TILINGS, FUNDAMENTAL COCYCLES AND FUNDAMENTAL GROUPS OF SYMBOLIC $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. We prove that certain topologically mixing two-dimensional shifts of finite type have a 'fundamental' 1-cocycle with the property that every continuous 1-cocycle on the shift space with values in a discrete group is continuously cohomologous to a homomorphic image of the fundamental cocycle. These fundamental cocycles are closely connected with representations of the shift space by Wang tilings and the tiling groups of J.H. Conway, J.C. Lagarias and W. Thurston, and they determine the projective fundamental groups of the shift spaces introduced by W. Geller and J. Propp.

1. INTRODUCTION

Recent investigations of certain classes of expansive and mixing $\mathbb{Z}^d$-actions with $d \geq 2$ have revealed an intrinsic algebraic structure of higher-dimensional shifts of finite type. One way of realising this structure is by representing any higher-dimensional shift of finite type (*SFT*) in terms of Wang tilings (cf. [2], [23], [32], [8] and Section 4). Following [5] and [30] one can then define, for every set T of Wang tiles (and hence for every *SFT*), a finitely presented group, the 'tiling group' of T (cf. Section 4); such groups were originally introduced by J.H. Conway and J.C. Lagarias in [5] as a way of deciding whether a region in the plane can be tiled by a given set of polygonal tiles (cf. [30]). Unfortunately, a given *SFT* can be represented by many different Wang tilings, and the corresponding tiling groups may differ considerably. This makes it desirable to determine whether the tiling groups of representations of the same *SFT* by different Wang tilings have anything in common.

In dimension two W. Geller and J. Propp provided a first answer to this question in [7] by associating with each two-dimensional *SFT* a 'projective fundamental group' which is an invariant of topological conjugacy, and which is not only conceptually, but also — in interesting examples — computationally related to tiling groups.

A different approach to constructing a topological conjugacy invariant connected with tiling groups is based on work by J. Kammeyer ([10], [11], [12]) in which she proved that every continuous cocycle on the full 2-dimensional n-shift with values in a finite group G is *trivial*, i.e. continuously cohomologous to a homomorphism (see Sections 2–4 for the relevant definitions). Kammeyer's result was extended to a more general class of higher-dimensional *SFT*'s in [25], where it was also shown that such shifts are generally quite particular about the discrete groups in which they have non-trivial first cohomology. The paper [25] also pointed out the connection between cohomology and Wang tilings, which is explored further in this paper. From the definition of a tiling group it is clear that every set of Wang tiles generates not only a group, but also a continuous 1-cocycle for the shift-action σ of $\mathbb{Z}^2$ on the tiling space (or *Wang shift*) with values in the tiling group, the 'tiling cocycle' (Section 4). Every representation of a *SFT* as a Wang shift leads to a corresponding tiling cocycle, and every continuous 1-cocycle for the shift-action of $\mathbb{Z}^2$ on the *SFT* is a homomorphic image of such a tiling cocycle for a suitable representation of the *SFT* as a Wang shift (Theorem 4.2). Representations of the *SFT* as different Wang shifts may, of course, lead to quite different tiling cocycles. However, in some interesting examples (such as the domino tilings, the three-coloured chessboards, the square ice model, or the lozenge tilings) there exists a nontrivial 'fundamental' cocycle c^* with the property that *every* continuous 1-cocycle with values in a discrete group (and in particular every tiling cocycle) is continuously cohomologous to the composition of c^* with a group homomorphism (Theorems 6.7, 7.1, 8.1 and 9.1).

These fundamental cocycles have an immediate interpretation as obstructions to the 'patching of holes' in the spirit of [5] and [30] (cf. Section 5). Suppose that $X \subset A^{\mathbb{Z}^2}$ is a two-dimensional *SFT* with (finite) alphabet A. If $z \in A^{\mathbb{Z}^2 \smallsetminus Q}$ is an allowed partial configuration, where $Q \subset \mathbb{Z}^2$ is a rectangular 'hole' of finite size, then it may not be possible to extend the configuration z to all of $\mathbb{Z}^2$, i.e. to find a point $x \in X$ whose projection onto the coordinates $\mathbb{Z}^2 \smallsetminus Q$ coincides with z. If we represent X as a Wang shift W_T and denote by $\Gamma(T)$ the tiling group of W_T (cf. Section 4), then the 'word' in $\Gamma(T)$ obtained by running along the edges of the tiles around the rectangular hole Q has to be equal to the identity if the tiling corresponding to z can be extended to an element of X (cf. [5], [30]). However, different representations of X by Wang shifts may give different answers: for example, the tiles (7.1) represent the *SFT* $X^{(3)}$ of three-coloured chessboards, but give no information about the possibility of filling such a hole Q since the associated tiling cocycle is a homomorphism. In contrast, the representation of $X^{(3)}$ in terms of the tiles (7.3) *does* yield a nontrivial obstruction. Would one obtain further information by using other sets of Wang tiles to represent $X^{(3)}$?

It turns out that any fundamental cocycle expresses the total information about the patching of holes that can be extracted from *all* Wang shifts representing a *SFT* X (cf. Proposition 5.2): by choosing Wang tiles T corresponding to a sufficiently high n-block representation of X in the sense of Theorem 4.2 (cf. (4.7)–(4.8)) we obtain a tiling cocycle c_T such that the fundamental cocycle is a homomorphic image of c_T, and *every* continuous cocycle — in particular every tiling cocycle of a Wang shift representing X — is continuously cohomologous to a homomorphic image of c_T. It follows that none of these cocycles can yield any obstructions beyond those obtained from c_T (or, indeed, from the fundamental cocycle). However, even a fundamental cocycle may not provide *complete* information about the patching of holes (Example 5.3).

The fundamental cocycles discussed here are also related to the projective fundamental groups of *SFT*'s introduced by W. Geller and J. Propp in [7] (cf. Section 5). If a *SFT* X is *cohomologically trivial* in the sense that every continuous cocycle on X with values in a discrete group is trivial then the projective fundamental group of X is also trivial (Corollary 5.8); more generally, if X has a fundamental cocycle $c^* \colon \mathbb{Z}^2 \times X \longmapsto G^*$, where G^* is some discrete group, then the projective fundamental group of [7] is (isomorphic to) an explicitly identified subgroup of G^* (Theorem 5.5 and the Remarks 5.6, 6.10, 7.6, 8.6 and 9.2).

This paper is organised as follows. Section 2 introduces the basic definitions of *SFT*'s, their cohomology, and the notion of a fundamental cocycle (Definition 2.3). Section 3 discusses a Livshitz-type result about cohomological triviality needed for the subsequent examples (Proposition 3.1 and its corollaries). Section 4 defines Wang tilings and their tiling groups and tiling cocycles, shows that every continuous cocycle on a *SFT* X with values in a discrete group is a homomorphic image of a tiling cocycle of X (Theorem 4.2), and proves that the existence of a fundamental cocycle on X is equivalent to the statement that all continuous cocycles on X depend (essentially) only on a fixed range of coordinates (Corollary 4.4). Section 5 investigates the connection of tiling groups, tiling cocycles and fundamental cocycles with extension problems and the projective fundamental groups of *SFT*'s. The Sections 6–9 deal with specific examples. In Section 6 we discuss domino tilings and prove that the tiling cocycle of a natural representation of the dominoes by Wang tiles is fundamental in the above sense (Theorem 6.7). This cocycle takes values in the group $G \times \mathbb{Z}^2$, where change

$$G = \left\{ \begin{pmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ 0 & 1 & a_{(2,3)} \\ 0 & 0 & a_{(3,3)} \end{pmatrix} : a_{(1,3)} \in \mathbb{Z}, a_{(1,1)}, a_{(3,3)} \in \{1,-1\}, \right.$$
$$\left. a_{(1,2)}, a_{(2,3)}, a_{(1,1)} + a_{(1,2)}, a_{(3,3)} + a_{(2,3)} \in \{0,1\} \right\}.$$

As a consequence of Corollary 3.3 we also obtain that the dominoes have no nontrivial cocycles with values in an abelian group (Theorem 6.6). In Section 7 we study the *SFT* consisting of all colourings of $\mathbb{Z}^2$ by 3 colours such that no two adjacent lattice points have the same colour. For these three-coloured 'chessboards' we obtain a fundamental cocycle with range $\mathbb{Z}^3$ (Theorem 7.1); if one uses more than three colours the first cohomology becomes trivial by Example 4.4 in [25]. Section 8 deals with the 'square ice' model, which is a three-to-one factor of the three-coloured chessboards and has a fundamental cocycle with values in $\mathbb{Z}^3$ (Theorem 8.1). Section 9 discusses a subshift of the square ice model, the lozenge tilings described in [30] and in Example 4.6 in [25]. Again we obtain a fundamental cocycle with values in $\mathbb{Z}^3$ which is, in fact, the restriction to the lozenge tilings of the fundamental cocycle of the square ice model (Theorem 9.1). Section 10 gives a simple sufficient condition for a d-dimensional *SFT* X to have trivial cohomology with values in every discrete group (Theorem 10.3) and lists a number of examples of *SFT*'s satisfying this triviality condition. Theorem 10.3 is closely related to Theorem 3.2 in [25]. Section 11 deals with *SFT*'s which are factors of *SFT*'s with trivial cohomology, gives a sufficient condition under which they have fundamental cocycles (Theorem 11.1), and presents some examples. However, the behaviour of fundamental cocycles under factor maps is not well understood in general (Remark 11.2). Section 12 investigates a mixing two-dimensional *SFT* X (originally introduced by F. Ledrappier) for which the existence of a fundamental cocycle is not known, although we construct a continuous cocycle $c^*\colon \mathbb{Z}\times X \longmapsto G^*$ with values in a discrete group G^* such that every *known* continuous cocycle of X is continuously cohomologous to a homomorphic image of c^* (Theorem 12.4). In the final section we prove that no mixing one-dimensional *SFT* can have a fundamental cocycle (Theorem 13.1).

The main problem left open in this paper is which higher-dimensional *SFT*'s have fundamental cocycles; as general statements about higher-dimensional *SFT*'s are rather hard to come by one really has to look at additional examples in order to shed further light on this question. Another open question is whether there exists a *SFT* X with trivial projective fundamental group which is cohomologically nontrivial (cf. Corollary 5.8).

Finally a remark about exposition. In order to keep the presentation as simple as possible I have restricted the discussion to continuous cocycles on *SFT*'s with values in discrete groups; there are no statements about Hölder cocycles on general expansive and mixing $\mathbb{Z}^d$-actions, although such an extension is possible (cf. Proposition 3.1 in [25] and [14], [15], [26], [28]). In line with this aim I have also kept the detailed description of Wang tilings and tiling cocycles to dimension two (with brief comments on the higher-dimensional case in the Remarks 4.3 and 5.9 (2)). Many of the examples
change presented in this paper also appear in [25], but I have tried to make their description reasonably self-contained.

2. Shifts of finite type and their cohomology

Let A be a finite set, $d \geq 1$, and let $A^{\mathbb{Z}^d}$ be the set of all maps $x\colon \mathbb{Z}^d \longmapsto A$, furnished with the compact product topology. We write a typical point $x \in A^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{m}}) = (x_{\mathbf{m}},\, \mathbf{m} \in \mathbb{Z}^d)$, where $x_{\mathbf{m}} \in A$ denotes the value of x at $\mathbf{m}$. The *shift action* $\sigma\colon \mathbf{n} \mapsto \sigma^{\mathbf{n}}$ of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$ is defined by

$$(\sigma^{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}} \tag{2.1}$$

for every $x = (x_{\mathbf{m}}) \in A^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$. A subset $X \subset A^{\mathbb{Z}^d}$ is *shift-invariant* if $\sigma^{\mathbf{n}}(X) = X$ for every $\mathbf{n} \in \mathbb{Z}^d$, and a closed, shift-invariant subset $X \subset A^{\mathbb{Z}^d}$ is a *subshift*. If $X \subset A^{\mathbb{Z}^d}$ is a subshift we write $\sigma = \sigma^X$ for the restriction of the shift-action (2.1) to X. For any subset $S \subset \mathbb{Z}^d$ we denote by $\pi_S\colon A^{\mathbb{Z}^d} \longmapsto A^S$ the projection map which restricts every $x \in A^{\mathbb{Z}^s}$ to S. A subshift $X \subset A^{\mathbb{Z}^d}$ is a *shift of finite type* (*SFT*) if there exists a finite set $F \subset \mathbb{Z}^d$ such that

$$X = \{x \in A^{\mathbb{Z}^d} : \pi_F \cdot \sigma^{\mathbf{n}}(x) \in \pi_F(X) \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}. \tag{2.2}$$

A standard re-coding argument allows us to assume that

$$F = \{0,1\}^d \subset \mathbb{Z}^d \tag{2.3}$$

change in (2.2), by changing the alphabet A, if necessary (cf. [25]; a more restrictive definition of higher-dimensional *SFT*'s is considered in [21]–[22]).

If $X \subset A^{\mathbb{Z}^d}$ is a *SFT* it will sometimes be helpful to specify the shift-action of $\mathbb{Z}^d$ explicitly and to write (X, σ) instead of X. For example, if $\bar{\sigma}\colon \mathbf{n} \mapsto \bar{\sigma}^{\mathbf{n}} = \sigma^{2\mathbf{n}}$ is the *even* shift-action on a *SFT* $X \subset A^{\mathbb{Z}^d}$ then $(X, \bar{\sigma})$

can be viewed as a *SFT* with alphabet $A' = \pi_F(X) \subset A^F$, where F is defined in (2.3).

Definition 2.1. Let $X \subset A^{\mathbb{Z}^d}$ be a *SFT* of the form (2.2)–(2.3).

(1) The *SFT* X is (*topologically*) *mixing* if σ is topologically mixing on X, i.e. if there exists, for any two nonempty open sets $\mathcal{O}_1, \mathcal{O}_2 \subset X$, an integer $N \geq 0$ with

$$\mathcal{O}_1 \cap \sigma^{-\mathbf{m}}(\mathcal{O}_2) \neq \varnothing$$

whenever $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $\|\mathbf{m}\| = \max_{i=1,\dots,d} |m_i| \geq N$.

(2) Let $E \subset \mathbb{Z}^d$ be a subset. An element $z \in A^E$ is *allowed* if, for every $\mathbf{n} \in \mathbb{Z}^d$,

$$\pi_{E \cap (\mathbf{n}+F)}(z) \in \pi_{E \cap (\mathbf{n}+F)}(X).$$

The set of all allowed elements in A^E is denoted by $\Pi_X(E)$. The *SFT* X has the *extension property* if there exists an integer $\bar{r} \geq 1$ such that

$$\pi_Q(X) = \Pi_X(Q)$$

for every rectangle $Q = \prod_{1=1}^d \{a_i, \dots, b_i\} \subset \mathbb{Z}^d$ with $-\infty \leq a_i < b_i \leq \infty$ and $b_i - a_i \geq \bar{r}$ for $i = 1, \dots, d$. In other words, X has the extension property if every allowed configuration on every sufficiently large rectangle can be extended to a point in X.

(3) Put, for every $r \geq 1$ and $i = 1, \dots, d$,

$$\mathbf{B}(r) = \{\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d : \|\mathbf{m}\| \leq r\},$$
$$\mathbf{S}(r, i) = \mathbf{B}(r) + \mathbb{Z}\mathbf{e}^{(i)} = \{\mathbf{m} + k\mathbf{e}^{(i)} : \mathbf{m} \in \mathbf{B}(r),\ k \in \mathbb{Z}\},$$
$$X(r, i) = \pi_{\mathbf{S}(r,i)}(X) \subset A^{\mathbf{S}(r,i)},$$

where $\mathbf{e}^{(i)} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^d$ is the i-th unit vector. We write

$$T_{(r,i)} \colon X(r, i) \longmapsto X(r, i)$$

for the homeomorphism of $X(r, i)$ induced by the shift $\sigma^{\mathbf{e}^{(i)}}$, i.e.

$$T_{(r,i)} \cdot \pi_{\mathbf{S}(r,i)} = \pi_{\mathbf{S}(r,i)} \cdot \sigma^{\mathbf{e}^{(i)}},$$

and set, for every $L \geq 1$,

$$P_X(i, L) = \{x \in X : \sigma^{L\mathbf{e}^{(i)}}(x) = x\},$$
$$P_X(r, i, L) = \{x \in X(r, i) : (T_{(r,i)})^L(x) = x\}.$$

If the *SFT* X has the extension property we say that X has the *periodic extension property in the direction* i if there exist integers $\bar{r} \geq 1$, $\bar{L} \geq 1$ with

$$\pi_{\mathbf{S}(r,i)}(P_X(i, L)) = P_X(r, i, L)$$

for all $r \geq \bar{r}$, $L \geq \bar{L}$. If X has the periodic extension property in every direction $i = 1, \dots, d$ we say that X has the *periodic extension property*.

Remarks 2.2. Let $d \geq 1$, and let $X \subset A^{\mathbb{Z}^d}$ be a *SFT* satisfying (2.2)–(2.3).

(1) If $d = 1$ then X has the periodic extension property.

(2) If $d \geq 1$ and X has the extension property then $(X(r,i), T_{(r,i)})$ is a one-dimensional *SFT* of the form (2.2)–(2.3) with alphabet $\pi_{\mathbf{B}(r,i)}(X) \subset A^{\mathbf{B}(r,i)}$ for every $r \geq \bar{r}$ and $i = 1, \dots, d$, where

$$\mathbf{B}(r,i) = \{\mathbf{m} = (m_1, \dots, m_d) \in \mathbf{B}(r) : m_i = 0\}.$$

For the following discussion we fix a finite alphabet A and a subshift $X \subset A^{\mathbb{Z}^d}$. Let G be a discrete group with identity element 1_G. A map $c\colon \mathbb{Z}^d \times X \longmapsto G$ is a *continuous cocycle* for the shift-action σ of $\mathbb{Z}^d$ on X defined in (2.1) if $c(\mathbf{n}, \cdot)\colon X \longmapsto G$ is continuous for every $\mathbf{n} \in \mathbb{Z}^d$ and

$$c(\mathbf{m}+\mathbf{n}, x) = c(\mathbf{m}, \sigma^{\mathbf{n}}(x))c(\mathbf{n}, x) \tag{2.4}$$

for all $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. The cocycle c is a *homomorphism* if $c(\mathbf{n}, \cdot)$ is constant for every $\mathbf{n} \in \mathbb{Z}^d$, and c is a *coboundary* if there exists a Borel map $b\colon X \longmapsto G$ such that

$$c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}(x))^{-1} b(x) \tag{2.5}$$

for all $x \in X$ and $\mathbf{n} \in \mathbb{Z}^d$. The map b in (2.5) is the *cobounding function* of c. Two continuous cocycles $c, c'\colon \mathbb{Z}^d \times X \longmapsto G$ are *cohomologous*, with a Borel measurable *transfer function* $b\colon X \longmapsto G$, if $c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}(x))^{-1} c'(\mathbf{n}, x) b(x)$ for all $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. The cocycles $c, c'\colon \mathbb{Z}^d \times X \longmapsto G$ are *continuously cohomologous* if they are cohomologous with a continuous transfer function. Following [14] and [25] we call a cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ *trivial* if it is continuously cohomologous to a homomorphism.

We write $Z_c^1(X, G)$ for the set of continuous cocycles on X with values in G, denote by $B_c^1(X, G) \subset Z_c^1(X, G)$ the subset of coboundaries and write

$$H_c^1(X, G) = \{[c] : c \in Z_c^1(X, G)\}$$

for the space of cohomology classes

$$[c] = \{c' \in Z_c^1(X, G) : c' \text{ is cohomologous to } c\}.$$

If G is abelian, the set $Z_c^1(X, G)$ is a group under pointwise addition, the coboundaries $B_c^1(X, G) \subset Z_c^1(X, G)$ form a subgroup, and the first cohomology

$$H_c^1(X, G) = Z_c^1(X, G)/B_c^1(X, G)$$

is a group.

Definition 2.3. Let A be a finite set, $d \geq 1$, and let $X \subset A^{\mathbb{Z}^d}$ be a *SFT*. A continuous cocycle $c^*\colon \mathbb{Z}^d \times X \longmapsto G^*$ with values in a discrete group G^* is *fundamental* if the following is true: for every discrete group G and every continuous cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ there exists a group homomorphism $\theta\colon G^* \longmapsto G$ such that c is continuously cohomologous to the cocycle

$$\theta \cdot c^*\colon \mathbb{Z}^2 \times X \longmapsto G,$$

defined by $\theta \cdot c^*(\mathbf{n}, x) = \theta(c^*(\mathbf{n}, x))$ for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$.

The paper [25] contains examples of two-dimensional *SFT*'s for which every continuous cocycle with values in a discrete group is trivial. If X is such a *SFT*, then the homomorphism $c\colon \mathbb{Z}^d \times X \longmapsto \mathbb{Z}^d$ with $c(\mathbf{n}, x) = \mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$ is obviously fundamental in the sense of Definition

2.3. The existence of nontrivial fundamental cocycles is, of course, more interesting (cf. Theorems 6.7, 7.1, 8.1 and 9.1). Note that no one-dimensional, mixing *SFT* can have a fundamental cocycle (cf. Theorem 13.1).

Remark 2.4 (Nonuniqueness of fundamental cocycles). If X is a d-dimensional *SFT* with a fundamental cocycle $c^*\colon \mathbb{Z}^d \times X \longmapsto G^*$ then the composition of c^* with any group automorphism of G^* is again fundamental. Furthermore, Theorem 4.2 allows us to write c^*, for every sufficiently large $r \geq 1$, as a homomorphic image of the tiling cocycle $c_{T_X^{(r)}}$ of the set $T_X^{(r)}$ of Wang tiles corresponding to the $(2r+1)$-block representation of X (cf. (4.7)–(4.8)), and $c_{T_X^{(r)}}$ is therefore — by definition — again fundamental. This indicates that a d-dimensional *SFT* may have several non-cohomologous fundamental cocycles.

3. Livshitz' theorem

For our investigation of cohomological rigidity we have to introduce the notion of a (1-)cocycle on an equivalence relation (cf. [6], [14], [25]–[26]). If $R \subset X \times X$ is an equivalence relation on X we denote by

$$R(x) = \{x' \in X : (x, x') \in R\}$$

the equivalence class of a point $x \in X$. An equivalence relation $R \subset X \times X$ is *Borel* if R is a Borel subset of $X \times X$. A Borel map $a\colon R \longmapsto G$ on a Borel equivalence relation R is a (1-)*cocycle* if

$$a(x, x')a(x', x'') = a(x, x'') \tag{3.1}$$

for every $x \in X$ and $x', x'' \in R(x)$. Two cocycles $a, a'\colon R \longmapsto G$ are *cohomologous* if there exists a Borel map $b\colon X \longmapsto G$ with

$$a'(x, x') = b(x)^{-1} a(x, x') b(x') \tag{3.2}$$

for every $(x, x') \in R$, and a cocycle $a\colon R \longmapsto G$ is a *coboundary* if it is cohomologous to the constant cocycle $a' \equiv 1_G$.

For every integer $r \geq 0$ and $i = 1, \dots, d$ we set

$$\begin{aligned} \Delta_X(r, i) = \{(x, x') \in X \times X : x_{\mathbf{n}} \neq x'_{\mathbf{n}} \text{ for only} \\ \text{finitely many } \mathbf{n} \in \mathbf{S}(r, i)\}, \end{aligned} \tag{3.3}$$

where $\mathbf{S}(r, i)$ is defined in Definition 2.2. Then $\Delta_X(r, i)$ is a Borel equivalence relation on X. If G is a discrete group and $h\colon X \longmapsto G$ a continuous map, then there exists a smallest integer $r = r(h)$ with

$$h(x) = h(x') \text{ whenever } x, x' \in X \text{ and } \pi_{\mathbf{B}(r)}(x) = \pi_{\mathbf{B}(r)}(x'), \tag{3.4}$$

and we can define, for every $r \geq r(h)$ and $i = 1, \dots, d$, cocycles $a_h^{\pm}\colon \Delta_X(r, i) \longmapsto G$ on the Borel equivalence relation $\Delta_X(r, i)$ on X as follows: for every

$(x,x')\in\Delta_X(r,i)$ and $l\ge 1$ we set

$$
\begin{aligned}
a_h^+(x,x')^{(l)} &= \left(\prod_{k=0}^{l-1} h(\sigma^{k\mathbf{e}^{(i)}}(x))^{-1}\right)\cdot\left(\prod_{k=0}^{l-1} h(\sigma^{k\mathbf{e}^{(i)}}(x'))^{-1}\right)^{-1}\\
&= h(x)^{-1}\cdots h(\sigma^{(l-1)\mathbf{e}^{(i)}}(x))^{-1}\\
&\qquad\cdot h(\sigma^{(l-1)\mathbf{e}^{(i)}}(x'))\cdots h(x'),\\
a_h^-(x,x')^{(l)} &= \left(\prod_{k=1}^{l} h(\sigma^{-k\mathbf{e}^{(i)}}(x))\right)\cdot\left(\prod_{k=1}^{l} h(\sigma^{-k\mathbf{e}^{(i)}}(x'))\right)^{-1}\\
&= h(\sigma^{-\mathbf{e}^{(i)}}(x))\cdots h(\sigma^{-l\mathbf{e}^{(i)}}(x))\\
&\qquad\cdot h(\sigma^{-l\mathbf{e}^{(i)}}(x'))^{-1}\cdots h(\sigma^{-\mathbf{e}^{(i)}}(x'))^{-1}.
\end{aligned}
\tag{3.5}
$$

Since there exists, for every $(x,x')\in\Delta_X(r,i)$, an integer $L\ge 1$ with

$$
\begin{aligned}
a_h^+(x,x')^{(l)} &= a_h^+(x,x')^{(L)},\\
a_h^-(x,x')^{(l)} &= a_h^-(x,x')^{(L)}
\end{aligned}
$$

whenever $l\ge L$, we obtain well-defined Borel maps $a_h^{\pm}\colon \Delta_X(r,i)\longmapsto G$ by setting

$$
\begin{aligned}
a_h^+(x,x') &= \lim_{l\to\infty} a_h^+(x,x')^{(l)},\\
a_h^-(x,x') &= \lim_{l\to\infty} a_h^-(x,x')^{(l)}
\end{aligned}
\tag{3.6}
$$

for every $(x,x')\in\Delta_X(r,i)$, and (3.5)–(3.6) guarantee that these maps satisfy the cocycle equations

$$
\begin{aligned}
a_h^+(x,x')a_h^+(x',x'') &= a_h^+(x,x''),\\
a_h^-(x,x')a_h^-(x',x'') &= a_h^-(x,x'')
\end{aligned}
\tag{3.7}
$$

for all $x\in X$ and $x',x''\in\Delta_X(r,i)(x)$.

