COMMON HOMOCLINIC POINTS OF COMMUTING TORAL AUTOMORPHISMS

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Abstract. The points homoclinic to 0 under a hyperbolic toral automorphism form the intersection of the stable and unstable manifolds of 0. This is a subgroup isomorphic to the fundamental group of the torus. Suppose that two hyperbolic toral automorphisms commute so that they determine a $\mathbb{Z}^2$-action, which we assume is irreducible. We show, by an algebraic investigation of their eigenspaces, that they either have exactly the same homoclinic points or have no homoclinic point in common except 0 itself. We prove the corresponding result for a compact connected abelian group, and compare the two proofs.

1. Introduction

Let $n \geq 2$, and let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a linear ergodic automorphism of the n-torus $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. A point $x \in X$ is *homoclinic* for A if $\lim_{|n|\to\infty} \delta(A^n x, 0) = 0$, where δ is the usual Euclidean metric on X. If the automorphism A is nonhyperbolic then A has no nonzero homoclinic points (cf. e.g. [4]). If A is hyperbolic we can describe the set $\Delta_A \subset X$ of homoclinic points of A as follows: let A act linearly on $\mathbb{R}^n$ and denote by S_A and U_A the stable (= contracting) and unstable (= expanding) subspaces of A, i.e.

$$\begin{aligned} S_A &= \{w \in \mathbb{R}^n : \lim_{n\to\infty} A^n w = 0\}, \\ U_A &= \{w \in \mathbb{R}^n : \lim_{n\to\infty} A^{-n} w = 0\}, \\ \mathbb{R}^n &= S_A \oplus U_A. \end{aligned}$$

If $\pi\colon \mathbb{R}^n \longmapsto \mathbb{T}^n$ is the quotient map, then

$$\Delta_A = \{\pi(S_A \cap (U_A + \mathbf{m})) : \mathbf{m} \in \mathbb{Z}^n\},$$

and it is well-known and easy to see that Δ_A is a dense subgroup of X.

Now suppose that A, B are two commuting linear hyperbolic automorphisms of $X = \mathbb{T}^n$. We write

$$\alpha\colon \mathbf{n} = (n_1, n_2) \mapsto \alpha^{\mathbf{n}} = A^{n_1} B^{n_2} \tag{1.1}$$

for the $\mathbb{Z}^2$-action generated by A and B and conclude from Theorem 4.1 in [4] that α has no nonzero homoclinic point, i.e. that $x = 0$ is the only point in X with

$$\lim_{\mathbf{n}\to\infty} \alpha^{\mathbf{n}} x = 0.$$

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It may happen, however, that there exist nonzero points $x \in X$ which are homoclinic both for A and B. In fact, if the action α in (1.1) is *irreducible* in the sense that X has no proper subtorus which is invariant both under A and B, then we shall see that either $\Delta_A \cap \Delta_B = \{0\}$ or $\Delta_A = \Delta_B$; more generally, if $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$ are chosen so that $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ are hyperbolic, then either $\Delta_{\alpha^{\mathbf{m}}} = \Delta_{\alpha^{\mathbf{n}}}$ or $\Delta_{\alpha^{\mathbf{m}}} \cap \Delta_{\alpha^{\mathbf{n}}} = \{0\}$, and both possibilities occur for suitable choices of $\mathbf{m}$ and $\mathbf{n}$. If the $\mathbb{Z}^2$-action α is not irreducible then we can obviously not expect such a clear-cut dichotomy.

In order to make the above statements more precise we fix commuting linear hyperbolic automorphisms $A, B \in \mathrm{GL}(n, \mathbb{Z})$ of $\mathbb{T}^n$ and assume that the $\mathbb{Z}^2$-action α generated by A, B via (1.1) is irreducible. Then it is easy to see that A and B are simultaneously diagonalisable over the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers; in particular there exists a basis $w_1, \dots, w_n$ of $\overline{\mathbb{Q}}^n$ consisting of common eigenvectors of A and B (cf. Lemma 2.1). For every $k = 1, \dots, n$ we define a homomorphism η_k from $\mathbb{Z}^2$ into the multiplicative group of positive real numbers by setting $\eta_k(\mathbf{n}) = \|\alpha^{\mathbf{n}}(w_k)\|/\|w_k\|$ for every $\mathbf{n} \in \mathbb{Z}^2$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$ (in other words, $\eta_k(\mathbf{n})$ is the absolute value of the eigenvalue of $\alpha^{\mathbf{n}}$ for the eigenvector w_k). For $k = 1, \dots, n$ we set

