INVARIANT COCYCLES HAVE ABELIAN RANGES

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ABSTRACT. Let R be a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$, and let V be an ergodic strongly asymptotically central automorphism of R. We prove that every V-invariant cocycle $c \colon R \longrightarrow G$ with values in a Polish group G takes values in an abelian subgroup of G.

The hypotheses of this result are satisfied, for example, if A is a finite set, $X \subset A^{\mathbb{Z}}$ a closed, shift-invariant subset, V is the shift, μ a shift-invariant and ergodic probability measure on X, Δ_X the two-sided tail-equivalence relation on X, $R \subset \Delta_X$ a shift-invariant subrelation which is μ-nonsingular, and $c \colon R \longrightarrow G$ a shift-invariant cocycle.

1. INTRODUCTION

Let R be a discrete nonsingular ergodic equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$ (for the definitions we refer to Section 2). A measure-preserving automorphism V of $(X, \mathcal{S}, \mu)$ is an *automorphism of* (R, μ) if $(V \times V)(R) = R$. We consider Borel cocycles $c \colon R \longmapsto G$ on R taking values in a Polish group G which are *invariant* under an automorphism V of R, i.e. which satisfy that $c(Vx, Vx') = c(x, x')$ for all $(x, x') \in R_{X \smallsetminus N}$, where $N \in \mathcal{S}$ is a μ-null set.

If the automorphism V is *weakly asymptotically central* (Definition 2.1), then every V-invariant cocycle has the following property: there exists a null set $N \in \mathcal{S}$ such the closure H in G of the set $\{c(x, x') : (x, x') \in R_{X \smallsetminus N}\}$ is a subgroup of G, and that c defines an ergodic skew product extension of R by H (for locally compact abelian groups G this was proved in [7, Theorem 2.3], and for Polish groups in [3]). Since this kind of automatic ergodicity of cocycles is an unusual property which has interesting applications in certain probabilistic exchangeability and tail triviality results (cf. [3] and [6]–[8]), one is led to ask whether this phenomenon can also occur nontrivially if the group G is nonabelian.

In this paper we show that weakly asymptotically central automorphisms may have invariant cocycles with nonabelian ranges (Example 3.3 (2)), but that a mild strengthening of the condition of weak asymptotic centrality which, according to Theorem 2.5, is satisfied by the basic examples of weakly asymptotically central automorphisms, forces the range of any invariant cocycle to be abelian (Theorem 3.2). This indicates that in the

nonabelian case one is naturally led to replacing the notion of a *V-invariant cocycle* by that of a *V-covariant pair of cocycles*, as was done in [3] and [8].

2. Automorphisms of equivalence relations

Let $(X, \mathcal{S})$ be a standard Borel space and $\mathrm{Aut}(X, \mathcal{S})$ the group of Borel automorphisms of X. A Borel set $R \subset X \times X$ is a *discrete Borel equivalence relation* on X if R is an equivalence relation whose *equivalence classes* $R(x) = \{y \in X : (x, y) \in R\}$, $x \in X$, are all countable.

Let R be a discrete Borel equivalence relation on X. The *full group* $[R]$ of R is the group of all $W \in \mathrm{Aut}(X, \mathcal{S})$ with $Wx \in R(x)$ for every $x \in X$. According to [2] there exists a countable subgroup $\Gamma \subset [R]$ with

$$R = R[\Gamma] = \{(\gamma x, x) : \gamma \in \Gamma, x \in X\}. \tag{2.1}$$

Conversely, if $\Gamma \subset \mathrm{Aut}(X, \mathcal{S})$ is a countable group, then (2.1) defines a discrete Borel equivalence relation $R[\Gamma]$ on X.

From (2.1) it follows that the *saturation*

$$R(B) = \bigcup_{x \in B} R(x) = \bigcup_{\gamma \in \Gamma} \gamma(B) \tag{2.2}$$

of every set $B \in \mathcal{S}$ lies in $\mathcal{S}$, and we write

$$\mathcal{S}^R = \{R(B) : B \in \mathcal{S}\} \subset \mathcal{S} \tag{2.3}$$

for the sigma-algebra of R-saturated Borel sets.

