1. Definition.
Conditions for (F) spaces where $(U_p)_{p\in\mathbb{N}}$ is a decreasing basis of closed absolutely convex 0-nbhds and $\|\_\|_p := \sup\{|x'(x)| : x \in U_p\}$:
We always assume $p, q, k \in \mathbb{N}$, $d, r, C > 0$ and $0 < \mu < 1$.

(Ω)	$:\Leftrightarrow$	$\forall p\ \exists q\ \forall k\ \exists d\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	[Vogt1977?]
$(\widetilde{\Omega})$	$:\Leftrightarrow$	$\forall p\ \exists q\ \exists d\ \forall k\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	[Vogt1983,5.2]
$(\overline{\Omega})$	$:\Leftrightarrow$	$\exists d\ \forall p\ \exists q\ \forall k\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	[Wagner1980,1.9], [VW1980], [VW1981]
		$\forall d\ \forall p\ \exists q\ \forall k\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	
$(\overline{\overline{\Omega}})$	$:\Leftrightarrow$	$\forall p\ \exists q\ \forall d\ \forall k\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	[Vogt1983,4.1]

Then $(\overline{\overline{\Omega}}) \Rightarrow (\overline{\Omega}) \Rightarrow (\widetilde{\Omega}) \Rightarrow (\Omega)$, since:

$$\exists q\ \forall d \Rightarrow\ \forall d\ \exists q,\ \ \exists d\ \forall p \Rightarrow\ \forall p\ \exists d,\ \ \exists d\ \forall k \Rightarrow\ \forall k\ \exists d.$$

For a bounded subset $B \subseteq E$ one defines:

$(\widetilde{\Omega}_B)$	$:\Leftrightarrow$	$\forall p\ \exists q\ \exists d\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_B \|\_\|_p^d$	[HK2003]
$(\overline{\Omega}_B)$	$:\Leftrightarrow$	$\exists d\ \forall p\ \exists q\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_B \|\_\|_p^d$	[Vogt1982,1.4]
		$\forall d\ \forall p\ \exists q\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_B \|\_\|_p^d$	
$(\overline{\overline{\Omega}}_B)$	$:\Leftrightarrow$	$\forall p\ \exists q\ \forall d\ \exists C : \|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d$	[Vogt1983,4.4]

2. Remark.
Note that in all these conditions we may assume w.l.o.g. that $p \le q \le k$, since q has to be choosen and $\|\_\|_q \le \|\_\|_{q'}$ for $q > q'$, and since validity of the conditions for all $k \ge q$ implies it for $k' \le q \le k$, since then $\|\_\|_k \le \|\_\|_{k'}$.

3. Remark.
Note that for $\mu = \frac{1}{1+d}$:

$$\|\_\|_q^{1+d} \le C\,\|\_\|_k \|\_\|_p^d \quad \Leftrightarrow \quad \|\_\|_q \le C\,\|\_\|_k^{\mu} \|\_\|_p^{1-\mu}$$

4. Remark.
Note that only μ near 1 (and hence d near 0) are relevant, since for $\mu' < \mu$ (and $\|y\|_p \neq 0$) we get

$$\|y\|_k^{\mu}\|y\|_p^{1-\mu} = \|y\|_p \underbrace{\Big(\frac{\|y\|_k}{\|y\|_p}\Big)}_{\le 1}^{\mu} \le \|y\|_p \Big(\frac{\|y\|_k}{\|y\|_p}\Big)^{\mu'} = \|y\|_k^{\mu'} \|y\|_p^{1-\mu'}$$

5. Lemma.
Let $a, b > 0$ *and* $\alpha, \beta \geq 0$. *Then* $\inf\{r^a\alpha + \frac{1}{r^b}\beta : r > 0\} = \frac{a+b}{a}\left(\frac{a}{b}\right)^{\frac{b}{a+b}} \alpha^{\frac{b}{a+b}} \beta^{\frac{a}{a+b}}$