$$H_k' = \{\mathbf{m} \in \mathbb{Z}^2 : |\eta_k(\mathbf{m})| = 1\}$$

and observe that H_k' is the intersection of $\mathbb{Z}^2$ with a hyperplane (cf. Remark 3.3) $H_k \subset \mathbb{R}^2$ and that

$$E_\alpha = \mathbb{Z}^2 \smallsetminus \bigcup_{k=1}^n H_k' = \{\mathbf{m} \in \mathbb{Z}^2 : \alpha^{\mathbf{m}} \text{ is hyperbolic}\} \tag{1.2}$$

is therefore nonempty. We shall prove the following result.

Theorem 1.1. *Let A, B be commuting linear hyperbolic automorphisms of a finite-dimensional torus $X = \mathbb{T}^n$, and let α be the $\mathbb{Z}^2$-action generated by A, B (cf. (1.1)). For every $\mathbf{m} = (m_1, m_2) \in E_\alpha$ (cf. (1.2)) we denote by $\Delta_{\alpha^{\mathbf{m}}}$ the group of homoclinic points of $\alpha^{\mathbf{m}}$. If $\mathbf{m}, \mathbf{n} \in E_\alpha$, then*

$$\Delta_{\alpha^{\mathbf{m}}} \cap \Delta_{\alpha^{\mathbf{n}}} = \begin{cases} \Delta_{\alpha^{\mathbf{m}}} & \text{if either } \mathbf{m}, \mathbf{n} \text{ or } \mathbf{m}, -\mathbf{n} \text{ are not separated} \\ & \quad \text{by any of the hyperplanes } H_k,\, k = 1, \dots, m, \\ \{0\} & \text{otherwise.} \end{cases}$$

Theorem 1.1 is a special case of a more general result for whose statement we need a few definitions. Let $\alpha \colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ be a $\mathbb{Z}^2$-action by continuous automorphisms of an infinite compact, connected, abelian group X. The action α is called *ergodic* if it is so with respect to the normalised Haar measure λ_X of X, *expansive* if there exists an open neighbourhood $N(0)$ of the identity element $0 \in X$ with $\bigcap_{\mathbf{n} \in \mathbb{Z}^2} \alpha^{\mathbf{n}}(N(0)) = \{0\}$, and *irreducible* or *almost minimal* if every closed, α-invariant subgroup $Y \subsetneq X$ is finite. Every expansive $\mathbb{Z}^2$-action by automorphisms of X is ergodic by Corollary 3.10 in [6].

Theorem 1.2. *Let α be an expansive, ergodic and irreducible $\mathbb{Z}^2$-action by automorphisms of a compact, connected, abelian group X. Put*

$$E_\alpha = \{\mathbf{m} \in \mathbb{Z}^2 : \alpha^{\mathbf{m}} \text{ is expansive}\}.$$

Then the following is true.

(1) *There exist finitely many hyperplanes $H_1, \dots, H_m$ in $\mathbb{R}^2 \supset \mathbb{Z}^2$ such that*

$$E_\alpha = \mathbb{Z}^2 \smallsetminus \bigcup_{k=1}^{m} H_k = \{\mathbf{m} \in \mathbb{Z}^2 : \alpha^{\mathbf{m}} \text{ is expansive}\}.$$

(2) *For every $\mathbf{m} \in \mathbb{Z}^2$, the group of homoclinic points $\Delta_{\alpha^{\mathbf{m}}}$ of $\alpha^{\mathbf{m}}$ is dense in X if and only if $\mathbf{m} \in E_\alpha$, and $\Delta_{\alpha^{\mathbf{m}}} = \{0\}$ otherwise.*

(3) *If $\mathbf{m}, \mathbf{n} \in E_\alpha$ then*

$$\Delta_{\alpha^{\mathbf{m}}} \cap \Delta_{\alpha^{\mathbf{n}}} = \begin{cases} \Delta_{\alpha^{\mathbf{m}}} & \text{if either } \mathbf{m}, \mathbf{n} \text{ or } \mathbf{m}, -\mathbf{n} \text{ are not separated} \\ & \quad \text{by any of the hyperplanes } H_k,\ k = 1, \dots, m, \\ \{0\} & \text{otherwise.} \end{cases}$$