For every $C \in \mathcal{S}$ we write

$$R_C = R \cap (C \times C) \tag{2.4}$$

for the equivalence relation *induced* by R on C.

A sigma-finite measure μ on $\mathcal{S}$ is *quasi-invariant* under R (or R is *μ-nonsingular*) if $\mu(R(B)) = 0$ for every $B \in \mathcal{S}$ with $\mu(B) = 0$. The measure μ is *conservative* under R if there exists, for every $A \in \mathcal{S}$ with $\mu(A) > 0$, an element $(x, y) \in R_A$ with $x \neq y$, and μ is *ergodic* under R (or R is μ-ergodic) if either $\mu(B) = 0$ or $\mu(X \smallsetminus B) = 0$ for every $B \in \mathcal{S}^R$.

Finally we denote by

$$\mathrm{Aut}(R) = \{V \in \mathrm{Aut}(X, \mathcal{S}) : (Vx, Vy) \in R \text{ if and only if } (x, y) \in R\} \tag{2.5}$$

the *automorphism group* of R. If μ is a probability measure on $\mathcal{S}$ which is quasi-invariant under R we set

$$\mathrm{Aut}(R, \mu) = \{V \in \mathrm{Aut}(R) : \mu \text{ is quasi-invariant under } V\}. \tag{2.6}$$

Definition 2.1. Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$ and μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R. An element $V \in \mathrm{Aut}(R, \mu)$ is *weakly asymptotically central* if it preserves μ and

$$\lim_{|n| \to \infty} \mu(B \triangle V^n W V^{-n} B) = \lim_{|n| \to \infty} \mu(V^{-n} B \triangle W V^{-n} B) = 0 \tag{2.7}$$

for every $W \in [R]$ and $B \in \mathcal{S}$. The automorphism V is *strongly asymptotically central* if it preserves μ and

$$\lim_{|n| \to \infty} \mu(\{x \in X : W V^{-n} W' V^n x = V^{-n} W' V^n W x\}) = 1 \tag{2.8}$$

for all $W, W' \in [R]$.

Remark 2.2. In [7]–[8] weakly asymptotically central automorphisms were called *asymptotically central*. The terminology chosen here is consistent with the weak and strong topology on the set of ergodic transformations: if V is a weakly asymptotically central automorphism of (R, μ), then

$$\lim_{|n| \to \infty} \mu(WV^{-n}W'V^n B \triangle V^{-n}W'V^n WB) = 0 \qquad (2.9)$$

for every $B \in \mathcal{S}$, i.e. W and $V^{-n}W'V^n$ commute asymptotically in the weak topology. If V is strongly asymptotically central, then (2.8) shows that W and $V^{-n}W'V^n$ commute asymptotically in the strong topology (note that, although W and W' need not be measure-preserving, the distortions of μ by the transformations W and $V^{-n}W'V^n$, $n \in \mathbb{Z}$, are bounded in the sense that the measures μW and $\mu V^{-n}W'V^n$ are uniformly absolutely continuous with respect to μ).

Proposition 2.3. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$ and μ a probability measure on $\mathcal{S}$ which is quasi-invariant and conservative under R. Then every strongly asymptotically central automorphism V of (R, μ) is weakly asymptotically central.*

Proof. Since μ is conservative, $R(x)$ is infinite for μ-a.e. $x \in X$, and we can find, for every $B \in \mathcal{S}$, an element $W \in [R]$ with $\mu(B \triangle \{x \in X : Wx = x\}) = 0$. Equation (2.8) implies that

$$\lim_{|n| \to \infty} \mu(B \triangle V^{-n}W'V^n B) = 0$$

for every $W' \in [R]$. As $B \in \mathcal{S}$ is arbitrary this proves that V is weakly asymptotically central. □

We continue with an example which shows that Proposition 2.3 fails without the hypothesis of conservativity.

Example 2.4. (*A strongly, but not weakly, asymptotically central automorphism*). Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with Borel field $\mathcal{S}$ and Lebesgue measure μ,

$$R = \left\{ (x,x), \left(x, x + \tfrac{1}{2}\right) : x \in X \right\},$$

and let $Vx = x + \alpha$ be an irrational rotation on X. Then $V \in \mathrm{Aut}(R, \mu)$, $\mu V^{-1} = \mu$, but V is not weakly asymptotically central: if $Wx = x + \tfrac{1}{2}$ for every $x \in X$, then W commutes with V, and

$$\mu(B \triangle V^n W V^{-n} B) = \mu(B \triangle WB) = \tfrac{1}{2}$$

for every $n \in \mathbb{Z}$, where $B = [0, \tfrac{1}{4}) \subset X$. On the other hand, V is strongly asymptotically central, since $[R]$ is abelian.