Proof. Let $f(r) := r^a\alpha + \frac{1}{r^b}\beta = \left(r^{a+b}\alpha + \beta\right) r^{-b}$. Then $f'(r) = a\,\alpha\, r^{a-1} - b\,\frac{1}{r^{b+1}}\,\beta$ and hence $f'(r) = 0 \Leftrightarrow r^{a+b}\alpha = \frac{b}{a}\,\beta$. Thus

$$\begin{aligned} f(r) &\geq \left(\frac{b}{a}\beta + \beta\right)\left(\frac{b\beta}{a\alpha}\right)^{-\frac{b}{a+b}} \\ &= \alpha^{\frac{b}{a+b}} \beta^{1-\frac{b}{a+b}} \left(1 + \frac{b}{a}\right)\left(\frac{a}{b}\right)^{-\frac{b}{a+b}} \\ &= \alpha^{\frac{b}{a+b}} \beta^{\frac{a}{a+b}} \frac{a+b}{a}\left(\frac{a}{b}\right)^{\frac{b}{a+b}}. \end{aligned}$$

Note that $f(r) \to +\infty$ for $r \searrow 0$ if $\alpha > 0$ and for $r \nearrow +\infty$ if $\beta > 0$, hence the infimum is attained if $\alpha, \beta > 0$. Otherwise $f(r) \to 0$ for $r \to 0$ or $r \to +\infty$, hence the statement is valid in this case as well. □

6. Bipolar Theorem.
Let E be a lcs which is not necessarily Hausdorff and $A \subseteq E$ be absolutely convex. Then $\overline{A} = (A^o)_o$.

Proof. We reduce this statement by passing to the Hausdorffication $\tilde{E} := E/\overline{\{0\}}$ of E. Note that for absolutely convex subsets $A \subseteq E$ we have $\overline{\{A\}} = \bigcap_U A + U$, where U runs through the (closed absolutely convex) 0-nbhds:

We have $\overline{A} \subseteq A+U$ and hence $\overline{A}$ lies in this intersection, since $x \in \overline{A} \Rightarrow x+U \cap A \neq \emptyset$ $\Rightarrow \exists u \in U, a \in A : x + u = a$, i.e. $x = a + (-u)$ with $a \in A$ and $-u \in U$, since U is balanced.

Conversely, let $z \in \bigcap_U A + U$. Then for each U exists an $a \in A$ and a $u \in U$ with $z = a+u$ and hence $z+(-u) \in A$. Since $-u \in U$ we have $z+U \cap A \neq \emptyset$, i.e. $z \in \overline{A}$.

In particular $\overline{\{0\}} = \bigcap_U U$ and this equals $\bigcap_p p^{-1}(0)$, where p runs through all continuous seminorms on E: In fact $\overline{\{0\}} \subseteq p^{-1}(0)$ since this inverse image is closed and contains 0. And, conversely, let $z \notin \overline{\{0\}}$. Then there exists a 0-neighborhood U, which we may assume to be of the form $U = \{x : p(x) < \varepsilon\}$ for some seminorm p and $\varepsilon > 0$, such that $0 \notin z + U$, i.e. $-z \notin U$ and hence $p(z) = p(-z) \geq \varepsilon > 0$ in contradiction to $z \in p^{-1}(0)$.

Thus $\overline{\{0\}}$ is a linear subspace of E (since the kernel of a seminorm is so) and hence $\tilde{E}$ is a welldefined locally convex space with respect to the quotiient topology induced by $\pi : E \to \tilde{E}$. Hence $\pi^{-1}(\tilde{U})$ is a absolutely convex closed 0-nbhd for each such nbhd $\tilde{U}$ in $\tilde{E}$. Conversely let U be a 0-nbhd in E. Then $\overline{\{0\}} \subseteq U$ and hence $\pi^{-1}(\pi(U)) = U + \overline{\{0\}} \subseteq \bigcap_V U + V = \overline{U} = U$. Thus $\tilde{U} := \pi(U)$ is such a nbhd in $\tilde{E}$ and $\mathcal{U} = \{\pi^{-1}(\tilde{U}) : \tilde{U} \in \tilde{\mathcal{U}}\}$.