If we call two elements $\mathbf{m}, \mathbf{n} \in E_\alpha$ *equivalent* when they are not separated by any of the hyperplanes H_k, then this relation partitions the set E_α into finitely many cones, each one of which is called a *Weyl chamber* of α in [2]–[3] (cf. also [1]). In this terminology Theorems 1.1–1.2 say that $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ share their homoclinic points if and only if $\mathbf{m}, \mathbf{n}$ or $\mathbf{m}, -\mathbf{n}$ lie in the same Weyl chamber of E_α; if not, then they have no nonzero common homoclinic point.

Although Theorem 1.1 is a consequence of Theorem 1.2 we shall prove it separately in Section 2 with a direct argument. The proof of Theorem 1.2 uses a little more algebraic machinery and will occupy Section 3.

2. The proof of Theorem 1.1

Let $A, B \in \mathrm{GL}(n, \mathbb{Z})$ be commuting linear hyperbolic automorphisms of $X = \mathbb{T}^n$ such that the resulting $\mathbb{Z}^2$-action on X is irreducible. We denote by $\mathbb{K} \subset \mathbb{C}$ the smallest subfield containing all eigenvalues of A and B.

Lemma 2.1. *The matrices A and B are simultaneously diagonalisable over $\mathbb{K}$, i.e. there exists a basis $w_1, \dots, w_n$ of $\mathbb{K}^n$ consisting of common eigenvectors of A and B (considered as acting linearly on $\mathbb{K}^n$).*

Proof. The assertion of the lemma is equivalent to the corresponding statement for the transposes $A^\top$ and $B^\top$ of A and B. The matrix $A^\top$ has an eigenvector in $\mathbb{K}^n$ with eigenvalue a_0, say, and the subspace $W = \{w \in \mathbb{K}^n : Aw = a_0 w\}$ is invariant under $B^\top$. Since $B^\top$ has an eigenvector in W there exists a *common* eigenvector $w \in \mathbb{K}^n$ of $A^\top$ and $B^\top$ with eigenvalues $a_0, b_0 \in \mathbb{K}$ under $A^\top$ and $B^\top$, respectively.

We write Γ for the Galois group of the extension $\mathbb{K} : \mathbb{Q}$ and let Γ act diagonally on $\mathbb{K}^m$ for every $m \geq 1$. For every $\gamma \in \Gamma$, $\gamma(w)$ is a common eigenvector of $A^\top$ and $B^\top$ with eigenvalues $(a, b) = (\gamma(a_0), \gamma(b_0))$. Put, for every $(a, b) \in E = \Gamma(a_0, b_0) \subset \mathbb{K}^2$,

$$V_{(a,b)} = \{w \in \mathbb{K}^n : A^\top w = aw, B^\top w = bw\} \neq \{0\} \tag{2.1}$$

and set

$$V = \bigoplus_{(a,b) \in E} V_{(a,b)}. \tag{2.2}$$

Then $\{0\} \neq V \subset \mathbb{K}^n$, and the lemma is proved if we can show that

$$V = \mathbb{K}^n. \tag{2.3}$$

Suppose that $V \neq \mathbb{K}^n$. We denote by $R_2 = \mathbb{Z}[u_1, u_2]$ the ring of polynomials with integral coefficients in the variables u_1, u_2, write $\mathfrak{p} = \{f \in R_2 : f(a,b) = 0\}$ for the ideal in R_2 consisting of all polynomials vanishing at (a,b), and set

$$\mathfrak{p}(A^\top, B^\top) = \{f(A^\top, B^\top) : f \in \mathfrak{p}\}.$$

Hilbert's Nullstellensatz implies that

$$V_{\mathfrak{p}} = \{(c_1, c_2) \in \mathbb{C}^2 : f(c_1, c_2) = 0 \text{ for every } f \in \mathfrak{p}\} = E$$

and hence that

$$\{0\} \neq V = \{w \in \mathbb{K}^n : Cw = 0 \text{ for every } C \in \mathfrak{p}(A^\top, B^\top)\} \neq \mathbb{K}^n.$$

Hence

$$\{0\} \neq \{w \in \mathbb{Q}^n : Cw = 0 \text{ for every } C \in \mathfrak{p}(A^\top, B^\top)\} \neq \mathbb{Q}^n,$$

and the annihilator of

$$M = \{w \in \mathbb{Z}^n : Cw = 0 \text{ for every } C \in \mathfrak{p}(A^\top, B^\top)\}$$

in $X = \widehat{\mathbb{Z}^n}$ is a proper subtorus of X which is invariant under A and B, contrary to our assumption of irreducibility.