We turn to the construction of explicit examples of strongly asymptotically central automorphisms. Here the following theorem is useful.

Theorem 2.5. *Let Γ be a countable abelian group of Borel automorphisms of a standard Borel space $(X, \mathcal{S})$, and let $V \in \mathrm{Aut}(X, \mathcal{S})$ with $V^{-1}\Gamma V = \Gamma$.*
Suppose that $R \subset R[\Gamma]$ is a V-invariant subrelation (i.e. $V \in \mathrm{Aut}(R)$) and μ a V-invariant probability measure on $\mathcal{S}$ which is quasi-invariant under

R. If $V \in \mathrm{Aut}(R, \mu)$ is weakly asymptotically central and mixing then it is strongly asymptotically central.

Proof. Let $W, W' \in [R]$, and put, for every $\gamma, \gamma' \in \Gamma$,

$$C_\gamma = \{x \in X : Wx = \gamma x\}, \quad C'_{\gamma'} = \{x \in X : W'x = \gamma' x\}.$$

We fix $\gamma, \gamma' \in \Gamma$ for the moment. Since V is weakly asymptotically central so is V^{-1}, and

$$\lim_{|n| \to \infty} \mu(V^{-n}W'^{-1}V^n C_\gamma \triangle C_\gamma) = \lim_{|n| \to \infty} \mu(W^{-1}V^{-n}C'_{\gamma'} \triangle V^{-n}C'_{\gamma'})$$

$$= \lim_{|n| \to \infty} \mu(V^{-n}W'^{-1}V^n W^{-1} C_\gamma \triangle W^{-1} C_\gamma) \tag{2.10}$$

$$= \lim_{|n| \to \infty} \mu(V^{-n}W^{-1}V^n W'^{-1} C'_{\gamma'} \triangle W'^{-1} C'_{\gamma'}) = 0.$$

Furthermore, if

$$x \in E(n) = C_\gamma \cap V^{-n}W'^{-1}V^n C_\gamma \cap V^{-n}C'_{\gamma'} \cap W^{-1}V^{-n}C'_{\gamma'},$$

then

$$WV^{-n}W'V^n x = \gamma V^{-n}\gamma' V^n x = V^{-n}\gamma' V^n \gamma x = V^{-n}W'V^n Wx,$$

since γ and $V^{-n}\gamma'V^n$ lie in Γ and therefore commute. Equation (2.10) implies that $\lim_{|n| \to \infty} \mu((C_\gamma \cap V^{-n}C'_{\gamma'}) \smallsetminus E(n)) = 0$ and hence that, for every $\varepsilon > 0$,

$$\mu(\{x \in C_\gamma \cap V^{-n}C'_{\gamma'} : WV^{-n}W'V^n x = V^{-n}W'V^n Wx\}) \geq \mu(C_\gamma)\mu(C'_{\gamma'}) - \varepsilon$$

for every $n \in \mathbb{Z}$ with $|n|$ sufficiently large (depending on ε). By varying $\gamma, \gamma' \in \Gamma$ and $\varepsilon > 0$ we obtain that W and $V^{-n}W'V^n$ commute asymptotically as $|n| \to \infty$. Since $W, W' \in [R]$ were arbitrary this shows that V is strongly asymptotically central. $\qquad \square$

Remark 2.6. If the equivalence relation R in Theorem 2.5 is ergodic then every weakly asymptotically central automorphism V of (R, μ) is mixing by Theorem 2.3 in [7].

Examples 2.7. (*Examples of strongly asymptotically central automorphisms*).

(1) Let A be a finite set, and let σ be the shift on the compact space $Y = A^{\mathbb{Z}}$ defined by

$$(\sigma y)_n = y_{n+1} \tag{2.11}$$

for every $y = (y_n) \in Y$. We denote by

$$\Delta_Y = \{(y, y') \in Y \times Y : y_n \neq y'_n \text{ for only finitely many } n \in \mathbb{Z}\} \tag{2.12}$$

the *Gibbs* (or *two-sided tail*) *equivalence relation* on Y.