Since all p and all $\ell \in E^*$ are continuous, they map $\overline{\{0\}}$ into the closure $\{0\}$ of the image $\{0\}$, hence factor to continuous seminorms and linear functions over $\pi : E \to \tilde{E}$. Thus $\pi^* : \tilde{E}^* \to E^*$ is a bijections. Let $A \subseteq E$ be absolutely convex and $\tilde{A} = \pi(A) \subseteq \tilde{E}$. Then

$$\begin{aligned}
\pi^*(\tilde{A}^o) &= \pi^*(\{\tilde{\lambda} \in \tilde{E}^* : \ \forall \tilde{x} \in \tilde{A} : |\tilde{\lambda}(\tilde{x})| \leq 1\}) \\
&= \{\tilde{\lambda} \circ \pi : \tilde{\lambda} \in \tilde{E}^* : \ \forall x \in A : |\tilde{\lambda}(\pi(x))| \leq 1\} \\
&= \{\lambda \in E^* : \ \forall x \in A : |\lambda(x)| \leq 1\} = A^o
\end{aligned}$$

Thus

$$\begin{aligned}
(A^o)_o &= \{x \in E : \ \forall \lambda \in A^o : |\lambda(x)| \leq 1\} \\
&= \{x \in E : \ \forall \lambda \in \pi^*(\tilde{A}^o) : |\lambda(x)| \leq 1\} \\
&= \{x \in E : \ \forall \lambda \in \tilde{A}^o : |\lambda(\pi(x))| = \pi^*(\lambda)(x)| \leq 1\} \\
&= \{x \in E : \pi(x) \in (\tilde{A}^o)_o\} = \pi^{-1}((\tilde{A}^o)_o).
\end{aligned}$$

For the closures we obtain

$$\begin{aligned}
\overline{A} &= \bigcap_{U \in \mathcal{U}} A + U = \bigcap_{\tilde{U} \in \tilde{\mathcal{U}}} A + \pi^{-1}(\tilde{U}) \\
&\overset{!}{=} \bigcap_{\tilde{U} \in \tilde{\mathcal{U}}} \pi^{-1}(\tilde{A} + \tilde{U}) = \pi^{-1}\Big(\bigcap_{\tilde{U} \in \tilde{\mathcal{U}}} \tilde{A} + \tilde{U} \Big) \\
&= \pi^{-1}(\overline{\tilde{A}}) \\
&\xlongequal{\text{Bipolar Theorem for } \tilde{E}} \pi^{-1}((\tilde{A}^o)_o) = (A^o)_o,
\end{aligned}$$

since $\pi^{-1}(\tilde{A}+\tilde{U}) \supseteq \pi^{-1}(\tilde{A})+\pi^{-1}(\tilde{U}) \supseteq A+\pi^{-1}(\tilde{U})$ and, conversely, $\pi(z) \in \tilde{A}+\tilde{U}$ $\Rightarrow\ \exists \tilde{a} \in \tilde{A} : \pi(z)-\tilde{a} \in \tilde{U}$, hence $\exists a \in A : \tilde{a} = \pi(a)$ und $z-a \in \pi^{-1}(\pi(z)-\pi(a)) \in \pi^{-1}(\tilde{U})$, i.e. $z \in A+\pi^{-1}(\tilde{U})$. $\square$

7. Interpolation Inequalities.
E $\mathbb{K}$-vs, $\|\text{-}\|_0 \le \|\text{-}\|_1 \le \|\text{-}\|_2$ SN on E, $U_j := \{x : \|x\|_j \le 1\}$, $a, b > 0$ and $\mu := \frac{a}{a+b}$. Then t.f.s.a.e.:

1. $\exists C_1 : \|\text{-}\|_1^* \le C_1 (\|\text{-}\|_0^*)^{1-\mu} (\|\text{-}\|_2^*)^{\mu}$ *on* $(E, \|\text{-}\|_0)^*$;
2. $\exists C_2 \;\; \forall r : \|\text{-}\|_1^* \le C_2 \left(r^a \|\text{-}\|_0^* + \frac{1}{r^b} \|\text{-}\|_2^* \right)$ *on* $(E, \|\text{-}\|_0)^*$;

2'. $\exists C_2' \;\; \forall r : \|\text{-}\|_1^* \le r^a \|\text{-}\|_0^* + C_2' \frac{1}{r^b} \|\text{-}\|_2^*$ *on* $(E, \|\text{-}\|_0)^*$;
2". $\exists C_2'' \;\; \forall r : \|\text{-}\|_1^* \le C_2'' r^a \|\text{-}\|_0^* + \frac{1}{r^b} \|\text{-}\|_2^*$ *on* $(E, \|\text{-}\|_0)^*$;
3. $\exists C_3 \;\; \forall r : U_1 \subseteq C_3 \left(r^a U_0 + \frac{1}{r^b} U_2 \right)$;

