ON THE COHOMOLOGY OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS WITH VALUES IN COMPACT LIE GROUPS

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Abstract. Let $d \geq 2$, and let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X. We prove the following: if the entropy $h(\alpha_{\mathbf{n}})$ is finite for every $\mathbf{n} \in \mathbb{Z}^d$, then there exist nontrivial Hölder 1-cocycles for α with values in certain compact Lie groups, even though α need not have any nontrivial Hölder cocycles with values in abelian groups. On the other hand, if α has completely positive entropy, then there exists an expansive and mixing 'cover' $\bar{\alpha}$ of α with completely positive entropy such that $h(\bar{\alpha}) = h(\alpha)$ and $\bar{\alpha}$ has no nontrivial Hölder 1-cocycles with values in any complete metric group which is compact or abelian. For a specific class of such actions $\bar{\alpha} = \alpha$, so that α itself is cohomologically trivial in this sense.

1. Introduction

Let α be a mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let (G, γ) be a complete metric group with a bi-invariant metric γ in which G has diameter 1 (γ is *bi-invariant* if $\gamma(gg_1, gg_2) = \gamma(g_1, g_2) = \gamma(g_1 g, g_2 g)$ for all $g, g_1, g_2 \in G$; a metrizable group G has such a metric if it is discrete, abelian or compact, for example). A continuous map $c\colon \mathbb{Z}^d \times X \longmapsto G$ is a continuous 1-*cocycle* (or simply a *cocycle*) for α if

$$c(\mathbf{m}, \alpha_{\mathbf{n}}(x))c(\mathbf{n}, x) = c(\mathbf{m} + \mathbf{n}, x) \tag{1.1}$$

for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. Two cocycles $c, c'\colon \mathbb{Z}^d \times X \longmapsto G$ for α are *measurably cohomologous* if there exists a Borel map $b\colon X \longmapsto G$ such that, for every $\mathbf{n} \in \mathbb{Z}^d$,

$$c'(\mathbf{n}, x) = b(\alpha_{\mathbf{n}}(x))^{-1} c(\mathbf{n}, x) b(x) \tag{1.2}$$

λ_X-a.e., where λ_X is the normalised Haar measure of X. The map b is the *transfer function* of (c, c'); if b is continuous then c and c' are said to be *continuously cohomologous* or simply *cohomologous*. If a cocycle c is (measurably) cohomologous to the constant cocycle $c'(\mathbf{n}, x) = 1_G$, $\mathbf{n} \in \mathbb{Z}^d, x \in X$, where 1_G is the identity element in G, then c is a (measurable) *coboundary* and the map b in (1.2) is the *cobounding function* of c. A cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ is a *homomorphism* if $c(\mathbf{n}, \cdot)$ is constant for every $\mathbf{n} \in \mathbb{Z}^d$, and *trivial* if it is cohomologous to a homomorphism.

Under our assumptions on α it is well known that there exist nontrivial cocycles for α with values in $\mathbb{R}$ or in the multiplicative circle group $\mathbb{S} = \{z \in$

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$\mathbb{C} : |z| = 1\}$: in both cases the set of nontrivial cocycles is residual in the set of all continuous cocycles. If we impose stronger continuity conditions on the cocycles, such as Hölder continuity, interesting differences begin to emerge between the cases $d = 1$ and $d > 1$. In order to describe the appropriate notion of Hölder continuity we assume from now on that α is expansive and proceed as in [4] and [13]: fix a metric δ on X, write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ for the Euclidean norm and inner product on $\mathbb{Z}^d \subset \mathbb{R}^d$, and put $\mathbf{B}(r) = \{\mathbf{m} \in \mathbb{Z}^d : \|\mathbf{m}\| \leq r\}$ for every $r \geq 0$. For every continuous function $h \colon X \longmapsto G$ and every $\varepsilon, r \geq 0$, we set

$$\omega_r^{\delta,\gamma}(h,\alpha,\varepsilon) = \sup_{\{(x,x') \in X \times X : \max_{\mathbf{m} \in \mathbf{B}(r)} \delta(\alpha_{\mathbf{m}}(x), \alpha_{\mathbf{m}}(x')) < \varepsilon\}} \gamma(h(x), h(x')). \tag{1.3}$$

The function h has *α-summable variation* if there exists an $\varepsilon > 0$ such that

$$\omega^{\delta,\gamma}(h,\alpha,\varepsilon) = \sum_{r=1}^{\infty} \omega_r^{\delta,\gamma}(h,\alpha,\varepsilon) < \infty, \tag{1.4}$$

and h is *α-Hölder* if there are constants $\varepsilon, \omega' > 0$ and $\omega \in (0,1)$ with

$$\omega_r^{\delta,\gamma}(h,\alpha,\varepsilon) < \omega' \omega^r \tag{1.5}$$

for every $r > 0$. These notions obviously depend on γ, but are independent of the metric δ on X, and every α-Hölder function has α-summable variation. If the group G is discrete, and if $\gamma(g, g') = 1$ for $g \neq g'$ and 0 otherwise, then a function $h \colon X \longmapsto G$ is α-Hölder if and only if it is continuous.

If there is no danger of confusion we suppress the prefix α- and simply speak of Hölder functions and functions with summable variation.

Note that the Hölder structure defined by α is a purely topological notion: if one adopts the analogous definition of Hölder functions for any continuous, expansive $\mathbb{Z}^d$-action T on a compact space Y (cf. [4], [12]), and if T' is a second continuous, expansive $\mathbb{Z}^d$-action on a compact space Y' which is topologically conjugate to T, then any homeomorphism $\psi \colon Y \longmapsto Y'$ implementing this topological conjugacy carries the set of T-Hölder functions on Y to the set of T'-Hölder functions on Y'. Furthermore, if Y is a compact manifold, then the Hölder structure defined by an expansive $\mathbb{Z}^d$-action T on Y coincides with the familiar one.

We return to our expansive and mixing $\mathbb{Z}^d$-action α by automorphisms of X. A cocycle $c \colon \mathbb{Z}^d \times X \longmapsto G$ for α has *summable variation* or is *Hölder* if $c(\mathbf{n}, \cdot)$ has summable variation or is Hölder for every $\mathbf{n} \in \mathbb{Z}^d$. It is not difficult to verify that, if two cocycles $c, c' \colon \mathbb{Z}^d \times X \longmapsto G$ with summable variation are measurably cohomologous, then they are continuously cohomologous; moreover, if c and c' are Hölder, then the (continuous) transfer function is again Hölder (cf. [4], [12]). We write $Z_H^1(\alpha, G) \subset Z_{sv}^1(\alpha, G) \subset Z_c^1(\alpha, G)$ for the sets of Hölder cocycles, cocycles with summable variation, and continuous cocycles with values in G, denote by $B_c^1(\alpha, G) \subset Z_c^1(\alpha, G)$ the subset of coboundaries, and write

$$H_c^1(\alpha, G) = \{[c] : c \in Z_c^1(\alpha, G)\}$$

for the space of cohomology classes

$$[c] = \{c' \in Z_c^1(\alpha, G) : c' \text{ is cohomologous to } c\}.$$

Similarly we set $B^1_{sv}(\alpha, G) = B^1_c(\alpha, G) \cap Z^1_{sv}(\alpha, G)$, $B^1_H(\alpha, G) = B^1_c(\alpha, G) \cap Z^1_H(\alpha, G)$, and denote by

$$\begin{aligned} H^1_{sv}(\alpha, G) &= \{[c]_{sv} = [c] \cap Z^1_{sv}(\alpha, G) : c \in Z^1_{sv}(\alpha, G)\}, \\ H^1_H(\alpha, G) &= \{[c]_H = [c] \cap Z^1_H(\alpha, G) : c \in Z^1_H(\alpha, G)\} \end{aligned}$$

the *cohomology with summable variation* and the *Hölder cohomology* of α with values in G. If G is abelian, the sets $Z^1_c(\alpha, G)$, $Z^1_{sv}(\alpha, G)$ and $Z^1_H(\alpha, G)$ are groups under pointwise addition, the coboundaries $B^1_c(\alpha, G) \subset Z^1_c(\alpha, G)$, $B^1_{sv}(\alpha, G) \subset Z^1_{sv}(\alpha, G)$ and $B^1_H(\alpha, G) \subset Z^1_H(\alpha, G)$ form subgroups, and the first cohomologies

$$\begin{aligned} H^1_c(\alpha, G) &= Z^1_c(\alpha, G)/B^1_c(\alpha, G), \\ H^1_{sv}(\alpha, G) &= Z^1_{sv}(\alpha, G)/B^1_{sv}(\alpha, G), \\ H^1_H(\alpha, G) &= Z^1_H(\alpha, G)/B^1_H(\alpha, G) \end{aligned}$$

are groups.

Livshitz' theorem ([8]) shows that the $\mathbb{Z}$-action defined by a single expansive and ergodic (hence mixing) automorphism of a compact, abelian group X has an abundance of nontrivial Hölder cocycles: for every connected, abelian group G the set of trivial Hölder cocycles is a proper, closed subgroup of $Z^1_H(\alpha, G)$. If $d \geq 2$, the situation becomes much more complex. In [4] it was shown that, if $d \geq 2$, and if α is an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, then every cocycle $c\colon \mathbb{Z}^d \times X \longmapsto \mathbb{R}$ with summable variation is trivial. If we replace $\mathbb{R}$ by $\mathbb{S}$, the cohomology is nontrivial in some, but not in all cases, and can be calculated explicitly ([13]). For the shift-action of $\mathbb{Z}^d$ on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^d}$ even more is true: for every complete metric group (G, γ) such that γ is bi-invariant, every cocycle $c\colon \mathbb{Z}^d \times (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^d} \longmapsto G$ with summable variation is trivial (cf. [2], [3], [12]; in [12] the group G was assumed to be locally compact, but this assumption was not used in the proof).

In [12] it became apparent that any understanding of the first cohomology of a higher dimensional shift of finite type has to take into account the cohomologies with values in nonabelian groups, and that there are some very curious connections between these cohomologies and the intrinsic complexity of the shift space. The results in [4], [12], [13] and this paper are beginning to reveal a similar picture for $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups. Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X. The main result in [13] states that every cocycle $c \in Z^1_{sv}(\alpha, \mathbb{S})$ is cohomologous to a cocycle $a\colon \mathbb{Z}^d \times X \longmapsto \mathbb{S}$ with the property that, for every $\mathbf{n} \in \mathbb{Z}^d$, the map $a(\mathbf{n}, \cdot)\colon X \longmapsto \mathbb{S}$ is a constant multiple of an element in the dual group $\hat{X}$ of X, and Theorem 11.1 in [13]) yields examples of $\mathbb{Z}^d$-actions α with $H^1_{sv}(\alpha, \mathbb{S}) = \{0\}$. The absence of nontrivial cocycles with values in abelian groups need not imply that every Hölder cocycle with values in, say, a compact Lie group is trivial. In fact we shall prove that, if the action α is sufficiently 'small', one can always find nontrivial Hölder cocycles with values in certain compact Lie groups (Theorem 3.1). There do exist, however, expansive and mixing $\mathbb{Z}^d$-actions by automorphisms of compact, connected, abelian groups for which every cocycle $c \in Z^1_{sv}(\alpha, G)$ is trivial whenever G is any compact group G with a fixed bi-invariant metric (Theorem 4.2). Example 4.8 shows that $\mathbb{Z}^d$-actions quite closely related to those featuring in Theorem 4.2 can still have nontrivial Hölder cohomology with values in compact Lie groups. Theorem 4.1 states that every expansive and mixing $\mathbb{Z}^d$-action α by automorphisms of a compact, abelian group X has an expansive, mixing, and cohomologically trivial 'cover' $\bar{\alpha}$ with completely positive entropy $h(\alpha) = h(\bar{\alpha})$. The proofs of the Theorems 4.2 and 4.1 also show that these actions have no nontrivial Hölder cocycles with values in abelian groups (cf. Remark 4.9).

