mathbb{R}^m$, then this is not sufficient in general as the following example shows.

20.5 Example.

We consider the 1-form

$$\omega(x, y)(v, w) := \frac{-yv + xw}{x^2 + y^2} \quad \text{i.e.} \quad \omega(x, y) := -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

on $M := \mathbb{R}^2 \setminus \{0\}$ from [86, 3.10]. Because of $\frac{\partial}{\partial y}(\frac{-y}{x^2+y^2}) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2})$, we have $d\omega = 0$. Suppose there were an f with $f' = \omega$, i.e.

$$f'(x, y) = (\partial_1 f(x, y), \partial_2 f(x, y)) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$

If $(x_0, y_0) \in S^1$ is a point, where f attains a minimum on S^1, then

$$\begin{aligned} 0 = f'(x_0, y_0)(-y_0, x_0) &= \frac{-y_0}{x_0^2 + y_0^2} \cdot (-y_0) + \frac{x_0}{x_0^2 + y_0^2} \cdot x_0 \\ &= 1, \text{ a contradiction.} \end{aligned}$$

For the form ω we have $d\omega = 0$, but there is no antiderivative for ω on M. This discrepancy between forms ω with $d\omega = 0$ and those of the form $\omega = f' = df$ can be used to identify topological properties of M (in our example, M was not simply-connected). We will come to that later.

How should k-forms look like for arbitrary manifolds M?
Let $\omega : x \mapsto \omega(x)$ be a 1-form, then $d\omega$ would have to be a mapping $d\omega : x \mapsto d\omega(x)$, given on M, with values $d\omega(x) : T_xM \times T_xM \to \mathbb{R}$ that are bilinear and alternating (such a mapping is called 2-FORM). So $(d\omega)_x \in L^2_{\text{alt}}(T_xM, \mathbb{R})$. Analogously we will define k forms. Let us now summarize the necessary basic facts of multilinear algebra.

21. Multilinear algebra and tensors

21.1 Definition.

We first collect the (multi-)linear theory, for a more in-depth study see [53] and [136, Vol.I, Cap.7]. In the following, E, F, etc. denote finite-dimensional vector spaces over $\mathbb{R}$. We use $L^k(E_1, \ldots, E_k; F)$ (or $L(E_1, \ldots, E_k; F)$ for short) to denote the SPACE OF k-LINEAR MAPPINGS $E_1 \times \ldots \times E_k \to F$. This is a vector space of finite dimension $\dim(E_1) \cdot \ldots \cdot \dim(E_k) \cdot \dim(F)$.

Let $T : E_1 \times \ldots \times E_k \to \mathbb{R}$ be k-linear and $S : E_{k+1} \times \ldots \times E_{k+i} \to \mathbb{R}$ be i-linear. The TENSOR PRODUCT $T \otimes S$ of T with S is the $k + i$-linear function defined as follows:

$$\begin{aligned} &T \otimes S : E_1 \times \ldots \times E_{k+i} \to \mathbb{R} \\ &(T \otimes S)(v_1, \ldots, v_{k+i}) := T(v_1, \ldots, v_k)\, S(v_{k+1}, \ldots, v_{k+i}) \end{aligned}$$

Completely analogous, one can also define the tensor product $T_1 \otimes \cdots \otimes T_k$ of several multilinear functionals T_i.

21.2 The tensor product of vector spaces.

For finite-dimensional vector spaces $E_1, \ldots, E_k$, their tensor product is defined by

$$E_1 \otimes \cdots \otimes E_k := L^k(E_1^*, \ldots, E_k^*; \mathbb{R}).$$

Together with the k linear mapping

$$\otimes : E_1 \times \ldots \times E_k \to E_1 \otimes \cdots \otimes E_k, \quad (x_1, \ldots, x_k) \mapsto x_1 \otimes \cdots \otimes x_k,$$
$$\text{where } (x_1 \otimes \cdots \otimes x_k)(y_1^*, \ldots, y_k^*) := y_1^*(x_1) \cdot \ldots \cdot y_k^*(x_k),$$

it solves the following universal problem:

$$\begin{tikzcd} E_1 \times \ldots \times E_k \arrow[rr, "\otimes"] \arrow[dr, "k\text{-linear}"'] & & E_1 \otimes \cdots \otimes E_k \arrow[dl, dotted, "\exists !\ \text{linear}"] \\ & F & \end{tikzcd}$$

If $\{e_i^j : 1 \le i \le \dim E_j\}$ is a basis of E_j, then a basis of $E_1 \otimes \cdots \otimes E_k$ is given by

$$\{e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k : 1 \le i_1 \le \dim E_1, \ldots, 1 \le i_k \le \dim E_k\}.$$

Proof. We first show the statement about the basis. The set $\{e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k : i_1, \ldots, i_k\}$ is linearly independent, because from $\sum_{i_1,\ldots,i_k} \mu^{i_1,\ldots,i_k}\, e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k = 0$ the equation

$$\begin{aligned} 0 &= \Big(\sum_{i_1,\ldots,i_k} \mu^{i_1,\ldots,i_k}\, e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k\Big)(e_1^{j_1}, \ldots, e_k^{j_k}) \\ &= \sum_{i_1,\ldots,i_k} \mu^{i_1,\ldots,i_k}\, \Big(e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k\Big)(e_1^{j_1}, \ldots, e_k^{j_k}) \\ &= \sum_{i_1,\ldots,i_k} \mu^{i_1,\ldots,i_k}\, \underbrace{e_1^{j_1}(e_{i_1}^1)}_{\delta_{i_1}^{j_1}} \cdot \ldots \cdot \underbrace{e_k^{j_k}(e_{i_k}^k)}_{\delta_{i_k}^{j_k}} = \mu^{j_1,\ldots,j_k} \end{aligned}$$

follows by applying to $(e_1^{j_1}, \ldots, e_k^{j_k})$.

This set is also generating for $E_1 \otimes \cdots \otimes E_k := L(E_1^*, \ldots, E_k^*; \mathbb{R})$, because every k-linear $\mu : E_1^* \times \ldots \times E_k^* \to \mathbb{R}$ can be described on $(x^1, \ldots, x^k) \in E_1^* \times \ldots \times E_k^*$ as follows

$$\begin{aligned} \mu(x^1, \ldots, x^k) &= \mu\Big(\sum_{i_1} x_{i_1}^1 e_1^{i_1}, \ldots, \sum_{i_k} x_{i_k}^k e_k^{i_k}\Big) \\ &= \sum_{i_1} \cdots \sum_{i_k} x_{i_1}^1 \ldots x_{i_k}^k \cdot \mu(e_1^{i_1}, \ldots, e_k^{i_k}) \\ &= \sum_{i_1} \cdots \sum_{i_k} e_{i_1}^1(x^1) \ldots e_{i_k}^k(x^k) \cdot \mu(e_1^{i_1}, \ldots, e_k^{i_k}) \\ &= \sum_{i_1,\ldots,i_k} \mu^{i_1,\ldots,i_k} \cdot (e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k)(x^1, \ldots, x^k) \end{aligned}$$

where $\mu^{i_1,\ldots,i_k} := \mu(e_1^{i_1}, \ldots, e_k^{i_k}) \in \mathbb{R}$.

Thus, each multilinear map $\mu : E_1 \times \ldots \times E_k \to F$ can be unambiguously extended to a linear map $\tilde{\mu} : E_1 \otimes \cdots \otimes E_k \to F$ by defining its values on the basis

$$\tilde{\mu}(e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k) := \mu(e_{i_1}^1, \ldots, e_{i_k}^k),$$

so that the specified triangle commutes. □

21.3 Remarks.

1. We get the following natural isomorphisms (the second one by induction):
$$\begin{aligned}
(E_1\otimes\dots\otimes E_k)^* &\cong L(E_1,\dots,E_k;\mathbb{R})\cong L(E_1^{**},\dots,E_k^{**};\mathbb{R})\\
&= E_1^*\otimes\dots\otimes E_k^*\\
(\dots(E_1\otimes E_2)\otimes\dots\otimes E_k) &\cong L((E_1\otimes\dots\otimes E_{k-1})^*,E_k^*;\mathbb{R})\\
&\cong L((E_1\otimes\dots\otimes E_{k-1})^*;L(E_k^*,\mathbb{R}))\\
&\cong L(E_1^*\otimes\dots\otimes E_{k-1}^*;L(E_k^*,\mathbb{R}))\\
&\cong L(E_1^*,\dots,E_{k-1}^*,L(E_k^*,\mathbb{R}))\\
&\cong L(E_1^*,\dots,E_{k-1}^*,E_k^*;\mathbb{R})\\
&\cong E_1\otimes\dots\otimes E_k\\
E_1\otimes E_2 &= L(E_1^*,E_2^*;\mathbb{R})\cong L(E_2^*,E_1^*;\mathbb{R})=E_2\otimes E_1\\
E_1\otimes\mathbb{R} &= L(E_1^*,\mathbb{R}^*;\mathbb{R})\cong L(E_1^*,\mathbb{R}^{**})\cong L(E_1^*,\mathbb{R})=E_1^{**}\cong E_1\\
L(E,F) &\cong L(E,F^{**})=L(E,L(F^*,\mathbb{R}))\cong L(E,F^*;\mathbb{R})\\
&\cong L(E^{**},F^*;\mathbb{R})=E^*\otimes F\\
L(E_1,\dots,E_k;F) &\cong L(E_1\otimes\dots\otimes E_k,F)\\
&\cong (E_1\otimes\dots\otimes E_k)^*\otimes F\cong E_1^*\otimes\dots\otimes E_k^*\otimes F.
\end{aligned}$$

2. For linear mappings $T_i : E_i\to F_i$ there exists a linear mapping $T_1\otimes\dots\otimes T_k : E_1\otimes\dots\otimes E_k\to F_1\otimes\dots\otimes F_k$ which is uniquely determined by the following diagram:
$$\begin{tikzcd}
E_1\times\ldots\times E_k \arrow[r, "\otimes", "k\text{-linear}"'] \arrow[d, "T_1\times\ldots\times T_k"', "\text{linear}"] & E_1\otimes\dots\otimes E_k \arrow[d, dotted, "T_1\otimes\dots\otimes T_k"', "\text{linear}"]\\
F_1\times\ldots\times F_k \arrow[r, "\otimes", "k\text{-linear}"'] & F_1\otimes\dots\otimes F_k.
\end{tikzcd}$$
Here $T_1\otimes\dots\otimes T_k$ is given on the basis $(e^1_{i_1}\otimes\dots\otimes e^k_{i_k})$ as follows:
$$\begin{aligned}
(T_1\otimes\dots\otimes T_k)(e^1_{i_1}\otimes\dots\otimes e^k_{i_k}) &= T_1(e^1_{i_1})\otimes\dots\otimes T_k(e^k_{i_k})\\
&= \sum_{j_1}(T_1)^{j_1}_{i_1}f^1_{j_1}\otimes\dots\otimes\sum_{j_k}(T_k)^{j_k}_{i_k}f^k_{j_k}\\
&= \sum_{j_1,\dots,j_k}(T_1)^{j_1}_{i_1}\dots(T_k)^{j_k}_{i_k}\,f^1_{j_1}\otimes\dots\otimes f^k_{j_k}
\end{aligned}$$

3. We have the following relationships between the tensor products we have just defined: For $T_i\in E_i^*$, the following tensor products
   1. $T_1\otimes\dots\otimes T_k\in L(E_1,\dots,E_k;\mathbb{R})\cong(E_1\otimes\dots\otimes E_k)^*$ of 21.1;
   2. $T_1\otimes\dots\otimes T_k\in E_1^*\otimes\dots\otimes E_k^*$ of 21.2;
   3. $T_1\otimes\dots\otimes T_k : E_1\otimes\dots\otimes E_k\to\mathbb{R}\otimes\dots\otimes\mathbb{R}$ by 2

   coincide up to the isomorphisms $(E_1\otimes\dots\otimes E_k)^*\cong E_1^*\otimes\dots\otimes E_k^*$ and $\mathbb{R}\otimes\dots\otimes\mathbb{R}\cong\mathbb{R}$ from 1.

4. $\bigotimes E := \coprod_{m=0}^\infty(\bigotimes_{i=1}^m E)$ is a graded, associative algebra with 1, the so-called TENSOR ALGEBRA over E. An algebra is called GRADED if $A=\coprod_{k\in\mathbb{N}}A_k$ and the multiplication restricts to $A_k\times A_l$ in A_{k+l}. The elements $\omega\in A_k$ are called HOMOGENEOUS OF DEGREE k. We put $\bigotimes^0 E := \bigotimes_{i\in\emptyset}E=\mathbb{R}$, because $\prod_{i\in\emptyset}E^*=\{\emptyset\}$ and every $f:\{\emptyset\}\to\mathbb{R}$ is 0-linear. The unity in $\bigotimes E$ is then $1\in\mathbb{R}=\bigotimes^o E\subseteq\bigotimes E$.

5. The tensor algebra has the following universal property: For each linear map $f: E\to A$, where A is an associative algebra with 1, there is a unique algebra

homomorphism $\tilde f : \bigotimes E \to A$, which coincide with f on $\bigotimes^1 E = E$:

$$\begin{tikzcd}
E \arrow[r, "\cong"] \arrow[dr, "f", "\text{linear}"'] & \bigotimes^1 E \arrow[r, hook] & \bigotimes E \arrow[dl, dotted, "\tilde f"', "\text{Alg-Homo}"] \\
& A &
\end{tikzcd}$$

21.4 Definition. Wedge product of alternating mappings.

By $L^k_{\mathrm{alt}}(E,F)$ we denote the subspace of $L^k(E,F)$ formed by the alternating k-linear mappings, where a mapping $T : E \times \ldots \times E \to F$ is called ALTERNATING if $\pi^{**}(T) := T \circ \pi^* = \mathrm{sgn}(\pi) \cdot T$ holds for all permutations $\pi \in S_k := \{\pi : \{1,\ldots,n\} \to \{1,\ldots,n\} : \pi \text{ is bijectiv}\}$, i.e.

$$\begin{aligned}
T(v_{\pi(1)},\ldots,v_{\pi(k)}) &= T(v \circ \pi) = T(\pi^* v) = (T \circ \pi^*)(v) = \big(\pi^{**}(T)\big)(v) \\
&= (\mathrm{sgn}(\pi) \cdot T)(v) = \mathrm{sgn}(\pi) \cdot T(v_1,\ldots,v_k) \quad \forall\, v_1,\ldots,v_k \in E,
\end{aligned}$$

where we consider $(v_1,\ldots,v_k)$ as mapping $v : \{1,\ldots,k\} \to E$.

The projection $\mathrm{alt} : L^k(E,F) \to L^k_{\mathrm{alt}}(E,F) \subseteq L^k(E,F)$, called ALTERNATOR, onto this subspace is given by

$$\mathrm{alt}(T)(v_1,\ldots,v_k) := \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sgn}(\pi) \cdot T(v_{\pi(1)},\ldots,v_{\pi(k)}),$$
$$\text{i.e. } \mathrm{alt}(T) = \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sgn}(\pi) \cdot \pi^{**}(T).$$

For alternating multilinear functionals T and S one defines the OUTER PRODUCT or WEDGE-PRODUCT by:

$$\begin{aligned}
(T \wedge S)(v_1,\ldots,v_{k+i}) &:= \tfrac{(k+i)!}{k!\,i!}\,\mathrm{alt}(T \otimes S)(v_1,\ldots,v_{k+i}) = \\
&= \tfrac{1}{k!\,i!} \sum_{\pi} \mathrm{sgn}\,\pi \cdot T(v_{\pi(1)},\ldots,v_{\pi(k)}) \cdot S(v_{\pi(k+1)},\ldots,v_{\pi(k+i)}) \\
&= \tfrac{1}{k!\,i!} \sum_{\pi_1,\pi_2} \sum_{\sigma \text{ piecew.}\nearrow} \mathrm{sgn}\,\sigma\ \mathrm{sgn}\,\pi_1\ \mathrm{sgn}\,\pi_2 \cdot T\big(v_{\sigma(\pi_1(1))},\ldots,v_{\sigma(\pi_1(k))}\big) \cdot \\
&\qquad\qquad \cdot S\big(v_{\sigma(\pi_2(k+1))},\ldots,v_{\sigma(\pi_2(k+i))}\big) \\
&= \sum_{\sigma(1)<\cdots<\sigma(k)} (-1)^{\sum_{j\le k}(\sigma(j)-j)} \cdot T(v_{\sigma(1)},\ldots,v_{\sigma(k)}) \cdot \\
&\qquad\qquad \cdot S(v_1,\ldots,\widehat{v_{\sigma(1)}},\ldots,\widehat{v_{\sigma(k)}},\ldots,v_{k+i}).
\end{aligned}$$

In this calculation, we have decomposed the permutations π of $\{1,\ldots,k+i\}$ uniquely as $\sigma \circ (\pi_1 \sqcup \pi_2)$, where π_1 is any permutation of $\{1,\ldots,k\}$, π_2 is of $\{k+1,\ldots,k+i\}$, and σ is one of $\{1,\ldots,k+i\}$, which is strictly monotone increasing on $\{1,\ldots,k\}$ and $\{k+1,\ldots,k+i\}$. So $\sigma(1) < \cdots < \sigma(k)$ is the monotone arrangement of $\{\pi(1),\ldots,\pi(k)\}$ and $\sigma(k+1) < \cdots < \sigma(k+i)$ is that of $\{\pi(k+1),\ldots,\pi(k+i)\}$. Thus $\pi_1 = \sigma^{-1} \circ \pi|_{\{1,\ldots,k\}}$ and $\pi_2 = \sigma^{-1} \circ \pi|_{\{k+1,\ldots,k+i\}}$. We have $\mathrm{sgn}(\sigma) = (-1)^{\sum_{j\le k}(\sigma(j)-j)}$ because in order to restore the natural order of $\sigma(1),\ldots,\sigma(k+i)$ we have to exchange the $\sigma(j) - j$ many smaller numbers in $\{\sigma(k+1),\ldots\}$ with $\sigma(j)$ for all $1 \le j \le k$.

If T, S, and R are linear, then

$$(T \wedge S)(w,v) = T(w)\,S(v) - T(v)\,S(w) = \det \begin{pmatrix} T(w) & S(w) \\ T(v) & S(v) \end{pmatrix}$$

and thus

$$
\begin{aligned}
2((T\wedge S)\wedge R)(w,v,u) = {}&\\
&= (T\wedge S)(w,v)\,R(u) - (T\wedge S)(v,w)\,R(u)\\
&+ (T\wedge S)(v,u)\,R(w) - (T\wedge S)(w,u)\,R(v)\\
&+ (T\wedge S)(u,w)\,R(v) - (T\wedge S)(u,v)\,R(w)\\
&= \Big(T(w)\,S(v) - T(v)\,S(w)\Big)\,R(u) - \Big(T(v)\,S(w) - T(w)\,S(v)\Big)\,R(u)\\
&+ \Big(T(v)\,S(u) - T(u)\,S(v)\Big)\,R(w) - \Big(T(w)\,S(u) - T(u)\,S(w)\Big)\,R(v)\\
&+ \Big(T(u)\,S(w) - T(w)\,S(u)\Big)\,R(v) - \Big(T(u)\,S(v) - T(v)\,S(u)\Big)\,R(w)\\
&= 2\,T(w)\,S(v)\,R(u) + 2\,T(v)\,S(u)\,R(w) + 2\,T(u)\,S(w)\,R(v)\\
&- 2\,T(v)\,S(w)\,R(u) - 2\,T(w)\,S(u)\,R(v) - 2\,T(u)\,S(v)\,R(w)\\
&= 2\ \det\begin{pmatrix} T(w) & S(w) & R(w)\\ T(v) & S(v) & R(v)\\ T(u) & S(u) & R(u)\end{pmatrix}
\end{aligned}
$$

Therefore, all factors disappear in the 3-fold product of 1-forms. This is the reason for choosing the factor $\frac{(k+i)!}{k!i!}$, respectively $\frac{1}{k!i!}$, see also 21.6.2. Similar to the above formula for $T\wedge S$, we can directly define a wedge product of several multilinear alternating functionals.
Note that

$$T\wedge S = (-1)^{ki}\,S\wedge T,$$

because

$$
\begin{aligned}
&(T\wedge S)(v_1,\dots,v_{k+i}) :=\\
&= \frac{1}{k!i!}\sum_{\pi}\operatorname{sgn}(\pi)\cdot T(v_{\pi(1)},\dots,v_{\pi(k)})\cdot S(v_{\pi(k+1)},\dots,v_{\pi(k+i)})\\
&= \frac{1}{k!i!}\sum_{\pi'}\operatorname{sgn}(\pi'\circ\sigma)\cdot T(v_{\pi'(\sigma(1))},\dots,v_{\pi'(\sigma(k))})\cdot S(v_{\pi'(\sigma(k+1))},\dots,v_{\pi'(\sigma(k+i))})\\
&= \operatorname{sgn}(\sigma)\,\frac{1}{k!i!}\sum_{\pi'}\operatorname{sgn}(\pi')\cdot S(v_{\pi'(1)},\dots,v_{\pi'(i)})\cdot T(v_{\pi'(i+1)},\dots,v_{\pi'(i+k)})\\
&= (-1)^{ki}\,(S\wedge T)(v_1,\dots,v_{k+i}),
\end{aligned}
$$

where $\pi = \pi'\circ\sigma$ and σ is the permutation that swaps block $(1,\dots,k)$ with $(k+1,\dots,k+i)$ and has sign $(-1)^{ik}$, i.e.

$$\sigma(j) := \begin{cases} j+i & \text{for } j\le k\\ j-k & \text{for } j>k.\end{cases}$$

21.5 Lemma (The outer product of a vector space).

The k-fold outer product of the vector space E is defined by $\bigwedge^k E := L^k_{alt}(E^*,\mathbb{R})$ and $\wedge : E\times\ldots\times E \to \bigwedge^k E \subseteq \bigotimes^k E = L^k(E^*,\mathbb{R})$ is the following alternating

k-linear mapping:

$$\begin{aligned}
\wedge : (v_1, \dots, v_k) &\mapsto v_1 \wedge \dots \wedge v_k \qquad \textit{with} \\
(v_1 \wedge \dots \wedge v_k)(w^1, \dots, w^k) &:= \sum_\pi \operatorname{sgn}(\pi)\, w^{\pi(1)}(v_1) \cdot \ldots \cdot w^{\pi(k)}(v_k) \\
&= k!\, \operatorname{alt}(v_1 \otimes \dots \otimes v_k)(w^1, \dots, w^k), \\
\textit{also} \quad v_1 \wedge \dots \wedge v_k &:= k!\, \operatorname{alt}(v_1 \otimes \dots \otimes v_k).
\end{aligned}$$

The outer product solves the following universal problem:

$$\begin{tikzcd}
E \times \ldots \times E \arrow[rr, "\wedge"] \arrow[dr, "k\text{-linear, alt.}"'] & & \bigwedge^k E \arrow[dl, dotted, "\exists\,!\ \text{linear}"] \\
& F &
\end{tikzcd}$$

If $\{e_i\}_{i=1}^m$ is a basis of E (i.e. $m = \dim E$), then $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < \dots < i_k \le m\}$ is a basis of $\bigwedge^k E$, so $\dim \bigwedge^k E = \binom{m}{k}$. In particular for $k = \dim E$ the vector $e_1 \wedge \dots \wedge e_k$ spans $\bigwedge^k E$ and

$$(e_1 \wedge \dots \wedge e_k)(w^1, \dots, w^k) = \sum_\pi \operatorname{sgn}(\pi)\, w_1^{\pi(1)} \cdot \ldots \cdot w_k^{\pi(k)} = \det(w^1, \dots, w^k).$$

Proof. The map $\wedge : E \times \ldots \times E \to \bigwedge^k E$ is given by $E \times \ldots \times E \xrightarrow{\otimes} \bigotimes^k E \xrightarrow{k!\,\operatorname{alt}} \bigwedge^k E$, therefore $(e_{i_1} \wedge \dots \wedge e_{i_k})_{i_1 < \dots < i_k}$ is a generating system for $\bigwedge^k E = L^k_{\operatorname{alt}}(E; \mathbb{R})$. These vectors are also linearly independent, because

$$\begin{aligned}
0 &= \Big(\sum_{i_1 < \dots < i_k} \mu^{i_1, \dots, i_k}\, e_{i_1} \wedge \dots \wedge e_{i_k} \Big)(e^{j_1}, \dots, e^{j_k}) \\
&= \sum_{i_1 < \dots < i_k} \mu^{i_1, \dots, i_k}\, \big(e_{i_1} \wedge \dots \wedge e_{i_k}\big)(e^{j_1}, \dots, e^{j_k}) \\
&= \mu^{j_1, \dots, j_k}
\end{aligned}$$

follows for $j_1 < \dots < j_k$.

Thus, each alternating multilinear map $\mu : E \times \ldots \times E \to F$ can be unambiguously considered as linear map $\tilde\mu : \bigwedge^k E \to F$ by

$$\tilde\mu(e^1_{i_1} \wedge \dots \wedge e^k_{i_k}) := \mu(e^1_{i_1}, \dots, e^k_{i_k})$$

so that the indicated triangle commutes. □

21.6 Remarks.

1. The following identities hold:

$$L^k_{\operatorname{alt}}(E, F) \cong L\Big(\bigwedge^k E, F\Big) \text{ and } \Big(\bigwedge^k E\Big)^* \cong L^k_{\operatorname{alt}}(E, \mathbb{R}) \cong L^k_{\operatorname{alt}}(E^{**}, \mathbb{R}) = \bigwedge^k E^*$$

2. For each linear mapping $T : E \to F$ there exists a linear mapping $\bigwedge^k T : \bigwedge^k E \to \bigwedge^k F$ which is uniquely determined by the following diagram:

$$\begin{tikzcd}
E \times \ldots \times E \arrow[r, "\wedge"] \arrow[d, "T \times \ldots \times T"'] & \bigwedge^k E \arrow[d, dotted, "\exists\,!\ \bigwedge^k T"] \\
F \times \ldots \times F \arrow[r, "\wedge"] & \bigwedge^k F
\end{tikzcd}$$

This makes $\bigwedge^k$ into a functor.
In fact, $\bigwedge^k T$ is given on the basis $(e_{i_1} \wedge \dots \wedge e_{i_k})$ as follows:

$$\begin{aligned}(\bigwedge^k T)(e_{i_1} \wedge \dots \wedge e_{i_k}) &:= T(e_{i_1}) \wedge \dots \wedge T(e_{i_k}) \\ &= \sum_{j_1} T^{j_1}_{i_1} f_{j_1} \wedge \dots \wedge \sum_{j_k} T^{j_k}_{i_k} f_{j_k} \\ &= \sum_{j_1,\dots,j_k} T^{j_1}_{i_1} \cdot \ldots \cdot T^{j_k}_{i_k}\; f_{j_1} \wedge \dots \wedge f_{j_k} \\ &= \sum_{j_1<\dots<j_k} \sum_{\pi} T^{j_{\pi(1)}}_{i_1} \cdot \ldots \cdot T^{j_{\pi(k)}}_{i_k}\; f_{j_{\pi(1)}} \wedge \dots \wedge f_{j_{\pi(k)}} \\ &= \sum_{j_1<\dots<j_k} \sum_{\pi} T^{j_{\pi(1)}}_{i_1} \cdot \ldots \cdot T^{j_{\pi(k)}}_{i_k}\; \operatorname{sgn}(\pi)\; f_{j_1} \wedge \dots \wedge f_{j_k} \\ &= \sum_{j_1<\dots<j_k} \det((T^{j_r}_{i_s})_{r,s})\; f_{j_1} \wedge \dots \wedge f_{j_k}.\end{aligned}$$

3. For $m = \dim(E)$, the space $\bigwedge E := \coprod_{i=0}^m \bigwedge^i E$ is a graded-commutative associative algebra with $1 \in \bigwedge^0 E := \mathbb{R}$, the so-called OUTER ALGEBRA over E.
   A graded algebra $A = \coprod_{k\in\mathbb{N}} A_k$ is called GRADED-COMMUTATIVE if
   $$a \in A_k,\ b \in A_j \Rightarrow a \cdot b = (-1)^{kj}\, b \cdot a.$$
   We have $\dim(\bigwedge E) = \sum_{i=0}^m \binom{m}{i} = 2^m$.

22. Vector bundle constructions

22.1 Definition (Tensor fields and differential forms).

Let M be an m-dimensional manifold and $x \in M$. As vector space E we now use the tangential space T_xM of M at x. Then $E^* = (T_xM)^*$, and we form the tensor product

$$\underbrace{T_xM \otimes \dots \otimes T_xM}_{p \text{ times}} \otimes \underbrace{(T_xM)^* \otimes \dots \otimes (T_xM)^*}_{q \text{ times}} = L^{p+q}(T_x^*M, \dots, T_xM; \mathbb{R}).$$

The elements of this vector space are referred to as p-FOLD CONTRAVARIANT, q-FOLD COVARIANT VECTORS or tensors. A basis of T_xM is given by $(\frac{\partial}{\partial u^i})_{i=1}^m$, where $(u^1, \dots, u^m)$ are local coordinates around x of M. The dual basis of $(T_xM)^*$ we have denoted $(du^i)_{i=1}^m$. By 21.3, we get as basis of the tensor product:

$$\Big(\frac{\partial}{\partial u^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \dots \otimes du^{j_q}\Big)_{i_1,\dots,i_p,j_1,\dots,j_q=1,\dots,m}$$

Analogously we form $\Lambda^k(T_xM)^* \cong L^k_{\text{alt}}(T_xM, \mathbb{R})$. The elements of this outer product are called k FORMS and

$$(du^{i_1} \wedge \dots \wedge du^{i_k})_{i_1<\dots<i_k}$$

forms a basis.

Let us now vary the point $x \in M$, so we consider mappings

$$\omega : M \ni x \mapsto \omega(x) \in \underbrace{T_xM \otimes \dots \otimes T_xM}_{p \text{ times}} \otimes \underbrace{(T_xM)^* \otimes \dots \otimes (T_xM)^*}_{q \text{ times}}.$$

These are called p-FOLD CONTRAVARIANT AND q-FOLD COVARIANT TENSOR FIELDS.

A mapping

$$\omega : M \ni x \mapsto \omega(x) \in \Lambda^k(T_xM)^*$$

is called DIFFERENTIAL FORM OF DEGREE k.

In order to be able to speak of the smoothness of a tensor field (or a differential form) we should make the family of vector spaces

$$\Big(\underbrace{T_xM \otimes \cdots \otimes T_xM}_{p \text{ times}} \otimes \underbrace{(T_xM)^* \otimes \cdots \otimes (T_xM)^*}_{q \text{ times}}\Big)_{x\in M}$$

into a manifold or even better a vector bundle over M.

We proceed here analogously to the construction of the cotangent bundle from the tangent bundle.

22.2 Direct sum of vector bundles.

Let $E \xrightarrow{p} M$ and $F \xrightarrow{q} M$ be two vector bundles over M, and φ^E a trivialization of E over $U \subset M$ and φ^F one of F over the (w.l.o.g.) same U. With $\psi^E : U \cap V \to GL(\mathbb{R}^k)$ and $\psi^F : U \cap V \to GL(\mathbb{R}^l)$ we denote the transition functions for two such vector bundle charts over U and V. We now make the disjoint union $E \oplus F := \bigsqcup_{x\in M}(E_x \oplus F_x)$ into a vector bundle. As vector bundle charts we use fiber-wise

$$\varphi_x^{E\oplus F} := \varphi_x^E \oplus \varphi_x^F : \mathbb{R}^{k+l} \cong \mathbb{R}^k \oplus \mathbb{R}^l \xrightarrow{\cong} E_x \oplus F_x.$$

The transition functions $\psi^{E\oplus F} : U \cap V \to GL(\mathbb{R}^{k+l})$ are then given by

$$\psi^{E\oplus F}(x) := \psi^E(x) \oplus \psi^F(x) \in GL(\mathbb{R}^k) \times GL(\mathbb{R}^l) \hookrightarrow GL(\mathbb{R}^{k+l}).$$

The matrix representation of $\psi^{E\oplus F}(x)$ is $\begin{pmatrix} [\psi^E(x)] & 0 \\ 0 & [\psi^F(x)] \end{pmatrix}$. So $\psi^{E\oplus F}$ is smooth and hence $E \oplus F \to M$ is a vector bundle, the so-called WHITNEY SUM of E and F.

22.3 Tensor product of vector bundles.

Analogous to the direct sum, we make the disjoint union $E \otimes F := \bigsqcup_{x\in M}(E_x \otimes F_x)$ into a vector bundle, the so-called TENSOR PRODUCT of E and F. As vector bundle maps we use fiber-wise

$$\varphi_x^{E\otimes F} := \varphi_x^E \otimes \varphi_x^F : \mathbb{R}^{kl} \cong \mathbb{R}^k \otimes \mathbb{R}^l \xrightarrow{\cong} E_x \otimes F_x.$$

The transition functions $\psi^{E\otimes F} : U \cap V \to GL(\mathbb{R}^{kl})$ are then given by

$$\psi^{E\otimes F}(x) := \psi^E(x) \otimes \psi^F(x) \in GL(\mathbb{R}^{kl}) \subset L(\mathbb{R}^k \otimes \mathbb{R}^l, \mathbb{R}^k \otimes \mathbb{R}^l).$$

The matrix representation of $\psi^{E\otimes F}(x)$ is $(a_i^r b_j^s)_{(i,j),(r,s)}$ by 21.3.2, where (a_i^r) is the matrix of $\psi^E(x)$ and (b_j^s) is that of $\psi^F(x)$. So $\psi^{E\otimes F}$ is smooth and $E \otimes F \to M$ is a vector bundle.

22.4 Outer product of a vector bundle.

Finally, we make the disjoint union $\bigwedge^p E := \bigsqcup_{x\in M} \bigwedge^p E_x$ into a vector bundle, the so-called p-FOLD OUTER PRODUCT of E. As vector bundle charts we use fiber-wise

$$\varphi_x^{\bigwedge^p E} := \bigwedge^p(\varphi_x^E) : \mathbb{R}^{\binom{k}{p}} \cong \bigwedge^p \mathbb{R}^k \xrightarrow{\cong} \bigwedge^p E_x.$$

The transition functions $\psi^{\bigwedge^p E} : U \cap V \to GL(\mathbb{R}^{\binom{k}{p}})$ are then given by

$$\psi^{\bigwedge^p E}(x) := \bigwedge^p(\psi^E(x)) \in GL(\mathbb{R}^{\binom{k}{p}}) \subset L\Big(\bigwedge^p \mathbb{R}^k, \bigwedge^p \mathbb{R}^k\Big).$$

The matrix representation of $\psi^{\bigwedge^p E}(x)$ is $(\det((a^{j_r}_{i_s})_{r,s}))_{i_1<\dots<i_p,j_1<\dots<j_p}$ by 21.6.2, hence the transition functions are smooth and $\bigwedge^p E$ a vector bundle.

More generally, one has the following construction:

22.5 Theorem (Functorial vector bundle constructions).

Let $\mathcal{F}$ be an assignment which associates a finite-dimensional vector space to each family of $(k+i)$ finite-dimensional vector spaces in a FUNCTORIAL *way.*

Functorial means that each $(k+i)$ tuple of linear mappings
$T_j : F_j \to E_j$ for $j \le k$ "contravariant in front variables"
$T_j : E_j \to F_j$ for $k < j$ "covariant in the back variables"
a linear mapping

$$\mathcal{F}(T_1, \dots, T_{k+i}) : \mathcal{F}(E_1, \dots, E_{k+i}) \to \mathcal{F}(F_1, \dots, F_{k+i})$$

is associated, which is compatible with composition and identity and smoothly depends on $T_1, \dots, T_{k+i}$.

Then, for $(k+i)$ many vector bundles $p_j : E_j \to M$ a natural vector bundle structure on $\mathcal{F}(E_1, \dots, E_{k+i}) := \bigsqcup_x \mathcal{F}(E_1|_x, \dots, E_{k+i}|_x)$ is given.

An example of such a functor is the direct sum $\oplus$; which, when applied to vector bundles, yields the Whitney sum.

Another is the dual-space functor, which maps the tangent bundle $\pi : TM \to M$ to the cotangent bundle $T^*M = \bigsqcup_x (T_xM)^* \to M$.

Other examples are the tensor product and the outer product, as well as combinations of them, such as $\bigwedge^k T^*M = L^k_{\text{alt}}(TM, \mathbb{R}) = \big(\bigwedge^k TM\big)^*$.

Proof. The vector bundle maps $\mathcal{F}(\psi_1, \dots, \psi_{k+i})$ are obtained from those for E_i fiber-wise by the following formula:

$$\begin{aligned}\mathcal{F}(\psi_1, \dots, &\psi_{k+i})|_x = \\ &= \mathcal{F}\Big((\psi_1)_x^{-1}, \dots, (\psi_k)_x^{-1}, (\psi_{k+1})_x, \dots, (\psi_{k+i})_x\Big) \\ &= \mathcal{F}(\psi_1, \dots, \psi_{k+i})|_x : \mathcal{F}(\mathbb{R}^{N_1}, \dots, \mathbb{R}^{N_{k+i}}) \xrightarrow{\cong} \mathcal{F}(E_1|_x, \dots, E_{k+i}|_x)\end{aligned}$$

The associated transition functions are then the composition of the following three mappings:

$$\begin{aligned}(\psi_1|_x, \dots, \psi_{k+i}|_x) : U \cap V \to GL(\mathbb{R}^{N_1}) \times \ldots \times GL(\mathbb{R}^{N_{k+i}})\\ \operatorname{inv} \times \ldots \times \operatorname{id} : GL(\mathbb{R}^{N_1}) \times \ldots \times GL(\mathbb{R}^{N_{k+i}}) \to GL(\mathbb{R}^{N_1}) \times \ldots \times GL(\mathbb{R}^{N_{k+i}})\\ \mathcal{F} : GL(\mathbb{R}^{N_1}) \times \ldots \times GL(\mathbb{R}^{N_{k+i}}) \to GL\Big(\mathcal{F}\big(\mathbb{R}^{N_1}, \dots, \mathbb{R}^{N_{k+i}}\big)\Big).\end{aligned}$$ □

23. Differential forms

23.1 Definition (Smooth tensor fields and differential forms).

The VECTOR SPACE OF THE SMOOTH p-FOLD CONTRAVARIANT AND q-FOLD COVARIANT TENSOR FIELDS or in short p-q TENSOR FIELDS, i.e. the smooth sections of the vector bundle

$$\underbrace{TM \otimes \dots \otimes TM}_{p \text{ times}} \otimes \underbrace{(TM)^* \otimes \dots \otimes (TM)^*}_{q \text{ times}} \to M$$

is also denoted $\mathcal{T}_p^q(M) := C^\infty(M \leftarrow \bigotimes^p TM \otimes \bigotimes^q T^*M)$. In particular, the 0-0 tensor fields are just the real valued functions, the 1-0 tensor fields are the vector fields and the 0-1 tensor fields are the 1-forms.
Locally, each tensor field Φ can be written as

$$\Phi = \sum_{\substack{i_1,\dots,i_p=1\\ j_1,\dots,j_q=1}}^{\dim(M)} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \tfrac{\partial}{\partial u^{i_1}} \otimes \dots \otimes \tfrac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \dots \otimes du^{j_q}.$$

We know that Φ is smooth if and only if all components of $\Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q}$ are smooth real-valued functions.

The vector space of the smooth differential forms of degree p, i.e. smooth sections of the vector bundle $\bigwedge^p(TM)^*$, is denoted $\Omega^p(M)$. Similarly to 19.8 we can describe this space differently:

$$\begin{aligned}
\Omega^p(M) &:= C^\infty\Big(M \leftarrow \bigwedge^p (TM)^*\Big)\\
&\cong C^\infty\Big(M \leftarrow (\bigwedge^p TM)^*\Big)\\
&\cong \Big\{\omega : \bigwedge^p TM \to \mathbb{R} : \omega_x \in L\Big(\bigwedge^p T_xM, \mathbb{R}\Big) \,\forall\, x\Big\}\\
&\cong \Big\{\omega : \bigoplus^p TM \to \mathbb{R} : \omega_x \in L^p_{\text{alt}}(T_xM, \mathbb{R}) \,\forall\, x\Big\}
\end{aligned}$$

Because of $\bigwedge^0(TM)^* = M \times \mathbb{R}$, the space $\Omega^0(M)$ of 0-forms coincides with $C^\infty(M, \mathbb{R})$.
Each differential form ω of degree k can be written locally as

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1,\dots,i_p}\, du^{i_1} \wedge \dots \wedge du^{i_p}.$$

Again, ω is smooth if and only if all its local components $\omega_{i_1,\dots,i_k}$ smooth. Since

$$\begin{aligned}
(du^{i_1} \wedge \dots \wedge du^{i_k}) \left(\frac{\partial}{\partial u^{j_1}}, \dots, \frac{\partial}{\partial u^{j_k}}\right) &\overset{\boxed{21.5}}{=\!=}\\
&= \sum_\pi \operatorname{sgn}(\pi)\, du^{i_1}\left(\frac{\partial}{\partial u^{j_{\pi(1)}}}\right) \cdot \ldots \cdot du^{i_k}\left(\frac{\partial}{\partial u^{j_{\pi(k)}}}\right)\\
&= \begin{cases} \operatorname{sgn}(\pi) & \text{if a permutation } \pi \text{ exists with } j_{\pi(k)} = i_k \,\forall\, k,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

we obtain the following formula for $j_1 < \dots < j_k$:

$$\begin{aligned}
\omega\left(\tfrac{\partial}{\partial u^{j_1}}, \dots, \tfrac{\partial}{\partial u^{j_k}}\right) &= \Big(\sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \cdot du^{i_1} \wedge \dots \wedge du^{i_k}\Big) \left(\tfrac{\partial}{\partial u^{j_1}}, \dots, \tfrac{\partial}{\partial u^{j_k}}\right)\\
&= \omega_{j_1 \dots j_k}.
\end{aligned}$$

23.2 Remark.

Because of

$$\begin{aligned}
&\underbrace{T_xM \otimes \dots \otimes T_xM}_{p \text{ times}} \otimes \underbrace{(T_xM)^* \otimes \dots \otimes (T_xM)^*}_{q \text{ times}} \cong\\
&\qquad\qquad \cong L(\underbrace{(T_xM)^*, \dots, (T_xM)^*}_{p \text{ times}}, \underbrace{T_xM, \dots, T_xM}_{q \text{ times}}; \mathbb{R})
\end{aligned}$$

we can apply a p-q tensor field

$$\Phi = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \Phi^{i_1,\dots,i_p}_{j_1\dots,j_q} \tfrac{\partial}{\partial u^{i_1}} \otimes \dots \otimes \tfrac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \dots \otimes du^{j_q}$$

pointwise to p cotangent vectors $\omega^1, \dots, \omega^p$ and q tangential vectors $\xi_1, \dots, \xi_q$:

$$\begin{aligned}
&\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q) = \\
&\quad = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot (\tfrac{\partial}{\partial u^{i_1}} \otimes \dots \otimes du^{j_q})\Big(\sum_{r_1} \omega^1_{r_1} du^{r_1}, \dots, \sum_{s_q} \xi_q^{s_q} \tfrac{\partial}{\partial u^{s_q}}\Big) \\
&\quad = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q\\ r_1,\dots,r_p\\ s_1,\dots,s_q}} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot \omega^1_{r_1} \delta^{r_1}_{i_1} \cdot \dots \cdot \xi_q^{s_q} \delta^{j_q}_{s_q} \\
&\quad = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot \omega^1_{i_1} \cdot \dots \cdot \omega^p_{i_p} \cdot \xi_1^{j_1} \cdot \dots \cdot \xi_q^{j_q}.
\end{aligned}$$

Theorem (Tensor fields as $C^\infty(M,\mathbb{R})$-multilinear maps).

The mapping from above provides a linear isomorphism of the space of smooth p-q tensor fields on M with the following space of $C^\infty(M,\mathbb{R})$-multilinear mappings:

$$\mathcal{T}^q_p(M) \cong \operatorname{Hom}_{C^\infty(M,\mathbb{R})}(\Omega^1(M), \dots, \mathfrak{X}(M); C^\infty(M,\mathbb{R}))$$

Proof. We proceed analogously to the proof of 19.11: Obviously, each tensor field Φ acts on 1-forms $\omega^1, \dots, \omega^p \in \Omega^1(M)$ and on vector fields $\xi_1, \dots, \xi_q \in \mathfrak{X}(M)$ as $C^\infty(M,\mathbb{R})$-linear mapping, via

$$\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)(x) := \Phi_x(\omega^1(x), \dots, \omega^p(x), \xi_1(x), \dots, \xi_q(x)),$$

and because of the local formula from above

$$\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q) = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot \omega^1_{i_1} \cdot \dots \cdot \omega^p_{i_p} \cdot \xi_1^{j_1} \cdot \dots \cdot \xi_q^{j_q},$$

we have $\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q) \in C^\infty(M,\mathbb{R})$.

Conversely, let $\Phi : \Omega^1(M) \times \dots \times \mathfrak{X}(M) \to C^\infty(M,\mathbb{R})$ be a $C^\infty(M,\mathbb{R})$-multilinear map. If one of the vector fields or 1-forms σ locally vanishes around $x \in M$, so does $\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)$, because $f \in C^\infty(M,\mathbb{R})$ is chosen so that $f = 1$ on the carrier of that section σ and $f(x) = 0$. Then $f \cdot \sigma = \sigma$ and because of the $C^\infty(M,\mathbb{R})$-linearity we have

$$\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)(x) = f(x) \cdot \Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)(x) = 0.$$

Thus, we obtain the local formula

$$\Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q) = \sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}} \Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} \cdot \omega^1_{i_1} \cdot \dots \cdot \omega^p_{i_p} \cdot \xi_1^{j_1} \cdot \dots \cdot \xi_q^{j_q},$$

with $\Phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} := \Phi(\frac{\partial}{\partial u^{i_1}}, \dots, du^{j_q})$, whose right-hand side at x depends only on the value of the 1-forms and vector fields at this point. So

$$\Phi_x(\omega^1|_x, \dots, \xi_q|_x) := \Phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)(x)$$

defines a smooth tensor field, the required inverse mapping to Φ. $\square$

23.3 Theorem (Differential forms as $C^\infty(M,\mathbb{R})$-multilinear mappings).
There is a linear isomorphism of $\Omega^k(M)$ with $\{\omega : \mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M) \to C^\infty(M,\mathbb{R}) :$ ω being k linear alternating and $C^\infty(M,\mathbb{R})$ homogeneous $\}$.

Proof. ($\Rightarrow$) Obviously $\omega(\xi_1,\ldots,\xi_k)|_p = \omega_p(\xi_1|_p,\ldots,\xi_k|_p)$ is k-linear and alternating.

Moreover, the mapping ω is also $C^\infty(M,\mathbb{R})$ homogeneous:

$$\begin{aligned}\omega(f\,\xi_1,\ldots,\xi_k)|_p &= \omega_p(f_p\,\xi_1|_p,\ldots,\xi_k|_p) = f(p)\,\omega_p(\xi_1|_p,\ldots,\xi_k|_p)\\ &= f\cdot\omega(\xi_1,\ldots,\xi_k)|_p\end{aligned}$$

Furthermore, $M \xrightarrow{(\xi_1,\ldots,\xi_k)} TM\oplus\cdots\oplus TM \xrightarrow[\boxed{23.1}]{\omega} \mathbb{R}$ is smooth, that is $\omega(\xi_1,\ldots,\xi_k)\in C^\infty(M,\mathbb{R})$.