Suppose that $X \subset Y$ is a closed, shift-invariant subset, μ a shift-invariant and mixing probability measure on X, and let $\Delta_X = \Delta_Y \cap (X \times X)$ be the *Gibbs relation on X*. According to Lemma 2.3 in [8] there exists a shift-invariant Borel set $B \subset X$ with $\mu(B) = 1$ such that the relation $R = \Delta_B \cup \{(x, x) : x \in X\} \subset \Delta_X$ is μ-nonsingular and $\mathcal{S}^R = \mathcal{S}^{\Delta_X}$, where $\mathcal{S}$ is the Borel field of X (cf. (2.3)–(2.4)). We claim that the restriction $V = \sigma|_X$ of σ to X is a strongly asymptotically central automorphism of (R, μ).

In order to verify this we view μ as a shift-invariant probability measure on Y and consider the shift-invariant equivalence relation

$$R' = R \cup \{(y,y) : y \in Y\} \subset \Delta_Y.$$

Then μ is shift-invariant and quasi-invariant under R'.

Since the exact nature of the set A is irrelevant, we may assume for convenience that $A = \mathbb{Z}_{/k} = \mathbb{Z}/k\mathbb{Z}$ and $Y = \mathbb{Z}_{/k}^{\mathbb{Z}}$ for some $k \geq 2$. Denote by $\Gamma = \sum_{\mathbb{Z}} \mathbb{Z}_{/k}$ the countable dense abelian subgroup of Y consisting of all elements with only finitely many nonzero coordinates and note that $\Delta_Y = R[\Gamma]$ in the sense of (2.1). As $\sigma^{-1}\Gamma\sigma = \Gamma$ and σ is obviously weakly asymptotically central on (R', μ) (cf. [7]–[8]), Theorem 2.5 implies that σ is strongly asymptotically central on (R', μ). This shows that the same is true for V on (R, μ).

(2) Let $\alpha \in \mathrm{GL}(k, \mathbb{Z})$ be a hyperbolic automorphism of the k-torus $X = \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $k \geq 2$, and let $\mathcal{S}$ be the Borel field and

$$\Delta_\alpha(X) = \left\{ x \in X : \lim_{|n| \to \infty} \alpha^n x = 0 \right\}$$

the *homoclinic group* of α. Theorem 4.2 in [5] shows that $\Delta_\alpha(X)$ is a countable dense subgroup of X, and we denote by

$$R(\Delta) = R[\Delta_\alpha(X)]$$

the *homoclinic equivalence relation* on X (cf. (2.1)).

Suppose that μ is an α-invariant probability measure on X. Following Lemma 2.3 in [8] we choose an α-invariant Borel set $B \subset X$ with $\mu(B) = 1$ such that μ is quasi-invariant under $R = R(\Delta)_B \cup \{(x,x) : x \in X\} \subset R(\Delta)$ and $\mathcal{S}^R = \mathcal{S}^{R(\Delta)}$. We claim that α is a weakly asymptotically central automorphism of (R, μ).

Indeed, let $A \in \mathcal{S}$ and $y \in \Delta_\alpha(X)$ be chosen so that $A \cup (A + y) \subset B$. Let $C \subset A$ be a Borel set with $\mu(C) > 0$, and write 1_C for the indicator function of C. For every $k \geq 1$ we choose a continuous map $f_k \colon X \longrightarrow \mathbb{R}$ with $0 \leq f_k \leq 1$ and $\|f_k - 1_C\|_1 = \int |f_k - 1_C| \, d\mu < \frac{1}{k}$.