3'. $\exists C_3' \;\; \forall r : U_1 \subseteq r^a U_0 + C_3' \frac{1}{r^b} U_2$
3". $\exists C_3'' \;\; \forall r : U_1 \subseteq C_3'' r^a U_0 + \frac{1}{r^b} U_2$

Proof. (1$\Leftrightarrow$2) By the lemma above

$$\inf\{r^a \|y\|_0^* + \frac{1}{r^b} \|y\|_2^* : r > 0\} = \frac{a+b}{a} \left(\frac{a}{b}\right)^{\frac{b}{a+b}} (\|y\|_0^*)^{\frac{b}{a+b}} (\|y\|_2)^{\frac{a}{a+b}},$$

hence (1$\Leftrightarrow$2) with $C_1 = C_2 \cdot \frac{a+b}{a} \left(\frac{a}{b}\right)^{\frac{b}{a+b}}$.

(2$\Leftrightarrow$2') and (3$\Leftrightarrow$3'), via $C' = C^{1+\frac{b}{a}}$ and $r' = C^{1/a} r$.

(2$\Leftrightarrow$2") and (3$\Leftrightarrow$3"), via $C'' = C^{1+\frac{a}{b}}$ and $r'' = C^{-1/b} r$.

(2$\Rightarrow$3)

$$\begin{aligned}
(2) \Rightarrow \forall r > 0, y \in (E, \|\text{-}\|_0)^* &\overset{(24.5)}{=\!=\!=} (E^*)_{U_o^o} : \\
\sup_{x \in U_1} |y(x)| &\le C_2 \left(r^a \sup_{x \in U_0} |y(x)| + \frac{1}{r^b} \sup_{x \in U_2} |y(x)| \right) \\
&= C_2 \sup\Big\{ |y(x)| : x \in r^a U_0 + \frac{1}{r^b} U_2 \Big\} \\
\xRightarrow{\text{Bipolarensatz}} \forall r > 0 : U_1 &\subseteq C_2 \, \overline{(r^a U_0 + \frac{1}{r^b} U_2)}^{\|\text{-}\|_0} \\
\text{Let } C_3 > C_2 \Rightarrow \forall r > 0 : U_1 &\subseteq C_2 \, \overline{(r^a U_0 + \frac{1}{r^b} U_2)}^{\|\text{-}\|_0} \subseteq C_3 \, (r^a U_0 + \frac{1}{r^b} U_2)
\end{aligned}$$

(3⇒2) This follows, since for absolutely convex subsets C, A, B we have:

$$
\begin{aligned}
C \subseteq A + B \Rightarrow \ \forall x^* \in E^* : \|x^*\|_C &= \sup\{|x^*(x)| : x \in C\} \\
&\le \sup\{|x^*(a+b)| : a \in A,\, b \in B\} = \|x^*\|_{A+B} \\
&= \sup\{|x^*(a)| : a \in A\} + \sup\{|x^*(b)| : b \in B\} \\
&= \|x^*\|_A + \|x^*\|_B,
\end{aligned}
$$

since $|x^*(a+b)| \le |x^*(a)| + |x^*(b)|$ and hence

$$
\begin{aligned}
\sup\{|x^*(a+b)| : a \in A,\, b \in B\} &\le \sup\{|x^*(a)| + |x^*(b)| : a \in A,\, b \in B\} \\
&= \sup\{|x^*(a)| : a \in A\} + \sup\{|x^*(b)| : b \in B\}
\end{aligned}
$$

and conversely

$$
0 \le |x^*(a_0)| + |x^*(b_0)| = \lambda x^*(a_0) + \mu x^*(b_0) = x^*(\lambda a_0 + \beta b_0) \le |x^*(a+b)|,
$$

with appropriately choosen $|\lambda| = 1 = |\mu|$ and hence $a := \lambda a_0 \in A$ und $b := \mu b_0 \in B$, since the sets are assumed to be balanced. □