($\Leftarrow$) Let $\omega : \mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M)\to C^\infty(M,\mathbb{R})$ be k-linear alternating and $C^\infty(M,\mathbb{R})$-homogeneous. We have to show that $\omega(\xi_1,\ldots,\xi_k)|_p$ depends only on $\xi_1|_p,\ldots,\xi_k|_p$, because then we can define: $\omega_p(\xi_1|_p,\ldots,\xi_k|_p) := \omega(\xi_1,\ldots,\xi_k)|_p$.

Let $\xi_1 = 0$ be locally at p, $f\in C^\infty(M,\mathbb{R})$ with $f(p)=0$ and $f=1$ where $\xi_1\neq 0$. Then $f\cdot\xi_1 = \xi_1$ holds and thus as before

$$\omega(\xi_1,\ldots,\xi_k)|_p = \omega(f\xi_1,\ldots,\xi_k)|_p = f(p)\,\omega(\xi_1,\ldots,\xi_k)|_p = 0.$$

Let $\xi_1 = \sum_{i=1}^m \xi_1^i\,\frac{\partial}{\partial x^i}$ locally. Then

$$\omega(\xi_1,\ldots,\xi_k) = \omega\Big(\sum_i \xi_1^i\,\frac{\partial}{\partial x^i},\xi_2,\ldots,\xi_k\Big) = \sum_i \xi_1^i\,\omega\Big(\frac{\partial}{\partial x^i},\xi_2,\ldots,\xi_k\Big)$$

and since $\xi_1|_p = 0$ all $\xi_1^i|_p = 0$ and thus $\omega(\xi_1,\ldots,\xi_k)|_p = 0$ holds.

Let $\omega = \sum_I \omega_I dx^I$ be a local representation of ω, with $dx^I := dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ for $I=(i_1,\ldots,i_k)$ with $i_1<\cdots<i_k$. Then $\omega_I(p) = \omega(\frac{\partial}{\partial x^{i_1}},\ldots,\frac{\partial}{\partial x^{i_k}})|_p$ is smooth at p, so $\omega\in\Omega^k(M)$. □

24. Differential forms on Riemannian manifolds

24.1 Remarks on duality.

For open $M\subset\mathbb{R}^m$ we can identify the tangent bundle and the cotangent bundle, because $TM = M\times\mathbb{R}^m$ and $T^*M = M\times(\mathbb{R}^m)^*$. Thus, both the vector fields and the 1-forms on M coincide with mappings $M\to\mathbb{R}^m$. For general manifolds M, however, there is no such canonical isomorphism between T_xM and $(T_xM)^*$. We will now describe manifolds for which there is such a thing. In which way is a finite-dimensional vector space E and its dual space E^* isomorphic? Since they have the same dimension, they are isomorphic. But to give such an isomorphism, one uses a basis of E and takes as images the vectors of the dual basis of E^*. If one chooses another basis on E, then also the isomorphism changes (see below). So we can not proceed like this on a manifold, because in T_xM we have no distinguished basis.

A second way to obtain such an isomorphism is to use an inner product $\langle\cdot,\cdot\rangle$ on E. Then this bilinear form induces a linear mapping $\sharp : E\to L(E,\mathbb{R}) = E^*$ given by $v\mapsto\langle v,\cdot\rangle$. This mapping is injective because $\forall\, w : \langle v,w\rangle = 0 \;\Rightarrow\; v=0$. For dimensional reasons, it is thus an isomorphism. The inverse mapping is denoted $\flat := \sharp^{-1} : E^*\to E$. For $\xi := \flat\omega$ and thus $\omega = \sharp\xi$ we have $\langle\flat\omega,\eta\rangle = \langle\xi,\eta\rangle = \sharp(\xi)(\eta) = \omega(\eta)$.

How does $\sharp$ look like in coordinates? Let (e_i) be an orthonormal(!) basis of E and (e^i) the corresponding dual basis. Then $\sharp(e_i)(e_j) := \langle e_i, e_j\rangle = \delta_{i,j} = e^i(e_j)$, so $\sharp$ maps the basis (e_i) to the dual basis (e^i).

If (g_i) is any basis of E and (g^i) is the associated dual basis of E^*, then:

$$\sharp(g_i)(g_k) = \langle g_i, g_k\rangle =: g_{i,k} = \sum_j g_{i,j}\, g^j(g_k) \quad \Rightarrow$$

$$\sharp(g_i) = \sum_j g_{i,j} g^j \text{ and } \sharp(v) = \sharp\Big(\sum_i v^i g_i\Big) = \sum_i v^i \sum_j g_{i,j} g^j = \sum_j \Big(\sum_i g_{i,j} v^i\Big) g^j$$

If we denote with v^i the coordinates of the vector $v \in E$ with respect to the basis (g_i) and with v_j the coordinates of the associated dual vector $\sharp(v) \in E^*$ with respect to the dual basis (g^i), then

$$v_j = \sum_i g_{i,j} v^i.$$

Let $M \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$. Then T_xM is a subspace of $\mathbb{R}^n$ and thus inherits the usual inner product of $\mathbb{R}^n$. So $(T_xM)^*$ is isomorphic to T_xM by virtue of the isomorphism $\sharp : T_xM \to (T_xM)^*$. Hence we also obtain a fiber-linear bijection of the bundles $TM \to M$ and $T^*M \to M$. In coordinates it is given by

$$\tfrac{\partial}{\partial u^i} \mapsto \sum_j g_{i,j} du^j$$

where $g_{i,j} := \langle g_i, g_j\rangle$ with $g_i := \frac{\partial}{\partial u^i}$ and $g^i = du^i$. Since the g_i are smooth functions $M \supseteq U \to \mathbb{R}^n$, all coefficients $g_{i,j} : M \supseteq U \to \mathbb{R}$ are smooth, and hence TM and T^*M are isomorphic.

Thus, also the smooth sections correspond to each other, i.e. $\mathfrak{X}(M) \cong \Omega^1(M)$. The vector field corresponding to an exact 1-form df is called GRADIENT FIELD $\operatorname{grad}(f)$ of f. For open submanifolds $M \subseteq \mathbb{R}^M$, the coordinate representation of $\operatorname{grad}(f)$ is obtained from that of df by transposition, but this is not true for general manifolds M.

24.2 Tensor fields on Riemannian manifolds.

We already know that $\Omega^0(M) = C^\infty(M, \mathbb{R})$. We want to describe $\Omega^1(M)$ now differently. Let first E be a finite-dimensional vector space with an inner product. Then we have the isomorphism $\sharp : E \xrightarrow{\cong} E^*$, $v \mapsto \langle v, \cdot\rangle$, by 24.1. Its inverse is denoted $\flat := \sharp^{-1}$. If $(e_i)_{i=1}^m$ is an orthonormal basis of E and $(e^i)_{i=1}^m$ is the dual basis of E^*, then:

$$\sharp : x = \sum_i x^i e_i \in E \mapsto \sum_i x^i e^i \in E^*.$$

For Riemannian manifolds (M, g) we thus have isomorphisms $\sharp : T_xM \cong (T_xM)^*$. A basis in the tangent space is given by $\frac{\partial}{\partial u^i}$, and this is mapped by 24.1 to $\sharp(\frac{\partial}{\partial u^i}) = \sum_j g_{j,i}\, du^j$. More generally, $\xi \in T_xM$ corresponds to $\omega \in (T_xM)^*$ as follows:

$$\xi = \sum_i \xi^i \frac{\partial}{\partial u^i} \in T_xM \xmapsto{\sharp} \omega = \sum_i \omega_i\, du^i \in (T_xM)^*.$$

where $\omega_i = \sum_j g_{i,j}\xi^j$, $\xi^i = \sum_j g^{i,j}\omega_j$, $g_{i,j} := \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\rangle$, and $(g^{i,j}) := (g_{i,j})^{-1}$.
It follows that $TM \cong T^*M$ canonically, and thus the space of the vector fields $\mathfrak{X}(M)$ is canonically isomorphic to the space of the 1-forms $\Omega^1(M)$.

More generally $\bigotimes^p TM \otimes \bigotimes^q T^*M \cong \bigotimes^{p+q} TM \cong \bigotimes^{p+q} T^*M$ and hence

$$\mathcal{T}_p^q(M) \cong \mathcal{T}_{p+q}^0(M) \cong \mathcal{T}_0^{p+q}(M)$$

24.3 Volume form.

Let E be a finite-dimensional, oriented linear space with an inner product. If $(e_i)_{i=1}^m$ is a positive oriented orthonormal basis of E, we define $\det \in L^m_{\text{alt}}(E;\mathbb{R})$ by $\det(e_1,\dots,e_m) := 1$. To show that this definition does not depend on the chosen basis, we choose arbitrary vectors $g_i \in E$ and consider the map $A : E \to E$, which maps e_j to $g_j := \sum_i a^i_j e_i$. Then

$$\begin{aligned}\det(g_1,\dots,g_m) &= \det\Big(\sum_{j_1} a_1^{j_1} e_{j_1},\dots,\sum_{j_m} a_m^{j_m} e_{j_m}\Big)\\ &= \sum_{j_1,\dots,j_m} a_1^{j_1}\cdot\ldots\cdot a_m^{j_m}\ \underbrace{\det(e_{j_1},\dots,e_{j_m})}_{\substack{=0 \text{ if } j_1,\dots,j_m \text{ isn't}\\ \text{a permutation of } 1,\dots,m}}\\ &= \sum_{j \text{ permutation}} a_1^{j(1)}\cdot\ldots\cdot a_m^{j(m)}\,\operatorname{sgn}(j)\,\underbrace{\det(e_1,\dots,e_m)}_{=1}\\ &= \det((a_i^j)_{i,j}).\end{aligned}$$

Thus, if $(g_i)_i$ is an orthonormal positively oriented basis, then $[A] \in SO(n)$, hence $\det(g_1,\dots,g_m) = \det[A] = 1$.

Since we want to apply this construction to the tangent space of an oriented Riemannian manifold (where we do not have an orthonormal basis but only a positively oriented basis $(\frac{\partial}{\partial u^j})_j$), we also need a formula for the determinant for such a basis (g_j): For this, we again consider the inner products

$$g_{i,j} := \langle g_i, g_j\rangle = \Big\langle \sum_k a_i^k e_k, \sum_l a_j^l e_l\Big\rangle = \sum_{k,l} a_i^k a_j^l \underbrace{\langle e_k, e_l\rangle}_{\delta_{k,l}} = \sum_k a_i^k a_j^k,$$

i.e. $(g_{i,j})_{i,j} = [A\cdot A^t]$ and furthermore

$$\det((g_{i,j})_{i,j}) = \det([A]\cdot[A]^t) = (\det[A])^2$$

and finally (because of $\det[A] > 0$)

$$\det(g_1,\dots,g_m) = \det[A] = \sqrt{\det((g_{i,j})_{i,j})} =: \sqrt{G}.$$

For each oriented (see [86, 34.3]) Riemannian manifold (M,g) of dimension m, we have $\det \in L^m_{\text{alt}}(T_xM,\mathbb{R})$ and we define the VOLUME FORM $\text{vol}_M \in \Omega^m(M)$ of the manifold by

$$\text{vol}_M(x) := \det \in L^m_{\text{alt}}(T_xM;\mathbb{R}).$$

We want to calculate this volume form by means of local coordinates $(u^1,\dots,u^m)$. The $g_i := \frac{\partial}{\partial u^i}$ form a basis in T_xM, which we may assume is positively oriented by using an orientation-preserving chart $\varphi = (u^1,\dots,u^m)^{-1}$. Then $\text{vol} = \text{vol}_{1,\dots,m}\cdot du^1\wedge\ldots\wedge du^m$ with

$$\begin{aligned}\text{vol}(\tfrac{\partial}{\partial u^1},\dots,\tfrac{\partial}{\partial u^m}) &= \big(\text{vol}_{1,\dots,m}\cdot du^1\wedge\ldots\wedge du^m\big)\,(\tfrac{\partial}{\partial u^1},\dots,\tfrac{\partial}{\partial u^m})\\ &= \text{vol}_{1,\dots,m}\cdot\sum_\pi \operatorname{sgn}(\pi)\,\underbrace{du^1(\tfrac{\partial}{\partial u^{\pi(1)}})}_{\delta_{\pi(1),1}}\cdot\ldots\cdot\underbrace{du^m(\tfrac{\partial}{\partial u^{\pi(m)}})}_{\delta_{\pi(m),m}} = \text{vol}_{1,\dots,m},\end{aligned}$$

since π must be the identity, see also 23.1. Because of the above calculation

$$\text{vol}(\tfrac{\partial}{\partial u^1},\dots,\tfrac{\partial}{\partial u^m}) = \det(g_1,\dots,g_m) = \sqrt{G}$$

where $G := \det((g_{i,j})_{i,j})$ and $g_{i,j} := \langle g_i, g_j\rangle = g(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j})$.

We obtain the following isomorphism for orientable Riemannian manifolds of dimension m:

$$C^\infty(M,\mathbb{R}) \xrightarrow{\cong} \Omega^m(M), \quad f \mapsto f\cdot \mathrm{vol}_M\,.$$

25. Graded derivations

25.1 Lemma (Algebra of differential forms).

The space $\Omega(M) := \bigoplus_k \Omega^k(M)$ is a graded commutative algebra with respect to the point-wise wedge product (see 21.4)

$$(\alpha\wedge\beta)_x(\xi_1,\dots,\xi_{k+i}) = \\ = \frac{1}{k!\,i!}\sum_\pi \operatorname{sgn}\pi\cdot\alpha_x(\xi_{\pi(1)},\dots,\xi_{\pi(k)})\cdot\beta_x(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+i)}),$$

for $\alpha\in\Omega^k(M)$, $\beta\in\Omega^i(M)$ and $\xi_j\in T_xM$. For paracompact manifolds M, this algebra is generated by $\{f,\,df : f\in C^\infty(M,\mathbb{R})\}$.

Note that $f\cdot\omega = f\wedge\omega$ for $f\in C^\infty(M,\mathbb{R})=\Omega^0(M)$ and $\omega\in\Omega(M)$.

Proof. Since the fibers $\bigwedge T_x^*M = \bigoplus_k \bigwedge^k T_x^*M$ are graded commutative algebras, $\Omega(M) = C^\infty(M\leftarrow\bigwedge T^*M)$ is also a graded commutative algebra. Locally $\Omega(M)$ is generated by $\{f,\,df : f\in C^\infty(M,\mathbb{R})\}$, because $\omega = \sum_{i_1<\dots<i_k}\omega_{i_1,\dots,i_k}du^{i_1}\wedge\dots\wedge du^{i_k}$. To get that globally, we use a finite atlas of M. For connected paracompact manifolds, such an atlas exists by 9.8. We choose a partition $\{f_1,\dots,f_N\}$ of unity which is subordinate to the associated vector bundle atlas of T^*M. Then $\omega = \sum_j f_j\omega$ and $f_j\omega = \sum_{i_1<\dots<i_k}\omega_{j;i_1,\dots,i_k}du^{i_1}\wedge\dots\wedge du^{i_k}$, where $(u^1,\dots,u^m)$ are local coordinates on a neighborhood W_i of $\mathrm{supp}(f_j)$, which have been extended to global smooth functions on M and the coefficients $\omega_{j;i_1,\dots,i_k}$ are global smooth functions with carrier in W_i. □

25.2 Pull-back of forms.

Let $f: M\to N$ be smooth, $T_pf: T_pM\to T_{f(p)}N$ the tangential mapping and $(T_pf)^*: T^*_{f(p)}N\to T_p^*M$ its adjoint. If p is not determined by $f(p)$, i.e. f is not injective, or there fails to exist p with $f(p)=q$ for some q, i.e. f is not surjective, then the $(T_pf)^*$ cannot be collected into a mapping $T^*f: T^*N\to T^*M$. However, by 19.11,

$$\begin{tikzcd} \bigwedge^k TM \arrow[r, "\bigwedge^k Tf"] \arrow[d] & \bigwedge^k TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

can be used to define

$$\begin{tikzcd} \Omega^k(M) \arrow[d, equal] & & \Omega^k(N) \arrow[ll, "f^*"'] \arrow[d, equal] \\ C^\infty\Big(M\leftarrow\big(\bigwedge^k TM\big)^*\Big) & & C^\infty\Big(N\leftarrow\big(\bigwedge^k TN\big)^*\Big) \arrow[ll, "(\bigwedge^k Tf)^*"] \end{tikzcd}$$

The form $f^*(\omega)$ is called the PULL-BACK along f of ω.

$$(f^*\omega)_p(\xi_1\wedge\dots\wedge\xi_k) := \omega_{f(p)}\Big(\big(\bigwedge^k Tf\big)(\xi_1\wedge\dots\wedge\xi_k)\Big)$$

or, by using the isomorphism $\left(\bigwedge^k T_pM\right)^* \cong L^k_{\text{alt}}(T_pM;\mathbb{R})$,

$$(f^*\omega)_p(\xi_1,\dots,\xi_k) := \omega_{f(p)}(T_pf\cdot\xi_1|_p,\dots,T_pf\cdot\xi_k|_p).$$

The so-defined $f^*:\Omega(N)\to\Omega(M)$ is an algebra homomorphism - as one easily shows - and the following holds: $(f_1\circ f_2)^* = {f_2}^*\circ {f_1}^*$ for the composite of mappingss f_1 and f_2.

By means of the ismorphism $(\bigwedge^k TM)^* \cong \bigwedge^k(T^*M)$ one can define the pull-back f^* for $\omega_1,\dots,\omega_k\in\Omega^1(M)$ equivalently by $f^*(\omega_1\wedge\dots\wedge\omega_k) := f^*(\omega_1)\wedge\dots\wedge f^*(\omega_k)$, where $f^*(\omega_j)$ is the pulled-back 1-form defined in $\boxed{19.12}$.

Let $(u^i)_{i=1}^m$ be local coordinates on M and $(v^j)_{j=1}^n$ local coordinates on N. Then $\omega\in\Omega^k(N)$ can be written locally as

$$\omega = \sum_{j_1<\ldots<j_k} \omega_{j_1,\dots,j_k}\, dv^{j_1}\wedge\ldots\wedge dv^{j_k}.$$

The pulled-back form must have a local representation of the form

$$f^*(\omega) = \sum_{i_1<\ldots<i_k} \eta_{i_1,\dots,i_k}\, du^{i_1}\wedge\ldots\wedge du^{i_k}.$$

We now calculate the local coefficients $\eta_{i_1,\dots,i_k}$ of $f^*(\omega)$:

$$\begin{aligned}
\eta_{i_1,\dots,i_k}(x) &= f^*(\omega)_x\left(\tfrac{\partial}{\partial u^{i_1}},\dots,\tfrac{\partial}{\partial u^{i_k}}\right)\\
&= \omega_{f(x)}\Big(\big(\bigwedge^k T_xf\big)\big(\tfrac{\partial}{\partial u^{i_1}}\wedge\dots\wedge\tfrac{\partial}{\partial u^{i_k}}\big)\Big)\\
&\overset{\boxed{21.6.2}}{=\!=} \omega_{f(x)}\Big(\sum_{j_1<\dots<j_k}\det\Big(\Big(\frac{\partial(v^{j_s}\circ f)}{\partial u^{i_t}}\Big)_{t,s}\Big)\tfrac{\partial}{\partial v^{j_1}}\wedge\dots\wedge\tfrac{\partial}{\partial v^{j_k}}\Big)\\
&= \sum_{j_1<\dots<j_k}\det\Big(\Big(\frac{\partial(v^{j_s}\circ f)}{\partial u^{i_t}}\Big)_{t,s}\Big)\,\omega_{f(x)}\Big(\tfrac{\partial}{\partial v^{j_1}},\dots,\tfrac{\partial}{\partial v^{j_k}}\Big)\\
&= \sum_{j_1<\dots<j_k}\det\Big(\Big(\frac{\partial(v^{j_s}\circ f)}{\partial u^{i_t}}\Big)_{t,s}\Big)\,\omega_{j_1,\dots,j_k}(f(x)).
\end{aligned}$$

Thus

$$f^*(\omega) = \sum_{\substack{i_1<\dots<i_k\\ i_1,\dots,i_k=1\dots m}}\ \sum_{\substack{j_1<\dots<j_k\\ j_1,\dots,j_k=1\dots n}} \omega_{j_1,\dots,j_k}\rho^{j_1,\dots,j_k}_{i_1,\dots,i_k}\, du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$\text{Wobei}\qquad \rho^{j_1,\dots,j_k}_{i_1,\dots,i_k} := \det\left(\frac{\partial(v^{j_1},\dots,v^{j_k})}{\partial(u^{i_1},\dots,u^{i_k})}\right) = \det\begin{pmatrix}\frac{\partial v^{j_1}}{\partial u^{i_1}} & \dots & \frac{\partial v^{j_1}}{\partial u^{i_k}}\\ \vdots & \ddots & \vdots\\ \frac{\partial v^{j_k}}{\partial u^{i_1}} & \dots & \frac{\partial v^{j_k}}{\partial u^{i_k}}\end{pmatrix}$$

$$\text{with}\qquad \frac{\partial v^j}{\partial u^i} := \frac{\partial}{\partial u^i}(v^j\circ f),$$

25.3 Corollary (Pull-back of volume forms).

Let $f: M\to N$ smooth, $\dim M = m = \dim N$ and $(x^1,\dots,x^m)$ local coordinates on M and $(y^1,\dots,y^m)$ such on N. Then:

$$f^*(g\cdot dy^1\wedge\dots\wedge dy^m) = (g\circ f)\cdot\det\left(\left(\frac{\partial(y^j\circ f)}{\partial x^i}\right)^m_{i,j=1}\right)\cdot dx^1\wedge\dots\wedge dx^m.$$

Proof. This is a special case of 25.2. As m-form, $f^*(g \cdot dy^1 \wedge \dots \wedge dy^m) = h \cdot dx^1 \wedge \dots \wedge dx^m$ for a smooth function h. By applying this to $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ we get:

$$\begin{aligned}
h &= f^*\big(g \cdot dy^1 \wedge \dots \wedge dy^m\big)\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right)\\
&\overset{25.2}{=\!=} f^*(g) \cdot (dy^1 \wedge \dots \wedge dy^m)\left(Tf\frac{\partial}{\partial x^1}, \dots, Tf\frac{\partial}{\partial x^m}\right)\\
&= (g \circ f) \cdot (dy^1 \wedge \dots \wedge dy^m)\left(\sum_{i_1} \frac{\partial(y^{i_1} \circ f)}{\partial x^1}\frac{\partial}{\partial y^{i_1}}, \dots, \sum_{i_m} \frac{\partial(y^{i_m} \circ f)}{\partial x^1}\frac{\partial}{\partial y^{i_m}}\right)\\
&= (g \circ f) \cdot \sum_{i_1, \dots, i_m} (dy^1 \wedge \dots \wedge dy^m)\left(\frac{\partial}{\partial y^{i_1}}, \dots, \frac{\partial}{\partial y^{i_m}}\right) \cdot \frac{\partial(y^{i_1} \circ f)}{\partial x^1} \cdots \frac{\partial(y^{i_m} \circ f)}{\partial x^m}\\
&= (g \circ f) \cdot \sum_\pi \operatorname{sgn}(\pi) \prod_{j=1}^m \frac{\partial(y^{\pi(j)} \circ f)}{\partial x^j} = (g \circ f) \cdot \det\left(\left(\frac{\partial(y^i \circ f)}{\partial x^j}\right)_{i,j=1}^m\right) \qquad \square
\end{aligned}$$

25.4 Remark.

In particular, for $f = \mathrm{id}$ and $g = 1$ we get by 25.3:

$$dy^1 \wedge \dots \wedge dy^m = \det\left(\left(\frac{\partial y^j}{\partial x^i}\right)_{i,j}\right) \cdot dx^1 \wedge \dots \wedge dx^m.$$

In 17.1 we considered the commutative algebra $A := C^\infty(M, \mathbb{R})$. We identified the space $\operatorname{Der}(A)$ of its derivations with the space $\mathfrak{X}(M)$ of the vector fields on M and we found the structure of a Lie algebra on $\operatorname{Der}(A)$. We now want to apply similar ideas to the graded commutative algebra $A := \Omega(M)$ of differential forms on M.

25.5 Definition (Graded derivation).

An mapping $D : \Omega(M) \to \Omega(M)$ is called graded derivation of degree d, if D is linear, for all k the summand $\Omega^k(M)$ is mapped into $\Omega^{d+k}(M)$ and for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega(M)$ the product rule $D(\omega \wedge \eta) = D\omega \wedge \eta + (-1)^{d\,k}\omega \wedge D\eta$ holds.

With $\operatorname{Der}_d(\Omega(M))$ we denote the vector space of all graded derivations of $\Omega(M)$ of degree d, and with $\operatorname{Der}(\Omega(M))$ we denote the direct sum $\coprod_{d\in\mathbb{Z}} \operatorname{Der}_d(\Omega(M))$.

More generally, for a smooth mapping $g : N \to M$, a map $D : \Omega(M) \to \Omega(N)$ is called graded derivation over g^*, if D is linear, for all k the summand $\Omega^k(M)$ is mapped into $\Omega^{d+k}(N)$, and for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega(M)$ the product rule $D(\omega \wedge \eta) = D(\omega) \wedge g^*(\eta) + (-1)^{d\,k} g^*(\omega) \wedge D\eta$ holds.

25.6 Lemma (Uniqueness of graded derivations).

Let $g : N \to M$ be smooth. Each graded derivation $D : \Omega(M) \to \Omega(N)$ over the algebra homomorphism $g^* : \Omega(M) \to \Omega(N)$ is uniquely determined by the values $D(f)$ and $D(df)$ for all $f \in C^\infty(M, \mathbb{R})$.

Proof. Since f and df generate the algebra $\Omega(M)$, this immediately follows from 25.1. However, if we do not want to use dimension theory here, we can also show this as follows:

If we had two such derivations, we consider the difference D. We have to show: $\forall\, f : D(f) = 0,\ D(df) = 0 \Rightarrow D = 0$.

We first claim that the derivation D is a local operator: In fact let $\omega \in \Omega(M)$ be locally 0 around $g(x)$. Then we choose a $f \in C^\infty(M,\mathbb{R})$ with $f(g(x)) = 1$ and $f\,\omega = 0$, and get

$$0 = D(0) = D(f\,\omega) = \underbrace{D(f)}_{=0} \wedge g^*(\omega) + g^*(f)\cdot D(\omega)$$

And at $x \in N$ we have $0 = f(g(x))\cdot D(\omega)(x) = D(\omega)(x)$. Since D is a local and linear operator, we may replace ω by its local representation:

$$\begin{aligned} D(\omega) &= D\Big(\sum_{i_1<\dots<i_p} \omega_{i_1,\dots,i_p} du^{i_1}\wedge\dots\wedge du^{i_p}\Big)\\ &= \sum_{i_1<\dots<i_p}\Big(\underbrace{D(\omega_{i_1,\dots,i_p})}_{=0}\wedge g^*(du^{i_1}\wedge\dots\wedge du^{i_p})\\ &\qquad + \sum_{k=1}^p \pm g^*(\omega_{i_1,\dots,i_p})\,g^*(du^{i_1})\wedge\dots\wedge\underbrace{D(du^{i_k})}_{=0}\wedge\dots\wedge du^{i_p}\Big) = 0. \qquad \square\end{aligned}$$

25.7 Examples of graded derivations.

From 25.6 follows that $\operatorname{Der}_d(\Omega(M)) = \{0\}$ for $d < -1$, because $D(\Omega^k(M)) \subseteq \Omega^{k+d}(M) = \{0\}$ for $k+d<0$ and in particular for $k \in \{0,1\}$.

We want to determine $\operatorname{Der}_{-1}(\Omega(M))$ next. Let $D : \Omega(M) \to \Omega(M)$ be a graded derivation of degree $d = -1$. Then $D(C^\infty(M,\mathbb{R})) = \{0\}$ and the linear mapping $D\circ d : C^\infty(M,\mathbb{R}) \to \Omega^1(M) \to \Omega^0(M) = C^\infty(M,\mathbb{R})$ satisfies $(D\circ d)(f\cdot g) = D(g\cdot df + f\cdot dg) = D(g)\wedge df + (-1)^{0\cdot d} g\cdot D(df) + D(f)\wedge dg + (-1)^{0\cdot d} f\cdot D(dg) = (D\circ d)f\cdot g + f\cdot(D\circ d)g$, because of $D(g) = 0 = D(f)$. So $D\circ d$ is a derivation on $C^\infty(M,\mathbb{R})$ and is thus given by a vector field ξ, i.e. $D(f) = 0$ and $D(df) = \xi(f) = df(\xi)$ for all $f \in C^\infty(M,\mathbb{R})$. We will show in 25.8 that we can define a graded derivation i_ξ of degree $d = -1$ by $(i_\xi\omega)(\xi_1,\dots,\xi_k) := \omega(\xi,\xi_1,\dots,\xi_k)$ to each vector field $\xi \in \mathfrak{X}(M)$.

Now to $\operatorname{Der}_0(\Omega(M))$. Let $D : \Omega(M) \to \Omega(M)$ be a graded derivation of degree $d = 0$. Then D acts on $\Omega^0(M) = C^\infty(M,\mathbb{R})$ as derivation, so it is given there by a vector field ξ, i.e.

$$D(f) = \xi(f) = \mathcal{L}_\xi(f) := \frac{d}{dt}\Big|_{t=0} (\mathrm{Fl}^\xi_t)^* f \text{ (see 17.9 and 17.10)} .$$

However, the last expression $\frac{d}{dt}\big|_{t=0}(\mathrm{Fl}^\xi_t)^*\omega$ also makes sense for $\omega \in \Omega(M)$ and we will show in 25.9 that this defines a derivation $\mathcal{L}_\xi$ of degree $d = 0$ on $\Omega(M)$ for each vector field $\xi \in \mathfrak{X}(M)$. We will also show in 25.10 that these are those derivations of degree 0 which additionally satisfy $D(df) = d(Df)$. To get a global formula for $\mathcal{L}_\xi$, we differentiate the function $\omega(\xi_1,\dots,\xi_k)$ (for $\omega \in \Omega^k(M)$ and $\xi_i \in \mathfrak{X}(M)$) in the direction ξ at $x \in M$ and get:

$$\begin{aligned}
(\xi \cdot \omega(\xi_1,\dots,\xi_k))_x &= \frac{\partial}{\partial t}\Big|_{t=0} (\omega(\xi_1,\dots,\xi_k) \circ \mathrm{Fl}^\xi_t)_x \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \omega_{\mathrm{Fl}^\xi_t(x)}(\xi_1,\dots,\xi_k) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \omega_{\mathrm{Fl}^\xi_t(x)}(T\,\mathrm{Fl}^\xi_t \cdot T\,\mathrm{Fl}^\xi_{-t} \cdot \xi_1,\dots, T\,\mathrm{Fl}^\xi_t \cdot T\,\mathrm{Fl}^\xi_{-t} \cdot \xi_k) \\
&\overset{\boxed{25.2}}{=\!=} \frac{\partial}{\partial t}\Big|_{t=0} \big((\mathrm{Fl}^\xi_t)^*\omega\big)_x \Big(T\,\mathrm{Fl}^\xi_{-t} \cdot \xi_1|_{\mathrm{Fl}^\xi_t(x)},\dots, T\,\mathrm{Fl}^\xi_{-t} \cdot \xi_k|_{\mathrm{Fl}^\xi_t(x)}\Big) \\
&\overset{\boxed{17.5}}{=\!=} \frac{\partial}{\partial t}\Big|_{t=0} \big((\mathrm{Fl}^\xi_t)^*\omega\big)_x \Big((\mathrm{Fl}^\xi_t)^*(\xi_1)(x),\dots,(\mathrm{Fl}^\xi_t)^*(\xi_k)(x)\Big) \\
&= \Big(\frac{\partial}{\partial t}\Big|_{t=0} \big((\mathrm{Fl}^\xi_t)^*\omega\big)_x\Big)\ (\xi_1(x),\dots,\xi_k(x)) \\
&\qquad + \sum_{j=1}^k \omega_x\Big(\xi_1(x),\dots, \frac{\partial}{\partial t}\Big|_{t=0}\Big((\mathrm{Fl}^\xi_t)^*(\xi_i)(x)\Big),\dots,\xi_k(x)\Big) \\
&\overset{\boxed{17.10.2}}{=\!=\!=} (\mathcal{L}_\xi\omega)_x\ (\xi_1(x),\dots,\xi_k(x)) \\
&\qquad + \sum_{i=1}^k \omega_x\Big(\xi_1(x),\dots,[\xi,\xi_i](x),\dots,\xi_k(x)\Big),
\end{aligned}$$

and thus

$$\mathcal{L}_\xi\omega(\xi_1,\dots,\xi_k) = \xi\cdot\omega(\xi_1,\dots,\xi_k) - \sum_{i=1}^k \omega\Big(\xi_1,\dots,\xi_{i-1},[\xi,\xi_i],\xi_{i+1},\dots,\xi_k\Big).$$

In particular,

$$\begin{aligned}
(\mathcal{L}_\xi df)(\eta) &= \xi\cdot(df(\eta)) - df([\xi,\eta]) = \xi\cdot(\eta\cdot f) - [\xi,\eta]\cdot f = \eta\cdot(\xi\cdot f) = d(\xi\cdot f)(\eta) \\
&= (d\mathcal{L}_\xi f)(\eta).
\end{aligned}$$

Finally, we want to describe a distinguished derivation of degree $d = 1$. By $\boxed{20.4}$, we hope that by considering the deviation of the derivative of a 1-form (or more generally a k-form) from being symmetric, we are able to recognize whether the form is the derivative of a function (or $k-1$-form). First, we consider the case where $M = U$ is open in a vector space E. Then a k-form on U is a mapping

$$\omega : U \to L^k_{\mathrm{alt}}(E;\mathbb{R})$$

and its derivative is

$$\omega' : U \to L(E, L^k_{\mathrm{alt}}(E;\mathbb{R})).$$

If we compose this with the alternator

$$L(E, L^k_{\mathrm{alt}}(E;\mathbb{R})) \subseteq L(E, L^k(E;\mathbb{R})) \cong L^{k+1}(E,\mathbb{R}) \xrightarrow{\mathrm{alt}} L^{k+1}_{\mathrm{alt}}(E,\mathbb{R}),$$

we get the deviation dw from $\omega'(x)$ being symmetric for all $x \in U$.

So

$$\begin{aligned}
d\omega(x)(\xi_0,\dots,\xi_k) &:= \frac{1}{(k+1)!}\sum_\sigma \mathrm{sgn}(\sigma)\,\omega'(x)(\xi_{\sigma(0)})(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)}) \\
&= \frac{1}{k+1}\sum_{i=0}^k (-1)^i\omega'(x)(\xi_i)(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k).
\end{aligned}$$

In order to obtain a global formula for d on arbitrary manifolds M we replace the vectors $\xi_i \in E$ with vector fields $\xi_i \in \mathfrak{X}(E)$ and differentiate $\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)$ at the position $x \in M$ in direction $\xi_i(x)$ and obtain:

$$\begin{aligned}\Big(\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)\Big)'(x)(\xi_i(x)) &= \omega'(x)(\xi_i(x))(\xi_0(x), \dots, \widehat{\xi_i(x)}, \dots, \xi_k(x))\\ &\quad + \sum_{j<i} \omega(x)(\dots, \xi_j'(x)\cdot\xi_i(x), \dots, \widehat{\xi_i(x)}, \dots)\\ &\quad + \sum_{j>i} \omega(x)(\dots, \widehat{\xi_i(x)}, \dots, \xi_j'(x)\cdot\xi_i(x), \dots)\end{aligned}$$

And inserting into the formula above yields

$$\begin{aligned}(k+1)d\omega(x)(\xi_0, \dots, \xi_k) &:= \sum_{i=0}^{k} (-1)^i \omega'(x)(\xi_i(x))(\xi_0(x), \dots, \widehat{\xi_i(x)}, \dots, \xi_k(x))\\ &= \sum_{i=0}^{k} (-1)^i (\xi_i \cdot \omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k))(x)\\ &\quad - \sum_{j<i} (-1)^{i+j} \omega(x)(\xi_j'(x)\cdot\xi_i(x), \xi_0(x), \dots, \widehat{\xi_j(x)}, \dots, \widehat{\xi_i(x)}, \dots)\\ &\quad - \sum_{j>i} (-1)^{i+j-1} \omega(x)(\xi_j'(x)\cdot\xi_i(x), \xi_0(x), \dots, \widehat{\xi_i(x)}, \dots, \widehat{\xi_j(x)}, \dots)\\ &\overset{17.2}{=} \Big(\sum_{i=0}^{k} (-1)^i \xi_i \cdot \omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)\\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k)\Big)(x).\end{aligned}$$

Because of the annoying factor $(k+1)$ we will replace dw by $(k+1)\, d\omega$ in the future.

25.8 Lemma (Lie algebra of graded derivations).

The space $\mathrm{Der}(\Omega(M))$ *is a graded Lie algebra with respect to the pointwise vector space operations and the graded commutator as Lie bracket:*

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{d_1\, d_2} D_2 \circ D_1 \text{ for } D_i \in \mathrm{Der}_{d_i}(\Omega(M))$$

In detail this means:

1. *The bracket* $[\cdot, \cdot] : \mathrm{Der}_{d_1}(\Omega(M)) \times \mathrm{Der}_{d_2}(\Omega(M)) \to \mathrm{Der}_{d_1+d_2}(\Omega(M))$ *is bilinear for all* d_1, d_2.
2. *It is* graded anticommutative*:* $[D_1, D_2] + (-1)^{d_1\, d_2}[D_2, D_1] = 0$.
3. $[D_0, \cdot]$ *is a graded derivation with respect to* $[\cdot, \cdot]$*, i.e. the* graded Jacobi identity

$$[D_0, [D_1, D_2]] = [[D_0, D_1], D_2] + (-1)^{d_0 d_1}[D_1, [D_0, D_2]]$$

holds, or equivalently it is cyclically symmetric

$$0 = (-1)^{d_0 d_2}[D_0, [D_1, D_2]] + (-1)^{d_1 d_0}[D_1, [D_2, D_0]] + (-1)^{d_2 d_1}[D_2, [D_0, D_1]].$$

Proof. We give the proof for any graded-commutative algebra A instead of $\Omega(M)$. Claim: $[D_1, D_2] \in \operatorname{Der}_{d_1+d_2}(A)$ for $D_i \in \operatorname{Der}_{d_i}(A)$ for $i = 1, 2$.

$$\begin{aligned}
[D_1, D_2](X \cdot Y) &= \Big(D_1 \circ D_2 - (-1)^{d_1 d_2} D_2 \circ D_1\Big)(X \cdot Y) \\
&= D_1\Big(D_2 X \cdot Y + (-1)^{x d_2} X \cdot D_2 Y\Big) \\
&\quad - (-1)^{d_1 d_2} D_2\Big(D_1 X \cdot Y + (-1)^{x d_1} X \cdot D_1 Y\Big) \\
&= D_1 D_2 X \cdot Y + (-1)^{d_1(d_2+x)} D_2 X \cdot D_1 Y \\
&\quad + (-1)^{x d_2} D_1 X \cdot D_2 Y + (-1)^{x d_2 + x d_1} X \cdot D_1 D_2 Y \\
&\quad - (-1)^{d_1 d_2} D_2 D_1 X \cdot Y - (-1)^{d_1 d_2 + (d_1+x) d_2} D_1 X \cdot D_2 Y \\
&\quad - (-1)^{d_1 d_2 + x d_1} D_2 X \cdot D_1 Y - (-1)^{d_1 d_2 + d_1 x + d_2 x} X \cdot D_2 D_1 Y \\
&= [D_1, D_2] X \cdot Y + (-1)^{x(d_1+d_2)} X \cdot [D_1, D_2] Y \qquad \text{for } x := \deg(X).
\end{aligned}$$

Clearly, $[\_, \_]$ is bilinear and graded anticommutative.

Remains to show the graded Jacobi identity:

$$\begin{aligned}
&[D_0, [D_1, D_2]] - [[D_0, D_1], D_2] - (-1)^{d_0 d_1}[D_1, [D_0, D_2]] = \\
&= [D_0, D_1 D_2 - (-1)^{d_1 d_2} D_2 D_1] - [D_0 D_1 - (-1)^{d_0 d_1} D_1 D_0, D_2] \\
&\quad - (-1)^{d_0 d_1}[D_1, D_0 D_2 - (-1)^{d_0 d_2} D_2 D_0] \\
&= D_0 D_1 D_2 - (-1)^{d_1 d_2} D_0 D_2 D_1 \\
&\quad - (-1)^{(d_1+d_2) d_0} D_1 D_2 D_0 + (-1)^{d_1 d_2 + d_0(d_1+d_2)} D_2 D_1 D_0 \\
&\quad - D_0 D_1 D_2 + (-1)^{d_0 d_1} D_1 D_0 D_2 \\
&\quad + (-1)^{(d_0+d_1) d_2} D_2 D_0 D_1 - (-1)^{d_0 d_1 + (d_0+d_1) d_2} D_2 D_1 D_0 \\
&\quad - (-1)^{d_0 d_1} D_1 D_0 D_2 + (-1)^{d_0 d_1 + d_0 d_2} D_1 D_2 D_0 \\
&\quad + (-1)^{d_0 d_1 + (d_0+d_2) d_1} D_0 D_2 D_1 - (-1)^{d_0(d_1+d_2) + (d_0+d_2) d_1} D_2 D_0 D_1 \\
&= 0. \qquad \square
\end{aligned}$$

25.9 Theorem (The basic graded derivations).

Let $\xi \in \mathfrak{X}(M)$.

(ι_ξ) *By $\iota_\xi(f) := 0$, $\iota_\xi(df) := \xi \cdot f$ a graded derivation ι_ξ of degree -1, the insertion operator is specified.*

($\mathcal{L}_\xi$) *By $\mathcal{L}_\xi(f) := \xi \cdot f$, $\mathcal{L}_\xi(df) := d(\xi \cdot f)$ a graded derivation $\mathcal{L}_\xi$ of degree 0 is specified, the Lie derivative.*

(d) *By $d(f) := df$, $d(df) := 0$, a graded derivation d of degree $+1$, the outer derivative, is specified.*

Global formulas for these graded derivations are given for $\omega \in \Omega^k(M)$ and $\xi_i \in \mathfrak{X}(M)$ by:

$$(\iota_{\xi_0}\omega)(\xi_1, \dots, \xi_{k-1}) := \omega(\xi_0, \xi_1, \dots, \xi_{k-1}) \qquad \text{for } k \geq 1$$

$$(\mathcal{L}_{\xi_0}\omega)(\xi_1, \dots, \xi_k) := \xi_0 \cdot \omega(\xi_1, \dots, \xi_k) - \sum_{i=1}^{k} \omega\Big(\xi_1, \dots, \xi_{i-1}, [\xi_0, \xi_i], \xi_{i+1}, \dots, \xi_k\Big)$$

$$(d\omega)(\xi_0, \dots, \xi_k) := \sum_{i=0}^{k} (-1)^i \xi_i \cdot \omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)$$

$$+\sum_{i<j}(-1)^{i+j}\omega\Big([\xi_i,\xi_j],\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_k\Big).$$

The graded commutators are given by the following table:

$[D_1,D_2]$	ι_η	$\mathcal{L}_\eta$	d
ι_ξ	0	$-\iota_{[\xi,\eta]}$	$-\mathcal{L}_\xi$
$\mathcal{L}_\xi$	$\iota_{[\xi,\eta]}$	$\mathcal{L}_{[\xi,\eta]}$	0
d	$\mathcal{L}_\eta$	0	0

If $\eta\in\mathfrak{X}(N)$ is related to $\xi\in\mathfrak{X}(M)$ with respect to a smooth mapping $g:M\to N$, i.e. $Tg\circ\xi=\eta\circ g$ is satisfied, then

$$g^*\circ\iota_\eta=\iota_\xi\circ g^*,\quad g^*\circ\mathcal{L}_\eta=\mathcal{L}_\xi\circ g^*,\quad g^*\circ d=d\circ g^*$$

Furthermore:

$$\iota_{f\xi}\omega=f\,\iota_\xi\omega,\quad \mathcal{L}_{f\xi}\omega=f\,\mathcal{L}_\xi\omega+df\wedge\iota_\xi\omega,\quad (\mathcal{L}_\xi\omega)(x)=\tfrac{d}{dt}|_{t=0}(\mathrm{Fl}^\xi_t)^*\omega|_x$$

Proof. The proof is done in 15 steps, where we use the global formulas as defintion for ι_ξ, $\mathcal{L}_\xi$, and d.

1. Claim. $\iota_\xi\omega\in\Omega^{k-1}(M)$:
Obviously, $\iota_\xi\omega$ is alternating and $k-1$-linear and

$$\begin{aligned}\iota_\xi\omega(f\xi_1,\dots,\xi_{k-1})&=\omega(\xi,f\xi_1,\dots,\xi_{k-1})=f\,\omega(\xi,\xi_1,\dots,\xi_{k-1})\\&=f\,\iota_\xi\omega(\xi_1,\dots,\xi_{k-1}).\end{aligned}$$

2. Claim. $\iota_\xi\in\mathrm{Der}_{-1}(\Omega(M))$: Let $\alpha\in\Omega^{k+1}$ and $\beta\in\Omega^l$, then

$$\begin{aligned}&\iota_{\xi_0}(\alpha\wedge\beta)(\xi_1,\dots,\xi_{k+l})=\\&\quad=(\alpha\wedge\beta)(\xi_0,\xi_1,\dots,\xi_{k+l})\\&\quad=\frac{1}{(k+1)!\,l!}\sum_\pi\mathrm{sgn}(\pi)\;\alpha(\xi_{\pi(0)},\dots,\xi_{\pi(k)})\;\beta(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+l)})\\&\quad=\sum_{\pi\text{ piecewise}\uparrow}\mathrm{sgn}(\pi)\;\alpha(\xi_{\pi(0)},\dots,\xi_{\pi(k)})\;\beta(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+l)})\\&\quad=\sum_{\pi(0)=0}\mathrm{sgn}(\pi)\;\alpha(\xi_{\pi(0)},\dots,\xi_{\pi(k)})\;\beta(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+l)})\\&\qquad+\sum_{\pi(k+1)=0}\mathrm{sgn}(\pi)\;\alpha(\xi_{\pi(0)},\dots,\xi_{\pi(k)})\;\beta(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+l)})\\&\quad=\Big(\iota_{\xi_0}\alpha\wedge\beta\Big)(\xi_1,\dots,\xi_{k+l})\\&\qquad+\sum_{\pi'\text{ piecewise}\uparrow}(-1)^{k+1}\,\mathrm{sgn}(\pi')\;\alpha(\xi_{\pi'(1)},\dots,\xi_{\pi'(k+1)})\;\beta(\xi_0,\xi_{\pi'(k+2)},\dots,\xi_{\pi'(k+l)})\\&\quad=\Big(\iota_{\xi_0}\alpha\wedge\beta+(-1)^{k+1}\alpha\wedge\iota_{\xi_0}\beta\Big)(\xi_1,\dots,\xi_{k+l}).\end{aligned}$$

Where $\pi':=\pi\circ(k+1,k,\dots,1,0)$ is the permutation which maps 0 to 0, $i+1$ to $\pi(i)$ for $i\le k$, and i to $\pi(i)$ for $i>k+1$.