Our choice of B implies that $0 < \frac{d\mu T_y}{d\mu} < \infty$ μ-a.e. on A, where $T_y x = x + y$ is translation by $y \in \Delta_\alpha(X)$ on X. Since $\frac{d\mu T_{\alpha^n y}}{d\mu} = \frac{d\mu T_y}{d\mu} \circ \alpha^{-n}$ for every $n \in \mathbb{Z}$, we can find, for every $\varepsilon > 0$, a $K \geq 1$ with

$$\|f_k \circ T_{-\alpha^n y} - 1_{C + \alpha^n y}\|_1 = \int_B |f_k - 1_C| \cdot \left(\frac{d\mu T_y}{d\mu} \circ \alpha^{-n} \right) d\mu < \varepsilon$$

for every $k \geq K$ and $n \in \mathbb{Z}$. As

$$\lim_{|n| \to \infty} \|f_k - f_k \circ T_{-\alpha^n y}\|_1 = 0$$

for every $k \geq 1$ by continuity, we obtain that

$$\lim_{|n| \to \infty} \mu(C \triangle (C + \alpha^n y)) = 0 \tag{2.13}$$

for every Borel set $C \subset A$.

From (2.13) one concludes easily that

$$\lim_{|n| \to \infty} \mu(C \triangle \alpha^n W \alpha^{-n} C) = 0$$

for every $W \in [R]$ and $C \in \mathcal{S}$, i.e. that α is weakly asymptotically central on (R, μ). If μ is, in addition, mixing under α then Theorem 2.5 shows that α is strongly asymptotically central.

(3) Let M be a smooth manifold, $U \subset M$ an open set, $\phi \colon U \longrightarrow M$ a C^1-diffeomorphism onto its image, and $X \subset U$ a compact locally maximal hyperbolic ϕ-invariant subset (for terminology we refer to [4]). We fix a metric δ on X and denote by

$$R(\Delta) = \left\{ (x, x') \in X \times X : \lim_{|n| \to \infty} \delta(\phi^n(x), \phi^n(x')) = 0 \right\}$$

the *homoclinic equivalence relation* of ϕ on X. If $\psi \colon M \longrightarrow \mathbb{R}$ is a C^1-function and μ_ψ the Gibbs measure on X arising from ψ, then μ_ψ is quasi-invariant under $R(\Delta)$ and ϕ is a strongly asymptotically central automorphism of $(R(\Delta), \mu_\psi)$. The easiest way to verify this is probably to choose a Markov partition of X and to use Example (1) above.

Example 2.8. (*A weakly, but not strongly, asymptotically central automorphism which has nonabelian invariant cocycles*). Let G be a finite nonabelian group, $A = G^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$, and let σ be the shift on the compact space $X = A^{\mathbb{Z}}$ defined as in (2.11). We write every $x \in X$ as $x = (x_{n,k})$ with $x_{n,k} \in G$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, and denote by μ the normalized Haar measure on X.

For every $g \in G$ we define a transformation T_g on X by

$$(T_g x)_{n,k} = \begin{cases} g x_{n,k} & \text{if } |n| \le k, \\ x_{n,k} & \text{otherwise.} \end{cases}$$

The equivalence relation R generated by the transformations $\{\sigma^{-n} T_g \sigma^n : g \in G, \, n \in \mathbb{Z}\}$ is discrete and μ-nonsingular (in fact, μ is *invariant* under R — cf. [1] and [2]), and σ is a measure-preserving automorphism of (R, μ).

For every $g \in G$ and every cylinder set $C \subset X = G^{\mathbb{Z} \times \mathbb{N}}$,

$$\lim_{|n| \to \infty} \mu(C \triangle \sigma^{-n} T_g \sigma^n(C)) = 0,$$

which implies that σ is weakly asymptotically central.

On the other hand, if g, g' do not commute, then

$$T_g \sigma^{-n} T_{g'} \sigma^n x \ne \sigma^{-n} T_{g'} \sigma^n T_g x$$

for every $n \in \mathbb{Z}$ and $x \in X$, which shows that σ is not strongly asymptotically central.

3. Ranges of invariant cocycles

Definition 3.1. Let G be a Polish (i.e. complete separable metric) group with identity element $1 = 1_G$ and Borel field $\mathcal{B}_G$, R a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$ and $V \in \text{Aut}(X)$. A Borel map $c \colon R \longrightarrow G$ is a *cocycle* on R if

$$c(x, x') c(x', x'') = c(x, x'') \tag{3.1}$$

for every $(x, x'), (x, x'') \in R$. A cocycle $c \colon R \longrightarrow G$ is *V-invariant* if

$$c(V x, V y) = c(x, y) \tag{3.2}$$

for every $(x, y) \in R$.