If one recalls that Zimmer's cocycle rigidity theorem (cf. [15]) is a statement about the triviality of certain measurable cocycles for finite measure preserving, ergodic actions of certain semisimple Lie groups, then one can view the results in this paper (as well as some of those in [5], [6], [4] and [12]) as analogues of Zimmer's theorem in the following sense: although it is well known that one cannot expect any rigidity statements about measurable cocycles for actions of abelian groups, analogous rigidity results *do* exist for certain actions of $\mathbb{Z}^d$, $d > 1$, but this time in the category of Hölder continuous functions. In this language one can express Theorem 3.1 as a statement about Zimmer-type rigidity of all Hölder cocycles for certain $\mathbb{Z}^d$-actions with values in compact Lie groups, whereas Theorem 4.2 states that the $\mathbb{Z}^d$-actions discussed there do have nonrigid (=nontrivial) cocycles (albeit of a very special form) with values in compact Lie groups.

2. Algebraic background

We begin with the algebraic description of $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups in [11] which will be needed in Section 4, and which allows the construction of examples of such actions with specified properties (like ergodicity, mixing, expansiveness, positive or zero entropy, etc.).

Let $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. We write every $f \in \mathfrak{R}_d$ as $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}}$ with $u^{\mathbf{m}} = u_1^{m_1} \cdot \dots \cdot u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $\sum_{\mathbf{m} \in \mathbb{Z}^d} |c_f(\mathbf{m})| < \infty$.

Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, metrizable, abelian group X. The additively written dual group $\mathfrak{M} = \hat{X}$ is a countable module over the ring $\mathfrak{R}_d$ with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \widehat{\alpha_{\mathbf{m}}}(a) \tag{2.1}$$

for every $f \in \mathfrak{R}_d$ and $a \in \mathfrak{M}$; here $\widehat{\alpha_{\mathbf{m}}}$ is the automorphism of $\mathfrak{M} = \hat{X}$ dual to $\alpha_{\mathbf{m}}$. In particular,

$$\widehat{\alpha_{\mathbf{m}}}(a) = u^{\mathbf{m}} \cdot a \tag{2.2}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. The module $\mathfrak{M}$ is Noetherian whenever α is expansive ((4.10) and Proposition 5.4 in [11]). Conversely, if $\mathfrak{M}$ is a countable $\mathfrak{R}_d$-module, and if

$$\widehat{\alpha^{\mathfrak{M}}_{\mathbf{m}}}(a) = u^{\mathbf{m}} \cdot a \tag{2.3}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$, then we obtain a $\mathbb{Z}^d$-action

$$\alpha^{\mathfrak{M}} \colon \mathbf{m} \mapsto \alpha^{\mathfrak{M}}_{\mathbf{m}} \tag{2.4}$$

on the compact, abelian group

$$X^{\mathfrak{M}} = \widehat{\mathfrak{M}} \tag{2.5}$$

dual to the $\mathbb{Z}^d$-action $\widehat{\alpha^{\mathfrak{M}}} \colon \mathbf{m} \mapsto \widehat{\alpha^{\mathfrak{M}}_{\mathbf{m}}}$ on $\mathfrak{M}$. If $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{R}_d$, then the action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is mixing (with respect to Haar measure) if and only if multiplication by $u^{\mathbf{m}} - 1$ on $\mathfrak{R}_d/\mathfrak{a}$ is injective for every non-zero $\mathbf{m} \in \mathbb{Z}^d$, and expansive if and only if $V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d = \varnothing$, where $V_{\mathbb{C}}(\mathfrak{a}) = \{c \in \mathbb{C}^d : f(c) = 0 \text{ for all } f \in \mathfrak{a}\}$. If the ideal $\mathfrak{a}$ is prime, then $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is mixing if and only if $\mathfrak{a} \cap \{u^{\mathbf{m}} - 1 : \mathbf{m} \in \mathbb{Z}^d\} = \{0\}$.

If $\mathfrak{M}$ is an arbitrary $\mathfrak{R}_d$-module, then a prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated with* $\mathfrak{M}$ if $\mathfrak{p} = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$ for some $a \in \mathfrak{M}$, and $\mathfrak{M}$ is *associated with a prime ideal* $\mathfrak{p} \subset \mathfrak{R}_d$ if $\mathfrak{p}$ is the only prime ideal associated with $\mathfrak{M}$. The set of (distinct) prime ideals associated with a Noetherian $\mathfrak{R}_d$-module $\mathfrak{M}$ is finite.

For an arbitrary Noetherian $\mathfrak{R}_d$-module $\mathfrak{M}$ with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ the following conditions are equivalent.

(i) $\alpha^{\mathfrak{M}}$ is expansive and mixing (with respect to the normalised Haar measure on $X^{\mathfrak{M}} = \widehat{\mathfrak{M}}$);

(ii) $\alpha^{\mathfrak{R}_d/\mathfrak{p}_j}$ is expansive and mixing for every $j = 1, \dots, m$.

Another property of $\alpha^{\mathfrak{M}}$ determined by the behaviour of the actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}_j}$, where $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ are the associated primes of $\mathfrak{M}$, is completely positive entropy. If $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal then the topological entropy $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}})$ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ (which is equal to the metric entropy with respect to Haar measure by [7] or Theorem 13.3 in [11]) is positive if and only if $\mathfrak{p}$ is principal and $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is mixing (cf. [7], or Corollary 18.5, Theorem 19.5 and Corollary 6.12 in [11]). In general, if $\mathfrak{M}$ is a Noetherian $\mathfrak{R}_d$-module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, then $\alpha^{\mathfrak{M}}$ has completely positive entropy if and only if $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}_j}) > 0$ for every $j = 1, \dots, m$.

One can realise explicitly the $\mathbb{Z}^d$-actions of the form $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal (cf. (2.3)–(2.5)). Write

$$(\sigma_{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}} \tag{2.6}$$

for the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$, put

$$f(\sigma) = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \sigma_{\mathbf{m}} \colon \mathbb{T}^{\mathbb{Z}^d} \longmapsto \mathbb{T}^{\mathbb{Z}^d} \tag{2.7}$$

for every $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \in \mathfrak{R}_d$, and identify $\mathfrak{R}_d$ with the dual group $\widehat{\mathbb{T}^{\mathbb{Z}^d}}$ of $\mathbb{T}^{\mathbb{Z}^d}$ by setting

$$\langle f, x \rangle = e^{2\pi i (f(\sigma)(x))_{\mathbf{0}}} \tag{2.8}$$

for every $f \in \mathfrak{R}_d$ and $x \in \mathbb{T}^{\mathbb{Z}^d}$. A closed subgroup $X \subset \mathbb{T}^{\mathbb{Z}^d}$ is shift-invariant if and only if its annihilator $X^{\perp} = \mathfrak{a} \subset \mathfrak{R}_d$ is an ideal, in which case

$$X = X^{\mathfrak{R}_d/\mathfrak{a}} = \{x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)(x) = 0_{\mathbb{T}^{\mathbb{Z}^d}} \text{ for every } f \in \mathfrak{a}\} \tag{2.9}$$

and $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is the restriction of σ to $X^{\mathfrak{R}_d/\mathfrak{a}} \subset \mathbb{T}^{\mathbb{Z}^d}$.

More generally, if α is an expansive $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, then ([11], (4.10), Proposition 5.4 and Example 5.2 (4)) allow us to regard X as a closed, shift-invariant subgroup of $(\mathbb{T}^m)^{\mathbb{Z}^d}$ for some $m \geq 1$, and α as the restriction to X of the shift-action σ on $(\mathbb{T}^m)^{\mathbb{Z}^d}$ (cf. (2.6)). We write a typical point $x \in X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}})$ with $x_{\mathbf{n}} = (x_{\mathbf{n}}^{(1)}, \dots, x_{\mathbf{n}}^{(m)}) \in \mathbb{T}^m$ for every $\mathbf{n} \in \mathbb{Z}^d$. Every character in

$$X^\perp \subset \widehat{(\mathbb{T}^m)^{\mathbb{Z}^d}} = \sum_{\mathbb{Z}^d} \mathbb{Z}^m \cong (\mathfrak{R}_d)^m$$

is of the form

$$\langle h, x \rangle = \prod_{i=1}^{m} \langle h^{(i)}, x^{(i)} \rangle \tag{2.10}$$

for every $x = (x^{(1)}, \dots, x^{(m)}) \in X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$, where $h = (h^{(1)}, \dots, h^{(m)}) \in (\mathfrak{R}_d)^m$, and where $\langle h^{(i)}, x^{(i)} \rangle$ is defined by (2.8) for $i = 1, \dots, m$. The shift-invariance of X guarantees that

$$X^\perp = \{h \in (\mathfrak{R}_d)^m : \langle h, x \rangle = 1 \text{ for every } x \in X\}$$

is a submodule of $(\mathfrak{R}_d)^m$, and hence Noetherian. In particular there exist finitely many elements $h_j = (h_j^{(1)}, \dots, h_j^{(m)})$, $j = 1, \dots s$, which generate $X^\perp$ as an $\mathfrak{R}_d$-module, and which therefore satisfy that

$$X = \{x \in (\mathbb{T}^m)^{\mathbb{Z}^d} : \langle u^{\mathbf{m}} h_j, x \rangle = 1 \text{ for every } \mathbf{m} \in \mathbb{Z}^d\}.$$

For every $t \in \mathbb{T}$ and $\mathbf{t} = (t^{(1)}, \dots, t^{(m)}) \in \mathbb{T}^m$ we set

$$|t| = \min\{|t + k| : k \in \mathbb{Z}\}, \qquad |\mathbf{t}| = \max_{i=1,\dots,m} |t^{(i)}|. \tag{2.11}$$