3. Claim. $d\omega \in \Omega^{k+1}(M)$:
Obviously $d\omega$ is $(k+1)$-linear and alternating. The $C^\infty(M,\mathbb{R})$ homogeneity is shown as follows:

$$
\begin{aligned}
d\omega(f\xi_0,\dots,\xi_k) &= (f\xi_0)\cdot\omega(\xi_1,\dots,\xi_k)\\
&\quad+\sum_{i>0}(-1)^i\xi_i\cdot f\omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)\\
&\quad+\sum_{j>i=0}(-1)^j\omega([f\xi_0,\xi_j],\widehat{\xi_0},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&\quad+\sum_{j>i>0}(-1)^{i+j}\omega([\xi_i,\xi_j],f\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&= f\cdot(\xi_0\cdot\omega(\xi_1,\dots,\xi_k))\\
&\quad+\sum_{i>0}(-1)^i\xi_i(f)\,\omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)\\
&\qquad+f\cdot\sum_{i>0}(-1)^i\xi_i\cdot\omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)\\
&\quad+f\cdot\sum_{j>i=0}(-1)^j\omega([\xi_0,\xi_j],\widehat{\xi_0},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&\qquad+\sum_{j>i=0}(-1)^j\xi_j(f)\,\omega(\xi_0,\widehat{\xi_0},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&\quad+f\cdot\sum_{j>i>0}(-1)^{i+j}\omega([\xi_i,\xi_j],\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&= f\cdot d\omega(\xi_0,\dots,\xi_k)\\
&\quad+\sum_{i>0}(-1)^i\xi_i(f)\,\omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)\\
&\quad-\sum_{j>i=0}(-1)^j\xi_j(f)\,\omega(\xi_0,\widehat{\xi_0},\dots,\widehat{\xi_j},\dots,\xi_k)\\
&= f\cdot d\omega(\xi_0,\dots,\xi_k).
\end{aligned}
$$

4. Claim. $d(f\omega) = df \wedge \omega + f \cdot d\omega$:

$$\begin{aligned}
d(f\omega)(\xi_0,\dots,\xi_k) &= \\
&= \sum_i (-1)^i \xi_i \cdot (f\omega)(\xi_0,\dots,\widehat{\xi_i},\dots\xi_k) \\
&\quad + \sum_{j>i} (-1)^{i+j} f\omega([\xi_i,\xi_j],\xi_1,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_k) \\
&= \sum_i (-1)^i \xi_i(f) \cdot \omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k) \\
&\quad + f \sum_i (-1)^i \xi_i \cdot \omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k) \\
&\quad + f \sum_{j>i} (-1)^{i+j} \omega([\xi_i,\xi_j],\xi_1,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_k) \\
&= \sum_i (-1)^i df(\xi_i) \cdot \omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k) + f \cdot d\omega(\xi_0,\dots,\xi_k) \\
&= (df \wedge \omega + f \cdot d\omega)(\xi_0,\dots,\xi_k).
\end{aligned}$$

5. Claim. d is a local operator:
Let $\omega|_U = 0$ and $x \in U$. Then there is an $f \in C^\infty(M,\mathbb{R})$ with $\operatorname{Trg} f \subseteq U$, $f(x) = 1$ and $df(x) = 0$. Consequently, $f\omega = 0$ and thus

$$0 = d(f\omega)(x) \overset{\boxed{4}}{=\!=} df(x) \wedge \omega(x) + f(x) \cdot d\omega(x) = d\omega(x).$$

6. Claim. d and ι_ξ satisfy the initial conditions:
This follows immediately by inserting into the global formulas:

$$\begin{aligned}
\iota_{\xi_0}(f) &:= 0 \\
\iota_{\xi_0}(df) &= df(\xi_0) = \xi_0 \cdot f \\
df(\xi_0) &= \sum_{i=0}^{0} \xi_0 \cdot f(\widehat{\xi_0}) + \sum_{\emptyset} = \xi_0 \cdot f \\
d(df)(\xi_0,\xi_1) &= \xi_0 \cdot df(\xi_1) - \xi_1 \cdot df(\xi_0) + (-1)^{0+1} df([\xi_0,\xi_1]) \\
&= \xi_0 \cdot (\xi_1 \cdot f) - \xi_1 \cdot (\xi_0 \cdot f) - [\xi_0,\xi_1] \cdot f = 0
\end{aligned}$$

7. Claim. $d(du^I) = 0$, where $du^I := du^{i_1} \wedge \dots \wedge du^{i_{k-1}}$ for $I := (i_1,\dots i_{k-1})$:

$$\begin{aligned}
d(du^I)(\tfrac{\partial}{\partial u^{j_1}},\dots,\tfrac{\partial}{\partial u^{j_k}}) &= \sum_{i=1}^{k} (-1)^i \tfrac{\partial}{\partial u^{j_i}} \Big(du^I (\tfrac{\partial}{\partial u^{j_1}},\dots,\widehat{\tfrac{\partial}{\partial u^{j_i}}},\dots \tfrac{\partial}{\partial u^{j_k}}) \Big) \\
&\quad + \sum_{l>i} (-1)^{i+l} du^I \big(\big[\tfrac{\partial}{\partial u^{j_i}},\tfrac{\partial}{\partial u^{j_l}}\big],\dots,\widehat{\tfrac{\partial}{\partial u^{j_i}}},\dots \widehat{\tfrac{\partial}{\partial u^{j_l}}},\dots \big) \\
&= 0, \qquad \text{because } \big[\tfrac{\partial}{\partial u^i},\tfrac{\partial}{\partial u^l}\big] = 0 \text{ and } du^i(\tfrac{\partial}{\partial u^j}) \text{ are constant.}
\end{aligned}$$

$$\Rightarrow \quad d\Big(\sum_I \omega_I du^I\Big) \overset{\boxed{4}}{=\!=} \sum_I d\omega_I \wedge du^I + \omega_I \wedge d(du^I) \overset{\boxed{7}}{=\!=} \sum_{i,I} \tfrac{\partial \omega_I}{\partial u^i} du^i \wedge du^I$$

8. Claim. $d \in \mathrm{Der}_{+1}(\Omega(M))$: Let $\alpha = \sum_I \alpha_I\, du^I$ and $\beta = \sum_J \beta_J\, du^J$ then

$$\begin{aligned}
d(\alpha \wedge \beta) &= \sum_{I,J} d(\alpha_I \beta_J\, du^I \wedge du^J) \\
&= \sum_{I,J,i} \tfrac{\partial(\alpha_I\beta_J)}{\partial u^i}\, du^i \wedge du^I \wedge du^J \\
&= \sum_{I,i} \tfrac{\partial \alpha_I}{\partial u^i}\, du^i \wedge du^I \wedge \sum_J \beta_J\, du^J + (-1)^{|I|} \sum_I \alpha_I\, du^I \wedge \sum_{J,i} \tfrac{\partial \beta_J}{\partial u^i}\, du^i \wedge du^J \\
&= d\alpha \wedge \beta + (-1)^{|I|} \alpha \wedge d\beta.
\end{aligned}$$

9. Claim. The formulas for the commutators of insertion operators ι_ξ and d hold: Because of 25.6 it is sufficient to check the "initial values":

$$\begin{gathered}
[\iota_\xi, \iota_\eta] = 0 \text{ as this is a derivation of degree -2,} \\
[d,d](f) = (d \circ d - (-1)^{1\cdot 1} d \circ d)(f) = 2d(df) = 0 \\
[d,d](df) = 2(d \circ d)(df) = 2d(d(df)) = 2d(0) = 0
\end{gathered}$$

10. Claim. The commutator $[d, \iota_\xi]$ results in the global formula of $\mathcal{L}_\xi$, so $\mathcal{L}_\xi \in \mathrm{Der}_0(\Omega(M))$:

$$\begin{aligned}
([d, \iota_{\xi_0}]\omega)(\xi_1, \dots, \xi_k) &= d(\iota_{\xi_0}\omega)(\xi_1, \dots, \xi_k) - (-1)^{1\cdot(-1)} \iota_{\xi_0}(d\omega)(\xi_1, \dots, \xi_k) \\
&= \sum_i (-1)^{i-1} \xi_i \cdot \iota_{\xi_0}\omega(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k) \\
&\quad + \sum_{0<i<j} (-1)^{i-1+j-1} \iota_{\xi_0}\omega([\xi_i, \xi_j], \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k) \\
&\quad + d\omega(\xi_0, \xi_1, \dots, \xi_k) \\
&= -\sum_{i>0} (-1)^i \xi_i \cdot \omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) \\
&\quad + \sum_{0<i<j} (-1)^{i+j} \omega(\xi_0, [\xi_i, \xi_j], \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k) \\
&\quad + \sum_{i \geq 0} (-1)^i \xi_i \cdot \omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) \\
&\quad\quad + \sum_{0 \leq i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k) \\
&= \xi_0 \cdot \omega(\xi_1, \dots, \xi_k) + \sum_{j>0} (-1)^j \omega([\xi_0, \xi_j], \xi_1, \dots, \widehat{\xi_j}, \dots, \xi_k) \\
&= (\mathcal{L}_{\xi_0}\omega)(\xi_1, \dots, \xi_k).
\end{aligned}$$

11. Claim. The initial condition holds for $\mathcal{L}_\xi$:

$$\begin{gathered}
\mathcal{L}_\xi = d \circ \iota_\xi + \iota_\xi \circ d \quad \Rightarrow \\
\mathcal{L}_\xi(f) = d(\iota_\xi f) + \iota_\xi(df) = 0 + \xi \cdot f \\
\mathcal{L}_\xi(df) = d(\iota_\xi df) + \iota_\xi(d^2 f) = d(\xi \cdot f) + 0.
\end{gathered}$$

12. Claim. The formulas for commutators with $\mathcal{L}_\xi$ hold:
Again, all we need to do is check the initial values (or we use the Jacobi identity):

$$\begin{aligned}
[\mathcal{L}_\xi, \iota_\eta](f) &= 0 = \iota_{[\xi,\eta]}(f), \text{ because the degree is } -1\\
[\mathcal{L}_\xi, \iota_\eta](df) &= \mathcal{L}_\xi(\eta \cdot f) - \iota_\eta(d(\xi \cdot f))\\
&= \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f) = [\xi, \eta](f) = \iota_{[\xi,\eta]}(df)\\
[\mathcal{L}_\xi, \mathcal{L}_\eta](f) &= \xi \cdot \eta \cdot f - \eta \cdot \xi \cdot f = [\xi, \eta](f) = \mathcal{L}_{[\xi,\eta]}(f)\\
[\mathcal{L}_\xi, \mathcal{L}_\eta](df) &= \mathcal{L}_\xi(d(\eta \cdot f)) - \mathcal{L}_\eta(d(\xi \cdot f))\\
&= d(\xi \cdot \eta \cdot f - \eta \cdot \xi \cdot f) = d([\xi, \eta] \cdot f) = \mathcal{L}_{[\xi,\eta]}(df)\\
[\mathcal{L}_\xi, d](f) &= \mathcal{L}_\xi(df) - d(\xi \cdot f) = 0\\
[\mathcal{L}_\xi, d](df) &= \mathcal{L}_\xi(ddf) - d(d(\xi \cdot f)) = 0.
\end{aligned}$$

13. Claim. The relations involving g^* hold:

For d we have the following:

$$\begin{aligned}
(g^* \circ d)(f)(\xi|_p) &= g^*(df)(\xi|_p) \overset{\boxed{25.2}}{=\!=} df|_{g(p)}(Tg \cdot \xi|_p)\\
&= d(f \circ g)(\xi|_p) = d(g^*(f))(\xi|_p) = (d \circ g^*)(f)(\xi|_p)\\
(g^* \circ d)(df) &= g^*(ddf) = g^*(0) = 0 = d^2(g^* f)\\
&= d((d \circ g^*)(f)) = d((g^* \circ d)(f)) = (d \circ g^*)(df).
\end{aligned}$$

For ι we have:

$$\begin{aligned}
(g^* \circ \iota_\eta)(df) &= g^*(df(\eta)) = g^*(\eta(f)) = \eta(f) \circ g\\
&\overset{\boxed{17.3}}{=\!=} \xi(f \circ g) = d(f \circ g)(\xi) = \iota_\xi(g^*(df)) = (\iota_\xi \circ g^*)(df)
\end{aligned}$$

or direct

$$\begin{aligned}
(g^* \circ \iota_\eta)\omega_p(\xi_1, \dots, \xi_k) &= g^*(\iota_\eta \omega)(\xi_1, \dots, \xi_k)\\
&= (\iota_\eta \omega)|_{g(p)}(T_p g \cdot \xi_1|_p, \dots, T_p g \cdot \xi_k|_p)\\
&= \omega|_{g(p)}(\eta|_{g(p)}, T_p g \cdot \xi_1|_p, \dots, T_p g \cdot \xi_k|_p)\\
&= \omega|_{g(p)}(T_p g \cdot \xi|_p, T_p g \cdot \xi_1|_p, \dots, T_p g \cdot \xi_k|_p)\\
&= (g^* \omega)_p(\xi|_p, \xi_1|_p, \dots, \xi_k|_p)\\
&= (\iota_\xi(g^* \omega))(\xi_1, \dots, \xi_k)_p\\
&= (\iota_\xi \circ g^*)\omega_p(\xi_1, \dots, \xi_k).
\end{aligned}$$

For $\mathcal{L}$ this follows by applying the commutation relation:

$$\begin{aligned}
g^* \circ \mathcal{L}_\eta &= g^* \circ (d \circ \iota_\eta + \iota_\eta \circ d)\\
&= g^* \circ d \circ \iota_\eta + g^* \circ \iota_\eta \circ d\\
&= d \circ g^* \circ \iota_\eta + \iota_\xi \circ g^* \circ d\\
&= d \circ \iota_\xi \circ g^* + \iota_\xi \circ d \circ g^*\\
&= (d \circ \iota_\xi + \iota_\xi \circ d) \circ g^*\\
&= \mathcal{L}_\xi \circ g^*.
\end{aligned}$$

14. Claim. The homogeneity formulas for $\iota_{f\xi}$ and $\mathcal{L}_{f\xi}$ hold:

$$\begin{aligned}
\iota_{f\xi}\omega(\xi_1,\dots,\xi_{k-1}) &= \omega(f\xi,\xi_1,\dots,\xi_{k-1}) = f\cdot\omega(\xi,\xi_1,\dots,\xi_{k-1})\\
&= f\cdot\iota_\xi\omega(\xi_1,\dots,\xi_{k-1})\\
\mathcal{L}_{f\xi}\omega &= [d,\iota_{f\xi}]\omega = d(\iota_{f\xi}\omega)+\iota_{f\xi}(d\omega)\\
&= d(f\cdot\iota_\xi\omega)+f\cdot\iota_\xi(d\omega)\\
&= df\wedge\iota_\xi\omega+f\cdot d(\iota_\xi\omega)+f\cdot\iota_\xi(d\omega)\\
&= df\wedge\iota_\xi\omega+f\cdot\mathcal{L}_\xi\omega.
\end{aligned}$$

15. Claim. $\mathcal{L}_\xi$ is the Lie derivative from 17.9:
Both sides define a derivation of degree 0, so it is enough to test on functions and exact 1 forms:

$$\begin{aligned}
\tfrac{d}{dt}|_{t=0}(\mathrm{Fl}^\xi_t)^*f &= \tfrac{d}{dt}|_{t=0}f\circ\mathrm{Fl}^\xi_t = \xi f = \mathcal{L}_\xi f,\\
\tfrac{d}{dt}|_{t=0}(Fl^\xi_t)^*(df_p)(\eta_p) &= \tfrac{d}{dt}|_{t=0}(df)_{\mathrm{Fl}^\xi_t(p)}(T\,\mathrm{Fl}^\xi_t\cdot\eta_p)\\
&= \tfrac{d}{dt}|_{t=0}(T\,\mathrm{Fl}^\xi_t\cdot\eta_p)f = \tfrac{d}{dt}|_{t=0}df(T\,\mathrm{Fl}^\xi_t(\eta_p)) = \tfrac{d}{dt}|_{t=0}\eta_p(f\circ\mathrm{Fl}^\xi_t)\\
&= \eta_p\big(\tfrac{d}{dt}|_{t=0}(f\circ\mathrm{Fl}^\xi_t)\big) = \eta_p(\xi f) = d(\xi\, f)(\eta_p)\\
&= \mathcal{L}_\xi(df)(\eta_p).
\end{aligned}$$

This ends the proof of 25.9. □

25.10 The Frölicher-Nijenhuis and Nijenhuis-Richardson bracket.

We now want to describe general graded derivations in more detail. For this we call a graded derivation $D\in\mathrm{Der}_k(\Omega(M))$ ALGEBRAIC, when it vanishes on the 0-forms $\Omega^0(M)=C^\infty(M,\mathbb{R})$. Obviously, the graded commutator of two algebraic graded derivatiion is itself algebraic, so they form a graded Lie subalgebra. For such derivations D,

$$D(f\,\omega)=D(f)\wedge\omega+(-1)^{0\cdot k}f\cdot D(\omega)=f\cdot D(\omega)$$

holds. Consequently, D is a local operator and even tensorial, i.e. $D(\omega)_x$ depends only on ω_x (Remark: Apply d to local representations of ω). By 25.6, D is uniquely determined by $D|_{\Omega^1(M)}:\Omega^1(M)\to\Omega^{k+1}(M)\subseteq\Omega(M)$, and this is fiberwise an element of

$$\begin{aligned}
L\Big(T^*_xM,\bigwedge^{k+1}T^*_xM\Big)&\cong T_xM\otimes\bigwedge^{k+1}T^*_xM\cong T_xM\otimes(\bigwedge^{k+1}T_xM)^*\cong\\
&\cong L\Big(\bigwedge^{k+1}T_xM,T_xM\Big)\cong L^{k+1}_{\mathrm{alt}}(T_xM;T_xM)
\end{aligned}$$

which smoothly depends on $x\in M$, i.e. is a vector-valued $k+1$ form

$$\begin{aligned}
K\in\Omega^{k+1}(M;TM) &:= C^\infty\Big(M\leftarrow L\Big(\bigwedge^{k+1}TM,TM\Big)\Big)\\
&\cong C^\infty\Big(M\leftarrow L\Big(T^*M,\bigwedge^{k+1}T^*M\Big)\Big).
\end{aligned}$$

Conversely, let $K \in \Omega^{k+1}(M;TM)$ be arbitrary. Then we can define an algebraic derivation $\iota_K \in \operatorname{Der}_k(\Omega(M))$ by the following formula:

$$(\iota_K\omega)(X_1,\dots,X_{k+l}) :=$$
$$= \frac{1}{(k+1)!(l-1)!}\sum_\sigma \operatorname{sgn}(\sigma)\;\omega(K(X_{\sigma(1)},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+l)}),$$

where $\omega \in \Omega^l(M)$ with $l \geq 1$ and $X_1,\dots,X_{k+l} \in \mathfrak{X}(M)$. One shows $\iota_K\omega \in \Omega^{k+l}(M)$ and $\iota_K \in \operatorname{Der}_k(\Omega(M))$ as in 25.9.1 and 25.9.2.

The mapping $\iota : \Omega^{*+1}(M;TM) \to \operatorname{Der}_*(\Omega(M))$ defines a linear isomorphism onto the Lie subalgebra of algebraic derivations, thus makes $\Omega^{*+1}(M,TM) := \coprod_{k\in\mathbb{Z}}\Omega^{k+1}(M,TM)$ itself a graded Lie algebra (whose bracket is also called Nijenhuis-Richardson bracket.

We define a graded derivation $\mathcal{L}_K := [\iota_K, d]$ for $K \in \Omega^k(M;TM)$. The mapping $\mathcal{L} : \Omega(M;TM) \to \operatorname{Der}(\Omega(M))$ is injective because $\mathcal{L}_K f = [\iota_K, d]f = \iota_K(df) \pm d(\iota_K f) = df \circ K$ for all $f \in C^\infty(M,\mathbb{R})$.

$$\begin{tikzcd}
\{D : D|_{C^\infty} = 0\} \arrow[r, hook] & \operatorname{Der}_k(\Omega(M)) & \{D : [D,d] = 0\} \arrow[l, hook'] \\
\Omega^{k+1}(M;TM) \arrow[u, "\iota"', two heads, tail] & & \Omega^k(M;TM) \arrow[u, "\mathcal{L}", two heads, tail]
\end{tikzcd}$$

Proposition.

Each $D \in \operatorname{Der}_k(\Omega(M))$ has a unique representation $D = \iota_L + \mathcal{L}_K$ with $L \in \Omega^{k+1}(M;TM)$ and $K \in \Omega^k(M;TM)$. The image of $\mathcal{L}$ is the Lie subalgebra of all D with $[D,d] = 0$. The mapping $\mathcal{L}$ thus induces a graded Lie algebra structure (the Frölicher-Nijenhuis bracket*) on $\Omega^*(M;TM)$.*

Proof. For fixed vector fields $X_i \in \mathfrak{X}(M)$, the assignment $f \mapsto D(f)(X_1,\dots,X_k)$ describes a derivation $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$. Consequently, there is a vector field $K(X_1,\dots,X_k) \in \mathfrak{X}(M)$ with $D(f)(X_1,\dots,X_k) = K(X_1,\dots,X_k)(f) = df(K(X_1,\dots,X_k))$. Obviously, $K \in \Omega^k(M;TM)$ and the defining equation for K is $D(f) = df \circ K = \mathcal{L}_K(f)$ for $f \in C^\infty(M,\mathbb{R})$. So $D - \mathcal{L}_K$ is algebraic, i.e. $D = \mathcal{L}_K + \iota_L$ for an $L \in \Omega^{k+1}(M;TM)$. We have

$$0 = [\iota_K, 0] = [\iota_K, [d,d]] = [[\iota_K, d], d] + (-1)^{k-1}[d, [\iota_K, d]] = 2[\mathcal{L}_K, d]$$

Thus, $0 = [D,d]$ for $D := \mathcal{L}_K + \iota_L$ if and only if $0 = [\iota_L, d] = \mathcal{L}_L$, i.e. $L = 0$. The uniqueness of K and L in the decomposition results from the injectivity of i and $\mathcal{L}$ and because 0 is the only algebraic derivation commuting with d. $\square$

We have $df \circ [X,Y] = \mathcal{L}_{[X,Y]}f = [\mathcal{L}_X, \mathcal{L}_Y]f$ for the Frölicher Nijenhuis bracket $[X,Y]$ for $X,Y \in \Omega^*(M,TM)$.

25.11 Differential forms on $\mathbb{R}^3$.

For open $M \subseteq \mathbb{R}^m$ we know that $\mathfrak{X}(M) \cong C^\infty(M,\mathbb{R}^m)$ by virtue of the mapping $\xi = \sum f_i \frac{\partial}{\partial x^i} \hookleftarrow (f_i)_{i=1}^m = f$, where x^i are the standard coordinates. Likewise, $\Omega^m(M) \cong C^\infty(M,\mathbb{R})$ by virtue of $* : f \cdot \operatorname{vol}_M \hookleftarrow f$, with $\operatorname{vol}_M = dx^1 \wedge \cdots \wedge dx^m$. In summary, we have the following isomorphisms in case $m = 3$:

1. $\Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3,\mathbb{R})$
2. $\Omega^1(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ via the basis dx^1, dx^2, dx^3
3. $\Omega^2(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ via the basis $dx^2 \wedge dx^3$, $dx^3 \wedge dx^1$, $dx^1 \wedge dx^2$

4. $\Omega^3(\mathbb{R}^3) \cong C^\infty(\mathbb{R}^3,\mathbb{R})$ via the basis $dx^1 \wedge dx^2 \wedge dx^3$

How does d look like with respect to these bases?

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3,\mathbb{R}) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & C^\infty(\mathbb{R}^3,\mathbb{R}^3) \arrow[r, "\mathrm{rot}"] \arrow[d, "\cong"'] & C^\infty(\mathbb{R}^3,\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\cong"'] & C^\infty(\mathbb{R}^3,\mathbb{R}) \arrow[d, "\cong"'] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[d, mapsto] & (f_1,f_2,f_3) \arrow[d, mapsto] & (f_1,f_2,f_3) \arrow[d, mapsto] & f \arrow[d, mapsto] \\
f & \sum_i f_i dx^i & \begin{array}{c} f_1\,dx^2\wedge dx^3 \\ +f_2\,dx^3\wedge dx^1 \\ +f_3\,dx^1\wedge dx^2 \end{array} & f\,dx^1\wedge dx^2\wedge dx^3
\end{tikzcd}$$

The operator d is given by the following formulas:

$$d : \Omega^0(\mathbb{R}^3) \ni f \mapsto df = \sum \tfrac{\partial f}{\partial x^i} dx^i$$

$$\begin{aligned} d : \Omega^1(\mathbb{R}^3) \ni \sum f_i dx^i \mapsto \sum_i df_i \wedge dx^i &= \sum_{i,j} \tfrac{\partial f_i}{\partial x^j}\, dx^j \wedge dx^i \\ &= (\tfrac{\partial f_3}{\partial x^2} - \tfrac{\partial f_2}{\partial x^3})\, dx^2 \wedge dx^3 + (\tfrac{\partial f_1}{\partial x^3} - \tfrac{\partial f_3}{\partial x^1})\, dx^3 \wedge dx^1 \\ &\quad + (\tfrac{\partial f_2}{\partial x^1} - \tfrac{\partial f_1}{\partial x^2})\, dx^1 \wedge dx^2 \end{aligned}$$

$$\begin{aligned} d : \Omega^2(\mathbb{R}^3) \ni & f_1\, dx^2 \wedge dx^3 + f_2\, dx^3 \wedge dx^1 + f_3\, dx^1 \wedge dx^2 \mapsto \\ &\mapsto \left(\sum \tfrac{\partial f_i}{\partial x^i}\right)\, dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

It coincides up to the vertical isomorphisms from above with:

$$\begin{aligned} \mathrm{grad}\, f &:= \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right) \\ \mathrm{rot}(f_1,f_2,f_3) &:= \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3}, \frac{\partial f_1}{\partial x^3} - \frac{\partial f_3}{\partial x^1}, \frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}\right) \\ \mathrm{div}(f_1,f_2,f_3) &:= \frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3}. \end{aligned}$$

From $d^2 = 0$ follow the well-known results from vector analysis:

$$(\mathrm{rot} \circ \mathrm{grad}) f = 0 \quad \text{and} \quad (\mathrm{div} \circ \mathrm{rot})(f_1,f_2,f_3) = 0.$$

And the Poincaré Lemma 26.5.6 implies:

$\mathrm{rot}(f_1,f_2,f_3) = 0 \Rightarrow \exists\, g$, s.t. $\mathrm{grad}\, g = (f_1,f_2,f_3)$ holds locally.

$\mathrm{div}(f_1,f_2,f_3) = 0 \Rightarrow \exists\, (g_1,g_2,g_3)$, s.t. $\mathrm{rot}(g_1,g_2,g_3) = (f_1,f_2,f_3)$ holds locally.

26. Cohomology

We now try to describe image of $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

26.1 Definition of Cohomology.

Let $d : \Omega(M) \to \Omega(M)$ be the outer derivative.

1. $Z^k(M) := \{\omega \in \Omega^k(M) : d\omega = 0\}$, the SPACE OF THE CLOSED DIFFERENTIAL FORMS (or COCYCLES).

2. $B^k(M) := \{d\omega : \omega \in \Omega^{k-1}(M)\}$, the SPACE OF THE EXACT DIFFERENTIAL FORMS (or COBOUNDARIES).
3. $H^k(M) := Z^k(M)/B^k(M)$, the k-TH DE-RHAM COHOMOLOGY of M. This is well-defined, since $B^k(M) \subseteq Z^k(M)$ holds because of $d \circ d = 0$.
4. $H(M) := \bigoplus_k H^k(M)$, the DE-RHAM COHOMOLOGY of M.
5. $b_k(M) := \dim(H^k(M)) \in \mathbb{N} \cup \{+\infty\}$, the k-TH BETTI NUMBER.
6. $f_M(t) := \sum_k b_k t^k$, the POINCARÉ POLYNOMIAL. This is well-defined if all Betti numbers are finite.
7. $\chi(M) := f_M(-1) = \sum_k (-1)^k b_k$, the EULER CHARACTERISTIC OF A MANIFOLD of M.

26.2 Definition (Cohomology functor).

If $g : M \to N$ smooth, then $g^*(d\omega) = d(g^*\omega)$ for $g^* : \Omega(N) \to \Omega(M)$ holds by 25.9. Thus, the restrictions $g^* : Z^k(N) \to Z^k(M)$ and $g^* : B^k(N) \to B^k(M)$ exist and the following definition of a linear mapping

$$g^* : H(N) \to H(M), \quad [\omega] \mapsto [g^*\omega].$$

makes sense

$$\begin{tikzcd} M \arrow[d, "g"] & B^k(M) \arrow[r, hook] & Z^k(M) \arrow[r, two heads] & H^k(M) \\ N & B^k(N) \arrow[r, hook] \arrow[u, "g^*"] & Z^k(N) \arrow[r, two heads] \arrow[u, "g^*"] & H^k(N) \arrow[u, "g^*"] \end{tikzcd}$$

26.3 Theorem (Cohomology axioms).

The cohomology has the following properties:

1. $H^0(\{*\}) = \mathbb{R}$, $H^k(\{*\}) = 0$ *for* $k \neq 0$ *(*DIMENSION AXIOM*).*
2. $f, g : M \to N$ *smooth,* $f \sim g \Rightarrow f^* = g^*$ *(*HOMOTOPY AXIOM*).*
3. *Let be* $M = \bigsqcup_\alpha M_\alpha \Rightarrow H^k(M) = \prod_\alpha H^k(M_\alpha)$ *(*DISJOINT UNION AXIOM*).*
4. *If* $M = U \cup V$ *is open with* $U, V \subseteq M$*, then there are linear maps* δ_k *that make the following long sequence exact:*

$$\ldots \to H^k(M) \xrightarrow{(i_U^*, i_V^*)} H^k(U) \oplus H^k(V) \xrightarrow{j_U^* - j_V^*} H^k(U \cap V) \xrightarrow{\delta_k}$$
$$\xrightarrow{\delta_k} H^{k+1}(M) \longrightarrow H^{k+1}(U) \oplus H^{k+1}(V) \longrightarrow H^{k+1}(U \cap V) \to \ldots$$

with the inclusions $i_U : U \hookrightarrow U \cup V$*,* $i_V : V \hookrightarrow U \cup V$*,* $j_U : U \cap V \hookrightarrow U$ *and* $j_V : U \cap V \hookrightarrow V$*. This sequence is called* MAYER-VIETORIS SEQUENCE *and* δ_k *is called* CONNECTING HOMOMORPHISM*.*

A sequence of linear mappings $\cdots \xrightarrow{f_k} E_k \xrightarrow{f_{k+1}} \cdots$ is called EXACT if $\operatorname{im} f_k = \ker f_{k+1}$ for all k.

Proof.

(1) is obvious since $\Omega^k(\{*\}) = \{0\}$ for $k \neq 0$ and $\Omega^0(\{*\}) = C^\infty(\{*\}, \mathbb{R}) = \mathbb{R}$.

(2) Let $H \in C^\infty(M \times \mathbb{R}, N)$ be a (smooth) homotopy from f to g, i.e. $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in M$. For $\omega \in \Omega^k(N)$ we have $H^*\omega \in \Omega^k(M \times \mathbb{R})$.

Let $j_t : M \to M \times \mathbb{R}$ be defined by $j_t(x) := (x,t)$. Then $H \circ j_0 = f$ and $H \circ j_1 = g$ and thus

$$g^* - f^* = (H \circ j_1)^* - (H \circ j_0)^* = (j_1^* - j_0^*) \circ H^*.$$

For $\varphi \in \Omega^k(M \times \mathbb{R})$ the mapping $t \mapsto j_t^*\varphi \in \Omega^k(M)$ is a smooth curve into $\Omega^k(M)$ and thus

$$(j_1^* - j_0^*)\varphi = \int_0^1 \frac{d}{dt} j_t^* \varphi\, dt = \int_0^1 j_t^*(\mathcal{L}_\xi \varphi) dt,$$

where $\xi := \frac{\partial}{\partial t} \in \mathfrak{X}(M \times \mathbb{R})$ denotes the unit vector field in direction $\{0\} \times \mathbb{R}$ because

$$\begin{aligned} & j_{t+s} = \mathrm{Fl}_s^\xi \circ j_t \text{ for } t,s \in \mathbb{R} \ \Rightarrow \\ \Rightarrow\ & \frac{d}{dt} j_t^* \varphi = \frac{d}{ds}\Big|_{s=0} (j_{t+s})^* \varphi = \frac{d}{ds}\Big|_{s=0} (\mathrm{Fl}_s^\xi \circ j_t)^* \varphi \\ & = \frac{d}{ds}\Big|_{s=0} \Big(j_t^* \circ (\mathrm{Fl}_s^\xi)^* \Big) \varphi = j_t^* \Big(\frac{d}{ds}\Big|_{s=0} (\mathrm{Fl}_s^\xi)^* \varphi \Big) = j_t^*(\mathcal{L}_\xi \varphi). \end{aligned}$$

Thus we define a fiber integration I_0^1 by

$$I_0^1 : \Omega^k(M \times \mathbb{R}) \to \Omega^k(M),\ I_0^1(\varphi) := \int_0^1 j_t^* \varphi\, dt.$$

Then

$$\begin{aligned} (d \circ I_0^1)(\varphi) &= d\Big(\int_0^1 j_t^* \varphi dt \Big) = \int_0^1 d(j_t^* \varphi)\, dt = \int_0^1 j_t^*(d\varphi) dt \\ &= I_0^1(d\varphi) = (I_0^1 \circ d)(\varphi); \\ (j_1^* - j_0^*)\varphi &= \int_0^1 j_t^*(\mathcal{L}_\xi \varphi)\, dt = I_0^1(\mathcal{L}_\xi \varphi) = I_0^1\big((d \circ \iota_\xi + \iota_\xi \circ d)\varphi \big). \end{aligned}$$

We now define the homotopy operator $G : \Omega^k(N) \to \Omega^{k-1}(M)$ by $G := I_0^1 \circ \iota_\xi \circ H^*$, i.e.

$$\begin{tikzcd} \Omega^k(N) \arrow[r, dotted, "G"] \arrow[d, "H^*"'] & \Omega^{k-1}(M) \\ \Omega^k(M \times \mathbb{R}) \arrow[r, "\iota_\xi"] & \Omega^{k-1}(M \times \mathbb{R}) \arrow[u, "I_0^1"'] \end{tikzcd}$$

Then

$$\begin{aligned} g^* - f^* &= (j_1^* - j_0^*) \circ H^* \\ &= I_0^1 \circ (d \circ \iota_\xi + \iota_\xi \circ d) \circ H^* \\ &= (d \circ I_0^1 \circ \iota_\xi + I_0^1 \circ \iota_\xi \circ d) \circ H^* \\ &= d \circ (I_0^1 \circ \iota_\xi \circ H^*) + (I_0^1 \circ \iota_\xi \circ H^*) \circ d = d \circ G + G \circ d. \end{aligned}$$

and thus $g^*\omega - f^*\omega = d(G\omega) + G(d\omega) = d(G\omega)$ is exact if $d\omega = 0$. So $g^* - f^* = 0 : H(N) \to H(M)$.

(3) is obvious since $\Omega^k(\bigsqcup_\alpha M_\alpha) \cong \prod_\alpha \Omega^k(M_\alpha)$ and d respects this decomposition.

(4) We show first that

$$0 \to \Omega^k(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_U^* - j_V^*} \Omega^k(U \cap V) \to 0$$

is exact.
The mapping $f := (i_U^*, i_V^*)$ is clearly injective with $\mathrm{im}(i_U^*, i_V^*) = \ker(j_U^* - j_V^*)$.
The mapping $g := (j_U^* - j_V^*)$ is surjective: Let $\{h_U, h_V\}$ be a partition of unity subordinated to the covering $\{U, V\}$ and let $\varphi \in \Omega(U \cap V)$. With $\varphi_U := h_V \varphi \in \Omega(U)$

(and $\varphi_U|_{U\setminus \operatorname{Trg}(h_V)} := 0$), respectively $\varphi_V := -h_U\varphi \in \Omega(V)$ (and $\varphi_V|_{V\setminus \operatorname{Trg}(h_U)} := 0$), we have:

$$(j_U^* - j_V^*)(\varphi_U, \varphi_V) = \varphi_U|_{U\cap V} - \varphi_V|_{U\cap V} = (h_V + h_U)\varphi = \varphi.$$

So the sequence is exact.

To get the long exact sequence in cohomology we can use the following general result. □

26.4 Theorem.

Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence of CHAIN MAPPINGS*, that is C, C', and C'' are* CHAIN COMPLEXES *(i.e. $\mathbb{Z}$-graded vector spaces with so-called* BOUNDARY OPERATORS*, that is linear operators ∂ of degree +1 which satisfy $\partial^2 = 0$) and connecting linear maps f and g of degree 0 which intertwine the boundary operators.*
Then we obtain a long exact sequence in homology:

$$\dots \xrightarrow{\partial_*} H_q(C') \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\partial_*} H_{q+1}(C') \xrightarrow{H_{q+1}(f)} \dots$$

where the q-th HOMOLOGY *$H_q(C)$ of the chain complex C is defined as above by*

$$H_q(C) := \ker(\partial_q : C_q \to C_{q+1}) / \operatorname{im}(\partial_{q-1} : C_{q-1} \to C_q).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C'_q \arrow[r, "f"] \arrow[d, "\partial"'] & C_q \arrow[r, "g"] \arrow[d, "\partial"'] & C''_q \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C'_{q+1} \arrow[r, "f"] & C_{q+1} \arrow[r, "g"] & C''_{q+1} \arrow[r] & 0
\end{tikzcd}$$

Let $\partial_*[z''] := [(f^{-1} \circ \partial \circ g^{-1})(z'')]$ for $z'' \in C''$ with $\partial z'' = 0$.

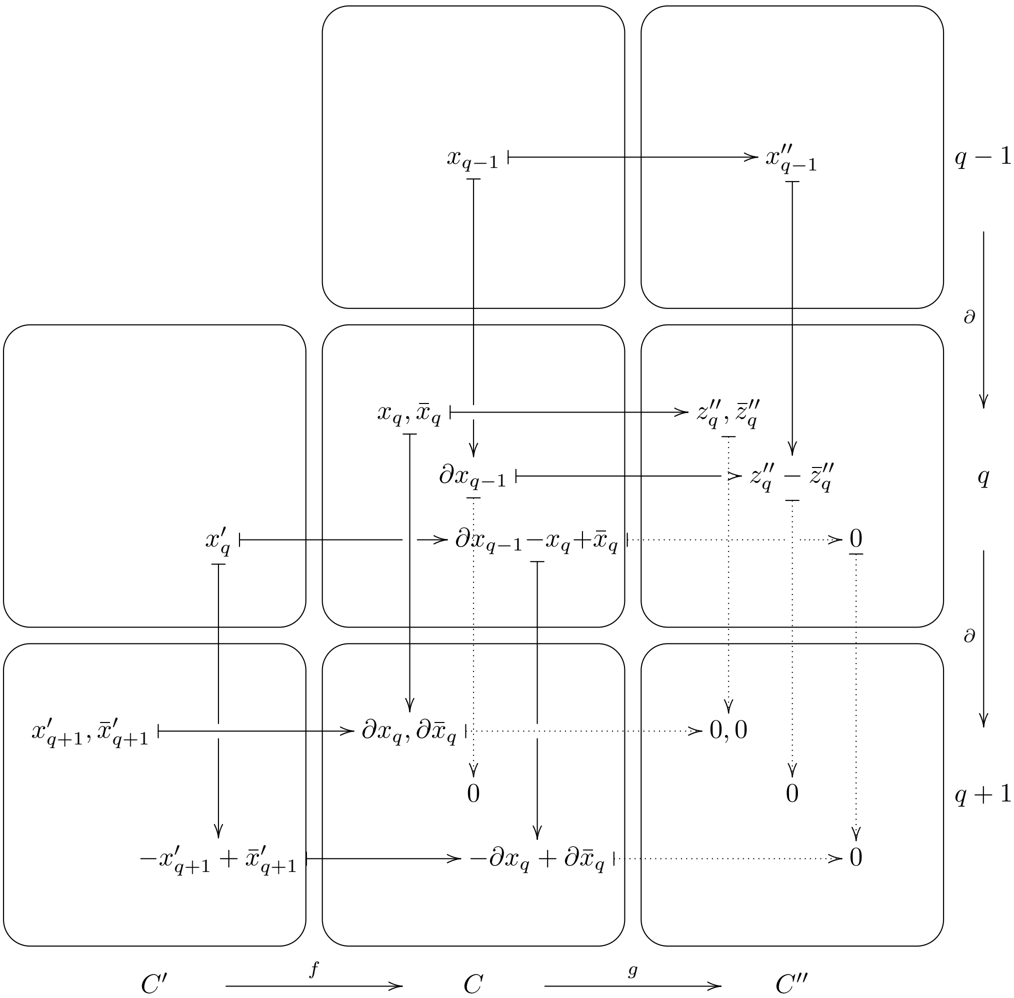
We show first that it is possible to choose elements in the corresponding inverse images, and then we show that the resulting class does not depend on any of the choices.

So let $z''_q \in C''_q$ be a cycle, i.e. $\partial z''_q = 0$. Since g is onto we find $x_q \in C_q$ with $g x_q = z''_q$. Since $g\partial x_q = \partial g x_q = \partial z''_q = 0$, we find $x'_{q+1} \in C'_{q+1}$ with $f x'_{q+1} = \partial x_q$. And hence $x'_{q+1} \in f^{-1}\partial g^{-1} z''_q$.

Furthermore $f\partial x'_{q+1} = \partial f x'_{q+1} = \partial\partial x_q = 0$. Since f is injective we get $\partial x'_{q+1} = 0$ and hence we may form the class $[x'_{q+1}] =: \partial_*[z''_q]$.

$$\begin{tikzcd}
 & x_q \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"'] & z''_q \arrow[d, mapsto, "\partial"'] \\
x'_{q+1} \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\partial"'] & \partial x_q \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\partial"'] & 0 \\
\partial x'_{q+1} \arrow[r, mapsto, "f"] & 0 &
\end{tikzcd}$$

Now the independence of all choices:



Let $[z''_q] = [\bar{z}''_q]$, i.e. $\exists\ x''_{q-1} : \partial x''_{q-1} = z''_q - \bar{z}''_q$. Select $x_q, \bar{x}_q \in C_q$ as before, such that $gx_q = x''_q$ and $g\bar{x}_q = \bar{x}''_q$. As before, we choose $x'_{q+1}, \bar{x}'_{q+1} \in C'_{q+1}$ with $fx'_{q+1} = \partial x_q$ and $f\bar{x}'_{q+1} = \partial \bar{x}_q$. We have to show that $[x'_{q+1}] = [\bar{x}'_{q+1}]$. For this we choose $x_{q-1} \in C_{q-1}$ with $gx_{q-1} = x''_{q-1}$. Then $g\partial x_{q-1} = \partial g x_{q-1} = \partial x''_{q-1} = z''_q - \bar{z}''_q = g(x_q - \bar{x}_q)$ and therefore an $x'_q \in C_q$ exists with $fx'_q = x_q - \bar{x}_q - \partial x_{q-1}$. Further, $f\partial x'_q = \partial f x'_q = \partial(x_q - \bar{x}_q - \partial x_{q-1}) = \partial x_q - \partial \bar{x}_q - 0 = f(x'_{q+1} - \bar{x}'_{q+1})$. Since f is injective, we have $x'_{q+1} - \bar{x}'_{q+1} = \partial x'_q$, i.e. $[x'_{q+1}] = [\bar{x}'_{q+1}]$.

Exactness at $H_q(C')$:
($\subseteq$) $f_*\partial_*[z''] = [ff^{-1}\partial g^{-1} z''] = [\partial g^{-1} z''] = 0$.
($\supseteq$) Let $\partial z' = 0$ and $0 = f_*[z'] = [fz']$, i.e. $\exists\, x$: $\partial x = fz'$. Then $x'' := gx$ satisfies $\partial x'' = \partial g x = g\partial x = gfz' = 0$ and $\partial_*[x''] = [f^{-1}\partial g^{-1} g x] = [f^{-1}\partial x] = [z']$.

Exactness at $H_q(C)$:
($\subseteq$) is obvious, since $g \circ f = 0$.
($\supseteq$) Let $\partial z = 0$ with $0 = g_*[z] = [gz]$, i.e. $\exists\, x''$: $\partial x'' = gz$. Then $\exists\, x$: $gx = x''$. Therefore, $gz = \partial x'' = \partial g x = g\partial x \Rightarrow \exists\, x'$: $fx' = z - \partial x \Rightarrow f\partial x' = \partial f x' = \partial(z - \partial x) = 0 \Rightarrow \partial x' = 0$ and $f_*[x'] = [fx'] = [z - \partial x] = [z]$.