8. Lemma. Characterizing $\overline{\Omega}$.
The following statements are equivalent (where $a, b > 0$):

$$
\begin{aligned}
(\overline{\Omega}) \quad :\Leftrightarrow \quad & (1) && \forall p\ \exists q\ \forall k\ \exists C : \|\text{-}\|_q^2 \le C\, \|\text{-}\|_k \|\text{-}\|_p \\
\Leftrightarrow \quad & (2) && \exists d\ \forall p\ \exists q\ \forall k\ \exists C : \|\text{-}\|_q^{1+d} \le C\, \|\text{-}\|_k \|\text{-}\|_p^d \\
\Leftrightarrow \quad & (3) && \forall d\ \forall p\ \exists q\ \forall k\ \exists C : \|\text{-}\|_q^{1+d} \le C\, \|\text{-}\|_k \|\text{-}\|_p^d \\
\Leftrightarrow \quad & (2') && \exists \mu\ \forall p\ \exists q\ \forall k\ \exists C : \|\text{-}\|_q \le C\, \|\text{-}\|_k^\mu \|\text{-}\|_p^{1-\mu} \\
\Leftrightarrow \quad & (3') && \forall \mu\ \forall p\ \exists q\ \forall k\ \exists C : \|\text{-}\|_q \le C\, \|\text{-}\|_k^\mu \|\text{-}\|_p^{1-\mu} \\
\Leftrightarrow \quad & (1'') && \forall p\ \exists q\ \forall k\ \exists C\ \forall r : \|\text{-}\|_q \le C \left(r\, \|\text{-}\|_k + \frac{1}{r}\, \|\text{-}\|_p \right) \\
\Leftrightarrow \quad & (2'') && \exists a, b\ \forall p\ \exists q\ \forall k\ \exists C\ \forall r : \|\text{-}\|_q \le C \left(r^a\, \|\text{-}\|_k + \frac{1}{r^b}\, \|\text{-}\|_p \right) \\
\Leftrightarrow \quad & (3'') && \forall a, b\ \forall p\ \exists q\ \forall k\ \exists C\ \forall r : \|\text{-}\|_q \le C \left(r^a\, \|\text{-}\|_k + \frac{1}{r^b}\, \|\text{-}\|_p \right) \\
\Leftrightarrow \quad & (1''') && \forall p\ \exists q\ \forall k\ \exists C\ \forall r : U_q \subseteq C \left(r\, U_k + \frac{1}{r} U_p \right) \\
\Leftrightarrow \quad & (2''') && \exists a, b\ \forall p\ \exists q\ \forall k\ \exists C\ \forall r : U_q \subseteq C \left(r^a U_k + \frac{1}{r^b} U_p \right) \\
\Leftrightarrow \quad & (3''') && \forall a, b\ \forall p\ \exists q\ \forall k\ \exists C\ \forall r : U_q \subseteq C \left(r^a U_k + \frac{1}{r^b} U_p \right)
\end{aligned}
$$

Proof.

((1)⇒(2)) $d := 1$.

((2)⇒(3)) We show first that (2) holds for $d' := \frac{d^2}{1+2d} \leq \frac{d}{2}$ as well:

$$\exists d\ \ \forall p\ \ \exists q\ \ \forall k\ \ \exists C : \|\text{-}\|_q^{1+d} \leq C\,\|\text{-}\|_k \|\text{-}\|_p^d$$

$$\forall q\ \ \exists q'\ \ \forall k'\ \ \exists C' : \|\text{-}\|_{q'}^{1+d} \leq C'\,\|\text{-}\|_{k'} \|\text{-}\|_q^d$$

$$\Rightarrow \forall p\ \ \exists q, q'\ \ \forall k, k'\ \ \exists C, C' :$$

$$\|\text{-}\|_{q'}^{1+d} \leq C'\,\|\text{-}\|_{k'}\|\text{-}\|_q^d = C'\,\|\text{-}\|_{k'}(\|\text{-}\|_q^{1+d})^{\frac{d}{1+d}} \leq C'\,\|\text{-}\|_{k'}\,(C\,\|\text{-}\|_k\|\text{-}\|_p^d)^{\frac{d}{1+d}}$$

$$\Rightarrow \forall p\ \ \exists q'\ \ \forall k = k'\ \ \exists C, C' :$$

$$\|\text{-}\|_{q'}^{1+d} \leq C'\,C^{\frac{d}{1+d}}\,\|\text{-}\|_k^{\frac{1+2d}{1+d}}\,\|\text{-}\|_p^{\frac{d^2}{1+d}}$$

$$\Rightarrow \forall p\ \ \exists q'\ \ \forall k\ \ \exists C, C' :$$

$$\underbrace{(\|\text{-}\|_{q'}^{1+d})^{\frac{1+d}{1+2d}}}_{\|\text{-}\|_{q'}^{1+\frac{d^2}{1+2d}}} \leq \underbrace{(C'\,C^{\frac{d}{1+d}})^{\frac{1+d}{1+2d}}}_{=:C''}\,\|\text{-}\|_k\,\underbrace{\|\text{-}\|_p^{\frac{d^2}{1+d}\cdot\frac{1+d}{1+2d}}}_{\|\text{-}\|_p^{\frac{d^2}{1+2d}}}$$