Put

$$\varepsilon = \Bigl(10 \sum_{j=1}^{s} \sum_{i=1}^{m} \sum_{\mathbf{n} \in \mathbb{Z}^d} |c_{h_j^{(i)}}(\mathbf{n})|\Bigr)^{-1} \tag{2.12}$$

and

$$N = \max_{j=1,\dots,s} \max_{i=1,\dots,m} \max\{\|\mathbf{n}\| : \mathbf{n} \in \mathbb{Z}^d \text{ and } c_{h_j^{(i)}}(\mathbf{n}) \neq 0\}, \tag{2.13}$$

where

$$h_j^{(i)} = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{h_j^{(i)}}(\mathbf{n}) u^{\mathbf{n}}$$

for every i, j. The expansiveness of α guarantees the existence of an integer $M \geq 0$ with the following property: if $x \in X$ and $|x_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n} \in \mathbf{B}(M+N)$, and if $y_{\mathbf{n}}^{(i)} \in \mathbb{R}$ satisfies that $|y_{\mathbf{n}}^{(i)}| < \varepsilon$ and $y_{\mathbf{n}}^{(i)} \pmod 1 = x_{\mathbf{n}}^{(i)}$ for every $\mathbf{n} \in \mathbf{B}(M+N)$, then

$$\sum_{i=1}^{m} \sum_{\mathbf{m} \in \mathbb{Z}^d} c_{h_j^{(i)}}(\mathbf{m}) y_{\mathbf{m}+\mathbf{n}}^{(i)} = 0 \tag{2.14}$$

for every $i = 1, \dots, m$, $j = 1, \dots, s$ and $\mathbf{n} \in \mathbf{B}(M)$, and

$$\max_{i=1,\dots,m} |y_{\mathbf{0}}^{(i)}| < \varepsilon/2. \tag{2.15}$$

A recursive application of (2.14)–(2.15) implies that

$$\max_{i=1,\dots,m} |y_{\mathbf{k}}^{(i)}| < 2^{-l}\varepsilon \tag{2.16}$$

whenever $|x_{\mathbf{n}}^{(i)}| < \varepsilon$ for every $\mathbf{n} \in \mathbf{k} + \mathbf{B}(l(N+M))$. We have proved the following proposition.

Proposition 2.1. *Let $m \geq 1$, and let $X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$ be a closed, shift-invariant subgroup such that the restriction $\alpha = \sigma^X$ of the shift-action σ on $(\mathbb{T}^m)^{\mathbb{Z}^d}$ to X is expansive. In the notation of* (2.11) *there exist constants $\varepsilon, \eta \in (0,1)$ and $C > 0$ with the following property for every $\mathbf{k} \in \mathbb{Z}^d$ and $L \geq 0$: if $x \in X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$ satisfies that*

$$\max_{\mathbf{n} \in \mathbf{k}+\mathbf{B}(L)} |x_{\mathbf{n}}| < \varepsilon,$$

then

$$|x_{\mathbf{k}}| < C\eta^L.$$

The final assertion in the following corollary was used without proof in [13].

Corollary 2.2. *Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, $k \geq 1$, and let $\eta\colon X \longmapsto \mathbb{T}^k$ be a continuous group homomorphism. If (G, γ) is a complete metric group, where γ is bi-invariant, and if $h\colon \mathbb{T}^k \longmapsto G$ is a map which is Hölder or has summable variation, then the composition $h \cdot \eta\colon X \longmapsto G$ is Hölder or has summable variation. In particular, every character χ of X is Hölder with respect to the usual* (*Euclidean*) *metric on $\mathbb{S}$.*

Proof. As explained above we may assume that X is a closed, shift-invariant subgroup of $(\mathbb{T}^m)^{\mathbb{Z}^d}$ for some $m \geq 1$, and that α is the restriction to X of the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{T}^m)^{\mathbb{Z}^d}$. There exist a finite subset $F \subset \mathbb{Z}^d$ and a continuous group homomorphism $\eta'\colon (\mathbb{T}^m)^F \longmapsto \mathbb{T}^k$ with

$$\eta = \eta' \cdot \pi_F,$$

where π_F is the coordinate projection which sends each $x = (x_{\mathbf{n}}) \in X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$ to its coordinates in F. Proposition 2.1 yields constants $\varepsilon, \eta \in (0,1)$ and $C > 0$ such that $\max_{\mathbf{n} \in F} |x_{\mathbf{n}}| < C\eta^L$ whenever $x = (x_{\mathbf{n}}) \in X$ and $|x_{\mathbf{m}}| < \varepsilon$ for every $\mathbf{m} \in \mathbf{B}(L)$ and implies the appropriate continuity property of $h \cdot \eta$. The last assertion follows by taking $G = \mathbb{S}$ and $\eta = \chi$. □

3. Cohomological nontriviality

If α is a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X we denote by $h(\alpha)$ and $h(\alpha_{\mathbf{n}})$ the topological entropies of α and $\alpha_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. In this section we prove the following result.

Theorem 3.1. *Let $d \geq 2$, and let α be an expansive and mixing $\mathbb{Z}^d$-action on a compact, abelian group X with the property that each $\alpha_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, has finite entropy. Then there exists a compact Lie group G for which $Z_H^1(\alpha, G)$ contains nontrivial elements.*

The proof of Theorem 3.1 requires three preliminary results.

Lemma 3.2. *Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, $Y \subset X$ a closed, α-invariant subgroup such that the restriction α^Y of α to Y is nonergodic, and let α^Z be the $\mathbb{Z}^d$-action induced by α on the quotient group $Z = X/Y$. Then α^Z is expansive, and there exist a compact Lie group G and a nontrivial Hölder cocycle $c\colon \mathbb{Z}^d \times Z \longmapsto G$.*

Proof. The expansiveness of α^Z follows from Corollary 6.15 in [11]. Since α^Y is nonergodic, there exists a nontrivial element $\chi \in \hat{Y}$ which has finite orbit $\{\chi = \chi_1, \dots, \chi_m\}$, say, under the dual action $\widehat{\alpha^Y}$ (cf. [1] or Lemma 1.2 in [11]). We extend each χ_i to a character $\bar{\chi}_i \in \hat{X}$ and set

$$D(x) = \begin{pmatrix} \bar{\chi}_1(x) & 0 & \dots & 0 & 0 \\ 0 & \bar{\chi}_2(x) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{\chi}_{m-1}(x) & 0 \\ 0 & 0 & \dots & 0 & \bar{\chi}_m(x) \end{pmatrix} \in U(m)$$

for every $x \in X$. For every $\mathbf{n} \in \mathbb{Z}^d$, the characters $(\chi_1 \cdot \alpha^Y_{\mathbf{n}}, \dots, \chi_m \cdot \alpha^Y_{\mathbf{n}})$ are a permutation of $(\chi_1, \dots, \chi_m)$. Hence there exist a unique permutation matrix $P(\mathbf{n})$ and characters $\psi_1^{(\mathbf{n})}, \dots, \psi_m^{(\mathbf{n})}$ in $Y^\perp \subset \hat{X}$ such that

$$D(\alpha_{\mathbf{n}}(x)) = Q(\mathbf{n}, x) P(\mathbf{n}) D(x) P(\mathbf{n})^{-1} \tag{3.1}$$

for every $\mathbf{n} \in \mathbf{Z}^d$ and $x \in X$, where

$$Q(\mathbf{n}, x) = \begin{pmatrix} \psi_1^{(\mathbf{n})}(x) & 0 & \dots & 0 & 0 \\ 0 & \psi_2^{(\mathbf{n})}(x) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \psi_{m-1}^{(\mathbf{n})}(x) & 0 \\ 0 & 0 & \dots & 0 & \psi_m^{(\mathbf{n})}(x) \end{pmatrix}.$$

If

$$c(\mathbf{n}, x + Y) = Q(\mathbf{n}, x) P(\mathbf{n}) \in U(m)$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$, then $c(\mathbf{m} + \mathbf{n}, z) = c(\mathbf{m}, \alpha^Z_{\mathbf{n}}(z)) c(\mathbf{n}, z)$ for every $z \in Z$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, so that $c \in Z^1_c(\alpha^Z, U(m))$. The Hölder continuity of c follows from the fact that every character of Z is Hölder continuous by Corollary 2.2.

If $c \in Z^1_H(\alpha^Z, U(m))$ were trivial, then we could find a continuous map $b\colon Z \longmapsto U(m)$ and a homomorphism $a\colon \mathbb{Z}^d \longmapsto U(m)$ with

$$c(\mathbf{n}, z) = b(\alpha^Z_{\mathbf{n}}(z))^{-1} a(\mathbf{n}) b(z) \tag{3.2}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $z \in Z$. Put

$$\Xi = \{\mathbf{n} \in \mathbb{Z}^d : P(\mathbf{n}) = 1_{U(m)}\} = \ker(P(\cdot)),$$
$$V = X \times \mathbb{Z}^d/\Xi,$$

set $\overline{\mathbf{m}} = \mathbf{m} + \Xi \in \mathbb{Z}^d/\Xi$ for every $\mathbf{m} \in \mathbb{Z}^d$, and define a continuous $\mathbb{Z}^d$-action T on V by

$$T_{\mathbf{m}}(x, \overline{\mathbf{n}}) = (\alpha_{\mathbf{m}}(x), \overline{\mathbf{m}} + \overline{\mathbf{n}})$$

for every $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. The map $b' \colon V \longmapsto U(m)$, defined by

$$b'(x, \overline{\mathbf{n}}) = D(x)P(\mathbf{n}),$$

satisfies that

$$b'(T_{\mathbf{m}}(x, \overline{\mathbf{n}})) \cdot b'(x, \overline{\mathbf{n}})^{-1} = c(\mathbf{m}, \theta(x)) \tag{3.3}$$

for every $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, where $\theta \colon X \longmapsto Z$ is the quotient map (this is also a verification that c is a cocycle). A comparison of (3.2) and (3.3) reveals that

$$b \cdot \theta(\alpha_{\mathbf{m}}(x))^{-1} a(\mathbf{m}) b \cdot \theta(x) = b'(T_{\mathbf{m}}(x, \overline{\mathbf{n}})) b'(x, \overline{\mathbf{n}})^{-1} \tag{3.4}$$

for every $\mathbf{m}, \mathbf{n}$ and x, so that the map

$$(x, \overline{\mathbf{n}}) \mapsto b''(x, \overline{\mathbf{n}}) = b(\theta(x)) b'(x, \overline{\mathbf{n}})$$

from V to $U(m)$ cobounds the homomorphism a with respect to the $\mathbb{Z}^d$-action T on V. In order to prove that $a(\mathbf{n}) = 1_{U(m)}$ for every $\mathbf{n} \in \Xi$ we define a $\mathbb{Z}^d$-action $\mathbf{n} \mapsto V_{\mathbf{n}}$ on the Hilbert space $\mathcal{H} = L^2(X, \lambda_X, \mathbb{C}^m)$ of square-integrable functions $\psi \colon X \longmapsto \mathbb{C}^m$ by setting, for every $\psi \in \mathcal{H}$, $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$,