For $d=1$ the equivalence relation $\Delta_X(0,1)=\Delta_X$ coincides with the *Gibbs equivalence relation* of the *SFT* X (cf. [25], equation (2.4)). The following result is essentially due to Livshitz and is valid under more general assumptions (G only needs to be a complete metric group with a bi-invariant metric and $h\colon X\longmapsto G$ a function with summable variation).

Proposition 3.1. *Let A be a finite set, $X\subset A^{\mathbb{Z}}$ a mixing* SFT*, and G a discrete group.*

(1) *Suppose that $h\colon X\longmapsto G$ is a continuous map, and that the cocycles $a_h^{\pm}\colon\Delta_X\longmapsto G$ in* (3.6) *are equal. Then there exists a continuous function $b\colon X\longmapsto G$ such that the map*

$$(b\cdot\sigma)^{-1}hb\colon x\mapsto b(\sigma(x))^{-1}h(x)b(x)\in G$$

is constant on X.

(2) *If $h_i\colon X\longmapsto G$, $i=1,2$, are continuous maps, and if there exists a Borel map $b\colon X\longmapsto G$ with*

$$b(\sigma(x))^{-1}h_1(x)b(x)=h_2(x) \tag{3.8}$$

for μ-a.e. $x \in X$, where μ is the measure of maximal entropy on X, then the map b in (3.8) *may be chosen to be continuous.*

The proof of Proposition 3.1 is a minor variation of that of Proposition 3.1 in [25] and will be omitted. The assertion (2) of Proposition 3.1 was obtained independently by W. Parry (private communication — cf. also [31] for the abelian case).

Corollary 3.2. *Let $d \geq 1$, $i \in \{1, \dots, d\}$, A a finite set, $X \subset A^{\mathbb{Z}^d}$ a mixing* SFT *with the extension property, G a discrete group, $h\colon X \longmapsto G$ a continuous map, and $r \geq r(h)$. If the cocycles $a_h^{\pm}\colon \Delta_X(r, i) \longmapsto G$ in* (3.6) *are equal then there exists a continuous function $b\colon X \longmapsto G$ such that the map $(b \cdot \sigma^{\mathbf{e}^{(i)}})^{-1} hb$ is constant on X.*

Proof. Let $h = c(\mathbf{e}^{(i)}, \cdot)\colon X \longmapsto G$, and let $r \geq \max(r(h), 1)$ (cf. (3.4)). By Remark 2.2 $(X(r,i), T_{(r,i)})$ is — for sufficiently large r — a (one-dimensional) mixing *SFT*, $X(r,i)$ coincides with the set of all allowed elements in $A^{\mathbf{S}(r,i)}$, and

$$(\pi_{\mathbf{S}(r,i)} \times \pi_{\mathbf{S}(r,i)})(\Delta_X(r,i)) = \Delta_{X(r,i)}$$

(cf. Definition 2.1). Now apply Proposition 3.1. □

Corollary 3.3. *Let $d > 1$, $i \in \{1, \dots, d\}$, A a finite set, $X \subset A^{\mathbb{Z}^d}$ a mixing* SFT *with the periodic extension property in the direction $i \in \{1, \dots, d\}$, G a discrete group and $c\colon \mathbb{Z}^d \times X \longmapsto G$ a continuous cocycle.*

(1) *If G is abelian, and if the restriction of σ to $P_X(i, L)$ is topologically transitive for infinitely many $L \geq 1$, then c is trivial;*
(2) *If G is finite and the restriction of σ to $P_X(i, L)$ is topologically transitive for all sufficiently large $L \geq 1$, then c is trivial.*

Proof. For every $L \geq 1$ and $x \in P_X(i, L)$ we set

$$w_L(x) = c(L\mathbf{e}^{(i)}, x) = h(\sigma^{(L-1)\mathbf{e}^{(i)}}) \cdots h(x)$$

and conclude from (2.4) that $w_L(\sigma^{\mathbf{m}}(x))$ is conjugate to $w_L(x)$ for every $x \in P_X(i, L)$. If σ is topologically transitive on $P_X(i, L)$, then the continuity of the map $x \mapsto w_L(x)$ implies that the set $\{w_L(x) : x \in P_X(i, L)\}$ lies in a single conjugacy class of G whenever $L \geq L_0$, say.

Now suppose that G is abelian and that there exists an increasing sequence $(L_k,\ k \geq 1)$ such that σ is topologically transitive on $P_X(i, L_k)$ for every $k \geq 1$. In particular the map $x \mapsto w_{L_k}(x)$ is constant on $P_X(i, L_k)$ for every $k \geq 1$.

For the following discussion we fix a sufficiently large $r \geq 1$, assume that $(x, x') \in \Delta_X(r, i)$, and put $y = \pi_{\mathbf{S}(r,i)}(x)$, $y' = \pi_{\mathbf{S}(r,i)}(x')$. Then there exists an integer $M > r$ such that $y_{\mathbf{n}} = y'_{\mathbf{n}}$ whenever $\mathbf{n} = (n_1, \dots, n_d) \in \mathbf{S}(r, i)$ and $\|\mathbf{n}\| = \max_{j=1,\dots,d} |n_j| = |n_i| \geq M$. As $X(r, i)$ is a mixing *SFT* there exists an integer $k \geq 1$ such that $L_k \geq 2M + 4r$ and periodic points $z, z' \in X(r, i)$ with (large) period L_k such that $z_{\mathbf{n}} = y_{\mathbf{n}} = x_{\mathbf{n}}$ and $z'_{\mathbf{n}} = y'_{\mathbf{n}} = x'_{\mathbf{n}}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbf{S}(r, i)$ with $|n_i| \leq M + r$, and $z_{\mathbf{n}} = z'_{\mathbf{n}}$ whenever $M + r \leq |n_i| \leq L_k - M - r$. Finally we use the periodic extension property to find points $\bar{z}, \bar{z}' \in P_X(i, L_k)$ with $\bar{z}_{\mathbf{n}} = z_{\mathbf{n}}$, $\bar{z}'_{\mathbf{n}} = z'_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbf{S}(r, i)$.

We write $h'\colon X(r,i) \longmapsto G$ for the well-defined map with $h' \cdot \pi_{\mathbf{S}(r,i)} = h$. Then the constancy of the map $x \mapsto w_{L_k}(x)$ on $P_X(i, L_k)$ guarantees that

$$\begin{aligned} a_h^+(x,x') &= a_h^+(x,x')^{(L_k)} \\ &= h(x)^{-1} \cdots h(\sigma^{(L_k-1)\mathbf{e}^{(i)}}(x))^{-1} h(\sigma^{(L_k-1)\mathbf{e}^{(i)}}(x')) \cdots h(x') \\ &= h'(z)^{-1} \cdots h'(T_{(r,i)}^{L_k-1}(z))^{-1} h'(T_{(r,i)}^{L_k-1}(z')) \cdots h'(z') \\ &= h(\bar{z})^{-1} \cdots h(\sigma^{(L_k-1)\mathbf{e}^{(i)}}(\bar{z}))^{-1} h(\sigma^{(L_k-1)\mathbf{e}^{(i)}}(\bar{z}')) \cdots h(\bar{z}') \\ &= a_h^+(\bar{z},\bar{z}')^{(L_k)} = a_h^-(\bar{z},\bar{z}')^{(L_k)} = a_h^-(x,x')^{(L_k)} = a_h^-(x,x'). \end{aligned} \tag{3.9}$$

We interrupt the proof of the abelian case for a moment and prove (3.9) under assumption (2). If $\bar{x} \in P_X(i,L)$ with $L \geq L_0$, and if $m \geq 1$ satisfies that $w_{mL}(\bar{x}) = w_L(\bar{x})^m = 1_G$, then $w_{kmL}(x)$ is conjugate — and hence equal — to $w_{kmL}(\bar{x}) = w_{mL}(\bar{x})^k = 1_G$ for every $x \in P_X(kmL)$. This allows us to repeat the calculation (3.9) with $L_k = kmL$, $k \geq 1$, and to conclude that $a_h^+(x,x') = a_h^-(x,x')$ for all $(x,x') \in \Delta_X(r,i)$ if either of the conditions (1) or (2) are satisfied.

According to Proposition 3.1 this guarantees the existence of a continuous map $b\colon X \longmapsto G$ for which $(b \cdot \sigma^{\mathbf{e}^{(i)}})^{-1} hb$ is constant, and we define a cocycle $c'\colon \mathbb{Z}^d \times X \longmapsto G$ by

$$c'(\mathbf{n}, \cdot) = (b \cdot \sigma^{\mathbf{n}})^{-1} c(\mathbf{n}, \cdot) b$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

By construction there exists a $g_0 \in G$ with $c'(\mathbf{e}^{(i)}, x) = g_0$ for every $x \in X$. For any $\mathbf{m} \in \mathbb{Z}^d$ the map $c'(\mathbf{m}, \cdot)\colon X \longmapsto G$ is continuous, and there exists a finite partition $\mathcal{O}_1, \dots, \mathcal{O}_s$ of X into nonempty, open subsets of X on each of which $c'(\mathbf{m}, \cdot)$ is constant and equal to g_j, $j = 1, \dots, s$, say. As $\sigma^{\mathbf{e}^{(i)}}$ is topologically mixing we can find, for every $j = 1, \dots, s$ and every sufficiently large $k \geq 1$, an element $x \in \mathcal{O}_j \cap \sigma^{-k\mathbf{e}^{(i)}}(\mathcal{O}_j)$ with $c'(k\mathbf{e}^{(i)}, x) = g_0^k$. The cocycle equation (2.4) yields that

$$c'(\mathbf{m}, \sigma^{k\mathbf{e}^{(i)}}(x)) c'(k\mathbf{e}^{(i)}, x) = g_j g_0^k = g_0^k g_j = c'(k\mathbf{e}^{(i)}, \sigma^{\mathbf{m}}(x)) c'(\mathbf{m}, x)$$

for every $j = 1, \dots, s$, and hence that g_0 commutes with $c'(\mathbf{m}, x)$ for every $x \in X$. It follows that $c'(\mathbf{m}, \cdot)$ is invariant under $\sigma^{k\mathbf{e}^{(i)}}$, and the topological transitivity of $\sigma^{\mathbf{e}^{(i)}}$ implies that $c'(\mathbf{m}, \cdot)$ is constant for every $\mathbf{m} \in \mathbb{Z}^d$. $\square$

4. Wang tiles

Suppose that T is a nonempty, finite set of distinct, closed 1×1 squares (tiles) with coloured edges such that no horizontal edge has the same colour as a vertical edge: such a set T is called a collection of *Wang tiles* (cf. [23], [32]). For each $\tau \in T$ we denote by $\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau)$ the colours of the right, top, left and bottom edges of τ, and we write $\mathcal{C}(T) = \{\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau) : \tau \in T\}$ for the set of colours occurring on the tiles in T. A Wang tiling w by T is a covering of $\mathbb{R}^2$ by translates of copies of elements of T with non-overlapping interiors such that the following conditions are satisfied:

(i) every corner of every tile in w lies in $\mathbb{Z}^2 \subset \mathbb{R}^2$,

(ii) two tiles of w are only allowed to touch along edges of the same colour, i.e. $\mathsf{r}(\tau) = \mathsf{l}(\tau')$ whenever τ, τ' are horizontally adjacent tiles with τ to the left of τ', and $\mathsf{t}(\tau) = \mathsf{b}(\tau')$ if τ, τ' are vertically adjacent with τ' above τ.

We identify each such tiling w with the point

$$w = (w_{\mathbf{n}}) \in T^{\mathbb{Z}^2},$$

where $w_{\mathbf{n}}$ is the unique element of T whose translate covers the square $\mathbf{n} + [0,1]^2 \subset \mathbb{R}^2$, $\mathbf{n} \in \mathbb{Z}^2$. The set $W_T \subset T^{\mathbb{Z}^2}$ of all Wang tilings by T is obviously a *SFT* of the form (2.2)–(2.3), and is called the *Wang shift* of T. Conversely, if A is a finite set and $X \subset A^{\mathbb{Z}^2}$ a *SFT* of the form (2.2) with $F = \{0,1\}^2 \subset \mathbb{Z}^2$ we set $T = \pi_F(X) \subset A^F$ and consider each

$$\tau = \boxed{\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \\ x_{(0,0)} & x_{(1,0)} \end{smallmatrix}} \in T \tag{4.1}$$

as a unit square with the 'colours' $\left[\begin{smallmatrix} x_{(0,0)} & x_{(1,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \end{smallmatrix}\right]$ along its bottom and top horizontal edges, and $\left[\begin{smallmatrix} x_{(0,1)} \\ x_{(0,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(1,1)} \\ x_{(1,0)} \end{smallmatrix}\right]$ along its left and right vertical edges. With this interpretation we obtain a one-to-one correspondence between the points $x = (x_{\mathbf{n}}) \in X$ and the Wang tilings $w = (w_{\mathbf{n}}) = (\pi_F \cdot \sigma^{\mathbf{n}}(x)) \in T^{\mathbb{Z}^2}$.

This correspondence allows us to regard each *SFT* as a Wang shift and vice versa. However, the correspondence is a bijection only up to topological conjugacy: if we start with a *SFT* $X \subset A^{\mathbb{Z}^2}$ of the form (2.2)–(2.3), view it as the Wang shift $W_T \subset T^{\mathbb{Z}^2}$ with $T = \pi_F(X)$, and then interpret W_T as a *SFT* as above, we do not end up with X, but with the 2-block representation of X. We simplify terminology by introducing the following definition.

Definition 4.1. Let A be a finite set and $X \subset A^{\mathbb{Z}^2}$ a *SFT*, T a set of Wang tiles, and W_T the associated Wang shift. We say that W_T *represents* X if the shift-action σ' of $\mathbb{Z}^2$ on the *SFT* W_T is topologically conjugate to the shift-action σ of $\mathbb{Z}^2$ on X. Two Wang shifts are *equivalent* if they are topologically conjugate as *SFT*'s.

Let T be a collection of Wang tiles, and let W_T be the Wang shift of T. Following [30] we write

$$\Gamma(T) = \langle \mathcal{C}(T) | \mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau),\ \tau \in T \rangle \tag{4.2}$$

for the free group generated by the colours occurring on the edges of elements in T, together with the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in T$. The countable, discrete group $\Gamma(T)$ is called the *tiling group* of T (or of the Wang shift W_T). From the definition of $\Gamma(T)$ is is clear that the map $\lambda \colon \Gamma(T) \longmapsto \mathbb{Z}^2$ change
given by

$$\lambda(\mathsf{b}(\tau)) = \lambda(\mathsf{t}(\tau)) = (1,0), \qquad \lambda(\mathsf{l}(\tau)) = \lambda(\mathsf{r}(\tau)) = (0,1), \tag{4.3}$$

for every $\tau \in T$ is a group homomorphism whose kernel is denoted by

$$\Gamma_0(T) = \ker(\lambda). \tag{4.4}$$

We view $W_T \subset T^{\mathbb{Z}^2}$ as a *SFT* and define the *tiling cocycle* $c_T\colon \mathbb{Z}^2 \times W_T \longmapsto \Gamma(T)$ of W_T (or, more precisely, for the shift-action σ of $\mathbb{Z}^2$ on W_T) by setting

$$c_T((1,0),w) = \mathsf{b}(w_{\mathbf{0}}), \quad c_T((0,1),w) = \mathsf{l}(w_{\mathbf{0}}) \tag{4.5}$$

for every Wang tiling $w \in W_T \subset T^{\mathbb{Z}^2}$, and by using (2.4) to extend c_T to a map from $\mathbb{Z}^2 \times W_T$ to $\Gamma(T)$ (the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in T$, in (4.2) are precisely what is needed to allow such an extension). At this level of generality it is not even clear that the space W_T is nonempty or consists of more than one point, so that it is not meaningful to ask whether this tiling cocycle is nontrivial. However, if X is a *SFT* then the following observation shows that every continuous cocycle on X with values in a discrete group is a homomorphic image of the tiling cocycle of some representation of X as a Wang shift.

Theorem 4.2. *Let A be a finite set, $X \subset A^{\mathbb{Z}^2}$ a* SFT*, G a discrete group, and $c\colon \mathbb{Z}^2 \times X \longmapsto G$ a continuous cocycle. Then there exists a set T of Wang tiles with the following properties.*

1. *There exists a shift-commuting homeomorphism $\psi\colon W_T \longmapsto X$ (i.e. W_T represents X);*
2. *There exists a group homomorphism ϕ from the tiling group $\Gamma(T)$ of T to G such that*
$$c(\mathbf{m}, \psi(w)) = \phi \cdot c_T(\mathbf{m}, w) \tag{4.6}$$
for every $w \in W_T$ and $\mathbf{m} \in \mathbb{Z}^2$.

Proof. We assume without loss in generality that X is of the form (2.2)–(2.3) and choose an integer $r \geq 1$ such that the functions $c((1,0),\cdot), c((0,1),\cdot)\colon X \longmapsto G$ depend only on the coordinates in $\mathbf{B}(r-1)$ (cf. Definition 2.1 and (3.4)). Put

$$\begin{aligned} \mathbf{B}(r) &= \{-r, \dots r\}^2 \subset \mathbb{Z}^2, \\ \mathbf{B}(r)_{\mathsf{b}} &= \{-r, \dots, r\} \times \{-r, \dots, r-1\}, \\ \mathbf{B}(r)_{\mathsf{l}} &= \{-r, \dots, r-1\} \times \{-r, \dots, r\}, \\ \mathbf{B}(r)_{\mathsf{t}} &= \{-r, \dots, r\} \times \{-r+1, \dots, r\}, \\ \mathbf{B}(r)_{\mathsf{r}} &= \{-r+1, \dots, r\} \times \{-r, \dots, r\}, \end{aligned} \tag{4.7}$$

and regard each $\tau \in T_X^{(r)} = \pi_{\mathbf{B}(r)}(X)$ as a unit square with the colours

$$\begin{aligned} \mathsf{b}(\tau) &= \pi_{\mathbf{B}(r)_{\mathsf{b}}}(\tau), \qquad & \mathsf{l}(\tau) &= \pi_{\mathbf{B}(r)_{\mathsf{l}}}(\tau), \\ \mathsf{t}(\tau) &= \pi_{\mathbf{B}(r)_{\mathsf{t}}}(\tau), \qquad & \mathsf{r}(\tau) &= \pi_{\mathbf{B}(r)_{\mathsf{r}}}(\tau) \end{aligned} \tag{4.8}$$

on the bottom, left, top and right edges of τ (cf. (4.1)). Our choice of r implies that

$$\begin{aligned} c((1,0),x) &= c((1,0),x'), \\ c((0,1),x) &= c((0,1),x'), \\ c((1,0),\sigma^{(0,1)}(x)) &= c((1,0),\sigma^{(0,1)}(x')) \\ c((0,1),\sigma^{(1,0)}(x)) &= c((0,1),\sigma^{(1,0)}(x')) \end{aligned}$$

whenever $\pi_{\mathbf{B}(r)}(x) = \pi_{\mathbf{B}(r)}(x')$, so that we may set

$$\mathsf{b}'(\tau) = c((1,0),x), \qquad \mathsf{l}'(\tau) = c((0,1),x),$$
$$\mathsf{t}'(\tau) = c((1,0),\sigma^{(0,1)}(x)), \qquad \mathsf{r}'(\tau) = c((0,1),\sigma^{(1,0)}(x))$$

for every $\tau \in T_X^{(r)}$ and $x \in X$ with $\pi_{\mathbf{B}(r)}(x) = \tau$. The cocycle equation (2.4) shows that

$$\mathsf{t}'(\tau)\mathsf{l}'(\tau) = \mathsf{r}'(\tau)\mathsf{b}'(\tau)$$

for every $\tau \in T_X^{(r)}$. The isomorphism of the shift-actions of $\mathbb{Z}^2$ on X and $W_{T_X^{(r)}}$ is obvious, since $W_{T_X^{(r)}}$ is nothing but the $(2r+1)$-block representation of X, and the definition of the tiling group $\Gamma(T_X^{(r)})$ implies that the map

$$\mathsf{t}(\tau) \mapsto \mathsf{t}'(\tau), \qquad \mathsf{l}(\tau) \mapsto \mathsf{l}'(\tau),$$
$$\mathsf{r}(\tau) \mapsto \mathsf{r}'(\tau), \qquad \mathsf{b}(\tau) \mapsto \mathsf{b}'(\tau)$$

induces a group homomorphism $\phi\colon \Gamma(T_X^{(r)}) \longmapsto G$ satisfying (4.6). □

Remark 4.3 (Higher-dimensional Wang tiles). The definition of Wang tiles and tiling cocycles given here can be extended very easily to dimension $d > 2$. For $d > 2$ a set T of d-dimensional Wang tiles is a finite set of d-dimensional unit cubes $[0,1]^d \subset \mathbb{R}^d$ with coloured faces such that distinct cubes are coloured differently and the the sets of colours occurring on non-parallel faces of elements of T are disjoint. A *Wang tiling* is a covering of $\mathbb{R}^d$ by integer translates of (copies of) tiles in T whose colours on overlapping faces coincide, and the *Wang shift* W_T of T is the set of all Wang tilings obtained from T.

Define, for each 1-dimensional edge of an element $\tau \in T$, the 'colour' of that edge as the set of colours of all the faces of τ containing that edge (note that each of these sets has exactly $d-1$ distinct elements). The *tiling group* $\Gamma(T)$ is the free group generated by the set of all colours of edges of elements in T, together with the relations resulting from the condition that, for every $\tau \in T$, every path along edges of τ leading from the vertex $(0,\dots,0) \in [0,1]^d$ to the diametrically opposite vertex $(1,\dots,1) \in [0,1]^d$ results in the same word in $\Gamma(T)$ (with multiplication written from right to left as for $d = 2$). With this definition of $\Gamma(T)$ the *tiling cocycle* $c_T\colon \mathbb{Z}^d \times W_T \longmapsto \Gamma(T)$ for the shift-action σ of $\mathbb{Z}^d$ on the Wang shift W_T is defined as in (4.5) by setting, for every $w \in W_T$ and $i = 1,\dots,d$, $c_T(\mathbf{e}^{(i)}, w)$ equal to the 'colour' of the edge $\mathbf{e}^{(i)}$ of the tile $w_{\mathbf{0}}$ in w covering $[0,1]^d$. With this definition one can easily prove the analogue of Theorem 4.2 for every $d \geq 2$.

Since all explicit examples of Wang tilings in this paper are in dimension two we leave the details for $d > 2$ to the reader.

Let $d \geq 1$, A a finite set, and let $X \subset A^{\mathbb{Z}^d}$ be a *SFT*. If G is a discrete group, $r \geq 0$ and $c\colon \mathbb{Z}^d \times X \longmapsto G$ a continuous cocycle we say that *c depends only on the coordinates in* $\mathbf{B}(r)$ if the map $c(\mathbf{e}^{(i)}, \cdot)\colon X \longmapsto G$ depends, for every $i = 1,\dots,d$, only on the coordinates in $\mathbf{B}(r)$, i.e. if

$$\begin{gathered} c(\mathbf{e}^{(i)}, x) = c(\mathbf{e}^{(i)}, x'), \;\; i = 1,\dots,d, \\ \text{whenever } \; x, x' \in X \; \text{ and } \; \pi_{\mathbf{B}(r)}(x) = \pi_{\mathbf{B}(r)}(x'). \end{gathered} \tag{4.9}$$

The following corollary of Theorem 4.2 shows that X has a fundamental cocycle $c^*\colon \mathbb{Z}^d \times X \longmapsto G^*$ if and only if all continuous cocycles c on X with values in discrete groups depend essentially only on a fixed range of coordinates.

Corollary 4.4. *Let $d \geq 1$, A a finite set, and let $X \subset A^{\mathbb{Z}^d}$ be a* SFT. *The following conditions are equivalent.*

(1) *There exists a continuous cocycle $c^*\colon \mathbb{Z}^d \times X \longmapsto G^*$ with values in a discrete group G^* which is fundamental in the sense of Definition* 2.3*;*
(2) *There exists an integer $r \geq 0$ with the following property: if G is a discrete group and $c\colon \mathbb{Z}^d \times X \longmapsto G$ a continuous cocycle, then c is continuously cohomologous to a cocycle $c'\colon \mathbb{Z}^d \times X \longmapsto G$ which depends only on the coordinates in $\mathbf{B}(r)$ in the sense of* (4.9).

Proof. Suppose that there exists a fundamental cocycle $c^*\colon \mathbb{Z}^d \times X \longmapsto G^*$. Since G^* is discrete we can find an integer $r \geq 0$ such that c^* only depends on the coordinates in $\mathbf{B}(r)$, and (2) follows from Definition 2.3. Conversely, if there exists an $r \geq 0$ satisfying (2), then the proof of Theorem 4.2 and, if $d > 2$, Remark 4.3, show that every continuous cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ with values in a discrete group G is continuously cohomologous to a homomorphic image of the tiling cocycle $c_{T_X^{(r+1)}}\colon \mathbb{Z}^d \times X \longmapsto \Gamma(T_X^{(r+1)})$. According to Definition 2.3 this means that $c_{T_X^{(r+1)}}$ is fundamental. □

5. Cocycles, extensions of configurations, and fundamental groups

As mentioned in the introduction, tiling groups were originally introduced as obstructions to extending certain partial tilings of $\mathbb{R}^2$ by polygonal tiles (cf. [5] and [30]). In order to describe a simple version of this extension problem for *SFT*'s we assume that A is a finite set and $X \subset A^{\mathbb{Z}^2}$ is a *SFT* satisfying (2.2)–(2.3).