Exactness at $H_q(C'')$:
($\subseteq$) We have $\partial_* g_*[z] = [f^{-1}\partial g^{-1} g z] = [f^{-1}\partial z] = [f^{-1}0] = 0$.
($\supseteq$) Let $\partial z'' = 0$ and $0 = \partial_*[z'']$, i.e. $\exists\, x'$: $\partial x' = z'$, where $z' \in f^{-1}\partial g^{-1} z''$, i.e. $\exists\, x$:

$gx = z''$ and $fz' = \partial x$. Then $\partial(x - fx') = fz' - f(\partial x') = 0$ and $g(x - fx') = z'' - 0$, i.e. $g_*[x - fx'] = [z'']$. □

26.5 Remarks.

1. The De-Rham cohomology is uniquely determined by the properties of Proposition 26.3, see [**146**, Kap.5].
2. For the 0-th cohomology we have:
$$\begin{aligned} H^0(M) &= \{f \in C^\infty(M,\mathbb{R}) : df = 0\} \\ &= \{f \in C^\infty(M,\mathbb{R}) : f \text{ is locally constant}\} = \mathbb{R}^\mu, \end{aligned}$$
where μ is the number of connected components of M.
3. If $k < 0$ or $k > \dim M$ then $\Omega^k(M) = 0$ and thus $H^k(M) = 0$.
4. Let M and N be HOMOTOPY EQUIVALENT, i.e. there are smooth mappings $f : M \to N$ and $g : N \to M$ with $f \circ g \sim \mathrm{id}_N$ and $g \circ f \sim \mathrm{id}_M$. The mappings $H(f) := f^* : H(N) \to H(M)$ and $H(g) := g^* : H(M) \to H(N)$ are then inverse isomorphisms. For example, the open Möbius strip is homotopy equivalent to the cylinder.
5. In particular, if $A \subseteq M$ is a DEFORMATION RETRACT, i.e. a homotopy $h : M \times \mathbb{R} \to M$ exists with $h(\_, 1) = \mathrm{id}_M$, $h(M \times \{0\}) \subseteq A$, and $h(\_, 0)|_A = \mathrm{id}_A$, then $H(M) \cong H(A)$ holds. For example, the base space of any vector bundle is a deformation retract of the total space (into which it is embedded as zero section).
6. If M is CONTRACTIBLE, i.e. a point $p \in M$ exists which is a deformation retract of M, then $H(M) \cong H(\{p\})$, i.e. any closed k-form ($k \neq 0$) is exact (this is the promised POINCARÉ LEMMA). If $M = \mathbb{R}^m$ - or more general $M \subseteq \mathbb{R}^m$ is star-shaped (with respect to 0) - then $h : M \times \mathbb{R} \to M$, $(x,t) \mapsto t \cdot x$, is a contraction from M to a point, so M is contractible. Therefore, locally every manifold is contractible.
7. If M is simply connected, then $H^1(M) = 0$. To show this, proceed as follows: Let $\omega \in \Omega^1(M)$ with $d\omega = 0$. We want to find an $f \in \Omega^0(M) = C^\infty(M,\mathbb{R})$ with $df = \omega$. For this we choose a point $x_0 \in M$ and for every other point $x \in M$ choose a curve c which connects x_0 with x and define
$$f(x) := \int_c \omega = \int_0^1 c^*(\omega).$$
This definition does not depend on the choice of the curve, because the composition with a second reversed curve provides a closed curve c which has to be homotopic to the constant curve konst_{x_0}, so $[c^*(\omega)] = [(\mathrm{konst}_{x_0})^*(\omega)]$. The two forms on S^1 thus differ only by an exact form dg and thus
$$\int_c \omega = \int_0^1 c^*(\omega) = \int_0^1 (\mathrm{konst}_{x_0})^*(\omega) = \int_0^1 0 = 0.$$
Since locally a smooth f with $df = \omega$ always exists by 20.4 and $\int_c df = f(c(1)) - f(c(0))$ holds, the above-defined f differs from it only by an additive constant, hence is smooth as well and is the antiderivative we are looking for

because

$$df|_{c(0)}(c'(0)) = (f\circ c)'(0) = \frac{d}{dt}\Big|_{t=0} f(c(t)) = \frac{d}{dt}\Big|_{t=0}\Big(f(c(0)) + \int_0^1 c(t\_)^*(\omega)\Big)$$
$$= \frac{d}{dt}\Big|_{t=0}\int_0^1 \omega_{c(t\,s)}(t\,c'(t\,s))\,ds = \int_0^1 \frac{\partial}{\partial t}\Big|_{t=0}\omega_{c(t\,s)}(t\,c'(t\,s))\,ds$$
$$= \int_0^1 \omega_{c(0)}(c'(0))\,ds + 0 = \omega_{c(0)}(c'(0))$$

8. Let $\ldots \to E_i \xrightarrow{T_i} E_{i+1} \to \ldots$ be an exact sequence of finite-dimensional vector spaces with $E_i = \{0\}$ for almost all i, then adding
$$\dim E_i = \dim(\ker T_i) + \dim(\operatorname{im}(T_i)) = \dim(\ker T_i) + \dim(\ker T_{i+1})$$
yields the identity
$$\sum_i (-1)^i \dim E_i = 0$$
The Mayer-Vietoris sequence implies the following for the Euler characteristic:
$$\chi(U\cup V) + \chi(U\cap V) = \chi(U) + \chi(V)$$

9. Since $\mathbb{R}^m$ is contractible, its Euler characteristic is that of a point, thus is 1 by the dimension axiom 26.3.1. Furthermore, $\chi(S^0) = 2$ because of 26.3.3. By 8, for each point $* \in M$ in an m-dimensional manifold M we have
$$\begin{aligned}\chi(M) &= \chi(M\setminus\{*\}) + \chi(\mathbb{R}^m) - \chi(\mathbb{R}^m\setminus\{*\})\\ &= \chi(M\setminus\{*\}) + \chi(\{*\}) - \chi(S^{m-1})\\ &= \chi(M\setminus\{*\}) + 1 - \chi(S^{m-1}).\end{aligned}$$
For the spheres we thus get
$$\chi(S^m) = \chi(\mathbb{R}^m) + 1 - \chi(S^{m-1}) = 2 - \chi(S^{m-1})$$
and in particular $\chi(S^1) = 2 - \chi(S^0) = 0$ and hence $\chi(M) = \chi(M\setminus\{*\}) + 1$ for $\dim(M) = 2$.
Compact 2-dimensional connected manifolds M_g of genus g are obtained by glueing g handles (by 1.2) or g Möbius strips (by 1.4) to S^2, so recursivly we have
in the non-orientable case:
$$\begin{aligned}\chi(M_g) &= \chi((M_{g-1}\setminus\{*\})\cup \text{ Möb}) = \chi(M_{g-1}\setminus\{*\}) + \chi(\text{ Möb}) - \chi(S^1)\\ &= \chi(M_{g-1}) - 1 = 2-(g-1)-1 = 2-g,\end{aligned}$$
and in the orientable case:
$$\begin{aligned}\chi(M_g) &= \chi((M_{g-1}\setminus\{*_-,*_+\})\cup S^1\times\mathbb{R})\\ &= \chi(M_{g-1}) - 2 + \chi(S^1\times\mathbb{R}) - \chi(S^1\sqcup S^1)\\ &= \chi(M_{g-1}) - 2 = 2 - 2(g-1) - 2 = 2-2g\end{aligned}$$

10. The Euler characteristic χ of a manifold can also be calculated by triangulating it, that is, decomposing it into simplexes. If γ_i is the number of simplexes of dimension i, then: $\chi = \sum_i (-1)^i\gamma_i$. In particular, for each polyhedron, the number of vertices minus the number of edges plus the number of faces is equal to $2 = \chi(S^2)$ (see algebraic topology).

11. Another way to calculate the Euler characteristic is through Morse functions, that is, functions $f : M \to \mathbb{R}$ whose critical points are not degenerated, i.e. the Hessian matrix is definite. If $\beta_k(f)$ denotes the number of critical points

in which the Hessian matrix has exactly k negative eigenvalues, then the Morse (in)equalities hold, see [65, S.161–162]

$$\beta_k(M) \le \beta_k(f)$$
$$\sum_k (-1)^k \beta_k(f) = \chi(M).$$

12. More generally, for a vector field ξ having only isolated zeros (e.g., the gradient field of a Morse function), one can define an index $\operatorname{ind}_x(\xi)$ at those points, see 29.27 or [65, S.133]. And then $\chi(M) = \sum_{\xi(x)=0} \operatorname{ind}_x(\xi)$ holds by a result of Hopf, see 29.29 or [65, S.164].

    If there is a nowhere vanishing vector field, the Euler characteristic $\chi(M)$ must be 0. This proves the Hairy Ball Theorem 29.11.

    Conversely, one can show that on every compact, oriented, continuous manifold with $\chi(M) = 0$ there exists a vector field without zeros, see [65, S.137].

13. *The cohomology of spheres S^n for $n \ge 1$ is:*

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } 0 < k < n \\ \mathbb{R} & \text{for } k = n \\ 0 & \text{for } n < k \end{cases}$$

    So the Poincaré polynomial has the form: $f_{S^n}(t) = 1 + t^n$ and - as we have already seen in 9 -, and the Euler characteristic is $\chi(S^{2n-1}) = 0$ and $\chi(S^{2n}) =$ 2.

    (i) $H^0(S^n) = \mathbb{R}$ follows from 2.

    (ii) $H^k(S^n) = 0$ for $k > n$ always holds, cf. 3.

    (iii) Remains to show: $H^k(S^n) \cong H^{k+1}(S^{n+1})$ for $0 < k$ and $H^1(S^n) = 0$ for $n > 1$ and $H^1(S^1) = \mathbb{R}$.
    We have $S^{n+1} = U \cup V$ with $U := \{x \in S^{n+1} : -1 < \langle x, a\rangle\}$ and $V := \{x \in S^{n+1} : +1 > \langle x, a\rangle\}$ for fixed $a \in S^{n+1}$. So $U \cap V \cong S^n \times \mathbb{R}$ and thus $H(U \cap V) \cong H(S^n)$ by 5. The Mayer-Vietoris sequence (for $k > 0$) is

$$\begin{array}{c}
\vdots \\
\downarrow \\
H^k(U) \oplus H^k(V) \cong H^k(\{*\}) \oplus H^k(\{*\}) \\
\downarrow \\
H^k(U \cap V) \cong H^k(S^n) \\
\downarrow \delta \\
H^{k+1}(U \cup V) = H^{k+1}(S^{n+1}) \\
\downarrow \\
H^{k+1}(U) \oplus H^{k+1}(V) \cong H^{k+1}(\{*\}) \oplus H^{k+1}(\{*\}) \\
\downarrow \\
\vdots
\end{array}$$

So $H^k(S^n) \xrightarrow[\cong]{\delta} H^{k+1}(S^{n+1})$ for all $k > 0$. The beginning of the sequence looks like this:

$$
\begin{array}{c}
0 \\
\downarrow \\
H^0(S^{n+1}) \cong \mathbb{R} \\
\downarrow \\
H^0(\{*\}) \oplus H^0(\{*\}) \cong \mathbb{R}^2 \\
\downarrow \\
H^0(S^n) \cong \begin{cases} \mathbb{R}^2 & \text{for } n = 0 \\ \mathbb{R} & \text{for } n > 0 \end{cases} \\
\downarrow \\
H^1(S^{n+1}) \\
\downarrow \\
0
\end{array}
$$

Thus, $H^1(S^1) = \mathbb{R}$ and $H^1(S^{n+1}) = 0$ for $n > 0$ because of $\boxed{8}$.

VII. Integration

In this chapter we develop the integration on oriented manifolds. The integrable objects are the differential forms of maximal degree. We prove the Theorem of Stokes and introduce manifolds with boundary. In the case of an oriented Riemannian manifold, we have a distinguished differential form of maximal degree, the volume form. Finally, we use the integration to further analyze the cohomology.

27. Orientability

In order to be able to perform integrations on manifolds we need an orientation concept, as, for example, the formula $\int_a^b f = -\int_b^a f$ for the ordinary Riemann integral over an interval $[a, b]$ shows.

27.1 Definition (Orientability).

A manifold M is called ORIENTABLE :⇔ there exists a compatible atlas $\mathcal{A}$, such that all the chart changes are orientation preserving, that is, the determinant of their Jakobi matrix in each point is positive (cf. [**86**, 34.3]).

A vector bundle $E \to M$ is called ORIENTABLE :⇔ a vector bundle atlas $\mathcal{A}$ exists whose transition functions have values in $GL_+(\mathbb{R}^n) := \{T \in GL(n) : \det(T) > 0\}$.

27.2 Proposition (Orientable vector bundles).

Let $E \to M$ be a vector bundle. The following statements are equivalent:

1. *The vector bundle $E \to M$ is orientable.*
2. *An orientation can be choosen on each fiber E_x and a vector bundle atlas, whose vector bundle charts are fiberwise orientation preserving.*
3. *An orientation may be chosen on each fiber E_x such that each vector bundle chart over a connected domain is either fiberwise orientation preserving or orientation reversing throughout.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) Let $\mathcal{A}$ be a vector bundle atlas whose transition functions are fiberwise orientation preserving. We define an orientation on E_x by taking the induced orientation of $\mathbb{R}^k$ for any VB chart $\varphi \in \mathcal{A}$ around x. This is well-defined and all VB-charts $\psi \in \mathcal{A}$ are orientation preserving, because if ψ would induce a different orientation - thus be orientation reversing at x - then the transition function $\varphi^{-1} \circ \psi$ would also be orientation reversing at x.

($\boxed{2} \Rightarrow \boxed{3}$) Let ψ be any VB chart around x with connected domain and φ an (orientation preserving) VB chart around x of the atlas $\mathcal{A}$ given by $\boxed{2}$. The transition function $\varphi^{-1} \circ \psi$ has values in $GL(\mathbb{R}^k)$ and therefore locally values in the open subset GL_+ or GL_-. Thus, ψ is either locally orientation preserving or is local orientation-interchanging, and thus the same holds on the connected domain of definition.

($\boxed{2} \Leftarrow \boxed{3}$) is clear.

($\boxed{1} \Leftarrow \boxed{2}$) The transition functions of orientation preserving VB charts are obviously orientiation preserving. □

27.3 Lemma (Orientable manifolds).

A manifold M is orientable $\Leftrightarrow$ $TM \to M$ is an orientable vector bundle.

Proof. ($\Rightarrow$) The transition functions of the vector bundle atlas of $TM \to M$ induced by M are exactly the derivatives of the chart changes of M.

($\Leftarrow$) Consider the VB atlas of $TM \to M$ induced by the charts of M and choose an orientation on the fibers of TM as in $\boxed{27.2.3}$: If $\tilde{\varphi}$ is a orientation reversing vector bundle chart of $TM \to M$ induced by a chart φ of M, then replace φ by the reparameterized chart $\psi := \varphi \circ j$, with $j : \mathbb{R}^k \to \mathbb{R}^k$,

$$j := \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & +1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & +1 \end{pmatrix}.$$

The resulting atlas on M then has only orientation preserving chart changes, so provides an orientation on M. □

27.4 Example.

Let G be a Lie group, i.e. a manifold with a smooth group structure: Then $TG \to G$ is a globally trivial vector bundle (see [**86**, 67.2]) and since each trivial vector bundle is orientable, $TG \to G$ and G are orientable.

27.5 Remark.

If M_i are orientable manifolds then obviously also $\prod M_i$ and $\coprod M_i$ are orientable. For example, all tori $T^n := (S^1)^n$ are orientable because S^1 is a Lie group.

The converse also holds, because open submanifolds of orientable manifolds are obviously orientable (hence orientability of $\coprod_i M_i$ implies that of M_i) and if $M \times N$ is orientable and $N \neq \emptyset$ then we choose a point $y \in N$ and a contractible (and thus orientable) neighborhood V of y. Thus $M \times V$ is orientable as an open submanifold, and we can coherently orient $T_x M$ using the orientations of $T_{(x,y)}(M \times V) \cong T_x M \times T_y V$ and of $T_y V$.

27.6 Definition (Transversal mappings).

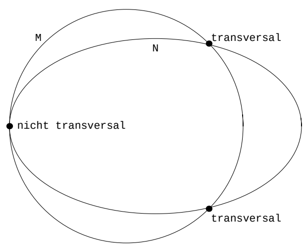
Two smooth mappings $f_i : M_i \to N$ for $i \in \{1, 2\}$ between manifolds are called TRANSVERSAL if $\operatorname{im}(T_{x_1} f_1) + \operatorname{im}(T_{x_2} f_2) = T_y N$ for all $(x_1, x_2) \in M_1 \times M_2$ with $f_1(x_1) = y = f_2(x_2) \in N$.

If f_2 is the inclusion of a submanifold, then f_1 is said to be TRANSVERSAL to M_2 in this situation, i.e. if $\operatorname{im}(T_x f_1) + T_y M_2 = T_y N$ is $y := f(x) \in M_2$ for all $x \in M_1$s.

If both f_i are inclusions of submanifolds, they are said to intersect each other TRANSVERSALLY in this situation, i.e. if $T_x M_1 + T_x M_2 = T_x N$ is for all $x \in M_1 \cap M_2$.

27.7 Example.

Let $M_2 := S^1$ in $N := \mathbb{R}^2$, $M := S^1$, and $f : M \to N$ be as shown in the image.
It is not required for transversality that the sum is a direct sum; For example, the identity $f : M := N \to N$ is transversal to each submanifold $M_2 \subseteq N$ with $\operatorname{im}(T_x f) \cap T_{f(x)} M_2 = T_{f(x)} M_2 \neq \{0\}$.



27.8 Proposition (Pull-back of manifolds).

Let $f_i \in C^\infty(M_i, N)$ for $i \in \{1,2\}$ with f_1 being transversal to f_2.

Then the PULL-BACK

$$M_1 \times_N M_2 := M_1 \times_{(f_1,N,f_2)} M_2 := \Big\{(x_1,x_2) \in M_1 \times M_2 : f_1(x_1) = f_2(x_2)\Big\}$$

is a regular submanifold of $M_1 \times M_2$ and has the following universal property:
For each pair of smooth mappings $g_i : X \to M_i$ with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a uniquely determined smooth mapping $g : X \to M_1 \times_N M_2$ with $\mathrm{pr}_i \circ g_i = g$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& M_1 \times_N M_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & M_2 \arrow[d, "f_2"] \\
& M_1 \arrow[r, "f_1"'] & N
\end{tikzcd}$$

Proof. By 2.4, it suffices to describe $M_1 \times_N M_2$ locally by a regular equation. So let $(x_1^0, x_2^0) \in M_1 \times_N M_2$. By replacing N locally at $f_1(x_1^0) = f_2(x_2^0) =: y$ by an open subset in $\mathbb{R}^n$, we have the local equation $f(x_1,x_2) := f_1(x_1) - f_2(x_2) = 0$ for the pull-back, and it is regular because $\operatorname{im}(T_{(x_1^0,x_2^0)} f) = \operatorname{im}(T_{x_1^0} f_1) + \operatorname{im}(T_{x_2^0} f_2) = T_y N$.

By definition, $f_1 \circ \mathrm{pr}_1 = f_2 \circ \mathrm{pr}_2$ holds on $M_1 \times_N M_2$. The mapping $g = (g_1, g_2) : X \to M_1 \times M_2$ is the only (smooth) mapping with $\mathrm{pr}_i \circ g = g_i$. Due to $f_1 \circ g_1 = f_2 \circ g_2$ its has values in the submanifold $M_1 \times_N M_2$ and is therefore also smooth into the pull-back by 11.5. □

By 10.1 the tangent space $T_{(x_1,x_2)}(M_1 \times_N M_2)$ is $\ker T_{(x_1,x_2)} f = \{(v_1,v_2) \in T_{x_1} M_1 \times T_{x_2} M_2 : T_{x_1} f_1 \cdot v_1 = T_{x_2} f_2 \cdot v_2\}$.

27.9 Corollary (Inverse images of submanifolds).

Let $f \in C^\infty(M,N)$ be transversal to a regular submanifold K of N.
Then $f^{-1}(K)$ is a regular submanifold of M and diffeomorphic to $M \times_N K$.

In particular, the intersection $M_1 \cap M_2$ of two transversally intersecting submanifolds M_1 and M_2 of N is itself a submanifold.

Proof. Obviously $\mathrm{pr}_1 : M \times_N K \to f^{-1}(K) \subseteq M$ is a bijection because

$$x \in \mathrm{pr}_1(M \times_N K) \Leftrightarrow \exists\, y \in K : (x,y) \in M \times_N K \Leftrightarrow f(x) = y \in K \Leftrightarrow x \in f^{-1}(K)$$

$$\begin{tikzcd}
M \times_N K \arrow[rr, hook] \arrow[d, dotted, two heads, tail] \arrow[dr, "\mathrm{pr}_1"] & & M \times K \arrow[d, hook, "M \times \mathrm{incl}"'] \\
f^{-1}(K) \arrow[r, hook] & M \arrow[r, hook, "(M,f)"] & M \times N
\end{tikzcd}$$

We have embeddings $M \times_N K \subseteq M \times K$, $M \times K \subseteq M \times N$, and $(M,f) : M \hookrightarrow M \times N$. Hence $f^{-1}(K) \cong M \times_N K$ is a regular submanifold of M by 11.10. □

We now give an example showing that this result is “stronger” than 11.12.2:

27.10 The Möbius strip as a zero set.

Consider the Möbius strip Möb $\subseteq \mathbb{R}^3$. This can not be the zero set of a regular $\mathbb{R}$-valued function, otherwise the Möbius strip would be orientable (see 27.41). We now want to represent Möb as the inverse image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ under a map from the full torus to $\mathbb{P}^2$, which is transversal to $\mathbb{P}^1$.
For this we use the following mapping:

$$p : \mathbb{R} \times D^2 \to \text{full-torus} \subseteq \mathbb{R}^3 \qquad \text{where} \qquad D^2 = \{(t,s) \in \mathbb{R}^2 : t^2 + s^2 < 1\}$$
$$p : \begin{pmatrix} \varphi \\ t \\ s \end{pmatrix} \mapsto \begin{pmatrix} \cos 2\varphi(1 + t \cdot \cos\varphi - s \cdot \sin\varphi) \\ \sin 2\varphi(1 + t \cdot \cos\varphi - s \cdot \sin\varphi) \\ t \cdot \sin\varphi + s \cdot \cos\varphi \end{pmatrix}$$

This is the composite

$$\begin{pmatrix} \varphi \\ t \\ s \end{pmatrix} \xmapsto{\Psi_1} \begin{pmatrix} \varphi \\ t \cdot \cos\varphi - s \cdot \sin\varphi \\ t \cdot \sin\varphi + s \cdot \cos\varphi \end{pmatrix} =: \begin{pmatrix} \varphi \\ \bar{t} \\ \bar{s} \end{pmatrix} \xmapsto{\Psi_2} \begin{pmatrix} \cos 2\varphi(1+\bar{t}) \\ \sin 2\varphi(1+\bar{t}) \\ \bar{s} \end{pmatrix}$$

with respected derivatives

$$\Psi_1' = \begin{pmatrix} 1 & 0 & 0 \\ * & \cos\varphi & -\sin\varphi \\ * & \sin\varphi & \cos\varphi \end{pmatrix} \qquad \text{and} \qquad \Psi_2' = \begin{pmatrix} -2\sin 2\varphi(1+\bar{t}) & \cos 2\varphi & 0 \\ 2\cos 2\varphi(1+\bar{t}) & \sin 2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus p is a local diffeomorphism (even a covering map) with

$$p^{-1}(\text{Möb}) = \mathbb{R} \times (-1,1) \times \{0\} \text{ (compare with 1.3)}$$

The mapping p is also the composite

$$\begin{pmatrix} \varphi \\ t \\ s \end{pmatrix} \xmapsto{\exp \times D^2} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ t \\ s \end{pmatrix} =: \begin{pmatrix} x \\ y \\ t \\ s \end{pmatrix} \xmapsto{\bar{p}} \begin{pmatrix} (x^2 - y^2)(1 + tx - sy) \\ 2xy \cdot (1 + tx - sy) \\ ty + sx \end{pmatrix}$$

Since $p = \bar{p} \circ (\exp \times D^2)$ is a local diffeomorphism, $\exp \times D^2 : \mathbb{R} \times D^2 \to S^1 \times D^2$ is a submersion (even the universal covering), and since $\bar{p} : S^1 \times D^2 \to \text{full-torus} \subseteq \mathbb{R}^3$ is smooth and every point has two inverse images with respect to $\bar{p}$, the mapping $\bar{p}$ is a two-fold covering map. In addition, $\bar{p}^{-1}(\text{Möb}) = S^1 \times (-1,1) \times \{0\}$ holds.
Let now the smooth map $f : S^1 \times D^2 \to S^2$ be given by

$$f : (x, y; t, s) \mapsto \frac{1}{\sqrt{1+s^2}}\,(x, y, s)$$

then $f^{-1}(S^1) = S^1 \times (-1,1) \times \{0\} = \bar{p}^{-1}(\text{Möb})$, where $S^1 \subseteq S^2$ is the equator $(s = 0)$.
Since the canonical mapping $q : S^2 \to \mathbb{P}^2$ is by definition a covering and

$$\begin{aligned}
&\bar{p}(x, y; t, s) = \bar{p}(x_1, y_1; t_1, s_1) \\
\Rightarrow\quad &(x, y; t, s) = \pm(x_1, y_1; t_1, s_1) \\
\Rightarrow\quad &f(x, y; t, s) = \frac{1}{\sqrt{1+s^2}}\,(x, y, s) = \pm\frac{1}{\sqrt{1+s^2}}\,(x_1, y_1, s_1) = \pm f(x_1, y_1; t_1, s_1) \\
\Rightarrow\quad &q(f(x, y; t, s) = q(f(x_1, y_1; t_1, s_1)),
\end{aligned}$$

the mapping $\bar f$: full-torus $\to \mathbb{P}^2$ and its restriction to Möb can be well-defined by the following commutative diagram:

$$\begin{tikzcd}
 & & S^1\times\mathbb{R} \arrow[r, hook] & \mathbb{R}^3\setminus\{0\} \arrow[dr, "r"] & \\
\mathbb{R}\times D^2 \arrow[r, two heads, "\exp\times D^2"'] \arrow[dddd, tail, "\psi_1"'] \arrow[ddddr, two heads, "p"] & S^1\times D^2 \arrow[ur, "S^1\times\mathrm{pr}_2"] \arrow[rrr, "f"'] \arrow[dddd, two heads, "\bar p"] & & & S^2 \arrow[dddd, two heads, "q"] \\
 & & S^1\times(-1,1)\times\{0\} \arrow[ul, hook] \arrow[r, dotted, two heads, "f"] \arrow[d, two heads, "\bar p"'] & S^1 \arrow[ur, hook] \arrow[d, two heads, "q|S^1"] & \\
 & & \text{Möb} \arrow[r, dotted, "\bar f|\text{Möb}"] \arrow[ddl, hook] & \mathbb{P}^1 \arrow[ddr, hook] & \\
 & & & & \\
\mathbb{R}\times D^2 \arrow[r, two heads, "\psi_2"] & \text{full-torus} \arrow[rrr, dotted, "\bar f"] & & & \mathbb{P}^2
\end{tikzcd}$$

and

$$\begin{aligned}
\bar f^{-1}(\mathbb{P}^1) &= \bar p(\bar p^{-1}(\bar f^{-1}(\mathbb{P}^1))) = \bar p((\bar f\circ\bar p)^{-1}(\mathbb{P}^1)) = \bar p((q\circ f)^{-1}(\mathbb{P}^1))\\
&= \bar p(f^{-1}(q^{-1}(\mathbb{P}^1))) = \bar p(f^{-1}(S^1)) = \bar p(S^1\times(-1,1)\times\{0\}) = \text{Möb}.
\end{aligned}$$

Moreover, f is transversal to $S^1\subseteq S^2$: In fact, take $v := (v_0, v_1, v_2)\in T_{(x,y,0)}S^2\subseteq\mathbb{R}^3$. Then $v\perp(x,y,0) = f(x,y,t,0)$, that is $(v_0,v_1)\perp(x,y)$, i.e. $(v_0,v_1)\in T_{(x,y)}S^1$. Because of $\frac{d}{ds}|_{s=0}\frac{1}{\sqrt{1+s^2}} = 0)$ and $\frac{d}{ds}|_{s=0}\frac{s}{\sqrt{1+s^2}} = 1$ we have $T_{(x,y,t,0)}f(0,0;t,v_2) = (0,0,v_2)$, so $T_{f(x,y,t,0)}S^2 = T_{(x,y,0)}S^1 + \operatorname{im} T_{(x,y,t,0)}f$.

Since q is a covering map, $\bar f$ is also transversal to $\mathbb{P}^1$. □

Now recall the concepts of vector bundles (see 14.5), vector bundle homomorphisms (see 14.8) and vector subbundles (see 14.9).

27.11 Proposition (Image of a vector bundle monomorphism).

Let $q : F\to M$ and $p : E\to M$ be two VB, and $f : F\rightarrowtail E$ a VB-monomorphism (i.e. a fiber-wise injective VB-homomorphism) over id_M, i.e. the following diagram commutes:

$$\begin{tikzcd}
F \arrow[rr, "f"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\
 & M &
\end{tikzcd}$$

Then $f(F)$ is a subvector bundle of E and is isomorphic to $F\to M$ via f:

$$\begin{tikzcd}
F \arrow[r, tail, two heads, "\cong", "f"'] \arrow[dr, "q"'] & f(F) \arrow[r, hook] \arrow[d, "p|_{f(F)}"] & E \arrow[dl, "p"] \\
 & M &
\end{tikzcd}$$

In particular, every fibre-wise bijective VB-homomorphism is a VB-isomorphism.

Proof. Since locally both VB are trivial, we assume, without restriction of generality, that $F = M\times\mathbb{R}^k$ and $E = M\times\mathbb{R}^l$, and thus $f : M\times\mathbb{R}^k\to M\times\mathbb{R}^l$ has the form $f(x,v) = (x, f_x(v))$. Since $f_{x_0} : \mathbb{R}^k\to\mathbb{R}^l$ is injective and linear, we can furthermore assume $f_{x_0} = \mathrm{incl} : \mathbb{R}^k\to\mathbb{R}^l$. Let pr be the projection $\mathbb{R}^l = \mathbb{R}^k\times\mathbb{R}^{l-k}\twoheadrightarrow\mathbb{R}^k$ with kernel $\mathbb{R}^{l-k}$, i.e. $\mathrm{pr}\circ\mathrm{incl} = \mathrm{id}\in GL(k)$. As VB-chart on E we should use

$\psi : M \times \mathbb{R}^k \times \mathbb{R}^{l-k} \ni (x; v, w) \mapsto (x; f_x(v) + (0, w)) \in M \times \mathbb{R}^l$ because commutativity of the diagram

$$\begin{tikzcd}
F \arrow[r, tail, two heads, "f"] \arrow[d, equal] & f(F) \arrow[r, hook] & E \\
M \times \mathbb{R}^k \arrow[r, equal, "\sim"] & M \times \mathbb{R}^k \times \{0\} \arrow[r, hook] \arrow[u, "\cong"', "\psi|_{M\times\mathbb{R}^k\times\{0\}}"] & M \times \mathbb{R}^k \times \mathbb{R}^{l-k}, \arrow[u, "\cong"', "\psi"]
\end{tikzcd}$$

implies that ψ trivializes the image of f, i.e. $f(F)$ corresponds to $M \times \mathbb{R}^k \times \{0\}$. So it remains to show that ψ is a local diffeomorphism: The mapping $x \mapsto \mathrm{pr} \circ f_x$ is a mapping from M to $L(k, k)$ with $x_0 \mapsto \mathrm{id}$, and thus has locally near x_0 values in the open subset $GL(k) \subseteq L(k, k)$. Without loss of generality, M is this neighborhood of x_0. Since $z = f_x(v) + (0, w) \Rightarrow \mathrm{pr}(z) = (\mathrm{pr} \circ f_x)(v) \Rightarrow v = (\mathrm{pr} \circ f_x)^{-1}(\mathrm{pr}(z))$, the inverse mapping to ψ is given by

$$\psi^{-1} : (x, z) \mapsto \Big(x, \Big((\mathrm{pr} \circ f_x)^{-1} \circ \mathrm{pr}\Big)(z), z - f_x\Big(\Big((\mathrm{pr} \circ f_x)^{-1} \circ \mathrm{pr}\Big)(z)\Big)\Big). \quad \square$$

27.12 Corollary (Tangent bundle of a submanifold).

Let $\mathrm{incl} : A \subseteq M$ *be a regular submanifold.*
Then $(T\,\mathrm{incl})(TA) \cong TA$ *is a subbundle of the* $TM|_A$.

Proof.

$$\begin{tikzcd}
TA \arrow[r, tail, two heads, "T\,\mathrm{incl}", "\cong"'] \arrow[dr, "\pi_A"'] & (T\,\mathrm{incl})(TA) \arrow[r, hook] \arrow[d] & TM|_A \arrow[r, hook, "\mathrm{incl}"] \arrow[dl, "\pi_M"'] & TM \arrow[dl, "\pi_M"'] \\
& A \arrow[r, hook] & M &
\end{tikzcd}$$

Apply 27.11 to the VB-monomorphism $T\,\mathrm{incl} : TA \rightarrowtail TM|_A$. $\square$

27.13 Corollary (Tangent bundles of sums and products).

The following diagrams describe vector bundle isomorphisms:

$$\begin{tikzcd}
T(\prod M_i) \arrow[r, "(T\,\mathrm{pr}_i)_i", "\cong"'] \arrow[d, "\pi_{\prod M_i}"'] & \prod TM_i \arrow[d, "\prod \pi_{M_i}"'] \\
\prod M_i \arrow[r, equal] & \prod M_i
\end{tikzcd}
\qquad
\begin{tikzcd}
T(\bigsqcup M_i) \arrow[d, "\pi_{\bigsqcup M_i}"'] & \bigsqcup TM_i \arrow[l, "\bigcup T\,\mathrm{incl}_i"', "\cong"] \arrow[d, "\bigsqcup \pi_{M_i}"'] \\
\bigsqcup M_i \arrow[r, equal] & \bigsqcup M_i
\end{tikzcd}$$

$\square$

27.14 Lemma (Pull-back bundle).

Let $p : E \twoheadrightarrow M$ *be a VB and* $f : N \to M$ *be smooth. Then, on the pull-back* $f^*E := N \times_M E$, *there exists a distinct VB structure* $f^*p := \mathrm{pr}_1|_{f^*E} : f^*E \twoheadrightarrow N$ *for which* $p^*f := \mathrm{pr}_2|_{f^*E} : f^*E \to E$ *is a fiberwise bijective VB-homomorphism over* f.

This bundle has the following universal property: For any VB $q : F \to N$ *and VB-homomorphism* $\bar{f} : F \to E$ *over* f, *there exists a unique VB-homomorphism* $f^+ : F \to f^*E$ *over* id_N, *which makes the adjacent diagram commutative:*

$$\begin{tikzcd}
F \arrow[dr, dotted, "f^+"] \arrow[drr, bend left, "\bar{f}"] \arrow[ddr, bend right, "q"'] & & \\
& f^*E \arrow[r, "p^*f"'] \arrow[d, two heads, "f^*p"] & E \arrow[d, two heads, "p"] \\
& N \arrow[r, "f"'] & M
\end{tikzcd}$$

Note that the fiber $(f^*E)_x := (f^*p)^{-1}(x)$ of $f^*p : f^*E \twoheadrightarrow N$ over $x \in N$ is given by $\{(x, v) : v \in p^{-1}(f(x)) = E_{f(x)}\}$ and is bijectively mapped to $E_{f(x)}$ by

$p^*f = \mathrm{pr}_2$, so $f^*E \cong \bigsqcup_{x\in M} E_{f(x)}$ can be considered as a reparametrization by virtue of $f : N \to M$ of the bundle $E = \bigsqcup_{y\in N} E_y$.

Proof. Since $p : E \twoheadrightarrow M$ is submersive, f and p are transversal to each other and thus $f^*E := N \times_M E$ is a submanifold of $N \times E$ with the universal property for smooth maps q and $\bar f$ by 27.8.

In order to show that $f^*p : f^*E \to N$ is a fiber bundle, we need to find local trivializations. For this purpose, let $\psi : U \times \mathbb{R}^k \to E|_U$ be a local trivialization of $E|_U$.

Note that the pull-back of an open subset $U \subseteq M$ under a smooth map $p : E \to M$ is given by $E|_U = p^{-1}(U)$, as a direct proof of the universal property shows. Explicitly, the diffeomorphism $p^{-1}(U) \cong U \times_M E$ is given by $z \mapsto (p(z), z)$ and $z \leftarrow\!\shortmid (u, z)$, cf. 27.15.

$$\begin{tikzcd} E|_U \arrow[r, hook] \arrow[d, "p|_{p^{-1}(U)}"] & E \arrow[d, "p"] \\ U \arrow[r, hook] & M \end{tikzcd}$$

Furthermore, the pullback of a trivial bundle $\mathrm{pr}_1 : M \times \mathbb{R}^k \to M$ along $f : N \to M$ is the trivial bundle $\mathrm{pr}_1 : N \times \mathbb{R}^k \to N$, as a direct proof of the universal property also shows. Explicitly, the diffeomorphism $N\times\mathbb{R}^k \cong N\times_M(M\times\mathbb{R}^k)$ is given by $(x, v) \mapsto (x, (f(x), v))$ and $(x, v) \leftarrow\!\shortmid (x, (y, v))$.

$$\begin{tikzcd} N \times \mathbb{R}^k \arrow[r, "f\times\mathbb{R}^k"] \arrow[d, "\mathrm{pr}_1"] & M \times \mathbb{R}^k \arrow[d, "\mathrm{pr}_1"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Considering a rectangle which is split vertically into two squares, with the right square being a pull-back, then the rectangle is a pull-back if and only if the left square is one, as a simple diagram chasing shows, see [88, 3.8.3] and [88, 3.8.4].

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[rr, bend left] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

We apply all this now to $\mathrm{incl}\circ f$ and $f \circ \mathrm{incl}$:

$$\begin{tikzcd} f^{-1}(U)\times\mathbb{R}^k \arrow[r, equal, "\sim"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] \arrow[rrrrr, bend left, "f\times\mathbb{R}^k"] & f^*E|_{f^{-1}(U)} \arrow[rrr] \arrow[dr, hook] \arrow[dd] & & & E|_U \arrow[r, equal, "\sim"] \arrow[dl, hook'] \arrow[dd] & U\times\mathbb{R}^k \arrow[ddl, bend left, "\mathrm{pr}_1"] \\ & & f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] & & \\ & & N \arrow[r, "f"'] & M & & \\ & f^{-1}(U) \arrow[ur, hook, "\mathrm{incl}"] \arrow[rrr, "f"] & & & U \arrow[ul, hook', "\mathrm{incl}"'] & \end{tikzcd}$$

The fact that the constructed bundle chart as well as p^*f and f^+ are fiber-linear follows also from this diagram. $\square$

27.15 Lemma (Restriction as a pull-back).

If $p : E \to M$ is a vector bundle and A is a regular submanifold of M, then $E|_A := p^{-1}(A) \cong \mathrm{incl}^* E$ holds.

Proof. Since p is a submersion it is transversal to A and hence $p^{-1}(A) \cong E\times_M A = \mathrm{incl}^* E$ by 27.9. $\square$

27.16 Lemma (Splitting exact vector bundle sequences).

Let $E^0 \xrightarrow{i} E^1 \xrightarrow{p} E^2$ be a short exact sequence of VB over a paracompact manifold M, i.e. i and p are VB-homomorphisms over id_M and fiberwise i is injective, p is surjective and $\mathrm{im}(i_x) = \ker(p_x)$.
Then $E^1 \cong E^0 \oplus E^2$.

Proof. By 27.11, $i : E^0 \to E^1$ induces an isomorphism onto a vector subbundle, i.e. without loss of generality, i is the inclusion of a subbundle.
We now construct a right-inverse vector bundle homomorphism $j : E^2 \to E^1$ to $p : E^1 \to E^2$:

Locally $E^1|_U \cong U \times \mathbb{R}^m$ and $E^2|_U \cong U \times \mathbb{R}^k$ for suitable m, $k \in \mathbb{N}$. Under the isomorphism i, the bundle E^0 corresponds to $U \times \mathbb{R}^n \times \{0\}$ for an $n \leq m$. Thus, the local representation of p induces an isomorphism $U \times \mathbb{R}^{m-n} \cong U \times \{0\} \times \mathbb{R}^{m-n} \xrightarrow{\cong} U \times \mathbb{R}^k$ and its inverse is a local right inverse to p.

$$\begin{tikzcd}
E^0 \arrow[r, hook, "i"] & E^1 \arrow[r, two heads, "p"] & E^2 \\
E^0|_U \arrow[u, hook] \arrow[r, hook, "i"] & E^1|_U \arrow[u, hook] \arrow[r, two heads, "p"] & E^2|_U \arrow[u, hook] \arrow[l, dotted, bend left, "j"] \\
U \times \mathbb{R}^n \arrow[u, "\cong"] \arrow[r, hook, "U\times\mathrm{incl}_1"'] & U \times \mathbb{R}^m \arrow[u, "\cong"] \arrow[r, two heads] & U \times \mathbb{R}^k \arrow[u, "\cong"'] \\
& U \times \mathbb{R}^{m-n} \arrow[u, hook, "U\times\mathrm{incl}_2"] \arrow[ur, dotted, "\cong"'] &
\end{tikzcd}$$

By means of a partition of unity, which is subordinate to the covering with these trivializing neighborhoods U, we can glue these local right-inverses and clearly get a right-inverse $j : E^2 \to E^1$ to p. The isomorphism $E^0 \times_M E^2 \cong E^1$ is then given by $(z^0, z^2) \mapsto i(z^0) + j(z^2)$ and has as inverse $z \mapsto (i^{-1}(z - j(p(z))), p(z))$ because $z - j(p(z)) \in \operatorname{Ker}(p) = \operatorname{im}(i)$. □

27.17 Tangent bundle of a vector bundle.

We now want to examine the tangent bundle $\pi_E : TE \to E$ (the total space) of a vector bundle $p : E \to M$ in more detail. In particular, we are interested in $T^2M = T(TM)$.

Locally E is given by $M \times \mathbb{R}^k$ and thus TE by $T(M \times \mathbb{R}^k) = TM \times \mathbb{R}^k \times \mathbb{R}^k$. On the other hand, the pull-back bundle $p^*(TM) = E \times_M TM$ is locally given by $TM \times \mathbb{R}^k$ and the pull-back $p^*(E) = E \times_M E$ locally by $M \times \mathbb{R}^k \times \mathbb{R}^k$.

$$\begin{tikzcd}
TU \times \mathbb{R}^k \arrow[r, two heads, "\pi\times\mathbb{R}^k"'] \arrow[d, hook] & U \times \mathbb{R}^k \arrow[d, hook] & U \times \mathbb{R}^k \times \mathbb{R}^k \arrow[l, two heads, "\mathrm{pr}_1"'] \arrow[d, hook] \\
p^*(TM) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & E \arrow[d, "p"] & p^*(E) \arrow[l, two heads] \arrow[d] \arrow[dl, phantom, "\mathrm{pb}"] \\
TM \arrow[r, two heads, "\pi_M"'] & M & E \arrow[l, two heads, "p"]
\end{tikzcd}$$

Thus, TE is locally isomorphic to $(TM \times \mathbb{R}^k) \times_{M\times\mathbb{R}^k} (M \times \mathbb{R}^k \times \mathbb{R}^k)$ and thus to $p^*(TM) \times_E p^*(E)$. In order to make these local isomorphisms global, we will construct a natural short exact sequence $p^*(E) \to TE \to p^*(TM)$ of vector bundles over E and then apply 27.16 to them.

Theorem.

Let $p : E \to M$ a VB. Then there is a short exact sequence of VB over E:

$$E \times_M E =: p^*(E) \xrightarrow{\mathrm{vl}_E} TE \xrightarrow{(\pi, Tp)} p^*(TM) := E \times_M TM.$$

Thus, according to 27.16, $TE \cong (E \times_M E) \times_E (E \times_M TM) \cong E \times_M E \times_M TM$, however no natural isomorphism exists, see 27.19.
The image of vl_E is called the VERTICAL SUBBUNDLE VE of TE and the image of some right-inverse to (π, Tp) can be viewed as selection of a HORIZONTAL SUBBUNDLE of TE.

Proof.

$$\begin{tikzcd}
TE \arrow[dr, dotted, "(\pi,Tp)"] \arrow[drr, bend left, "Tp"] \arrow[ddr, bend right, "\pi_E"'] & & \\
& p^*(TM) \arrow[r, two heads, "\pi^*p"'] \arrow[d, "p^*\pi"] & TM \arrow[d, "\pi_M"] \\
& E \arrow[r, two heads, "p"'] & M
\end{tikzcd}$$

We can define a vector bundle homomorphism $(\pi, Tp) : TE \to p^*(TM)$ by the adjacent diagram. Locally TE is given by $TM\times\mathbb{R}^k\times\mathbb{R}^k$ and $p^*(TM)$ by $(M\times\mathbb{R}^k)\times_M TM \cong TM\times\mathbb{R}^k$ and $Tp : TE \to TM$ by $(\xi, v, w) \mapsto \xi$. Thus, (π, Tp) is described locally by $(\xi, v, w) \mapsto (\xi, v)$. In particular, (π, Tp) is fiberwise surjective.

The fiberwise kernel of (π, Tp) over E consists of those vectors which are mapped to 0 vectors by Tp, i.e. the tangent vectors to curves in the fibers $p^{-1}(x)$ of E. Therefore, we define a VB-homomorphism (the so-called vertical lift) $\mathrm{vl}_E : p^*(E) = E \times_M E \to TE$ by $p^*(E) = E \times_M E \ni (v_x, w_x) \mapsto \frac{d}{dt}|_{t=0} v_x + t\, w_x \in TE$. With respect to the local descriptions $M\times\mathbb{R}^k\times\mathbb{R}^k$ of $p^*(E)$ and $TM\times\mathbb{R}^k\times\mathbb{R}^k$ of TE it is given by $(x, v, w) \mapsto \frac{d}{dt}|_{t=0}(x, v + tw) = (0_x, v, w)$. Fiberwise it is thus injective and

$$\mathrm{im}\Big((\mathrm{vl}_E)_{(x,v)}\Big) = \Big\{(0_x, v; w) : w \in \mathbb{R}^k\Big\} = \ker\Big((\pi, Tp)_{(x,v)}\Big),$$

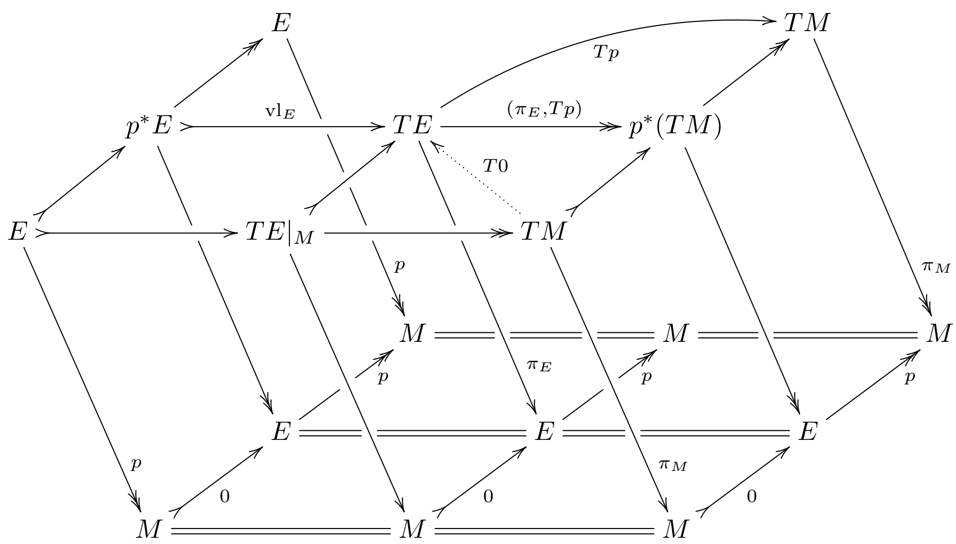
so the sequence is exact. □

27.18 Proposition.

Let $p : E \to M$ be a vector bundle. Then $TE|_M := 0^*(TE) \cong E\oplus TM$ (canonically) as vector bundles over M, where $0 : M \hookrightarrow E$ denotes the zero section.

We will apply this in [**86**, 62.8] to $E := TM \to M$ and in [**86**, 62.9] to $(TN)^\perp \to N$.

Proof.