$$(V_{\mathbf{n}} \psi)(x) = a(\mathbf{n}) \psi(\alpha_{\mathbf{n}}(x)).$$

For every $\zeta \in \mathbb{C}^m$, the map $\psi_\zeta(x) = b''(x, \overline{\mathbf{0}})\zeta$ satisfies that $V_{\mathbf{n}} \psi_\zeta = \psi_\zeta$ for every $\mathbf{n} \in \Xi$. Choose an orthonormal basis $\zeta^{(1)}, \dots, \zeta^{(m)}$ of $\mathbb{C}^m$ in which every $a(\mathbf{n})$ is diagonal, denote by $P^{(i)} \colon \mathcal{H} \longmapsto \mathcal{H}$ the projections onto the subspaces $\{\psi \zeta^{(i)} : \psi \in L^2(X, \lambda_X)\} \subset \mathcal{H}$, and observe that $P^{(i)} V_{\mathbf{n}} = V_{\mathbf{n}} P^{(i)}$ and

$$V_{\mathbf{n}} \psi_\zeta^{(i)} = \chi^{(i)}(\mathbf{n}) (\psi_\zeta^{(i)} \cdot \alpha_{\mathbf{n}}) = \psi_\zeta^{(i)} \ \ \lambda_X\text{-a.e.},$$

for every $\zeta \in \mathbb{C}^m$, $i = 1, \dots, m$ and $\mathbf{n} \in \Xi$, where $\psi_\zeta^{(i)} = P^{(i)} \psi_\zeta$ and $a(\mathbf{n}) \zeta^{(i)} = \chi^{(i)}(\mathbf{n}) \zeta^{(i)}$. By varying ζ we obtain nonzero eigenfunctions for the Ξ-action α with eigenvalues $\chi^{(i)} \in \widehat{\Xi} \subset \widehat{\mathbb{Z}^d}$, $i = 1, \dots, m$. As the restriction of α to Ξ is mixing, $\chi^{(i)}(\mathbf{n}) = 1$ for every i and $\mathbf{n} \in \Xi$, so that $a(\mathbf{n}) = 1_{\mathrm{U}(m)}$ for every $\mathbf{n} \in \Xi$, as claimed.

The triviality of a on Ξ in (3.4) implies that the map $x \mapsto b(\theta(x)) b'(x, \overline{\mathbf{n}})$ is invariant under each $\alpha_{\mathbf{n}}$, $\mathbf{n} \in \Xi$, and hence constant, for every $\mathbf{n} \in \mathbb{Z}^d$. In particular, $b'(\cdot, \overline{\mathbf{0}})$ must be invariant under translation by Y, which implies that each of the characters χ_i is constant on Y. This contradiction to our choice of $\chi = \chi_1$ shows that c cannot be trivial. □

Proposition 3.3. *Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let*

$$\mathrm{Fix}(\alpha_{\mathbf{n}}) = \{x \in X : \alpha_{\mathbf{n}}(x) = x\}$$

for every $\mathbf{n} \in \mathbb{Z}^d$. If the restriction $\alpha^{\mathrm{Fix}(\alpha_{\mathbf{n}})}$ of α to the closed, α-invariant subgroup $\mathrm{Fix}(\alpha_{\mathbf{n}})$ is nonergodic for some $\mathbf{n} \in \mathbb{Z}^d$, then there exist a compact Lie group G with a nontrivial cocycle $c \in Z^1_H(\alpha, G)$.

Proof. Let $\mathbf{n} \in \mathbb{Z}^d$ be an element such that $Y = \mathrm{Fix}(\alpha_{\mathbf{n}})$ is nonergodic. We set $Z = X/Y$ and use Lemma 3.2 to construct a nontrivial cocycle $c \in Z^1_H(\alpha^Z, U(m))$ for some $m \geq 1$, where α^Z is the $\mathbb{Z}^d$-action on Z induced by α. As α is mixing, $\alpha_{\mathbf{n}}$ is ergodic, and the map $x \mapsto \alpha_{\mathbf{n}}(x) - x = \eta(x)$

from X to X is surjective and induces a continuous group isomorphism $\phi\colon Z = X/\ker(\eta) \longmapsto X$. The cocycle $c'\colon \mathbb{Z}^d \times X \longmapsto U(m)$, defined by

$$c'(\mathbf{m}, x) = c(\mathbf{m}, \phi(x))$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $x \in X$, has the desired properties. □

Lemma 3.4. *Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X such that $h(\alpha_{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$. If $\mathrm{Fix}(\alpha_{\mathbf{n}}) \neq \{0\}$ for some nonzero $\mathbf{n} \in \mathbb{Z}^d$, then the restriction of α to $\mathrm{Fix}(\alpha_{\mathbf{n}})$ is nonergodic.*

Proof. Suppose that, for some nonzero $\mathbf{n} \in \mathbb{Z}^d$, $Y = \mathrm{Fix}(\alpha_{\mathbf{n}}) \neq \{0\}$ and α is ergodic on Y. By Theorem 6.5 in [11] there exists an element $\mathbf{m} \in \mathbb{Z}^d$ such that the restriction $\alpha_{\mathbf{m}}^Y$ of $\alpha_{\mathbf{m}}$ to Y is ergodic and $h(\alpha_{\mathbf{m}}^Y) > 0$ (cf. e.g. Theorem 19.2 in [11]).

We define $\eta\colon X \longmapsto X$ by $\eta(x) = \alpha_{\mathbf{n}}(x) - x$ and note that η is surjective, since $\alpha_{\mathbf{n}}$ is ergodic, and that Y is equal to the kernel $\ker(\eta)$. Yuzvinskii's addition formula ([14]) guarantees that, for every $k \geq 1$, the restriction of $\alpha_{\mathbf{m}}$ to $\ker(\eta^k)$ has entropy $kh(\alpha_{\mathbf{m}}^Y)$, so that $h(\alpha_{\mathbf{m}}) > kh(\alpha_{\mathbf{m}}^Y)$ for every $k \geq 1$. Since $h(\alpha_{\mathbf{m}}) < \infty$ by assumption we obtain a contradiction. □

Proof of Theorem 3.1. Let $\mathbf{n} \in \mathbb{Z}^d$ be a nonzero element. Theorem 12.1 in [11] guarantees that

$$\mathrm{Per}(\alpha_{\mathbf{n}}) = \bigcup_{k \geq 1} \mathrm{Fix}(\alpha_{k\mathbf{n}}) \tag{3.5}$$

is dense in X, and Lemma 3.4 shows that $\alpha^{\mathrm{Fix}(\alpha_{k\mathbf{n}})}$ is nonergodic for some $k \geq 1$. Now apply Proposition 3.3. □

Remarks 3.5. (1) The cocycle c in Lemma 3.2 and Theorem 3.1 takes values in an abelian group G if and only if χ is invariant under α. Such invariant characters exist whenever

$$\mathrm{Fix}(\alpha) = \{x \in X : \alpha_{\mathbf{m}}(x) = x \ \text{ for every } \ \mathbf{m} \in \mathbb{Z}^d\} = \mathrm{Fix}(\alpha_{\mathbf{n}}) \neq \{0\}$$

for some $\mathbf{n} \in \mathbb{Z}^d$.

(2) The last part of the proof of Lemma 3.2 also shows that the cocycle c constructed there is not cohomologous to a cocycle taking values in $U(m')$ for some $m' < m$.

A more careful version of Lemma 3.4 reveals that the groups $\mathrm{Fix}(\alpha_{\mathbf{n}})$ are, in fact, finite for every nonzero $\mathbf{n} \in \mathbb{Z}^d$. If we fix a nonzero $\mathbf{n} \in \mathbb{Z}^d$, then the denseness of the group $\mathrm{Per}(\alpha_{\mathbf{n}})$ in X (cf. (3.5)) shows that the cardinalities $|\mathrm{Fix}(\alpha_{k!\mathbf{n}})| \to \infty$ as $k \to \infty$. In particular we can find, for sufficiently large $k \geq 1$, an element $\chi \in \widehat{\mathrm{Fix}(\alpha_{k!\mathbf{n}})}$ with arbitrarily large orbit under $\alpha^{\mathrm{Fix}(\alpha_{k!\mathbf{n}})}$. Hence there exist arbitrarily large integers $m \geq 1$ for which we can find nontrivial cocycles $c \in Z_H^1(\alpha, U(m))$ which are not cohomologous to cocycles taking values in $U(m')$ for $m' < m$.

(3) Every cocycle $c \in Z_H^1(\alpha^Z, U(m))$ constructed in Lemma 3.2 (and in particular the cocycles $c \in Z_H^1(\alpha, U(m))$ obtained in Theorem 3.1) has the property that there exists an integer $k \geq 1$ such that $c(k\mathbf{m}, x)$ is a diagonal element of $U(m)$ for every $\mathbf{m} \in \mathbb{Z}^d$: it suffices to choose k such that $\alpha_{k\mathbf{m}}^Y$ fixes χ for every $\mathbf{m} \in \mathbb{Z}^d$. Furthermore, the skew product extension of α defined

by any of the cocycles appearing in Theorem 3.1 has the property that the restriction to any one of its ergodic components is essentially isomorphic the original action α (apart from a possible translation term arising from a homomorphism). Does every Hölder cocycle $c\colon \mathbb{Z}^d \times Z \longmapsto U(m)$ of α have this property?

(4) The condition in Lemma 3.4 that $\alpha^{\mathrm{Fix}(\alpha_{\mathbf{n}})}$ be nonergodic for some $\mathbf{n} \in \mathbb{Z}^d$ can be replaced by the condition that α acts nonergodically on the kernel of the surjective group homomorphism

$$h(\alpha) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_h(\mathbf{n})\alpha_{\mathbf{n}}\colon X \longmapsto X$$

for some $h \in \mathfrak{R}_d$, but I don't know whether this condition is really weaker than the one in Lemma 3.4. Can one construct nontrivial cocycles with values in compact Lie groups even if α is ergodic on the kernel of each such homomorphism?

(5) The nontrivial cocycles for Ledrappier's example described in [12] or [9] are special cases of the construction in Theorem 3.1.

4. Cohomological triviality

In this section we prove the following results.

Theorem 4.1. *Let α be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X with completely positive entropy. Then there exists an expansive and mixing $\mathbb{Z}^d$-action $\bar{\alpha}$ on a compact, abelian group $\bar{X}$ and a continuous, surjective group homomorphism $\psi\colon \bar{X} \longmapsto X$ with the following properties:*

(1) *$\psi \cdot \bar{\alpha}_{\mathbf{n}} = \alpha_{\mathbf{n}} \cdot \psi$ for every $\mathbf{n} \in \mathbb{Z}^d$;*
(2) *$h(\alpha) = h(\bar{\alpha})$;*
(3) *If G is a compact group with a bi-invariant metric γ, then every cocycle $c\colon \mathbb{Z}^d \times \bar{X} \longmapsto G$ for $\bar{\alpha}$ with summable variation is trivial.*

For certain expansive and mixing $\mathbb{Z}^d$-actions α this 'cover' $\bar{\alpha}$ coincides with α. If $\mathfrak{M}$ is an $\mathfrak{R}_d$-module we recall the definition of the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ on $X^{\mathfrak{M}} = \widehat{\mathfrak{M}}$ in (2.3)–(2.9).