Thus we have (2) also for $d' := \frac{d^2}{1+2d} \leq \frac{d}{2}$.

By induction we get (2) for some $d < 1$ and hence also for $d = 1$, since $\|\text{-}\|_q \leq C\,\|\text{-}\|_k \leq \left(\frac{\|\text{-}\|_p}{\|\text{-}\|_q}\right)^d \leq C\,\|\text{-}\|_k \leq \left(\frac{\|\text{-}\|_p}{\|\text{-}\|_q}\right)^{d'}$ for $d \leq d'$ and $\|\text{-}\|_p \geq \|\text{-}\|_q$.

((3)⇒(1)) $\forall \Rightarrow\ \exists$.

((2)⇔(2')) and ((3)⇔(3')) by remark (3).

((2')⇔(2")) and ((3')⇔(3")) by the interpolation inequalities (7) (1⇔2).

((2")⇔(2"')) and ((3")⇔(3"')) by the interpolation inequalities (7) (2⇔3).

((3")⇒(1")⇒(2")) and ((3"')⇒(1"')⇒(2"')) are trivial ($a = 1, b := 1$). □

9. Lemma.
[Vogt1982] E (FS).

- $(\overline{\Omega}) \Leftrightarrow\ \exists B \subseteq E$ *bd.* $(\overline{\Omega}_B)$.
- $(\widetilde{\Omega}) \Leftrightarrow\ \exists B \subseteq E$ *bd.* $(\widetilde{\Omega}_B)$.
- $(\overline{\overline{\Omega}}) \Leftrightarrow\ \exists B \subseteq E$ *bd.* $(\overline{\overline{\Omega}}_B)$.

Proof. $(\overline{\Omega})$ $(\Leftarrow)$ is obvious, since $\forall k\ \ \exists D_k : B \subseteq D_k \cdot U_k$ and hence $\|\text{-}\|_B^* \le D_k \, \|\text{-}\|_k^*$.
$(\Rightarrow)$ Let $(U_k)_k$ be a decreasing basis of absolutely convex closed 0-nbhds. E has the following property $(\overline{\Omega})$:

$$E \in (\overline{\Omega}) \quad \Leftrightarrow \quad \forall \mu\ \ \forall p\ \ \exists q\ \ \forall k\ \ \exists C\ \ \forall r : U_q \subseteq r^\mu U_k + \frac{C}{r^{1-\mu}} U_p.$$

Now

$$
\begin{aligned}
& U_q \subseteq r^\mu U_k + \frac{C_\mu}{r^{1-\mu}} U_p \\
\Leftrightarrow\quad & \forall x \in U_q \;\; \exists y \in U_k \;\; \exists z \in U_p : x = r^\mu y + \frac{C_\mu}{r^{1-\mu}} z \\
\Leftrightarrow\quad & \forall x \in U_q \;\; \exists y \in U_k : x - r^\mu y \in \frac{C_\mu}{r^{1-\mu}} U_p \\
\Leftrightarrow\quad & \forall x \in U_q \;\; \exists y \in U_k : \|x - r^\mu y\|_p \le \frac{C_\mu}{r^{1-\mu}} \\
\Rightarrow\quad & \forall x \in U_q : d_p(x, r^\mu U_k) \le \frac{C_\mu}{r^{1-\mu}} \\
\Leftrightarrow\quad & \varepsilon(r,k) := r^{1-\mu} \sup\Big\{ d_p(x, r^\mu U_k) : x \in U_q \Big\} \le C_\mu
\end{aligned}
$$

and conversely

$$
\begin{aligned}
& \varepsilon(r,k) := r^{1-\mu} \sup\Big\{ d_p(x, r^\mu U_k) : x \in U_q \Big\} < C_\mu \\
\Rightarrow\quad & \forall x \in U_q \;\; \exists y \in U_k : \|x - r^\mu y\|_p \le \frac{C_\mu}{r^{1-\mu}} \\
\Rightarrow\quad & U_q \subseteq r^\mu U_k + \frac{C_\mu}{r^{1-\mu}} U_p.
\end{aligned}
$$