Let $E \subset \mathbb{Z}^2$ be a finite set, and let $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus E) \subset A^{\mathbb{Z}^2 \smallsetminus E}$ be an allowed element (cf. Definition 2.1). How can one tell whether z has an allowed extension to all of $\mathbb{Z}^2$, i.e. whether there exists an element $x \in X$ with $\pi_{\mathbb{Z}^2 \smallsetminus E}(x) = z$? A weaker form of this question is whether there exists a finite set $E' \supset E$ and an element $x \in X$ with $\pi_{\mathbb{Z}^2 \smallsetminus E'}(x) = \pi_{\mathbb{Z}^2 \smallsetminus E'}(z)$, i.e. whether z can be extended after possibly 'correcting' finitely many coordinates.

For simplicity we call an allowed element $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus E)$ *weakly extensible* if there exists an $x \in X$ and a finite set E' with $E \subset E' \subset \mathbb{Z}^2$ with $\pi_{\mathbb{Z}^2 \smallsetminus E'}(x) = \pi_{\mathbb{Z}^2 \smallsetminus E'}(z)$, and *extensible* if we can assume in addition that $E' = E$.

We shall concentrate on weak extensibility and may therefore take $E \subset \mathbb{Z}^2$ to be a finite rectangle. In order to avoid certain technical complications we also assume that X has the extension property (Definition 2.1).

Let $c\colon \mathbb{Z}^2 \times X \longmapsto G$ be a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on X with values in a discrete group G and put

$$r_c = \max(r(c((1,0),\cdot)), r(c((0,1),\cdot)))$$

(cf. (3.4)). Suppose furthermore that $\mathbf{n}^{(0)}, \mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)} = \mathbf{n}^{(0)}$ are the vertices of a rectangular loop $\mathfrak{P}$ in $\mathbb{Z}^2$ with the following properties:

(a) $\mathbf{n}^{(1)} - \mathbf{n}^{(0)}$ and $\mathbf{n}^{(3)} - \mathbf{n}^{(2)}$ are multiples of $(1,0)$, and $\mathbf{n}^{(2)} - \mathbf{n}^{(1)}$ and $\mathbf{n}^{(3)} - \mathbf{n}^{(0)}$ multiples of $(0,1)$,

(b) if $\bar{r}$ is the integer appearing in Definition 2.1 (2), then each edge $\mathbf{n}^{(j-1)} + t(\mathbf{n}^{(j)} - \mathbf{n}^{(j-1)})$, $j = 1, \dots, 4$, of $\mathfrak{P}$ has distance $\geq \max(r_c, \bar{r})$ from the rectangle Q. In other words,

$$\begin{aligned} \delta(\mathfrak{P}, Q) &= \min_{\mathbf{m} \in Q} \min_{j=1,\dots,4} \min_{0 \leq t \leq 1} \|\mathbf{m} - \mathbf{n}^{(j-1)} + t(\mathbf{n}^{(j)} - \mathbf{n}^{(j-1)})\| \\ &\geq \max(r_c, \bar{r}) \end{aligned} \tag{5.1}$$

with $\|\mathbf{k}\| = \max(|k_1|, |k_2|)$ for every $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$.

Under these assumptions the extension property of X guarantees that each of the values $c(\mathbf{n}^{(j)} - \mathbf{n}^{(j-1)}, \sigma^{\mathbf{n}^{(j-1)}}(z))$, $j = 1, \dots, 4$, is well defined, and we set

$$\begin{aligned} c(\mathfrak{P}, z) = c(\mathbf{n}^{(4)} - \mathbf{n}^{(3)}, \sigma^{\mathbf{n}^{(3)}}(z)) c(\mathbf{n}^{(3)} - \mathbf{n}^{(2)}, \sigma^{\mathbf{n}^{(2)}}(z)) \\ \cdot c(\mathbf{n}^{(2)} - \mathbf{n}^{(1)}, \sigma^{\mathbf{n}^{(1)}}(z)) c(\mathbf{n}^{(1)} - \mathbf{n}^{(0)}, \sigma^{\mathbf{n}^{(0)}}(z)). \end{aligned} \tag{5.2}$$

The following proposition is an immediate consequence of the cocycle equation (2.4) and the extension property of X.

Lemma 5.1. *Let $Q \subset \mathbb{Z}^2$ be a finite rectangle, and let $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus Q)$. For every rectangular loop $\mathfrak{P} \subset \mathbb{Z}^2$ satisfying* (a)–(b) *which does not contain Q in its interior we have that $c(\mathfrak{P}, z) = 1_G$. If $\mathfrak{P}$ and $\mathfrak{P}'$ are rectangular loops satisfying* (a)–(b) *which contain the rectangle Q in their interior, then $c(\mathfrak{P}, z) = c(\mathfrak{P}', z)$. Finally, if z is weakly extensible, then*

$$c(\mathfrak{P}, z) = 1_G. \tag{5.3}$$

for every rectangular loop $\mathfrak{P} \subset \mathbb{Z}^2$ satisfying (a)–(b).

In view of the the last assertion in Lemma 5.1 we set

$$c(Q, z) = c(\mathfrak{P}, z) \tag{5.4}$$

for any rectangular loop $\mathfrak{P} \subset \mathbb{Z}^2$ which satisfies (a)–(b) and contains Q in its interior. If $c = c_T$ is the tiling cocycle of a Wang shift W_T representing X, then there exists a rectangular hole $Q' \supset Q$ such that the allowed element $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus Q)$ determines uniquely a partial tiling z' of $\mathbb{R}^2 \smallsetminus Q'$, and the value $c_T(\mathfrak{P}, z) \in \Gamma_0(T) \subset \Gamma(T)$ for any rectangular loop $\mathfrak{P}$ around the hole Q' is equal to the word in the tiling group formed by the product of the colours on the edges of the tiles touching Q' in anti-clockwise direction starting from the bottom left hand corner of Q' (written from right to left, and with the colours along the top and left edges of Q' inverted — cf. (4.4)). This is the form in which the obstruction $c(Q, z)$ appears in [5] and [30]. The following statement clarifies the dependence of $c(Q, z)$ on the cocycle $c\colon \mathbb{Z}^2 \times X \longmapsto G$.

change

Proposition 5.2. *Let A be a finite set, $X \subset A^{\mathbb{Z}^2}$ a* SFT *of the form* (2.2)–(2.3) *with the extension property, $Q \subset \mathbb{Z}^2$ a finite rectangle, $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus Q)$ an allowed configuration, and $c\colon \mathbb{Z}^2 \times X \longmapsto G$ a continuous cocycle with values in a discrete group G.*

(1) *If* $c' \colon \mathbb{Z}^2 \times X \longmapsto G$ *is a second continuous cocycle which is continuously cohomologous to* c *then* $c(Q,z) = h^{-1}c'(Q,z)h$ *for some* $h \in G$*;*
(2) *If* c *is trivial then* $c(Q,z) = 1_G$*;*
(3) *If the cocycle* c *is fundamental* (*Definition* 2.3)*, and if* $c(Q,z) = 1_G$*, then* $c'(Q,z) = 1_{G'}$ *for every continuous cocycle* $c' \colon \mathbb{Z}^2 \times X \longmapsto G'$ *with values in a discrete group* G'*.*

Proof. The assertion (1) is clear from the definition of cohomology, (2) follows from (1) and triviality, and (3) from (1) and Definition 2.3. □

By varying the representation of a *SFT* X as a Wang system W_T we may obtain different obstructions to the weak extensibility of a given allowed configuration $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus Q)$. If the tiling cocycle c_T is trivial, then $c_T(\mathfrak{P}, z) = 1_G$, irrespective of whether z is weakly extensible or not: for example, the tiling (7.1) leads to a representation of the *SFT* $X^{(3)}$ with a trivial tiling cocycle, whereas the representation of $X^{(3)}$ in terms of (7.3) has a nontrivial tiling cocycle. The representations of a *SFT* X by the Wang systems $W_{T_X^{(n)}}$, $n \geq 1$, in (4.7)–(4.8) yields infinitely many different tiling cocycles, each of which may conceivably give a different obstruction to our extension problem for elements $z \in \Pi_X(\mathbb{Z}^2 \smallsetminus Q)$. However, if there exists a fundamental cocycle $c^* \colon \mathbb{Z}^2 \times X \longmapsto G^*$, then c^* contains all the available cohomological information about the weak extensibility of z. In
change some examples the fundamental cocycle can even be used to find sufficient conditions for z to be extensible ([5], [30]). In general, however, the cohomological obstructions discussed here only lead to necessary, but not to sufficient conditions, as Example 5.3 below shows.

Example 5.3. Let W_{T_L} be the Wang shift defined by the tiles

$$T_L = \{ \quad \}$$


in (9.1). From the description of W_{T_L} at the beginning of Section 9 we know that $W_{T_L} \subset W_{T_I}$ is the set of all Wang tilings which do not contain a copy of the tile , and Theorem 9.1 shows that W_{T_L} has a fundamental cocycle c^* which is the restriction to $W_{T_L} \subset W_{T_I}$ of the fundamental cocycle c_{T_I} of the square ice shift W_{T_I} in (8.1).

If we can find a rectangle $Q \subset \mathbb{Z}^2$ and a partial tiling $z \in \Pi_{W_L}(\mathbb{Z}^2 \smallsetminus Q) \subset \Pi_{W_I}(\mathbb{Z}^2 \smallsetminus Q)$ which is weakly extensible in W_{T_I}, but not in W_{T_L}, then

$$c_{T_L}(\mathfrak{P}, z) = c_{T_I}(\mathfrak{P}, z) = 1_{\Gamma(T_I)} = 1_{\Gamma(T_L)}$$

for every appropriate rectangular loop $\mathfrak{P} \subset \mathbb{Z}^2$, so that the cohomological information available is insufficient to determine whether $z \in \Pi_{W_{T_L}}(\mathbb{Z}^2 \smallsetminus Q)$ can be extended weakly in W_{T_L}. The following picture describes such a rectangle Q and a point $z \in \Pi_{W_{T_L}}(\mathbb{Z}^2 \smallsetminus Q)$ which can be extended in W_{T_I}, but not even weakly in W_{T_L}.

Consider the allowed tiling in Figure 1 of the set $E = [-3,4]^2 \smallsetminus (0,1)^2 \subset \mathbb{R}^2$ by the Wang tiles T_L, with the open square $(0,1)^2$ shaded grey for convenience.

We define an allowed tiling z of $\mathbb{R}^2 \smallsetminus (0,1)^2$ with the tiles T_L by extending the tiling in Figure 1 to $\mathbb{R}^2 \smallsetminus (0,1)^2$ by first tiling the rectangles

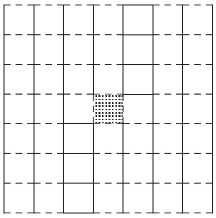


FIGURE 1. An allowed tiling of E

$[-3,4]\times[4,\infty)$ and $[-3,4]\times(-\infty,-3]$ with copies of the top and bottom rows and in Figure 1, respectively, and then tiling the rest of $\mathbb{R}^2 \smallsetminus (0,1)^2$ with copies of the tile . A little bit of thought reveals that, for any finite, open rectangle Q with integral vertices and $(0,1)^2 \subset Q \subset \mathbb{R}^2$, any extension of $\pi_{\mathbb{R}^2 \smallsetminus Q}(z)$ to a tiling $x \in W_{T_I}$ has to contain a copy of the tile , so that z cannot be extended weakly to a point in W_T.

We leave the extension problem and turn to the connection between the cohomology of a *SFT* X and its *projective fundamental group* $\pi_1^{\mathrm{proj}}(X,*)$ introduced in [7]. Let A be a finite set, and let $X \subset A^{\mathbb{Z}^2}$ be a *SFT* satisfying (2.2)–(2.3). For every $r \geq 1$ we consider the Wang tiles $T_X^{(r)}$ defined in (4.7)–(4.8) and observe that there exists, for every $r \geq 2$, a well defined, surjective homomorphism $\eta^{(r)}\colon \Gamma(T_X^{(r)}) \longmapsto \Gamma(T_X^{(r-1)})$ of the corresponding tiling groups with

$$\begin{aligned} \eta^{(r)}(\pi_{\mathbf{B}(r)_\mathsf{b}}(x)) &= \pi_{\mathbf{B}(r-1)_\mathsf{b}}(x), & \eta^{(r)}(\pi_{\mathbf{B}(r)_\mathsf{l}}(x)) &= \pi_{\mathbf{B}(r-1)_\mathsf{l}}(x), \\ \eta^{(r)}(\pi_{\mathbf{B}(r)_\mathsf{t}}(x)) &= \pi_{\mathbf{B}(r-1)_\mathsf{t}}(x), & \eta^{(r)}(\pi_{\mathbf{B}(r)_\mathsf{r}}(x)) &= \pi_{\mathbf{B}(r-1)_\mathsf{r}}(x) \end{aligned} \tag{5.5}$$

and

$$\eta^{(r)}(c_{T_X^{(r)}}(\mathbf{n},x)) = c_{T_X^{(r-1)}}(\mathbf{n},x) \tag{5.6}$$

for every $x \in X$ and $\mathbf{n} \in \mathbb{Z}^2$. The sequence of homomorphisms $\eta^{(r)}$, $r \geq 2$, in (5.5)–(5.6) allows us to define the *projective tiling group* $\Gamma(X)$ and the *projective tiling cocycle* $c_X\colon \mathbb{Z}^2 \times X \longmapsto \Gamma(X)$ as the projective limits

$$\begin{gathered} \Gamma(X) = \operatorname*{proj\,lim}_{r\to\infty} \Gamma(T_X^{(r)}) \subset \prod_{r\geq 1} \Gamma(T_X^{(r)}), \\ c_X(\mathbf{n},x) = \operatorname*{proj\,lim}_{r\to\infty} c_{T_X^{(r)}}(\mathbf{n},x). \end{gathered} \tag{5.7}$$

If we furnish $\Gamma(X)$ with the product topology inherited from $\prod_{r\geq 1}\Gamma(T_X^{(r)})$, then the cocycle $c_X\colon \mathbb{Z}^2 \times X \longmapsto \Gamma(X)$ is continuous (note, however, that $\Gamma(X)$ is not discrete).

Let $c\colon \mathbb{Z}^2 \times X \longmapsto G$ be a continuous cocycle with values in a discrete group G and set

$$r_c = \max(r(c((1,0),\cdot)), r(c((0,1),\cdot)))$$

as in (3.4). Assume that $\mathfrak{P} = (\mathbf{0} = \mathbf{n}^{(0)}, \dots, \mathfrak{n}^{(k)} = \mathbf{0})$ is a finite sequence of points in $\mathbb{Z}^2$ with

$$\mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} \in \{\pm(1,0), \pm(0,1)\} \tag{5.8}$$

for every $j = 1, \dots, k$. We fix an element $\bar{x} \in X$ and an integer $s \geq r_c$ and choose, for every $j = 0, \dots, k$, a tile $\tau^{(j)} \in T_X^{(s)}$ such that $\tau^{(0)} = \tau^{(k)} = \pi_{\mathbf{B}(r)}(\bar{x})$ and the sequence $\mathfrak{T} = (\tau^{(0)}, \dots, \tau^{(k)})$ is consistent with $\mathfrak{P}$ in the sense that

$$\begin{aligned} \mathsf{r}(\tau^{(j-1)}) &= \mathsf{l}(\tau^{(j)}) \text{ if } \mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} = (1,0), \\ \mathsf{t}(\tau^{(j-1)}) &= \mathsf{b}(\tau^{(j)}) \text{ if } \mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} = (0,1), \\ \mathsf{l}(\tau^{(j-1)}) &= \mathsf{r}(\tau^{(j)}) \text{ if } \mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} = -(1,0), \\ \mathsf{b}(\tau^{(j-1)}) &= \mathsf{t}(\tau^{(j)}) \text{ if } \mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} = -(0,1). \end{aligned} \tag{5.9}$$

We regard the pair $(\mathfrak{P}, \mathfrak{T})$ as a path in $\mathbb{Z}^2 \times T_X^{(s)}$ beginning and ending in the point $(\mathbf{0}, \pi_{\mathbf{B}(s)}(\bar{x}))$, choose elements $x^{(j)} \in X$ with $\pi_{E^{(s)}}(x^{(j)}) = \tau^{(j)}$ for $j = 0, \dots, k$, and set

$$c(\mathfrak{P}, \mathfrak{T}) = g^{(k-1)} \cdots g^{(0)},$$

where

$$g^{(j)} = \begin{cases} c(\mathbf{n}^{(j+1)} - \mathbf{n}^{(j)}, x^{(j)}) & \text{if } \mathbf{n}^{(j+1)} - \mathbf{n}^{(j)} \in \{(1,0),(0,1)\} \\ c(\mathbf{n}^{(j)} - \mathbf{n}^{(j+1)}, x^{(j+1)})^{-1} & \text{if } \mathbf{n}^{(j+1)} - \mathbf{n}^{(j)} \in \{-(1,0),-(0,1)\} \end{cases}$$

for $j = 0, \dots, k-1$.

Following [7] one can define a natural notion of 'homotopy' between paths in $\mathbb{Z}^2 \times T_X^{(s)}$ which begin and end in $(\mathbf{0}, \pi_{\mathbf{B}(s)}(\bar{x}))$. As a consequence of that definition and the cocycle equation (2.4) one obtains that

$$c(\mathfrak{P}, \mathfrak{T}) = c(\mathfrak{P}', \mathfrak{T}')$$

whenever $(\mathfrak{P}, \mathfrak{T})$ and $(\mathfrak{P}', \mathfrak{T}')$ are paths in $\mathbb{Z}^2 \times T_X^{(s)}$ which are homotopic in this sense.

By varying $(\mathfrak{P}, \mathfrak{T})$ over the set of all finite paths of arbitrary length in $\mathbb{Z}^2 \times T_X^{(s)}$ which begin and end in $(\mathbf{0}, \pi_{\mathbf{B}(s)}(\bar{x}))$ we obtain a subgroup

$$\begin{aligned} \Gamma_{\bar{x}}^{(s)}(c) = \{c(\mathfrak{P}, \mathfrak{T}) : k \geq 0,\ (\mathfrak{P}, \mathfrak{T}) = ((\mathbf{n}^{(j)}, \tau^{(j)}),\ j = 0, \dots, k) \\ \text{and } \mathbf{n}^{(0)} = \mathbf{n}^{(k)} = \mathbf{0},\ \tau^{(0)} = \tau^{(k)} = \pi_{E^{(s)}}(\bar{x})\} \subset G, \end{aligned}$$

and we set

$$\Gamma_{\bar{x}}(c) = \bigcap_{s \geq r_c} \Gamma_{\bar{x}}^{(s)}(c) \subset G.$$

The following proposition is an immediate consequence of the definition of the subgroup $\Gamma_{\bar{x}}(c) \subset G$.

Proposition 5.4. *Let A be a finite set, $X \subset A^{\mathbb{Z}^2}$ a* SFT *of the form* (2.2)–(2.3)*, and let $\bar{x} \in X$. If $c, c' \colon \mathbb{Z}^2 \longmapsto G$ are cohomologous continuous cocycles with values in a discrete group G, then there exists an element $h \in G$ with*

$$\Gamma_{\bar{x}}(c) = h^{-1} \Gamma_{\bar{x}}(c') h.$$

In particular, if c is trivial, then $\Gamma_{\bar{x}}(c) = \{1_G\}$.

If there exists a fundamental cocycle $c^* \colon \mathbb{Z}^2 \times X \longmapsto G^*$ on X, then $\Gamma_{\bar{x}}(c^*)$ determines not only the group $\Gamma_{\bar{x}}(c)$ for *every* continuous cocycle c, but also the projective fundamental group $\pi_1^{\mathrm{proj}}(X, \bar{x})$ of X defined in [7].

Theorem 5.5. *Let A be a finite set, and let $X \subset A^{\mathbb{Z}^2}$ a* SFT *of the form* (2.2)–(2.3). *If there exists a fundamental cocycle $c^* \colon \mathbb{Z}^2 \times X \longmapsto G^*$ with values in a discrete group G^* then $\Gamma_{\bar{x}}(c)$ is, for every continuous cocycle $c \colon \mathbb{Z}^2 \times X \longmapsto G$ with values in a discrete group G and every $\bar{x} \in X$, a homomorphic image of $\Gamma_{\bar{x}}(c^*)$, and*

$$\pi_1^{\mathrm{proj}}(X, \bar{x}) \cong \Gamma_{\bar{x}}(c^*).$$

Proof. We fix a point $\bar{x} \in X$. If $c \colon \mathbb{Z}^2 \times X \longmapsto G$ is a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on X then there exists, by definition, a group homomorphism $\eta \colon G^* \longmapsto G$ such that the cocycle $\eta \cdot c^* \colon \mathbb{Z}^2 \times X \longmapsto G$ is cohomologous to c. For every sufficiently large s and every path $(\mathfrak{P}, \mathfrak{T})$ in $\mathbb{Z}^2 \times T_X^{(s)}$ of the form (5.8)–(5.9) which begins and ends in $(\mathbf{0}, \pi_{\mathbf{B}(s)}(\bar{x}))$,

$$\eta(c^*(\mathfrak{P}, \mathfrak{T})) = c(\mathfrak{P}, \mathfrak{T}),$$

so that

$$\Gamma_{\bar{x}}(c) = \eta(\Gamma_{\bar{x}}(c^*)),$$

as claimed.

If we apply these observations to the tiling cocycles $c_{T_X^{(r)}} \colon \mathbb{Z}^2 \times X \longmapsto \Gamma(T_X^{(r)})$ in (4.7)–(4.8) we obtain, for every $r \geq 1$, a homomorphism $\theta^{(r)} \colon G^* \longmapsto \Gamma(T_X^{(r)})$ such that $\theta^{(r)} \cdot c^*$ is cohomologous to $c_{T_X^{(r)}}$ and

$$\theta^{(r)}(c^*) \cong \Gamma_{\bar{x}}(c_{T_X^{(r)}}).$$

Conversely, if r is sufficiently large, then (4.6) yields a group homomorphism $\phi^{(r)} \colon \Gamma(T_X^{(r)}) \longmapsto G^*$ with $\phi^{(r)} \cdot c_{T_X^{(r)}} = c^*$ and

$$\phi^{(r)}(c_{T_X^{(r)}}(\mathfrak{P}, \mathfrak{T})) = c^*(\mathfrak{P}, \mathfrak{T})$$

for every appropriate loop $(\mathfrak{P}, \mathfrak{T})$ in $\mathbb{Z}^2 \times T_X^{(s)}$ with s sufficiently large, and hence with

$$\phi^{(r)}(\Gamma_{\bar{x}}(c_{T_X^{(s)}})) = \Gamma_{\bar{x}}(c^*).$$

Finally, if $\eta^{(r)} \colon \Gamma(T_X^{(r)}) \longmapsto \Gamma(T_X^{(r-1)})$ are the group homomorphisms appearing in (5.5)–(5.6), then

$$\eta^{(r)}(c_{T_X^{(r)}}(\mathfrak{P}, \mathfrak{T})) = c_{T_X^{(r-1)}}(\mathfrak{P}, \mathfrak{T})$$

for every appropriate loop $(\mathfrak{P}, \mathfrak{T})$ in $\mathbb{Z}^2 \times T_X^{(r)}$, and

$$\eta^{(r)}(\Gamma_{\bar{x}}(c_{T_X^{(r)}})) = \Gamma_{\bar{x}}(c_{T_X^{(r-1)}}).$$

From the definitions of these homomorphisms it is now clear that

$$\Gamma_{\bar{x}}(c^*) \cong \pi_1^{\mathrm{proj}}(X, \bar{x}).$$

□

Remark 5.6. Let A be a finite set and $X \subset A^{\mathbb{Z}^2}$ a *SFT* of the form (2.2)–(2.3). Then X is *projectively connected* if there exists, for every $r \geq 1$ and $\tau, \tau' \in T_X^{(r)}$, a finite 'path' $(\mathfrak{P}, \mathfrak{T})$ in $\mathbb{Z}^2 \times T_X^{(r)}$ which begins in $(\mathbf{n}^{(0)}, \tau^{(0)}) = (\mathbf{0}, \tau)$, ends in $(\mathbf{n}^{(k)}, \tau^{(k)}) = (\mathbf{n}^{(k)}, \tau')$, and satisfies the conditions (5.8)–(5.9) above (cf. [7]).

Note that every topologically mixing *SFT* $X \subset A^{\mathbb{Z}^2}$ is projectively connected. If X is projectively connected and $c\colon \mathbb{Z}^2 \times X \longmapsto G$ a continuous cocycle, then $\Gamma_{\bar{x}}(c)$, $\Gamma_{\bar{x}}(c^*)$ and $\pi_1^{\mathrm{proj}}(X, \bar{x})$ do not depend on the specific point $\bar{x} \in X$ (at least up to isomorphism), so that we can suppress the reference to $\bar{x}$ and write $\Gamma_*(c)$, $\Gamma_*(c^*)$ and $\pi_1^{\mathrm{proj}}(X, *)$ for these groups.

Definition 5.7. Let $d \geq 2$ and A a finite set. A *SFT* $X \subset A^{\mathbb{Z}^d}$ is *cohomologically trivial* if every continuous cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ for the shift-action σ of $\mathbb{Z}^d$ with values in a discrete group G is trivial.

Corollary 5.8 (of Proposition 5.4 and Theorem 5.5)**.** *Let A be a finite set and $X \subset A^{\mathbb{Z}^2}$ a mixing* SFT *of the form* (2.2)–(2.3)*. If X is cohomologically trivial then*

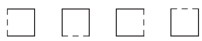
$$\Gamma_*(X) = \pi_1^{\mathrm{proj}}(X, *) = \{1\}.$$

Remarks 5.9. (1) If X is a mixing two-dimensional *SFT* with an explicitly determined fundamental cocycle $c^*\colon \mathbb{Z}^2 \times X \longmapsto G^*$ of X, then the calculation of $\pi_1^{\mathrm{proj}}(X, *) \cong \Gamma_*(c^*)$ is usually quite straightforward (cf. the Remarks 6.10, 7.6, 8.6 and 9.2).

(2) For a d-dimensional *SFT* $X \subset A^{\mathbb{Z}^d}$ with $d > 2$ the definitions of the projective tiling group $\Gamma(X)$ and the projective tiling cocycle $c_X\colon \mathbb{Z}^d \times X \longmapsto \Gamma(X)$ are completely analogous (cf. Remark 4.3). Furthermore, if $c\colon \mathbb{Z}^d \times X \longmapsto G$ is a continuous cocycle for the shift-action σ of $\mathbb{Z}^d$ on X with values in a discrete group G, then one can again define the group $\Gamma_{\bar{x}}(c) \subset G$ for every $\bar{x} \in X$ and obtains exact analogues of Proposition 5.4 and Theorem 5.5. The (obvious) details are left to the reader.