Theorem 4.2. *Let $d \geq 2$, and let $f \in \mathfrak{R}_d$ be a Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/f\mathfrak{R}_d}$ on $X = X^{\mathfrak{R}_d/f\mathfrak{R}_d}$ is expansive and mixing. If G is a compact group with a given bi-invariant metric γ, then every cocycle $c \in Z^1_{sv}(\alpha, G)$ is trivial.*

For the constant polynomial $f = 2$, Theorem 4.2 yields the cohomological triviality of the higher-dimensional two-shift first observed by J. Kammeyer in [2].

Although this paper is mainly concerned with cocycles taking values in compact groups we remark in passing that an insignificant variation (in fact, simplification) of the proofs of the Theorems 4.1–4.2 yields the identical result for cocycles with values in any complete metric, abelian group (G, γ) (Remark 4.9).

For the proof of Theorem 4.2 we have to introduce a group of homoclinic points of α discussed in [4] and [12].

Definition 4.3. Let α be a mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let δ be a metric on X. For every nonzero element $\mathbf{n} \in \mathbb{Z}^d$ and every ξ with $0 < \xi < 1$, consider the cones

$$C^+(\mathbf{n}, \xi) = \{\mathbf{m} \in \mathbb{Z}^d : \langle \mathbf{m}, \mathbf{n} \rangle \geq \xi \|\mathbf{m}\| \|\mathbf{n}\|\},$$
$$C^-(\mathbf{n}, \xi) = \{\mathbf{m} \in \mathbb{Z}^d : \langle \mathbf{m}, \mathbf{n} \rangle \leq -\xi \|\mathbf{m}\| \|\mathbf{n}\|\},$$

and denote by

$$\Delta_\alpha(\mathbf{n}, \xi) = \left\{ x \in X : \lim_{\substack{\mathbf{k} \to \infty \\ \mathbf{k} \in C^+(\mathbf{n}, \xi') \cup C^-(\mathbf{n}, \xi')}} \alpha_{\mathbf{k}}(x) = 0_X \ \text{ for some } \xi' \in (0, \xi) \right\}$$

the group of $(\mathbf{n}, \xi)$-*homoclinic points*. Note that

$$\alpha_{\mathbf{m}}(\Delta_\alpha(\mathbf{n}, \xi)) = \Delta_\alpha(\mathbf{n}, \xi)$$

for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, $\mathbf{n} \neq \mathbf{0}$, and $\xi \in (0, 1)$.

Let $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$ and $\xi \in (0, 1)$. The $\mathbb{Z}^d$-action α has *weak* $(\mathbf{n}, \xi)$-*specification* if $\Delta_\alpha(\mathbf{n}, \xi)$ is dense in X, and if there exist, for every $\varepsilon > 0$, constants $s' \geq 1, t' \geq 0$, with the following property: for every $r \geq 0$, and for every $x \in \Delta_\alpha(\mathbf{n}, \xi)$ with $\delta(\alpha_{\mathbf{m}}(x), 0_X) < \varepsilon$ for every $\mathbf{m} \in \mathbf{B}(s'r + t')$, one can find a $y \in \Delta_\alpha(\mathbf{n}, \xi)$ with

$$\delta(\alpha_{\mathbf{k}}(y), \alpha_{\mathbf{k}}(x)) < \varepsilon \ \text{ for all } \ \mathbf{k} \in C^+(\mathbf{n}, \xi) + \mathbf{B}(r),$$
$$\delta(\alpha_{\mathbf{k}}(y), 0_X) < \varepsilon \ \text{ for all } \ \mathbf{k} \in C^-(\mathbf{n}, \xi) + \mathbf{B}(r).$$

The action α has *weak* $\mathbf{n}$-*specification* if it has weak $(\mathbf{n}, \xi)$-specification for some $\xi \in (0, 1)$.

An element $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ is *primitive* if $\gcd\{n_1, \dots, n_d\} = 1$; more generally, a subgroup $\Gamma \subset \mathbb{Z}^d$ is *primitive* if the group $\mathbb{Z}^d/\Gamma$ is torsion-free. If α is an expansive and mixing $\mathbb{Z}^d$-action of the form $\alpha = \alpha^{\mathfrak{R}_d/f\mathfrak{R}_d}$ for some $f \in \mathfrak{R}_d$ we denote by $\mathcal{S}(f) = \{\mathbf{m} \in \mathbb{Z}^d : c_{f_i}(\mathbf{m}) \neq 0\}$ the *support* of f. For every primitive subgroup $\Gamma \cong \mathbb{Z}^{d-1}$ of $\mathbb{Z}^d$ we choose a primitive element $\mathfrak{n}(\Gamma) \in \mathbb{Z}^d$ with $\langle \mathfrak{n}(\Gamma), \mathbf{m} \rangle = 0$ for every $\mathbf{m} \in \Gamma$ and write

$$\Phi_\Gamma(f)^+ = \bigl\{\mathbf{n} \in \mathcal{S}(f) : \langle \mathbf{n}, \mathfrak{n}(\Gamma) \rangle = \max_{\mathbf{k} \in \mathcal{S}(f)} \langle \mathbf{k}, \mathfrak{n}(\Gamma) \rangle \bigr\},$$
$$\Phi_\Gamma(f)^- = \bigl\{\mathbf{n} \in \mathcal{S}(f) : \langle \mathbf{n}, \mathfrak{n}(\Gamma) \rangle = \min_{\mathbf{k} \in \mathcal{S}(f)} \langle \mathbf{k}, \mathfrak{n}(\Gamma) \rangle \bigr\}$$

for the 'faces' of $\mathcal{S}(f)$ parallel to Γ. Put

$$\begin{aligned} \Psi(f) = \{\Gamma : \Gamma \cong \mathbb{Z}^{d-1} \ &\text{is a primitive subgroup of } \mathbb{Z}^d \\ &\text{and } |\Phi_\Gamma(f)^+| = |\Phi_\Gamma(f)^-| = 1\}, \end{aligned} \tag{4.1}$$

where $|S|$ is, as usual, the cardinality of a set S. Note that $\Psi(f)$ consists of all primitive $(d-1)$-dimensional subgroups of $\mathbb{Z}^d$ which do not contain any elements parallel to an edge of the convex hull of $\mathcal{S}(f)$.

More generally, if $\mathfrak{M}$ is a Noetherian $\mathfrak{R}_d$-module such that the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ has completely positive entropy, and if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ are the associated primes of $\mathfrak{M}$, then each $\mathfrak{p}_i$ is principal, and we choose nonzero Laurent polynomials $f_i \in \mathfrak{R}_d$ with $\mathfrak{p}_i = f_i\mathfrak{R}_d$ for every $i = 1, \dots, m$ (cf. Theorem

20.8 and Corollary 18.5 in [11]). Define $\mathcal{S}(f_i)$ and $\Psi(f_i)$ as above, and note that the set

$$\Psi(\mathfrak{M}) = \bigcap_{i=1}^{m} \Psi(f_i) \tag{4.2}$$

is infinite.

The following lemma is an elaboration of Remark 3.8 (2) in [4]. Recall that, if (X,δ) is a metric space, then a set $Y \subset X$ is *ε-dense* if $\inf_{y\in Y}\delta(x,y) < \varepsilon$ for every $x \in X$.

Lemma 4.4. *Let $f \in \mathfrak{R}_d$ be an irreducible Laurent polynomial such that $\alpha = \alpha^{\mathfrak{R}_d/f\mathfrak{R}_d}$ is expansive and mixing. The following is true for every nonzero $\mathbf{n} \in \bigcup_{\Gamma\in\Psi(f)}\Gamma$:*

(1) *α has weak $\mathbf{n}$-specification;*
(2) *For every $k \geq 1$, the restriction of α to the closed, invariant subgroup*
$$\mathrm{Fix}(\alpha_{k\mathbf{n}}) = \{x \in X = X^{\mathfrak{R}_d/f\mathfrak{R}_d} : \alpha_{k\mathbf{n}}(x) = x\}$$
is ergodic;
(3) *If δ is an invariant metric on X then there exists, for every $\varepsilon > 0$, an integer $K \geq 1$ such that $\mathrm{Fix}(\alpha_{k\mathbf{n}})$ is ε-dense in X for every $k \geq K$.*

Proof. If the support $\mathcal{S}(f)$ consists of a single point then we can multiply f by a monomial to achieve that $f = p$ for some rational prime $p \geq 2$. In this case $X \cong (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$, α is the d-dimensional p-shift, and all assertions of the lemma are obvious.

Assume therefore that $|\mathcal{S}(f)| > 1$. Let $\Gamma \in \Psi(f)$ be a primitive subgroup, and let $\mathbf{k} \in \mathbb{Z}^d$ be an element with $\{l\mathbf{k} + \mathbf{n} : k \in \mathbb{Z},\, \mathbf{n} \in \Gamma\} = \mathbb{Z}$. According to the definition of $\Psi(f)$ we may multiply f by a monomial and assume that there exists an integer $L \geq 0$ and Laurent polynomials $\phi_f^{(j)}$ such that $\mathcal{S}(\phi_f^{(j)}) \subset \Gamma$ for $j = 0,\dots,L$, $\phi_f^{(0)} = mu^{\mathbf{m}}$, $\phi_f^{(L)} = m'u^{\mathbf{m}'}$ with $m,m' \in \mathbb{Z}$, $mm' \neq 0$, $\mathbf{m},\mathbf{m}' \in \Gamma$, and

$$f = \sum_{j=0}^{L} u^{j\mathbf{k}}\phi_f^{(j)}.$$

If

$$\bar{\Gamma} = \{l\mathbf{k} + \mathbf{n} : 0 \leq l \leq L-1,\, \mathbf{n} \in \Gamma\},$$

then the definition of $X = X^{\mathfrak{R}_d/f\mathfrak{R}_d}$ in (2.9) shows that the projection $\pi_{\bar{\Gamma}} \colon X \longmapsto \mathbb{T}^{\bar{\Gamma}}$, which restricts every $x = (x_{\mathbf{m}}) \in X \subset \mathbb{T}^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a surjective group homomorphism with zero-dimensional kernel. In fact, if $\Gamma' = \Gamma - \mathbf{k}$, $\Gamma'' = \Gamma + L\mathbf{k}$, then

$$\pi_{\Gamma'}(\pi_{\bar{\Gamma}}^{-1}(y)) \cong (\mathbb{Z}/m\mathbb{Z})^{\mathbb{Z}^{d-1}}, \qquad \pi_{\Gamma''}(\pi_{\bar{\Gamma}}^{-1}(y)) \cong (\mathbb{Z}/m'\mathbb{Z})^{\mathbb{Z}^{d-1}}. \tag{4.3}$$