By restricting the exact sequence of VB over E from 27.17 to the zero section, we get by 27.15 an exact sequence $p^*(E)|_M \to TE|_M \to p^*(TM)|_M$ of VB over M. Because of $p \circ 0 = \mathrm{id}$ in the side faces in the above diagram, $p^*(E)|_M \cong E$ and $p^*(TM)|_M \cong TM$ and a canonical right-inverse to $Tp|_{TE|_M}$ is given by $T0$, so the sequence $E \to TE|_M \to TM$ splits canonically, that is $TE|_M \cong E \oplus TM$ as bundles over M. □

Moreover, $Tp : TE \to TM$ as well as $\pi_M^* p : p^*(TM) \to TM$ are vector bundles and $(\pi_E, Tp) : TE \to p^*(TM)$ is a vector bundle homomorphism over TM.
So we may ask, if the right-inverse of 27.17 can also be choosen fiberwise linear over TM.

$$\begin{tikzcd}
TE \arrow[drr, bend left, "\pi_E"] \arrow[dr, "(\pi,Tp)"'] \arrow[ddr, bend right, "Tp"'] & & \\
& p^*(TM) \arrow[r, "p^*\pi_M"] \arrow[d, "\pi_M^* p"] & E \arrow[d, "p"] \\
& TM \arrow[r, "\pi_M"] & M
\end{tikzcd}$$

27.19 Proposition. Linear connection of a vector bundle.

Let $p : E \to M$ be a vector bundle. An isomorphism $TE \cong p^*(E) \times_E p^*(TM)$ is equivalently described by each of the following mappings:

1. *A horizontal lift, i.e. a vector bundle homomorphism $C : p^*(TM) \to TE$ over E being right-inverse to $(\pi, Tp) : TE \to p^*(TM) = E \times_M TM$ and which is also fiberwise linear over TM.*
2. *A linear connection, i.e. vector bundle homomorphism $\Phi : TE \to VE := p^*(E)$ over E being a left-inverse to the vertical lift $\mathrm{vl}_E : VE \hookrightarrow TE$ and which is also fiberise linear over $\pi_M : TM \to M$.*
3. *A connector, i.e. a vector bundle homomorphism $K : TE \to E$ over p and over π_M with $K \circ \mathrm{vl}_E = \mathrm{pr}_2 : E \times_M E \to TE \to E$.*
4. *A covariant derivation $\nabla : \Omega^0(M; E) = \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M; E)$ with $\nabla(f \cdot s) = f \cdot \nabla s + df \otimes s$ for all $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$. ($\nabla$ may be extended to a so-called outer covariant derivative $\Omega^*(M; E) \to \Omega^{*+1}(M; E)$ with the same formula as in 25.9, see, e.g., [**105**, 37.29]).*
5. *A covariant derivative $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, which is $C^\infty(M, \mathbb{R})$-linear in the first variable and $\mathbb{R}$-linear in the second variable, and satisfies $\nabla_\xi(f \cdot s) = f \cdot \nabla_\xi s + \xi(f) \cdot s$ for all $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$.*

Proof. The morphisms in 1 - 3 are described by the following diagram:

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "p"'] & M & TM \arrow[l, "\pi_M"'] \arrow[r, equal] & TM \arrow[dr, "\pi_M"] & \\
M \arrow[r, phantom, "\mathrm{pb}"] & p^*(E) \arrow[u, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[r, "\cong", "\mathrm{vl}_E"'] & VE \arrow[u, "\pi_M \circ Tp"] \arrow[d, "\pi_E"] \arrow[r, hook] & TE \arrow[u, "Tp"'] \arrow[d, "\pi_E"'] \arrow[l, dotted, bend left, "\Phi"] \arrow[ull, dotted, "K"'] \arrow[r, "(\pi_E,Tp)"] & p^*(TM) \arrow[u, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] \arrow[l, dotted, bend left, "C"] \arrow[r, phantom, "\mathrm{pb}"] & M \\
& E \arrow[ul, "p"] \arrow[r, equal] & E \arrow[r, equal] & E \arrow[r, equal] & E \arrow[ur, "p"'] &
\end{tikzcd}$$

With respect to local trivializations $E|_U \cong U \times \mathbb{R}^k$, $TU \cong U \times \mathbb{R}^m$ and thus $T(E|_U) \cong (E|_U) \times \mathbb{R}^m \times \mathbb{R}^k \cong U \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^k$, the mappings of the above diagram have the following form

$$\begin{tikzcd}
& (x,w) \arrow[r, mapsto, "p"] \arrow[dl, mapsto, "p"'] & x & (x,y) \arrow[l, mapsto, "\pi_M"'] \arrow[r, equal] & (x,y) \arrow[dr, mapsto, "\pi_M"] & \\
x & (x,v;x,w) \arrow[u, mapsto, "\mathrm{pr}_2"] \arrow[d, mapsto, "\mathrm{pr}_1"] \arrow[r, mapsto, "\cong", "\mathrm{vl}_E"'] & (x,v,0,w) \arrow[u, mapsto, "\pi_M \circ Tp"] \arrow[d, mapsto, "\pi_E"] & (x,v,y,\bar w) \arrow[u, mapsto, "Tp"'] \arrow[d, mapsto, "\pi_E"'] \arrow[l, dotted, mapsto, "w:=\bar w-\Gamma_x(v,y)"'] \arrow[ull, dotted, mapsto, "w:=\bar w-\Gamma_x(v,y)"'] \arrow[r, mapsto, "(\pi_E,Tp)"] & (x,v;x,y) \arrow[u, mapsto, "\mathrm{pr}_2"'] \arrow[d, mapsto, "\mathrm{pr}_1"'] \arrow[l, dotted, mapsto, bend left, "\bar w:=\Gamma_x(v,y)"] & x \\
& (x,v) \arrow[ul, mapsto, "p"] \arrow[r, equal] & (x,v) \arrow[r, equal] & (x,v) \arrow[r, equal] & (x,v) \arrow[ur, mapsto, "p"'] &
\end{tikzcd}$$

where $\Gamma_x : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ is bilinear and $x \mapsto \Gamma_x$ is smooth.

($\boxed{1}\Leftrightarrow\boxed{2}$) A short exact sequence $E^0 \xrightarrow{i} E^1 \xrightarrow{p} E^2$ splits (i.e. $E^1 \cong E^0 \oplus E^2$) if and only if p has a right-inverse s, or if i has a left-inverse q: The relation between s and q is given by $q(z_1) := i^{-1}(z_1 - s(p(z_1)))$ and $s(p(z_2)) := z_2 - i(q(z_2))$.

($\boxed{2}\Leftrightarrow\boxed{3}$) Since $\mathrm{vl}_E : p^*(E) \to TE$ is a VB isomorphism onto the vertical bundle, a projection $TE \to VE := \operatorname{im} \mathrm{vl}$ is uniquely given by its second component $K : TE \xrightarrow{\Phi} VE \cong E \oplus E \xrightarrow{\mathrm{pr}_2} E$.

($\boxed{3}\Rightarrow\boxed{5}$) Using the connector $K : TE \to E$, we obtain a section $\nabla_\xi s : M \xrightarrow{\xi} TM \xrightarrow{Ts} TE \xrightarrow{K} E$ of $p : E \to M$ for $\xi \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, because $p \circ \nabla_\xi s = p \circ K \circ Ts \circ \xi = \pi \circ Tp \circ Ts \circ \xi = \pi \circ T \operatorname{id} \circ \xi = \pi \circ \xi = \operatorname{id}$.

With respect to local trivializations $E|_U \cong U \times \mathbb{R}^k$, we have $TU \cong U \times \mathbb{R}^m$ and thus $T(E|_U) \cong (E|_U) \times \mathbb{R}^m \times \mathbb{R}^k \cong U \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^k$. So ∇ has the following form:

$$\begin{aligned}
\xi(x) &= (x, y(x)), \quad s(x) = (x, \bar{s}(x)),\\
Ts(x,y) &= \Big(x, \bar{s}(x), y, \bar{s}'(x)(y)\Big), \quad K(x,v,y,\bar{w}) = \Big(x, \bar{w} - \Gamma_x(v,y)\Big)\\
\Rightarrow \quad \nabla_\xi s(x) &= (K \circ Ts \circ \xi)(x) = K\Big(x, \bar{s}(x), y(x), \bar{s}'(x)\big(y(x)\big)\Big)\\
&= \Big(x, \bar{s}'(x)\big(y(x)\big) - \Gamma_x\big(\bar{s}(x), y(x)\big)\Big)
\end{aligned}$$

($\boxed{4}\Leftarrow\boxed{5}$) We have $(\nabla s)(\xi_x) := (\nabla_\xi s)(x)$. Because of the $C^\infty(M,\mathbb{R})$-linearity of $\xi \mapsto \nabla_\xi s$, this defines $\nabla s \in \Omega^1(M;E)$.

($\boxed{3}\Leftarrow\boxed{4}$) Any non-vertical vector $\sigma \in TE$ can be written as $\sigma = Ts \cdot \xi$ for some $\xi \in T_xM$ and $s \in \Gamma(E)$ with $x = \pi_M(Tp \cdot \sigma)$ and $s(x) = \pi_E(\sigma) \in E_x$. Because of $p \circ s = \operatorname{id}$, the vector $\xi = Tp(Ts \cdot \xi) = Tp(\sigma)$ is uniquely determined.
We define $K(\sigma) := (\nabla s)(\xi) \in E_x \subseteq E$. This definition depends only on σ and not on s: The required product rule implies that ∇ is a local operator. Let a local trivialization $U \times \mathbb{R}^k \cong E|_U$ be described by local sections $g_i : U \ni x \mapsto g_i(x) \in E_x$. Thus $s = \sum_i \bar{s}^i \cdot g_i$ locally with certain coefficient functions $\bar{s}^i \in C^\infty(M,\mathbb{R})$. Then

$$\nabla s(\xi) = \nabla\Big(\sum_i \bar{s}^i \cdot g_i\Big)(\xi) = \sum_i \Big(\bar{s}^i(x) \cdot \nabla g_i(\xi) + d\bar{s}^i(\xi) \cdot g_i(x)\Big).$$

and thus depends only on $\bar{s}^i(x)$ and on $d\bar{s}^i(\xi)$, i.e. only on $Ts \cdot \xi = \sigma$.
With respect to local trivializations $E|_U \cong U \times \mathbb{R}^k$, we have $TU \cong U \times \mathbb{R}^m$ and thus $T(E|_U) \cong (E|_U) \times \mathbb{R}^m \times \mathbb{R}^k \cong U \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^k$, $\nabla s(\xi)$ and thus K have the following form:

$$\begin{aligned}
\sigma &= (x,v,y,w) = Ts \cdot \xi, \qquad \text{with } \xi = (x,y) \text{ and } s(x) = (x, \bar{s}(x)),\\
(x,v,y,w) &= Ts(x,y) = \Big(x, \bar{s}(x), y, \bar{s}'(x)(y)\Big), \quad w = \bar{s}'(x)(y) = ((\bar{s}^i)'(x)(y))_i\\
\Gamma_x(v,y) &:= -\sum_i v^i \cdot \nabla g_i(x,y) \in \mathbb{R}^k \text{ is bilinear in } v \text{ and } y\\
\Rightarrow \quad \nabla s(\xi) &= \big(x, -\Gamma_x(v,y) + \bar{s}'(x)(y)\big) \text{ according to the above formula}\\
K(x,v,y,w) &= \nabla s(\xi) = \big(x, \bar{s}'(x)(y) - \Gamma_x(v,y)\big) = \Big(x, w - \Gamma_x(v,y)\Big).
\end{aligned}$$

So we can extend K uniquely to VE (i.e. $y = 0$) using this local formula. Hence $K \circ \mathrm{vl}_E = \mathrm{pr}_2 : E \times_M E \cong VE \subseteq TE \to E$. □

27.20 Proposition.

Each vector bundle $p : E \to M$ is isomorphic to a vector subbundle of a trivial bundle $\mathrm{pr}_1 : M \times \mathbb{R}^s \to M$ with appropriate $s \in \mathbb{N}$.

Proof.

$$\begin{tikzcd}
 & & & M\times\mathbb{R}^s \arrow[ddl] \arrow[dr, hook, "f|_{M\times\mathbb{R}^s}"] \arrow[d, phantom, "\mathrm{pb}"] & \\
E \arrow[r, hook] \arrow[d, "p"] & E\oplus TM \arrow[r, "\cong"', "\boxed{27.18}"] \arrow[d] & TE|_M \arrow[r, hook] \arrow[d] \arrow[ur, dotted, "(0^*\pi_E, Tf|_{TE|_M})"] & TE \arrow[r, hook, "Tf"] \arrow[d, "\pi_E"] & \mathbb{R}^s\times\mathbb{R}^s \arrow[d, "\mathrm{pr}_1"] \\
M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, hook, "0"] & E \arrow[r, hook, "f"] & \mathbb{R}^s
\end{tikzcd}$$

Let $f : E \to \mathbb{R}^s$ be an embedding (or just an immersion) of the manifold E into some $\mathbb{R}^s$. Then, $Tf : TE \rightarrowtail \mathbb{R}^s \times \mathbb{R}^s$ is a vector bundle monomorphism over $f : E \to \mathbb{R}^s$ and thus $Tf \circ \iota : E \hookrightarrow TE|_M \hookrightarrow TE \rightarrowtail \mathbb{R}^s \times \mathbb{R}^s$ is a vector bundle monomorphism over $M \to E \to \mathbb{R}^s$, so $(p, \mathrm{pr}_2 \circ Tf \circ \iota) : E \rightarrowtail M \times \mathbb{R}^s = (f \circ 0)^*(\mathbb{R}^s \times \mathbb{R}^s)$ is a vector bundle monomorphism over id_M. □

We will show in 27.36) that such a vector bundle monomorphism exists already for $s = \dim(E)$.

27.21 Corollary (Existence of inverse bundles). [65, p.100, 4.3.3].

Let $E \to M$ be a vector bundle of fiber dimension k over an m-dimensional manifold. Then a vector bundle $F \to M$ (with fiber dimension m) exists such that $E \oplus F$ is a trivial bundle (with fiber dimension $k + m$).

Proof. Let $\varphi : E \rightarrowtail M \times \mathbb{R}^s$ be a VB-monomorphism over id_M to 27.20 (or 27.36 for $s = k + m$). For F it is sufficient to take $\varphi(E)^\perp \subseteq M \times \mathbb{R}^s$. □

27.22 K-Theory.

For a fixed manifold M we consider the set of isomorphism classes of finite dimensional vector bundles over M. With respect to the Whitney sum $E_1 \oplus E_2$ and the tensor product $E_1 \otimes E_2$, they form a semiring with 0-element $[\varepsilon_0]$ and 1-element $[\varepsilon_1]$, where ε_k denotes the trivial bundle $\mathrm{pr}_1 : M \times \mathbb{R}^k \to M$. As in the construction of $\mathbb{Z}$ from the semiring $\mathbb{N}$, one looks at equivalence classes of pairs (E, F) of vector bundles over M with respect to the equivalence relation $(E_1, F_1) \equiv (E_2, F_2) :\Leftrightarrow E_1 \oplus F_2 \cong F_1 \oplus E_2$ and adds these by adding the representants via $(E_1, F_1) + (E_2, F_2) := (E_1 \oplus E_2, F_1 \oplus F_2)$ and multiplies them by $(E_1, F_1) \cdot (E_2, F_2) := (E_1 \otimes E_2 \oplus F_1 \otimes F_2, E_1 \otimes F_2 \oplus F_2 \otimes E_1)$. However, since isomorphic classes do not satisfy the (additive) cancellation rules (for example, $TS^2 \oplus \varepsilon_1 \cong \varepsilon_3 \cong \varepsilon_2 \oplus \varepsilon_1$, but TS^2 is not trivial, and since $E \to [(E, 0)]_\equiv$ is injective, the cancellation rule can not be valid on the image), we have to use the coarser equivalence relation $(E_1, F_1) \approx (E_2, F_2) :\Leftrightarrow \exists\, F : E_1 \oplus F_2 \oplus F \cong F_1 \oplus E_2 \oplus F$ instead. Then, the set of all $\approx$ equivalence classes of pairs of vector bundles over M becomes a commutative ring with 1, denoted $K(M)$. With respect to the pull-back along smooth mappings $f : N \to M$, the assignment K becomes a contra-variant functor. Vector bundles over 1-point manifolds $M = \{x\}$ are vector spaces, and pairs of such are $\approx$-equivalent if and only if the difference in dimension of the components is equal. So $K(\{x\}) \cong \mathbb{Z}$.

A second possibility to determine the K-theory $K(M)$ is to consider the equivalence relation $E_1 \sim E_2 :\Leftrightarrow \exists\, k_1, k_2 : E_1 \oplus \varepsilon_{k_1} \cong E_2 \oplus \varepsilon_{k_2}$. The set of these equivalence classes forms a commutative ring $\tilde{K}(M)$ with respect to the addition $([E_1]_\sim, [E_2]_\sim) \mapsto [E_1 \oplus E_2]_\sim$ and the multiplication $([E_1]_\sim, [E_2]_\sim) \mapsto [E_1 \otimes E_2]_\sim$. The neutral element is obviously given by $[\varepsilon_0]_\sim$. The additive inverse (see 27.21) to $[E]_\sim$ we find as follows: By 27.20 there exists an $s \in \mathbb{N}$, s.t. E is isomorphic to a subset of ε_s. Thus, $E \oplus E^\perp \cong \varepsilon_s \sim \varepsilon_0$ holds, where $E^\perp$ denotes the orthogonal complementary bundle to this subbundle in ε_s. We have $\tilde{K}(\{x\}) = \{0\}$, which is why $\tilde{K}$ is called the reduced K theory.

We now describe a ring homomorphism $K(M) \to \tilde{K}(M)$:
Let given (E_1, F_1). Then, as before, there is an $s \in \mathbb{N}$ and an $F_1^\perp$ with $F_1 \oplus F_1^\perp \cong \varepsilon_s$. Thus $(E_1, F_1) \approx (E_1 \oplus F_1^\perp, \varepsilon_s)$. We map this to $[E_1 \oplus F_1^\perp]_\sim$. This map is well defined, because $(E_1, \varepsilon_{s_1}) \approx (E_2, \varepsilon_{s_2})$, that is $\exists\, F : E_1 \oplus \varepsilon_{s_2} \oplus F \cong E_2 \oplus \varepsilon_{s_1} \oplus F$, implies $E_1 \oplus \varepsilon_{s_2} \oplus \varepsilon_s \cong E_1 \oplus \varepsilon_{s_2} \oplus F \oplus F^\perp \cong E_2 \oplus \varepsilon_{s_1} \oplus F \oplus F^\perp \cong E_2 \oplus \varepsilon_{s_1} \oplus \varepsilon_s$, i.e. $E_1 \sim E_2$. This homomorphism is obviously surjective and its kernel consists exactly of those classes $[(E, \varepsilon_k)]_\approx$ for which $E \sim \varepsilon_0$, that is $\exists\, l, n \in \mathbb{N} : E \oplus \varepsilon_l \cong \varepsilon_n$, thus $(E, \varepsilon_k) \approx (E \oplus \varepsilon_l, \varepsilon_k \oplus \varepsilon_l) \approx (\varepsilon_n, \varepsilon_{k+l})$. The mapping $(n, m) \mapsto (\varepsilon_n, \varepsilon_m)$ factors to a ring monomorphism $\mathbb{Z} \hookrightarrow K(M)$ with left inverses $K(\mathrm{incl}) : K(M) \to K(\{x\}) \cong \mathbb{Z}$. Thus, the kernel is isomorphic to $\mathbb{Z}$, so the short exact sequence of Abelian groups $\mathbb{Z} \hookrightarrow K(M) \twoheadrightarrow \tilde{K}(M)$ splits, i.e.

$$K(M) \cong \tilde{K}(M) \oplus \mathbb{Z}.$$

27.23 Definition (Universal vector bundle).

Let $p : E \to M$ be a vector bundle with fiber dimension k (a so-called k-plane bundle) over M and $f : E \rightarrowtail M \times \mathbb{R}^s$ a vector bundle monomorphism as in 27.20 over id_M. We will show in 27.35 that such a monomorphism exists if $s \geq k + m$. From this we obtain a map $g : M \to G(k, s)$ (a so called classifying mapping for the vector bundle $p : E \to M$) with values in the Graßmann manifold of the k-planes in $\mathbb{R}^s$ by mapping x to the image of the mapping of $f_x : \mathbb{R}^k \cong E_x \to RR^s$: Since E is locally $M \times \mathbb{R}^k$, the mapping f has as local description a smooth mapping $M \to L_k(k, S)$, $x \mapsto f_x$, and hence the composite g with $\mathrm{im} : L_k(k, s) \to G(k, s)$ (see [**98**, 15.1]) is smooth.
Now consider the universal vector bundle $E(k, s) \to G(k, s)$, where

$$E(k, s) := \{(\varepsilon, v) \in G(k, s) \times \mathbb{R}^s : v \in \varepsilon\},$$

a submanifold of the product (see [**98**, 20.7]), and the fibrewise bijective vector bundle homomorphism

$$\gamma : E \to E(k, s), \quad v \mapsto \Big(g\big(p(v)\big), f_{p(v)}(v)\Big),$$

which is obviously smooth into tyhe product and has values in $E(k, s)$.

$$\begin{tikzcd}
E \arrow[r, "f"] & M \times \mathbb{R}^s \arrow[r, two heads, "\mathrm{pr}_2"] & \mathbb{R}^s \\
E_x \arrow[u, hook] \arrow[rr, tail, two heads, "f_x"] & & g(x) := f_x(E_x) \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times \mathbb{R}^s \arrow[r, two heads, "\mathrm{pr}_2"] & \mathbb{R}^s & \\
E \arrow[u, "f"] \arrow[r, dotted, tail, "\gamma"] \arrow[d, "p"] & E(k, s) \arrow[u] \arrow[r, hook] \arrow[d] & G(k, s) \times \mathbb{R}^s \arrow[ul, "\mathrm{pr}_2"'] \arrow[dl, "\mathrm{pr}_1"] \\
M \arrow[r, tail, "g"] & G(k, s) &
\end{tikzcd}$$

It is not hard to show that $E \cong g^*(E(k, s))$ (see [**98**, 20.8]), hence the name universal VB: By the universal property of the pull-back $g^*(E(k, s))$ the VB-monomorphism γ over g induces a VB-monomorphism $(p, \gamma) : E \rightarrowtail G^*(E(k, s))$ over id_M which is easily seen to be fibrewise onto, hence a VB-isomorphism by 27.11.

We now wish to show that for $s > m + k$ the formation of pull-backs provides a bijection between homotopy classes of maps $M \to G(k, s)$ and isomorphism classes of VB of fiber dimension k over M.

Since homotopies are mappings $f : M \times I \to N$ we should treat vector bundles over the (see 28.6) manifold $M \times I$ (with boundary $M \times \{0, 1\}$). Of course, we could also extend the homotopy to $M \times \mathbb{R} \to N$ in order to avoid manifolds with boundary, but $I := [0, 1]$ has the advantage of being compact.

27.24 Lemma. [65, p.89, 4.1.1].

Let $p : E \to M \times I$ be a vector bundle. Then for each $x \in M$ there is a neighborhood $U \subseteq M$, so that $E|_{U\times I}$ is trivial.

Proof. Since I is compact, there exist $0 = t_0 < \cdots < t_n = 1$ and neighborhoods U_i of x, such that E is trivial on a neighborhood of $U_i \times [t_i, t_{i+1}]$. Let $U := \bigcap U_i$. We show by induction on i that E is also trivial on a neighborhood of $U \times [0, t_i]$. It suffices to consider case $i = 2$. So be φ^j be trivializations along neighborhoods of $U \times [t_j, t_{j+1}]$. On the intersection for $j = 0$ and $j = 1$ - a neighborhood of $U \times \{t_1\}$ - we can consider the transition function $z \mapsto (\varphi^1_z)^{-1} \circ \varphi^0_z \in GL(k)$. We can expand its germ (by inflating the domain of definition) to the germ of an mapping $g : U \times [t_1, t_2] \to GL(k)$. This allows us to extend φ^0 by $(z, v) \mapsto \varphi^1(z, g(z) \cdot v)$ to a trivialization on a neighborhood of $U \times [t_0, t_2]$. □

To achieve this result globally on M we need the following two results:

27.25 Lemma (Homotopy extension property for germs). [65, p.90, 4.1.3].

Let the adjacent commutative diagram be given with $U \subseteq N$ open and $A \subseteq U$ closed in N. Then there is an $\tilde{h} : N \times I \to M$ with $\tilde{h}|_{A\times I} = h|_{A\times I}$ and $\tilde{h}|_{N\times\{0\}} = f$.

$$\begin{tikzcd}
U\times\{0\} \arrow[r, hook] \arrow[d, hook] & U\times I \arrow[d, "h"] \\
N\times\{0\} \arrow[r, "f"] & M
\end{tikzcd}$$

Proof. Let $\rho : N \to [0, 1]$ be C^∞ with $\rho|_A = 1$ and $\operatorname{supp}(\rho) \subseteq U$. Then $\tilde{h} : N \times I \to M$ defined by

$$\tilde{h}(x, t) := \begin{cases} f(x, 0) & \text{for } x \notin \operatorname{supp}(\rho) \\ h(x, \rho(x)\, t) & \text{for } x \in U \end{cases}$$

satisfies the desired. □

$$\begin{tikzcd}
U\times\{0\} \arrow[rrr, hook] \arrow[dd, hook] & & & U\times I \arrow[ddd, "h"] \\
& A\times\{0\} \arrow[r, hook] \arrow[ul, hook] \arrow[dl, hook] & A\times I \arrow[ur, hook] \arrow[d, hook] & \\
N\times\{0\} \arrow[rr, hook] \arrow[drrr, bend right, "f"'] & & N\times I \arrow[dr, dotted, "\tilde{h}"] & \\
& & & M
\end{tikzcd}$$

27.26 Globalization lemma. [65, p.53, 2.2.11].

Let X be a set and $\mathcal{B}$ a set of subsets of X that contains X and is closed under taking unions. Furthermore, let Φ be a functor of the inclusion-ordered category $\mathcal{B}$ to the category of sets, so there is a mapping $\Phi(B_2) \to \Phi(B_1)$ for $B_2 \supseteq B_1$ in $\mathcal{B}$. Assume that the functor is continuous, i.e. if $\mathcal{B}' \subseteq \mathcal{B}$ is linearly ordered, then

$$\Phi(\bigcup \mathcal{B}') = \varprojlim_{B\in\mathcal{B}'} \Phi(B) := \Big\{x \in \prod_{B\in\mathcal{B}'} \Phi(B) : x_{B'} \overset{\Phi}{\longmapsto} x_B \text{ for all } B' \supseteq B \text{ in } \mathcal{B}'\Big\};$$

and is locally extendable, i.e.

$$\forall\, x \in X\ \exists\, B_x \in \mathcal{B}\ \forall\, B \in \mathcal{B} : \Phi(B \cup B_x) \twoheadrightarrow \Phi(B) \text{ is onto and } x \in B_x.$$

Then $\Phi(X) \twoheadrightarrow \Phi(B)$ is onto for all $B \in \mathcal{B}$.
If the functor Φ is in addition not trivial, i.e. $\exists\, B \in \mathcal{B}$ with $\Phi(B) \neq \emptyset$, then $\Phi(B) \neq \emptyset$ for each $B \in \mathcal{B}$.

Proof. Let $B_0 \in \mathcal{B}$ and $y_0 \in \Phi(B_0)$. Then $\mathcal{M} := \{(B, y) : B_0 \subseteq B \in \mathcal{B},\ y \in \Phi(B),\ y \mapsto y_0\}$ is partially ordered by $(B_1, y_1) \preceq (B_2, y_2) :\Leftrightarrow B_1 \subseteq B_2$ and $y_2 \mapsto y_1$.

Each linearly ordered subset $\mathcal{M}_0$ of $\mathcal{M}$, because of the continuity of Φ, has a maximal element (B_∞, y_∞) with $B_\infty := \bigcup \mathcal{B}_0$, where $B_\infty := \bigcup_{M\in\mathcal{M}_0} \mathrm{pr}_1(M)$ and $y_\infty := (\mathrm{pr}_2(M))_{M\in\mathcal{M}_0} \in \varprojlim_{B\in\mathcal{B}_0} \Phi(B) = \Phi(B_\infty)$.

According to Zorn's Lemma, there is a maximal element of $\mathcal{M}$, which we again refer to as (B_∞, y_∞). Suppose $B_\infty \subset X$. Let $x \in X \setminus B_\infty$. Since Φ is locally extendable a $B_x \in \mathcal{B}$ exists with $x \in B_x$ and $\Phi(B \cup B_x) \twoheadrightarrow \Phi(B)$ onto for all $B \in \mathcal{B}$. Let $B' := B_\infty \cup B_x \supseteq B_\infty \cup \{x\} \supset B_\infty$ and y' be an inverse image in $\Phi(B')$ of $y_\infty \in \Phi(B_\infty)$. Then (B', y') contradicts the maximality.

If $\Phi(B_0) \neq \emptyset$, then also $\Phi(X) \neq \emptyset$ because $\Phi(X) \twoheadrightarrow \Phi(B_0)$ is onto, and hence $\Phi(B) \neq \emptyset$ for each $B \in \mathcal{B}$, since by assumption there is a mapping $\Phi(X) \to \Phi(B)$. □

27.27 Theorem. [**65**, p.90, 4.1.5].

Each vector bundle $p : E \to M \times I$ is isomorphic to $E|_{M\times\{0\}} \times I$.

In the following we will briefly write $E|_M = \mathrm{ins}_0^*(E)$ instead of $E|_{M\times\{0\}}$, if it is clear that this pull-back is meant with respect to the insertion ins_0. The isomorphism in the theorem can be chosen as identity on $E|_M$:

$$\begin{tikzcd}
E \arrow[rr, tail, two heads, "\cong", "\varphi"'] \arrow[dd] & & E|_M \times I \arrow[rr, tail, two heads, "\cong", "\varphi^{-1}\times I"'] \arrow[dd] & & E|_M \times I \arrow[dd] & \\
& E|_M \arrow[ul, hook] \arrow[rr, tail, two heads, "\cong", "\varphi"'] \arrow[dd] & & E|_M \arrow[ul, hook] \arrow[rr, tail, two heads, "\cong", "\varphi^{-1}"'] \arrow[dd] & & E|_M \arrow[ul, hook] \arrow[dd] \\
M \times I \arrow[rr, equal] & & M \times I \arrow[rr, equal] & & M \times I & \\
& M \arrow[ul, hook, "\mathrm{ins}_0"] \arrow[rr, equal] & & M \arrow[ul, hook] \arrow[rr, equal] & & M \arrow[ul, hook]
\end{tikzcd}$$

Proof. We consider a locally finite covering $\mathcal{A}$ with closed sets $A \subseteq M$, for which E is trivial along a neighborhood of $A \times I$: Coverings with such sets exist because of 27.24 and they can be chosen locally finite because M is paracompact. Let $\mathcal{B}$ be the set of all (automatically closed, see [**98**, EX6]) unions of subsets of $\mathcal{A}$.

For $B \in \mathcal{B}$, we consider pairs $(f, \tilde{B})$, where $\tilde{B} \subseteq M$ is an open neighborhood of B, and $f : E|_{\tilde{B}\times I} \cong E|_{\tilde{B}} \times I$ is a VB isomorphism whose restriction to $E|_{\tilde{B}\times\{0\}}$ is the identity. With $\Phi(B)$ we denote the set of germs of such VB isomorphisms, i.e. the equivalence classes of such pairs, where $(f_1, \tilde{B}_1) \sim (f_2, \tilde{B}_2)$ if a neighborhood $\tilde{B} \subseteq \tilde{B}_1 \cap \tilde{B}_2$ of B exists with $f_1|_{\tilde{B}\times I} = f_2|_{\tilde{B}\times I}$. Obviously the functor Φ is continuous and it is locally extendable: Namely let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then we have to show that $\Phi(B \cup A) \to \Phi(B)$ is onto, i.e. that the germ of a vector bundle isomorphism $f : E \cong E|_M \times I$ over $B \times I$ can be extended to one over $(B \cup A) \times I$. Let $f : E|_{\tilde{B}\times I} \cong E|_{\tilde{B}} \times I$ be the vector bundle isomorphism with open $\tilde{B} \supseteq B$. By assumption, $E|_{\tilde{A}\times I} \cong (\tilde{A} \times I) \times \mathbb{R}^k$ for an open $\tilde{A} \supseteq A$. In order to extend f to a germ over $(B \cup A) \times I$, it suffices to extend the restricted germ of f over $(B \cap A) \times I$ to one over A. With $E|_{\tilde{A}\times I}$ also $E|_{\tilde{A}}$, $E|_{(\tilde{A}\cap\tilde{B})\times I}$, and $E|_{\tilde{A}\cap\tilde{B}} \times I$ are trivial bundles, and the restriction of f between the latter two bundles is a VB isomorphism, which is thus described by a map $g : (\tilde{A} \cap \tilde{B}) \times I \to GL(k)$ with $g(z, 0) = \mathrm{id}\ \forall\ z$. By 27.25, its germ on $(A \cap B) \times I$ can be expanded to a map $\tilde{g} : \tilde{A} \times I \to GL(k)$, which in turn defines a VB isomorphism $E|_{\tilde{A}\times I} \cong E_{\tilde{A}} \times I$, that extends the germ from f to $(B \cap A) \times I$.

Because of 27.24 the functor Φ is not trivial and hence, by 27.26, $\Phi(M) \neq \emptyset$ and $f : E \to E|_M \times I$ is a VB-isomorphism for (each) $(f, M) \in \Phi(M)$. □

27.28 Proposition (The pull-back is homotopy invariant). [**65**, p.97, 4.2.4]**.**

Let $f_0, f_1 : N \to M$ be homotopic and $p : E \to M$ be a vector bundle. Then $f_0^* E \cong f_1^* E$ as vector bundles over N.

Proof. Let $H : N \times I \to M$ be a homotopy between f_0 and f_1. For $j \in \{0,1\}$, we have $H^*E \cong (H^*E)|_{N\times\{j\}} \times I \cong \mathrm{ins}_j^*(H^*E) \times I = f_j^* E \times I$ by 27.27 and thus

$$f_0^* E \cong (f_0^* E \times I)|_N \cong (f_1^* E \times I)|_N \cong f_1^* E. \quad \square$$

27.29 Corollary. [**65**, p.97, 4.2.5]**.**

Every vector bundle over a contractible space is trivial.

Proof. For contractible spaces M, the identity id on M is homotopic to a constant mapping const_{x_0}. By 27.28 $E = \mathrm{id}^* E \cong \mathrm{const}_{x_0}^* E = M \times E_{x_0}$, thus is trivial. $\square$

27.30 Definition (Function space topologies).

For $0 \le r < \infty$ we need two types of topologies on the space $C^r(M,N)$ of C^r-mappings between manifolds M and N, which we assume to be σ-compact:

The COARSE (or COMPACT OPEN) C^r-TOPOLOGY has as neighborhood basis of $f \in C^r(M,N)$ the sets

$$\Big\{ g \in C^r(M,N) : g(K) \subseteq \operatorname{im}\psi \text{ and } \forall\, x \in K \;\forall\, k \le r: \\ \Big\| (\psi^{-1} \circ g \circ \varphi)^{(k)}(x) - (\psi^{-1} \circ f \circ \varphi)^{(k)}(x) \Big\| < \varepsilon \Big\},$$

where φ is a chart of M, ψ is one of N, $K \subseteq \operatorname{im}\varphi$ is compact with $f(K) \subseteq \operatorname{im}\psi$, and $\varepsilon > 0$.

The FINE or WHITNEY C^r-TOPOLOGY has as neighborhood basis of $f \in C^\infty(M,N)$ the sets

$$\Big\{ g \in C^\infty(M,N) : g(K_i) \subseteq \operatorname{im}\psi_i \text{ and } \forall\, i \;\forall\, x \in K_i \;\forall\, k \le r: \\ \Big\| (\psi_i^{-1} \circ g \circ \varphi_i)^{(k)}(x) - (\psi_i^{-1} \circ f \circ \varphi_i)^{(k)}(x) \Big\| < \varepsilon_i \Big\},$$

where φ_i are charts of M whose images form a locally finite family, ψ_i are charts of N, $K_i \subseteq \operatorname{im}\varphi_i$ are compact with $f(K_i) \subseteq \operatorname{im}\psi_i$, and $\varepsilon_i > 0$.

The coarse (resp. fine) C^∞-topology on $C^\infty(M,N)$ is defined as initial topology with respect to the corresponding C^r-topologies for $r \in \mathbb{N}$.

On $C^r(M,N)$, for $0 \le r \le \infty$, the course C^r-topology is completely-metrizable, see [**65**, p.62, 2.4.4.a], and the fine C^∞-topology has the Baire property, see [**65**, p.62, 2.4.4.b]. Recall that a subset $B \subseteq X$ of a topological space X is called

- NOWHERE DENSE iff the interior of its closure is empty;
- MEAGER iff it is a countable union of nowhere dense subsets, or equivalently, iff it is contained in the countable union of closed sets with empty interior;
- RESIDUAL iff it is the complement of a meager subset, or equivalently, if contains a countable intersection of open and dense subsets of X.

A topological space is called BAIRE iff every residual subset is dense, or equivalently, iff every meager subset has empty interior.

27.31 Transversality Theorem of Thom. [65, p.74] **or** [47, 4.12]**.**

Let M and N be smooth manifolds and $L \subseteq N$ be a regular submanifold.

1. *Then, the set $\pitchfork^L(M,N)$ of the smooth mappings $M \to N$ being transversal to L is residual and hence dense in $C^\infty(M,N)$ with respect to the coarse and also the Whitney C^∞-topologies.*
2. *If L is closed in N and A is a closed subset of M, then the set $\pitchfork_A^L(M,N)$ of smooth mappings $M \to N$ being transversal to L along A is open and dense in $C^\infty(M,N)$ with respect to the Whitney C^∞-topology and for compact A also with respect to the coarse C^∞-topology.*

Here a mapping $f \in C^\infty(M,N)$ is called TRANSVERSAL to L along A if $T_{f(x)}N = T_{f(x)}L + \operatorname{im} T_x f$ for each $x \in A$ with $f(x) \in L$. For the proof we need some preparation:

27.32 Globalization lemma for denseness. [65, p..75, 3.2.2]**.**

For manifolds M and N and $r \in \mathbb{N}$ let a mapping Φ be given, which associates to each tuple (A,U,V) of open subsets $U \subseteq M$, $V \subseteq N$ and closed $A \subseteq M$ with $A \subseteq U$ a subset $\Phi_A(U,V) \subseteq C^\infty(U,V)$ with the following properties:

Functorality: *For such tuples (A,U,V) and (A',U',V') with $A' \subseteq A$, $U' \subseteq U$ and $V' \subseteq V$ we have $\{f|_{U'} : f \in \Phi_A(U,V) \text{ und } f(U') \subseteq V'\} \subseteq \Phi_{A'}(U',V')$.*

Localization: *For each tuple (A,U,V), we have that $f \in C^\infty(U,V)$ is in $\Phi_A(U,V)$ if there are such tuples (A_i,U_i,V_i) and $f_i \in \Phi_{A_i}(U_i,V_i)$ with $A \subseteq \bigcup_i A_i$ and $f = f_i$ on a neighborhood of A_i for all i.*

Local open and denseness: *There are open coverings $\mathcal{U}$ of M and $\mathcal{V}$ of N, such that for tuple each (A,U,V) as above but with A compact and U and V contained in elements of $\mathcal{U}$ and $\mathcal{V}$ the set $\Phi_A(U,V)$ is open and dense in $C^\infty(U,V)$ in the coarse C^r-topology.*

Then for closed subsets $A \subseteq M$ we have:

1. *$\Phi_A(M,N)$ is open and dense in $C^\infty(M,N)$ for the Whitney C^r-topology.*
2. *If A is compact, then $\Phi_A(M,N)$ is open (and dense) for the coarse C^r-topology.*

Proof. By assumption, open coverings $\mathcal{U}$ of M and $\mathcal{V}$ of N exist with $\Phi_K(U,V) \subseteq C^\infty(U,V)$ open and dense with respect to the coarse topology provided $K \subseteq U$ is compact and U and V are contained in elements of $\mathcal{U}$ and $\mathcal{V}$.

Openness: Let $f \in \Phi_A(M,N)$. The sets $U \cap f^{-1}(V)$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$ form a covering of M and can therefore be refined to a locally finite countable covering $\{U_i : i \in \Lambda\}$ because of paracompactness, and if A is compact already a finite index set Λ is enough to cover A. We may choose compact $K_i \subseteq U_i$ which also cover A. Let V_i be the corresponding $V \in \mathcal{V}$ with $U_i \subseteq U \cap f^{-1}(V)$. Put

$$\mathcal{C} := \{g \in C^\infty(M,N) : g|_{U_i} \in \Phi_{K_i}(U_i,V_i) \text{ for all } i\}.$$

Because of localization, $\mathcal{C} \subseteq \Phi_A(M,N)$, and due to functorality, $f \in \mathcal{C}$. By the openness of $\Phi_{K_i}(U_i,V_i) \subseteq C^\infty(U_i,V_i)$ in the coarse C^r-topology, $\mathcal{C} \subseteq \Phi_A(M,N) \subseteq C^\infty(U,V)$ is also open in the coarse C^r-topology (for finite Λ), respectively in the Whitney C^r-topology.

Denseness Let $f \in C^\infty(M,N)$ and

$$\mathcal{C} := \Big\{ g \in C^\infty(M,N) : g(K_i) \subseteq V_i \text{ and } \forall\, i\ \forall\, x \in K_i,\ \forall\, k \le r: \\ \Big\| (\psi_i^{-1} \circ g \circ \varphi_i)^{(k)}(x) - (\psi_i^{-1} \circ f \circ \varphi_i)^{(k)}(x) \Big\| < \varepsilon_i \Big\},$$

a typical neighborhood in the Whitney topology, i.e. φ_i are charts of M whose images $U_i := \operatorname{im} \varphi_i$ form a locally finite family and are w.l.o.g. subsets of elements of $\mathcal{U}$; the $K_i \subseteq U_i$ are compact and w.l.o.g. a covering of A; ψ_i are charts of N with $f(K_i) \subseteq V_i := \operatorname{im}(\psi_i)$ and w.l.o.g. the V_i are subsets of elements of $\mathcal{V}$ and $\operatorname{dom} \psi_i \subseteq \mathbb{R}^n$ are convex; and finally $\varepsilon_i > 0$.

For fixed $i \in \Lambda$, let $U := U_i \cap f^{-1}(V_i)$, i.e. $K_i \subseteq U$. Let $\rho \in C^\infty(M,[0,1])$ with $\rho = 1$ locally around K_i and $\operatorname{supp}(\rho)$ compact in U. For $g \in C^\infty(U, V_i)$ the mapping

$$\Gamma(g) : x \mapsto \begin{cases} f(x) + \rho(x)(g(x) - f(x)) & \text{for } x \in U \\ f(x) & \text{for } x \notin \operatorname{supp}(\rho) \end{cases}$$

in well-defined in $C^\infty(M,N)$, where we identified V_i with the convex domain of ψ_i. If $g \to f|_U$ with respect to the coarse topology, then $\Gamma(g) \to f$ in the Whitney C^r-topology because the net only varies on the compact set $\operatorname{supp}(\rho) \subseteq U$. Since $\Phi_{K_i}(U, V_i)$ is dense by the local denseness property, we may choose $g \in \Phi_{K_i}(U, V_i)$ close enough to f and hence $\Gamma(g) \in \mathcal{C}$. Because of $g = \Gamma(g)$ locally around K_i, we have $\Gamma(g) \in \Phi_{K_i}(M,N)$ by the localization property. Thus, $\Phi_{K_i}(M,N)$ is (open and) dense in $C^\infty(M,N)$ with respect to the Whitney topology, hence $\bigcap_i \Phi_{K_i}(M,N) \subseteq \Phi_A(M,N)$ is residual and thus dense by the Baire property (see [**65**, p.62, 2.4.4.b]). □

27.33 Lemma. [**65**, p.76, 3.2.3]**.**

Let K be a compact subset of a manifold M and $1 \le r < \infty$. Then

$$\pitchfork_K^{\mathbb{R}^k}(M, \mathbb{R}^n) := \Big\{ f \in C^\infty(M, \mathbb{R}^n) : f \text{ is transversal along } K \text{ to } \mathbb{R}^k \subseteq \mathbb{R}^n \Big\}$$

is open and dense in $C^\infty(M,\mathbb{R}^n)$ with respect to the coarse C^r-topology.

Note that the same proof works for the closed half space $\{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 \ge 0\}$ instead of $\mathbb{R}^k$.

Proof. Let $\operatorname{pr} : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^k$ be the canonical projection. For $x \in M$, the map $f \in C^\infty(M,\mathbb{R}^n)$ is transversal to $\mathbb{R}^k$ along $\{x\}$ if and only if $f(x) \notin \mathbb{R}^k$ or x is a regular point of $\operatorname{pr} \circ f$.

Openness: Let f be transversal to $\mathbb{R}^k$ along K. Then every point $x \in K$ has a neighborhood K_x s.t. $f(K_x) \cap \mathbb{R}^k = \emptyset$ or each point in K_x is a regular point of $\operatorname{pr} \circ f$. Thus there is a finite covering with such compact sets K_i. The set of all $g \in C^\infty(M,\mathbb{R}^n)$, satisfying this condition on K_i, is open in the coarse C^1-topology and contained in $\pitchfork_{K_i}^{\mathbb{R}^k}(M,\mathbb{R}^n)$. So $\pitchfork_K^{\mathbb{R}^k}(M,\mathbb{R}^n) = \bigcap_i \pitchfork_{K_i}^{\mathbb{R}^k}(M,\mathbb{R}^n)$ is open.

Denseness: Let $f \in C^\infty(M,\mathbb{R}^n)$. Then, by Proposition 11.15 of Sard, there is a sequence $y_i \to 0$ in $\mathbb{R}^n$ for which $\operatorname{pr}(y_i)$ is a regular value of $\operatorname{pr} \circ f$. Then the sequence of functions $f_i := f - y_i \in C^\infty(M,\mathbb{R}^n)$ converges to f in the coarse topology, and f_i is transversal to $\mathbb{R}^k$ along K, because if $f_i(x) \in \mathbb{R}^k$, i.e. $\operatorname{pr}(f(x)) = \operatorname{pr}(y_i)$, then x is a regular point of $\operatorname{pr} \circ f$. □

Proof of 27.31. Let $L \subseteq N$ be a regular submanifold and **closed for the moment**. Then

$$\Phi_A(U,V) := \pitchfork_A^L(U,V) := \Big\{ f \in C^\infty(U,V) : f \text{ is transversal along } A \text{ to } L \cap V \Big\}$$

fulfills the requirements of 27.32:

Functorality: Let (A, U, V) and (A', U', V') be corresponding tuples with $A' \subseteq A$, $U' \subseteq U$ and $V' \subseteq V$.
Then obviously $\{f|_{U'} : f \in \pitchfork_A^L(U, V) \text{ und } f(U') \subseteq V'\} \subseteq \pitchfork_{A'}^L(U', V')$.

Localization: Let (A, U, V) as well as (A_i, U_i, V_i) be corresponding tuples with $A \subseteq \bigcup_i A_i$. Furthermore, let $f \in C^\infty(U, V)$ and $f_i \in \Phi_{A_i}(U_i, V_i)$ with $f = f_i$ locally around A_i. Then $f \in \Phi_A(U, V)$, because let $f(x) \in L$ for an $x \in A$, then $x \in A_i \subseteq U_i$ for some i and $\operatorname{im}(T_x f) = \operatorname{im}(T_x f_i)$ together with $T_{f(x)}(L \cap V) = T_{f_i(x)}(L \cap V_i)$ generates $T_{f(x)}V$.

Local open and denseness: Let $U \subseteq M$ open, V a chart that describes L as submanifold or has empty intersection with it (i.e. w.l.o.g. $V \subseteq \mathbb{R}^n$ is open and $L \cap V = \mathbb{R}^k \cap V$) and let $A \subseteq U$ be compact. According to Lemma [**94**, 27.33], $\pitchfork_A^{\mathbb{R}^k}(U, \mathbb{R}^n)$ is open and dense in $C^\infty(U, \mathbb{R}^n)$ with respect to the coarse C^r-topology for $1 \leq r < \infty$, so also $\Phi_A(U, V) = \pitchfork_A^{\mathbb{R}^k}(U, V) = \pitchfork_A^{\mathbb{R}^k}(U, \mathbb{R}^n) \cap C^\infty(U, V)$ in the topological subspace $C^\infty(U, V)$ of $C^\infty(U, \mathbb{R}^n)$. Denseness can be seen as follows: Approximate $f \in C^\infty(U, V)$ by some $g \in \pitchfork_A^{\mathbb{R}^k}(U, \mathbb{R}^n)$, which we may assume to have locally around A values in V, and then modify g outside of A so that it is still in this neighborhood of f but has there also values in V.