6. Dominoes

Let T_D be the set of Wang tiles



(6.1)

with the colours $\mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v}$ on the solid horizontal, broken horizontal, solid vertical and broken vertical edges. The Wang shift $X = W_{T_D} \subset T_D^{\mathbb{Z}^2}$ has the periodic extension property (cf. Section 4, [13], [29], [3], [9] and Example 4.2 in [25]). Every tile in T_D can be viewed als a half-domino, and by deleting all broken edges one sees that every element of X determines a tiling of the plane by dominoes — hence the name of this *SFT*.

Proposition 6.1. *Let*

$$\Gamma = \Gamma(T_D) = \langle \mathsf{h}, \mathsf{H}, \mathsf{v}, \mathsf{V} | \mathsf{HV} = \mathsf{vH}, \mathsf{hV} = \mathsf{VH}, \mathsf{Hv} = \mathsf{VH}, \mathsf{HV} = \mathsf{Vh} \rangle \tag{6.2}$$

change *be the tiling group of T_D (cf. Section 4). Then*

$$\Gamma = M_3 \times \mathbb{Z}^2,$$

where

$$M_3 = \Bigl\{ \begin{pmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ 0 & 1 & a_{(2,3)} \\ 0 & 0 & a_{(3,3)} \end{pmatrix} : a_{(1,3)} \in \mathbb{Z}, a_{(1,1)}, a_{(3,3)} \in \{1, -1\},$$

$$* \qquad a_{(1,2)}, a_{(2,3)}, a_{(1,1)} + a_{(1,2)}, a_{(3,3)} + a_{(2,3)} \in \{0, 1\} \Bigr\}.$$

Proof. Let Z be the centre of Γ. A simple calculation shows that $\mathsf{v}^2 = \mathsf{V}^2 \in Z$ and $\mathsf{h}^2 = \mathsf{H}^2 \in Z$. Denote by $C = \langle \mathsf{h}^2, \mathsf{v}^2 \rangle$ the central subgroup of Γ generated by h^2 and v^2, put $\Gamma' = \Gamma/C$, and write $\gamma' \in \Gamma'$ for the image under the quotient map $\Gamma \longmapsto \Gamma'$ of an element $\gamma \in \Gamma$. Then $\mathsf{h}'^2 = \mathsf{H}'^2 = \mathsf{v}'^2 = \mathsf{V}'^2 = 1_{\Gamma'}$, and one can check directly that there exists a surjective group isomorphism $\xi\colon \Gamma' \longmapsto M_3$ with change

$$\begin{aligned} \xi(\mathsf{h}') &= \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \xi(\mathsf{H}') &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \xi(\mathsf{v}') &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, & \xi(\mathsf{V}') &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{6.3}$$

and

$$\xi(\mathsf{H}')\xi(\mathsf{h}') = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \xi(\mathsf{v}')\xi(\mathsf{V}'). \tag{6.4}$$

From (6.2)–(6.4) one also obtains that the the group $C' = \{(\mathsf{h}'\mathsf{H}')^n : n \in \mathbb{Z}\} \subset \Gamma'$ is normal and that $C' \cong \xi(C') \cong \mathbb{Z}$. We set $\Gamma'' = \Gamma'/C'$ and denote by $\mathsf{h}'', \mathsf{H}'', \mathsf{v}'', \mathsf{V}''$ the images of $\mathsf{h}', \mathsf{H}', \mathsf{v}', \mathsf{V}'$ under the quotient map $\Gamma' \longmapsto \Gamma''$. Then $\mathsf{h}'' = \mathsf{H}''$, $\mathsf{v}'' = \mathsf{V}''$, and the relations occurring in the definition of Γ imply that $\Gamma'' \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Since the same is true for $\xi(\Gamma')/\xi(C')$ we conclude that ξ is faithful, as claimed.

As the centre of $M_3 = \xi(\Gamma')$ is trivial, the same is true for Γ'. Hence $C = Z$, and (4.3)–(4.4) imply that $\Gamma_0 \cong M_3$ and $\Gamma \cong M_3 \times \mathbb{Z}^2$. □ change

We define the tiling cocycle $c_{T_D}\colon \mathbb{Z}^2 \times X \longmapsto \Gamma$ by (4.5), identify $\Gamma_0(T_D)$ change
with M_3 (cf. (4.4)), and denote by $\bar{c}_{T_D} = \xi \cdot c_{T_D} \colon \mathbb{Z}^2 \times X \longmapsto M_3 = \Gamma_0(T_D)$ the composition of c_{T_D} with the group homomorphism $\xi\colon \Gamma \longmapsto M_3$ in (6.3). Then

$$\begin{aligned} c_{T_D}((1,0),x) &= \xi(\mathsf{b}(x_{\mathbf{0}})) \times (1,0), & c_{T_D}((0,1),x) &= \xi(\mathsf{l}(x_{\mathbf{0}})) \times (0,1), \\ \bar{c}_{T_D}((1,0),x) &= \xi(\mathsf{b}(x_{\mathbf{0}})), & \bar{c}_{T_D}((0,1),x) &= \xi(\mathsf{l}(x_{\mathbf{0}})), \end{aligned} \tag{6.5}$$

where $\mathsf{b}(x_{\mathbf{0}}) \in \{\mathsf{h}, \mathsf{H}\}$ and $\mathsf{l}(x_{\mathbf{0}}) \in \{\mathsf{v}, \mathsf{V}\}$ are the colours on the bottom and left edge of the tile $x_{\mathbf{0}}$.

Lemma 6.2. *The cocycle $\bar{c}_{T_D}$ has the property that*

$$\bar{c}_{T_D}(2\mathbf{m}, x) \in M_3' = \Bigl\{ \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \Bigr\} \cong \mathbb{Z} \tag{6.6}$$

for every $\mathbf{m} \in \mathbb{Z}^2$.

Proof. An elementary calculation shows that

$$\bar{c}_{T_D}((2,0),x) \in M_3', \qquad \bar{c}_{T_D}((0,2),x) \in M_3'$$

for every $x \in X$, and the proof is completed by applying (2.4). □

In view of Lemma 6.2 we define a cocycle $\gamma\colon \mathbb{Z}^2 \times X \longmapsto \mathbb{Z}$ for the even shift-action $\mathbf{m} \mapsto \bar{\sigma}^{\mathbf{m}} = \sigma^{2\mathbf{m}}$ of $\mathbb{Z}^2$ on X by

$$\bar{c}_{T_D}(2\mathbf{n}, x) = \left(\begin{smallmatrix} 1 & 0 & \gamma(\mathbf{n},x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \tag{6.7}$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$ (cf. (6.6)).

Proposition 6.3. *The skew-product action $\bar{\sigma}_{(\gamma)}$ of $\mathbb{Z}^2$ on $X \times \mathbb{Z}$, defined by*

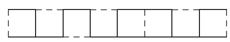
$$\bar{\sigma}_{(\gamma)}^{\mathbf{m}}(x,k) = (\bar{\sigma}^{\mathbf{m}}(x), \gamma(\mathbf{m}, x) + k)$$

for every $\mathbf{m} \in \mathbb{Z}^2$, $k \in \mathbb{Z}$ and $x \in X$, is topologically mixing.

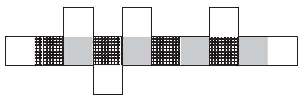
Proof. For every $m \geq 1$ and $x \in X \subset A^{\mathbb{Z}^2}$, the value $\gamma((m,0),x)$ only depends on the bottom edges of the coordinates $(x_{(0,0)}, \dots, x_{(2m-1,0)})$ of x (cf. (6.1)), and has the following geometrical interpretation: the coordinates $(x_{(0,0)}, \dots, x_{(2m-1,0)})$ determine a tiling of the rectangle $R = [0, 2m] \times [0,1] \subset \mathbb{R}^2$ by the half-dominoes (6.1), and we complete these half-dominoes to dominoes. Denote by $e(2m,x)$ and $o(2m,x)$ the numbers of squares $[0,1]^2 + (k,0)$, $0 \leq k \leq 2m-1$ in R with k even or odd, which are covered by top halves of vertical dominoes. Then

$$\gamma((m,0),x) = o(2m,x) - e(2m,x).$$

The following picture illustrates this fact: let $m = 4$ and

$(x_{(0,0)}, \dots, x_{(7,0)}) =$  ,

complete these half-dominoes to dominoes, and colour the even and odd unit squares in the rectangle R dark and light grey for better visibility. We obtain the picture



and observe that $\gamma((4,0),x) = o(8,x) - e(8,x) = 0 - 1 = -1$.

The value of $\gamma((0,m),x)$, which depends on the coordinates $(x_{(0,0)}, \dots, x_{(0,2m-1)})$ of x, is obtained by completing the half-dominoes in the corresponding tiling of the rectangle $R' = [0,1] \times [0,2m]$ to dominoes, and by taking the difference between the numbers of even and odd squares in R' which form right halves of horizontal dominoes.

In order to prove that $\bar{\sigma}^{\gamma}$ is topologically mixing it suffices to show the following: if $\mathcal{O}_1, \mathcal{O}_2$ are nonempty cylinder sets in X and $k \in \mathbb{Z}$, then

$$\mathcal{O}_1 \cap \bar{\sigma}^{-\mathbf{m}}(\mathcal{O}_2) \cap \{x \in X : \gamma(\mathbf{m}, x) = k\} \neq \varnothing \tag{6.8}$$

for all $\mathbf{m} \in \mathbb{Z}^2$ with $\|\mathbf{m}\| = \max\{|m_1|, |m_2|\}$ sufficiently large.

By decreasing the sets $\mathcal{O}_i$, if necessary, we may assume that there exists an integer $r \geq 0$ with $\mathcal{O}_i = \pi_S^{-1}(\mathbf{a}^{(i)})$ for some $\mathbf{a}^{(1)}, \mathbf{a}^{(2)} \in A^S$, where $S = \{-r, \dots, r-1\}^2 \subset \mathbb{Z}^2$. The coordinates $\mathbf{a}^{(i)} \in A^S$ determine a half-domino tiling of $S' = [-r, r]^2 \subset \mathbb{R}^2$, and we extend these half-dominoes to dominoes in the usual manner. This partial domino tiling can be extended to an exact tiling of the square $S'' = [-2r, 2r]^2$ by dominoes (Remark 6.5), and by decreasing $\mathcal{O}_i$ further we assume that each $\mathcal{O}_i$ corresponds to an exact domino tiling of S''.

We shall prove (6.8) whenever $\mathbf{m} = (m_1, m_2)$ with $m_1 > 12r + 2|k|$; the case where m_1 is small but m_2 is large is dealt with similarly.

Extend the exact domino tiling of S'' corresponding to $\mathcal{O}_2$ to a tiling of the vertical strip $V = [-2r, 2r] \times \mathbb{R} \subset \mathbb{R}^2$ by filling the rest of the strip with vertical dominoes, and shift this tiling by $2\mathbf{m}$ to a tiling of the strip $V + 2\mathbf{m}$. We extend this tiling to a domino tiling of the strip $\bar{V} = [2r, 2m_1 + 2r] \times \mathbb{R}$ by covering the rectangle $[2r, 2m_1 - 2r] \times \mathbb{R}$ with infinite columns of vertical dominoes. For each of these columns we have two choices, since the bottom edges of the dominoes in the strip can be either at even or at odd levels. By combining this tiling of $\bar{V}$ with that of S'' determined by $\mathbf{a}^{(1)}$ we obtain a tiling of $S'' \cup \bar{V}$ which, in turn, determines a nonempty subset W of X; this set W will, of course, depend on the choices we have made in filling the vertical strip $[2r, 2m_1 - 2r] \times \mathbb{R}$ with columns of vertical dominoes. If all the dominoes in these columns have their bottom edges at even levels, then $\gamma(\mathbf{m}, x)$ is an integer which is independent of $\mathbf{m}$ and $x \in W$, and which depends only on the two cylinder sets $\mathcal{O}_1, \mathcal{O}_2$. The description of γ given above also shows that $|\gamma(\mathbf{m}, x)| \leq 4r$. Every time we move one of the columns of dominoes in the strip $[2r, 2m_1 - 2r] \times \mathbb{R}$ up or down one unit, the resulting value of $\gamma(\mathbf{m}, x)$, $x \in W$, changes by ± 1, and by adjusting at most $|k| + 4r$ even or odd columns we can guarantee that $\gamma(\mathbf{m}, x) = k$ for every x in the nonempty set $W \subset \mathcal{O}_1 \cap \bar{\sigma}^{-\mathbf{m}}(\mathcal{O}_2)$. $\square$

Corollary 6.4. *The skew-product action $\sigma_{(\bar{c}_{T_D})}$ of $\mathbb{Z}^2$ on $X \times M_3$, defined by*

$$\sigma^{\mathbf{m}}_{(\bar{c}_{T_D})}(x, a) = (\sigma^{\mathbf{m}}(x), \bar{c}_{T_D}(\mathbf{m}, x)a)$$

for every $\mathbf{m} \in \mathbb{Z}^2$, $a \in M_3$ and $x \in X$, is topologically transitive, but not mixing.

Proof. The topological transitivity follows from Proposition 6.3 and the definition of $\bar{c}_{T_D}$. As $\bar{c}_{T_D}(\mathbf{n}, x) \in M_3'$ if and only if $\mathbf{m} \in 2\mathbb{Z}^2$, $\sigma^{\bar{c}_{T_D}}$ cannot be mixing. $\square$

Remark 6.5 (Extending overtilings to exact tilings). The following proof of the fact that every overtiling of a $2r \times 2r$ square in $\mathbb{R}^2$ by dominoes can be extended to an exact tiling of the concentric $4r \times 4r$ square is due to O. Hryniv (*overtiling* means that the square is covered by dominoes, but that some half-dominoes may stick out). Take an overtiling of the square $[-r, r]^2 \subset \mathbb{R}^2$, divide the tiling into four disjoint regions by drawing a polygonal geodesic (shortest connection) along edges of dominoes from the centre to each of the four vertices of the overtiled square, and extend the overtiling of $[-r, r]^2$ to a partial tiling of $[-2r, 2r]^2$ by reflecting each of four regions of the tiling along the appropriate edge of the square $[-r, r]^2$. The example of an overtiling of $[-4, 4]^2$ in the top left picture of Figure 2 may help to understand what is going on.

As the top right picture in Figure 2 shows, the partial tiling of $[-2r, 2r]^2$ obtained by reflection is naturally divided into four regions. In the bottom left picture of Figure 2 we reflect each of the 'geodesics' from the centre to the four corners of the original square with respect to the appropriate corner,

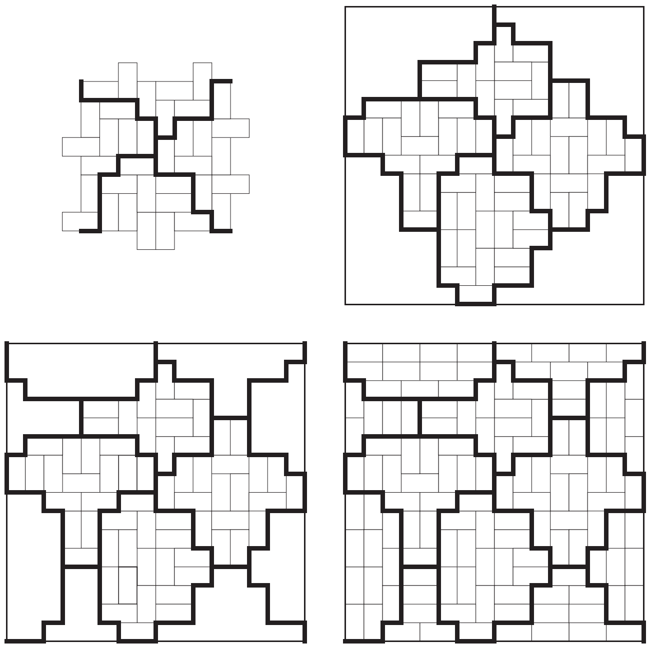


Figure 2. Extending an overtiling to an exact tiling

thereby obtaining altogether twelve disjoint regions of the square $[-2r, 2r]^2$, four of which are already tiled. The remaining eight regions can be tiled exactly by dominoes parallel to the edge of the square $[-2r, 2r]^2$ bounding the region (cf. the bottom right picture of Figure 2). This gives an exact tiling of $[-2r, 2r]^2$.

The following results show that there are no nontrivial cocycles on $X = W_{T_D}$ with values in abelian groups, and that the tiling cocycle $c_{T_D}\colon \mathbb{Z}^2 \times X \longmapsto \Gamma_{T_D} = \mathbf{M}_3$ is essentially the only continuous cocycle on X with values in a discrete group.

Theorem 6.6. *If G is a discrete, abelian group, then every continuous cocycle $c\colon \mathbb{Z}^2 \times X \longmapsto G$ for the shift-action σ of $\mathbb{Z}^2$ on the domino shift $X = W_{T_D}$ is trivial.*

Theorem 6.7. *Let G be a discrete group, and let $c\colon \mathbb{Z}^2 \times X \longmapsto G$ be a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on the domino shift $X = W_{T_D}$. Then there exist a continuous map $b\colon X \longmapsto G$ and a homomorphism $\theta\colon \Gamma \longmapsto G$ such that*

$$b(\sigma^{\mathbf{n}}(x))^{-1} c(\mathbf{n}, x) b(x) = \theta(c_{T_D}(\mathbf{n}, x))$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$. In other words, the tiling cocycle c_{T_D} is fundamental (*cf. Definition* 2.3).

For the proof of Theorem 6.6 we take $L \geq 1$ and write

$$P_X(1, 2L+1) = \{x \in X : \sigma^{(2L+1,0)}(x) = x\}$$

for the set of points with horizontal period $2L+1$. Then $P_X(1, 2L+1) \subset X$ is invariant under σ. Furthermore, if $Q = \{(0,0), \dots, (2L,0)\} \subset \mathbb{Z}^2$, then

$(P_X(1,2L+1),\sigma^{(0,1)})$ is a *SFT* with alphabet $\mathcal{A} = \pi_Q(P_X(1,2L+1)) \subset T_D^Q \cong T_D^{2L+1}$.

Lemma 6.8. *The* SFT $(P_X(1,2L+1),\sigma^{(0,1)})$ *in* $\mathcal{A}^{\mathbb{Z}}$ *is irreducible and has period* 2.

Proof. For every $a = \{(a_0,\dots,a_{2L})\} \in \mathcal{A} \subset T_D^{2L+1}$ we write

$$u(a) = |\{k : 0 \le k \le 2L,\, \mathsf{b}(a_k) = \mathsf{H},\, \mathsf{l}(a_k) = \mathsf{r}(a_k) = \mathsf{V}\}|,$$
$$d(a) = |\{k : 0 \le k \le 2L,\, \mathsf{t}(a_k) = \mathsf{H},\, \mathsf{l}(a_k) = \mathsf{r}(a_k) = \mathsf{V}\}|,$$

for the number of coordinates in a which are bottom and top halves of vertical dominoes, respectively. Fix an element $a^{(0)} \in \mathcal{A}$ and consider an allowed string $(a^{(-1)},a^{(0)},a^{(1)}) \in \mathcal{A}^3$ for the *SFT* $Y(2L{+}1)$ with the property that $a^{(1)}$ and $a^{(-1)}$ contain as many horizontal dominoes as possible. Then $d(a^{(1)}) = u(a^{(0)})$, $u(a^{(-1)}) = d(a^{(0)})$, and we claim that

$$u(a^{(1)}) \le \max(u(a^{(0)})-1,0),\;\; d(a^{(-1)}) \le \max(d(a^{(0)})-1,0). \tag{6.9}$$

For the proof of the first inequality in (6.9) we set $m = u(a^{(0)})$, denote by $0 \le l_0 < l_1 < \dots < l_{m-1} \le 2L$ the coordinates with $\mathsf{b}(a_{l_i}) = \mathsf{H}$ and $\mathsf{l}(a_{l_i}) = \mathsf{r}(a_{l_i}) = \mathsf{V}$, and set $d_j = l_{j+1\,(\mathrm{mod}\,m)} - l_j$ for $j = 0,\dots,m-1$. According to our choice of $a^{(1)}$, each of the intervals $[l_j+1, l_{j+1\,(\mathrm{mod}\,m)}]$ is filled with horizontal dominoes if d_j is odd, or with horizontal dominoes and a single bottom half of a vertical domino, if d_j is even, and therefore $u(a^{(1)}) \le u(a^{(0)})$. Since $\sum_{j=0}^{m-1} d_j = 2L+1$ is odd, at least one of the d_j must be odd, so that $u(a^{(1)}) \le \min(u(a^{(0)})-1,0)$, as claimed. The other inequality in (6.9) is proved similarly.

By applying (6.9) repeatedly we obtain an allowed string $(a^{(-m')},\dots, a^{(0)},\dots,a^{(m)})$, $m,m' \ge 0$, in $\mathcal{A}^{m+m'+1}$ with $d(a^{(-m')}) = u(a^{(m)}) = 0$, and we can extend $(a^{(-m')},\dots,a^{(m)})$ to an allowed string $(a^{(-m'-2)},\dots,a^{(m+1)}) \in \mathcal{A}^{m+m'+4}$ such that

$$u(a^{(-m'-2)}) = u(a^{(m+1)}) = 2L+1,$$

i.e. $\bar{a} = a^{(-m'-2)} = a^{(m'+1)}$ consists entirely of bottom halves of vertical dominoes. It follows that the *SFT* $P_X(1,2L+1) \subset \mathcal{A}^{\mathbb{Z}}$ is irreducible with period ≤ 2.

Finally, since $u(a^{(i)}) + d(a^{(i)}) \pmod 2 = 1$ and $d(a^{(i+1)}) = u(a^{(i)})$ we see that $u(a^{(i)}) = u^(a^{(i+2)}) \pmod 2$ for every i. This proves that $P_X(1,2L+1)$ has period 2. $\square$

Proof of Theorem 6.6. Apply Lemma 6.8 and Corollary 3.3 (1). $\square$

We turn to the proof of Theorem 6.7 and assume from now on that G is a discrete group and $c\colon \mathbb{Z}^2 \times X \longmapsto G$ a continuous cocycle. For the definition of the cocycle $\gamma\colon \mathbb{Z}^2 \times X \longmapsto G$ for the even shift-action $\bar{\sigma}$ of $\mathbb{Z}^2$ on X we refer to (6.7).

Lemma 6.9. *There exist a continuous map* $b\colon X \longmapsto G$ *and commuting elements* $h_1,h_2,h \in G$ *such that the cocycle* $c^*\colon \mathbb{Z}^2 \times X \longmapsto G$*, defined by*

$$c^*(\mathbf{m},x) = b(\sigma^{\mathbf{m}}(x))^{-1} c(\mathbf{m},x) b(x) \tag{6.10}$$

satisfies that

$$c^*(2\mathbf{m}, x) = h_1^{m_1} h_2^{m_2} h^{\gamma(\mathbf{m},x)} \tag{6.11}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ *and* $x \in X$.

Proof. Put $r = \max(r(c((1,0),\cdot)), r(c((0,1),\cdot)), 1)$ (cf. (3.4)),

$$Q = \{-2r, \dots, 2r-1\}^2 \subset \mathbb{Z}^2,$$

and denote by

$$\mathcal{O} \subset X \tag{6.12}$$

the cylinder set corresponding to the exact tiling of the rectangle $[-2r, 2r]^2 \subset \mathbb{R}^2$ by horizontal dominoes. For every $L \geq 0$ we set

$$S(2L) = \{0, \dots, 2L-1\} \times \mathbb{Z} \subset \mathbb{Z}^2,$$
$$B(2L) = \{x \in X : \pi_Q \cdot \sigma^{\mathbf{m}}(x) \in \mathcal{O} \text{ for every } \mathbf{m} \in \{0, 2L\} \times \mathbb{Z}\},$$
$$Y(2L) = \pi_{S(2L)}(B(2L)),$$

and denote by T the vertical shift on $Y(2L)$ induced by $\bar{\sigma}^{(0,1)} = \sigma^{(0,2)}$, i.e.

$$(Ty)_{(k_1,k_2)} = y_{(k_1,k_2+2)}$$

for every $y \in Y(2L) \subset \mathcal{O}^{S(2L)}$ and $(k,l) \in S(2L)$. In terms of dominoes, $Y(2L)$ is the set of all exact domino tilings of the infinite vertical strip $[0, 2L] \times \mathbb{R} \subset \mathbb{R}^2$ such that the two strips $[0, 2r] \times \mathbb{R}$ and $[2L-2r, 2L] \times \mathbb{R}$ are tiled entirely and exactly by horizontal dominoes. Note that $(Y(2L), T)$ is a *SFT* of the form (2.2)–(2.3) with alphabet

$$\mathcal{A}(2L) = \pi_{Q(2L,2)}(B(2L)) \subset T_D^{Q(2L,2)},$$

where

$$Q(m,n) = \{0, \dots, m-1\} \times \{0, \dots, n-1\} \subset \mathbb{Z}^2 \tag{6.13}$$

for every $m, n \geq 0$.

The *SFT* $(Y(2L), T)$ is not transitive. In order to determine the transitive components we set, for every $y \in Y(2L)$ and $x \in B(2L) \subset X$ with $\pi_{S(2L)}(x) = y$,

$$\gamma_L(y) = \gamma((L,0), x) \in \mathbb{Z}. \tag{6.14}$$

One can interpret $\gamma_L(y)$ as in the proof of Proposition 6.3 by viewing y as a tiling of the rectangle $[0, 2L] \times \mathbb{R}$ by the half-dominoes (6.1), and by completing these half-dominoes to dominoes: if $e(y)$ and $o(y)$ are the numbers of the even and odd unit squares in the rectangle $[0, 2L] \times [0, 1]$ which form top halves of vertical dominoes, then $\gamma_L(y) = o(y) - e(y)$.