We write β for the Γ-action

$$(\beta_{\mathbf{n}}(y))_{\mathbf{m}} = y_{\mathbf{m}+\mathbf{n}},\ \mathbf{n} \in \Gamma,\ \mathbf{m} \in \bar{\Gamma},$$

on $Y = \mathbb{T}^{\bar{\Gamma}}$. Fix a primitive element $\mathbf{n} \in \Gamma$, and let $k \geq 1$. Then

$$\pi_{\bar{\Gamma}}(\mathrm{Fix}(\alpha_{k\mathbf{n}})) = \mathrm{Fix}(\beta_{k\mathbf{n}}),$$

and $\mathrm{Fix}(\alpha_{k\mathbf{n}})$ is easily seen to be connected. The expansiveness of the restriction $\alpha^{\mathrm{Fix}(\alpha_{k\mathbf{n}})}$ of α to $\mathrm{Fix}(\alpha_{k\mathbf{n}})$ implies that $\alpha^{\mathrm{Fix}(\alpha_{k\mathbf{n}})}$ is ergodic and proves (2) (cf. Corollary 6.14 in [11]). The assertions (1) and (3) are obvious from (4.3) and the surjectivity of $\pi_{\bar{\Gamma}}$. □

Lemma 4.5. *Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module such that $\alpha = \alpha^{\mathfrak{M}}$ is expansive, mixing, and has completely positive entropy. We write $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ for the set of associated primes of $\mathfrak{M}$, note that each $\mathfrak{p}_i$ is principal* (Theorem 20.8 *and* Corollary 18.5 *in* [11]), *and choose $f_i \in \mathfrak{R}_d$ with $\mathfrak{p}_i = f_i\mathfrak{R}_d$ for $i = 1, \dots, m$. There exist a Noetherian $\mathfrak{R}_d$-module $\mathfrak{N}$ and an injective module-homomorphism $\hat{\phi}\colon \mathfrak{M} \longmapsto \mathfrak{N}$ with the following properties:*

1. *$\mathfrak{N} = \mathfrak{N}^{(1)} \oplus \dots \oplus \mathfrak{N}^{(m)}$, where each $\mathfrak{N}^{(j)}$ has a prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \dots \subset \mathfrak{N}_{r_j}^{(j)} = \mathfrak{N}^{(j)}$ with $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{p}_j$ for $k = 1, \dots, r_j$;*
2. *$\mathfrak{N}$ has the same set of associated primes as $\mathfrak{M}$;*
3. *$h(\alpha^{\mathfrak{N}}) = h(\alpha)$;*
4. *$\alpha^{\mathfrak{N}}$ is expansive and mixing.*

Furthermore, if

$$\Psi(\mathfrak{M}) = \bigcap_{i=1}^{m} \Psi(f_i)$$

and $\mathbf{0} \neq \mathbf{n} \in \bigcup_{\Gamma \in \Psi(\mathfrak{M})} \Gamma$, then the following is true:

5. *$\alpha^{\mathfrak{N}}$ has weak $\mathbf{n}$-specification,*
6. *For every $k \geq 1$, the restriction of $\alpha^{\mathfrak{N}}$ to the closed, invariant subgroup*
$$\mathrm{Fix}(\alpha_{k\mathbf{n}}^{\mathfrak{N}}) = \{x \in X^{\mathfrak{N}} : \alpha_{k\mathbf{n}}^{\mathfrak{N}}(x) = x\}$$
is ergodic;
7. *If δ is a metric on $X^{\mathfrak{N}}$ then there exists, or every $\varepsilon > 0$, an integer $K \geq 1$ such that $\mathrm{Fix}(\alpha_{k\mathbf{n}}^{\mathfrak{N}})$ is ε-dense in $X^{\mathfrak{N}}$ for every $k \geq K$.*

Proof. Lemma 3.3 in [4] implies the existence of a Noetherian module $\mathfrak{N}$ and a module homomorphism $\hat{\phi}\colon \mathfrak{M} \longmapsto \mathfrak{N}$ satisfying (1)–(2). Condition (3) follows from Corollary 18.5 and Proposition 18.6 in [11] and the definition of $\mathfrak{N}$ in [4], and (4) is a consequence of (2).

The Corollaries 3.5 and 3.7 in [4] imply (5), (6) is proved by applying Lemma 4.4 repeatedly, and (7) is obvious from the structure of $\mathfrak{N}$ (or from Theorem 12.1 in [11]). □

For the following discussion we assume the hypotheses of Lemma 4.5, choose $\mathbf{n} \in \Gamma \in \bigcup_{\Gamma \in \Psi(\mathfrak{M})} \Gamma$ and $\xi \in (0,1)$ such that $\bar{\alpha} = \alpha^{\mathfrak{N}}$ has weak $(\mathbf{n}, \xi)$-specification, and put $\bar{X} = X^{\mathfrak{N}} = \widehat{\mathfrak{N}}$. If $\mathfrak{M} = \mathfrak{R}_d/f\mathfrak{R}_d$ for some $f \in \mathfrak{R}_d$ then Lemma 4.4 allows us to set $\mathfrak{N} = \mathfrak{M} = \mathfrak{R}_d/f\mathfrak{R}_d$, $\bar{\alpha} = \alpha$ and $\bar{X} = X$.

Suppose that (G, γ) is a complete metric group with identity element 1_G and a bi-invariant metric γ in which G has diameter 1. If $h\colon \bar{X} \longmapsto G$ is a function with summable variation we define cocycles $a_h^{\pm}\colon \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) \times \bar{X} \longmapsto G$ for the action of $\Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ on $\bar{X}$ by translation as follows: for every $y \in$

$\Delta_{\bar{\alpha}}(\mathbf{n},\xi)$ and $x \in \bar{X}$,

$$\begin{aligned} a_h^+(y,x) &= \left(\prod_{k=0}^{\infty} h(\bar{\alpha}_{k\mathbf{n}}(y+x))^{-1}\right) \cdot \left(\prod_{k=0}^{\infty} h(\bar{\alpha}_{k\mathbf{n}}(x))^{-1}\right)^{-1}, \\ a_h^-(y,x) &= \left(\prod_{k=1}^{\infty} h(\bar{\alpha}_{-k\mathbf{n}}(y+x))\right) \cdot \left(\prod_{k=1}^{\infty} h(\bar{\alpha}_{-k\mathbf{n}}(x))\right)^{-1}. \end{aligned} \tag{4.4}$$

In order to explain the equations (4.4) we fix $y \in \Delta_{\bar{\alpha}}(\mathbf{n},\xi)$ for the moment. Since h has summable variation there exists, for every $\varepsilon > 0$, an integer $N \geq 0$ with

$$\sum_{|k|\geq N} \gamma(h(\bar{\alpha}_{k\mathbf{n}}(y+x)), h(\bar{\alpha}_{k\mathbf{n}}(x))) < \varepsilon$$

for every $x \in \bar{X}$. For every $K \geq 1$ we set

$$\begin{aligned} a_h^+(y,x)^{(K)} &= \left(\prod_{k=0}^{K-1} h(\bar{\alpha}_{k\mathbf{n}}(y+x))^{-1}\right) \cdot \left(\prod_{k=0}^{K-1} h(\bar{\alpha}_{k\mathbf{n}}(x))^{-1}\right)^{-1} \\ &= h(y+x)^{-1} \cdots h(\bar{\alpha}_{(K-1)\mathbf{n}}(y+x))^{-1} \\ &\qquad \cdot h(\bar{\alpha}_{(K-1)\mathbf{n}}(x)) \cdots h(x), \\ a_h^-(y,x)^{(K)} &= \left(\prod_{k=1}^{K} h(\bar{\alpha}_{-k\mathbf{n}}(y+x))\right) \cdot \left(\prod_{k=1}^{K} h(\bar{\alpha}_{-k\mathbf{n}}(x))\right)^{-1} \\ &= h(\bar{\alpha}_{-\mathbf{n}}(y+x)) \cdots h(\bar{\alpha}_{-K\mathbf{n}}(y+x)) \\ &\qquad \cdot h(\bar{\alpha}_{-K\mathbf{n}}(x))^{-1} \cdots h(\bar{\alpha}_{-\mathbf{n}}(x))^{-1}. \end{aligned} \tag{4.5}$$

The bi-invariance of the metric γ guarantees that

$$\gamma(a_h^+(y,x)^{(K)}, a_h^+(y,x)^{(K')}) + \gamma(a_h^-(y,x)^{(K)}, a_h^-(y,x)^{(K')}) < \varepsilon$$

whenever $K, K' > N$ and $x \in \bar{X}$, so that the infinite products in (4.4) converge uniformly to continuous maps $a_h^{\pm}(y,\cdot)\colon \bar{X} \longmapsto G$. From the (4.4) it is also clear that the maps $a_h^{\pm}\colon \Delta_{\bar{\alpha}}(\mathbf{n},\xi) \longmapsto G$ satisfy the cocycle equations

$$\begin{aligned} a_h^+(y_1, y_2+x)a_h^+(y_2,x) &= a_h^+(y_1+y_2,x), \\ a_h^-(y_1, y_2+x)a_h^-(y_2,x) &= a_h^-(y_1+y_2,x) \end{aligned} \tag{4.6}$$

for all $y_1, y_2 \in \Delta_{\bar{\alpha}}(\mathbf{n},\xi)$ and $x \in \bar{X}$. The following result is a minor modification of Proposition 3.1 in [12]; we include its proof for completeness.

Lemma 4.6. *Let $\bar{\alpha}$ be an expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group $\bar{X}$ which has weak $(\mathbf{n},\xi)$-specification for some nonzero $\mathbf{n} \in \mathbb{Z}^d$ and some $\xi \in (0,1)$, and let $h\colon \bar{X} \longmapsto G$ be a function with summable variation. If the cocycles $a_h^{\pm}\colon \Delta_{\bar{\alpha}}(\mathbf{n},\xi) \times \bar{X} \longmapsto G$ in (4.4)–(4.6) are equal, then there exists a continuous function $b\colon \bar{X} \longmapsto G$ such that the map*

$$x \mapsto b(\alpha_{\mathbf{n}}(x))^{-1} \cdot h(x) \cdot b(x)$$

is constant on $\bar{X}$. If h is Hölder, then b is also Hölder.

Proof. We fix an $\varepsilon > 0$ satisfying (1.4) and put, for every $r > 0$,

$$\Delta^+(r) = \{y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) : \delta(\bar{\alpha}_{\mathbf{k}}(y), 0_X) < \varepsilon \text{ for every } \mathbf{k} \in C^-(\mathbf{n}, \xi) + \mathbf{B}(r)\},$$
$$\Delta^-(r) = \{y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) : \delta(\bar{\alpha}_{\mathbf{k}}(y), 0_X) < \varepsilon \text{ for every } \mathbf{k} \in C^+(\mathbf{n}, \xi) + \mathbf{B}(r)\},$$

and note that $\gamma(a_h^{\pm}(y,x), 1_G) \leq 4\xi^{-1} \cdot \sum_{k \geq r} \omega_k^{\delta,\gamma}(h) < \infty$ for all $y \in \Delta^{\mp}(r)$ and $x \in \bar{X}$.