This contradiction proves both (2.3) and the lemma. □

Lemma 2.2. *If an element $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^n$ is expressed as*

$$\mathbf{n} = \sum_{(a,b)\in E} w_{(a,b)}$$

with $w_{(a,b)} \in V_{(a,b)}$ for every $(a,b) \in E$ (cf. (2.1)–(2.3)), then $w_{(a,b)} \neq 0$ for every $(a,b) \in E$.

Proof. Suppose that $\sum_{(a,b)\in E} w_{(a,b)} = \mathbf{n} \in \mathbb{Z}^n$ with $w_{(a',b')} \neq 0$ and $w_{(a'',b'')} = 0$ for two eigenvalue pairs $(a',b'), (a'',b'') \in E$. We choose an element $\gamma \in \Gamma$ with $\gamma(a') = a'', \gamma(b') = b''$ and obtain that

$$\mathbf{0} = \sum_{(a,b)\in E} w_{(a,b)} - \gamma(w_{(a,b)})$$

has a nonzero component in $V_{(a'',b'')}$, which is absurd. □

This argument is a special case of that in [5].

Lemma 2.3. *For every homoclinic point $x \in \Delta_A \subset \mathbb{T}^n$ there exists a point $y \in (\mathbb{R} \cap \mathbb{K})^n$ with $\pi(y) = x$, where $\pi\colon \mathbb{R}^n \longmapsto \mathbb{T}^n$ is the quotient map.*

Proof. Let $a_1, \dots, a_r$ and $a_{r+1}, \dots, a_n$ be the expanding and contracting eigenvalues of A, counted with multiplicity, and put $A_s = \prod_{i=1}^r (A - a_i I)$, $A_u = \prod_{i=r+1}^n (A - a_i I)$, where I is the $n \times n$ identity matrix. Then $A_s S_A = A_u U_A = \{0\}$. Every $x \in \Delta_A$ is of the form $x = \pi(w)$ with $\{w\} = S_A \cap (U_A + \mathbf{m})$ for some $\mathbf{m} \in \mathbb{Z}^n$. If we write $\mathfrak{a}_1, \dots \mathfrak{a}_n \in (\mathbb{R} \cap \mathbb{K})^n$ and $\mathfrak{b}_1, \dots, \mathfrak{b}_n \in$

$(\mathbb{R} \cap \mathbb{K})^n$ for the row vectors of A_s and A_u, respectively, then w is the unique solution in $\mathbb{R}^n$ of the inhomogeneous system of linear equations

$$\begin{aligned} \langle \mathfrak{a}_1, w\rangle = 0, \dots, \langle \mathfrak{a}_n, w\rangle = 0, \\ \langle \mathfrak{b}_1, w - \mathbf{m}\rangle = 0, \dots, \langle \mathfrak{b}_n, w - \mathbf{m}\rangle = 0, \end{aligned} \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. As the coefficients of the system (2.4) all lie in $\mathbb{K}$, we conclude that the system (2.4) also has a unique solution in $\mathbb{K}^n$, i.e. that $w \in (\mathbb{R} \cap \mathbb{K})^n$. □

Proof of Theorem 1.1. We let A, B act linearly on $\mathbb{R}^n$ and write $\mathbb{R}^n = S_A \oplus U_A = S_B \oplus U_B$ for the splittings of $\mathbb{R}^n$ into the stable (= contracting) and unstable (= expanding) subspaces of A and B. If $S_A = S_B$ and $U_A = U_B$ then it is clear that $\Delta_A = \Delta_B$. If $S_A = U_B$ and $U_A = S_B$ we can invert B and again arrive at the conclusion that $\Delta_A = \Delta_B$.