Thus, by 27.32, the set $\pitchfork_A^L(M, N)$ is open and dense in the Whitney C^r-topology on $C^\infty(M, N)$ and if A is compact also in the coarse one. Since the corresponding C^∞-topologies are the union of these C^r-topologies, the result follows for them as well. This proves 27.31.2.

If L is **not closed**, then we choose a countable family of compact submanifolds L_i with boundary (the image of compact balls under charts) with $L = \bigcup_i L_i$. Then $\pitchfork_A^{L_i}(M, N)$ are open and dense in the Whitney topology by the first part of the proof, so $\pitchfork_A^L(M, N) = \bigcap_i \pitchfork_A^{L_i}(M, N)$ is residual.
Concerning the coarse topology, we use a covering of A with countably many compact sets of A_j. Then $\pitchfork_A^{L_i}(M, N) = \bigcap_j \pitchfork_{A_j}^{L_i}(M, N)$, where $\pitchfork_{A_j}^{L_i}(M, N) \subseteq C^\infty(M, N)$ is open and dense in coarse topology by what was shown above. □

27.34 Lemma. [**19**, p.160, 14.8].

Let $f : M \to \mathbb{R}^n$ be continuous and, as germ on a closed subset $A \subseteq M$, smooth. Then, arbitrarily close to f (in the fine C^0-topology) there is a smooth mapping $\tilde{f}$ with $\tilde{f} = f$ on A.

Proof. Let $\varepsilon : M \to \mathbb{R}$ be continuous with $\varepsilon(x) > 0$ for all x. Let f be C^∞ on a neighborhood U_∞ of A. For each $x \notin A$ let $U_x := \{x' \in M \setminus A : |f(x') - f(x)| < \varepsilon(x')\}$, an open neighborhood of x. Let $\{\rho_x : x \in \{\infty\} \cup (X \setminus A)\}$ be a partition of unity subordinated to the covering $\{U_\infty\} \cup \{U_x : x \in M \setminus A\}$ and let $\tilde{f} := \rho_\infty \cdot f + \sum_{x \neq \infty} \rho_x \cdot f(x)$. Clearly, $\tilde{f}$ is C^∞, $\tilde{f} = f$ on A, and for $x' \notin A$:

$$\begin{aligned} |\tilde{f}(x') - f(x')| &= \Big|\rho_\infty(x') f(x') + \sum_{x \neq \infty} \rho_x(x')\, f(x) - f(x') \sum_x \rho_x(x')\Big| \\ &\leq \sum_{\{x \neq \infty \,:\, x' \in U_x\}} |f(x') - f(x)| \cdot \rho_x(x') \leq \varepsilon(x') \sum_x \rho_x(x') = \varepsilon(x'). \quad \square \end{aligned}$$

The VB-monomorphisms $\varphi : M \times \mathbb{R}^k \rightarrowtail M \times \mathbb{R}^s$ obviously correspond bijectively to the smooth mappings $\varphi^\vee : M \to L_k(k, s)$. For these the following holds:

27.35 Theorem. [**65**, p.78, 3.2.6].

Let M be a m-dimensional manifold and $A \subseteq M$ closed.
Then any smooth germ $M \supseteq A \to L_k(k,s)$ with $s \geq m+k$ can be extended to a smooth mapping $M \to L_k(k,s)$.

Proof. By the extension theorem of Tietze f can be extended to a continuous mapping $\tilde f : M \to L(\mathbb{R}^k, \mathbb{R}^s) =: L(k,s)$. By 27.34 we may choose $\tilde f$ smooth. For $x \in A$ let $2\varepsilon_x := d(f(x), L(k,s) \setminus L_k(k,s)) > 0$ and $U_x := \{y \in M : |\tilde f(y) - \tilde f(x)| < \varepsilon_x\} := \tilde f^{-1}(U_{\varepsilon_x}(f(x)))$. There is a locally finite countable refinement by relatively compact charts U_i. Choose $\rho \in C^\infty(M,[0,1])$ with $\rho = 1$ locally around A and $\operatorname{supp}\rho \subseteq \bigcup_i U_i$ and choose compact subsets $K_i \subseteq U_i$, s.t. $\operatorname{supp}\rho \subseteq \bigcup_i K_i$. We have $L(k,s) \setminus L_k(k,s) = \bigcup_{r<k} L_r(k,s)$, where $L_r(k,s)$ is a $r(k+s-r)$-dimensional submanifold of $L(k,s)$ by 3.8. By 27.31.1 we find $g : M \to L(k,s)$ in the Whitney-C^0-neighborhood $\{g : |g(x) - \tilde f(x)| < \varepsilon_i \ \forall\, x \in K_i\}$ being transversal to $L_r(k,s)$ for all $r < k$. Because of

$$\begin{aligned}\dim L(k,s) - \dim L_r(k,s) &= ks - r(k+s-r) = (k-r)(s-r)\\ &\geq (k-(k-1))(s-(k-1)) = s-k+1 \geq m+1 > m\end{aligned}$$

this is only possible if g does not meet the set $\bigcup_{r<k} L_r(k,s)$, i.e. $g(M) \subseteq L_k(k,s)$.
Thus $\tilde g := \rho \cdot \tilde f + (1-\rho)\cdot g : M \to L_k(k,s)$ is the desired mapping:
In fact, $\tilde g = \tilde f = f$ on A and $\tilde g = g$ on $M \setminus \bigcup_i K_i$. Whereas, for $y \in K_i \subseteq U_i \subseteq U_{x_i}$ we have also $\tilde g(y) \in L_k(k,s)$ because

$$|\tilde g(y) - f(x_i)| \leq |(\tilde g - \tilde f)(y)| + |\tilde f(y) - \tilde f(x_i)| = (1-\rho(y))|(g-\tilde f)(y)| + |\tilde f(y) - \tilde f(x_i)| < 2\varepsilon_i.$$

Note that on order that $\tilde g$ is not only an extension of $f|_A$ but even of the germ of f along A, we only have to replace A by a closed neighborhood contained in the domain of f. □

27.36 Proposition (Extending germs of VB-monos). [65, p.99, 4.3.1].

Let $E \to M$ be a vector bundle with fiber dimension k and m-dimensional base M. Let furthermore, A be closed and $U \supseteq A$ open in M and $\varphi : E|_U \rightarrowtail U \times \mathbb{R}^s$ be a VB-monomorphism with $k+m \leq s$. Then, there exists a VB-monomorphism $\tilde\varphi : E \rightarrowtail M \times \mathbb{R}^s$ which coincides with φ on a neighborhood of A.

Proof. If $E \to M$ is trivial, then φ defines a mapping $U \to L_k(k,s)$. Because of $s \geq m+k$, there is an extension $M \to L_k(k,s)$ of the germ of this mapping by 27.35. Thus, we get a VB-monomorphism $E = M \times \mathbb{R}^k \rightarrowtail M \times \mathbb{R}^s$ over id_M.

The general case follows by applying the globalization lemma 27.26: Namely, let $\mathcal{B}_0$ be a locally finite covering of M by closed sets $B_0 \subseteq M$, for which $E|_{U_0}$ is trivial for some neighborhood U_0 of B_0. Let $\mathcal{B}$ be the set of all (automatically closed) unions of sets from $\mathcal{B}_0$. For $B \in \mathcal{B}$, let $\Phi(B)$ be the set of germs of VB-monomorphisms over B, i.e. there exists a neighborhood $V \subseteq M$ of B and a VB-homomorphism $\tilde\varphi : E|_V \rightarrowtail V \times \mathbb{R}^s$, such that it coincides over a neighborhood $W \subseteq U \cup V$ of $A \cap B$ with φ. By shrinking U and V we may assume that $W = U \cap V$ and hence $\tilde\varphi$ on $E|_V$ extends via φ on $E|_U$ to a VB-monomorphism $E|_{U\cup V} \rightarrowtail (U \cup V) \times \mathbb{R}^s$: In fact, $A' := A \setminus W$ and $B' := B \setminus W$ are disjoint closed subsests, so there exist disjoint open neighborhoods U' and V' by normality. Hence $U'' := U' \cup W$ and $V'' := V' \cup W$ are open neighborhoods of A and B (e.g. $U'' \supseteq A' \cup W = (A \setminus W) \cup W \supseteq A$) and $U'' \cap V'' = (U' \cap V') \cup W = \emptyset \cup W = W$.

The functor Φ is obviously not trivial and continuous. It is also **locally extendable**: Namely let $x \in B_0 \in \mathcal{B}_0$, i.e. $E|_{U_0}$ is trivial for some neighborhood U_0 of B_0. Let $B \in \mathcal{B}$ and $\tilde\varphi \in \Phi(B)$, i.e. $\tilde\varphi$ has as representant a VB-monomorphism

$\tilde\varphi : E|_{V\cup U} \rightarrowtail (V\cup U)\times\mathbb{R}^s$ for neighborhoods U of A and V of B which extends the germ $\varphi : E|_U \rightarrowtail U\times\mathbb{R}^s$. Since $E|_{U_0}$ is trivial, we may extend the germ of the VB-monomorphism $\tilde\varphi$ over the closed set $(A\cup B)\cap B_0$ to a VB-monomorphism over U_0 by the special case and as before we may assume that it agrees with $\tilde\varphi$ on the intersection of neighborhoods of $A\cup B$ and of B_0, thus describes together with $\tilde\varphi$ the germ of a VB-homomorphism over $A\cup B\cup B_0$, i.e. an element in $\Phi(B\cup B_0)$ with image $\tilde\varphi\in\Phi(B)$.

Hence, the result follows from 27.26. □

27.37 Proposition. [65, p.100, 4.3.2]**.**

Let $E\to M\times I$ be a vector bundle with fiber dimension k and m-dimensional M. For $s > k+m$ and $i\in\{0,1\}$, let $\varphi_i : E|_{M\times\{i\}} \rightarrowtail (M\times\{i\})\times\mathbb{R}^s$ be a VB-monomorphism. Then there is an extension $\varphi : E \rightarrowtail M\times I\times\mathbb{R}^s$ of $\varphi_0\cup\varphi_1$ to a VB-monomorphism over $M\times I$.

Proof. By 27.27, we may extend each φ_i to a VB-monomorphism

$$E \cong E|_{M\times\{i\}}\times I \xrightarrow{\varphi_i\times I} (M\times\{i\})\times\mathbb{R}^s\times I \cong (M\times I)\times\mathbb{R}^s.$$

This gives us an extension $\varphi : E|_U \rightarrowtail U\times\mathbb{R}^s$ to a VB-monomorphism over the open neighborhood $U := M\times(I\setminus\{\frac{1}{2}\})$ of $A := M\times\{0,1\}\subseteq M\times I$. Its germ over A can be extended into a global VB-monomorphism by 27.36. □

27.38 Theorem. [65, p.100, 4.3.4]**.**

Let M be an m-dimensional manifold.
For $s\ge k+m$ and each vector bundle $E\to M$ with fiber dimension k, there is a classifying map $g : M\to G(k,s)$, i.e. $E\cong g^*(E(k,s))$.
For $s>k+m$ the assignment $g\mapsto g^*(E(k,s))$ is a bijection between the homotopy classes of maps $g\in C^\infty(M,G(k,s))$ and isomorphism classes of vector bundles $E\to M$ of fiber dimension k.

Proof. By 27.28, taking the pull-back is a **well-defined** mapping $g\mapsto g^*(E(k,s))$ for the corresponding classes, i.e. homotopic mappings induce isomorphic pull-backs.

By 27.36 there exists a VB-monomorphism $E\rightarrowtail M\times\mathbb{R}^s$ for $s\ge m+k$ and thus by 27.23 a classifying map $g : M\to G(k,s)$. This shows the **surjectivity**.

Injectivity: Let two bundles over M be isomorphic and g a classifying mapping of one of the two bundles. Then, of course, this also classifies the other bundle. Remains to show that each two classifying maps $g^i : M\to G(k,s)$ of a vector bundle $E\to M$ are homotopic. As such they induce VB-monomorphisms

$$\varphi^i : E\cong (g^i)^*(E(k,s)) = M\times_{G(k,s)} E(k,s) \hookrightarrow M\times_{G(k,s)} G(k,s)\times\mathbb{R}^s \cong M\times\mathbb{R}^s$$

over M with $g^i(x) = \operatorname{im}(\varphi^i_{(x,i)})$. By 27.37 the VB-monomorphism

$$\varphi^0\cup\varphi^1 : (E\times I)|_{M\times\{0,1\}} = E\times\{0,1\} \rightarrowtail (M\times\{0,1\})\times\mathbb{R}^s = ((M\times I)\times\mathbb{R}^s)|_{M\times\{0,1\}}$$

extends to a VB-monomorphism $\varphi : E\times I \rightarrowtail (M\times I)\times\mathbb{R}^s$ which induces a homotopy $g : M\times I\to G(k,s)$ between the g^i by virtue of $g(x,t) := \operatorname{im}(\varphi_{(x,t)} : E_x = (E\times I)_{(x,t)} \rightarrowtail \mathbb{R}^s)$. □

Now we return to the investigation of orientability.

27.39 Remarks.

1. If two of the following three vector bundles are orientable, so is the third one: $E_1 \to M$, $E_2 \to M$, $E_1 \oplus E_2 \to M$. From fiberwise orientations of two of these bundles it is easy to construct an orientation on the third.

2. Let $E_0 \xrightarrow{i} E_1 \xrightarrow{p} E_2$ be a short exact sequence of vector bundles. If two of the bundles are orientable, then also the third one (use that every short exact sequence of vector bundles splits by 27.16, that is $E_1 \cong E_0 \oplus E_2$, and then apply 1).

3. If a vector bundle $E \to M$ is orientable and $f : N \to M$ is smooth, then the induced bundle $f^*(E) \to N$ is orientable (choose the orientation of $E_{f(x)}$ on $(f^*(E))_x \cong E_{f(x)}$).

4. If 2 of the following 3 objects are orientable, so is the third one: $E \to M$ as a vector bundle, E as a manifold, M as a manifold: Use the short exact sequence $p^*(E) \to TE \to p^*(TM)$ from 27.17, as well as 2, 3, and the fact that $E \to M$ is the $0 : M \hookrightarrow E$-induced bundle to $p^*(E) \to E$ and $TM \to M$ is the $0 : M \hookrightarrow E$-induced bundle to $p^*(TM) \to E$, so M is orientable (by 27.3) if and only if $p^*(TM) \to E$ is it, and E if and only if $p^*(E) \to E$ is.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & p^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ M \arrow[r, "0"] \arrow[rr, bend right, "\mathrm{id}"'] & E \arrow[r, "p"] & M \end{tikzcd} \qquad \begin{tikzcd} TM \arrow[r] \arrow[d, "\pi_M"'] & p^*(TM) \arrow[r] \arrow[d] & TM \arrow[d, "\pi_M"] \\ M \arrow[r, "0"] \arrow[rr, bend right, "\mathrm{id}"'] & E \arrow[r, "p"] & M \end{tikzcd}$$

5. TM is always orientable as a manifold; but as a vector bundle $TM \to M$ is orientable if and only if M is orientable as a manifold (by 27.3):

   In the case of $E = TM$ and $p = \pi_M$, the sequence from 27.17 used in 4 is reduced to $\pi^*(TM) \to T(TM) \to \pi^*(TM)$, so as a vector bundle $T^2M \to TM$ is isomorphic to $\pi^*(TM) \oplus \pi^*(TM)$ and the sum of identical bundles is always orientable: If one chooses the same orientation on the fibers of the two summands, this obviously results in an orientation on the sum, which is independent on the respective choice.

27.40 Examples.

1. Vector bundles with one-dimensional fibers are orientable if and only if they are trivial (a trivialization is given by oriented unit (with respect to some metric) vectors). The vector bundle Möb $\to S^1$ is not orientable (since it is not trivial), nor is the Möbius band as manifold after remark 27.39.4. All 1-dimensional manifolds are orientable.

2. Each complex vector bundle is orientable, because $GL(\mathbb{C}^n) \subseteq GL_+(\mathbb{R}^{2n})$ by [**86**, 14.14] or also because $GL(\mathbb{C}^n)$ is connected and contains the orientation preserving identity. The tangent bundle of a complex manifold is a complex vector bundle and therefore orientable. Thus, every complex manifold itself is orientable according to 27.3.

3. A 2-dimensional manifold is orientable if and only if it carries a complex structure, see [**29**].

4. Let $E \to M$ be a vector bundle with simply connected base manifold M, then the vector bundle $E \to M$ is orientable (but not necessarily trivial as $TS^2 \to S^2$ shows): Along curves in the basis we can prolong the orientation of the fibers independently on the choosen curve, since the base space is simply connected. In particular, any simply connected manifold is orientable.

27.41 Lemma (Orientability of inverse images).

Let $f : M \to N$ be smooth and transversal to a regular submanifold $L \subseteq N$. If M and $(TL)^\perp \to L$ are orientable, so is $f^{-1}(L)$.
If L is a single point or both L and N are orientable, then the normal bundle $(TL)^\perp \to L$ is orientable.

Proof. By definition the fibers of the normal bundle $(TL)^\perp$ are T_yN/T_yL for $y \in L$, so we have the short exact sequence $TL \hookrightarrow (TN)|_L \twoheadrightarrow (TL)^\perp$ of VB over L.

If L and N are orientable, then also the bundles $TL \to L$ and $TN \to N$ and thus by 27.39.3 also the pull-back bundle $(TN)|_L$ and finally $(TL)^\perp \to L$ as well by 27.39.2.

The special case for $L = \{y\}$ follows, since $(TL)^\perp$ is trivial as VB over the single-point space and thus orientable, or also because we can replace N with an (oriented) chart neighborhood.

$$\begin{tikzcd}
 & & TM \arrow[rr, "Tf"] \arrow[ddd] & & TN \arrow[ddd] \\
 & TM|_{f^{-1}L} \arrow[ur, hook] \arrow[rr] \arrow[ddd] & & TN|_L \arrow[ur, hook] \arrow[ddd] & \\
T(f^{-1}L) \arrow[ur, hook] \arrow[rr] \arrow[ddr] & & TL \arrow[ur, hook] \arrow[ddr] & & \\
 & & M \arrow[rr, "f"] & & N \\
 & f^{-1}L \arrow[ur, hook] \arrow[rr, "f|_{f^{-1}L}"] & & L \arrow[ur, hook] &
\end{tikzcd}$$

The map is $T(f^{-1}L) \to (TM)|_{f^{-1}L}$ a VB monomorphism over $f^{-1}L$ and furthermore $(TM)|_{f^{-1}L} \xrightarrow{Tf} TN|_L \to (TL)^\perp$ a VB epimorphism over $f|_{f^{-1}L} : f^{-1}L \to L$, because for $x \in f^{-1}(L)$, $y = f(x)$ and $\dot w \in (TL)^\perp_y := T_yN/T_yL$ there exists a $w \in T_yN$ with $\dot w = [w]$ and, because of transversality, there exist $v \in T_xM$ and $v' \in T_yL$ with $w = T_xf \cdot v + v'$. So $[T_xf \cdot v] = [T_xf \cdot v + v'] = [w] = \dot w$. Fiberwise the kernel of this epimorphism is $\{v \in T_xM : [T_xf \cdot v] = 0\} = \{v \in T_xM : T_xf \cdot v \in T_{f(x)}L\} = (T_xf)^{-1}(T_{f(x)}L) = T_x(f^{-1}L)$ by 27.9.

Thus, $T(f^{-1}L) \rightarrowtail (TM)|_{f^{-1}L} \twoheadrightarrow f^*((TL)^\perp)$ is a short exact sequence of vector bundles with orientable (according to 27.39.3) vector bundles $(TM)|_{f^{-1}L}$ and $f^*((TL)^\perp)$. Thus $T(f^{-1}L) \to f^{-1}L$ is orientaable, i.e. $f^{-1}(L)$ is orientable. □

27.42 Examples.

1. Each S^n is orientable by 27.41.
2. All compact 2-dimensional manifolds in $\mathbb{R}^3$ are orientable, see the Classification Theorem 1.2 and exercise [**98**, 1].
3. $\mathbb{P}^1 \cong S^1$ is orientable. The projective surface $\mathbb{P}^2$ contains a Möbius strip as an open part, so it (and all the other surfaces of 1.4) is not orientable. In general $\mathbb{P}^n$ is orientable $\Leftrightarrow$ n is odd, see 6.12.5 and 27.44.2.

27.43 The orientation covering.

For manifolds M let $M^{or} := \{(p, \omega) : p \in M,\ \omega \text{ is an orientation on } T_pM\}$. We define an atlas $\mathcal{A}$ for M^{or} using charts $\varphi_\pm : \operatorname{dom}\varphi \to M^{or}$, $x \mapsto (\varphi(x), \pm\omega)$, where φ is a chart of M and ω is the orientation induced by $T\varphi$ from the default orientation on $\mathbb{R}^n$. Then $\mathcal{A}$ is a C^∞ atlas on M^{or}, because when φ and ψ are charts of M, then $\varphi_+^{-1}\circ\psi_+$ (and also $\varphi_+^{-1}\circ\psi_-$, etc.) is defined precisely on the open set $\{x\in\mathbb{R}^m : (\varphi^{-1}\circ\psi)'(x)$ is orientation preserving (reversing)$\}$ and coincides there with $\varphi^{-1}\circ\psi$:

$$\begin{aligned}
&x\in\operatorname{dom}(\varphi_+^{-1}\circ\psi_+)\\
\Leftrightarrow\ &x\in\operatorname{dom}(\varphi^{-1}\circ\psi)\text{ s.t. } T_x\psi \text{ and } T_{(\varphi^{-1}\circ\psi)(x)}\varphi \text{ induce the same orientation}\\
\Leftrightarrow\ &x\in\operatorname{dom}(\varphi^{-1}\circ\psi)\text{ s.t. } (\varphi^{-1}\circ\psi)'(x) \text{ is orientation preserving.}
\end{aligned}$$

Obviously, $\mathrm{pr}_1 : M^{or}\to M$ is a two-fold covering map of M, the so-called ORIENTATION COVERING of M.

The manifold M^{or} is oriented, with the orientation on $T_{(p,\omega)}M^{or}\cong T_pM$ being just ω.

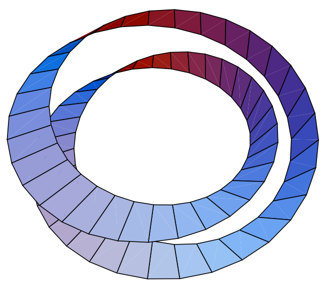
Furthermore M is orientable $\Leftrightarrow$ $M^{or}\cong M\times\{-1,1\}$, i.e. is trivial:

($\Leftarrow$) The embedding $M\hookrightarrow M\times\{-1,1\}\cong M^{or}$ is open so with M^{or} also M can be oriented.

($\Rightarrow$) If M can be orientated, then there is a specified distinguished orientation ω_p on T_pM. Thus $(p,\pm1)\mapsto(p,\pm\omega_p)$ provides a trivialization $M\times\{-1,1\}\cong M^{or}$.

27.44 Example.

(1) A two-fold twisted Möbius strip (i.e. a cylinder) is the orientation covering of the Möbius strip.



(2) $S^n=(\mathbb{P}^n)^{or}$ for n odd.

28. Integration and the Theorem of Stokes

28.1 Proposition.

M is orientable $\Leftrightarrow$ $\Lambda^{\dim M}T^*M$ is trivial as a vector bundle.

Proof. Let $m:=\dim M$.

($\Rightarrow$) It suffices to show the existence of a nowhere vanishing section $\omega\in\Omega^m(M)$ (which directly provides us with a global trivialization $\Phi : M\times\mathbb{R}\to\Lambda^mT^*M$, $(x,t)\mapsto t\cdot\omega_x$, of the one-dimensional bundle $\Lambda^mT^*M\to M$). On the image U of each orientation preserving chart $(u^1,\dots,u^m)^{-1}$ we can define $\omega_U\in\Omega^m(U)$ by $\omega_U(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^m}):=1$. Then $\omega_U(v_1,\dots,v_m)>0$ for any positive oriented basis by $\boxed{25.4}$. We choose a covering $\mathcal{U}$ of M with such open sets U and associated ω_U, and let $\{f_U : U \in \mathcal{U}\}$ be a subordinated partition of unity. We define $\omega \in \Omega^m(M)$ by $\omega := \sum_U f_U \cdot \omega_U \in \Omega^m(M)$. Then $\omega_x(v_1, \dots, v_m) > 0$ for every positive oriented basis of T_xM, thus, in particular, $\omega_x \neq 0$.

($\Leftarrow$) If $\Phi : M \times \mathbb{R} \to \Lambda^m T^*M$ is a global VB isomorphism, then $\omega := \Phi(\_ \times \{1\})$ is a nowhere vanishing m-form. We orientate T_xM by calling a basis $(v_i)_{i=1}^m$ of T_xM positively oriented if $\omega_x(v_1, \dots v_m) > 0$. Let $(u^1, \dots, u^m)^{-1}$ be a chart with connected domain. Since ω does not vanish anywhere, $\omega_p\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}\right) \neq 0$ and hence $\omega_p\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m}\right)$ is positive everywhere or negative everywhere, the vector bundle $TM \to M$ is orientable according to 27.2.3 and hence also M is orientable as manifold. □

28.2 Motivation.

We can not easily integrate functions $f : M \to \mathbb{R}$ over a manifold M. Let us take a look at the simplest case of 1-dimensional manifolds. If M is an interval in $\mathbb{R}$ with boundary points a and b, then the usual Riemann integral $\int_M f = \int_a^b f$ measures the oriented surface below the graph of f. In order to be able to define the integral for any (1-dimensional) manifold M, we definitely need an orientation on M. In this section, therefore, all manifolds are assumed to be oriented. Furthermore, we also have to be able to measure (infinitesimal) lengths (or volumes) on M. If M is a Riemann manifold, then we can do so using the volume form vol_M, which is in the one-dimensional Riemann case the arc element.

On abstract manifolds we need a substitute for the volume element. In the 1-dimensional case, this would be a 1-form $\omega \in \Omega^1(M)$ (which does not vanish at any point). Then we could define the integral $\int_M f \cdot \omega$ over M of f with respect to ω. But since $f \cdot \omega$ itself is a 1-form, it is sufficient (and necessary, since any form can be written as $f \cdot \omega$) to define $\int_M \omega$ for arbitrary 1-forms $\omega \in \Omega^1(M)$. If $c : [a, b] \to M$ is an orientation preserving global parameterization, then $\int_M \omega := \int_a^b \omega_{c(t)}(\dot{c}(t))\, dt$ is the path integral defined as usual.

On general oriented m-dimensional manifolds M we now want to define the integral $\int_M \omega$ for any m-form $\omega \in \Omega^m(M)$ with compact support.

28.3 Definition (Integration of differential forms).

Let M be an oriented m-dimensional manifold and let $\omega \in \Omega^m(M)$ have compact support.

1. If $M = U \subseteq \mathbb{R}^m$ is open then ω can be written as
$$\omega(x^1, \dots, x^m) = f(x^1, \dots, x^m)\, dx^1 \wedge \dots \wedge dx^m$$
with $f \in C^\infty(U, \mathbb{R})$. The integral is then defined as the usual Riemann integral:
$$\int_M \omega := \int_U f(x^1, \dots, x^m)\, d(x^1, \dots, x^m).$$
Note that for orientation preserving diffeomorphisms $g : \mathbb{R}^m \supseteq V \to U \subseteq \mathbb{R}^m$:
$$\int_{g(V)} \omega = \int_V g^*(\omega),$$
because if $\omega = f\, dx^1 \wedge \dots \wedge dx^m$, then
$$(g^*(\omega))(x) = (f \circ g)(x)\, \underbrace{\det(g'(x))}_{>0}\, dx^1 \wedge \dots \wedge dx^m$$
by 25.3 and hence the integrals coincide by the transformation formula for multidimensional integrals, see e.g. [82, 7.5.10].

2. If $\operatorname{supp}\omega \subseteq \varphi(U)$ for an orientation preserving chart $\varphi : \mathbb{R}^m \supseteq U \to \varphi(U) \subseteq M$, then we define:
$$\int_M \omega := \int_U \varphi^*(\omega).$$
This definition makes sense, because let $\operatorname{supp}\omega \subseteq \varphi(U)\cap\psi(V) =: W$ for charts φ and ψ with orientation preserving chart change $g := \varphi^{-1}\circ\psi : \psi^{-1}(W) \to \varphi^{-1}(W)$. Then
$$\int_U \varphi^*(\omega) \xlongequal{\operatorname{supp}(\varphi^*\omega)\subseteq\varphi^{-1}(W)} \int_{\varphi^{-1}(W)} \varphi^*(\omega) = \int_{g(\psi^{-1}(W))} \varphi^*(\omega) =$$
$$\xlongequal{\boxed{1}} \int_{\psi^{-1}(W)} \underbrace{g^*(\varphi^*(\omega))}_{(\varphi\circ g)^*(\omega)} = \int_{\psi^{-1}(W)} \psi^*(\omega) \xlongequal{\operatorname{supp}(\psi^*\omega)\subseteq\psi^{-1}(W)} \int_V \psi^*(\omega).$$
3. If $\operatorname{supp}\omega$ is arbitrary compact, we choose a finite open covering by chart neighborhoods of $\operatorname{supp}\omega$, as well as a partition of unity $\{h_i\}$, which is subordinate to this covering. Then each $h_i\cdot\omega$ has its support contained in some chart, and so we can define using $\boxed{2}$:
$$\int_M \omega = \int_M \Big(\sum_i h_i\Big)\omega := \sum_i \int_M h_i\,\omega.$$
Again, this definition makes sense, because if $\{g_j\}$ is a second partition of unity, which is subordinate to a finite covering of the support with chart neighborhoods. Then
$$\sum_i \int_M h_i\omega = \sum_i \int_M \Big(\sum_j g_j\Big) h_i\omega = \sum_i\sum_j \int_M g_j h_i\omega =$$
$$= \sum_j \int_M \Big(\sum_i h_i\Big) g_j\omega = \sum_j \int_M g_j\omega.$$

28.4 Remark (Densities).

If we want to integrate over non-orientable manifolds, we need something else than m forms. For this we define a one-dimensional vector bundle $\operatorname{vol}(M)$ by using $x \mapsto |\det\psi'(x)| \in GL(1)$ as transition functions for the chart changes ψ of M. Sections of $\operatorname{vol}(M)$ are called densities, which can then be integrated over M. If M is orientable, then $\operatorname{vol}(M) \cong \Lambda^m T^*M$.

28.5.

We now approach Stokes's Theorem: According to the fundamental theorem of Analysis (see, e.g., [**81**, 5.2.2]), $\int_a^b f'(x)dx = f(b) - f(a)$. In particular: $\int_{-\infty}^0 f' = \int_a^0 f' = f(0)$ if $\operatorname{supp} f$ is compact and $a \le \inf(\operatorname{supp} f)$.

Lemma (Theorem of Stokes for half-spaces).

Let $H = H^{m+1} := \{(t,x) : t \le 0, x \in \mathbb{R}^m\}$ be an $(m{+}1)$-dimensional half space. The subset $\partial H := \{(0,x) : x \in \mathbb{R}^m\} \cong \mathbb{R}^m$ is called the boundary of H. For each m-form ω on $\mathbb{R}^{m+1}$ with compact support we have
$$\int_H d\omega = \int_{\partial H} \omega := \int_{\partial H} \operatorname{incl}^*\omega,$$
where $\operatorname{incl} : \partial H \hookrightarrow H$ denotes the inclusion.

Proof. For $\omega \in \Omega^m(\mathbb{R}^{m+1})$ we have:

$$\omega = \sum_{i=0}^{m} \omega_i \, dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m;$$

$$d\omega = \sum_{i=0}^{m}\sum_{j=0}^{m} \frac{\partial \omega_i}{\partial x^j}\, dx^j \wedge dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m + 0$$

$$= \sum_{i=0}^{m} \frac{\partial \omega_i}{\partial x^i} (-1)^i \, dx^0 \wedge \dots \wedge dx^m \;\Rightarrow$$

$$\Rightarrow \int_H d\omega = \sum_{i=0}^{m} (-1)^i \int_H \frac{\partial \omega_i}{\partial x^i}\, d(x^0,\dots,x^m) \overset{\text{Fubini}}{=\!=}$$

$$= \int_{\mathbb{R}^m} \left(\int_{-\infty}^{0} \frac{\partial \omega_0}{\partial x^0}(x^0,x^1,\dots,x^m)\, dx^0 \right) d(x^1,\dots,x^m)$$

$$+ \sum_{i=1}^{m} (-1)^i \int_{H_i} \left(\int_{-\infty}^{+\infty} \frac{\partial \omega_i}{\partial x^i}\, dx^i \right) d(x^0,\dots,\widehat{x^i},\dots,x^m)$$

$$= \int_{\mathbb{R}^m} \omega_0(0,x^1,\dots,x^m)\, d(x^1,\dots,x^m) + 0,$$

where $H_i := \{(t,x^1,\dots,\widehat{x^i},\dots,x^m) : t \le 0\}$ and the second summand is 0 because $\operatorname{supp}\omega$ is compact. On the other hand,

$$\int_{\partial H} \omega := \int_{\partial H} \operatorname{incl}^* \omega$$

$$\overset{\boxed{25.2}}{=\!=} \int_{\partial H} \sum_{i=0}^{m} \omega_i(0,x^1,\dots,x^m) \det\Big(\tfrac{\partial(x^0,\dots,\widehat{x^i},\dots,x^m)}{\partial(x^1,\dots,x^m)} \Big)\, dx^1 \wedge \dots \wedge dx^m$$

$$= \int_{\mathbb{R}^m} \omega_0(0,x^1,\dots,x^m)\, d(x^1,\dots,x^m) + 0,$$

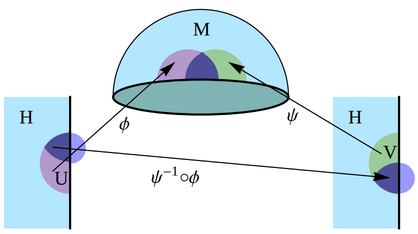
because

$$\det\Big(\frac{\partial(x^0,\dots,\widehat{x^i},\dots,x^m)}{\partial(x^1,\dots,x^m)} \Big) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

Now we want to transfer these considerations to spaces that only locally look like H:

28.6 Definition (Manifolds with boundry).

A C^∞ manifold with boundary is a set M together with an atlas $\mathcal{A}$ of injective maps $\varphi : U \to M$, where $U \subseteq H$ is open in a closed halfspace $H := \{(t,x) : t \le 0, x \in \mathbb{R}^{m+1}\}$, and the chart changes $\psi^{-1}\circ\varphi : \varphi^{-1}(\psi(V)) \to \psi^{-1}(\varphi(U))$ are defined on open subsets of half-spaces and are smooth.



A mapping between such subsets of half-spaces is called smooth if there is a smooth extension to open subsets of $\mathbb{R}^m$. As usual, we assume that the final topology induced by the atlas is Hausdorff and paracompact. The boundary of M (not in the topological sense) is then defined as

$$\partial M := \{p \in M : \exists\, \varphi, \text{ a chart at } p \text{ with } \varphi^{-1}(p) \in \partial H\}.$$

Since the chart change is a local (diffeomorphism and thus) homeomorphism of $\mathbb{R}^n$, it maps inner points (i.e. those in $H \setminus \partial H$) to inner points, and thus $p \in \partial M \Leftrightarrow \varphi^{-1}(p) \in \partial H$ for each φ chart at p. The boundary ∂M is a manifold (without boundary), with an atlas on ∂M given by the restrictions $\varphi|_{\partial M}$ of the charts φ of M. One can define $C^\infty(M,N)$, TM, T^*M, $\Lambda^k T^*M$ and $\Omega^k(M)$ as for manifolds without boundaries.

28.7 Definition (Inner tangent vector).

A vector $v \in T_pM := \mathrm{Der}_x(C^\infty(M,\mathbb{R}))$ is called an INNER TANGENT VECTOR if $p \notin \partial M$ or $T_p\varphi^{-1} \cdot v \in T_{\varphi^{-1}(p)}H = \mathbb{R} \times \mathbb{R}^m$ has 0-th component less than 0.

28.8 Lemma (Prolongation of manifolds with boundary).

Any manifold with boundary can be extended to a manifold without boundary, i.e. is a submanifold of same dimension:

Proof sketch. Using a partition of unity one finds a VF on M, which consists only of inner tangent vectors. By rescaling the vector field its flow can be made global (see [**86**, 62.11]) and thus $\mathrm{Fl}(1, .) : M \to M \setminus \partial M$ is an embedding of M into the manifold without boundary $M \setminus \partial M$. □

Simple examples of manifolds with boundary are the closed Möbius strip and the closed ball.

28.9 Lemma (Orientability of the boundary).

The boundary of each oriented manifold with boundary is canonically oriented.

Proof. For this it suffices to call a basis $(e_i)_{i=1}^m$ of $T_p(\partial M)$ positively oriented, if for an tangential vector e_0 pointing outwards (i.e. $-e_0$ is inner tangent vector) the basis $(e_0, \dots, e_m)$ is positively oriented in T_pM. □

28.10 Remark.

Let N be an oriented submanifold of codimension 1 of the $(n+1)$ dimensional oriented Riemannian manifold M, and let ν_x be the uniquely determined vector in T_xM for $x \in N$, so that $(\nu_x, e_1, \dots, e_n)$ is a positively oriented orthonormal basis in T_xM for an orthonormal basis $(e_1, \dots, e_n)$ from T_xN. In the case of the canonically oriented boundary $N = \partial M$ of a manifold M with boundary, ν is the outward-pointing unit normal vector, see 28.9. If ν is prolongated to a vector field of the same name on the whole M, then

$$\mathrm{vol}_N = \mathrm{inkl}^*(\iota_\nu(\mathrm{vol}_M)) \text{ on } N,$$

because $\mathrm{vol}_N(e_1, \dots, e_n) = 1 = \mathrm{vol}_M(\nu_N, e_1, \dots, e_n) = (\iota_{\nu_N} \mathrm{vol}_M)(e_1, \dots, e_n)$.

28.11 Theorem of Stokes.

Let M be an $(n+1)$-dimensional oriented manifold with canonically oriented boundary ∂M. For every $\omega \in \Omega^n(M)$ with compact support we have

$$\int_M d\omega = \int_{\partial M} \omega := \int_{\partial M} \mathrm{incl}^* \omega \quad (\textit{ where } \mathrm{incl} : \partial M \hookrightarrow M)$$

Proof. Let $\{h_i\}$ be a partition of unity subordinated to a covering by chart neighborhoods and put $\omega_i := h_i \cdot \omega$. Then

$$\omega = \sum_i \omega_i, \quad \text{where } \operatorname{supp}\omega_i \subseteq \operatorname{supp} h_i \quad \xRightarrow{\boxed{28.3.3}} \quad \int_{\partial M} \omega = \sum_i \int_{\partial M} \omega_i,$$

$$d\omega = \sum_i d\,\omega_i, \quad \text{where } \operatorname{supp}(d\,\omega_i) \subseteq \operatorname{supp}\omega_i \quad \xRightarrow{\boxed{28.3.3}} \quad \int_M d\omega = \sum_i \int_M d\,\omega_i.$$

Thus, the proof is reduced to the case already shown in 28.5, where $\operatorname{supp}\omega$ is in a chart neighborhood, i.e. w.l.o.g. M is the half-space H^{n+1} and $\omega \in \Omega^n(\mathbb{R}^{n+1})$. □

29. Applications of integration to cohomology

We now want to determine the highest cohomology $H^m(M)$ for each m-dimensional manifold M.

29.1 Definition. Cohomology with compact support.

By using the subspaces $\Omega^k_c(M) := \{\omega \in \Omega^k(M) : \operatorname{supp}\omega \text{ is compact}\}$ instead of $\Omega^k(M)$, we obtain the cohomology with compact support

$$\begin{aligned} Z^k_c(M) &:= \ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M)), \\ B^k_c(M) &:= \operatorname{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M)), \\ H^k_c(M) &:= Z^k_c(M)/B^k_c(M). \end{aligned}$$

Note that $B^k_c(M) \subset \{d\eta \in \Omega^k_c(M) : \eta \in \Omega^{k-1}(M)\}$ for $M = \mathbb{R}^n$: For $f \in C^\infty_c(\mathbb{R}^n)$ with $0 \neq f \geq 0$, the differential form $\omega := f\,dx^1 \wedge \cdots \wedge dx^n \in \Omega^n_c(\mathbb{R}^n)$ is exact by the Poincaré Lemma 26.5.6, but for no $\eta \in \Omega^{n-1}_c(\mathbb{R}^n)$ is $d\eta = \omega$, because by the theorem 28.11 of Stokes $0 < \int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} d\eta = \int_\emptyset \eta = 0$ would be a contradiction.

A direct generalization of this argument shows that for each m-dimensional orientable manifold M there is an $\omega_0 \in \Omega^m_c(M) = Z^m_c(M)$ with $\int_M \omega_0 = 1$ and $\omega_0 \notin B^m_c(M)$, i.e. $H^m_c(M) \neq 0$.
We now want to show that $H^m_c(M) \cong \mathbb{R}$ for all such connected M by finding for each $\omega \in \Omega^m_c(M)$ an $\eta \in \Omega^{m-1}_c(M)$ with $\omega = (\int_M \omega)\,\omega_0 + d\eta$, and thus $\int : \Omega^m_c(M) \to \mathbb{R}$ induces an isomorphism $H^m_c(M) \cong \mathbb{R}$, $[\omega] \mapsto \int_M \omega$.

29.2 Lemma.

Let $r : \mathbb{R}^{m+1} \setminus \{0\} \to S^m$ be the retraction $x \mapsto \frac{x}{\|x\|}$ and $\nu \in \mathfrak{X}(\mathbb{R}^{m+1})$ the vector field $x \mapsto x$. Then

$$\begin{aligned} (r^* \operatorname{vol}_{S^m})(x) &= \frac{1}{\|x\|^{m+1}}\, \iota_\nu \operatorname{vol}_{\mathbb{R}^{m+1}}(x) \\ &= \frac{1}{\|x\|^{m+1}} \sum_{i=0}^m (-1)^i x^i\, dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m. \end{aligned}$$

Proof. Since, for $x \neq 0$, the tangential space $T_x\mathbb{R}^{m+1}$ is generated by $T_x(\|x\|\,S^m)$ and ν_x, it suffices to test both sides on vectors $v_1, \dots, v_m$ from these two subspaces.

If $v_i = \nu_x$ is for at least one i, the left side is 0, because $T_x r \cdot \nu = \frac{d}{dt}|_{t=0} r(x + t\,x) = 0$ and also the right side:

$$\frac{1}{\|x\|^{m+1}} (\iota_\nu \operatorname{vol}_{\mathbb{R}^{m+1}})_x(\dots, v_i, \dots) = \frac{1}{\|x\|^{m+1}} \operatorname{vol}_{\mathbb{R}^{m+1}}(\nu_x, \dots, \nu_x, \dots) = 0.$$

Otherwise, if all $v_i \in T_x(\|x\|\, S^m)$, then $T_x r \cdot v_i = \frac{1}{\|x\|} v_i$, because $r|_{\|x\|\, S^m}$ is multiplication with the factor $\frac{1}{\|x\|}$. Thus both sides are the same, because by 28.10 (see also Exercise [98, 37])

$$\mathrm{vol}_{S^m} \overset{28.10}{=\!=\!=} \mathrm{incl}^*(\iota_\nu(\mathrm{vol}_{\mathbb{R}^{m+1}})) = \mathrm{incl}^*\Big(\sum_{i=0}^m (-1)^i x^i \, dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m\Big)$$

and hence

$$\begin{aligned}
(r^* \mathrm{vol}_{S^m})_x(v_1, \dots, v_m) &:= \\
&:= (\mathrm{vol}_{S^m})_{r(x)}\Big(\frac{1}{\|x\|} v_1, \dots, \frac{1}{\|x\|} v_m\Big) \\
&= \frac{1}{\|x\|^m}\Big(\sum_{i=0}^m (-1)^i \, r^i(x) \, dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m\Big)(v_1, \dots, v_m) \\
&= \frac{1}{\|x\|^{m+1}} (\iota_\nu \, \mathrm{vol}_{\mathbb{R}^{m+1}})_x(v_1, \dots, v_m). \quad \square
\end{aligned}$$

29.3 Lemma. Integration with respect to polar coordinates.

Let $B := \{x \in \mathbb{R}^{m+1} : \|x\| \le 1\}$ and $f \in C^\infty(B, \mathbb{R})$. Then

$$\int_B f = \int_B f \ \mathrm{vol}_{\mathbb{R}^{m+1}} = \int_{S^m} g \ \mathrm{vol}_{S^m} \quad \textit{with } g : x \mapsto \int_0^1 t^m f(t\, x)\, dt.$$

Proof. Let $h : S^m \times [0,1] \to \mathbb{R}$ be given by $h(y,t) := t^m f(t\, y)$ and $dt \wedge \mathrm{vol}_{S^m} := \mathrm{pr}_2^*(dt) \wedge \mathrm{pr}_1^*(\mathrm{vol}_{S^m}) \in \Omega^{m+1}(S^m \times [0,1])$. If we use the orientation induced by $dt \wedge \mathrm{vol}_{S^m} \neq 0$ (see 28.1) on $S^m \times [0,1]$, then

$$\int_{S^m} g \ \mathrm{vol}_{S^m} = \int_{S^m} \int_0^1 \underbrace{t^m f(t\, \_)}_{=h(\_,t)} \, dt \ \mathrm{vol}_{S^m} = \int_{S^m \times [0,1]} h \, dt \wedge \mathrm{vol}_{S^m}.$$

We have $B \setminus \{0\} \cong S^m \times (0,1]$ by $\varphi : x \mapsto (\frac{x}{\|x\|}, \|x\|)$ with inverse mapping $(y,t) \mapsto ty$. With $\rho(x) := \|x\|$ we get

$$\rho^*(dt) \overset{25.2}{=\!=\!=} \sum_i \frac{\partial \rho}{\partial x^i} dx^i = \frac{x^i}{\|x\|} dx^i \text{ and hence}$$

$$\begin{aligned}
\varphi^*(dt \wedge \mathrm{vol}_{S^m}) &= \varphi^*(\mathrm{pr}_2^*(dt) \wedge \mathrm{pr}_1^*(\mathrm{vol}_{S^m})) = (\mathrm{pr}_2 \circ \varphi)^*(dt) \wedge (\mathrm{pr}_1 \circ \varphi)^*(\mathrm{vol}_{S^m}) \\
&= \rho^*(dt) \wedge r^*(\mathrm{vol}_{S^m}) \\
&\overset{29.2}{=\!=\!=} \sum_{j=0}^m \frac{x^j}{\|x\|} dx^j \wedge \frac{1}{\|x\|^{m+1}} \sum_{i=0}^m (-1)^i x^i \, dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m \\
&= \frac{1}{\|x\|^{m+2}} \sum_{i=0}^m (x^i)^2 \, dx^0 \wedge \dots \wedge dx^m = \frac{1}{\|x\|^m} dx^0 \wedge \dots \wedge dx^m
\end{aligned}$$

and thus, for $x \neq 0$,

$$\begin{aligned}
\varphi^*(h \, dt \wedge \mathrm{vol}_{S^m})(x) &= h(\varphi(x)) \, \varphi^*(dt \wedge \mathrm{vol}_{S^m})(x) \\
&= \|x\|^m f(x) \frac{1}{\|x\|^m} dx^0 \wedge \dots \wedge dx^m = f(x) \, dx^0 \wedge \dots \wedge dx^m.
\end{aligned}$$

So

$$\int_B f = \lim_{\varepsilon\searrow 0}\int_{B\setminus\varepsilon B}\varphi^*(h\,dt\wedge \mathrm{vol}_{S^m}) = \lim_{\varepsilon\searrow 0}\int_{\varphi(B\setminus\varepsilon B)} h\,dt\wedge\mathrm{vol}_{S^m}$$
$$= \lim_{\varepsilon\searrow 0}\int_{S^m\times[\varepsilon,1]} h\,dt\wedge\mathrm{vol}_{S^m} = \int_{S^m\times[0,1]} h\,dt\wedge\mathrm{vol}_{S^m} = \int_{S^m} g\ \mathrm{vol}_{S^m}. \quad \square$$

29.4 Theorem.

For each connected orientable m-dimensional manifold M, the mapping $[\omega]\mapsto\int_M\omega$ is an isomorphism $H^m_c(M)\cong\mathbb{R}$.