The weak $(\mathbf{n}, \xi)$-specification of $\bar{\alpha}$ allows us to choose constants $s' \geq 1, t' \geq 0$ such that we can find, for every $r \geq 0$ and every $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ with $\delta(\bar{\alpha}_{\mathbf{m}}(x), 0_{\bar{X}}) < \varepsilon$ for every $\mathbf{m} \in \mathbf{B}(s'r + t')$, an element $y' \in \Delta^+(r)$ with $\delta(\bar{\alpha}_{\mathbf{m}}(y), \bar{\alpha}_{\mathbf{m}}(y')) < \varepsilon$ for every $\mathbf{m} \in C^+(\mathbf{n}, \xi) + \mathbf{B}(r)$. Since the metric γ is bi-invariant and $a_h^+ = a_h^-$ we obtain that

$$\begin{gathered}
\gamma(a_h^+(y', x), a_h^+(y, x)) \leq 4\xi^{-1} \sum_{k \geq r} \omega_k^{\delta,\gamma}(h) = C'(r), \text{ say,} \\
\gamma(a_h^+(y', x), 1_G) = \gamma(a_h^-(y', x), 1_G) \leq C'(r), \\
\gamma(a_h^+(y, x), 1_G) \leq 2C'(r),
\end{gathered} \tag{4.7}$$

for all $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ with $\delta(\bar{\alpha}_{\mathbf{m}}(y), 0_{\bar{X}}) < \varepsilon$ for every $\mathbf{m} \in \mathbf{B}(s'r + t')$. By varying r we see that

$$\lim_{\substack{y \to 0_{\bar{X}} \\ y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)}} \gamma(a_h^+(y, x), 1_G) = 0$$

uniformly in $x \in \bar{X}$. The cocycle equation (4.6) shows that the function $a_h^+ \colon \Delta_{\bar{\alpha}}(\mathbf{n}, \xi) \times \bar{X} \longmapsto G$ has a unique, continuous extension $\bar{a}_h^+ \colon \bar{X} \times \bar{X} \longmapsto G$ with $\bar{a}_h^+(y, x) = a_h^+(y, x)$ for all $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ and $x \in \bar{X}$, and that $\bar{a}_h^+$ satisfies the cocycle equation (4.6) for every $y_1, y_2, x \in \bar{X}$. We define $b \colon \bar{X} \longmapsto G$ by $b(x) = \bar{a}_h^+(x, 0_{\bar{X}})$ for every $x \in \bar{X}$, and obtain that $\bar{a}_h^+(x, x') = b(x + x') \cdot b(x')^{-1}$ for all $x, x' \in \bar{X}$.

If h is Hölder we choose $0 < \omega < 1$, $\omega' > 0$, such that (1.5) is satisfied. From (4.7) we see that there exists a positive constant $\bar{\omega}'$ such that $\gamma(a_h^+(y, x), 1_G) \leq \bar{\omega}' \omega^r$ for all $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ with $\delta(\bar{\alpha}_{\mathbf{m}}(y), 0_{\bar{X}}) < \varepsilon$ for every $\mathbf{m} \in B(s'r + t')$, and we conclude that $\gamma(\bar{a}_h^+(x, x'), 1_G) \leq \bar{\omega}' \omega^r$ for all $(x, x') \in \bar{X} \times \bar{X}$ with $\delta(\bar{\alpha}_{\mathbf{m}}(x), 0_{\bar{X}}) < \varepsilon$ for every $\mathbf{m} \in B(s'r + t')$. Hence $\gamma(b(x), b(x')) \leq 2\bar{\omega}' \omega^r$ whenever $\delta(\bar{\alpha}_{\mathbf{m}}(x), \bar{\alpha}_{\mathbf{m}}(x')) < \varepsilon$ for every $\mathbf{m} \in B(s'r + t')$, which implies that b is Hölder.

Let $\bar{\Delta} = \mathbb{Z} \times \bar{X}$ with the group operation $(n, y) \cdot (n', y') = (n + n', \bar{\alpha}_{n'\mathbf{n}}(y) + y')$, and let $\bar{\Delta}' \subset \bar{\Delta}$ be the subgroup consisting of all (n, y) with $n \in \mathbb{Z}$ and $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$. We write T for the action of $\bar{\Delta}$ on $\bar{X}$ given by $T_{(n,y)}(x) = \bar{\alpha}_{n\mathbf{n}}(x + y)$ and define a continuous map $\psi \colon \bar{\Delta} \times \bar{X} \longmapsto \mathbb{R}$ by setting, for every $(n, y) \in \bar{\Delta}$ and $x \in \bar{X}$,

$$\psi((n, y), x) = \begin{cases} h(\bar{\alpha}_{(n-1)\mathbf{n}}(x + y)) \cdots h((x + y))\bar{a}_h^+(y, x) & \text{if } n > 0, \\ \bar{a}_h^+(y, x) & \text{if } n = 0, \\ h(\bar{\alpha}_{-n\mathbf{n}}(x + y))^{-1} \cdots h(\bar{\alpha}_{-\mathbf{n}}(x + y))^{-1} \bar{a}_h^+(y, x) & \text{if } n < 0. \end{cases}$$

A straightforward calculation shows that ψ is a cocycle for the action T of $\bar{\Delta}$ on $\bar{X}$ in the sense of (1.1). In particular,

$$b(y + x)b(x)^{-1} = \bar{a}_h^+(y, x) = \psi((0, y), x)$$

$$
\begin{aligned}
&= \psi((-1,0), \bar{\alpha}_{\mathbf{n}}(x+y))\psi((0, \alpha_{\mathbf{n}}(y)), \alpha_{\mathbf{n}}(x))\psi((1,0), x) \\
&= h(x+y)^{-1}\bar{a}_h^{+}(\bar{\alpha}_{\mathbf{n}}(y), \bar{\alpha}_{\mathbf{n}}(x))h(x) \\
&= h(x+y)^{-1}b(\bar{\alpha}_{\mathbf{n}}(x+y))b(\bar{\alpha}_{\mathbf{n}}(x))^{-1}h(x)
\end{aligned}
$$

for all $x, y \in \bar{X}$, so that $(b \cdot \alpha_{\mathbf{n}})^{-1}hb$ is translation invariant on $\bar{X}$ and hence constant. □

We return to the setting of Lemma 4.5, assume that $c\colon \mathbb{Z}^d \times \bar{X} \longmapsto G$ is an element of $Z^1_{sv}(\bar{\alpha}, G)$, where G is a compact group with a bi-invariant metric γ, and set

$$h = c(\mathbf{n}, \cdot)\colon \bar{X} \longmapsto G. \tag{4.8}$$

According to (1.1),

$$
\begin{aligned}
h(\bar{\alpha}_{\mathbf{m}}(x))c(\mathbf{m}, x) &= c(\mathbf{n}, \bar{\alpha}_{\mathbf{m}}(x))c(\mathbf{m}, x) = c(\mathbf{m}+\mathbf{n}, x) \\
&= c(\mathbf{m}, \bar{\alpha}_{\mathbf{n}}(x))c(\mathbf{n}, x) = c(\mathbf{m}, \bar{\alpha}_{\mathbf{n}}(x))h(x)
\end{aligned}
\tag{4.9}
$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $x \in \bar{X}$.

Lemma 4.7. *For every $x \in \bar{X}$ and $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$, $a_h^{+}(y, x) = a_h^{-}(y, x)$.*

Proof. For every $k \geq 1$ and every $x \in \mathrm{Fix}(\alpha_{k\mathbf{n}})$ we denote by

$$w_h^{(k)}(x) = h(\bar{\alpha}_{(k-1)\mathbf{n}}(x))h(\bar{\alpha}_{(k-2)\mathbf{n}}(x)) \cdots h(\bar{\alpha}_{\mathbf{n}}(x))h(x)$$

the *h-weight* of x. The equation (4.9) implies that

$$w_h^{(k)}(\bar{\alpha}_{\mathbf{m}}(x)) = c(\mathbf{m}, x)w_h^{(k)}(x)c(\mathbf{m}, x)^{-1}$$

for every $x \in \mathrm{Fix}(\bar{\alpha}_{k\mathbf{n}})$ and $\mathbf{m} \in \mathbb{Z}^d$. Denote by $[g]$ the conjugacy class of an element $g \in G$, write $[G] = \{[g] : g \in G\}$ for the space of conjugacy classes in G with the quotient topology, and note that the ergodicity of $\bar{\alpha}^{\mathrm{Fix}(\bar{\alpha}_{k\mathbf{n}})}$ implies that the continuous map $x \mapsto [w_h^{(k)}(x)]$ from $\mathrm{Fix}(\bar{\alpha}_{k\mathbf{n}})$ to $[G]$ is constant. In particular, $w_h^{(k)}(x)$ is conjugate to $w_h^{(k)}(0_X) = h(0_X)^k$ for every $x \in \mathrm{Fix}(\alpha_{k\mathbf{n}})$.

As G is compact, we can find an increasing sequence $(k_j,\ j \geq 1)$ of positive integers with

$$\lim_{j\to\infty} h(0_X)^{k_j} = 1_G,$$

and there exists, for every $\varepsilon > 0$, an integer $J \geq 1$ with

$$\gamma(w_h^{(k_j)}(x), 1_G) < \varepsilon/3 \tag{4.10}$$

for every $j \geq J$ and $x \in \mathrm{Fix}(\bar{\alpha}_{k_j\mathbf{n}})$.