By using the cocycle equation (2.4) for the tiling cocycle $\bar{c}_{T_D}$ and elementary geometry we see that

$$\gamma_L(Ty) = \gamma_L(y)$$

for every $y \in Y(2L)$ and that the restriction of T to each of the closed, T-invariant subsets

$$Y(2L, k) = \{y \in Y(2L) : \gamma_L(y) = k\},\ k \in \mathbb{Z},$$

is an irreducible, aperiodic *SFT* whenever $Y(2L, k) \neq \varnothing$ (note that $Y(2L, k)$ may consist of a single fixed point).

We return to our cocycle $c\colon \mathbb{Z}^2 \times X \longmapsto G$ and define a cocycle $\bar{c}\colon \mathbb{Z}^2 \times X \longmapsto G$ for the even shift-action $\bar{\sigma}$ of $\mathbb{Z}^2$ on X by

$$\bar{c}(\mathbf{n}, x) = c(2\mathbf{n}, x)$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$. For every $y \in Y(2L)$ we choose an $x \in B(2L)$ with $\pi_{S(2L)}(x) = y$ and observe that, according to our choice of the integer r, $c((2L,0),x) = \bar{c}((L,0),x)$ and $c((0,1),x) = c((0,1),\sigma^{(2L,0)}(x)) = c((0,1),\sigma^{(0,l)}(x))$, $l \in \mathbb{Z}$, are independent of the specific choice of x. In view of this we set

$$\omega_L(y) = c((2L,0),x) = \bar{c}((L,0),x)$$

and note that $\omega_L\colon Y(2L) \longmapsto G$ is continuous.

The cocycle equation (2.4) shows that

$$\begin{aligned}\bar{c}((0,1),x)^n \bar{c}((L,0),x) &= \bar{c}((0,n),\bar{\sigma}^{(L,0)}(x))\bar{c}((L,0),x) = \bar{c}((L,n),x)\\ &= \bar{c}((L,0),\bar{\sigma}^{(0,n)}(x))\bar{c}((0,n),x)\\ &= \bar{c}((L,0),\bar{\sigma}^{(0,n)}(x))\bar{c}((0,1),x)^n\end{aligned}$$

for every $n \geq 0$. Since T is irreducible and aperiodic on $Y(2L,k)$ we conclude that $\bar{c}((0,1),x)$ commutes with $\bar{c}((L,0),x) = \omega_L(y)$, and that

$$\omega_L(y) = \bar{c}((L,0),x) = \bar{c}((L,0),\bar{\sigma}^{(0,n)}x) = \omega_L(T^n y)$$

for every $L \geq 2r$, $n \in \mathbb{Z}$ and $y \in Y(2L)$. The topological transitivity of T on $Y(2L,k)$ guarantees that the map $\omega_L\colon Y(2L,k) \longmapsto G$ is constant for every $L \geq r$ and $k \in \mathbb{Z}$ with $Y(2L,k) \neq \varnothing$.

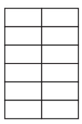
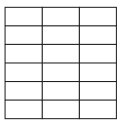
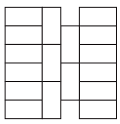
Any two elements $y \in Y(2L,k)$, $y' \in Y(2L',k')$, can be concatenated to an element $yy' \in Y(2L+2L',k+k')$ by horizontal juxtaposition, and

$$\omega_{L+L'}(yy') = \omega_{L'}(y')\omega_L(y).$$

As $y'y$ also lies in $Y(2L+2L',k+k')$ and $\omega_{L+L'}$ is constant on $Y(2L+2L',k+k')$ we conclude that the subgroup $\mathcal{H}$ of G generated by the set $\{\omega_L(y) : L \geq 2r, y \in Y(2L)\}$ is abelian, and is generated by the two elements

$$h_0' = \omega_{r+1}(y^{(4r+2,0)})\omega_r(y^{(4r,0)})^{-1}, \qquad h_1' = \omega_{r+1}(y^{(4r+2,1)})\omega_r(y^{(4r,0)})^{-1},$$

where $y^{(4r,0)} \in Y(4r,0)$ and $y^{(4r+2,0)} \in Y(4r+2,0)$ correspond to exact tilings of the strips $[0,4r]\times\mathbb{R}$ and $[0,4r+2]\times\mathbb{R}$ in $\mathbb{R}^2$ by horizontal dominoes, and $y^{(4r+2,1)}$ is the unique element in $Y(4r+2,1)$. For example, if $r=1$, then $y^{(4r,0)}$, $y^{(4r+2,0)}$ and $y^{(4r+2,1)}$ correspond to the tilings

 ,  ,  ,

of the rectangles $[0,4]\times[0,6]$ and $[0,6]\times[0,6]$, extended vertically with period 2 to the strips $[0,4]\times\mathbb{R}$ and $[0,6]\times\mathbb{R}$. With this choice of h_0', h_1' one checks easily that

$$\bar{c}((L,0),x) = \omega_L(y) = {h_0'}^L (h_1' {h_0'}^{-1})^{\gamma((L,0),x)}$$

for every $L \geq 0$, $y \in Y(2L)$, and $x \in B(2L)$ with $\pi_{S(2L)}(x) = y$. By setting $h = h_1' {h_0'}^{-1}$ we conclude that

$$\bar{c}((L,0),x) = \bar{\gamma}((L,0),x) \tag{6.15}$$

for every $L \geq 0$ and every $x \in \mathcal{O} \cap \bar{\sigma}^{(-L,0)}(\mathcal{O})$, where $\mathcal{O} \subset X$ is the cylinder set defined in (6.12), and where $\bar{\gamma}$ is the cocycle for $\bar{\sigma}$ defined by

$$\bar{\gamma}(\mathbf{m}, x) = {h'_0}^{m_1} h^{\gamma(\mathbf{m},x)} \tag{6.16}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X$.

In order to apply (6.15) we have to modify the cocycle $\bar{c}$ so that $\bar{c}((n,0),x) \in \mathcal{H}$ for every $n \in \mathbb{Z}$. First we observe that

$$\bar{c}((n,0),x) = \bar{c}((n,0),x')$$

for every $n \in \mathbb{Z}$ whenever $\pi_{\mathbf{S}(r,1)}(x) = \pi_{\mathbf{S}(r,1)}(x')$ (for notation we refer to Definition 2.1). This allows us to regard each $\bar{c}((n,0),\cdot)$ as a map from $X(r,1) = \pi_{\mathbf{S}(r,1)}(X)$ to G. As X has the extension property we may view $X(r,1)$ as a mixing *SFT* with respect to the homeomorphism S induced by the even shift $\bar{\sigma}^{(1,0)}$ on X (Remark 2.2) and define a cocycle $u \colon \mathbb{Z} \times X(r,1) \longmapsto G$ for S by $u(n, \pi_{\mathbf{S}(r,1)}(x)) = \bar{c}((n,0),x)$ for every $n \in \mathbb{Z}$ and $x \in X$. Then (6.15) can be written as

$$u(m,z) = v(m,z)$$

for every $m \in \mathbb{Z}$ and $z \in \mathcal{O}' \cap S^{-m}(\mathcal{O}') \subset X(r,1)$, where $\mathcal{O}' = \pi_{\mathbf{S}(r,1)}(\mathcal{O})$, $\pi_{\mathbf{S}(r,1)}(x) = z$ and $v(m,z) = \bar{\gamma}((m,0),x)$ (cf. (6.16)).

For every $m \geq 0$ and $z \in S^{-m}(\mathcal{O}') \smallsetminus \bigcup_{0 \leq k < m} S^{-k}(\mathcal{O}')$ we put

$$b'(z) = u(-m, S^m z) = u(m,z)^{-1}.$$

On the set

$$Z = \bigcap_{n \in \mathbb{Z}} \bigcup_{m \geq n} S^{-n}(\mathcal{O}') \subset X(r,1),$$

which satisfies that $\mu(Z) = 1$ for the measure of maximal entropy μ of $(X(r,1), S)$, we obtain that

$$\begin{aligned} b'(S^n z)^{-1} u(n,z) b'(z) &= u(m(S^n z), S^n z) u(n,z) u(-m(z), S^{m(x)} z) \\ &= u(m(S^n z) + n, S^{m(x)} z) \\ &= v(m(S^n z) + n, S^{m(x)} z) \in \mathcal{H}, \end{aligned} \tag{6.17}$$

where $m(z)$ is the smallest nonnegative integer such that $S^{m(z)} z \in \mathcal{O}'$. Proposition 3.1 (2) allows us to conclude from (6.17) that the μ-a.e. defined map $z \mapsto w(z) = b'(Sz)^{-1} u(1,z) b'(z) v(x)^{-1}$ from $X(r,1)$ to $\mathcal{H}$ satisfies that

$$w(n,z) = w(S^{n-1}z) \cdots w(z) = 1_G$$

whenever $n \geq 0$ and $z \in Z \cap \mathcal{O}' \cap S^{-n}(\mathcal{O}')$. According to Theorem 3.9 in [24] there exists a Borel map $b'' \colon X(r,1) \longmapsto \mathcal{H}$ with

$$w(z) = b''(Sz)^{-1} b''(z)$$

for μ-a.e. $z \in X(r,1)$. Hence

$$b'(Sz)^{-1} u(1,z) b'(z) = b''(Sz)^{-1} v(z) b''(z)$$

for μ-a.e. $z \in X(r,1)$, and by setting $b^* = b' {b''}^{-1}$ we see that

$$(b^* \cdot S)^{-1} u(1,\cdot) b^* = v(1,\cdot)$$

μ-a.e. on $X(r,1)$. Proposition 3.1 (2) implies the existence of a continuous map $b\colon X(r,1) \longmapsto G$ such that

$$(b \cdot S)^{-1} u(1,\cdot) b = v(1,\cdot),$$

and by regarding b as a map from X to G we obtain that

$$\bar{c}^*((m,0),x) = \bar{\gamma}((m,0),x)$$

for every $m \in \mathbb{Z}$ and $x \in X$, where c^* and $\bar{\gamma}$ are defined in (6.10) and (6.16), and where

$$\bar{c}^*(\mathbf{m},x) = c^*(2\mathbf{m},x)$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X$.

We have thus found two cocycles for the topologically mixing $\mathbb{Z}^2$-action $\bar{\sigma}$ on X, namely $\bar{\gamma}$ and $\bar{c}^*$, which coincide for $\mathbf{m} = (m,0)$, $m \in \mathbb{Z}$. The cocycle equation (2.4) implies that

$$\begin{aligned} \bar{c}^*((0,1),\bar{\sigma}^{(n,0)}(x))\bar{\gamma}((n,0),x) &= \bar{\gamma}((n,1),\bar{\sigma}^{(0,1)}(x))\bar{c}^*((0,1),x), \\ \bar{\gamma}((0,1),\bar{\sigma}^{(n,0)}(x))\bar{\gamma}((n,0),x) &= \bar{\gamma}((n,1),\bar{\sigma}^{(0,1)}(x))\bar{\gamma}((0,1),x), \end{aligned} \tag{6.18}$$

and hence that

$$t(\bar{\sigma}^{(n,0)}(x))^{-1}\bar{\gamma}((n,0),x)t(x) = \bar{\gamma}((n,0),x)$$

for every $n \in \mathbb{Z}$ and $x \in X$, where $t = \bar{\gamma}((0,1),\cdot)^{-1}\bar{c}^*((0,1),\cdot)\colon X \longmapsto G$ is continuous. As $\bar{\sigma}^{(\gamma)}$ is mixing by Proposition 6.3 we conclude from the definition of $\bar{\gamma}$ that $t(x)$ commutes with h_0' and h for every $x \in X$, and hence that there exists an abelian subgroup $\mathcal{H}' \subset G$ with $\bar{c}^*(\mathbf{m},x) \in \mathcal{H}'$ for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X$. A look at (6.18) and (6.16) yields that $\bar{c}^*$ and $\bar{\gamma}$ differ by a homomorphism, which is precisely the statement of the lemma. □

Proof of Theorem 6.7. Let $b\colon X \longmapsto G$ be the continuous function and h_1, $h_2, h \in G$ the commuting elements satisfying (6.10)–(6.11) in Lemma 6.9, and let $\mathcal{H} \subset G$ be the subgroup generated by $\{h_1, h_2, h\}$. Equation (2.4) shows that

$$c^*(\mathbf{m},\sigma^{2\mathbf{n}}(x))c^*(2\mathbf{n},x) = c^*(2\mathbf{n},\sigma^{\mathbf{m}}x)c^*(\mathbf{m},x) \tag{6.19}$$

for all $\mathbf{m},\mathbf{n} \in \mathbb{Z}^2$, $x \in X$. From Proposition 6.3 it is clear that we can find, for any specific value g of $c^*(\mathbf{m},\cdot)$ and every cylinder set $\mathcal{O} \subset X$ with $c^*(\mathbf{m},x) = g$ for every $x \in \mathcal{O}$, an element $\mathbf{n} \in \mathbb{Z}^2$ and points $x, x' \in \mathcal{O} \cap \sigma^{-2\mathbf{n}}(\mathcal{O})$ with $\gamma(\mathbf{n},x) = \gamma(\mathbf{n},x') + 1$. For simplicity we assume that $\mathcal{O}$ is determined by the coordinates in $\mathbf{B}(r)$ for some $r > \|\mathbf{m}\|$ (cf. Definition 2.1). A direct calculation shows that

$$\gamma(\mathbf{n},\sigma^{\mathbf{m}}(y)) = (-1)^{m_1+m_2}\gamma(\mathbf{n},y)$$

for every $y \in \mathcal{O} \cap \sigma^{-\mathbf{n}}(\mathcal{O})$, and (6.11) and (6.19) yield that $ghg^{-1} = h^{(-1)^{m_1+m_2}}$. By choosing $\mathbf{n}$ sufficiently large, but otherwise arbitrary, we can also find elements $x, x' \in X$ with the properties that

$$\begin{gathered} c^*(\mathbf{m},x) = c^*(\mathbf{m},x') = c^*(\mathbf{m},\sigma^{2\mathbf{n}}(x)) = c^*(\mathbf{m},\sigma^{2\mathbf{n}+\mathbf{m}}(x')) = g, \\ \gamma(\mathbf{n},x) = \gamma(\mathbf{n},x') = 0, \end{gathered}$$

and by substituting this in (6.19) for every large $\mathbf{n}$ we see that $gh_1g^{-1} = h_1$, $gh_2g^{-1} = h_2$.

We have proved that $\mathcal{H}$ is a normal subgroup of the group $\mathcal{G}$ generated by the values of c^*. For the quotient group $\mathcal{G}/\mathcal{H}$ we obtain that

$$(c^*(\mathbf{m},\sigma^{2\mathbf{n}}(x))\mathcal{H})(c^*(\mathbf{m},x)\mathcal{H}) = 1_{\mathcal{G}/\mathcal{H}}$$

for every $\mathbf{m},\mathbf{n}\in\mathbb{Z}^2$ and $x\in X$. For $\mathbf{n}\neq\mathbf{0}$, $\sigma^{2\mathbf{n}}$ is topologically mixing, so that $c^*(\mathbf{m},\cdot)\mathcal{H}\colon X\longmapsto\mathcal{G}/\mathcal{H}$ must be constant. As $c^*(2\mathbf{m},\cdot)\mathcal{H}=1_{\mathcal{G}/\mathcal{H}}$ and

$$\begin{aligned} c^*(\mathbf{m},\cdot)c^*(\mathbf{n},\cdot)\mathcal{H} &= c^*(\mathbf{m},\sigma^{\mathbf{n}}(\cdot))c^*(\mathbf{n},\cdot)\mathcal{H} \\ &= c^*(\mathbf{m}+\mathbf{n},\cdot)\mathcal{H} = c^*(\mathbf{n},\cdot)c^*(\mathbf{m},\cdot)\mathcal{H} \end{aligned}$$

for all $\mathbf{m},\mathbf{n}\in\mathbb{Z}^2$ we see that $\mathcal{G}/\mathcal{H}$ a quotient of $\mathbb{Z}^2/2\mathbb{Z}^2$ and hence abelian.

Let $x^*\in X$ be the fixed point of $\sigma^{(1,0)}$ with $\mathsf{b}(x^*_{\mathbf{0}})=\mathsf{H}$, and let

$$\begin{aligned} g_1 &= c^*((1,0),x^*), & g_2 &= c^*((0,1),x^*), \\ g_1' &= c^*((1,0),\sigma^{(0,1)}(x^*)), & g_2' &= c^*((0,1),\sigma^{(0,1)}(x^*)). \end{aligned}$$

Then one can check that

$$g_1^2 = {g_1'}^2, \qquad g_1'g_2 = g_2g_1, \qquad g_1g_2' = g_2'g_1', \qquad g_2'g_2 = h_2,$$

that $\mathcal{G}$ is generated by the elements $\{g_1,g_2,h_1,h_2,h\}$ with the relations

$$\begin{gathered} g_1^2 = h_1, \qquad h_2g_1 = g_1h_2, \qquad h_1g_2 = g_2h_1, \\ g_1^{-1}hg_1 = g_2^{-1}hg_2 = h^{-1}, \end{gathered}$$

and that there exist elements $a,b\in\mathcal{H}$ and $\alpha,\alpha',\beta,\beta'\in\mathcal{G}$ such that

$$\begin{gathered} g_1 = g_1'a, \qquad g_2 = g_2'b, \\ a = \alpha\alpha', \qquad \alpha = h_1^{m_1}h_2^{m_2}, \qquad \alpha' = h^{m_3}, \\ b = \beta\beta', \qquad \beta = h_1^{n_1}h_2^{n_2}, \qquad \beta' = h^{n_3} \end{gathered}$$

for some $m_1,m_2,m_3,n_1,n_2,n_3\in\mathbb{Z}$.

We increase the group $\mathcal{G}$ to a group $\overline{\mathcal{G}}$ by adding, if necessary, elements $\sqrt{\alpha}$, $\sqrt{h}$, $\sqrt{b}$ with the relations

$$\begin{gathered} (\sqrt{\alpha})^2 = \alpha, \qquad (\sqrt{h})^2 = h, \qquad (\sqrt{b})^2 = b, \\ \sqrt{\alpha}, \sqrt{h}, \sqrt{b}, h_1, h_2 \text{ commute}, \\ g_1^{-1}\sqrt{\alpha}g_1 = g_2^{-1}\sqrt{\alpha}g_2 = (\sqrt{\alpha})^{-1}, \\ g_1^{-1}\sqrt{h}g_1 = g_2^{-1}\sqrt{h}g_2 = (\sqrt{h})^{-1}, \\ g_1^{-1}\sqrt{b}g_1 = g_2^{-1}\sqrt{b}g_2 = (\sqrt{b}), \end{gathered}$$

and set

$$g_1^* = g_1(\sqrt{\alpha})^{-1}(\sqrt{h})^{m_3}, \qquad g_2^* = g_2\sqrt{b}.$$

A direct calculation shows that there exists a group homomorphism $\theta\colon\Gamma\longmapsto\overline{\mathcal{G}}$ with

$$\begin{aligned} \theta(\mathsf{H}) &= g_1^*\sqrt{h}, & \theta(\mathsf{h}) &= g_1^*(\sqrt{h})^{-1}, \\ \theta(\mathsf{V}) &= g_2^*, & \theta(\mathsf{v}) &= g_2^*h. \end{aligned}$$

Since

$$\theta(\mathsf{h})\theta(\mathsf{H}) = hh_1, \qquad \theta(\mathsf{V})\theta(\mathsf{v}) = hh_2,$$

(6.11) implies that the cocycle $c^{**} = \theta \cdot c_{T_D} \colon \mathbb{Z}^2 \times X \longmapsto \overline{\mathcal{G}}$ satisfies that

$$\theta \cdot c_{T_D}(2\mathbf{m}, x) = c^*(2\mathbf{m}, x)$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X$. Fix $\mathbf{m} \in \mathbb{Z}^2$ for the moment and apply the cocycle equation (2.4) to see that

$$\begin{aligned} c^*(\mathbf{m}, \sigma^{2\mathbf{n}}(x))c^*(2\mathbf{n}, x)c^*(\mathbf{m}, x)^{-1} &= c^*(2\mathbf{n}, \sigma^{\mathbf{m}}(x)) = c^{**}(2\mathbf{n}, \sigma^{\mathbf{m}}(x)) \\ &= c^{**}(\mathbf{m}, \sigma^{2\mathbf{n}}(x))c^*(2\mathbf{n}, x)c^{**}(\mathbf{m}, x)^{-1}, \end{aligned}$$

so that

$$\psi(\sigma^{2\mathbf{n}}(x))^{-1}c^*(2\mathbf{n}, x)\psi(x) = c^*(2\mathbf{n}, x)$$

with

$$\psi(x) = c^*(\mathbf{m}, x)^{-1}c^{**}(\mathbf{m}, x)$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$. As above we use topological mixing to conclude that $\psi(x)$ commutes with $c^*(2\mathbf{n}, x)$ for every $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$, and that ψ is therefore constant. This shows that there exists, for every $\mathbf{m} \in \mathbb{Z}^2$, an element $g_{\mathbf{m}} \in \overline{\mathcal{G}}$ with

$$c^*(\mathbf{m}, x) = g_{\mathbf{m}}c^{**}(\mathbf{m}, x)$$

for every $x \in X$, and that $g_{2\mathbf{m}} = 1_{\overline{\mathcal{G}}}$ for every $\mathbf{m} \in \mathbb{Z}^2$. If

$$\begin{aligned} \bar{g}_1 &= g_{(1,0)}g_1^*\sqrt{h}, & \bar{g}_1' &= g_{(1,0)}g_1^*(\sqrt{h})^{-1}, \\ \bar{g}_2 &= g_{(0,1)}g_2^*, & \bar{g}_2' &= g_{(0,1)}g_2^*h, \end{aligned}$$

one obtains a group homomorphism $\bar{\theta} \colon \Gamma \longmapsto \overline{\mathcal{G}}$ with

$$\begin{aligned} \bar{\theta}(\mathsf{H}) &= \bar{g}_1, & \bar{\theta}(\mathsf{h}) &= \bar{g}_1', \\ \bar{\theta}(\mathsf{V}) &= \bar{g}_2, & \bar{\theta}(\mathsf{v}) &= \bar{g}_2', \end{aligned}$$

and the cocycle $\bar{\theta} \cdot c_{T_D} \colon \mathbb{Z}^2 \times X \longmapsto \overline{\mathcal{G}}$ coincides with c^*. It follows that $\bar{\theta}(\Gamma) \subset \mathcal{G} \subset G$. This completes the proof of the theorem. $\square$

Remark 6.10. By using the fundamental cocycle $c_{T_D} \colon \mathbb{Z}^2 \times X \longmapsto \Gamma(T_D)$ to calculate the group $\Gamma_{\bar{x}}(c_{T_D}) \cong \pi_1^{\mathrm{proj}}(X, *)$ (cf. Theorem 5.5 and Remark 5.6) we obtain that

$$\pi_1^{\mathrm{proj}}(X, \bar{x}) \cong \pi_1^{\mathrm{proj}}(X, *) \cong \Gamma_*(c_{T_D}) \cong \Gamma_{\bar{x}}(c_{T_D}) \cong \left\{ \left(\begin{smallmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right) : m \in \mathbb{Z} \right\} \cong \mathbb{Z}$$

for every $\bar{x} \in X$ (cf. [7]).

7. Chessboards

Let $n \geq 3$, and let $X^{(n)}$ be the set of all colourings of the lattice $\mathbb{Z}^2$ with n colours $\{0, \dots, n-1\}$ so that no two horizontally or vertically adjacent lattice points have the same colour. Equivalently, $X^{(n)}$ can be described as the set of all colourings of an infinite chessboard with n colours in which adjacent squares are coloured differently (cf. [1], [19] and Examples 4.3–4.4 in [25]). It is easy to check that $X^{(n)} \subset \{0, \dots, n-1\}^{\mathbb{Z}^2}$ has the periodic extension property. If $n \geq 4$, then Example 4.4 in [25] shows that every continuous cocycle $c \colon \mathbb{Z}^2 \times X^{(n)} \longmapsto G$ for the shift-action σ of $\mathbb{Z}^2$ on $X^{(n)}$ with values in a discrete group G is trivial (cf. Example 10.6). However, there are nontrivial cocycles on $X^{(3)}$ with values in, say, $\mathbb{Z}$ (cf. [25]). In order to investigate the continuous cocycles on $X^{(3)}$ we consider the set T'_C of Wang tiles

 (7.1)

with the colours

$$\mathsf{h}_0 = \text{—}\ ,\quad \mathsf{h}_1 = \text{- - -}\ ,\quad \mathsf{h}_2 = \cdots\ ,\quad \mathsf{v}_0 = |\ ,\quad \mathsf{v}_1 = ¦\ ,\quad \mathsf{v}_2 = \vdots \tag{7.2}$$

on the horizontal and vertical edges. The Wang shift $W_{T'_C}$ is isomorphic to $X^{(3)}$: an explicit shift-commuting isomorphism is given by identifying the sets $\{0,1,2\}$ and $\{\mathsf{h}_0,\mathsf{h}_1,\mathsf{h}_2\}$, and by mapping each $(\tau^{\mathbf{n}}) \in W_{T'_C} \subset T^{\mathbb{Z}^2}$ to $(\mathsf{b}(\tau^{\mathbf{n}})) \in X^{(3)} \subset \{\mathsf{h}_0,\mathsf{h}_1,\mathsf{h}_2\}^{\mathbb{Z}^2}$. The tiling group $\Gamma(T'_C)$ is of the form

$$\begin{aligned}\Gamma(T'_C) = \{\mathsf{h}_i, \mathsf{v}_i,\ i = 0,1,2 | \mathsf{v}_1\mathsf{h}_0 = \mathsf{v}_2\mathsf{h}_0 = \mathsf{h}_1\mathsf{v}_0 = \mathsf{h}_2\mathsf{v}_0,\\ *\mathsf{v}_2\mathsf{h}_1 = \mathsf{v}_0\mathsf{h}_1 = \mathsf{h}_2\mathsf{v}_1 = \mathsf{h}_0\mathsf{v}_1,\ \ \mathsf{v}_0\mathsf{h}_2 = \mathsf{v}_1\mathsf{h}_2 = \mathsf{h}_0\mathsf{v}_2 = \mathsf{h}_1\mathsf{v}_2\}.\end{aligned}$$

Since $\mathsf{h}_0 = \mathsf{h}_1 = \mathsf{h}_2$, $\mathsf{v}_0 = \mathsf{v}_1 = \mathsf{v}_2$ and $\mathsf{h}_0\mathsf{v}_0 = \mathsf{v}_0\mathsf{h}_0$, $\Gamma(T'_C) \cong \mathbb{Z}^2$, and the tiling cocycle $c_{T'_C}\colon \mathbb{Z}^2 \times W_{T'_C} \longmapsto \Gamma(T)$ in (4.5) is a homomorphism.