In any other case (after replacing B by B^{-1} and interchanging A and B, if necessary) we have

$$\{0\} \neq U_A \cap U_B \neq U_A,\ U_A + U_B \neq \mathbb{R}^n. \tag{2.5}$$

If a point $x \in \mathbb{T}^n$ is homoclinic for both A and B then there exist elements $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{Z}^n$ and $w \in \mathbb{R}^n$ such that

$$\{w\} = S_A \cap (U_A - \mathbf{p}) = (S_B - \mathbf{q}) \cap (U_B - \mathbf{r}). \tag{2.6}$$

By Lemma 2.3, $w \in (\mathbb{R} \cap \mathbb{K})^n$, so that we may replace the subspaces S_A, S_B, U_A, U_B in (2.5)–(2.6) by the subspaces

$$S'_A = \bigoplus_{\{(a,b)\in E : |a|<1\}} V_{(a,b)}, \qquad U'_A = \bigoplus_{\{(a,b)\in E : |a|>1\}} V_{(a,b)},$$

$$S'_B = \bigoplus_{\{(a,b)\in E : |b|<1\}} V_{(a,b)}, \qquad U'_B = \bigoplus_{\{(a,b)\in E : |b|>1\}} V_{(a,b)}.$$

Then (2.6) shows that $\mathbf{q} \in S'_A + S'_B$ and $\mathbf{p} - \mathbf{r} \in U'_A + U'_B$. However, according to (2.5) there exist elements $(a, b), (a', b'), (a'', b'') \in E$ with $V_{(a,b)} \subset U'_A \cap U'_B$, $V_{(a',b')} \cap U'_A = V_{(a',b')} \cap U'_B = \{0\}$, $V_{(a'',b'')} \cap U'_B = \{0\}$, $V_{(a'',b'')} \subset U'_A$ and hence with $V_{(a,b)} \cap (S'_A + S'_B) = \{0\}$ and $V_{(a',b')} \cap (U'_A + U'_B) = \{0\}$, so that $\mathbf{q} = \mathbf{p} - \mathbf{r} = 0$ by Lemma 2.2. Then $w \in S'_A \cap S'_B$, $w + \mathbf{p} \in U'_A \cap U'_B$, and hence $\mathbf{p} \in (S'_A \cap S'_B) + (U'_A \cap U'_B)$. Since $V_{(a'',b'')} \cap ((S'_A \cap S'_B) + (U'_A \cap U'_B)) = \{0\}$, this is an expression of $\mathbf{p}$ as a sum of common eigenvectors of A, B with zero component in $V_{(a'',b'')}$. Lemma 2.2 yields that $\mathbf{p} = 0$, so that $w \in S'_A \cap U'_A = \{0\}$. □

3. The proof of Theorem 1.2

Let $\overline{\mathbb{Q}}^\times = \overline{\mathbb{Q}} \smallsetminus \{0\}$, $c = (c_1, c_2) \in (\overline{\mathbb{Q}}^\times)^2$, and let $\mathbb{K} = \mathbb{Q}(c)$ be the algebraic number field generated by c_1, c_2. We write $P^{\mathbb{K}}, P^{\mathbb{K}}_f$ and $P^{\mathbb{K}}_\infty$ for the sets of places (= equivalence classes of valuations), finite places and infinite places of $\mathbb{K}$, choose for each $v \in P^{\mathbb{K}}$ a valuation $|\cdot|_v$ in v, and denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to the valuation $|\cdot|_v$.

Proceeding as in Section 7 in [7] we set

$$P(c) = P^{\mathbb{K}}_\infty \cup \{v \in P^{\mathbb{K}}_f : |c_i|_v \neq 1 \text{ for some } i = 1, 2\}$$

and put

$$\begin{aligned} R_c &= \{a \in \mathbb{K} : |a|_v \le 1 \text{ for every } v \in P^{\mathbb{K}} \smallsetminus P(c)\}, \\ Z_c &= \prod_{v \in P(c)} \mathbb{K}_v, \qquad\qquad Y_c = Z_c / i_{P(c)}(R_c), \end{aligned} \tag{3.1}$$

where $i_{P(c)} \colon R_c \longmapsto Z_c$ is the diagonal embedding $a \mapsto (a_v,\, v \in P(c)) \in Z_c$ with $a_v = a$ for every $v \in P(c)$. Then $i_{P(c)}(R_c)$ is a discrete and co-compact subgroup of the locally compact abelian group Z_c and Y_c is (isomorphic to) the Pontryagin dual $\widehat{R_c}$ of R_c.