Let R be a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$ and $c \colon R \longrightarrow G$ a cocycle with values in a Polish group G. In Example 3.3 (2) we shall see that there may exist cocycles c on R which take values in a nonabelian group G and are invariant under a *weakly* asymptotically central automorphism V of R. On the other hand, Theorem 3.2 below shows that invariance of c under a *strongly* asymptotically central automorphism forces the range of c to be abelian.

Theorem 3.2. *Let R be a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$, V an ergodic automorphism of (R, μ) which is both weakly and strongly asymptotically central (cf. Theorem 2.5), and $c : R \longrightarrow G$ a V-invariant cocycle with values in a Polish group G. Then there there exist a μ-null set $N \in \mathcal{S}$ and a closed abelian subgroup $G_0 \subset G$ such that $c(x, y) \in G_0$ for every $(x, y) \in R_{X \smallsetminus N}$.*

Proof. By using a slight modification of the relevant argument in [2] we can find, for every neighbourhood $N(1) \subset G$, a countable collection of elements $\Gamma' \subset [R]$ with

$$R = \{(\gamma x, x) : \gamma \in \Gamma', \, x \in X\}, \qquad (3.3)$$

such that every $\gamma \in \Gamma'$ has the following property: there exist disjoint Borel sets $A_\gamma, B_\gamma \subset X$ and an element $g_\gamma \in G$ with

$$\gamma(A_\gamma) = B_\gamma, \ \ \gamma(B_\gamma) = A_\gamma$$
$$\gamma x = x \ \text{ for } \ x \in X \smallsetminus (A_\gamma \cup B_\gamma)$$
$$\gamma^2 x = x \ \text{ for every } \ x \in X, \qquad (3.4)$$
$$c(\gamma x, x) \in \begin{cases} g_\gamma N(1) & \text{if } x \in A_\gamma, \\ g_\gamma^{-1} N(1) & \text{if } x \in B_\gamma. \end{cases}$$

Let $N(1)$ be a symmetric neighbourhood of the identity in G, and let $\Gamma' \subset [R]$ be a set of nonsingular Borel automorphisms of $(X, \mathcal{S}, \mu)$ with the properties described in (3.3)–(3.4), and fix involutions $\gamma, \gamma' \in \Gamma'$ for the moment. We claim that there exist infinitely many $m \geq 0$ with

$$\mu(A_m(\gamma, \gamma')) > \tfrac{1}{4}\mu(A_\gamma)\mu(A_{\gamma'}), \qquad (3.5)$$

where

$$A_m(\gamma, \gamma') = A_\gamma \cap V^{-m}\gamma'V^m A_\gamma \cap V^{-m}A_{\gamma'} \cap \gamma V^{-m}A_{\gamma'}. \qquad (3.6)$$

Indeed, since V is ergodic,

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} \mu(A_\gamma \cap V^{-m}A_{\gamma'}) = \mu(A_\gamma)\mu(A_{\gamma'}),$$

and hence $\mu(A_\gamma \cap V^{-m}A_{\gamma'}) > \tfrac{3}{4}\mu(A_\gamma)\mu(A_{\gamma'})$ for infinitely many $m \geq 0$. Furthermore, since V is weakly asymptotically central,

$$\mu(A_\gamma \triangle V^{-m}\gamma'V^m A_\gamma) < \tfrac{1}{4}\mu(A_\gamma)\mu(A_{\gamma'})$$

and

$$\mu(A_{\gamma'} \triangle V^m\gamma V^{-m}A_{\gamma'}) < \tfrac{1}{4}\mu(A_\gamma)\mu(A_{\gamma'})$$

for all sufficiently large $m \geq 0$, and by combining these inequalities we obtain (3.5)–(3.6).