With a different representation of $X^{(3)}$ as a Wang shift we obtain a much more complicated tiling group and a nontrivial tiling cocycle. Let T''_C be the set of Wang tiles

$$\begin{array}{ccccccccc}\boxed{\begin{smallmatrix}1&0\\0&1\end{smallmatrix}} & \boxed{\begin{smallmatrix}1&2\\0&1\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&0\\0&1\end{smallmatrix}} & \boxed{\begin{smallmatrix}1&0\\0&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&0\\0&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&1\\0&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}0&1\\1&0\end{smallmatrix}} & \boxed{\begin{smallmatrix}0&2\\1&0\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&1\\1&0\end{smallmatrix}}\\ \boxed{\begin{smallmatrix}0&1\\1&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&0\\1&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&1\\1&2\end{smallmatrix}} & \boxed{\begin{smallmatrix}0&1\\2&0\end{smallmatrix}} & \boxed{\begin{smallmatrix}0&2\\2&0\end{smallmatrix}} & \boxed{\begin{smallmatrix}1&2\\2&0\end{smallmatrix}} & \boxed{\begin{smallmatrix}0&2\\2&1\end{smallmatrix}} & \boxed{\begin{smallmatrix}1&0\\2&1\end{smallmatrix}} & \boxed{\begin{smallmatrix}1&2\\2&1\end{smallmatrix}}\end{array} \tag{7.3}$$

with the colours $\mathsf{h}_{ij} = \left[\,i\;j\,\right]$ on the horizontal and $\mathsf{v}_i^j = \left[\begin{smallmatrix} j\\ i\end{smallmatrix}\right]$ on the vertical edges, where $i,j \in \{0,1,2\}$ and $i \neq j$ (cf. (4.1)). The isomorphism between the Wang shift $W_{T''_C}$ and the *SFT* $X^{(3)}$ is given by sending each $(\tau^{\mathbf{n}}) \in W_{T''_C} \subset {T''_C}^{\mathbb{Z}^2}$ to the point $(a_{\mathbf{n}}) \in \{0,1,2\}^{\mathbb{Z}^2}$ with

$$\tau^{\mathbf{n}} = \boxed{\begin{smallmatrix}c_{\mathbf{n}} & d_{\mathbf{n}}\\ a_{\mathbf{n}} & b_{\mathbf{n}}\end{smallmatrix}} \in T''_C$$

for every $\mathbf{n} \in \mathbb{Z}^2$.

The tiling group $\Gamma(T''_C)$ of $X^{(3)} = W_{T''_C}$ has the generators

$$\{\mathsf{h}_{ij},\, \mathsf{v}_i^j : 0 \le i,j \le 2,\, i \neq j\} \tag{7.4}$$

and the relations arising from (7.3): for example, the first tile in the second row of (7.3) leads to the relation $\mathsf{h}_{01}\mathsf{v}_1^0 = \mathsf{v}_2^1\mathsf{h}_{12}$.

The group $\Gamma(T''_C)$ is nonabelian: if F_3 is the free group on the three generators $\gamma_0,\gamma_1,\gamma_2$ then there exists a surjective group homomorphism $\phi\colon \Gamma(T''_C)$ with

$$\begin{aligned}&\phi(\mathsf{h}_{01}) = \phi(\mathsf{v}_0^1) = \gamma_1, && \phi(\mathsf{h}_{10}) = \phi(\mathsf{v}_1^0) = \gamma_1^{-1},\\ &\phi(\mathsf{h}_{12}) = \phi(\mathsf{v}_1^2) = \gamma_2, && \phi(\mathsf{h}_{21}) = \phi(\mathsf{v}_2^1) = \gamma_2^{-1},\\ &\phi(\mathsf{h}_{20}) = \phi(\mathsf{v}_2^0) = \gamma_3, && \phi(\mathsf{h}_{02}) = \phi(\mathsf{v}_0^2) = \gamma_3^{-1},\end{aligned} \tag{7.5}$$

and the homomorphic image $\phi \cdot c_{T''_C}\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto F_3$ of the tiling cocycle $c_{T''_C}$ is easily checked to be nontrivial.

While the Wang tiles (7.1) lead to a trivial tiling cocycle, the tiles (7.3) generate a tiling group which is too big. In order to find a more suitable candidate for a fundamental cocycle we set $\gamma_1 = \gamma_2 = \gamma_3$ in (7.5) and define a cocycle $\zeta\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}^3$ by

$$\zeta(\mathbf{m}, x) = (m_1, m_2, \bar{\gamma}(\mathbf{m}, x)) \tag{7.6}$$

with

$$\begin{aligned} \bar{\gamma}((1,0),x) &= \begin{cases} 1 & \text{if } \mathsf{b}(x_{\mathbf{0}}) \in \{[\,0\ 1\,], [\,1\ 2\,], [\,2\ 0\,]\}, \\ -1 & \text{otherwise,} \end{cases}, \\ \bar{\gamma}((0,1),x) &= \begin{cases} 1 & \text{if } \mathsf{l}(x_{\mathbf{0}}) \in \{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}2\\1\end{smallmatrix}], [\begin{smallmatrix}2\\0\end{smallmatrix}]\}, \\ -1 & \text{otherwise} \end{cases} \end{aligned} \tag{7.7}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X^{(3)}$. Since the cocycle $\bar{\gamma}$ is continuously cohomologous to a cocycle taking values in $3\mathbb{Z}$ we have to replace it by the closely related cocycle $\gamma'\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}$ in (7.9).

Represent $X^{(3)} \subset \{0,1,2\}^{\mathbb{Z}^2}$ as the set of allowed colourings of $\mathbb{Z}^2$ with the colours $\{0,1,2\}$, where 'allowed' means that no two adjacent lattice points have the same colour, and define a cocycle $\gamma\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}$ by setting

$$\begin{aligned} \gamma((1,0),x) &= \begin{cases} 1 & \text{if } (x_{(0,0)}, x_{(1,0)}) = (0,1), \\ -1 & \text{if } (x_{(0,0)}, x_{(1,0)}) = (1,0), \\ 0 & \text{otherwise,} \end{cases} \\ \gamma((0,1),x) &= \begin{cases} 1 & \text{if } (x_{(0,0)}, x_{(0,1)}) = (0,1), \\ -1 & \text{if } (x_{(0,0)}, x_{(0,1)}) = (1,0), \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{7.8}$$

for every $x = (x_{\mathbf{n}}) \in X^{(3)} = W_{T_C} \subset T_C^{\mathbb{Z}^2}$. Let $b'\colon X^{(3)} \longmapsto \mathbb{Z}$ be the map

$$b'(x) = \begin{cases} 1 & \text{if } x_{\mathbf{0}} = 2, \\ 0 & \text{otherwise} \end{cases}$$

and set

$$\gamma'(\mathbf{m}, x) = \gamma(\mathbf{m}, x) + b'(x) - b'(\sigma^{\mathbf{m}}(x)) \tag{7.9}$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X^{(3)}$. Then the cocycle $\gamma'\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}$ satisfies that

$$\gamma'(\mathbf{m}, x) - m_1 - m_2 = 0 \pmod 2 \tag{7.10}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X^{(3)}$.

Theorem 7.1. *Let G be a discrete group, and let $c\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto G$ be a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on the space $X^{(3)}$ of three-coloured chessboards. Then there exists a continuous map $b\colon X^{(3)} \longmapsto G$ and commuting elements $h, h_1, h_2 \in G$ such that*

$$b(\sigma^{\mathbf{m}}(x))^{-1} c(\mathbf{m}, x) b(x) = h_1^{(m_1+m_2-\gamma'(\mathbf{m},x))/2} h_2^{m_2} h^{\gamma'(\mathbf{m},x)}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X^{(3)}$. In other words, the cocycle $c^*\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}^3$, defined by

$$c^*(\mathbf{m}, x) = \left(\tfrac{m_1+m_2-\gamma'(\mathbf{m},x))}{2}, m_2, \gamma'(\mathbf{m}, x)\right)$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X^{(3)}$, is fundamental (cf. Definition 2.3).

We begin the proof of Theorem 7.1 with a proposition.

Proposition 7.2. *The shift-action σ of $\mathbb{Z}^2$ on $X^{(3)}$ is topologically mixing. Furthermore, if $k \in \mathbb{Z}$ and $\mathcal{O} \subset X^{(3)}$ is a nonempty open set, then*

$$\begin{aligned} \mathcal{O} \cap \sigma^{(\gamma')}_{(-2m,0)}(\mathcal{O}) \cap \{x \in X^{(3)} : \gamma'((2m,0),x) = 2k\} \neq \varnothing, \\ \mathcal{O} \cap \sigma^{(\gamma')}_{(-2m-1,0)}(\mathcal{O}) \cap \{x \in X^{(3)} : \gamma'((2m+1,0),x) = 2k+1\} \neq \varnothing \end{aligned} \tag{7.11}$$

whenever $|m|$ is sufficiently large. Finally, if the skew-product action $\sigma_{(\gamma')}$ of $\mathbb{Z}^2$ on $X^{(3)} \times \mathbb{Z}$ is defined as in Proposition 6.3 by

$$\sigma^{\mathbf{m}}_{(\gamma')}(x,l) = (\sigma^{\mathbf{m}}(x), \gamma'(\mathbf{m},x) + l)$$

for every $\mathbf{m} \in \mathbb{Z}^2$, $x \in X^{(3)}$ and $l \in \mathbb{Z}$, then $\sigma^{(1,0)}_{(\gamma')}$ is topologically transitive, but not mixing.

Proof. Every allowed colouring of a square $\{-r, \dots, r\}^2 \subset \mathbb{Z}^2$ can be extended to an allowed colouring of a larger square which contains only the colours 0 and 1 along its boundary (cf. Example 4.3 in [25]). If $\mathcal{O}_1$, $\mathcal{O}_2$ are nonempty cylinder sets in $X^{(3)}$ we may therefore decrease these sets, if necessary, and assume that there exists an integer $r \geq 0$ such that each $\mathcal{O}_i$ is determined by a colouring of the square $\{-r, \dots, r\}^2$ containing only the colours 0 and 1 along its boundary. Then it is clear that $\mathcal{O}_1 \cap \sigma^{-\mathbf{m}}(\mathcal{O}_2) \neq \varnothing$ whenever $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ with $\max(|m_1|, |m_2|) > 2r$, which shows that σ is mixing.

In order to prove (7.11) we fix $k \in \mathbb{Z}$ and assume that $m > 12r + 6|k| + 6$. By colouring each of the columns $\{j\} \times \mathbb{Z}$ in the rectangle $R = \{2r + 2, \dots, m - 2r - 2\} \times \mathbb{Z}$ with only two appropriately chosen colours we can construct an allowed colouring of the region

$$\{-2r-1, \dots, 2r+1\}^2 \cup R \cup (\{m-2r-1, \dots, m+2r+1\} \times \{-2r-1, \dots, 2r+1\})$$

such that any $x \in X^{(3)}$ with this partial colouring satisfies that $x \in \mathcal{O}_1 \cap \sigma^{(-m,0)}(\mathcal{O}_2)$ and

$$\gamma'((m,0),x) = \begin{cases} 2k & \text{if } m \text{ is even,} \\ 2k+1 & \text{if } m \text{ is odd.} \end{cases}$$

This proves (7.11) and implies that $\sigma^{(\gamma')}_{(1,0)}$ is topologically transitive. Note that $\sigma^{(\gamma')}_{(1,0)}$ is not mixing by (7.10). □

Let $L \geq 1$, and let

$$P_X(1,L) = \{x \in X^{(3)} : \sigma^{(L,0)}(x) = x\}$$

be the set of points with horizontal period L. Then $(P_X(1,L), \sigma^{(0,1)})$ is a *SFT* with alphabet $\mathcal{A}(L) = \pi_{Q(L,1)}(P_X(1,L)) \subset \{0,1,2\}^{Q(L,1)} \cong \{0,1,2\}^L$ (cf. (6.13)). Put

$$\gamma_L(a) = \gamma'((L,0),x) = \gamma((L,0),x)$$

for every $a \in \mathcal{A}(L)$ and $x \in P_X(1,L)$ with $\pi_{Q(L,1)}(x) = a$ and observe that

$$\gamma_L(a) \pmod 2 = \gamma((L,0),x) \pmod 2 = L \pmod 2,$$

$$\gamma((L,0),x)=\gamma'((L,0),x)=\gamma'((L,0),\sigma^{(0,1)}(x))=\gamma((L,0),\sigma^{(0,1)}(x))$$

(cf. (7.10) and (2.4)). It follows that

$$P_X^{(k)}(1,L)=\{x\in P_X(1,L):\gamma((L,0),x)=k\}$$

is invariant under $\sigma^{(0,1)}$ for every L,k, and is a *SFT* with respect to $\sigma^{(0,1)}$ with alphabet

$$\mathcal{A}(L,k)=\pi_{Q(L,1)}(P_X^{(k)}(1,L))=\{a\in\mathcal{A}(L):\gamma_L(a)=k\}$$

whenever it is nonempty.

Lemma 7.3. *If* $P_X^{(k)}(1,L)\neq\varnothing$ *then the* SFT $(P_X^{(k)}(1,L),\sigma^{(0,1)})$ *is irreducible and aperiodic.*

Proof. Fix k,L with $P_X^{(k)}(1,L)\neq\varnothing$, write every $a\in\mathcal{A}(L,k)\subset\{0,1,2\}^{Q(L,1)}\cong\{0,1,2\}^L$ as $a=(a_0,\dots,a_{L-1})$, and set

$$f_m(a)=\bar\gamma((m,0),x)$$

for every $m=0,\dots,L$ and $x\in P_X^{(k)}(1,L)$ with $x_{(j,0)}=a_j$ for $j=0,\dots,L-1$ (cf. (7.7)). Then $f_L(a)=3\gamma_L(a)$ for every $a\in\mathcal{A}(L,k)$.

Call a string $(a^{(0)},\dots,a^{(l)})\in\mathcal{A}(L,k)^{l+1}$ *allowed* if there exists an element $x\in P_X^{(k)}(1,L)$ with $\pi_{Q(L,1)}(\sigma^{(0,j)}(x))=a^{(j)}$ for $j=0,\dots,l$. Then a pair $(a^{(0)},a^{(1)})\in\mathcal{A}(L,k)^2$ is allowed if and only if $|f_m(a^{(0)})+a_0^{(0)}-f_m(a^{(1)})-a_0^{(1)}-\alpha|=1$ for every $m=0,\dots,L$, where α is the unique integer with $\alpha=0 \pmod 3$ and $|a_0^{(0)}-a_0^{(1)}-\alpha|=1$ (this interpretation is due to O. Hryniv). If we represent $a^{(0)}$ as an L-step walk on $\mathbb{Z}$ which starts in $a_0^{(0)}$ and moves by ±1 to the position $a_0^{(0)}+f_m(a^{(0)})$ at each time $m=1,\dots,L$, then the possible allowed 'successors' $a^{(1)}$ of $a^{(0)}$ correspond to all possible L-step walks on $\mathbb{Z}$ which start either in $\beta=a_0^{(0)}+1$ or $\beta=a_0^{(0)}-1$, end in $\beta+f_L(a^{(1)})=\beta+f_L(a^{(0)})$, and whose positions $\beta+f_m(a^{(1)})$ at each time $m=0,\dots,L$ have distance 1 from the position $a_0^{(0)}+f_m(a^{(0)})$ of the walk associated with $a^{(0)}$.

By using this observation we can construct, for every $a\in\mathcal{A}(L,k)$, an allowed sequence $(a^{(-m')},\dots,a^{(0)},\dots a^{(m)})$ such that $a^{(0)}=a$ and $a'=a^{(-m')}=a^{(m)}$ is the unique element in $\mathcal{A}(L,k)$ with

$$a'_m=\begin{cases} m \pmod 2 & \text{if } 0\le m\le L-3|k|,\\ m-L+3|k| \pmod 3 & \text{if } L-3|k|\le m\le L-1 \text{ and } k\ge 0,\\ -(m-L+3|k|) \pmod 3 & \text{if } L-3|k|\le m\le L-1 \text{ and } k\le 0\end{cases}$$

(note that $L-3|k|$ must be even by (7.10)). This shows that $P_X^{(k)}(1,L)$ is irreducible. The aperiodicity follows from the existence of allowed words in $\mathcal{A}(L,k)^3$ and $\mathcal{A}(L,k)^4$ beginning and ending in a'.

A second proof of Lemma 7.3 can be derived from the proof of Lemma 8.3. ∎

Put $r = 2\max(r(c(1,0),\cdot)), r(c((0,1),\cdot)),1)$ (cf. (3.4)) and denote by $\mathcal{O} \subset X^{(3)}$ the cylinder set defined by the unique allowed colouring of the rectangle $Q = \{-r,\dots,r\}^2$ with the colours $\{0,1\}$ in which the colour of $\mathbf{0}$ is equal to 0. Let $L \geq 0$, and let

$$\begin{aligned} Z(L) &= \{x \in P_X(1,L) : \sigma^{(0,n)}(x) \in \mathcal{O} \text{ for every } n \in \mathbb{Z}\}, \\ Z(L,k) &= \{y \in Z(L) : \gamma((L,0),y) = k\} \end{aligned} \tag{7.12}$$

for every $k \in \mathbb{Z}$. Then $Z(L)$ and $Z(L,k)$ are invariant under the even vertical shift $T = \sigma^{(0,2)}$ for every $k \in \mathbb{Z}$. Furthermore, if $Z(L,k) \neq \varnothing$, then $(Z(L,k),T)$ is a *SFT* with alphabet

$$\mathcal{A}(L,k)' = \pi_{Q(L,2)}(Z(L,k)) \subset \{0,1,0\}^{Q(L,2)}$$

(cf. (3.4)). An insignificant modification of the proof of Lemma 7.3 yields the following result.

Lemma 7.4. *If $Z(L,k) \neq \varnothing$ then the* SFT $(Z(L,k),T)$ *is irreducible and aperiodic.*

Proof of Theorem 7.1. For $L \geq 0$ we define $Z(L)$ by (7.12) and deduce from Lemma 7.4 almost exactly as in the proof of Lemma 6.9 that the set $\{c((L,0),y) : L \geq 1, y \in Z(L)\}$ generates an abelian subgroup $H \subset G$ with generators $g_2 g_1^{-1}$ and g_3, where

$$g_1 = \gamma((2r+2,0),y^{(1)}),\ \ g_2 = \gamma((2r+4,0),y^{(2)}),\ \ g_3 = \gamma((2r+3,0),y^{(3)}),$$

and where

$$y^{(1)} \in Z(2r+2) = Z(2r+2,0),\ \ y^{(2)} \in Z(2r+4,0),\ \ y^{(3)} \in Z(2r+3,1)$$

are determined by the conditions

$$y^{(1)}_{(r+1,j)} = y^{(2)}_{(r+3,j)} = y^{(2)}_{(r+2,j+1)} = \begin{cases} 1 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

$$y^{(3)}_{(r+1,j)} = \begin{cases} 1 & \text{if } j \text{ is even,} \\ 2 & \text{if } j \text{ is odd.} \end{cases}$$

We increase H to an abelian group $\bar{H}$ by adding, if necessary, an element h' with $h'^2 = g_2 g_1^{-1}$. Then Lemma 7.4 and the proof of Lemma 6.9 imply that

$$c((L,0),y) = h'^L h^{\gamma((L,0),y)} = h'^L h^{\gamma'((L,0),y)} \tag{7.13}$$

whenever $L \geq 1$ and $y \in Z(L) \neq \varnothing$, where $h = g_3 h'^{-2r-3}$. Put

$$\gamma^*(\mathbf{m},x) = h'^{m_1+m_2} h^{\gamma'(\mathbf{m},x)} \in H \tag{7.14}$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X^{(3)}$. From (7.10) and (7.13) it is clear that (7.14) defines a continuous cocycle $\gamma^* \colon \mathbb{Z}^2 \times X^{(3)} \longmapsto H$ with

$$c((L,0),y) = \gamma^*((L,0),y) \in H$$

whenever $L \geq 1$ and $y \in Z(L)$. Since H is abelian and $X^{(3)}$ has the periodic extension property we can apply Theorem 3.9 in [24] and Proposition 3.1

exactly as in the proof of Lemma 6.9 to obtain a continuous map $b\colon X^{(3)} \longmapsto H$ with

$$c^*(\mathbf{m}, x) = b(\sigma^{\mathbf{m}}(x))^{-1} c(\mathbf{m}, x) b(x) = \gamma^*(\mathbf{m}, x)$$

whenever $x \in X^{(3)}$ and $\mathbf{m} = (m, 0)$ for some $m \in \mathbb{Z}$.

From (7.11)) we obtain as in the proof of Theorem 6.7 that $c^*((0,1), x)$ commutes with $h'^2 = g_2 g_1^{-1}$ and g_3 for every $x \in X^{(3)}$, and that $c^*((0,1), \cdot)$ and $\gamma^*((0,1), \cdot)$ differ by a constant which commutes with H, i.e. that there exists an $h_2 \in G$ which commutes with H such that

$$c^*((0,1), x) = h_2 \gamma^*((0,1), x)$$

for every $x \in X$. After replacing h by hh' and h'^2 by h_1 we obtain the statement of the theorem. □

Remark 7.5. The cocycle $\gamma\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}$ in (7.8) is asymmetric in the symbols $\{0, 1, 2\}$. The symmetric version $\bar{\gamma}$ of γ in (7.7) is continuously cohomologous to 3γ (and hence also to $3\gamma'$).

Remark 7.6. By using the fundamental cocycle $c^*\colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}^3$ in Theorem 7.1 to calculate the group $\Gamma_{\bar{x}}(c^*) \cong \pi_1^{\mathrm{proj}}(X^{(3)}, \bar{x})$ in Remark 5.6 we obtain that

$$\pi_1^{\mathrm{proj}}(X^{(3)}, \bar{x}) \cong \pi_1^{\mathrm{proj}}(X^{(3)}, *) \cong \Gamma_*(c^*) \cong \Gamma_{\bar{x}}(c^*) \cong \mathbb{Z}$$

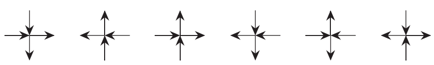
for every $\bar{x} \in X$, in accordance with [7].

8. Square ice

We continue with a *SFT* closely related to the three-coloured chessboards, the *square ice model*. Consider the set of Wang tiles T_I of the form

 (8.1)

with the colours $\mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v}$ on the solid horizontal, broken horizontal, solid vertical and broken vertical edges, let $Y = W_{T_I}$ be the Wang shift of T_I, and observe that Y has the periodic extension property. If we consider the configurations



obtained by drawing arrows orthogonal to the edges of the tiles (8.1) in such a way that the edges $\mathsf{H}, \mathsf{V}, \mathsf{h}, \mathsf{v}$ are crossed by arrows pointing down, right, up and left, respectively, then we obtain a shift-commuting isomorphism between the *SFT* $Y = W_{Y_I}$ and the 'square ice' model, which consists of all configurations of arrows between horizontally or vertically adjacent points in $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ with the property that each lattice point has exactly two arrows pointing towards it, and two pointing away from it. Furthermore, if we represent $X^{(3)}$ in the form (7.3), then the map of colours

$$[\,0\ 1\,], [\,1\ 2\,], [\,2\ 0\,] \mapsto \mathsf{H}, \qquad [\,1\ 0\,], [\,2\ 1\,], [\,0\ 2\,] \mapsto \mathsf{h},$$
$$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right] \mapsto \mathsf{V}, \qquad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right] \mapsto \mathsf{v}$$

induces a continuous, surjective, open, three-to-one, shift-commuting map $\psi\colon X^{(3)} \longmapsto Y$ (cf. [19], [7] or Example 4.5 in [25]). The tiling group

$$\Gamma(T_I) = \langle \mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v} | \mathsf{HV} = \mathsf{VH},\ \ \mathsf{hv} = \mathsf{vh},\ \ \mathsf{hV} = \mathsf{Vh}, \\ \mathsf{Hv} = \mathsf{vH},\ \ \mathsf{hV} = \mathsf{vH},\ \ \mathsf{Hv} = \mathsf{Vh}\rangle \tag{8.2}$$

is easily checked to be isomorphic to $\mathbb{Z}^3$, with generators

$$\mathsf{H},\ \ \mathsf{V},\ \ \mathsf{H}^{-1}\mathsf{h} = \mathsf{V}^{-1}\mathsf{v}.$$

We write $c_{T_I}: \mathbb{Z}^2 \times Y \longmapsto \Gamma(T_I)$ for the tiling cocycle of $Y = W_{T_I}$ and obtain the following result.

Theorem 8.1. *Let G be a discrete group, and let $c\colon \mathbb{Z}^2 \times Y \longmapsto G$ be a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on the space $Y = W_{T_I}$. Then there exist a continuous map $b\colon Y \longmapsto G$ and a group homomorphism $\theta\colon \Gamma(T_I) \longmapsto G$ such that*

$$b(\sigma^{\mathbf{m}}(x))^{-1}c(\mathbf{m}, x)b(x) = \theta(c_{T_I}(\mathbf{m}, x))$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in Y$. In other words, the tiling cocycle c_{T_I} is fundamental (*cf. Definition* 2.3).

The proof of Theorem 8.1 is quite similar to that of Theorem 7.1. According to (4.5), the tiling cocycle c_{T_I} is given by

$$c((1,0),x) = \mathsf{b}(x_{\mathbf{0}}) \in \{\mathsf{h},\mathsf{H}\}, \qquad c((0,1),x) = \mathsf{l}(x_{\mathbf{0}}) \in \{\mathsf{v},\mathsf{V}\},$$

and we define a second cocycle $\gamma\colon \mathbb{Z}^2 \times Y \longmapsto \mathbb{Z}$ by setting

$$\gamma((1,0),x) = \begin{cases} 1 & \text{if } \mathsf{b}(x_{\mathbf{0}}) = \mathsf{H}, \\ -1 & \text{otherwise,} \end{cases} \qquad \gamma((0,1),x) = \begin{cases} 1 & \text{if } \mathsf{l}(x_{\mathbf{0}}) = \mathsf{V}, \\ -1 & \text{otherwise} \end{cases}$$

for every $x \in Y$.