We define a $\mathbb{Z}^2$-action $\hat{\alpha}$ on R_c by setting

$$\hat{\alpha}^{\mathbf{n}}(a) = c^{\mathbf{n}} a$$

for every $a \in R_c$ and $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, where $c^{\mathbf{n}} = c_1^{n_1} c_2^{n_2}$. The $\mathbb{Z}^2$-action α_c on $Y_c \cong \widehat{R_c}$ dual to $\hat{\alpha}$ is given by

$$\alpha_c^{\mathbf{n}}((a_v,\, v \in P(c)) + i_{P(c)}(R_c)) = (c^{\mathbf{n}} a_v,\, v \in P(c)) + i_{P(c)}(R_c) \tag{3.2}$$

for every $\mathbf{n} \in \mathbb{Z}^2$ and $(a_v,\, v \in P(c)) + i_{P(c)}(R_c) \in Y_c$. As an application of Theorem 29.2 and Corollary 7.4 in [7] we obtain the following lemma.

Lemma 3.1. *Let β be an irreducible $\mathbb{Z}^2$-action by automorphisms of a compact, connected, abelian group X. Then there exist a point $c = (c_1, c_2) \in (\overline{\mathbb{Q}}^{\times})^2$ and continuous, surjective, finite-to-one group homomorphisms $\phi \colon Y_c \longmapsto X$, $\psi \colon X \longmapsto Y_c$ such that $\phi \cdot \alpha_c^{\mathbf{n}} = \beta^{\mathbf{n}} \cdot \phi$ and $\psi \cdot \beta^{\mathbf{n}} = \alpha_c^{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^2$.*

Proof. The only assertion which goes beyond the statements of Theorem 29.2 and Corollary 7.4 in [7] is the existence of the homomorphism $\psi \colon X \longmapsto Y_c$ with the required properties. The homomorphism $\phi \colon Y_c \longmapsto X$ allows us to regard $\hat{X}$ as a $\hat{\beta}$-invariant subgroup of finite index in R_c. Choose an integer $L \ge 1$ with $LR_c \subset \hat{X} \subset R_c$, denote by $\psi \colon X \longmapsto Y_c$ the surjective homomorphism dual to the inclusion $R_c \cong LR_c \subset \hat{X}$, and note that ψ is finite-to-one. □

Lemma 3.2. *Let β be an irreducible $\mathbb{Z}^2$-action by automorphisms of a compact, connected, abelian group X, and let α_c be the $\mathbb{Z}^2$-action defined in Lemma 3.1. The following conditions are equivalent.*

1. *β is expansive;*
2. *α_c is expansive;*
3. *The orbit of c under the diagonal action on $\mathbb{K}^2$ of the Galois group Γ of $\mathbb{K} : \mathbb{Q}$ does not contain any point $c' = (c_1', c_2')$ with $|c_1'| = |c_2'| = 1$.*

Proof. This is Proposition 7.2 (5) in [7]. □

Proof of Theorem 1.2. Let β be an expansive and irreducible $\mathbb{Z}^2$-action on a compact, connected abelian group X. Since the homomorphisms ϕ, ψ in Lemma 3.1 are finite-to-one, they are injective on the groups $\Delta_{\alpha_c^{\mathbf{n}}}$ and $\Delta_{\beta^{\mathbf{n}}}$ of homoclinic points for every $\mathbf{n} \in \mathbb{Z}^2$ for which $\beta^{\mathbf{n}}$ or, equivalently, $\alpha_c^{\mathbf{n}}$, is expansive. It follows that Theorem 1.2 holds for an expansive $\mathbb{Z}^2$-action β if and only if it holds for a corresponding action α_c. This observation allows us to assume without loss of generality that the $\mathbb{Z}^2$-action α in the statement of the theorem is of the form $\alpha = \alpha_c$ for some $c = (c_1, , c_2) \in (\overline{\mathbb{Q}}^{\times})^2$.