Proof. We have to show that $B^m_c(M)$ is the kernel of $\int_M:\Omega^m_c(M)\to\mathbb{R}$, i.e. for each $\omega\in\Omega^m_c(M)$ with $\int_M\omega=0$ there exists an $\eta\in\Omega^{m-1}_c(M)$ with $\omega=d\eta$.

Claim: *The theorem is valid for $M=\mathbb{R}$.*
Let $\omega\in\Omega^1_c(\mathbb{R})$ be such that $\int_\mathbb{R}\omega=0$. Because of the Poincaré Lemma 26.5.6, there is an $f\in C^\infty(\mathbb{R},\mathbb{R})$ with $\omega=df$. Since $\operatorname{supp}\omega=\operatorname{supp}f'$ is compact, there is an N, s.t. f is constant both on $(-\infty,-N]$ and $[N,+\infty)$. Because of $0=\int_\mathbb{R}\omega=\int_\mathbb{R}df=\int_{-\infty}^{+\infty}f'(t)dt=\int_{-N}^{+N}f'(t)dt=f(N)-f(-N)$ we have $f(N)=f(-N)$ and hence $g:=f-f(N)\in C^\infty_c$ and $\omega=dg$.

Claim: *If the theorem holds for S^m, then also for $\mathbb{R}^{m+1}$.*
Let $\omega=f\,dx^0\wedge\dots\wedge dx^m\in\Omega^{m+1}_c(\mathbb{R}^{m+1})$ be such that $\int_{\mathbb{R}^{m+1}}\omega=0$ and, w.l.o.g. $\operatorname{supp}(\omega)\subseteq B:=\{x\in\mathbb{R}^{m+1}:\|x\|<1\}$. Because of the Poincaré Lemma 26.5.6, there is an $\eta\in\Omega^m(\mathbb{R}^{m+1})$ with $\omega=d\eta$. By exercise [98, 36], w.l.o.g.

$$\eta(x)=\int_0^1 t^m f(tx)\,dt\cdot\sum_{i=0}^m(-1)^i x^i\,dx^0\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^m$$
$$\overset{(t=\frac{s}{\|x\|})}{=\!=}\frac{1}{\|x\|^{m+1}}\int_0^{\|x\|}s^m f\left(s\frac{x}{\|x\|}\right)ds\cdot\sum_{i=0}^m(-1)^i x^i\,dx^0\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^m$$
$$\overset{29.2}{=\!=}\int_0^{\|x\|}t^m f\left(t\frac{x}{\|x\|}\right)dt\cdot(r^*\mathrm{vol}_{S^m})(x).$$

Let $g:S^m\to\mathbb{R}$ be defined as in 29.3 by $g(x):=\int_0^1 t^m f(tx)\,dt$. There exists a $\lambda\in\Omega^{m-1}(S^m)$ with $g\ \mathrm{vol}_{S^m}=d\lambda$ because the theorem is assumed to be valid for S^m and

$$0=\int_{\mathbb{R}^{m+1}}\omega=\int_B f\overset{29.3}{=\!=}\int_{S^m}g\ \mathrm{vol}_{S^m}.$$

Since $f|_{\mathbb{R}^{m+1}\setminus B}=0$, we have

$$\eta(x)=\int_0^1 t^m f\left(t\frac{x}{\|x\|}\right)dt\cdot(r^*\mathrm{vol}_{S^m})(x)\text{ for }\|x\|>1.$$

So $\eta=(g\circ r)\cdot r^*\mathrm{vol}_{S^m}=r^*(g\ \mathrm{vol}_{S^m})=r^*(d\lambda)=d(r^*\lambda)$ on $\mathbb{R}^{m+1}\setminus B$.
Let $h\in C^\infty(\mathbb{R}^{m+1},[0,1])$ be such that $h=0$ near 0 and $h|_{\mathbb{R}^{m+1}\setminus B}=1$. Then $h\cdot r^*\lambda\in\Omega^{m-1}(\mathbb{R}^{m+1})$ and

$$\omega=d\eta=d(\eta-d(h\cdot r^*\lambda))\text{ with }(\eta-d(h\cdot r^*\lambda))|_{\mathbb{R}^{m+1}\setminus B}=(\eta-d(r^*\lambda))|_{\mathbb{R}^{m+1}\setminus B}=0.$$

Claim: *If the theorem holds for $\mathbb{R}^m$, then it does so for all m-dimensional connected M.*
Let $\omega_0\in\Omega^m_c(M)$ be such that $\int\omega_0=1$ and $\operatorname{supp}\omega_0\subseteq\operatorname{im}\psi_0$ for a chart $\psi_0:\mathbb{R}^m\to M$. We show that for $\omega\in\Omega^m_c(M)$ there is an $\eta\in\Omega^{m-1}_c(M)$ with $\omega=$

$(\int_M \omega) \cdot \omega_0 + d\eta$. From this it follows in particular that $\omega = d\eta$ for all ω with $\int \omega = 0$.

First, consider the case $\operatorname{supp}\omega \subseteq \operatorname{im}\psi$ for some chart $\psi : \mathbb{R}^m \to M$: There are finitely many charts $\psi_0, \dots, \psi_l : \mathbb{R}^m \to M$ with $\operatorname{im}\psi_i \cap \operatorname{im}\psi_{i+1} \neq \emptyset$, where ψ_0 is the chart from above for ω_0 and $\psi_l = \psi$. There are $\omega_i \in \Omega_c^m(M)$ with $\operatorname{supp}\omega_i \subseteq \operatorname{im}\psi_{i-1} \cap \operatorname{im}\psi_i$ and $\int \omega_i = 1$. Since the theorem is assumed to be valid for $\mathbb{R}^m \cong \operatorname{im}\psi_{i-1}$, there exist $\eta_i \in \Omega_c^{m-1}(M)$ with $\operatorname{supp}\eta_i \subseteq \operatorname{im}\psi_{i-1}$ and $\omega_i - \omega_{i-1} = d\eta_i$. Finally, there is, for $c := \int \omega$, also an η_{l+1} with $\omega - c\,\omega_l = d\eta_{l+1}$. Consequently

$$\omega = c\,\omega_l + d\eta_{l+1} = \ldots = c\,\omega_0 + c\sum_{i=1}^{l} d\eta_i + d\eta_{l+1} = c\,\omega_0 + d\Big(\eta_{l+1} + c\sum_{i=1}^{l}\eta_i\Big).$$

Now, if $\omega \in \Omega_c^m(M)$ is arbitrary and $\{f_i\}$ is a partition of unity subordinated to a covering with open sets diffeomorphic to $\mathbb{R}^m$, then $f_i\omega = c_i\omega_0 + d\eta_i$ for some $\eta_i \in \Omega_c^{m-1}(M)$ and thus $\omega = \sum_i f_i\omega = (\sum_i c_i)\,\omega_0 + d\sum_i \eta_i$ with $\int_M \omega = \sum_i c_i \int_M \omega_0 + \int_M d\sum_i \eta_i = \sum_i c_i$, where we only have to sum over the finitely many i with $f_i\omega \neq 0$. $\square$

29.5 Theorem (Highest cohomology).

For connected m-dimensional manifolds M without boundary, the following holds:

$$H^m(M) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is compact and orientable,} \\ 0 & \text{otherwise.} \end{cases}$$

$$H_c^m(M) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is orientable,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. According to 29.4, $H_c^m(M) \cong \mathbb{R}$ for all **orientable** M and thus $H^m(M) = H_c^m(M) \cong \mathbb{R}$ for all **orientable compact** M.

Next, let M be **orientable but not compact**:
Since M is not compact, a covering $\{\operatorname{im}\psi_i : i \in \mathbb{N}\}$ exists by charts ψ_i, s.t. every compact set meets only finitely many $\operatorname{im}\psi_i$ (see 9.3.3) and (by rearrangement we may assume w.l.o.g. that) $\operatorname{im}\psi_i \cap \operatorname{im}\psi_{i+1} \neq \emptyset$. Let $\{f_i : i \in \mathbb{N}\}$ be a subordinated partition of unity. We again choose $\omega_i \in \Omega_c^m(M)$ with $\operatorname{supp}\omega_i \subseteq \operatorname{im}\psi_i \cap \operatorname{im}\psi_{i+1}$ and $\int_M \omega_i = 1$. Now let $\omega \in \Omega^m(M)$ s.t. $\operatorname{supp}\omega \subseteq \operatorname{im}\psi_j$ for some j and put $c := \int_M \omega$. Then, for $i \geq j$, there are $\eta_i \in \Omega_c^{m-1}(M)$ with $\operatorname{supp}\eta_i \subseteq \operatorname{im}\psi_i$ and $\omega = c\,\omega_j + d\eta_j$ and $c\,\omega_{i-1} = c\,\omega_i + d\eta_i$ for $i > j$ by 29.4. Thus,

$$\omega = c\,\omega_j + d\eta_j = \cdots = c\,\omega_k + \sum_{i=j}^{k} d\eta_i = d\Big(\sum_{i=j}^{\infty}\eta_i\Big) = d\mu_j$$

where $\mu_j := \sum_{i=j}^{\infty}\eta_i$ is locally finite.
Now let $\omega \in \Omega^m(M)$ be arbitrary. Then $f_j\omega$ is as before, so there is a $\mu_j \in \Omega^{m-1}(M)$ with $f_j\omega = d\mu_j$ and $\operatorname{supp}\mu_j \subseteq \bigcup_{i\geq j}\operatorname{im}\psi_i$ by what we have just shown. So $\sum_j \mu_j$ is locally finite and

$$\omega = \sum_j f_j\omega = \sum_j d\mu_j = d\Big(\sum_j \mu_j\Big),$$

hence $H^m(M) = \{0\}$.

Finally, let M **be not orientable**:
Let $p : \tilde{M} \to M$ be a two-folded covering and let $\chi : \tilde{M} \to \tilde{M}$ be the automorphism, which exchanges the two points in each fiber.

We put

$$\begin{aligned}
\Omega^k_\pm(\tilde M) &:= \{\omega \in \Omega^k(\tilde M) : \chi^*\omega = \pm\omega\}\\
\Omega^k_{\pm,c}(\tilde M) &:= \{\omega \in \Omega^k_c(\tilde M) : \chi^*\omega = \pm\omega\}\\
H^k_\pm(\tilde M) &:= \{\omega \in \Omega^k_\pm(\tilde M) : d\omega = 0\}\,/\,\{d\eta : \eta \in \Omega^{k-1}_\pm(\tilde M)\}\\
H^k_{\pm,c}(\tilde M) &:= \{\omega \in \Omega^k_{\pm,c}(\tilde M) : d\omega = 0\}\,/\,\{d\eta : \eta \in \Omega^{k-1}_{\pm,c}(\tilde M)\}.
\end{aligned}$$

Claim:

$$\begin{aligned}
\Omega^k(\tilde M) &= \Omega^k_+(\tilde M) \oplus \Omega^k_-(\tilde M)\\
H^k(\tilde M) &= H^k_+(\tilde M) \oplus H^k_-(\tilde M)\\
p^* : H^k(M) &\xrightarrow{\cong} H^k_+(\tilde M)
\end{aligned}$$

and analogously for forms with compact supports:

Let $\omega \in \Omega^k(\tilde M)$, then

$$\begin{aligned}
\omega &= \frac{1}{2}\Big((\omega + \chi^*\omega) + (\omega - \chi^*\omega)\Big) \in \Omega^k_+ \oplus \Omega^k_-, \text{ where } \Omega^k_- \cap \Omega^k_+ = \{0\}\\
&\Rightarrow\ \Omega^k(\tilde M) = \Omega^k_-(\tilde M) \oplus \Omega^k_+(\tilde M) \text{ and}\\
d(\Omega^k_\pm) &\subseteq \Omega^{k+1}_\pm, \text{ because of } \chi^*(d\omega) = d(\chi^*\omega) = \pm d\omega \text{ for } \omega \in \Omega^k_\pm,\\
&\Rightarrow\ H^k(\tilde M) = H^k_-(\tilde M) \oplus H^k_+(\tilde M).
\end{aligned}$$

The mapping $p^* : \Omega^k(M) \to \Omega^k(\tilde M)$ is injective (since p is a surjective submersion) with image $p^*(\Omega^k(M)) = \Omega^k_+(\tilde M)$:
($\subseteq$) holds, since $\chi^* p^*\omega = (p \circ \chi)^*\omega = p^*\omega$.
($\supseteq$) Let $\omega \in \Omega^k_+(\tilde M)$ and let $U \subseteq \tilde M$ be such that $p|_U : U \to p(U)$ is a diffeomorphism. Put $\tilde\omega|_{p(U)} := ((p|_U)^{-1})^*\omega$. Then $\tilde\omega \in \Omega^k(M)$ is well defined (because $\chi^*\omega = \omega$) and $p^*\tilde\omega = \omega$.

Thus, $p^* : \Omega^k(M) \xrightarrow{\cong} \Omega^k_+(\tilde M) \hookrightarrow \Omega^k(\tilde M)$ holds. Because of $p^* \circ d = d \circ p^*$ it follows that $p^* : H^k(M) \xrightarrow{\cong} H^k_+(\tilde M) \hookrightarrow H^k(\tilde M)$ and analogously $p^* : H^k_c(M) \xrightarrow{\cong} H^k_{+,c}(\tilde M) \hookrightarrow H^k_c(\tilde M)$, which is the last statement claimed above.

We now apply this to the orientation covering $\tilde M := M^{\mathrm{or}}$ and the orientation reversing automorphism $\chi : (x, \pm o) \mapsto (x, \mp o)$. Let $\omega \in \Omega^m_{+,c}(M^{or})$. Then $\int \omega = 0$, because $\int \omega = \int \chi^*\omega \overset{\chi \text{ orient.reversing}}{=\!=\!=\!=\!=} -\int \omega$. Thus, for the oriented manifold M^{or}, there exists an $\eta \in \Omega^{m-1}_c(M^{or})$ with $\omega = d\eta$ and hence $\omega = \frac{1}{2}(\omega + \chi^*\omega) = \frac{1}{2}(d\eta + \chi^* d\eta) = d(\eta_+)$ with $\eta_+ := \frac{1}{2}(\eta + \chi^*\eta) \in \Omega^{m-1}_{+,c}(M^{or})$, i.e. $[\omega] = [d(\eta_+)] = 0 \in H^m_{+,c}(M^{or})$. Thus, $H^m_c(M) \cong H^m_{+,c}(M^{or}) = \{0\}$. And, for **compact** M, also $H^m(M) = H^m_c(M) = \{0\}$. If, on the other hand, M is **not compact**, then $H^m(M) = \{0\}$ follows from the orientable case, because $p^* : H^m(M) \hookrightarrow H^m(M^{or}) = \{0\}$. □

29.6 Example.

For the oriented two-fold covering map $S^n \to \mathbb{P}^n$ let χ be the antipodal map $x \mapsto -x$. Then $H^k(\mathbb{P}^n) \cong H^k_+(S^n) \subseteq H^k(S^n)$ because of the claim at the end of the proof of 29.5 and thus

$$H^k_c(\mathbb{P}^n) \cong H^k(\mathbb{P}^n) \cong \begin{cases} \mathbb{R} & \text{for } k = 0 \quad (\text{because } \mathbb{P}^n \text{ is connected})\\ 0 & \text{for } k \notin \{0, n\} \quad (\text{because } H^k(S^n) = 0)\\ 0 & \text{for } k = n \text{ even}\\ \mathbb{R} & \text{for } k = n \text{ odd} \end{cases}$$

In fact, χ is orientation preserving on S^n if and only if n is odd and thus $\int_{S^n} \pm\omega = \int_{S^n} \chi^*\omega = (-1)^{n+1}\int_{S^n}\omega$ for $\omega \in \Omega^n_\pm(S^n)$, so as in the proof of $\boxed{29.5}$ $H^n_+(S^n) = 0$ if n is even; and $H^n_-(S^n) = 0$ for odd n and therefore $H^n(\mathbb{P}^n) \cong H^n_+(S^n) = H^n(S^n) = \mathbb{R}$.

29.7 Fixed Point Theorem of Brouwer.

Let $f : B^n \to B^n := \{x \in \mathbb{R}^n : \|x\| \le 1\}$ be smooth, then there is an $x \in B^n$ with $f(x) = x$.

Proof. Indirectly: If $f(x) \neq x$ for all x, then there is a smooth retraction $r : B^n \to S^{n-1}$, i.e. $r|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, namely let $r(x)$ be the intersection point of the sphere S^{n-1} with well-defined straight half-line from $f(x)$ through x. For $n = 1$ this is not possible because of the intermediate value theorem. For $n > 1$, we extend r to a retraction of the same name $r : \mathbb{R}^n \to S^{n-1}$ using the radial projection. Since $r \circ \mathrm{incl} = \mathrm{id}_{S^{n-1}}$ we have:

$$\begin{tikzcd}
H^{n-1}(S^{n-1}) \arrow[r, "r^*"] \arrow[dd, equal] & H^{n-1}(\mathbb{R}^n) \arrow[r, "\mathrm{incl}^*"] \arrow[d, equal] & H^{n-1}(S^{n-1}) \arrow[dd, equal] \\
 & 0 \arrow[dr, dotted] & \\
\mathbb{R} \arrow[ur, dotted] \arrow[rr, "\mathrm{id}"] & & \mathbb{R}
\end{tikzcd}$$

This is a contradiction. ☐

50.9 Definition (Degree of mappings).

Let M and N be connected compact and oriented manifolds of equal dimension m and let $f : M \to N$ be smooth. The DEGREE $\deg f \in \mathbb{R}$ of f is defined by the adjacent diagram:

$$\begin{tikzcd}
H^m(M) \arrow[d, "\int"', "\cong"] & H^m(N) \arrow[l, "H^m(f)"'] \arrow[d, "\int"', "\cong"] \\
\mathbb{R} & \mathbb{R} \arrow[l, dotted, "\deg f"'] \\
\deg f \cdot t & t \arrow[l, mapsto]
\end{tikzcd}$$

$$\text{i.e.}\qquad \deg f \cdot \int \omega = \deg f \cdot \int [\omega] = \int H^m(f)[\omega] = \int [f^*\omega] = \int f^*\omega.$$

More generally, if M and N are oriented but not necessarily compact manifolds of equal dimension m and $f : M \to N$ is smooth and PROPER (i.e. the inverse image of compact sets is compact), we generalize the DEGREE $\deg f \in \mathbb{R}$ of a map via the following diagram:

$$\begin{tikzcd}
H^m_c(M) \arrow[d, "\int"', "\cong"] & H^m_c(N) \arrow[l, "H^m_c(f)"'] \arrow[d, "\int"', "\cong"] \\
\mathbb{R} & \mathbb{R} \arrow[l, dotted, "\deg f"'] \\
\deg f \cdot t & t \arrow[l, mapsto]
\end{tikzcd}$$

Note that $f^* : \Omega^k_c(N) \to \Omega^k_c(M)$ is well-defined for proper f and thus also $H^k_c(f) : H^k_c(N) \to H^k_c(M)$.

29.9 Proposition.

Let $f : M \to N$ be a proper smooth mapping between connected oriented m-dimensional manifolds and $y \in N$ a regular value of f. Then

$$\deg f = \sum_{x \in f^{-1}(y)} \mathrm{sign}_x f \in \mathbb{Z},$$

where

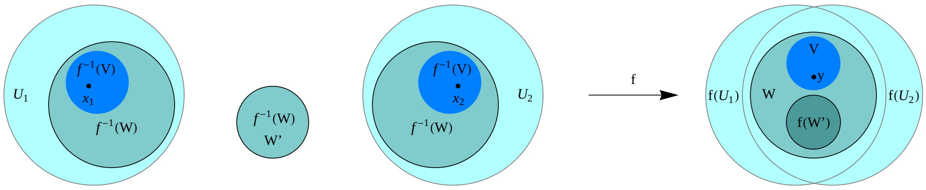
$$\mathrm{sign}_x f := \begin{cases} +1 & \text{if } T_x f : T_x M \to T_y N \text{ is orientation preserving,} \\ -1 & \text{if } T_x f : T_x M \to T_y N \text{ is orientation reversing.} \end{cases}$$

Note that according to Sard's theorem 11.15, such a regular value y always exists, and because f is proper, $f^{-1}(y)$ is finite.

Proof. Let $f^{-1}(y) = \{x_1, \dots, x_n\}$. We choose pairwise disjoint open coordinate neighborhoods U_i of x_i, s.t. $f : U_i \to f(U_i)$ is an orientation preserving or reversing diffeomorphism. We want to make $V := f(U_i)$ independent of i and $f^{-1}(V) = \bigsqcup_i U_i$. Let $W \subseteq \bigcap_i f(U_i)$ be a compact neighborhood of y. Then $W' := f^{-1}(W) \setminus \bigcup_i U_i \subseteq M \setminus f^{-1}(y)$ is compact and thus $f(W')$ is closed and does not contain y. Let $V \subseteq W \setminus f(W') \subseteq \bigcap_i f(U_i)$ be a neighborhood of y. Then

$$f^{-1}(V) \subseteq f^{-1}(W \setminus f(W')) = f^{-1}(W) \setminus f^{-1}(f(W')) \subseteq f^{-1}(W) \setminus W' \subseteq \bigcup_i U_i.$$

Because $f(f^{-1}V \cap U_i) \subseteq f(f^{-1}V) \subseteq V$ and $V \subseteq f(f^{-1}V \cap U_i)$ we may replace U_i by $f^{-1}(V) \cap U_i$ and obtain even $f(U_i) = V$. And since $f^{-1}V \subseteq \bigcup_i U_i$ we have $f^{-1}V = f^{-1}V \cap \bigcup U_i = \bigcup_i f^{-1}V \cap U_i$ and $f^{-1}(V) = \bigcup_i U_i$ after the replacement.



Now let $w \in \Omega_c^m(N)$ with $\operatorname{supp} \omega \subseteq V \subseteq \bigcap_i f(U_i)$ and $\int_M \omega = 1$. Then $[\omega]$ is a generator of $H_c^m(N) \cong \mathbb{R}$ with $\operatorname{supp} f^*(\omega) \subseteq f^{-1}(V) \subseteq \bigcup_i U_i$ and

$$\int_M f^*\omega = \sum_i \int_{U_i} f^*\omega = \sum_i \operatorname{sign}_{x_i} f \cdot \int_{f(U_i)} \omega = \sum_i \operatorname{sign}_{x_i} f \cdot \int_M \omega. \qquad \square$$

29.10 Corollary.

1. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
2. $f \sim g$ *between compact manifolds* $\Rightarrow \deg f = \deg g$.
3. f *is diffeomorphism* $\Rightarrow \deg f = \pm 1$;
   Furthermore, $\deg f = 1 \Leftrightarrow f$ *is orientation preserving.*
4. $\deg f \neq 0 \Rightarrow f$ *is surjective.*

Proof. 1 since $(f \circ g)^* = g^* \circ f^*$.

2 because then $H^k(f) = H^k(g)$ by 26.3.2.

3 follows from 1 using $\deg f \in \mathbb{Z}$ by 29.9.

4 Let f be not surjective. Then $\deg(f) = 0$ by 29.9, since every $y \in N \setminus f(M)$ is a regular value. We can also see this directly by choosing $\omega \in \Omega^m(N)$ s.t. $\operatorname{supp} \omega \subseteq N \setminus f(M)$ and $\int_N \omega = 1$. So $\deg f = \deg f \cdot \int_M \omega = \int_M f^*\omega = \int_M 0 = 0$. $\square$

29.11 Hairy Ball Theorem.

Let $\xi \in \mathfrak{X}(S^{2n})$. *Then there is an* $x \in S^{2n}$ *with* $\xi(x) = 0$.

Proof. Indirectly: Let $\xi(x) \neq 0$ for all x. Then there is a homotopy between the identity and the antipodal mapping σ (for this we connect x with $-x$ along the (half) circle in direction $\xi(x)$) and thus $1 = \deg(\mathrm{id}) = \deg(\sigma) = -1$ (see 29.6), a contradiction. $\square$

29.12 Mayer-Vietoris sequence for cohomology with compact support.

If $M = U \cup V$ with open $U, V \subseteq M$, then there are linear maps δ_k which make the following long sequence exact:

$$\dots \to H_c^k(U \cap V) \xrightarrow{(j'_U, -j'_V)} H_c^k(U) \oplus H_c^k(V) \xrightarrow{i'_U + i'_V} H_c^k(U \cup V) \xrightarrow{\delta_k}$$
$$\xrightarrow{\delta_k} H_c^{k+1}(U \cap V) \to H_c^{k+1}(U) \oplus H_c^{k+1}(V) \to H_c^{k+1}(U \cup V) \to \dots$$

with the inclusions $i_U : U \hookrightarrow U \cup V$, $i_V : V \hookrightarrow U \cup V$, $j_U : U \cap V \hookrightarrow U$ and $j_V : U \cap V \hookrightarrow V$ where the mappings i'_U, j'_U, etc. are given using extension by 0.

Cf. 26.3.4.

Proof. Because of 26.4, it suffices to show exactness of

$$0 \to \Omega_c^k(U \cap V) \longrightarrow \Omega_c^k(U) \oplus \Omega_c^k(V) \longrightarrow \Omega_c^k(U \cup V) \to 0.$$

Obviously, the first mapping is injective. The second is surjective, because $\omega = h_U\omega + h_V\omega$, where $\{h_U, h_V\}$ is a partition of unity, which is subordinate to $\{U, V\}$. The composition is obviously 0, and if $i'_U(\omega_1) + i'_V(\omega_2) = 0$, then $\operatorname{supp}\omega_1 = \operatorname{supp}(i'_U\omega_1) = \operatorname{supp}(i'_V\omega_2) = \operatorname{supp}\omega_2$, hence $\omega := \omega_1|_{U\cap V} \in \Omega_c^k(U \cap V)$ and $j'_U(\omega) = \omega_1$ and $j'_V(\omega) = -\omega_2$. □

29.13 Remark.

The cohomology H_c^* with compact supports is much harder to calculate than H^*, since the homotopy axiom does not hold for it. For example, $\mathbb{R}^m$ is homotopy-equivalent to $\{0\}$ and $H_c^0(\{0\}) = H^0(\{0\}) = \mathbb{R}$ but $H_c^0(\mathbb{R}^m) = \{0\}$, because every $f \in C_c^\infty(\mathbb{R}^m)$ with $df = 0$ must be constant and thus equal to 0. Another example is $H_c^2(S^1 \times \mathbb{R}) = \mathbb{R}$ because the cylinder $S^1 \times \mathbb{R}$ is orientable and 2-dimensional, but $H_c^2(S^1) = H^2(S^1) = \{0\}$.

29.14 Theorem. The long exact sequence of a pair.

Let $N \subseteq M$ be a compact submanifold. Then there is a long exact sequence in cohomology:

$$\dots \to H_c^k(M \setminus N) \longrightarrow H_c^k(M) \longrightarrow H_c^k(N) \xrightarrow{\delta_k}$$
$$\xrightarrow{\delta_k} H_c^{k+1}(M \setminus N) \to H_c^{k+1}(M) \to H_c^{k+1}(N) \to \dots$$

Proof. Note, that

$$0 \to \Omega_c^k(M \setminus N) \hookrightarrow \Omega_c^k(M) \xrightarrow{\text{incl}^*} \Omega_c^k(N) \to 0$$

is not exact at $\Omega_c^k(M)$, because $\ker(\text{incl}^*)$ contains all $\omega \in \Omega_c^k(M)$ which vanish on N, while the image of the extension operator $\Omega_c^k(M \setminus N) \hookrightarrow \Omega_c^k(M)$ consists of those $\omega \in \Omega_c^k(M)$, which vanish on an neighborhood of N. Therefore we replace $\Omega_c^k(N) = \Omega^k(N)$ by $\Omega^k(N \subseteq M)$, the space of germs on $N \subseteq M$ of smooth k-forms, i.e. $\Omega^k(N \subseteq M) := \bigcup_{U \supseteq N} \Omega^k(U)/\sim$, where U runs through the open neighborhoods of N in M and $\omega_1 \sim \omega_2 :\Leftrightarrow \omega_1 = \omega_2$ on a neighborhood of N in M.

Then

$$0 \to \Omega_c^k(M \setminus N) \hookrightarrow \Omega_c^k(M) \xrightarrow{\text{incl}^*} \Omega^k(N \subseteq M) \to 0$$

is obviously exact.

From 26.4 the existence of a long exact sequence in cohomology follows:

$$\dots \to H_c^k(M \setminus N) \longrightarrow H_c^k(M) \longrightarrow H^k(\Omega^*(N \subseteq M)) \xrightarrow{\delta_k} H_c^{k+1}(M \setminus N) \to \dots$$

Remains to show $H(\Omega^*(N \subseteq M)) \cong H(N)$: For this, let $p : M \supseteq U \to N$ be a tubular neighborhood according to [86, 62.9], i.e. U is open in M and p diffeomorphic to a vector bundle over N. If we choose a metric g on $p : U \to N$, then the sets $U_j := \{\xi : g(\xi, \xi) < \frac{1}{j^2}\}$ form a neighborhood basis of N and $0^* : H^k(U_j) \to H^k(N)$ is an isomorphism since U_j is homotopy equivalent to N. The restriction map $\Omega^k(N \subseteq M) \to \Omega^k(N)$ induces a mapping $H^k(\Omega^*(N \subseteq M)) \to H^k(N)$, which is surjective because the composition with $H^k(U_j) \to H^k(\Omega^*(N \subseteq M))$ is an isomorphism. It is also injective, because let $[\omega] \in \Omega^k(N \subseteq M)$ be a closed form with representant $\omega \in \Omega^k(U_j)$ and $\omega|_N \in \Omega^k(N)$ exact. Thus $\mathrm{incl}^*([\omega]) = [\omega|_N] = 0$ and hence $0 = [\omega] \in H^k(U_j)$, i.e. ω is exact and consequently also $[\omega] \in \Omega^k(N \subseteq M)$ is exact, so the cohomology class $[[\omega]]$ of $[\omega]$ in $H^k(\Omega^*(N \subseteq M))$ vanishes. □

29.15 Corollary.

Let M be a manifold with compact boundary ∂M. Then there is a long exact sequence:

$$\begin{aligned} \dots \to &H_c^k(M \setminus \partial M) \longrightarrow H_c^k(M) \longrightarrow H_c^k(\partial M) \xrightarrow{\delta_k} \\ \xrightarrow{\delta_k} &H_c^{k+1}(M \setminus \partial M) \to H_c^{k+1}(M) \to H_c^{k+1}(\partial M) \to \dots \end{aligned}$$ □

29.16 Corollary.

$$H_c^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } k = m \\ 0 & \text{for } k \neq m > 0 \end{cases}.$$

First proof. We apply 29.15 to the closed unit ball $M \subseteq \mathbb{R}^m$. Then $M \setminus \partial M \cong \mathbb{R}^m$, $\partial M = S^{m-1}$, and $H_c^k(M) = H^k(M) \cong H^k(\{*\}) = \{0\}$ for $k > 0$ and thus 29.15 yields the exact sequence

$$0 \to H^k(S^{m-1}) \xrightarrow{\delta_k} H_c^{k+1}(\mathbb{R}^m) \to 0 \qquad \text{for } k > 0$$

starting with

$$0 \to \mathbb{R} \to H^0(S^{m-1}) \xrightarrow{\delta_0} H_c^1(\mathbb{R}^m) \to 0$$

because $H_c^0(\mathbb{R}^m) = 0$, see 29.13. It follows that

$$H_c^k(\mathbb{R}^m) = H^{k-1}(S^{m-1}) \overset{26.5.13}{=\!=\!=} \begin{cases} \mathbb{R} & \text{for } 1 < k = m \\ 0 & \text{for } 1 < k \neq m \end{cases}$$

and

$$H_c^1(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } m = 1 \\ 0 & \text{for } m > 1. \end{cases}$$ □

Second proof. ($k = 0$) we have already seen in 29.13.

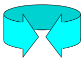
($0 < k < n$) We have $\omega \in \Omega_c^k(\mathbb{R}^n)$ with $d\omega = 0$. According to the Poincaré lemma, an $\eta \in \Omega^{k-1}(\mathbb{R}^n)$ exists with $d\eta = \omega$. Let B be a ball with $\operatorname{supp}\omega \subseteq B$. For $k = 1$ the form η is therefore constant outside B, say c, and thus $\eta - c \in \Omega_c^{n-1}(\mathbb{R}^n)$ with $d(\eta - c) = d\eta = \omega$. If $k > 1$, $\eta|_{\mathbb{R}^n \setminus B}$ is closed and, because of $\mathbb{R}^n \setminus B \cong \mathbb{R}^n \setminus \{0\}$ and $H^{k-1}(\mathbb{R}^n \setminus \{0\}) \cong H^{k-1}(S^{n-1}) = \{0\}$ by 26.5.13, there is an $\lambda \in \Omega^{k-2}(\mathbb{R}^n \setminus B)$ with $d\lambda = \eta|_{\mathbb{R}^n \setminus B}$. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $f = 1$ on $\mathbb{R}^n \setminus 2B$ and $f = 0$ on a neighborhood of B. Then $f\,\lambda \in \Omega^{k-2}(\mathbb{R}^n)$ is well-defined and $\eta - d(f\,\lambda) \in \Omega^{k-1}(\mathbb{R}^n)$ has compact support in $2B$ with $d(\eta - d(f\,\lambda)) = d\eta = \omega$. □

29.17 Preparation for the generalized curve theorem of Jordan.

Let $M \subseteq \mathbb{R}^{m+1}$ be a compact connected hypersurface. First we want to show that $\mathbb{R}^{m+1} \setminus M$ has at least 2 connected components. Let M be oriented for the moment. The WINDING NUMBER of M with respect to $p \notin M$ is defined by

$$w_M(p) := \deg(r_p|_M), \text{ where } r_p : x \mapsto \frac{1}{\|x-p\|}(x-p), \quad M \to S^m.$$

It is constant on the connected components of $\mathbb{R}^{m+1} \setminus M$: Namely, if $t \mapsto p(t)$ is a curve in $\mathbb{R}^{m+1} \setminus M$, then $(t,x) \mapsto r_{p(t)}(x)$ is a homotopy and thus $t \mapsto \deg(r_{p(t)}|_M) = w_M(p(t))$ is constant. Up to a diffeomorphism (with compact support), $0 \in M$ and M is locally around 0 a hyperplane $v^\perp$. We claim that $w_M(p) - w_M(q) = \pm 1$ for p, q close to 0 on different sides of $v^\perp$ and thus $\mathbb{R}^{m+1} \setminus M$ *has at least two connected components*. In fact, $[-1,1] \ni t \mapsto r_{tv}|_{M \setminus \{0\}}$ is a homotopy and for $x \in v^\perp$ with $\|x\| \leq \delta$ the image is a polar cap around v which degenerates for $t = 0$ to the equator and then mutates into the opposite polar cap.



We modify this homotopy to a homotopy H on the whole of M by keeping the images of points close to 0 fixed near the pole v and not allowing the remaining points to come close to the antipodal pole $-v$. Let y near $-v$ be a regular value for r_v. Then, the end value $H_1 : M \to S^m$ of the homotopy has one inverse image x (near 0) less than r_v, and thus $w_M(-v) = \deg(r_{-v}|_M) = \deg(H_{-1}) = \deg(H_1) = \deg(r_v|_M) - \operatorname{sign}_x(r_v) = w_M(v) \pm 1$ by 29.9.

Now let M and N be compact and connected but not necessarily orientable. We define the MOD-2 DEGREE of $f : M \to N$ by $\deg_2(f) := \sum_{x \in f^{-1}(y)} 1 \in \mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z})$, where y is some regular value of f.

We have to show that *this number modulo 2 does not depend on the choice of* y. For the moment, let f_0 and f_1 be smoothly homotopic via $H : [0,1] \times M \subseteq \mathbb{R} \times M \to N$ and let y be a joint regular value of f_0 and f_1. Without loss of generality, $H(t,x) = f_i(x)$ for t near $i \in \{0,1\}$ (replace H by $(t,x) \mapsto H(h(t),x)$, with h constant near 0 and near 1). By the proof of 29.9, all values close to regular values are themselves regular (and have the same number of inverse images). Thus we may assume (because of the Proposition 11.15 of Sard) that y is a regular value of H and also of f_0 and f_1. Hence $H^{-1}(y)$ is a 1-dimensional submanifold of $\mathbb{R} \times M$ that intersects $\{0,1\} \times M$ transversely. The trace $H^{-1}(y) \cap [0,1] \times M$ is thus a disjoint union of finitely many connected compact 1-dimensional manifolds with boundary contained in $\partial([0,1] \times M) = \{0,1\} \times M$ and thus the total number of boundary points $\{(i,x) : i \in \{0,1\}, f_i(x) = y\}$ is even, so

$$\sum_{x \in f_0^{-1}(y)} 1 \equiv -\sum_{x \in f_1^{-1}(y)} 1 \equiv \sum_{x \in f_1^{-1}(y)} 1 \mod 2.$$

Now, if y_0 and y_1 are both regular values of f, then there exists a diffeotopy h_1 on N (i.e. a diffeomorphism which can be connected to the identity by means of a homotopy consisting entirely of diffeomorphisms) with compact support which maps y_0 to y_1 (the equivalence classes of points with respect to diffeotopies are open: consider the flow Fl_t^ξ of the vector field $\xi = f \cdot \frac{\partial}{\partial u^1}$ with appropriate f with compact support) and thus $h_1 \circ f \sim f$ and $y_1 = h_1(y_0)$ is a regular value of f and of $h_1 \circ f$, hence $|f^{-1}(y_0)| = |(h_1 \circ f)^{-1}(y_1)| \equiv |f^{-1}(y_1)| \mod 2$ by what we have previously shown.

Now, if we define the winding number $w_M(p) \in \mathbb{Z}_2$ as before, but with $\deg_2$ instead of deg, we can proceed in the proof as above and get $w_p(M) \neq w_q(M)$ for points locally on different sides of M.

29.18 Generalized Curve Theorem of Jordan.

Let $M \subseteq \mathbb{R}^{n+1}$ be a compact connected hypersurface. Then M is orientable and $\mathbb{R}^{n+1} \setminus M$ has exactly 2 connected components and M is the boundary of both. In particular this holds for $M \cong S^n$.

Proof. The cohomology sequence 29.14 of the pair $M \subseteq \mathbb{R}^{n+1}$ is by 29.16 the following:

$$\to \underbrace{H_c^n(\mathbb{R}^{n+1})}_{=0} \to H^n(M) \xrightarrow{\delta} H_c^{n+1}(\mathbb{R}^{n+1} \setminus M) \to \underbrace{H_c^{n+1}(\mathbb{R}^{n+1})}_{\cong\mathbb{R}} \to \underbrace{H^{n+1}(M)}_{=0} \to$$

Thus, by 26.5.8, $1 + \dim H^n(M) = \dim(H_c^{n+1}(\mathbb{R}^{n+1} \setminus M)$ is the number of (by 29.17 at least 2) connected components of $\mathbb{R}^{n+1} \setminus M$. So $\dim H^n(M) \geq 1$ and thus M is orientable and $\dim H^n(M) = 1$ by 29.5. Hence $\mathbb{R}^{n+1} \setminus M$ has exactly 2 connected components.

Since, by the arguments in 29.17, near $x \in M$ there are points in both connected component of $\mathbb{R}^{n+1} \setminus M$ we get that M is the boundary of each component. □

29.19 Corollary.

Neither the projective plane nor the Klein bottle can be realized as a submanifold of $\mathbb{R}^3$.

Proof. Otherwise they would be orientable by 29.18. □

29.20 Example.

Even for orientable connected 2-dimensional manifolds M, the first cohomology $H^1(M)$ need not be finite dimensional. Let e.g. $U := \mathbb{C} \setminus \mathbb{Z} \subseteq \mathbb{R}^2$ and $V \subseteq \mathbb{C}$ be the union of the open balls around all $z \in \mathbb{Z}$ with radius $\frac{1}{3}$. Then $U \cap V \sim \bigsqcup_{\mathbb{Z}} S^1$ and thus the Mayer-Vietoris 26.3.4 sequence is

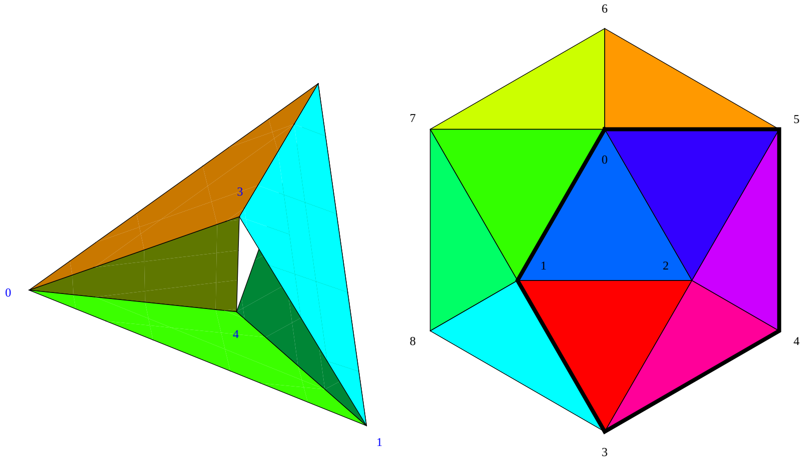
$$0 \to H^1(U) \oplus 0 \to H^1(U \cap V) \xrightarrow{\delta} 0.$$

So $H^1(U) \cong H^1(U \cap V) \cong H^1(\bigsqcup_{\mathbb{Z}} S^1) = \prod_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{\mathbb{Z}}$.

29.21 Definition.

The CUP PRODUCT $\cup : H^k(M) \times H_c^j(M) \to H_c^{k+j}(M)$ is defined by $[\alpha]\cup[\beta] := [\alpha\wedge\beta]$ and orientable manifolds M the POINCARÉ DUALITY $H^k(M) \to H_c^{m-k}(M)^*$ is the induced linear mapping via $H_c^m(M) \cong \mathbb{R}$.

A TRIANGULATION is a finite family $\{\sigma_i : i\}$ of diffeomorphic images of the STANDARD m-SIMPLEX $\Delta_m := \{x = (x^0, \dots, x^m) \in \mathbb{R}^{m+1} : \sum_i x^i = 1 \text{ and } \forall\, i : x^i \geq 0\}$, s.t. $\sigma_i \cap \sigma_j \neq \emptyset \Rightarrow \sigma_i \cap \sigma_j$ is a k-face of σ_i and of σ_j, where a k-FACE is the image of the subset of Δ_m formed by setting $m - k$ many coordinates x^i to 0. It can be shown with some effort that every smooth manifold has a triangulation, see [**122**] or [**154**].



A triangulation of the Möbius strip and the projective plane

29.22 Proposition.

Let M be a connected oriented manifold. Then the Poincaré duality is an isomorphism $H^k(M) \xrightarrow{\cong} H_c^{m-k}(M)^*$.

Proof (for triangulable manifolds). If $M = U \cup V$ with open U and V, s.t. the theorem for U, V and $U \cap V$ holds, then by the Mayer-Vietoris sequence 26.3.4 and the dual of Mayer-Vietoris sequence 29.12 for compact support

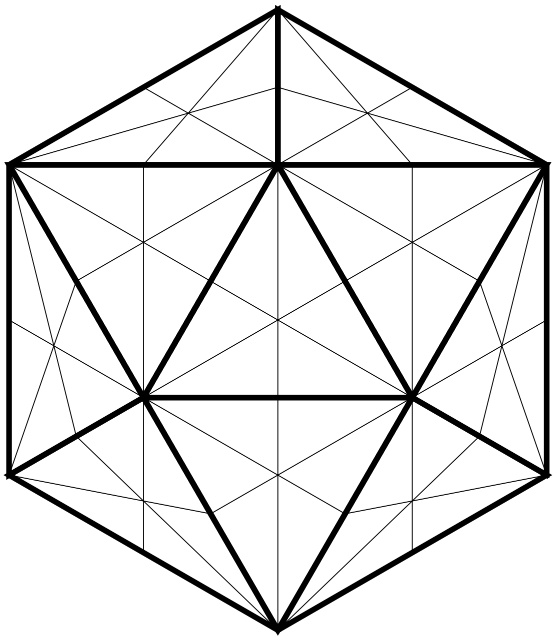
$$\begin{tikzcd}
H^{k-1}(U) \oplus H^{k-1}(V) \arrow[r] \arrow[d] & H^{k-1}(U \cap V) \arrow[r] \arrow[d] & H^k(M) \arrow[r] \arrow[d] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d] & H^k(U \cap V) \arrow[d] \\
H_c^{l+1}(U)^* \oplus H_c^{l+1}(V)^* \arrow[r] & H_c^{l+1}(U \cap V)^* \arrow[r] & H_c^l(M)^* \arrow[r] & H_c^l(U)^* \oplus H_c^l(V)^* \arrow[r] & H_c^l(U \cap V)^*
\end{tikzcd}$$

we get the result for M by means of the following 5-Lemma 29.23, because as exercise [**98**, EX30] shows the diagram to be commutative, and the dual of an exact sequence is also exact (as one easily shows).

To use this for a proof by induction, we choose on each face of the simplices of the triangulation an “inner” point, e.g. the barycenter. Recursively, we define a covering of M by disjoint unions U_k of open contractible subsets of M as follows:

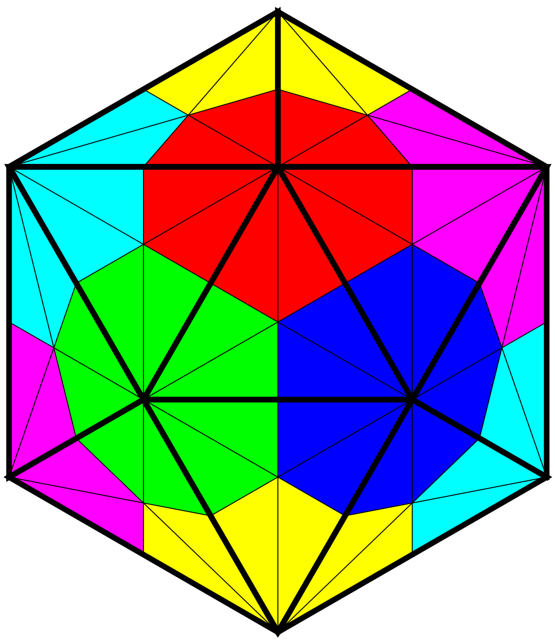
Let U_0 be the disjoint union of suitably chosen contractible neighborhoods of each vertex, which does not contain any of the other interior points.