Proposition 8.2. *The shift-action σ of $\mathbb{Z}^2$ on $Y = W_{T_I}$ is topologically mixing, and the skew-product transformation $\sigma_{(\gamma)}^{(1,0)}: Y \times \mathbb{Z} \longmapsto Y \times \mathbb{Z}$, defined as in Proposition* 6.3*, is topologically transitive, but not mixing.*

Proof. The first assertion is clear from Proposition 7.2 and the existence of the shift-commuting, surjective, three-to-one map $\psi\colon X^{(3)} \longmapsto Y$. The second assertion is equivalent to an elementary statement about the standard random walk on $\mathbb{Z}$ in which one moves at each step by ± 1. □

Lemma 8.3. *For every $L \geq 1$ and $-L \leq k \leq L$, put*

$$P_Y(1,L) = \{x \in Y : \sigma^{(L,0)}(x) = x\}, \\ P_Y^{(k)}(1,L) = \{y \in P_Y(1,L) : \gamma((L,0),y) = k\}.$$

Then $\sigma^{(0,1)}(P_Y^{(k)}(1,L)) = P_Y^{(k)}(1,L)$, and the SFT $(P_Y^{(k)}(1,L), \sigma^{(0,1)})$ *is irreducible and aperiodic whenever $P_Y^{(k)}(1,L) \neq \varnothing$.*

Proof. We write $\mathcal{A}(L) = \pi_{Q(L,1)}(P_Y(1,L)) \subset T_I^L \cong T_I^{Q(L,1)}$ for the alphabet of the *SFT* $(P_Y(1,L),\sigma^{(0,1)})$ (cf. (6.13)) and set, for every $a = (a_0,\dots,a_{L-1}) \in \mathcal{A}(L)$,

$$\mathsf{b}(a) = (\mathsf{b}(a_0),\dots,\mathsf{b}(a_{L-1})),$$
$$N(a) = |\{m : 0 \le m \le L-1 \text{ and } \mathsf{b}(a_m^{(0)}) = \mathsf{H}\}|.$$

Then

$$\{\mathsf{b}(a) : a \in \mathcal{A}(L)\} = \{\mathsf{h},\mathsf{H}\}^L,$$

and we put

$$\mathcal{A}(L,k) = \{a \in \mathcal{A}(L) : \gamma_L(a) = k\} = \{a \in \mathcal{A}(L) : 2N(a) - L = k\}$$

for every $k = -L,\dots,L$, where $\gamma_L(a) = \gamma((L,0),x)$ for every $a \in \mathcal{A}(L)$ and $x \in P_Y(1,L)$ with $\pi_{Q(L,1)}(x) = a$. As in the proof of Lemma 7.3 we call a string $(a^{(0)},\dots,a^{(l)}) \in \mathcal{A}(L,k)^{l+1}$ *allowed* if there exists a point $y \in P_X^{(k)}(1,L)$ with $\pi_{Q(L,1)}(\sigma^{(0,j)}(y)) = a^{(j)}$ for $j = 0,\dots,l$.

If we fix $a \in \mathcal{A}(L)$, then we can find elements $a' = (a'_0,\dots,a'_{L-1})$, $a'' = (a''_0,\dots,a''_{L-1})$ in $\mathcal{A}(L)$ such that (a,a') and (a,a'') are allowed,

$$\mathsf{l}(a'_0) = \mathsf{l}(a''_0) = \mathsf{r}(a'_{L-1}) = \mathsf{r}(a''_{L-1}) = \mathsf{v},$$
$$\mathsf{t}(a'_m) = \mathsf{b}(a'_m) \text{ for every } m = 0,\dots,L-1,$$
$$\alpha_{\mathsf{H}}(a'') \le \max(N(a),\alpha_{\mathsf{H}}(a)-1),$$

where

$$\alpha_{\mathsf{H}}(b) = \max\{m : 0 \le m \le L-1 \text{ and } \mathsf{t}(b_m) = \mathsf{H}\}$$

for every $b = (b_0,\dots,b_{L-1}) \in \mathcal{A}(L)$. Similarly we can find elements $b' = (b'_0,\dots,b'_{L-1})$, $b'' = (b''_0,\dots,b''_{L-1})$ in $\mathcal{A}(L)$ such that (a,b') and (a,b'') are allowed,

$$\mathsf{l}(b'_0) = \mathsf{l}(b''_0) = \mathsf{r}(b'_{L-1}) = \mathsf{r}(b''_{L-1}) = \mathsf{V},$$
$$\mathsf{t}(b'_m) = \mathsf{b}(b'_m) \text{ for every } m = 0,\dots,L-1,$$
$$\alpha_{\mathsf{h}}(b'') \le \max(L - N(a),\alpha_{\mathsf{h}}(a)-1),$$

where

$$\alpha_{\mathsf{h}}(b) = \max\{m : 0 \le m \le L-1 \text{ and } \mathsf{t}(b_m) = \mathsf{h}\}$$

for every $b = (b_0,\dots,b_{L-1}) \in \mathcal{A}(L,k)$. By applying this observation repeatedly we can find, for every $a^{(0)} \in \mathcal{A}(L,k)$, an allowed string $(a^{(-m')},\dots,a^{(0)},\dots a^{(m)})$ such that

$$a^{(-m')} = a^{(m)} = a^* = (a_0^*,\dots,a_{L-1}^*)$$

with

$$\mathsf{b}(a^*) = (\mathsf{b}(a_0^*),\dots,\mathsf{b}(a_{L-1}^*)) = (\mathsf{H},\dots,\mathsf{H},\mathsf{h},\dots\mathsf{h})$$

and $\mathsf{l}(a_i^*) = \mathsf{r}(a_i^*) = \mathsf{V}$ for every $i = 0,\dots,L-1$. Since the string (a^*,a^*) is allowed we have proved the lemma. □

As in the proof of Theorem 7.1 we have to verify an analogue of Lemma 7.4. Let $r = 2\max(r(c(1,0),\cdot)), r(c((0,1),\cdot)),1)$, and let $\mathcal{O} \subset Y$ be the cylinder corresponding to the allowed tiling of the square $[-r,r]^2 \subset \mathbb{R}^2$ in which we only use the tiles and , with the tile covering $[0,1]^2$ (the set $\mathcal{O} \subset Y$ is the image under the shift-commuting, open map $\psi \colon X^{(3)} \longmapsto Y$

described at the beginning of this section of the open subset $\mathcal{O} \subset X^{(3)}$ appearing in (7.12)). For every $L \geq 1$ and $k \in \mathbb{Z}$ we set

$$\begin{aligned} Z(L) &= \{x \in P_Y(1, L) : \sigma^{(0,2n)}(x) \in \mathcal{O} \text{ for every } n \in \mathbb{Z}\}, \\ Z(L,k) &= \{x \in Z(L) : \gamma((L,0),x) = k\} = Z(L) \cap P_Y^{(k)}(1,L). \end{aligned} \tag{8.3}$$

Then

$$\sigma^{(0,2)}(Z(L,k)) = Z(L,k)$$

for every L, k. The proof of Lemma 8.3 can be modified easily to yield the following result.

Lemma 8.4. *If $Z(L,k) \neq \varnothing$ then $(Z(L,k), \sigma^{(0,2)})$ is an irreducible and aperiodic* SFT *with alphabet $\pi_{Q(L,2)}(Z(L,k))$ (cf. (6.13)).*

Proof of Theorem 8.1. A slight adaptation of the proof of Theorem 7.1 shows that the set

$$\{\gamma((L,0),y) : L \geq 1, y \in Z(L)\}$$

generates an abelian subgroup of G with generators h_1, h such that

$$c((L,0),x) = h_1^L h^{\gamma((L,0),x)}$$

whenever $L \geq 1$ and $x \in Z(L)$. Furthermore there exists a continuous map $b \colon Y \longmapsto G$ and an element $h_2 \in G$ such that the group H generated by h_1, h_2, h is abelian and

$$b(\sigma^{\mathbf{m}}(x))^{-1} c(\mathbf{m}, x) b(x) = h_1^{m_1} h_2^{m_2} h^{\gamma(\mathbf{m},x)}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in Y$. We denote by $\theta \colon \Gamma(T_I) \longmapsto G$ the homomorphism with $\theta(\mathsf{H}) = h_1$, $\theta(\mathsf{V}) = h_2$ and $\theta(\mathsf{H}^{-1}\mathsf{h}) = h$ and obtain the assertion of the theorem. $\square$

Remark 8.5. The factor map $\psi \colon X^{(3)}$ allows us to define a cocycle

$$c(\mathbf{m}, x) = c_{T_I}(\mathbf{m}, \psi(x))$$

for the shift-action σ of $\mathbb{Z}^2$ on the three-coloured chessboards with values in $\Gamma(T_I) \cong \mathbb{Z}^3$. According to Theorem 7.1, there exist elements $h_1, h_2, h \in \Gamma(T_I)$ such that c is continuously cohomologous to the cocycle

$$c'(\mathbf{m}, x) = h_1^{(m_1+m_2-\gamma'(\mathbf{m},x))/2} h_2^{m_2} h^{\gamma'(\mathbf{m},x)}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $x \in X^{(3)}$, where $\gamma' \colon \mathbb{Z}^2 \times X^{(3)} \longmapsto \mathbb{Z}$ is defined in (7.9). A possible choice of h_1, h_2, h in terms of the generators $\mathsf{H}, \mathsf{V}, \mathsf{H}^{-1}\mathsf{h}$ of $\Gamma(T_I)$ is

$$h_1 = 1_{\Gamma(T_I)}, \;\; h_1 = 1_{\Gamma(T_I)}, \;\; h = \mathsf{H}^{-3}\mathsf{h}^3.$$

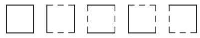
Remark 8.6. By using the fundamental cocycle $c_{T_I} \colon \mathbb{Z}^2 \times Y \longmapsto \mathbb{Z}^3$ to calculate the group $\Gamma_{\bar{y}}(c_{T_I}) \cong \pi_1^{\text{proj}}(Y, \bar{y})$ in Theorem 5.5 and Remark 5.6 we obtain that

$$\pi_1^{\text{proj}}(Y, \bar{y}) \cong \pi_1^{\text{proj}}(Y, *) \cong \Gamma_*(c_{T_I}) \cong \Gamma_{\bar{y}}(c_{T_I}) \cong \mathbb{Z}$$

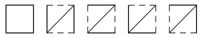
for every $\bar{y} \in Y$, as in [7].

9. Lozenges

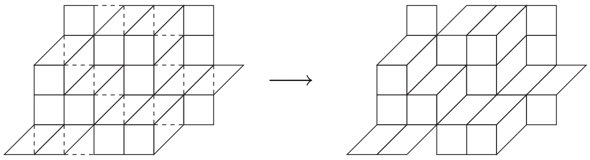
Denote by T_L the set of Wang tiles

 (9.1)

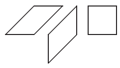
and write $X = W_{T_L}$ for the corresponding Wang shift. In Example 4.6 in [25] it was pointed out that W_{T_L} is isomorphic to the set of *lozenge tilings* discussed in [30]: re-draw the tiles (9.1) as



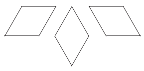
and delete the broken edges in the corresponding tilings as in the following picture:



The resulting patterns correspond to the set of all tilings of the plane by the tiles

 ,

which are, in turn, obtained by shearing the usual lozenges

 .

A comparison of (9.1) with (8.1) shows that $W_{T_L} \subset W_{T_I}$ is the set of all tilings which do not contain any translate of the tile , and that the Wang shift W_{T_I} is again a *SFT* with the periodic extension property. By using the same colours on the edges as in (8.1) we obtain the tiling group

$$\begin{aligned} \Gamma(T_L) = \langle \mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v} | \mathsf{HV} = \mathsf{VH},\ \ \mathsf{hV} = \mathsf{Vh},\ \ \mathsf{Hv} = \mathsf{vH}, \\ \mathsf{hV} = \mathsf{vH},\ \ \mathsf{Hv} = \mathsf{Vh} \rangle = \Gamma(T_I) \cong \mathbb{Z}^3. \end{aligned} \tag{9.2}$$

The tiling cocycle c_{T_L} is the restriction of $c_{T_I} \colon \mathbb{Z}^2 \times W_{T_I} \longmapsto \Gamma(T_I)$ to $\mathbb{Z}^2 \times W_{T_L}$. With a proof essentially identical to that of Theorem 8.1 we obtain:

Theorem 9.1. *Let G be a discrete group, and let $c \colon \mathbb{Z}^2 \times Y \longmapsto G$ be a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on the space $X = W_{T_L}$. Then there exist a continuous map $b \colon X \longmapsto G$ and a group homomorphism $\theta \colon \Gamma(T_L) \longmapsto G$ such that*

$$b(\sigma^{\mathbf{m}}(x))^{-1} c(\mathbf{m}, x) b(x) = \theta(c_{T_L}(\mathbf{m}, x))$$

for every $\mathbf{m} \in \mathbb{Z}^2$ and $x \in X$. In other words, the tiling cocycle c_{T_L} is fundamental (*cf. Definition* 2.3).

Remark 9.2. By using the fundamental cocycle c_{T_L} to calculate the fundamental group $\Gamma_{\bar{x}}(c_{T_L}) \cong \pi_1^{\mathrm{proj}}(X, \bar{x})$ in Theorem 5.5 and Remark 5.6 we obtain that

$$\pi_1^{\mathrm{proj}}(X, \bar{x}) \cong \pi_1^{\mathrm{proj}}(X, *) \cong \Gamma_*(c_{T_L}) \cong \Gamma_{\bar{x}}(c_{T_L}) \cong \mathbb{Z}$$

for every $\bar{x} \in X$ (cf. [7]).

10. Examples with trivial cohomology

Let $d \geq 2$, A a finite set and $X \subset A^{\mathbb{Z}^d}$ a *SFT*. A sufficient condition for cohomological triviality (cf. Definition 5.7) of X was given in Theorem 3.2 in [25]; here we present a simplified version of this condition which can be verified quite easily in specific examples.

Definition 10.1. Let $d \geq 2$, A a finite set, and let $X \subset A^{\mathbb{Z}^d}$ be a *SFT* satisfying (2.2)–(2.3).

(1) The set

$$\Delta_X = \{(x, x') \in X \times X : x_{\mathbf{n}} \neq x'_{\mathbf{n}} \text{ for only finitely many } \mathbf{n} \in \mathbb{Z}^d\} \subset X \times X$$

is the *Gibbs equivalence relation* of X.

(2) For every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ and every $r \geq 0$ we set

$$\begin{aligned} \langle \mathbf{m} \rangle &= \min_{i=1,\dots,d} |m_i|, \\ \mathbf{Q}(\mathbf{m}) &= \prod_{i=1}^{m} \{0, \dots, m_i\} \subset \mathbb{Z}^d, \\ \partial \mathbf{Q}(\mathbf{m})^{(r)} &= \{\mathbf{n} = (n_1, \dots, n_d) \in \mathbf{Q}(\mathbf{m}) : \\ &\qquad n_i \in \{0, \dots, r\} \cup \{m_i - r, \dots, m_i\} \\ &\qquad\qquad \text{for some } i = 1, \dots, d\}. \end{aligned} \tag{10.1}$$

For $\mathbf{m} \in \mathbb{N}^d$ and $r \geq 1$ with $\langle \mathbf{m} \rangle > r$ we call an element $z \in A^{\partial \mathbf{Q}(\mathbf{m})^{(r)}}$ *allowed* if $\pi_{\mathbf{k}+F}(z) \in \pi_{\mathbf{k}+F}(X) = \pi_F(X)$ for every $\mathbf{k} \in \mathbb{Z}^d$ with $\mathbf{k} + F \subset \partial \mathbf{Q}(\mathbf{m})^{(r)}$, where $F = \{0,1\}^d \subset \mathbb{Z}^d$. The *SFT* X has the *box extension property* if there exist integers $m^* > r^* \geq 0$ such that we can find, for every $r \geq 0$, every $\mathbf{m} \in \mathbb{N}^d$ with $\langle \mathbf{m} \rangle \geq m^*$, and every allowed element $z \in \partial \mathbf{Q}(\mathbf{m})^{(r^*+r)}$, an element $x \in X$ with $\pi_{\partial \mathbf{Q}(\mathbf{m})^{(r)}}(x) = \pi_{\partial \mathbf{Q}(\mathbf{m})^{(r)}}(z)$.

Remark 10.2. If a *SFT* $X \subset A^{\mathbb{Z}^d}$ has the box extension property then it also has the extension property. It follows that $X(r, i) = \Pi_X(\mathbf{S}(r, i))$, and that $(X(r, i), T_{(r,i)})$ is a mixing *SFT* for every $r \geq m^*$ and $i \in \{1, \dots, d\}$ (cf. Definition 2.1).

The following result is closely related to (and in fact a simplified version of) Theorem 3.2 in [25]. Its proof is left to the reader.

Theorem 10.3. *Let $d > 1$, A a finite set, and let $X \subset A^{\mathbb{Z}^d}$ be a mixing* SFT *with the box extension property. Then X is cohomologically trivial* (*cf. Definition* 5.7).

Remark 10.4. For $d > 2$ the box extension property in Definition 10.1 is unnecessarily strong. Indeed, if $d > 2$, A is a finite set and $X \subset A^{\mathbb{Z}^d}$ a mixing *SFT* of the form (2.2)–(2.3) with the extension property, then X is cohomologically trivial whenever the projection $Y = \pi_{\mathbf{H}(r)}(X)$ of X onto the coordinates in $\mathbf{H}(r)$ is, for every $r \geq 1$, a two-dimensional *SFT* with the box extension property. Here

$$\mathbf{H}(r) = \{\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d : |n_i| \leq r \text{ for } i = 3, \dots, d\}.$$

We illustrate Theorem 10.3 with a list of examples, most of which are taken from [25].

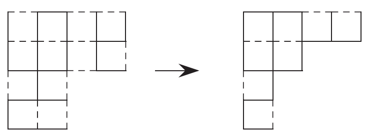
Example 10.5 (The full shift)**.** Let $d \geq 2$, A a finite set, and let $X = A^{\mathbb{Z}^d}$ be the d-dimensional *full shift* with alphabet A. As X has the box extension property with $m^* = 1, r^* = 0$, every continuous cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ with values in a discrete group G is trivial by Theorem 10.3 (cf. [10] and Example 4.1 in [25]).

Example 10.6 (Chessboards with $n \geq 4$ colours)**.** We define the n-coloured chessboards $X^{(n)}$ with $n \geq 3$ as in Section 7. If $n \geq 4$, then $X^{(n)}$ has the box extension property with $m^* = 8$ and $r^* = 0$ (cf. Example 4.4 in [25]), and is therefore cohomologically trivial.

Example 10.7 (The golden mean)**.** The d-dimensional golden mean (called the d-dimensional *hard core model* in [4]) is the subshift $X \subset \{0,1\}^{\mathbb{Z}^d}$ consisting of all configurations in which the 1's are isolated. In other words, X is the set of points $x = (x_{\mathbf{n}}) \in \{0,1\}^{\mathbb{Z}^d}$ with $x_{\mathbf{n}\pm\mathbf{e}^{(i)}} = 0$ for $i = 1, \dots, d$ whenever $x_{\mathbf{n}} = 1$. It is clear that X has the box extension property with $r^* = 0$, so that X is cohomologically trivial whenever $d > 1$ (cf. Theorem 10.3 and Example 4.7 in [25]).

Example 10.8 (The iceberg model)**.** Let M be a positive integer, put $A = \{-M, \dots, M\} \subset \mathbb{Z}$, and let $X \subset A^{\mathbb{Z}^2}$ be the *SFT* consisting of all $x \in A^{\mathbb{Z}^2}$ in which no positive coordinate is adjacent to a negative coordinate (cf. [4]). In other words, if $x \in X$, $\mathbf{n} \in \mathbb{Z}^2$ and $x_{\mathbf{n}} > 0$ ($x_{\mathbf{n}} < 0$) then $x_{\mathbf{m}} \geq 0$ ($x_{\mathbf{m}} \leq 0$) for all four neighbours $\mathbf{m}$ of $\mathbf{n}$. As in the preceding example one sees immediately that X has the box extension property with $r^* = 0$ and is therefore cohomologically trivial.

Example 10.9 (Long dominoes)**.** Consider the set of Wang tiles (6.1), augmented by the tiles . The resulting Wang shift X consists of all coverings of $\mathbb{R}^2$ by 'dominoes' of arbitrary length. The *SFT* X has the box extension property with $m^* = 5$ and $r^* = 2$, since we have to remove or modify some tiles if, for example, the configuration in one of the inner corners of $\partial\mathbf{Q}(\mathbf{m})^{(r)}$ looks like the following picture:


.

By Theorem 10.3, X is cohomologically trivial. The same conclusion holds for the set X' of all tilings of $\mathbb{R}^2$ by dominoes of length 2 or 3, i.e. by integer translates of rectangles of the form $[0,2] \times [0,1]$, $[0,3] \times [0,1]$, $[0,1] \times [0,2]$ and $[0,1] \times [0,3]$.

Example 10.10 (Dominoes in three dimensions)**.** Let X be the set of all tilings of $\mathbb{R}^3$ by integer translates of copies of the three-dimensional 'dominoes' $[0,2] \times [0,1] \times [0,1]$, $[0,1] \times [0,2] \times [0,1]$ and $[0,1] \times [0,1] \times [0,2]$ in $\mathbb{R}^3$.

By translating each element of X by $(\frac{1}{2}, \frac{1}{2})$ we obtain a one-to-one correspondence between X and the set of all partitions of $\mathbb{Z}^3$ into 'dimers', where each dimer is a subset of $\mathbb{Z}^3$ consisting of exactly two adjacent lattice points.

We can represent X as a three-dimensional Wang shift of the form (2.2)–(2.3) by cutting each of the three 'dominoes' into two unit cubes with suitably coloured faces as in dimension 2 (cf. Section 6 and Remark 4.3) and obtain that X is a mixing Wang shift with the extension property. Since X has the two-dimensional box extension property described in Remark 10.4 it is cohomologically trivial.

11. Factors of shifts with trivial cohomology

Theorem 11.1. *Suppose that X and Y are topologically mixing d-dimensional SFT's, and that $\phi\colon X \longmapsto Y$ is a continuous, surjective, constant-to-one, open, shift-commuting map. If X is cohomologically trivial* (*cf. Theorem* 10.3) *then Y has a fundamental cocycle* (*cf. Definition* 2.3).

Proof. The proof of Corollary 3.5 in [25] yields a finite group G, a subgroup $H \subset G$, and a continuous, nontrivial cocycle $c\colon \mathbb{Z}^d \times X \longmapsto G$ such that the shift-action σ of $\mathbb{Z}^d$ on X is topologically conjugate to the skew-product action

$$\bar{\sigma}_{(c)}^{\mathbf{n}}(y, gH) = (\bar{\sigma}^{\mathbf{n}}(y), c(\mathbf{n}, y)gH) \tag{11.1}$$

of $\mathbb{Z}^d$ on $Y \times (G/H)$, where $\bar{\sigma}$ is the shift-action of $\mathbb{Z}^d$ on Y. Every continuous cocycle $c'\colon \mathbb{Z}^d \times Y \longmapsto G'$, where G' is a discrete group, induces a cocycle $\bar{c}'\colon \mathbb{Z}^d \times X \longmapsto G$ by $\bar{c}'(\mathbf{n}, x) = c'(\mathbf{n}, \phi(x))$, and $\bar{c}'$ is trivial by hypothesis. In particular there exists a continuous map $b\colon Y \times (G/H) \longmapsto G'$ and a homomorphism $\eta\colon \mathbb{Z}^d \longmapsto G'$ with

$$c'(\mathbf{n}, y) = b(\bar{\sigma}^{\mathbf{n}}(y), c(\mathbf{n}, y)gH)^{-1}\eta(\mathbf{n})b(y, gH)$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $(y, gH) \in Y \times (G/H)$. Fix an open set $\mathcal{O} \subset Y$ with $b(y, gH) = b(y', gH)$ for all $g \in G$ and $y, y' \in \mathcal{O}$, and modify b by a constant and η by the appropriate conjugation so that $b(y, H) = 1_{G'}$ for every $y \in \mathcal{O}$.