Equation (3.2) shows that an automorphism $\alpha^{\mathbf{n}} = \alpha_c^{\mathbf{n}}$ is expansive on $X = Y_c$ if and only if $|c^{\mathbf{n}}|_v \neq 1$ for every $v \in P(c)$. We write $\eta\colon Z_c \longmapsto Y_c$ for the quotient map and denote by

$$\begin{aligned} S_{\mathbf{n}} &= \{(a_v,\, v \in P(c)) \in Z_c : a_v = 0 \ \text{ for every } \ v \in P(c) \ \text{ with } \ |c^{\mathbf{n}}|_v > 1\}, \\ U_{\mathbf{n}} &= \{(a_v,\, v \in P(c)) \in Z_c : a_v = 0 \ \text{ for every } \ v \in P(c) \ \text{ with } \ |c^{\mathbf{n}}|_v < 1\} \end{aligned}$$

the stable and unstable subspaces of $\alpha^{\mathbf{n}}$ in Z_c. An element $x \in X$ lies in $\Delta_{\alpha^{\mathbf{n}}}$ if and only if it is of the form $x = \eta(w)$ for some $w \in S_{\mathbf{n}} \cap (U_{\mathbf{n}} + \mathbf{a})$ for some $\mathbf{a} \in i_{P(c)}(R_c)$.

We complete the proof in the same way as that of Theorem 1.1. For every $v \in P(c)$, the map $\mathbf{n} \mapsto |c^{\mathbf{n}}|_v$ is a homomorphism from $\mathbb{Z}^2$ into the multiplicative group of positive real numbers, and we set $H_v' = \{\mathbf{n} \in \mathbb{Z}^2 : |c^{\mathbf{n}}|_v = 1\}$. If α_c is expansive, Lemma 3.2 shows that each of these homomorphisms is nontrivial, so that H_v' is the intersection with $\mathbb{Z}^2$ of a hyperplane $H_v \subset \mathbb{R}^2$. It follows that the set

$$E = \mathbb{Z}^2 \smallsetminus \bigcup_{v \in P(c)} H_v' = \{\mathbf{n} \in \mathbb{Z}^2 : \alpha_c^{\mathbf{n}} \ \text{ is expansive}\}$$

is nonempty.

If $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$ satisfy that either $\mathbf{m}$ and $\mathbf{n}$ or $\mathbf{m}$ and $-\mathbf{n}$ lie on the same side of H_v for every $v \in P(c)$, then the homoclinic points of $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ coincide. If $\mathbf{m}$ and $\mathbf{n}$ do not satisfy this condition then we can interchange $\mathbf{m}$ and $\mathbf{n}$ and replace $\mathbf{n}$ by $-\mathbf{n}$, if necessary, and assume that $S_{\mathbf{m}} + S_{\mathbf{n}} \neq Z_c$, $S_{\mathbf{m}} + U_{\mathbf{n}} \neq Z_c$ and $U_{\mathbf{m}} + U_{\mathbf{n}} \neq Z_c$. If a point $x \in Y_c$ is homoclinic both for $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ then there exist elements $w \in Z_c$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in i_{P(c)}(R_c)$ with $\eta(w) = x$ and

$$\{w\} = S_{\mathbf{m}} \cap (U_{\mathbf{m}} - \mathbf{a}) = (S_{\mathbf{n}} - \mathbf{b}) \cap (U_{\mathbf{n}} - \mathbf{c})$$

and, exactly as in the proof of Theorem 1.1, we obtain that $\mathbf{a} = \mathbf{b} = \mathbf{c} = w = 0$. □

Remark 3.3. The group Y_c in (3.1) is a finite-dimensional solenoid; in fact, Y_c is a finite-dimensional torus if and only if each c_i is an algebraic unit (cf. [6] and [7]). In the latter case Theorem 1.2 reduces to Theorem 1.1, and it may be useful to compare the two proofs in this case. Since each c_i is an algebraic unit, $P(c) = P_\infty^{\mathbb{K}}$ and R_c is the ring of algebraic integers in $\mathbb{K} = \mathbb{Q}(c)$. Rewriting the covering space $\mathbb{R}^n$ in the proof of Theorem 1.1 as $Z_c = \prod_{v \in P(c)} \mathbb{K}_v$ amounts to a change of basis. In this new basis Lemma 2.3 is obvious, since every homoclinic point $x \in Y_c$ is the image under η of a point $w = (a_v,\, v \in P_\infty(c)) \in Z_c$ with

$$a_v = \begin{cases} a & \text{if } |c^{\mathbf{n}}|_v < 1, \\ 0 & \text{otherwise} \end{cases}$$

for some $a \in R_c$. Lemma 2.2 is equally obvious, since every element of $i_{P(c)}(R_c)$ has a nonzero component in each $\mathbb{K}_v$, $v \in P(c)$.