The invariance of c and the cocycle equation (3.1) imply that, for every $m \geq 0$ and $x \in A_m(\gamma, \gamma')$,

$$c(V^{-m}\gamma'V^m\gamma x, x) = c(V^{-m}\gamma'V^m\gamma x, \gamma x)c(\gamma x, x)$$
$$= c(\gamma'y, y)c(\gamma x, x) \in g_{\gamma'}N(1)g_\gamma N(1),$$
$$c(\gamma V^{-m}\gamma'V^m x, x) = c(\gamma V^{-m}\gamma'V^m x, V^{-m}\gamma'V^m x)c(V^{-m}\gamma'V^m x, x)$$
$$= c(\gamma z, z)c(\gamma'z', z') = g_\gamma N(1)g_{\gamma'}N(1),$$

where $y = V^m\gamma x \in A_{\gamma'}$, $z = V^{-m}\gamma'V^m x \in A_\gamma$ and $z' = V^m x \in A'_\gamma$.

Finally we note that

$$\lim_{m\to\infty} \mu(\{x \in X : \gamma V^{-m}\gamma'V^m x = V^{-m}\gamma'V^m\gamma x\}) = 1$$

by (2.8), and hence that

$$g_\gamma N(1)g_{\gamma'}N(1) \cap g_{\gamma'}N(1)g_\gamma N(1) \neq \varnothing.$$

Since the elements $\gamma, \gamma' \in \Gamma'$ were arbitrary and $R = \{(\gamma x, x) : \gamma \in \Gamma', x \in X\}$, this proves the existence of a μ-null set $N \in \mathcal{S}$ with

$$c(x, y)N(1)^2 c(x', y')N(1)^2 \cap c(x', y')N(1)^2 c(x, y)N(1)^2 \neq \varnothing$$

for all $(x, y), (x', y') \in R_{X \smallsetminus N}$. As the neighbourhood $N(1)$ was arbitrary we obtain that there exists a null set $N \in \mathcal{S}$ such that $c(x, y)$ and $c(x', y')$ commute for all $(x, y), (x', y') \in R_{X \smallsetminus N}$. This proves the theorem. □

We conclude this section by pointing out the necessity of strong asymptotic centrality in Theorem 3.2.

Examples 3.3. (1) (*A cocycle with nonabelian range which is invariant under an ergodic automorphism*). Let G be a finite nonabelian group, $X = G^{\mathbb{Z}}$, and let $\mathcal{S}$ be the Borel field and μ the normalized Haar measure on X. We denote by Δ_X the Gibbs relation (2.12) on X and define, for every $g \in G$, a map $T_g \colon X \longrightarrow X$ by $(T_g x)_n = gx_n$ for every $x = (x_n) \in X$. Each T_g is an automorphism of Δ_X, and

$$R = \{(T_g x, x') : g \in G, (x, x') \in \Delta_X\} \supsetneq \Delta_X$$

is the equivalence relation generated by $\{T_g : g \in G\}$ and Δ_X. The measure μ is quasi-invariant (in fact, *invariant* — cf. [1] and [2]) and ergodic under R, and the shift $\sigma \colon X \longrightarrow X$ in (2.11) is a μ-preserving automorphism of R which is not weakly (and hence, by Proposition 2.3, not strongly) asymptotically central, since it commutes with T_g for every $g \in G$. The cocycle $c \colon R \longrightarrow G$ with

$$c(T_g x, x') = g$$

for every $g \in G$ and $(x, x') \in R$ is shift-invariant and has nonabelian range.

(2) (*A cocycle with nonabelian range which is invariant under a weakly asymptotically central automorphism*). Let G be a finite nonabelian group, $A = G^{\mathbb{N}}$, and let $X = A^{\mathbb{Z}}$ be the space and $R \subset X \times X$ the equivalence relation described in Example 2.8. As was shown there, the shift σ on X is

weakly, but not strongly asymptotically central. For every $x = (x_{n,k}) \in X$ and $g \in G$ we set

$$c'(T_g x, x) = \lim_{k \to \infty} (T_g x)_{0,k} x_{0,k}^{-1} = g \qquad (3.7)$$

and extend c' to a cocycle $c \colon R \longrightarrow G$ by (3.1). Since

$$c'(T_g x, x) = \lim_{k \to \infty} (T_g x)_{m,k} x_{m,k}^{-1}$$

for every $m \in \mathbb{Z}$ it is clear that

$$c(x, y) = \lim_{k \to \infty} x_{m,k} y_{m,k}^{-1}$$

for every $x = (x_{n,k})$, $y = (y_{n,k})$ and $m \in \mathbb{Z}$. This shows that the cocycle c is σ-invariant and has nonabelian range G.

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