Since $\bar{\sigma}_{(c)}$ is topologically mixing we can find an integer $N \geq 1$ such that

$$\mathcal{O} \cap \bar{\sigma}^{-\mathbf{n}}(\mathcal{O}) \cap \{y \in Y : c(\mathbf{n}, y) \in gH\} \neq \varnothing$$

for every $g \in G$ and every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ with $\|\mathbf{n}\| = \max_{i=1,\dots,d} |n_i| \geq N$. In particular we can find, for every $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > N$, every $h \in G$, and every $i \in \{1, \dots, d\}$, elements $y \in \mathcal{O} \cap \bar{\sigma}^{-\mathbf{n}}(\mathcal{O})$ and $y' \in \mathcal{O} \cap \bar{\sigma}^{-\mathbf{n}-\mathbf{e}^{(i)}}(\mathcal{O})$ with $c(y, \mathbf{n}) = c(y', \mathbf{n} + \mathbf{e}^{(i)}) = h$. By varying g we see that

$$\begin{gathered} b(y, hgH)^{-1}\eta(\mathbf{n})b(y, gH), \\ b(y', hgH)^{-1}\eta(\mathbf{n} + \mathbf{e}^{(i)})b(y', gH) \end{gathered}$$

are constant in g for every $h \in G$. Hence

$$\begin{aligned} \eta(\mathbf{n})^{-1}b(y, hgH)b(y, hH)^{-1}\eta(\mathbf{n}) &= b(y, gH)b(y, H)^{-1}, \\ \eta(\mathbf{n} + \mathbf{e}^{(i)})^{-1}b(y, hgH)b(y, hH)^{-1}\eta(\mathbf{n} + \mathbf{e}^{(i)}) & \\ = \eta(\mathbf{n} + \mathbf{e}^{(i)})^{-1}b(y', hgH)&b(y', hH)^{-1}\eta(\mathbf{n} + \mathbf{e}^{(i)}) \\ = b(y', gH)b(y', H)^{-1} &= b(y, gH)b(y, H)^{-1} \end{aligned}$$

for every $y \in \mathcal{O}$ and $g, h \in G$. We conclude that $\eta(\mathbf{n})$ commutes with $b(y, hgH)b(y, hH)^{-1}$ for every $y \in \mathcal{O}$, $g, h \in G$, $\mathbf{n} \in \mathbb{Z}^d$, and that the map $h \mapsto b(y, hgH)b(y, hH)^{-1}$ from G to G' is, for any fixed $y \in \mathcal{O}$, constant in h, and equal to $\theta(g)^{-1} = b(y, gH)b(y, H)^{-1} = b(y, gH)$, say. The resulting map $\theta \colon G \longmapsto G'$ is a group homomorphism satisfying that

$$c'(\mathbf{n}, y) = \theta(c(\mathbf{n}, y))\eta(\mathbf{n}) = \eta(\mathbf{n})\theta(c(\mathbf{n}, y))$$

whenever $\mathbf{n} \in \mathbb{Z}^d$ and $y \in \mathcal{O} \cap \sigma^{-\mathbf{n}}(\mathcal{O})$, and by arguing as in the proof of Lemma 6.9 we conclude that c' is continuously cohomologous to the cocycle

$$(\mathbf{n}, y) \mapsto \theta \cdot c(\mathbf{n}, y)\eta(\mathbf{n}). \tag{11.2}$$

Put $G^* = G \times \mathbb{Z}^d$ and define a cocycle $c^* \colon \mathbb{Z}^d \times Y \longmapsto G^*$ by

$$c^*(\mathbf{n}, y) = (c(\mathbf{n}, y), \mathbf{n})$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $y \in Y$. Then (11.2) implies that the cocycle c^* is fundamental. $\square$

Remark 11.2. Suppose that the map $\phi \colon X \longmapsto Y$ in Theorem 11.1 is continuous, surjective, constant-to-one, shift-commuting, but not open (at present no examples of such maps ϕ are known — cf. [7] for a more detailed discussion). Then the shift-action σ of $\mathbb{Z}^d$ on X is Borel conjugate to a skew-product of the form (11.1) over the shift-action $\bar{\sigma}$ of $\mathbb{Z}^d$ on Y, but the cocycle c appearing in (11.1) is Borel, but not continuous. By repeating the argument of Theorem 11.1 we obtain the following result. *Let $c' \colon \mathbb{Z}^d \times Y \longmapsto G'$ be a continuous cocycle on Y with values in a discrete group G'. Then there exist a group homomorphism $\theta \colon G \longmapsto G'$ and a Borel map $b \colon Y \longmapsto G'$ such that, for every $\mathbf{n} \in \mathbb{Z}^d$ and every fully supported, shift-invariant, ergodic probability measure μ on Y,*

$$c'(\mathbf{n}, y) = b(\bar{\sigma}^{\mathbf{n}}(y))^{-1}\theta(c(\mathbf{n}, y))b(y) \text{ for } \mu\text{-a.e. } y \in Y.$$

Does Y have a fundamental cocycle? More generally, let X and Y be topologically mixing d-dimensional *SFT*'s, and let $\phi \colon X \longmapsto Y$ be a continuous, surjective, shift-commuting map. If X possesses a fundamental cocycle $c \colon \mathbb{Z}^d \times X \longmapsto G$, is the same true for Y?

Example 11.3 (Factors of the full shift)**.** Let $n \geq 2$, $A = \mathbb{Z}/n\mathbb{Z}$, $X = (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}^2}$, denote by σ the shift-action of $\mathbb{Z}^2$ on X, and let $\Xi \subset X$ be a finite, shift-invariant subgroup of X. Then σ induces a continuous $\mathbb{Z}^d$-action on $Y = X/\Xi$, and Example 5.2 (4) and Theorem 3.8 in [27] imply that $(Y, \bar{\sigma})$ can be represented as (or is topologically conjugate to) a *SFT* with some finite alphabet A, and the Theorems 10.3 and 11.1 imply that Y has a fundamental cocycle.

Example 11.4 (Factors of n-coloured chessboards)**.** Let $n \geq 3$, and let

$$X^{(n)} \subset \{0, \dots, n-1\}^{\mathbb{Z}^2}$$

be the *SFT* defined in Section 7. As was described in Example 4.5 in [25], there exist a *SFT* $Y^{(n)}$ and a continuous, surjective, n-to-one, shift-commuting, open map $\phi\colon X^{(n)} \longmapsto Y^{(n)}$: let T be the set of Wang tiles

$$T = \left\{ \boxed{\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix}} : 1 \le a, b, c, d \le n-1 \text{ and } a + b = c + d \pmod{n} \right\}$$

with $\mathsf{b}(\tau) = a$, $\mathsf{r}(\tau) = b$, $\mathsf{t}(\tau) = c$ and $\mathsf{l}(\tau) = d$ for every

$$\tau = \boxed{\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix}} \in T,$$

and define $\phi\colon X^{(n)} \longmapsto Y^{(n)} = W_T$ by

$$\phi^{(n)}(x)_{\mathbf{m}} = \boxed{\begin{smallmatrix} & c_{\mathbf{m}} & \\ d_{\mathbf{m}} & & b_{\mathbf{m}} \\ & a_{\mathbf{m}} & \end{smallmatrix}}$$

for every $x = (x_{\mathbf{n}}) \in X^{(n)}$ and $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$, where

$$\begin{aligned} a_{\mathbf{m}} &= x_{(m_1+1,m_2)} - x_{(m_1,m_2)} \pmod{n}, \\ b_{\mathbf{m}} &= x_{(m_1+1,m_2+1)} - x_{(m_1+1,m_2)} \pmod{n}, \\ c_{\mathbf{m}} &= x_{(m_1+1,m_2+1)} - x_{(m_1,m_2+1)} \pmod{n}, \\ d_{\mathbf{m}} &= x_{(m_1,m_2+1)} - x_{(m_1,m_2)} \pmod{n}. \end{aligned}$$

If $n = 3$, $Y^{(3)} = Y$ is the 'square ice' *SFT* of Section 8. The *SFT* $Y^{(n)}$ has a fundamental cocycle by Theorem 8.1 (if $n = 3$) or by Theorem 10.3 (if $n \ge 4$).

Example 11.5 (More factors of full shifts)**.** This variation of Example 11.4 is taken from [7]. Let G be a finite group, and let $X = G^{\mathbb{Z}^2}$ be the full shift with alphabet G. Put

$$T^{(G)} = \left\{ \boxed{\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix}} : a, b, c, d \in G \text{ and } cd = ba \right\}$$

with $\mathsf{b}(\tau) = a$, $\mathsf{r}(\tau) = b$, $\mathsf{t}(\tau) = c$ and $\mathsf{l}(\tau) = d$ for every

$$\tau = \boxed{\begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix}} \in T^{(G)},$$

and define $\phi\colon X \longmapsto Y^{(G)} = W_{T^{(G)}}$ by

$$\phi(x)_{\mathbf{m}} = \boxed{\begin{smallmatrix} & c_{\mathbf{m}} & \\ d_{\mathbf{m}} & & b_{\mathbf{m}} \\ & a_{\mathbf{m}} & \end{smallmatrix}}$$

for every $x = (x_{\mathbf{n}}) \in X = G^{\mathbb{Z}^2}$ and $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$, where

$$\begin{aligned} a_{\mathbf{m}} &= x_{(m_1+1,m_2)} x_{(m_1,m_2)}^{-1}, \\ b_{\mathbf{m}} &= x_{(m_1+1,m_2+1)} x_{(m_1+1,m_2)}^{-1}, \\ c_{\mathbf{m}} &= x_{(m_1+1,m_2+1)} x_{(m_1,m_2+1)}^{-1}, \\ d_{\mathbf{m}} &= x_{(m_1,m_2+1)} x_{(m_1,m_2)}^{-1}. \end{aligned}$$

The tiling cocycle $c_{T^{(G)}}\colon \mathbb{Z}^2 \times Y^{(G)} \longmapsto G$ of $W_{T^{(G)}} = Y^{(G)}$ is fundamental by Theorem 10.3.

If G is abelian and $H \subset X = G^{\mathbb{Z}^2}$ is the shift-invariant subgroup of fixed points of the shift-action σ of $\mathbb{Z}^2$ on X then $Y^{(G)} \cong X/H$ (cf. Example 11.3).

12. Ledrappier's example

In this section we consider a two-dimensional *SFT* introduced by F. Ledrappier in [18]. Consider the closed, shift-invariant subgroup

$$X = \{x = (x_{\mathbf{n}}) \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : \\ x_{(n_1,n_2)} + x_{(n_1+1,n_2)} + x_{(n_1,n_2+1)} = 0 \pmod 2 \\ \text{for every } (n_1,n_2) \in \mathbb{Z}^2\}, \tag{12.1}$$

and denote by σ the shift-action (2.1) of $\mathbb{Z}^2$ on X. Then σ is mixing, and every continuous cocycle $c\colon \mathbb{Z}^2 \times X \longmapsto G$ with values in a discrete, abelian group X is trivial, but there exist nontrivial cocycles for certain nonabelian, discrete groups G (Proposition 5.5 in [25]).

Unlike the examples in the preceding sections, this *SFT* has a fairly complicated first cohomology. However, all known continuous cocycles on X with values in discrete groups arise in a particular way, and if this list of known cocycles is exhaustive (which seems possible), then X possesses a fundamental cocycle as a consequence of Corollary 4.4.

Denote by $R = (\mathbb{Z}/2\mathbb{Z})[u_1^{\pm1}, u_2^{\pm1}]$ the ring of Laurent polynomials in the commuting variables u_1, u_2 with coefficients in $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$. We write every $f \in R$ as $f = \sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) u^{\mathbf{n}}$ with $c_f(\mathbf{n}) \in \mathbb{Z}/2\mathbb{Z}$ and $u^{\mathbf{n}} = u_1^{n_1} u_2^{n_2}$ for every $\mathbf{n} = (n_1,n_2) \in \mathbb{Z}^2$, and with $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n}$. For every $f \in R$ we define a group homomorphism $f(\sigma)\colon X \longmapsto X$ by

$$f(\sigma) = \sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) \sigma^{\mathbf{n}}$$

and set

$$K_f = \ker f(\sigma) = \{x \in X : f(\sigma)(x) = 0_X\}. \tag{12.2}$$

Then $f(\sigma)\colon X \longmapsto X$ is surjective if and only if f is not divisible by $1+u_1+u_2$ or, equivalently, if f does not lie in the ideal $\mathfrak{p} = (1+u_1+u_2)R$; if $f \in \mathfrak{p}$ then $f(\sigma)(X) = \{0_X\}$ (cf. [17] and equation (5.9) in [27]).

Lemma 12.1. *Let $Y \subsetneq X$ be a closed, shift-invariant subgroup, and let $\sigma_{X/Y}$ be the $\mathbb{Z}^2$-action induced by σ on X/Y. The following conditions are equivalent.*

1. *There exists a continuous group isomorphism $\phi\colon X/Y \longmapsto X$ such that $\phi \cdot \sigma_{X/Y}^{\mathbf{n}} = \sigma^{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^2$;*
2. *$Y = K_f$ for some $f \in R \smallsetminus \mathfrak{p}$.*

Proof. This lemma is essentially contained in [17] (cf. also Remark 31.4 in [27]), and we restrict ourselves to a brief outline of the proof. Identify R with the dual group of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ by setting, for every $f \in R$ and $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$,

$$\langle f, x\rangle = e^{\pi i (f(\sigma)(x))_{\mathbf{0}}} = e^{\pi i \sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) x_{\mathbf{n}}}.$$

Then

$$X^{\perp} = \{f \in R : \langle f, x\rangle = 1 \ \text{for every} \ x \in X\} = \mathfrak{p},$$

and hence $\hat{X} = R/\mathfrak{p}$. Furthermore, the automorphism of $\hat{X} = R/\mathfrak{p}$ dual to $\sigma^{\mathbf{n}}$ is multiplication by $u^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^2$.

If $Y \subsetneq X$ is a closed, shift-invariant subgroup, then $\mathfrak{q} = Y^\perp \subset R$ is a subgroup which is invariant under multiplication by monomials, hence an ideal which strictly contains $\mathfrak{p}$, and $\widehat{X/Y} = \mathfrak{q}/\mathfrak{p}$.

Suppose that $\phi\colon X/Y \longmapsto X$ is a group isomorphism satisfying (1). Then the dual isomorphism $\hat{\phi}\colon R/\mathfrak{p} \longmapsto \mathfrak{q}/\mathfrak{p}$ commutes with multiplication by $u^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^2$ and is thus an R-module isomorphism. In particular, if $f \in R$ satisfies that $f + \mathfrak{p} = \hat{\phi}(1 + \mathfrak{p})$, then $f \notin \mathfrak{p}$, $\mathfrak{q} = fR + (1 + u_1 + u_2)R$, and $\hat{\phi}$ consists of multiplication by f. By translating this back to X we see that (1) and (2) are equivalent. □

Lemma 12.2. *For every $f \in R \smallsetminus \mathfrak{p}$ there exists a polynomial*

$$\psi_f = 1 + c_1 u_1 + \dots + c_{l-1} u_1^{l-1} + u_1^l \in (\mathbb{Z}/2\mathbb{Z})[u_1] \subset R$$

with the following properties.

(1) $\psi_f(1) = 1$;
(2) $K_{\psi_f} = K_f$;
(3) *If $\mathbf{Z} = \{(n,0) : n \in \mathbb{Z}\} \subset \mathbb{Z}^2$, and if $W = \pi_{\mathbf{Z}}(Y) \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbf{Z}}$ is the image of Y under projection onto the coordinates in $\mathbf{Z}$, then*

$$W = \ker \psi_f(\bar{\sigma}) \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbf{Z}},$$

where $\bar{\sigma}$ is the shift on $(\mathbb{Z}/2\mathbb{Z})^{\mathbf{Z}}$ defined by $\bar{\sigma}(z)_{(k,0)} = z_{(k+1,0)}$, and where

$$\psi_f(\bar{\sigma}) = \sum_{k \in \mathbb{Z}} c_{\psi_f}(k,0)\bar{\sigma}^k = \mathrm{id}_W + c_1\bar{\sigma} + \dots + c_{l-1}\bar{\sigma}^{l-1} + \bar{\sigma}^l;$$

(4) *There exists, for every $(a_0, \dots, a_{l-1}) \in (\mathbb{Z}/2\mathbb{Z})^l$, and for every $x \in X$, a unique element $y \in X$ with*

$$\begin{gathered} \sigma^{\mathbf{m}} \cdot f(\sigma)(y) = \psi_f(\sigma)(y) = x \text{ for some } \mathbf{m} \in \mathbb{Z}^2, \\ y_{(k,0)} = a_k \text{ for } k = 0, \dots, l-1. \end{gathered}$$

Proof. Let $f \in R \smallsetminus \mathfrak{p}$. After multiplying f by a monomial we may assume that f is a polynomial, and by replacing each power of u_2 by the corresponding power of $(1 + u_1)$ we obtain a polynomial $\psi \in (\mathbb{Z}/2\mathbb{Z})[u_1]$ with

$$\psi R + \mathfrak{p} = fR + \mathfrak{p}. \tag{12.3}$$

We multiply ψ by a power of u_1^{-1}, if necessary, and assume that in addition $\psi(0) = 1$. If $\psi(1) = 0$ the sum of the coefficients of ψ is even, and ψ is divisible by a power of $1 + u_1$. By setting $\psi = (1 + u_1)^l \psi_f$ with ψ_f not divisible by $1 + u_1$ we obtain a polynomial $\psi_f \in (\mathbb{Z}/2\mathbb{Z})[u_1] \subset R$ which satisfies the conditions (1) and (12.3).

As a consequence of (12.3) (with ψ_f replacing ψ) we know that $K_{\psi_f} = K_f$. Define $W \subset (\mathbb{Z}/2\mathbb{Z})^{\mathbf{Z}}$ as in (3) and observe that there exists, for every $w \in W$, a unique point $y \in K_{\psi_f} = K_f$ with

$$\begin{gathered} \pi_{\mathbf{Z}}(y) = w, \\ \pi_{\mathbf{Z}}(\sigma^{\mathbf{m}} y) \in W \text{ for every } \mathbf{m} \in \mathbb{Z}^2. \end{gathered}$$

This proves (3).

Finally, if we regard ψ_f as an element of R, then the definition of ψ_f shows that $u^{\mathbf{m}}f-\psi_f \in \mathfrak{p}$ for some $\mathbf{m}\in\mathbb{Z}^2$, and hence that $\sigma^{\mathbf{m}}\cdot f(\sigma)(y)=\psi_f(\sigma)(y)$ for every $y\in X$. As $f(\sigma)$ is surjective, the same holds for $\psi_f(\sigma)$, and we choose, for a given $x\in X$, an element $y\in\psi_f^{-1}(\{x\})$. According to (3) there exists a unique $z\in K_{\psi_f}=K_f$ with $z_{(k,0)}=y_{(k,0)}$ for $k=0,\dots,l-1$, and by replacing y with $y-z$ we have proved (4). □

For every Laurent polynomial $f\in R\smallsetminus\mathfrak{p}$ with $K_f\neq\{0_X\}$ one can construct a continuous cocycle $c^{(f)}$ for the shift-action σ of $\mathbb{Z}^2$ on X with values in a finite group, and all known cocycles of σ on X arise in this manner. In order to describe $c^{(f)}$ we define the polynomial $\psi_f=1+c_1u_1+\dots+c_{l-1}u_1^{l-1}+u_1^l\in(\mathbb{Z}/2\mathbb{Z})[u_1]\subset R$ according to Lemma 12.2 and set

$$Y=K_f=K_{\psi_f}.$$

Lemma 12.2 (4) implies that there exists, for every $x\in X$ and $a=(a_0,\dots,a_{l-1})\in(\mathbb{Z}/2\mathbb{Z})^l$, a unique point $y=\kappa(a,x)\in X$ with

$$\begin{aligned}\psi_f(y)&=x,\\ y_{(k,0)}&=a_k\ \text{ for }\ k=0,\dots,l-1.\end{aligned}\tag{12.4}$$

For $a=\mathbf{0}=(0,\dots,0)$ the map

$$x\mapsto\kappa'(x)=\kappa(\mathbf{0},x)\tag{12.5}$$

is obviously a continuous group homomorphism from X into X. The first equation in (12.4) implies that

$$\kappa'\cdot\sigma^{\mathbf{n}}(x)-\sigma^{\mathbf{n}}\cdot\kappa'(x)\in K_{\psi_f}=K_f\tag{12.6}$$

for every $\mathbf{n}\in\mathbb{Z}^2$ and $x\in X$. Put

$$\begin{gathered}\Phi_f=\{\mathbf{n}\in\mathbb{Z}^2:\sigma^{\mathbf{n}}(y)=y\ \text{ for every }\ y\in K_f\},\\ H_f=\mathbb{Z}^2/\Phi_f,\qquad G_f=H_f\times K_f,\end{gathered}$$

and furnish G_f with the group operation

$$(\overline{\mathbf{m}},x)(\overline{\mathbf{n}},y)=(\overline{\mathbf{m}+\mathbf{n}},x+\sigma^{\mathbf{m}}(y))$$

for every $\mathbf{m},\mathbf{n}\in\mathbb{Z}^2$ and $x,y\in K_f$, where $\overline{\mathbf{k}}=\mathbf{k}+\Psi_f\in H_f$ for every $\mathbf{k}\in\mathbb{Z}^2$. Then the map $c^{(f)}\colon\mathbb{Z}^2\times X\longmapsto G_f$, defined by

$$c^{(f)}(\mathbf{n},x)=(\overline{\mathbf{n}},\kappa'\cdot\sigma^{\mathbf{n}}(x))-\sigma^{\mathbf{n}}\cdot\kappa'(x)),\tag{12.7}$$

is a continuous cocycle for the shift-action σ of $\mathbb{Z}^2$ on X.

Lemma 12.3. *For every $f\in R\smallsetminus\mathfrak{p}$ with $K_f\neq\{0_X\}$,*

$$\begin{aligned}c^{(f)}((1,0),x)&=(\overline{(1,0)},\beta_f(x_{(0,0)})),\\ c^{(f)}((0,1),x)&=(\overline{(0,1)},\beta'_f(x_{(0,0)}))\end{aligned}$$

for every $x\in X$, where $\beta_f,\beta'_f\colon\mathbb{Z}/2\mathbb{Z}\longmapsto K_f$ are nonzero group homomorphisms.

Proof. A calculation shows that

$$\kappa' \cdot \sigma^{(1,0)}(x) = \sigma^{(1,0)} \cdot \kappa'(x)$$

if and only if $x_0 = 0$ (cf. (12.5)–(12.6)), so that

$$\ker(\kappa' \cdot \sigma^{(1,0)} - \sigma^{(1,0)} \cdot \kappa') = \{x \in X : x_{(0,0)} = 0\}.$$

This shows that $c^{(f)}((1,0),\cdot)$ is a function of $x_{(0,0)}$. Similarly one sees that

$$\sigma^{(0,1)} \cdot \kappa'(x) = \kappa' \cdot \sigma^{(0,1)}(x)$$

if and only if $x_{(0,0)} = 0$, that

$$\ker(\kappa' \cdot \sigma^{(0,1)} - \sigma^{(0,1)} \cdot \kappa') = \{x \in X : x_{(0,0)} = 0\},$$

and that $c^{(f)}((0,1),\cdot)$ is a function of $x_{(0,0)}$. □

Theorem 12.4. *There exists a continuous cocycle* $c^*\colon \mathbb{Z}^2 \times X \longmapsto G^*$ *with values in a discrete group* G^* *such that, for every* $f \in R \setminus \mathfrak{p}$*, the cocycle* $c^{(f)}\colon \mathbb{Z}^2 \times X \longmapsto G_f$ *in* (12.7) *is of the form* $c^{(f)} = \theta_f \cdot c^*$ *for some group homomorphism* $\theta_f\colon G^* \longmapsto G_f$*.*

Proof. According to Lemma 12.3, the cocycles $c^{(f)}$ all depend only on the coordinates in $\mathbf{B}(0)$ (cf. (4.9)), and the proof of Theorem 4.2 shows that each $c^{(f)}$ is a homomorphic image of $c_{T_X^{(1)}}$. □

Problem 12.5. Is the cocycle c^* fundamental? More generally, does every higher-dimensional mixing *SFT* possess a fundamental cocycle? The answer to this question is probably no, but I don't have an explicit counterexample. It is, however, not difficult to construct topologically transitive, but nonmixing two-dimensional *SFT*'s without fundamental cocycles (cf. e.g. Theorem 13.1).

13. One-dimensional shifts of finite type

Theorem 13.1. *Let* A *be a finite set and* $X \subset A^{\mathbb{Z}}$ *an aperiodic and mixing* SFT*. Then there is no fundamental cocycle for the shift-action* σ *of* $\mathbb{Z}$ *on* X*.*

Proof. Corollary 4.4 shows that it suffices to find continuous cocycles on X which depend essentially on arbitrarily many coordinates.

Assume without loss of generality that $A = \{0, \dots, n-1\}$, that X is of the form (2.2)–(2.3), and that the element $0 \in A$ has two distinct successors i^*, j^* (i.e. that $i^* \neq j^*$ and $(1,i^*), (1,j^*) \in \pi_{\{0,1\}}(X)$).

Let $M, N \geq 2$ be chosen so that there exist allowed strings

$$(i_0, \dots, i_N, \dots, i_{M+N-2}),$$
$$(j_0, \dots, j_N, \dots, j_{M+N-2})$$

in A^{M+N-1} with $i_0 = j_0$, $i_N \neq j_N$, and $i_{M+N-2} = j_{M+N-2} = 0$ (it is clear that there exist arbitrarily large integers M with this property). Denote by

G the free abelian group generated by the finite set $\pi_{\{0,\dots,M-1\}}(X) \subset A^M$ and define a continuous cocycle $c\colon \mathbb{Z} \times X \longmapsto G$ by setting

$$c(1,x) = h(x) = (x_0, \dots, x_{M-1}) \in G$$

for every $x \in X$.

We recall the definition of the Gibbs relation

$$\Delta_X = \{(x,y) \in X^2 : x_n = y_n \text{ whenever } |n| \text{ is sufficiently large}\}$$

in (3.3) and define a cocycle $a_h\colon \Delta_X \longmapsto G$ by setting

$$a_h(z,z') = \sum_{k \in \mathbb{Z}} h(\sigma^k(z)) - h(\sigma^k(z'))$$

for every $(z,z') \in \Delta_X$ (cf. (3.6)). Note that, if $c'\colon \mathbb{Z} \times X \longmapsto G$ is continuously cohomologous to c and $h' = c'(1,\cdot)$, then

$$a_{h'}(z,z') = a_h(z,z')$$

for every $(z,z') \in \Delta_X$.

Choose points $x, y, x', y' \in X$ with the following properties:

$$\begin{gathered} x_k = x'_k = y_k = y'_k \text{ for } k \le 0, \\ x_k = y_k \text{ and } x'_k = y'_k \text{ for } k \ge M+N-2, \\ x_{M+N-2} = y_{M+N-2} = x'_{M+N-2} = y_{M+N-2} = 0, \\ x_{M+N-1} = y_{M+N-1} = i^* \text{ and } x'_{M+N-1} = y'_{M+N-1} = j^*, \end{gathered}$$

where i^* and j^* are the distinct successors of 0 mentioned at the beginning of this proof. Since $(x,y), (x',y') \in \Delta_X$, $a_h(x,y)$ and $a_h(x',y')$ are well defined, and a direct calculation shows that

$$a_h(x,y) \ne a_h(x',y'). \tag{13.1}$$

If c were continuously cohomologous to a cocycle $c'\colon \mathbb{Z} \times X \longmapsto G$ for which $h' = c'(1,\cdot)$ depends only on $M-1$ successive coordinates (e.g. the coordinates $0, \dots, M-2$), we would obtain that

$$a_h(x,y) = a_{h'}(x,y) = a_{h'}(x',y') = a_h(x',y'),$$

contrary to (13.1). As explained above, this shows that there is no fundamental cocycle for the shift-action of $\mathbb{Z}$ on X. $\square$

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