The set U_k then consists of the disjoint union of suitably chosen open contractible neighborhoods of the choosen points on the faces of dimension k.

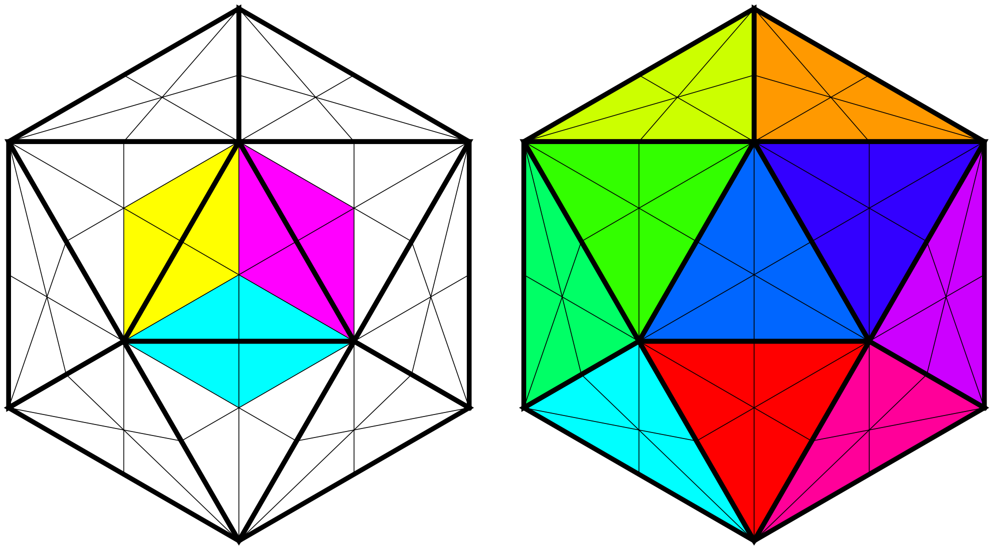


Explicitly, this can be achieved by looking at all the simplices that have the previously chosen inner points as vertices, for ascending ordered faces of a simplex of the triangulation.

Now we take as U_0 the union of all such “open” simplices which have one of the original vertices as a joint vertex.



And more generally, for U_k, we take the union of all such “open” simplices that each have one of the inner points of a k-simplex as a vertex.



Obviously $\sigma \subseteq \bigcup_{j\le k} U_j$ for each (closed) k-simplex σ and thus $\bigcup_{k=0}^m U_k = M$. Furthermore, $U_k \cong \bigsqcup_{\alpha_k} \mathbb{R}^m$ and, for $k>0$, $U_k \cap \bigcup_{j<k} U_j \cong \bigsqcup_{\alpha_k} S^{k-1}\times\mathbb{R}^{m-k+1}$.

Clearly, the Poincaré duality holds for $\mathbb{R}^m$ (because $H_0(\mathbb{R}^m) = \mathbb{R} = H_c^m(\mathbb{R}^m)^*$ and 0 otherwise), and it follows by induction (using the Mayer-Vietoris sequence) that it also holds for $S^m \times \mathbb{R}^k$ (See exercise [**98**, EX29]). Since $H^k(\bigsqcup_{j\in J} M_j) \cong \prod_{j\in J} H^k(M_j)$ and $H_c^k(\bigsqcup_{j\in J} M_j) \cong \bigoplus_{j\in J} H_c^k(M_j)$, it also holds for $U_k \cap \bigcup_{j<k} U_j$ and thus by induction for $\bigcup_{j\le k} U_j$ and hence also for $M = \bigcup_{j\le \dim(M)} U_j$. □

29.23 Five Lemma.

Let

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\varphi_4"] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\
B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r, "\psi_3"] & B_4 \arrow[r, "\psi_4"] & B_5
\end{tikzcd}$$

be a commutative diagram with exact horizontal rows. If all but the middle vertical arrow are isomorphisms, then also the middle one is an isomorphism.

Proof.
(f_3 is injective)

$$\begin{aligned}
f_3a_3 = 0 \Rightarrow 0 = \psi_3 f_3 a_3 &= f_4\varphi_3 a_3 \\
&\xRightarrow{f_4 \text{ inj.}} \varphi_3 a_3 = 0 \\
&\xRightarrow{\text{exact at } A_3} \exists\, a_2 : a_3 = \varphi_2 a_2 \\
&\Rightarrow 0 = f_3 a_3 = f_3\varphi_2 a_2 = \psi_2 f_2 a_2 \\
&\xRightarrow{\text{exact at } B_2} \exists\, b_1 : f_2 a_2 = \psi_1 b_1 \\
&\xRightarrow{f_1 \text{ surj.}} \exists\, a_1 : b_1 = f_1 a_1 \\
&\Rightarrow f_2 a_2 = \psi_1 f_1 a_1 = f_2 \varphi_1 a_1 \\
&\xRightarrow{f_2 \text{ inj.}} a_2 = \varphi_1 a_1 \\
&\xRightarrow{\text{exact at } A_2} a_3 = \varphi_2 a_2 = \varphi_2\varphi_1 a_1 = 0
\end{aligned}$$

$$
\begin{tikzcd}
a_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"'] & a_2 \arrow[r, "\varphi_2"] \arrow[d, tail, "f_2"'] & a_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & 0 \arrow[d, tail, "f_4"'] & \bullet \\
b_1 \arrow[r, "\psi_1"] & f_2(a_2) \arrow[r, "\psi_2"] & 0 \arrow[r, "\psi_3"] & 0 & \bullet
\end{tikzcd}
$$

(f_3 is surjective)

$$
\begin{aligned}
b_3 &\xRightarrow{f_4 \text{ surj.}} \exists\, a_4 : f_4 a_4 = \psi_3 b_3 \\
&\xRightarrow{\text{exact at } B_4} f_5 \varphi_4 a_4 = \psi_4 f_4 a_4 = \psi_4 \psi_3 b_3 = 0 \\
&\xRightarrow{f_5 \text{ inj.}} \varphi_4 a_4 = 0 \\
&\xRightarrow{\text{exact at } A_4} \exists\, a_3 : a_4 = \varphi_3 a_3 \\
&\Rightarrow \psi_3 f_3 a_3 = f_4 \varphi_3 a_3 = f_4 a_4 = \psi_3 b_3 \\
&\xRightarrow{\text{exact at } B_3} \exists\, b_2 : b_3 - f_3 a_3 = \psi_2 b_2 \\
&\xRightarrow{f_2 \text{ surj.}} \exists\, a_2 : b_2 = f_2 a_2 \\
&\Rightarrow b_3 = f_3 a_3 + \psi_2 b_2 = f_3 a_3 + \psi_2 f_2 a_2 = f_3(a_3 + \varphi_2 a_2)
\end{aligned}
$$

$$
\begin{tikzcd}
\bullet & a_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"'] & a_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & a_4 \arrow[r, "\varphi_4"] \arrow[d, "f_4"'] & \varphi_4(a_4) \arrow[d, tail, "f_5"'] \\
\bullet & b_2 \arrow[r, "\psi_2"] & b_3 \arrow[r, "\psi_3"] & \psi_3(b_3) \arrow[r, "\psi_4"] & 0
\end{tikzcd}
\qquad \square
$$

29.24 Proposition.

Let $\mathcal{K}$ be a triangulation of a compact manifold M and α_i be the number of i-simplices of $\mathcal{K}$. Then

$$\chi(M) = \sum_i (-1)^i \alpha_i.$$

Proof. We use the open sets $U_k \subseteq M$ constructed in the proof of $\boxed{29.22}$, which are disjoint unions of α_k many sets diffeomorphic to $\mathbb{R}^m$ and for which the $U_k \cap \bigcup_{j<k} U_j$ are disjoint unions of α_k many sets diffeomorphic to $S^{k-1} \times \mathbb{R}^{m-k+1} \sim S^{k-1}$. Thus, for the Euler characteristic we get:

$$
\begin{aligned}
\chi\Big(\bigcup_{j<k} U_j\Big) + \alpha_k = \chi\Big(\bigcup_{j<k} U_j\Big) + \chi(U_k) &\xlongequal{\boxed{26.5.8}} \chi\Big(\bigcup_{j\le k} U_j\Big) + \chi\Big(U_k \cap \bigcup_{j<k} U_j\Big) \\
&\xlongequal{\boxed{26.5.9}} \chi\Big(\bigcup_{j\le k} U_j\Big) + (1 + (-1)^{k-1})\alpha_k.
\end{aligned}
$$

So

$$\chi\Big(\bigcup_{j\le k} U_j\Big) = \chi\Big(\bigcup_{j<k} U_j\Big) + (-1)^k \alpha_k$$

and thus

$$\chi(M) = \chi\Big(\bigcup_{j\le m} U_j\Big) = \chi(\emptyset) + \sum_{j\le m} (-1)^j \alpha_j = \sum_{j\le m} (-1)^j \alpha_j. \quad \square$$

29.25 Definition (Thom and Euler class of a vector bundle).

Let $p : E \to M$ be an oriented k-plane bundle over an m-dimensional compact oriented connected manifold M. The cup product

$$\cup : H^m(E) \times H_c^k(E) \to H_c^{m+k}(E) \cong \mathbb{R}, \quad [\alpha] \cup [\beta] := [\alpha \wedge \beta]$$

induces by 29.21 and 29.22 the Poincaré duality

$$H_c^k(E) \xrightarrow{\cong} H^m(E)^*, \quad [\beta] \mapsto \Big([\alpha] \mapsto \int_E \alpha \wedge \beta\Big).$$

Since $0 : M \hookrightarrow E$ is a deformation retract with retraction p, $H^m(M) \cong H^m(E)$ via $[\gamma] \mapsto [p^*(\gamma)]$ and thus

$$H_c^k(E) \cong H^m(E)^* \cong H^m(M)^*$$

via

$$H_c^k(E) \ni [\beta] \mapsto \Big(H^m(E) \ni [\alpha] \mapsto \int_E \alpha \wedge \beta\Big) \mapsto \Big(H^m(M) \ni [\gamma] \mapsto \int_E p^*(\gamma) \wedge \beta\Big).$$

By 29.5 we have $\mathbb{R} \cong \mathbb{R}^* \cong H^m(M)^*$, via $1 \mapsto \int_M \Big(: H^m(M) \to \mathbb{R},\ [\gamma] \mapsto \int_M \gamma\Big)$. Thus, there exists a unique class $U(p) := [\tau] \in H_c^k(E)$, the so-called Thom class of the k-plane bundle p, defined by

$$\int_E p^*(\gamma) \wedge \tau = \int_M \gamma \text{ for all } [\gamma] \in H^m(M).$$

The Euler class $\chi(p) \in H^k(M)$ of the k-plane bundle p is then defined by

$$\chi(p) := 0^*(U(p)) = s^*(U(p)),$$

where $0 : M \hookrightarrow E$ is the 0-section, resp. s is any (obviously homotopic) section of p.

If $p : E \to M$ has a nowhere vanishing section s, then $\chi(p) = (K \cdot s)^*(U(p)) = 0$, where K was chosen such that $\operatorname{im}(K \cdot s) \cap \operatorname{supp}(\tau) = \emptyset$ for $[\tau] = U(p)$.

29.26 Proposition.

Let M be a connected compact oriented m-dimensional manifold. Furthermore, let $p : E \to M$ be an oriented k-plane bundle, and for $x \in M$ let $j_x : E_x \hookrightarrow E$ be the inclusion of the fiber E_x over x.
Then the Thom class $U(p) = [\tau]$ is the unique element from $H_c^k(E)$ with $\int_{E_x} j_x^*(\tau) = 1$ for all $x \in M$,

Proof. The Thom class $[\tau] \in H_c^k(E)$ is unquely defined by the implicite equation $\int_E p^*(\mu) \wedge \tau = \int_M \mu$ for all $\mu \in \Omega^m(M)$. Let $W \cong \mathbb{R}^m$ be an open subset of M for which $E|_W$ is trivial, that is, w.l.o.g. $E|_W = W \times \mathbb{R}^k$ and $p = \mathrm{pr}_1$, as well as $j_x : v \mapsto (x, v)$. Then there is a $K > 0$ with $\operatorname{supp}(\tau|_{p^{-1}(W)}) \subseteq W \times \{v : \|v\| < K\}$ and for the moment let $\operatorname{supp}(\mu) \subseteq W$. The contraction $\underline{\mathrm{H}} : W \times I \to W$ from W to $x \in W$ induces a smooth homotopy $H : W \times \mathbb{R}^k \times I \to W \times \mathbb{R}^k$, $(y, v, t) \mapsto (\underline{\mathrm{H}}(y, t), v)$ with $H_0 = \mathrm{id}$ and $H_1 = (\mathrm{konst}_x, \mathrm{pr}_2) = j_x \circ \mathrm{pr}_2$. We have $\operatorname{supp}(H^*\tau) \subseteq H^{-1}(\operatorname{supp} \tau|_{p^{-1}(W)}) \subseteq \{(y, v, t) : \|v\| < K\}$. Therefore, for $\lambda := G(\tau) \in \Omega^{k-1}(W \times \mathbb{R}^k)$, where $G := I_0^1 \circ \iota_\xi \circ H^*$ is the homotopy operator from the proof of the homotopy axiom 26.3.2 with $\xi := \frac{\partial}{\partial t}$, we have $\operatorname{supp}(\lambda) \subseteq \{(y, v) : \|v\| < K\}$, and thus

$$\mathrm{pr}_2^* j_x^* \tau - \tau = (H_1)^*(\tau) - (H_0)^*(\tau) \overset{26.3.2}{=\!=\!=} (dG + Gd)(\tau) = d\lambda.$$

Thus (compare with the proof of 29.3 or see exercise [98, EX26])

$$\int_{p^{-1}(W)} p^*(\mu)\wedge\tau = \int_{W\times\mathbb{R}^k} \operatorname{pr}_1^*\mu\wedge(\operatorname{pr}_2^* j_x^*\tau - d\lambda) = \int_W \mu\cdot\int_{\mathbb{R}^k} j_x^*\tau,$$

because $\operatorname{pr}_1^*\mu\wedge d\lambda = \pm d(\operatorname{pr}_1^*\mu\wedge\lambda)$ and hence $\int_{W\times\mathbb{R}^k}\operatorname{pr}_1^*\mu\wedge d\lambda = 0$. As a result, $\int_{\mathbb{R}^k} j_x^*\tau$ is independent on $x\in W$ and hence we denote this value by $\int_{\mathbb{R}^k} j^*\tau$.

Let $\mathcal{W}$ be a covering of M with trivializing open sets W as before and let $\{\lambda_W : W\in\mathcal{W}\}$ be a subordinated partition of 1. Then

$$\begin{aligned}\int_M \mu &= \int_E p^*(\mu)\wedge\tau = \sum_{W\in\mathcal{W}}\int_{p^{-1}(W)} p^*(\lambda_W\mu)\wedge\tau\\ &= \sum_{W\in\mathcal{W}}\int_W \lambda_W\mu\cdot\int_{\mathbb{R}^k} j^*\tau = \int_M\mu\cdot\int_{\mathbb{R}^k} j^*\tau,\end{aligned}$$

hence $\int_{\mathbb{R}^k} j^*\tau = 1$.

Uniqueness: For each $U\in H_c^k(E)$ there exists $c\in\mathbb{R}$ with $U = c\cdot[\tau]$ because $H_c^k(E)\cong\mathbb{R}$. If U has also the required property, then $j_x^*U = j_x^*(c[\tau]) = cj_x^*[\tau]$ and thus $c=1$ because of $c\int_{E_x} j_x^*[\tau] = \int_{E_x} j_x^*U = 1 = \int_{E_x} j_x^*[\tau]$. □

29.27 Definition. Index of vector fields in isolated zeros.

Let ξ be a vector field with isolated zero 0 on an open set $U\subseteq\mathbb{R}^m$. Then the INDEX of ξ at 0 is defined by

$$\operatorname{ind}_0(\xi) = \deg\Big(r\circ\xi\circ\iota : S^{m-1}\hookrightarrow U\setminus\{0\}\xrightarrow{\ \xi\ }\mathbb{R}^m\setminus\{0\}\twoheadrightarrow S^{m-1}\Big),$$

where $r(x) := \frac{1}{\|x\|}x$ and $\iota : S^{m-1}\hookrightarrow U\setminus\{0\}$ is the embedding of a sphere, contained together with its interior in $U\setminus(\xi^{-1}(0)\setminus\{0\})$.

This index is invariant under diffeomorphisms: In fact, if h is a (for now) orientation preserving diffeomorphism with $h(0)=0$, then id is smooth homotopic to $h'(0)$ (since $GL_+(\mathbb{R}^m)$ is connected, see [103, 1.10]) and furthermore

$$H(x,t) := \begin{cases}\frac{h(tx)}{t} & \text{for } t>0\\ h'(0)(x) & \text{for } t=0\end{cases}$$

is a smooth homotopy between $h'(0)$ and h locally around 0. Thus $r\circ\xi\sim r\circ h^*\xi$ near 0 and thus $\deg(r\circ\xi\circ\iota) = \deg(r\circ h^*\xi\circ\iota)$.

In order to obtain this also for non orientation preserving h it suffices to show this in particular for the linear isometry $h:(x^1,\dots,x^{m-1},x^m)\mapsto(x^1,\dots,x^{m-1},-x^m)$. For this we have $h^*\xi = h^{-1}\circ\xi\circ h$ and thus $r\circ h^*\xi\circ\iota = r\circ h^{-1}\circ\xi\circ h\circ\iota = h|_{S^{m-1}}^{-1}\circ r\circ\xi\circ\iota\circ h|_{S^{m-1}}$ so

$$\deg(r\circ h^*\xi\circ\iota) = \deg(h|_{S^{m-1}}^{-1}\circ r\circ\xi\circ\iota\circ h|_{S^{m-1}}) \overset{\boxed{29.10.1}}{=\!=} \deg(r\circ\xi\circ\iota).$$

The radial vector field $x\mapsto x$ obviously has index 1 at 0. More generally, a linear vector field ξ at $\mathbb{R}^m$ which is diagonalizable with k negative and $m-k$ positive eigenvalues has index $(-1)^k$, because up to linear orientation preserving isomorphisms, it is of the form

$$(x_1,\dots,x_m)\mapsto(-x_1,\dots,-x_k,x_{k+1},\dots,x_m)$$

and restricted to S^{m-1} has degree $(-1)^k$ (because, for example, e_1 is a regular value of $\xi\circ\iota = r\circ\xi\circ\iota$ with single inverse image $-e_1$ and $\det(T_{-e_1}\xi|_{S^{m-1}}) = (-1)^k$).

For a vector field ξ with isolated zero x on a manifold we define the INDEX $\operatorname{ind}_x\xi$ as $\operatorname{ind}_0\bar\xi$ for a chart representation $\bar\xi$ of ξ centered at x.

29.28 Proposition.

Let M be a compact oriented connected manifold and $[\mu] \in H^m(M)$ with $\int_M \mu = 1$. Let $\xi \in \mathfrak{X}(M)$ be a vector field with only isolated zeros. Then

$$\chi\big(\pi_M : TM \to M\big) = \Big(\sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi\Big) \cdot [\mu] \in H^m(M).$$

Proof. Let $\xi^{-1}(0) =: \{x_1, \dots, x_k\}$ and W_i be pairwise disjoint compact chart neighborhoods centered at x_i and diffeomorphic to the closed ball $D := \{v \in \mathbb{R}^m : \|v\| \le 1\}$. So $TM|_{W_i} \cong W_i \times \mathbb{R}^m$ is trivial. We extend the norm on $\mathbb{R}^m$ to a Riemannian metric g on M. Let $[\tau] \in H^k_c(TM)$ be the Thom class of $\pi_M : TM \to M$ and hence a $K > 0$ exists with $\operatorname{supp}(\tau) \subseteq \{\eta_x \in TM : g(\eta_x, \eta_x) < K^2\}$. Consider the scalar multiplication $K : TM \to TM$, $\xi \mapsto K\xi$. Then

$$\int_{E_x} j_x^*(K^*\tau) = \int_{E_x} K^*(j_x^*\tau) = \int_{E_x} j_x^*\tau = 1,$$

hence $[\tau] = [K^*\tau]$ by 29.25 and $K \cdot \operatorname{supp}(K^*\tau) = \operatorname{supp}(\tau)$. So we may assume w.l.o.g. that $\operatorname{supp}(\tau) \subseteq \{\eta_x \in TM : g(\eta_x, \eta_x) < 1\}$, i.e. $K = 1$.
Because of $\chi(\pi_M) := \xi^*([\tau])$ by 29.25 we have to show $\int_M \xi^*\tau = \sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi$. By stretching ξ (which does not change $\operatorname{ind}_x \xi$) we achieve $g(\xi_y, \xi_y) \ge K^2$ for all $y \notin \bigcup_i W_i$, i.e. $\operatorname{supp}(\xi^*\tau) \subseteq \xi^{-1}(\operatorname{supp}\tau) \subseteq \bigcup_{i=1}^k W_i$. Thus

$$\int_M \xi^*\tau = \sum_{i=1}^k \int_{W_i} \xi^*\tau$$

and it is enough to show $\int_{W_i} \xi^*\tau = \operatorname{ind}_{x_i} \xi$.

For the sake of simplicity we omit the index i in the remainder of the proof and denote with $\iota : D \to \iota(D) =: W$ a chart centered at $\{x\} := \xi^{-1}(0) \cap W$ and $\psi = T\iota \circ (\iota^{-1} \times \mathbb{R}^m) : W \times \mathbb{R}^m \to TM|_W$ the corresponding VB-chart. Let $\bar\tau := \psi^*\tau$ and let $\bar\xi := \psi^{-1} \circ \xi$ be the representation of ξ. Then $\bar\xi^*\bar\tau = \bar\xi^*\psi^*\tau = (\psi\circ\bar\xi)^*\tau = \xi^*\tau$. In the proof of 29.26 we have shown that $\operatorname{pr}_2^* j_x^*\bar\tau - \bar\tau = d\lambda$ for a $\lambda \in \Omega^{m-1}(W \times \mathbb{R}^m)$ with $\operatorname{supp}(\lambda) \subseteq \{(y, v) : \|v\| < 1\}$ because $\operatorname{supp}(\bar\tau) = \psi^{-1}(\operatorname{supp}\tau) \subseteq \{(y, v) : \|v\| < 1\}$. For all $y \in \partial W = \iota(S^{m-1})$ we have $g(\xi_y, \xi_y) \ge 1$ and thus $\bar\xi^*\lambda|_{\partial W} = 0$, hence

$$\int_W \bar\xi^* \operatorname{pr}_2^* j_x^*\bar\tau - \int_W \bar\xi^*\bar\tau = \int_W \bar\xi^* d\lambda = \int_W d\bar\xi^*\lambda \overset{\boxed{28.11}}{=\!=\!=} \int_{\partial W} \bar\xi^*\lambda = 0.$$

Because of the Poincaré lemma 26.5.6 for $W \cong D \subseteq \mathbb{R}^m$, $j_x^*\bar\tau = d\rho$ for some $\rho \in \Omega^{m-1}(\mathbb{R}^m)$. Since the retraction $r : \mathbb{R}^m \setminus \{0\} \to S^{m-1} \subseteq \mathbb{R}^m$ is homotopic to the identity we get

$$\begin{aligned}
\int_W \xi^*\tau &= \int_W \bar\xi^*\bar\tau = \int_W \bar\xi^* \operatorname{pr}_2^* j_x^*\bar\tau = \int_W \bar\xi^* \operatorname{pr}_2^* d\rho \overset{\boxed{28.11}}{=\!=\!=} \int_{\partial W} \bar\xi^* \operatorname{pr}_2^* \rho \\
&= \int_{\partial W} \bar\xi^*(W \times r)^* \operatorname{pr}_2^* \rho = \int_{\iota(S^{m-1})} (\operatorname{pr}_2 \circ (W \times r) \circ \bar\xi)^*\rho \\
&= \int_{S^{m-1}} (r \circ \operatorname{pr}_2 \circ \bar\xi \circ \iota)^*\rho = \deg(r \circ \operatorname{pr}_2 \circ \bar\xi \circ \iota) \cdot \int_{S^{m-1}} \rho = \operatorname{ind}_x \xi \cdot 1,
\end{aligned}$$

because $\operatorname{supp}(\bar\tau) \subseteq \{(y, v) : \|v\| < 1\}$ and thus

$$\int_{S^{m-1}} \rho = \int_{\partial D} \rho \overset{\boxed{28.11}}{=\!=\!=} \int_D d\rho = \int_D j_x^*\bar\tau = \int_{\mathbb{R}^m} j_x^*\bar\tau = \int_{T_xM} j_x^*\tau \overset{\boxed{29.26}}{=\!=\!=} 1. \quad \square$$

29.29 Theorem of Poincaré-Hopf.

Let M be a compact oriented connected manifold and let $\xi \in \mathfrak{X}(M)$ be a vector field with isolated zeros only. Then

$$\chi(M) = \sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi = \int_M \chi(\pi_M).$$

Proof (for triangulable manifolds). For $\mu \in \Omega^m(M)$ with $\int_M \mu = 1$, and for each vector field ξ, with isolated zeros only, we have by 29.28 that

$$\sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi \cdot [\mu] = \chi(\pi_M)$$

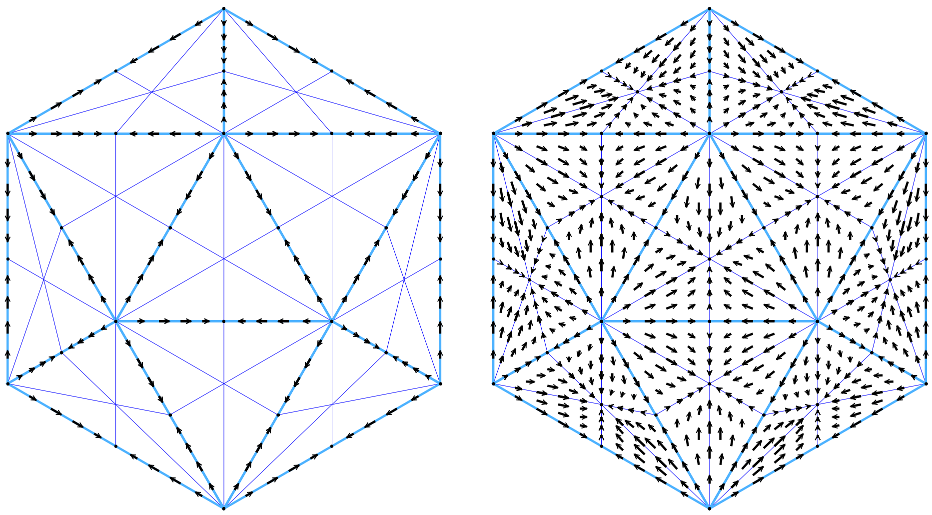
and thus

$$\sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi = \int_M \sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi \cdot \mu = \int_M \chi(\pi_M)$$

is independent on ξ. Therefore, it is sufficient to find such some $\xi \in \mathfrak{X}(M)$ with

$$\chi(M) = \sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi.$$

We use a finite triangulation and, as in the proof of 29.22, we choose an "inner" point on each face simplex. By 29.24, $\chi(M) = \sum_{k=0}^m (-1)^k \alpha_k$, where α_k denotes the number of k-simplices. Recursively we choose a vector field ξ with exactly these points as zeros, so that on each k-simplex it has the choosen inner points as sink.



According to what is said in 29.27, the index of ξ in these inner points of the k-simplices is just $(-1)^k$ and thus

$$\sum_{x\in\xi^{-1}(0)} \operatorname{ind}_x \xi = \sum_{k=0}^m (-1)^k \alpha_k = \chi(M). \quad \square$$

Bibliography

[1] J. F. Adams. Vector fields on spheres. *Ann. of Math.*, 75:603–632, 1962.

[2] J.W. Alexander. An example of a simply connected surface bounding a region which is not simply connected. *Proc.Nat.Acad.Sci. USA.*, 10:8–10, 1924.

[3] C. Bankwitz. Über die Fundamentalgruppe des inversen Knotens und des gerichteten Knotens. *Ann.of Math*, 31:129–130, 1930.

[4] J.L.M. Barosa and A.G. Colares. *Minimal Surfaces in* $\mathbb{R}^3$. Springer Lecture Notes 1195, 1986.

[5] Helga Baum. *Eichfeldtheorie*. Springer, Heidelberg, 2009.

[6] A.F. Beardon. *A primer on Riemann Surfaces*. Cambridge Univ.Press, London, 1984.

[7] Pierre Bérard. Domaines plans isospectraux à la Grodon-Webb-Wolpert: une preuve terre à terre. *Séminaire de Théorie spectrale er géoétrie*, 10(3):131–142, 1991-1992.

[8] Pierre Bérard. Transplantation et isospectralité. I. *Math. Ann.*, 292(3):547–560, 1992.

[9] Pierre Bérard. Transplantation et isospectralité. II. (Transplantation and isospectrality. II). *J. Lond. Math. Soc., II. Ser.*, 48(3):565–576, 1993.

[10] M. Berger. Les variétés Riemanniennes (1/4)-pincées. *Ann. Scuola Norm. Sup. Pisa (3)*, 14:161–170, 1960.

[11] M. Berger. *Geometry 1*. Springer, Berlin, 1987.

[12] M. Berger and B. Gostiaux. *Differential Geometry, Manifolds, Curves, and Surfaces*. Springer, New York, 1988.

[13] Marcel Berger and Bernard Gostiaux. *Géométrie différentielle: variétés, courbes et surfaces*. Mathématiques. [Mathematics]. Presses Universitaires de France, Paris, 1987.

[14] S. Bernstein. Sur un théorème de Géométrie et ses applications aux équations aux dérivées partielles du type elliptique. *Comm.de la Soc.M. Kharkov*, 15:38–45, 1915.

[15] Harald Biller. Characterizations of proper actions. *Math. Proc. Cambridge Philos. Soc.*, 136(2):429–439, 2004.

[16] O. Bonnet. Mémoire sur la théorie générale des surfaces. *Journ. de l'Ecole Polytechn.*, 19:1–146, 1848.

[17] Simon Brendl and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *Journal of the American Mathematical Society*, 22:287–307, 2009.

[18] Simon Brendl and Richard Schoen. Curvature, sphere theorems, and the ricci flow. *Bulletin of the American Mathematical Society*, 48:1–32, 2011.

[19] Th. Bröcker and K. Jänich. *Einführung in die Differentialtopologie*. Springer, Heidelberg, 1973.

[20] Brown. Smooth n-manifolds immerse in $\mathbb{R}^{2n-\alpha(n)}$. *Contemp. Math*, 12:73–79, 1982.

[21] Peter Buser. Isospectral Riemann surfaces. *Ann. Inst. Fourier (Grenoble)*, 36(2):167–192, 1986.

[22] Peter Buser, John Conway, Peter Doyle, and Klaus-Dieter Semmler. Some planar isospectral domains. *Internat. Math. Res. Notices*, (9):391ff., approx. 9 pp. (electronic), 1994.

[23] Eugenio Calabi and Maxwell Rosenlicht. Complex analytic manifolds without countable base. *Proc. Amer. Math. Soc.*, 4:335–340, 1953.

[24] Carathéodory. Über die gegenseitigen Beziehungen der Ränder bei konformen Abbildungen des Inneren einer Jordanschen Kurve auf einen Kreis. *Math. Annal.*, 73:305–320, 1913.

[25] E. Cartan. *Leçons sur la géométrie des espaces de Riemann*. Paris: Gauthier-Villars (Cahiers scientifiques publiés sous la direction de G. Julia, 2). VI, 273 p. , 1928.

[26] Catalan. *Journal de Mathém.*, 7:203, 1842.

[27] S. J. Chapman. Drums that sound the same. *Am. Math. Soc.*, 102.2:124–138, 1995.

[28] S.S. Chern. An elementary proof of the existence of isothermal parameters on a surface. *Proc. AMS.*, 6:771–782, 1955.

[29] S.S. Chern. *Complex manifolds without potential theory*. Van Nostrand, Princeton, 1967.

[30] Cohen. A proof of the immersion conjecture. *Proced. Math. Acad. Soc.*, 79:3390–3392, 1982.

[31] St. Cohn-Vossen. Zwei Sätze über die Starrheit der Eiflächen. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1927:125–134, 1927.

[32] J.H. Conway and N.J.A. Sloane. Four-dimensional lattices with the same theta series. *Int. Math. Res. Not.*, 4:93–96, 1992.
[33] M. Dehn. Über die Topologie des dreidimensionalen Raumes. *Math. Ann.*, 69:137–168, 1910.
[34] Ulrich Dierkes, Stefan Hildebrandt, Albrecht Küster, and Ortwin Wohlrab. *Minimal Surfaces I*, volume 295. Springer, Berlin, Heidelberg, New York, 1992.
[35] J. Dieudonné. *Foundations of Modern Analysis*. Academic Press, New York, 1960.
[36] Beno Eckmann. Systeme von richtungsfeldern in sphären und stetige lösungen komplexer linearer gleichungen. *Comment. Math. Helv.*, 15:1–26, 1943.
[37] N. V. Efimov. The impossibility in Euclidean 3-space of a complete regular surface with a negative upper bound of the Gaussian curvature. *Dokl. Akad. Nauk SSSR*, 150:1206–1209, 1963.
[38] R. Engelking. *Outline of General Topology*. North-Holland Publ. Comp., Amsterdam, 1968.
[39] H. Federer. *Geometric Measure Theory*. Springer, Berlin, 1969.
[40] C. Fefferman. The Bergman kernel of biholomorphic mappings of pseudo convex domains. *Inv. Math.*, 26:1–65, 1974.
[41] G.M. Fischer. On the group of all homeomorphisms of a manifold. *Trans. AMS*, 97:193–212, 1960.
[42] William Fulton and Joe Harris. *Representation Theory*. Springer-Verlag, 1991.
[43] Alexandre Gabard. A separable manifold failing to have the homotopy type of a CW-complex. *Arch. Math. (Basel)*, 90(3):267–274, 2008.
[44] Gauß. Disquisitiones generales circa superficies curvas. *Comm. Soc. Regiae Sci. Gottingensis Rec.*, 6, 1828.
[45] Oliver Giraud and Koen Thas. Hearing shapes of drums – mathematical and physical aspects of isospectrality. pages 1–42, 2011.
[46] Gleason. Groups without small subgroups. *Ann. of Math.*, 56:193–212, 1952.
[47] M. Golubistky and V. Guillemin. *Stable mappings and their singularities*. Springer, New York, 1973.
[48] Robert E. Gompf. Three exotic $\mathbb{R}^4$'s and other anomalies. *Journal of Differential Geom.*, 18.2:317–328, 1983.
[49] C. Gordon, D. Webb, and S. Wolpert. Isospectral plane domains and surfaces via riemannian orbifolds. *Invent. Math.*, 110:1–22, 1992.
[50] C. Gordon and E. Wilson. Isospektral deformations of compact solvmanifolds. *J. Diff. Geom.*, 19:241–256, 1984.
[51] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J.AMS.*, 2:371, 1989.
[52] Carolyn Gordon, David L. Webb, and Scott Wolpert. One cannot hear the shape of a drum. *Bull. Am. Math. Soc., New Ser.*, 27:134–138, 1992.
[53] Werner Greub. *Multilinear Algebra*. Springer,
[54] Werner Greub, Stephen Halperin, and Ray Vanstone. *Connections, Curvature and Cohomology I-III*. Academic Press 773, 76, New York, 1972.
[55] W. Gröbner. *Matrizenrechnung*. Bibliographisches Inst., Mannheim, 1966.
[56] H.W. Guggenheimer. *Differential Geometry*. Dover Publ., New York, 1963.
[57] J. Hadamard. Les surfaces á corbures opposées. *J.Math.Pures Appl.*, 4:27–73, 1889.
[58] Haeflinger and Reeb. Varietés a une dimension et structures feulletés de plan. *l'Enseignement Math*, 2:107–125, 1957.
[59] Philip Hartman and Louis Nirenberg. On spherical image maps whose Jacobians do not change sign. *Amer. J. Math.*, 81:901–920, 1959.
[60] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Acad. Press, New York, 1978.
[61] G. Herglotz. Über die Starrheit der Eiflächen. *Abh. Math.Sem.Univ.Hamburg*, 15:127–129, 1943.
[62] D. Hilbert. Über Flächen von konstanter Gaußscher Krümmung. *Trans.AMS.*, 2:87–99, 1901.
[63] S. Hildebrandt. Boundary behaviour of minimal surfaces. *Arch. Rational Mech. Anal.*, 35:47–82, 1969.
[64] J. Hilgert and K.-H. Neeb. *Lie-Gruppen und Lie-Algebren*. Vieweg, Braunschweig, 1991.
[65] M.W. Hirsch. *Differential Topology*. Springer, New York, 1976.
[66] H. Hopf and W. Rinow. Über den Begriff der vollständigen differentialgeometrischen Flächen. *Math. Ann.*, 116:749–766, 1938.
[67] Chuan-Chih Hsiung. *A First Course in Differential Geometry*. John Wiley &Sons, New York, 1981.
[68] I.M. James. Whitehead products and vector fields on spheres. *Proc. Cambridge*, 53:817–820, 1957.
[69] H. Jarchow. *Locally convex spaces*. Teubner, Stuttgart, 1981.

[70] Joris. Une c^∞-application non-immersive qui possede la propriete universelle des immersions. *Archiv Math.*, 39:267–277, 1982.
[71] M. Kac. Can one hear the shape of a drum? *Amer.Math.Monthly*, 73:1–23, 1966.
[72] M. Kervaire. A manifold which doesn't admit any differentiable structure. *Comm. Math. Helv.*, 34:257–270, 1960.
[73] Michel A. Kervaire. A manifold which doesn't admit a differentiable structure. *Comm. Math. Helv.*, 35:1–14, 1961.
[74] W. Klingenberg. *Eine Vorlesung über Differentialgeometrie*. Springer, Heidelberg, 1973.
[75] Wilhelm Klingenberg. Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. *Comment. Math. Helv.*, 35:47–54, 1961.
[76] S. Kobayashi. *On conjugate and cut loci*. The Mathem. Assoc. Am., Englewood Cliffs, N.J., 1967.
[77] S. Kobayashi. *Transformation groups in Differential Geometry*. Springer, Ergebnisse der Math. 70, Berlin, 1972.
[78] S. Kobayashi and K. Nomitzu. *Foundations of Differential Geometry*. Interscience, New York, 1969.
[79] A. Kriegl. *Topologie 1*. Vorlesung, Univ. Wien, 1999.
[80] A. Kriegl. *Analysis 1*. Vorlesung, Univ. Wien, 2004.
[81] A. Kriegl. *Analysis 2*. Vorlesung, Univ. Wien, 2004.
[82] A. Kriegl. *Analysis 3*. Vorlesung, Univ. Wien, 2004/05.
[83] A. Kriegl. *Funktionalanalysis 2*. Uni.Wien, Wien, 2005.
[84] A. Kriegl. *Funktional Analysis*. Vorlesung, Univ. Wien, 2006.
[85] A. Kriegl. *Differentialgeometrie 1*. Uni.Wien, Wien, 2007.
[86] A. Kriegl. *Differentialgeometrie*. Univ. Wien, 2007.
[87] A. Kriegl. *Differentialgeometrie 1*. Vorlesung, Univ. Wien, 2007.
[88] A. Kriegl. *Kategorien Theorie*. Vorlesung, Univ. Wien, 2008.
[89] A. Kriegl. *Differentialgeometrie 1*. Vorlesung, Univ. Wien, 2009.
[90] A. Kriegl. *Komplexe Analysis*. Vorlesung, Univ. Wien, 2011.
[91] A. Kriegl. *Algebraic Topology*. Lecture Notes, Univ. Vienna, 2014.
[92] A. Kriegl. *Proseminar Analysis auf Mannigfaltigkeiten*. Univ. Wien, SS 2013.
[93] A. Kriegl. *Höhere Funktional-Analysis*. Univ. Wien, SS 2014.
[94] A. Kriegl. *Analysis auf Mannigfaltigkeiten*. Univ. Wien, SS 2015.
[95] A. Kriegl. *Proseminar Analysis auf Mannigfaltigkeiten*. Univ. Wien, SS 2015.
[96] A. Kriegl. *Analysis auf Mannigfaltigkeiten*. Univ. Wien, SS 2018.
[97] A. Kriegl. *Analysis on Manifolds*. Univ. Wien, SS 2018.
[98] A. Kriegl. *Proseminar Analysis auf Mannigfaltigkeiten*. Univ. Wien, SS 2018.
[99] A. Kriegl. *Proseminar Analysis on Manifolds*. Univ. Wien, SS 2018.
[100] A. Kriegl. *Proseminar Lie-Gruppen*. Univ. Wien, WS 2008.
[101] A. Kriegl. *Proseminar Lie-Gruppen*. Univ. Wien, WS 2010.
[102] A. Kriegl. *Riemannsche Geometrie*. Univ. Wien, WS 2013.
[103] A. Kriegl. *Lie-Gruppen*. Univ. Wien, WS 2016.
[104] A. Kriegl. *Proseminar Lie-Gruppen*. Univ. Wien, WS 2016.
[105] Andreas Kriegl and Peter W. Michor. *The Convenient Setting of Global Analysis*. Am. Math. Soc., 1997.
[106] Lagrange. *Œvres Vol. 1*. Gauthier-Villars, Paris, 1868.
[107] Diane J. Lane. Paracompactness in perfectly normal, locally connected, locally compact spaces. *Proc. Amer. Math. Soc.*, 80(4):693–696, 1980.
[108] S. Lang. *Differentiable Manifolds*. Addison-Wesley, 1962.
[109] H. Liebmann. Eine neue Eigenschaft der Kugel. *Nachr. Kgl. Ges. Wiss. Göttingen. Math. Phys.*, 1899:44–55, 1899.
[110] William S. Massey. Surfaces of Gaussian curvature zero in Euclidean 3-space. *Tôhoku Math. J. (2)*, 14:73–79, 1962.
[111] Meusnier. Mémoire sur la curbure des surfaces. *Mémoires des savans étrangers*, 10:477–510, 1776.
[112] P. Michor. *Riemannsche Differentialgeometrie*. Vorlesung, Univ. Wien, WS 1988/89.
[113] P. W. Michor. *Topics in Differential Geometry*. AMS, Providence, Rhode Island, 2008.
[114] Peter W. Michor. A generalisation of Hamiltonian mechanics. *J. Geometry and Physics*, 2 (2):67–82, 1985.
[115] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math.*, 64:399–405, 1956.
[116] J. Milnor. Differentiable Structures on Spheres. *Ann. of Math.*, 81:962–972, 1959.
[117] J. Milnor. Eigenvalues of the Laplace operator on certain mf's. *Proc. Nat. Acad. Sci. USA*, 51:542, 1964.

[118] Montgomery and Zippin. Small subgroups of finite dimensional groups. *Ann. of. Math.*, 56:213–241, 1952.
[119] B. Morin and J-P. Petit. Let retournement de la sphére. *CRAS.*, 287:767–770, 791–794, 879–882, 1978.
[120] S. Mukhopadhyaya. New methods in the geometry of a plane arc. *Bull. Calcutta Math. Soc.*, 1:31–37, 1909.
[121] Munkres. Obstruction to the smoothing of piecewise differentiable homeomorphism. *Ann.of Math.*, 72:521–554, 1960.
[122] Munkres. *Elementary Differential Geometry*. Princeton University Press, 1966.
[123] Sumner Byron Myers. Connections between differential geometry and topology. I. Simply connected surfaces. *Duke Math. J.*, 1(3):376–391, 1935.
[124] J. Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math.*, 63:20–64, 1956.
[125] Nomitzu and Ozeki. The existence of complete Riemannian metrics. *Proc. AMS.*, 12:889–891, 1961.
[126] R. Osserman. Global Properties of minimal surfaces in e^3 and e^n. *Ann. of Math.*, 80:340–364, 1964.
[127] R. Osserman. *A Survey of Minimal Surfaces*. Van Nostrand, NewYork, 1969.
[128] R. Osserman. A proof of regularity everywhere of the classical solution to Plateaus's problem. *Ann.of Math*, 91:550–569, 1970.
[129] P. Painlevé. Sur la theorie de la Représentation conforme. *CRAS.*, 112:653–657, 1891.
[130] C.D. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots. *Ann. of Math.*, 66:1–26, 1957.
[131] E. Peschl. *Funktionentheorie*. Bibliographisches Inst., Mannheim, 1967.
[132] Alexandre Preissman. Quelques propriétés globales des espaces de Riemann. *Comment. Math. Helv.*, 15:175–216, 1943.
[133] Mary Ellen Rudin. Two nonmetrizable manifolds. *Topology and its Appl.*, 35:137–152, 1990.
[134] H.F. Scherk. Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *Crellés Journal f. reine und angew. Math.*, 13:185–208, 1835.
[135] J-P. Serre. *Complex Semi-simple Lie-Algebras*. Springer-Verlag, New York, 1987.
[136] M. Spivak. *A Comprehensive Introduction to Differential Geometry 1-4*. Publish or Perish, Berkeley, 1979.
[137] J. Stallings. The piecewise-linear structure of euclidean space. *Math. Proc. Camb. Phil. Soc.*, 58:481–488, 1962.
[138] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Ann. Math. (2)*, 121:169–186, 1985.
[139] Taubes. Gauge theory on asymptotically periodic 4-manifolds. *J.Diffgeom.*, 25:363–430, 1987.
[140] Jacques Tits. *Liesche Gruppen und Algebren*. Springer Hochschultext, Berlin, 1983.
[141] Trotter. Non-invertible knots exist. *Topology*, 2:341–358, 1964.
[142] V.S. Varadarajan. *Lie Groups, Lie Algebras and their Representation*. Springer Graduate Text, Berlin, 1984.
[143] L. Vietoris. Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven. *Arch. Math.*, 3:304–306, 1952.
[144] M.F. Vignéras. Variétés Riemanniennes isospektrales et non isométriques. *Ann. Math*, 112:21–32, 1980.
[145] R. Walter. *Differentialgeometry*. Bibliographisches Inst., Mannheim, 1978.
[146] F.W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Scott Foresman and Company, Illinois, 1971.
[147] F.W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, Graduate Texts in Mathematics 94, New York, 1983.
[148] S. Warschawski. Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformen Abbildungen. *Math.Z.*, 35:321–456, 1932.
[149] Weierstrass. *Monatsber. der Berlin. Akad.*, 1866.
[150] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978.
[151] H. Whitney. A function not constant on a connected set of critical points. *Duke Math. J.*, 1:514–517, 1935.
[152] H. Whitney. The selfintersections of a smooth n-manifold in $2n$-space. *Annals of Math.*, 45:220–446, 1944.
[153] H. Whitney. The singularities of a smooth n-manifold in (2n-1) space. *Annals of Math.*, 45:247–293, 1944.
[154] Hassler Whitney. *Geometric Integration Theory*. Princeton University Press, Princeton, N. J., 1957.

[155] David R. Wilkins. *A Course in Riemannian Geometry.* 2005.
[156] F. Xavier. The Gauss map of a complete, non-flat minimal surface cannot omit 7 points of the sphere. *Annals of Math.*, 113:211–214, 1981.
[157] H. Yamabe. On an arcwise connected subgroup of a Liegroup. *Osaka M.J.*, 2:14–15, 1950.
[158] K. Zindler. Über konvexe Gebilde I. *Monatsh. Math. Phys.*, 31:87–102, 1921.

Index