The discussion preceding Proposition 2.1 allows us to embed $\bar{X}$ as a closed, shift-invariant subgroup of $(\mathbb{T}^s)^{\mathbb{Z}^d}$ for some $s \geq 1$. In order to prove that $a_h^{+}(y, x) = a_h^{-}(y, x)$ for every given $x \in \bar{X}$ and $y \in \Delta_{\bar{\alpha}}(\mathbf{n}, \xi)$ we fix $r > 0$ for the moment and apply Lemma 4.5 (7) to find, for every sufficiently large $k \geq 1$, an element $x(k) \in \mathrm{Fix}(\bar{\alpha}_{2k\mathbf{n}})$ with $|x_{\mathbf{m}} - x(k)_{\mathbf{m}}| < \varepsilon/3$ for every $\mathbf{m} \in \mathbf{B}(r)$ (cf. (2.11)). Proposition 2.1 guarantees that, for every $\mathbf{k} \in \mathbb{Z}^d$, $|y_{\mathbf{k}+l\mathbf{n}}| \to 0$ with exponential speed as $|l| \to \infty$ (cf. (2.11) for notation). In particular, the series $\sum_{l\in\mathbb{Z}} \bar{\alpha}_{2lk\mathbf{n}}(y)$ converges to a point $y(k) \in \mathrm{Fix}(\bar{\alpha}_{2k\mathbf{n}})$, and by choosing r and $k > r$ sufficiently large and ε sufficiently small the following expressions become arbitrarily small (cf. (1.3)–(1.5) and (4.5)):

$$\begin{gathered}
\sum_{\mathbf{m}\in\mathbb{Z}^d\smallsetminus\mathbf{B}(r)} |y_{\mathbf{m}}|,\\
\sum_{|j|\le k} \gamma(h(\bar\alpha_{j\mathbf{n}}(y)), h(\bar\alpha_{j\mathbf{n}}(y(k)))),\\
\gamma(a_h^+(y,x), a_h^+(y,x(k))) + \gamma(a_h^-(y,x), a_h^-(y,x(k))),\\
\gamma(a_h^+(y,x(k)), a_h^+(y,x(k))^{(k)}) + \gamma(a_h^-(y,x(k)), a_h^-(y,x(k))^{(k)}),\\
\gamma(a_h^+(y,x(k))^{(k)}, a_h^+(y(k),x(k))^{(k)})\\
\qquad\qquad +\gamma(a_h^-(y,x(k))^{(k)}, a_h^-(y(k),x(k))^{(k)}).
\end{gathered}\tag{4.11}$$

The expressions $a_h^{\pm}(y(k),x(k))^{(k)}$ in (4.11) are defined as in (4.5) by

$$\begin{aligned}
a_h^+(y(k),x(k))^{(k)} &= h(y(k)+x(k))^{-1}\cdots h(\bar\alpha_{(k-1)\mathbf{n}}(y(k)+x(k)))^{-1}\\
&\qquad\cdot h(\bar\alpha_{(k-1)\mathbf{n}}(x(k)))\cdots h(x(k)),\\
a_h^-(y(k),x(k))^{(k)} &= h(\bar\alpha_{-\mathbf{n}}(y(k)+x(k)))\cdots h(\bar\alpha_{-k\mathbf{n}}(y(k)+x(k)))\\
&\qquad\cdot h(\bar\alpha_{-k\mathbf{n}}(x(k)))^{-1}\cdots h(\bar\alpha_{-\mathbf{n}}(x(k)))^{-1}.
\end{aligned}\tag{4.12}$$

From (4.10) we know that

$$\lim_{j\to\infty} \gamma(w_h^{(2k_j)}(\alpha_{-k_j\mathbf{n}}(x(k_j))), 1_G) + \gamma(w_h^{(2k_j)}(\alpha_{-k_j\mathbf{n}}(y(k_j)+x(k))), 1_G) = 0.$$

Hence the distance between the products

$$\begin{gathered}
h(\bar\alpha_{(k_j-1)\mathbf{n}}(y(k_j)+x(k_j)))\cdots h(\alpha_{-k_j\mathbf{n}}(y(k_j)+x(k_j))),\\
h(\bar\alpha_{(k_j-1)\mathbf{n}}(x(k_j)))\cdots h(\bar\alpha_{-k_j\mathbf{n}}(x(k_j)))
\end{gathered}\tag{4.13}$$

tends to 0 as $j\to\infty$. After re-ordering the products in (4.13) we see that

$$\lim_{j\to\infty} \gamma(a_h^+(y(k_j),x(k_j))^{(k_j)}, a_h^-(y(k_j),x(k_j))^{(k_j)}) = 0, \tag{4.14}$$

and by combining (4.11)–(4.13) we obtain that $a_h^+(y,x) = a^-(y,x)$ for every $x\in\bar X$ and $y\in\Delta_{\bar\alpha}(\mathbf{n},\xi)$. □

Proof of Theorem 4.1. Let $\mathfrak{M}=\hat X$ be the Noetherian $\mathfrak{R}_d$-module arising in (2.1)–(2.2), define $\mathfrak{N}$, $\bar\alpha=\alpha^{\mathfrak{N}}$ and $\bar X = X^{\mathfrak{N}}$ as in Lemma 4.5, and choose a nonzero $\mathbf{n}\in\mathbb{Z}^d$ and $\xi\in(0,1)$ such that $\bar\alpha$ has weak $(\mathbf{n},\xi)$-specification. Let G be a compact group with bi-invariant metric γ, $c\colon \mathbb{Z}^d\times X\longmapsto G$ a cocycle for $\bar\alpha$ with summable variation, and define $h\colon \bar X\longmapsto G$ and $a_h^{\pm}\colon \Delta_{\bar\alpha}(\mathbf{n},\xi)\times \bar X\longmapsto G$ by (4.8) and (4.4)–(4.6). Lemma 4.7 shows that $a_h^+ = a_h^-$, and Lemma 4.6 yields the existence of a continuous function $b\colon \bar X\longmapsto G$ for which the map

$$x\mapsto b(\bar\alpha_{\mathbf{n}}(x))^{-1}\cdot c(\mathbf{n},x)\cdot b(x)$$

is everywhere equal to some element $g\in G$. Furthermore, if c is Hölder, then b is Hölder.

We define a cocycle $c'\colon \mathbb{Z}^d \times \bar{X} \longmapsto G$ for $\bar{\alpha}$ by setting $c'(\mathbf{m},x) = b(\bar{\alpha}_{\mathbf{m}}(x))^{-1}c(\mathbf{m},x)b(x)$ for every $\mathbf{m} \in \mathbb{Z}^d$ and $x \in \bar{X}$. Then c' is cohomologous to c, and the cocycle equation (1.1) shows that

$$c'(\mathbf{m},x) = g^{-k} \cdot c'(\mathbf{m},\alpha_{k\mathbf{n}}(x)) \cdot g^k \tag{4.15}$$

for every $\mathbf{m} \in \mathbb{Z}^d, k \in \mathbb{Z}$, and $x \in X$. We fix $\mathbf{m} \in \mathbb{Z}^d$ and set $\psi = c'(\mathbf{m},\cdot)$. If ψ is not constant we can find $\varepsilon > 0$, $g_1, g_2 \in G$, and nonempty, open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, such that $\sup_{x\in\mathcal{O}_i} \gamma(\psi(x),g_i) < \varepsilon$ for $i = 1,2$, and $\gamma(g_1,g_2) > 3\varepsilon$. From (4.15) and the invariance of γ we see that $\gamma(\psi(x_1),\psi(x_2)) < \varepsilon$ whenever $k \geq 0$ and $x_1,x_2 \in \alpha_{-k\mathbf{n}}(\mathcal{O}_1)$. However, since $\alpha_{\mathbf{n}}$ is topologically mixing, we know that $\alpha_{-k\mathbf{n}}(\mathcal{O}_1)\cap\mathcal{O}_i \neq \varnothing$ for $i = 1,2$, and for all sufficiently large k. If $k > 0$ is large enough we can thus find elements $y_1,y_2 \in \alpha_{-k\mathbf{n}}(\mathcal{O}_1)$ such that $\gamma(\psi(y_i),g_i) < \varepsilon$ for $i = 1,2$, which is absurd. This contradiction implies that $c'(\mathbf{m},\cdot)$ must be constant. As $\mathbf{m} \in \mathbb{Z}^d$ was arbitrary, c' is a homomorphism. $\square$

Proof of Theorem 4.2. If $\hat{X} = \mathfrak{M} = \mathfrak{R}_d/f\mathfrak{R}_d$ for some $f \in \mathfrak{R}_d$, Lemma 4.4 allows us to set $\mathfrak{N} = \mathfrak{M}$ in the Lemmas 4.5 and 4.7. Hence $\bar{\alpha} = \alpha$ and $\bar{X} = X$ in Theorem 4.1, which proves our claim. $\square$

The following example shows that even a minor modification of the actions in Theorem 4.2 may introduce nontrivial cohomology with values in compact Lie groups, and that the $\mathbb{Z}^d$-action $\bar{\alpha}$ in Theorem 4.1 cannot, in general, be equal to α.

Example 4.8. As in Theorem 4.2 we assume that $\alpha = \alpha^{\mathfrak{R}_d/f\mathfrak{R}_d}$ and $X = X^{\mathfrak{R}_d/f\mathfrak{R}_d} = \widehat{\mathfrak{R}_d/f\mathfrak{R}_d}$ for some $f \in \mathfrak{R}_d$ such that α is expansive and mixing. For every subgroup $\Gamma \subset \mathbb{Z}^d$ with finite index, the set

$$\mathrm{Fix}_\alpha(\Gamma) = \{x \in X : \alpha_{\mathbf{n}}(x) = x \text{ for every } \mathbf{n} \in \Gamma\}$$

is finite (by expansiveness), and

$$\lim_{\langle\Gamma\rangle\to\infty} |\mathbb{Z}^d/\Gamma|^{-1} \log|\mathrm{Fix}_\alpha(\Gamma)| = h(\alpha) > 0,$$

where

$$\langle\Gamma\rangle = \min\{\|\mathbf{m}\| : \mathbf{0} \neq \mathbf{m} \in \Gamma\}$$

(cf. Theorems 21.1 (3) and 19.5 in [11]). In particular there exist subgroups $\Gamma \subset \mathbb{Z}^d$ for which $Y = \mathrm{Fix}_\alpha(\Gamma)$ is arbitrarily large, and obviously finite and α-invariant. Lemma 3.2 allows us to construct a nontrivial Hölder cocycle $c\colon \mathbb{Z}^d \times X/Y \longmapsto G$ for some compact Lie group G, and for the $\mathbb{Z}^d$-action $\alpha^{X/Y}$ induced by α on X/Y. The actions α and $\alpha^{X/Y}$ are Bernoulli with equal entropy (cf. [10]), but Theorem 4.2 implies that $\alpha^{X/Y}$ is not topologically conjugate to α (this can also be proved much more directly—cf. Theorem 5.9 in [11]).

Remarks 4.9. (1) For $d = 2$, the dichotomy expressed by the Theorems 3.1 and 4.1 is optimal, since the cohomological triviality assertion of Theorem 4.1 can be shown to be equivalent to α having completely positive entropy. However, if $d \geq 3$, there exist $\mathbb{Z}^d$-actions α by automorphisms of compact, abelian groups which have zero entropy, and for which every element $c \in$

$Z^1_{sv}(\alpha, G)$ is trivial, where (G, γ) is any complete metric group with a bi-invariant metric γ (cf. Example 5.4 in [12]).

(2) As was stated at the beginning of this section, the Theorems 4.1–4.2 remain correct if we assume that the group G is abelian and not necessarily compact (or, more generally, if $G = G_1 \times G_2$, where G_1 is compact and (G_2, γ_2) is a complete metric, abelian group). The only change in the proof occurs in Lemma 4.7, where we obtain directly that the h-weight $w_h^{(k)}(x)$ is constant (and hence equal to $w_h^{(k)}(0_X)$) for every $x \in \mathrm{Fix}(\bar{\alpha}_{k\mathbf{n}})$. Do the Theorems 4.1–4.2 hold for an arbitrary complete metric group (G, γ), where γ is bi-invariant (e.g. for a discrete group)?

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