Note that

$$\varepsilon(r^2, \frac{\mu}{2}) = (r^2)^{1-\frac{\mu}{2}} \sup\Big\{d_p(x, (r^2)^{\frac{\mu}{2}} U_k) : x \in U_q\Big\} = r\,\varepsilon(r, \mu)$$

and can deduce from the condition for $\mu/2$ and r^2 that for each k we have

$$\varepsilon(r, k) \to 0 \text{ for } r \to +\infty.$$

Thus $\forall k\ \ \exists r_k\ \ \forall r \geq r_k : \varepsilon(r, k) < \frac{1}{k}$ and w.l.o.g. we assume $k \leq r_k < r_{k+1} \ni \mathbb{N}$. Now let $k(n) := \max\{k : r_k \leq n\} \leq n < \infty$. Then $k(n) \nearrow \infty$ for $n \to \infty$, since otherwise there would exists a bound K and thus $r_{K+1} > n$ for all n. Let $\varepsilon(n) := \frac{1}{k(n)}$. Then $\varepsilon \to 0$ and since $r_{k(n)} \leq n$ we get $\varepsilon(n, k(n)) < \frac{1}{k(n)} = \varepsilon(n)$. By the equivalence above

$$U_q \subseteq n^\mu U_{k(n)} + \frac{\varepsilon(n)}{n^{1-\mu}}\, U_p.$$

Since E is (S) we find finite sets $Z_n \subseteq U_{k(n)}$ with $U_{k(n)} \subseteq Z_n + \frac{\varepsilon(n)}{n} U_p$ and hence

$$U_q \subseteq n^\mu U_{k(n)} + \frac{\varepsilon(n)}{n^{1-\mu}}\, U_p \subseteq \Big(n^\mu Z_n + \frac{\varepsilon(n)}{n^{1-\mu}} U_p\Big) + \frac{\varepsilon(n)}{n^{1-\mu}}\, U_p = n^\mu Z_n + \frac{2\varepsilon(n)}{n^{1-\mu}} U_p.$$

Let B be the absolutely convex hull of $\bigcup_n Z_n$. Then B ist bounded since it is contained in the absolutely convex hull of $U_{k(n)} \cup \bigcup_{j<n} Z_j$ and $k(n) \to \infty$. Furthermore

$$U_q \subseteq n^\mu Z_n + \frac{2\varepsilon(n)}{n^{1-\mu}} U_p \subseteq n^\mu B + \frac{2 \max_n |\varepsilon(n)|}{n^{1-\mu}} U_p.$$

Note that w.l.o.g. $q \geq p$, i.e. $U_q \subseteq U_p$ and hence for $0 < r \leq 1$:

$$U_q \subseteq U_p \subseteq r^\mu B + U_p \subseteq r^\mu B + \frac{1}{r^{1-\mu}} U_p.$$

Finally, if $n \leq r < n+1$ then for $C \geq 2 \max_n |\varepsilon(n)|$:

$$U_q \subseteq n^\mu B + \frac{C}{n^{1-\mu}} U_p \subseteq r^\mu B + \underbrace{(\frac{n+1}{n})^{1-\mu}}_{\leq 2} \frac{C}{r^{1-\mu}}\, U_p,$$

Thus for $C \geq \max\{4 \max_n |\varepsilon(n)|, 1\}$ we have

$$\forall r > 0 :\ U_q \subseteq r^\mu B + \frac{C}{r^{1-\mu}} U_p. \quad \square$$

($\widetilde{\Omega}$) and ($\overline{\overline{\Omega}}$) The proof for the corresponding result for $\overline{\overline{\Omega}}$ (see [Vogt1983,4.4]) and for $\tilde{\Omega}$ (see [DineenMeiseVogt1984,Proposition 3b] or [MeiseVogt1986,Lemma 3.6]) goes along the same lines. In the later case $\mu/2$ is the μ in the assumption and μ is one in the conclusion.