The proof of Theorem 1.2 in this section yields an explicit description of the walls H_v, $v \in P(c)$, of the Weyl chambers of the $\mathbb{Z}^2$-action α: they are the kernels of the homomorphism $(t_1, t_2) \mapsto |c_1|_v^{t_1} |c_2|_v^{t_2}$ from $\mathbb{R}^2$ to the multiplicative group of positive real numbers, corresponding to the places $v \in P(c)$. We illustrate this with some examples.

Examples 3.4. (1) Let A, B be commuting expansive automorphisms of the n-torus $\mathbb{T}^n$ for which the $\mathbb{Z}^2$-action α in (1.1) is irreducible. By comparing the proofs of the Theorems 1.1 and 1.2 we see that α is a finite-to-one factor of a $\mathbb{Z}^2$-action α_c for some $c = (c_1, c_2)$, where c_1 and c_2 are eigenvalues of A and B, respectively. We set $\mathbf{K} = \mathbb{Q}(c)$, write E for the orbit of c under the diagonal action of the Galois group, and note that each $(a, b) \in E$ corresponds to a place $v \in P_\infty^{\mathbf{K}}$ with $|c_1^{m_1} c_2^{m_2}|_v = |a^{m_1} b^{m_2}|$ for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$. The hyperplanes H_v, $v \in P(c) = P_\infty(c)$ are the kernels of the homomorphisms $(t_1, t_2) \mapsto |a|^{t_1} |b|^{t_2}$ from $\mathbb{R}^2$ to the multiplicative group of positive real numbers, with $(a, b) \in E$. Since these hyperplanes are lines with irrational slopes their intersection with $\mathbb{Z}^2$ is equal to $\{\mathbf{0}\}$, so that $\alpha^{\mathbf{n}}$ is expansive whenever $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^2$. The automorphisms $\alpha^{\mathbf{m}}, \alpha^{\mathbf{n}}$ corresponding to two elements $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$ have the same homoclinic points if and only if one of the line segments $t\mathbf{m} \pm (1 - t)\mathbf{n}$, $0 \leq t \leq 1$, does not intersect any H_v, $v \in P(c)$.

(2) Let $c = (2, 3)$. Then $\mathbb{K} = \mathbb{Q}$, $P(c) = \{\infty, 2, 3\}$ (where we are identifying the finite places of $\mathbb{Q}$ with the rational primes), $Z_c = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ (where $\mathbb{Q}_p$ denotes the p-adic integers), $R_c = \mathbb{Z}[\frac{1}{6}]$, and $Y_c = Z_c / i_{P(c)}(R_c)$ is the 6-adic solenoid. The hyperplanes H_v, $v \in P(c)$, are given by

$$\begin{aligned} H_\infty &= \{(t_1, t_2) \in \mathbb{R}^2 : 2^{t_1} 3^{t_2} = 1\}, \\ H_2 &= \{(t_1, t_2) \in \mathbb{R}^2 : t_1 = 0\}, \\ H_3 &= \{(t_1, t_2) \in \mathbb{R}^2 : t_2 = 0\}, \end{aligned}$$

and the set of expansive elements of the action α_c is given by

$$E = \{\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2 : n_1 n_2 \neq 0\}.$$

In particular, any two elements $\mathbf{m}, \mathbf{n}$ in the strictly positive quadrant of $\mathbb{Z}^2$ lead to automorphisms $\alpha^{\mathbf{m}}, \alpha^{\mathbf{n}}$ with the same homoclinic points, whereas the only homoclinic point common to $\alpha^{(1,1)}$ and $\alpha^{(1,-1)}$ is 0.

References

[1] M. Boyle and D. Lind, *Expansive subdynamics*, Trans. Amer. Math. Soc. 349 (1997), 55–102.

[2] A. Katok and R.J. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 131–156.

[3] A. Katok and R.J. Spatzier, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergod. Th. & Dynam. Sys. **16** (1996), 751–778.

[4] D. Lind and K. Schmidt, *Homoclinic points of algebraic $\mathbb{Z}^d$-actions*, Preprint Nr. 411, Erwin Schrödinger Institute, Vienna, 1996.

[5] A. Manning, *Irrationality of linear combinations of eigenvectors*, Proc. Indian Acad. Sci. Math. Sci. 105 (1995), 269–271.

[6] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480–496.

[7] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.

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